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Vector generation functions, q-spectral functions of hyperbolic geometry, and vertex operators for quantum affine algebras

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We investigate the concept of q-replicated argument in symmetric functions with its connection to spectral functions of hyperbolic geometry. This construction suffices for vector generation functions in the form of q-series and string theory. We hope that the mathematical side of the construction can be enriched by ideas coming from physics. *Published by AIP Publishing.* https://doi.org/10.1063/1.4994135

I. INTRODUCTION

In this paper, we discuss the multipartite (*vector*) generation functions and the vertex operators, and how it can be applied to derive some explicit results concerning the bosonic strings, symmetric functions, spectral functions of hyperbolic geometry, and the vertex operator traces.

A. The organization of the paper and our key results

In Sec. II, we begin with a brief review of the multipartite generation functions and derive some explicit results using the Bell polynomial technique.

We then turn to the polynomial ring $\Lambda(X)$ and its algebraic properties (Sec. III) and proceed to apply restricted specializations and q-series.

We consider a $2N$-piecewise string in Sec. IV A. A piecewise uniform bosonic string which consists of $2N$ parts of equal length, of alternating type I and type II material, is relativistic in the sense that the velocity of sound everywhere equals the velocity of light. The present section is a continuation of two earlier papers, one dealing with the Casimir energy of a $2N$-piece string[1] and the other dealing with the thermodynamic properties of a string divided into two (unequal) parts.[2]

Finally in Secs. IV B and IV C, we turn to symmetric functions with replicated variables and vertex operator traces in connection with the spectral functions of hyperbolic geometry. There have been interesting developments in physics, in string theory, and related subjects. During the last few years or so, the mathematical side has been greatly enriched by ideas from physics.

II. MULTIPARTITE GENERATING FUNCTIONS

For any ordered m-tuple or *multipartite* numbers of nonnegative integers (not all zeros), $(k_1, k_2, \ldots, k_m) = \overrightarrow{k}$, let us consider the (multi)partitions, i.e., distinct representations of $(k_1, k_2, \ldots, k_m)$ as sums of multipartite numbers. Let $\mathcal{C}_-^{(z,m)}(\overrightarrow{k}) = \mathcal{C}_-^m(z; k_1, k_2, \cdots, k_m)$ be the number of such multipartitions; in addition, introduce the symbol $\mathcal{C}_+^{(z,m)}(\overrightarrow{k}) = \mathcal{C}_+^{(m)}(z; k_1, k_2, \cdots, k_m)$. Their generating

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functions can be defined as[4]

$$\mathcal{F}(z;X) \stackrel{\text{def}}{=} \prod_{\vec{k}\geq 0} \left(1 - zx_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}\right)^{-1} = \sum_{\vec{k}\geq 0} \mathcal{C}_-^{(z,m)}(\vec{k})x_1^{k_1}x_2^{k_2}\cdots x_r^{k_m}, \tag{2.1}$$

$$\mathcal{G}(z;X) \stackrel{\text{def}}{=} \prod_{\vec{k}\geq 0} \left(1 + zx_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}\right) = \sum_{\vec{k}\geq 0} \mathcal{C}_+^{(z,m)}(\vec{k})x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}. \tag{2.2}$$

Then

$$\begin{aligned}\log \mathcal{F}(z;X) &= -\sum_{\vec{k}\geq 0} \log\left(1 - zx_1^{k_1}x_2^{k_2}\cdots x_r^{k_m}\right) = \sum_{\vec{k}\geq 0}\sum_{n=1}^{\infty} \frac{z^n}{n} x_1^{nk_1}x_2^{nk_2}\cdots x_m^{nk_m} \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n}(1-x_1^n)^{-1}(1-x_2^n)^{-1}\cdots(1-x_m^n)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n}\prod_{j=1}^{m}(1-x_j^n)^{-1}, \qquad (2.3) \\ \log \mathcal{G}(-z;X) &= \log \mathcal{F}(z;X). \qquad (2.4)\end{aligned}$$

Let $\beta_m(n) := \prod_{j=1}^m (1-x_j^n)^{-1}$, and finally

$$\mathcal{F}(z;X) = \sum_{\vec{k}\geq 0} \mathcal{C}_-^{(z,m)}(\vec{k})x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m} = \exp\left(\sum_{n=1}^{\infty}\frac{z^n}{n}\beta_m(n)\right), \tag{2.5}$$

$$\mathcal{G}(z;X) = \sum_{\vec{k}\geq 0} \mathcal{C}_+^{(z,m)}(\vec{k})x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m} = \exp\left(\sum_{n=1}^{\infty}\frac{(-z)^n}{n}\beta_m(n)\right). \tag{2.6}$$

A. The Bell polynomials

The Bell polynomials, first extensively studied by Bell (see, for example, Ref. 5), arise in the task of taking the nth derivative of a composite function. Namely, one can find a formula for the n-th derivative of h(t) = f(g(t)). If we denote $d^n h/dt^n = h_n$, $d^n f/dg^n = f_n$, and $d^n g/dt^n = g_n$, then we see that $h_1 = f_1, h_2 = f_1 g_2 + f_2 g_1^2, h_3 = f_1 g_3 + 3f_2 g_2 g_1 + f_3 g_1^3, \cdots$. By mathematical induction, we find that $h_n = f_1\alpha_{n1}(g_1, \ldots, g_n) + f_2\alpha_{n2}(g_1, \ldots, g_n) + \cdots + f_n\alpha_{nn}(g_1, \ldots, g_n)$, where $\alpha_{nj}(g_1, \ldots, g_n)$ is a homogeneous polynomial of degree j in $g_1, \ldots, g_n$.

As a result, the study of h_n may be reduced to the study of the *Bell polynomials*: $Y_n(g_1, g_2, \ldots, g_n) = \alpha_{n1}(g_1, \ldots, g_n) + \alpha_{n2}(g_1, \ldots, g_n) + \cdots + \alpha_{nn}(g_1, \ldots, g_n)$. Note that Y_n is a polynomial in n variables and the fact that g_j was originally a jth derivative is not necessary in the consideration.

Recurrence relations for the Bell polynomial $Y_n(g_1, g_2, \ldots, g_n)$ and generating function $\mathcal{B}(z)$ have the forms[4]

$$Y_{n+1}(g_1, g_2, \ldots, g_{n+1}) = \sum_{k=0}^{n}\binom{n}{k} Y_{n-k}(g_1, g_2, \ldots, g_{n-k})g_{k+1}, \tag{2.7}$$

$$\mathcal{B}(z) = \sum_{n=0}^{\infty} Y_n z^n/n! \Longrightarrow \log \mathcal{B}(z) = \sum_{n=1}^{\infty} g_n z^n/n!. \tag{2.8}$$

From the last formula, one can obtain the following explicit formula for the Bell polynomials (it is known as Faa di Bruno's formula):

$$Y_n(g_1, g_2, \ldots, g_n) = \sum_{\mathbf{k}\vdash n} \frac{n!}{k_1!\cdots k_n!}\prod_{j=1}^{n}\left(\frac{g_j}{j!}\right)^{k_j}. \tag{2.9}$$

If we let

$$\mathcal{F}(z;X) := 1 + \sum_{j=1}^{\infty} \mathcal{P}_j(x_1, x_2, \ldots, x_m) z^j, \qquad \mathcal{P}_j = 1 + \sum_{\vec{k}>0} P(\vec{k};j) x_1^{k_1} \cdots x_m^{k_m}, \tag{2.10}$$

$$\mathcal{G}(z;X) := 1 + \sum_{j=1}^{\infty} \mathcal{Q}_j(x_1, x_2, \ldots, x_r) z^j, \qquad \mathcal{Q}_j = 1 + \sum_{\vec{k}>0} Q(\vec{k};j) x_1^{k_1} \cdots x_m^{k_m}, \tag{2.11}$$

then the following result holds (see for details Ref. 4):

$$\mathcal{P}_j = \frac{1}{j!} Y_j \left(0!\beta_m(1),\ 1!\beta_m(2), \ldots,\ (j-1)!\beta_m(j)\right), \tag{2.12}$$

$$\mathcal{Q}_j = \frac{1}{(-1)^j j!} Y_j \left(-0!\beta_m(1),\ -1!\beta_m(2), \ldots,\ -(j-1)!\beta_m(j)\right). \tag{2.13}$$

B. Restricted specializations and q-series

Setting $X = (x_1, x_2, \ldots, x_r, 0, 0, \ldots) = (q, q^2, \ldots, q^r, 0, 0, \ldots)$ for finite additive manner, as a result, we get

$$\mathcal{F}(z;X) = \prod_{\vec{k}\geq 0} \left(1 - zq^{k_1+k_2+\cdots+k_r}\right)^{-1} = \exp\left(-\sum_{m=1}^{\infty} \frac{z^m}{m} \prod_{\ell=1}^{r} (1 - q^{\ell m})^{-1}\right), \tag{2.14}$$

$$\mathcal{G}(z;X) = \prod_{\vec{k}\geq 0} \left(1 + zq^{k_1+k_2+\cdots+k_r}\right) = \exp\left(-\sum_{m=1}^{\infty} \frac{(-z)^m}{m} \prod_{\ell=1}^{r} (1 - q^{\ell m})^{-1}\right). \tag{2.15}$$

1. Spectral functions of hyperbolic geometry

Let us begin by explaining the general lore for the characteristic classes and $\mathfrak{g}$-structure on compact groups.

Remark 2.1. Suppose $\mathfrak{g}$ is the Lie algebra of a Lie group G. Let us consider the pair (Γ, G) of Lie groups, where Γ is a closed subgroup of G with normalizer subgroup $N_\Gamma \subset G$. Then the pair (Γ, G) with the discrete quotient group N_Γ/Γ corresponds to the inclusion $\mathfrak{g} \hookrightarrow W_n$, where W_n is the Lie algebra of formal vector fields in $n = \dim G/\Gamma$ variables, while the homogeneous space G/Γ possesses a canonical $\mathfrak{g}$-structure ω. Combining this $\mathfrak{g}$-structure with the inclusion $\mathfrak{g} \hookrightarrow W_n$, one obtains a W_n-structure on the quotient space G/Γ for any discrete subgroup Γ of the Lie group G; this is precisely the W_n-structure which corresponds to the Γ-equivariant foliation of G by left cosets of Γ.[6] The homomorphism

$$\mathrm{char}_\omega : \Gamma^\sharp(W_n) \to \Gamma^\sharp(G/\Gamma, \mathbb{R}) \tag{2.16}$$

associated with characteristic classes of W_n-structures decomposes into the composition of two homomorphisms

$$\Gamma^\sharp(W_n) \to \Gamma^\sharp(\mathfrak{g}) \quad \text{and} \quad \Gamma^\sharp(\mathfrak{g}) \to \Gamma^\sharp(G/\Gamma, \mathbb{R}). \tag{2.17}$$

The first homomorphism is independent of Γ and is induced by the inclusion $\mathfrak{g} \hookrightarrow W_n$, while the second homomorphism is independent of Γ and corresponds to the canonical homomorphism that determines the characteristic classes of the canonical $\mathfrak{g}$-structure ω on G/Γ.

If the group G is semi-simple, then the Lie algebra $\mathfrak{g}$ is unitary and G contains a discrete subgroup Γ for which G/Γ is compact; for appropriate choice of Γ, the kernel of the homomorphism $\Gamma^\sharp(W_n) \to \Gamma^\sharp(G/\Gamma, \mathbb{R})$ coincides with the kernel of the homomorphism $\Gamma^\sharp(W_n) \to \Gamma^\sharp(\mathfrak{g})$.

In our applications, we shall consider a *compact hyperbolic* three-manifold G/Γ with $G = SL(2, \mathbb{C})$. By combining the characteristic class representatives of field theory, elliptic genera with the homomorphism char_ω, we can compute quantum partition functions in terms of the spectral functions of hyperbolic three-geometry.[7,8]

Let us introduce next the Ruelle spectral function $\mathcal{R}(s)$ associated with hyperbolic three-geometry.[7,8] The function $\mathcal{R}(s)$ is an alternating product of more complicated factors, each of which is the so-called Patterson-Selberg zeta-functions Z_{Γ^γ}; functions $\mathcal{R}(s)$ can be continued meromorphically to the entire complex plane $\mathbb{C}$,

$$\prod_{n=\ell}^{\infty}(1-q^{an+\varepsilon}) = \prod_{p=0,1} Z_{\Gamma^\gamma}(\underbrace{(a\ell+\varepsilon)(1-i\varrho(\vartheta))+1-a}_{s}+a(1+i\varrho(\vartheta)p)^{(-1)^p}$$
$$= \mathcal{R}(s=(a\ell+\varepsilon)(1-i\varrho(\vartheta))+1-a), \tag{2.18}$$

$$\prod_{n=\ell}^{\infty}(1+q^{an+\varepsilon}) = \prod_{p=0,1} Z_{\Gamma^\gamma}(\underbrace{(a\ell+\varepsilon)(1-i\varrho(\vartheta))+1-a+i\sigma(\vartheta)}_{s}+a(1+i\varrho(\vartheta)p)^{(-1)^p}$$
$$= \mathcal{R}(s=(a\ell+\varepsilon)(1-i\varrho(\vartheta))+1-a+i\sigma(\vartheta)), \tag{2.19}$$

where $q\equiv e^{2\pi i\vartheta}$, $\varrho(\vartheta)=\mathrm{Re}\,\vartheta/\mathrm{Im}\,\vartheta$, $\sigma(\vartheta)=(2\,\mathrm{Im}\,\vartheta)^{-1}$, a is a real number, and $\varepsilon, b\in\mathbb{C}$, $\ell\in\mathbb{Z}_+$.

Obviously, $\prod_{\ell=1}^{r}(1-q^{\ell m})^{-1}\equiv\prod_{\ell=1}^{\infty}(1-q^{\ell m})^{-1}\prod_{\ell=r+1}^{\infty}(1-q^{\ell m})$ and

$$\mathcal{F}(z;X) = \prod_{\vec{k}\geq 0}\left(1-zq^{k_1+k_2+\cdots+k_r}\right)^{-1}$$
$$= \exp\left(-\sum_{m=1}^{\infty}\frac{z^m}{m}\cdot\frac{\mathcal{R}(s=-im\varrho(\vartheta)(r+1)+mr+1)}{\mathcal{R}(s=-im\varrho(\vartheta)+1)}\right), \tag{2.20}$$

$$\mathcal{G}(z;X) = \prod_{\vec{k}\geq 0}\left(1+zq^{k_1+k_2+\cdots+k_r}\right)$$
$$= \exp\left(-\sum_{m=1}^{\infty}\frac{(-z)^m}{m}\cdot\frac{\mathcal{R}(s=-im\varrho(\vartheta)(r+1)+mr+1)}{\mathcal{R}(s=-im\varrho(\vartheta)+1)}\right). \tag{2.21}$$

2. Hierarchy

Setting $\mathcal{O}q^{k_0+k_1+\cdots+k_r}=\mathcal{O}_{\vec{k}}q^{k_0}$ with $\mathcal{O}_{\vec{k}}=\mathcal{O}q^{k_1+\cdots+k_r}$ $[\vec{k}=(k_1,\ldots,k_r)]$ we get

$$Z_2\left(\mathcal{O}_{\vec{k}},q\right)=\prod_{k_0=0}^{\infty}\left[1-\mathcal{O}_{\vec{k}}q^{k_0}\right]^{-1}=[(1-\mathcal{O}_{\vec{k}})\mathcal{R}(s=(k_1+\cdots+k_r)(1-i\varrho(\tau)))]^{-1}. \tag{2.22}$$

Therefore the infinite products $\prod_{k_r=0}^{\infty}\prod_{k_{r-1}=0}^{\infty}\cdots\prod_{k_1=0}^{\infty}\prod_{k_0=0}^{\infty}(1-q^{k_0+k_1+\cdots+k_r})^{-1}$ can be factorized as $\prod_{\vec{k}\geq\vec{0}}Z_2\left(\mathcal{O}_{\vec{k}},q\right)$. We can treat this factorization as a product of r copies, each of them is $Z_2\left(\mathcal{O}_{\vec{k}},q\right)$ and corresponds to a free two-dimensional conformal field theory (see Ref. 7 for similar results).

III. SYMMETRIC FUNCTIONS FOR QUANTUM AFFINE ALGEBRAS

A. The polynomial ring $\Lambda(X)$

Let $\mathbb{Z}[x_1,\ldots,x_n]$ be the polynomial ring, or the ring of formal power series, in n commuting variables $x_1,\ldots,x_n$. The symmetric group S_n acts on n letters of this ring by permuting the variables. For $\pi\in S_n$ and $f\in\mathbb{Z}[x_1,\ldots,x_n]$, we have $\pi f(x_1,\ldots,x_n)=f(x_{\pi(1)},\ldots,x_{\pi(n)})$. We are interested in the subring of functions invariant under this action, $\pi f=f$, that is to say the ring of symmetric polynomials in n variables: $\Lambda(x_1,\ldots,x_n)=\mathbb{Z}[x_1,\ldots,x_n]^{S_n}$. This ring may be graded by the degree of the polynomials so that $\Lambda(X)=\oplus_n\Lambda^{(n)}(X)$, where $\Lambda^{(n)}(X)$ consists of homogeneous symmetric polynomials in $x_1,\ldots,x_n$ of total degree n.

In order to work with an arbitrary number of variables, following Macdonald,[9] we define the ring of symmetric functions $\Lambda=\lim_{n\to\infty}\Lambda(x_1,\ldots,x_n)$ in its stable limit $(n\to\infty)$. There exist various bases of $\Lambda(X)$:

(i) $\mathbb{Z}$ bases for $\Lambda^{(n)}$ are provided by the monomial symmetric functions $\{m_\lambda\}$, where λ is any partition of n.

(ii) The other (integral and rational) bases for $\Lambda^{(n)}$ are provided by the partitions λ of n. There are the complete, elementary, and power sum symmetric functions: $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$, and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$, where for $\forall n \in \mathbb{Z}_+$,

$$h_n(X) = \sum_{i_1 \leq i_2 \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \qquad e_n(X) = \sum_{i_1 < i_2 \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \qquad p_n(X) = \sum_i x_i^n, \tag{3.1}$$

with the convention $h_0 = e_0 = p_0 = 1$ and $h_{-n} = e_{-n} = p_{-n} = 0$. Three of these bases are multiplicative, with $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$, and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$. The relationships between the various bases are mentioned at this stage by the transitions

$$p_\rho(X) = \sum_{\lambda \vdash n} \chi_\rho^\lambda s_\lambda(X) \quad \text{and} \quad s_\lambda(X) = \sum_{\rho \vdash n} \mathfrak{z}_\rho^{-1}\, \chi_\rho^\lambda\, p_\rho(X)\,. \tag{3.2}$$

For each partition λ, the Schur function is defined by

$$s_\lambda(X) \equiv s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) X^{\sigma(\lambda+\delta)}}{\prod_{i<j}(x_i - x_j)}, \tag{3.3}$$

where $\delta = (n-1, n-2, \ldots, 1, 0)$. In fact, both h_n and e_n are special Schur functions, $h_n = s_{(n)}$, $e_n = s_{(1^n)}$, and their generating functions are expressed in terms of the power-sum p_n,

$$\sum_{n\geq 0} h_n z^n = \exp(\sum_{n=1}^{\infty}(p_n/n) z^n), \qquad \sum_{n\geq 0} e_n z^n = \exp(-\sum_{n=1}^{\infty}(p_n/n)(-z)^n)\,. \tag{3.4}$$

The Jacobi-Trudi formula[9] expresses the Schur functions in terms of h_n or e_n: $s_\lambda = \det(h_{\lambda_i - i + j}) = \det(e_{\lambda' - i + j})$, where λ' is the conjugate of λ. An involution ω: $\Lambda \to \Lambda$ can be defined by $\omega(p_n) = (-1)^{n-1} p_n$. Then it follows that $\omega(h_n) = e_n$. Also we have $\omega(s_\lambda) = s_{\lambda'}$. χ_ρ^λ is the character of the irreducible representation of the symmetric groups S_n specified by λ in the conjugacy class specified by ρ. These characters satisfy the orthogonality conditions

$$\sum_{\rho \vdash n} \mathfrak{z}_\rho^{-1}\, \chi_\rho^\lambda\, \chi_\rho^\mu = \delta_{\lambda,\mu} \quad \text{and} \quad \sum_{\lambda \vdash n} \mathfrak{z}_\rho^{-1} \chi_\rho^\lambda \chi_\sigma^\lambda = \delta_{\rho,\sigma}\,. \tag{3.5}$$

The significance of the Schur function bases lies in the fact that with respect to the usual Schur-Hall scalar product $\langle \cdot \mid \cdot \rangle_{\Lambda(X)}$ on $\Lambda(X)$, we have

$$\langle s_\mu(X) \mid s_\nu(X) \rangle_{\Lambda(X)} = \delta_{\mu,\nu} \quad \text{and} \quad \text{therefore} \quad \langle p_\rho(X) \mid p_\sigma(X) \rangle_{\Lambda(X)} = \mathfrak{z}_\rho \delta_{\rho,\sigma}, \tag{3.6}$$

where $\mathfrak{z}_\lambda = \prod_i i^{m_i} m_i!$ for $\lambda = (1^{m_1}, 2^{m_2}, \cdots)$.

The ring, $\Lambda(X)$, of symmetric functions over X has a Hopf algebra structure, and two further algebraic, and two coalgebraic operations. For notation and basic properties, we refer the reader to Refs. 10 and 11 and references therein.

IV. SYMMETRIC FUNCTIONS OF A REPLICATED ARGUMENT

A. Example: 2N-piecewise string

In this section, we consider the bosonic *composite* string of length L in D-dimensional spacetime, which is assumed to be uniform and consists of two or more uniform pieces. Such a model was introduced in 1990.[1] The composite string was assumed to be divided into two pieces, of length L_I and L_{II}, and it was relativistic in the sense that the velocity of sound was everywhere required to be equal to the velocity of light. Various aspects of the relativistic piecewise uniform string model were studied in Ref. 2. In Ref. 3, scaling properties of the piecewise uniform string model were worked out. One may note, for instance, the paper of Lu and Huang[12] in which the model finds application in relation to the Green-Schwarz superstring.

The present paper focuses attention on the $2N$-piece string, made up of $2N$ parts of equal length, of alternating type I and type II material. The string of a total length L is relativistic; the velocity of sound is everywhere equal to the velocity of light $v_s = \sqrt{T_I/\rho_I} = \sqrt{T_{II}/\rho_{II}} = c$, where T_I and T_{II} are the tensions and ρ_I and ρ_{II} are the mass densities in the two pieces.

Our interest is the transverse oscillations $\psi = \psi(\sigma, \tau)$ of the string, where σ denoting as usual the position coordinate and τ the time coordinate of the string. Thus in the two regions, we have

$$\psi_I = \xi_I e^{i\omega(\sigma-\tau)} + \eta_I e^{-i\omega(\sigma+\tau)}, \qquad \psi_{II} = \xi_{II} e^{i\omega(\sigma-\tau)} + \eta_{II} e^{-i\omega(\sigma+\tau)}, \tag{4.1}$$

where ξ and η are appropriate constants. The junction conditions say that ψ and the transverse elastic force $T\partial\psi/\partial\sigma$ are continuous, i.e., at each of the $2N$ junctions

$$\psi_I = \psi_{II}, \qquad T_I \partial\psi_I/\partial\sigma = T_{II}\partial\psi_{II}/\partial\sigma\,. \tag{4.2}$$

Define $x \stackrel{\rm def}{=} T_I/T_{II}$ and also the symbols p_N and ϵ by $p_N \stackrel{\rm def}{=} \omega L/N$ and $\epsilon \stackrel{\rm def}{=} (1-x)/(1+x)$.

- The eigenfrequencies are determined from $\det(\mathbf{M}_{2N}(x, p_N) - 1\!\!1) = 0$. Here it is convenient to scale the resultant matrix $\boldsymbol{M}_{2N}$ as[13]

$$\mathbf{M}_{2N}(x, p_N) = \left(\frac{(1+x)^2}{4x}\right)^N \mathbf{m}_{2N}(\epsilon, p_N), \quad \mathbf{m}_{2N}(\epsilon, p_N) = \prod_{j=1}^{2N} \mathbf{m}^{(j)}(\epsilon, p_N), \tag{4.3}$$

$$\mathbf{m}^{(j)}(\epsilon, p_N) = \begin{pmatrix} 1, & \mp\epsilon e^{-ijp_N} \\ \mp\epsilon e^{ijp_N}, & 1 \end{pmatrix}, \tag{4.4}$$

for $j = 1, 2, \ldots (2N-1)$. The sign convention is to use $\pm$ for even/odd j.

- At the last junction, for $j = 2N$, the component matrix has a particular form (given an extra prime for clarity): $\mathbf{m}'_{2N}(\epsilon, p_N) = \begin{pmatrix} e^{-iNp_N}, & \epsilon e^{-iNp_N} \\ \epsilon e^{iNp_N}, & e^{iNp_N} \end{pmatrix}$. The recursion formula alluded to the above can be stated as

$$\mathbf{m}_{2N}(\epsilon, p_N) = \Omega^N(\epsilon, p_N), \qquad \Omega(\epsilon, p) = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \tag{4.5}$$

with $a = e^{-ip} - \epsilon^2$ and $b = \epsilon\,(e^{-ip} - 1)$. The obvious way to proceed now is to calculate the eigenvalues of Ω and express the elements of $\mathbf{M}_{2N}$ as powers of these. More details can be found in Ref. 13.

- Assume that $L = \pi$, in conformity with usual practice. Thus $p_N = \pi\omega/N$. We let $X^\mu(\sigma, \tau)$, with $\mu = 0, 1, 2, \ldots, (D-1)$ and specify the coordinates on the world sheet. For each of the eigenvalue branches, we can write X^μ on the form

$$X^\mu = x^\mu + \frac{p^\mu\tau}{\pi T_0} + X_I^\mu, \quad \text{region I}, \tag{4.6}$$

$$X^\mu = x^\mu + \frac{p^\mu\tau}{\pi T_0} + X_{II}^\mu, \quad \text{region II}, \tag{4.7}$$

where x^μ is the centre–of–mass position, p^μ is the total momentum of the string, and $T_0 = \frac{1}{2}(T_I + T_{II})$ is the mean tension. Further, X_I^μ and X_{II}^μ can be decomposed into oscillator coordinates,

$$X_I^\mu = \frac{i}{2}\ell_s \sum_{n\neq 0} \frac{1}{n}\left(\alpha_{nI} e^{i\omega(\sigma-\tau)} + \tilde{\alpha}_{nI} e^{-i\omega(\sigma+\tau)}\right), \tag{4.8}$$

$$X_{II}^\mu = \frac{i}{2}\ell_s \sum_{n\neq 0} \frac{1}{n}\left(\alpha_{nII} e^{i\omega(\sigma-\tau)} + \tilde{\alpha}_{nII} e^{-i\omega(\sigma+\tau)}\right). \tag{4.9}$$

Here ℓ_s is the fundamental string length, unspecified so far, and $\alpha_n, \tilde{\alpha}_n$ are oscillator coordinates of the right- and left-moving waves, respectively. A characteristic property of the composite string is that the oscillator coordinates have to be specified for each of the various branches.

A significant simplification can be obtained if, following Ref. 2, we limit ourselves to the case of extreme string rations only. It is clear that the eigenvalue spectrum has to be invariant under the transformation $x \to 1/x$. It is sufficient, therefore, to consider the tension ratio interval $0 < x \leq 1$ only. The case of extreme tensions corresponds to $x \to 0$. We consider only this case in the following.

Assume that the case $x \to 0$ corresponds to $T_I \to 0$. Also as $\epsilon \to 1$ and $\lambda_- = 0$, $\lambda_+ = \cos p_N - 1$ (the case of extreme tensions[2]). We obtain the remarkable simplification that all the eigenfrequency branches degenerate into one single branch determined by $\cos p_N = 1$. Thus the eigenvalue spectrum becomes $\omega_n = 2Nn$, $n = \pm 1, \pm 2, \pm 3, \ldots$. The junction conditions (4.2) permit all waves to propagate from region I to region II.

1. Oscillator coordinates in 2N-piecewise model

The case $x \to 0$ gives the equations $\xi_I + \eta_I = 2\xi_{II} = 2\eta_{II}$, which show that the right- and left-moving amplitudes ξ_I and η_I in region I can be chosen freely and the amplitudes ξ_{II} and η_{II} in region II are fixed. This means, in oscillator language, that α_n^μ and $\tilde{\alpha}_n^\mu$ can be chosen freely. Choosing the fundamental length equal to $\ell_s = (\pi T_I)^{-1/2}$, we can write the expansion (4.8) and (4.9) in both regions as (subscripts I and II on α_n's are omitted)

$$X_I^\mu = \frac{i}{2\sqrt{\pi T_I}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{2iNn(\sigma-\tau)} + \tilde{\alpha}_n^\mu e^{-2iNn(\sigma+\tau)}\right), \tag{4.10}$$

$$X_{II}^\mu = \frac{i}{2\sqrt{\pi T_I}} \sum_{n\neq 0} \frac{1}{n} \gamma_n^\mu e^{-2iNn\tau} \cos(2Nn\sigma), \tag{4.11}$$

where we have defined γ_n^μ as $\gamma_n^\mu = \alpha_n^\mu + \tilde{\alpha}_n^\mu, \quad n \neq 0$.

Recall that the string action is

$$S = -(1/2) \int d\tau d\sigma T(\sigma) \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \tag{4.12}$$

where $\alpha, \beta = 0, 1$ and $T(\sigma) = T_I$ in region I and $T(\sigma) = T_{II}$ in region II. The momentum conjugate to X^μ is $P^\mu(\sigma) = T(\sigma)\dot{X}^\mu$, and the Hamiltonian is accordingly

$$H = \int_0^\pi \left(P_\mu(\sigma)\dot{X}^\mu - \mathcal{L}\right) d\sigma = (1/2) \int_0^\pi T(\sigma)(\dot{X}^2 + X'^2) d\sigma, \tag{4.13}$$

where $\mathcal{L}$ is the Lagrangian. Some care has to be taken for the string constraint equation. In the classical theory for the uniform string, the constraint equation reads $T_{\alpha\beta} = 0$, $T_{\alpha\beta}$ being the energy-momentum tensor. However the situation is here more complicated since the junctions restrict the freedom, and one has to take the variations δX^μ. Thus we have to replace the strong condition $T_{\alpha\beta} = 0$ by a weaker condition. The most natural choice, which we will adopt, is to impose that $H = 0$ when applied to the physical states.

Let us introduce light cone coordinates $\sigma^- = \tau - \sigma$ and $\sigma^+ = \tau + \sigma$. The derivatives conjugate to $\sigma^\mp$ are $\partial_\mp = \frac{1}{2}(\partial_\tau \mp \partial_\sigma)$,

region I:

$$\partial_- X^\mu = \frac{N}{\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \alpha_n^\mu e^{2iNn(\sigma-\tau)}, \quad \partial_+ X^\mu = \frac{N}{\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \tilde{\alpha}_n e^{-2iNn(\sigma+\tau)}. \tag{4.14}$$

region II:

$$\partial_\mp X^\mu = \frac{N}{2\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \gamma_n^\mu e^{\pm 2in(\sigma\mp\tau)}, \tag{4.15}$$

where we have defined $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{p^\mu}{NT_{II}}\sqrt{\frac{T_I}{\pi}}$, $\gamma_0^\mu = 2\alpha_0^\mu$. Inserting these expressions into the Hamiltonian, we obtain[2]

$$H = \frac{1}{2} N^2 \sum_{-\infty}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) + \frac{N^2}{4x} \sum_{-\infty}^{\infty} \gamma_{-n} \cdot \gamma_n. \tag{4.16}$$

The momentum conjugate to X^μ is at any position on the string equal to $T(\sigma)\dot{X}^\mu$. We accordingly require the commutation rules

$$T_I[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = -i\delta(\sigma-\sigma')\eta^{\mu\nu}, \quad \text{region } I, \tag{4.17}$$

$$T_{II}[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = -i\delta(\sigma-\sigma')\eta^{\mu\nu}, \quad \text{region } II, \tag{4.18}$$

$\eta^{\mu\nu}$ being the D-dimensional flat metric. The other commutators vanish. The quantities to be promoted to Fock state operators are $\alpha_{\mp n}$ and $\gamma_{\mp n}$. We insert the expansions for X^μ and $\dot{X}^\mu$ for regions I and II and make use of the effective relationship $\sum_{n=-\infty}^{\infty} e^{2iNn(\sigma-\sigma')} = 2\sum_{n=-\infty}^{\infty}\cos 2Nn\sigma\cos 2Nn\sigma' \longrightarrow \frac{\pi}{N}\delta(\sigma-\sigma')$. We then get $[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n+m,0}\eta^{\mu\nu}$, region I (with a similar relation for $\tilde{\alpha}_n$), and $[\gamma_n^\mu, \gamma_m^\nu] = 4nx\,\delta_{n+m,0}\,\eta^{\mu\nu}$, region II.

Introduce annihilation and creation operators by

$$\alpha_n^\mu = \sqrt{n}\,a_n^\mu, \quad \alpha_{-n}^\mu = \sqrt{n}\,a_n^{\mu\dagger}, \quad \gamma_n^\mu = \sqrt{4nx}\,c_n^\mu, \quad \gamma_{-n}^\mu = \sqrt{4nx}\,c_n^{\mu\dagger}, \tag{4.19}$$

and find for $n \geq 1$ the standard form

$$[a_n^\mu, a_m^{\nu\dagger}] = \delta_{nm}\eta^{\mu\nu}, \quad [c_n^\mu, c_m^{\nu\dagger}] = \delta_{nm}\eta^{\mu\nu}. \tag{4.20}$$

Note that infinite dimensional Heisenberg algebras play a central role in applying symmetric function techniques to various problems in mathematical physics. Such algebra is generated by operators $\{\tilde{\alpha}_m^\mu, \alpha_n^\nu, \gamma_n^\nu | n, m \in \mathbb{Z}\}$ obeying the commutation relations of type (4.20). These algebras can be realized on the space of symmetric functions by the association

$$\alpha_{-n} = p_n(X), \quad \alpha_n = n\frac{\partial}{\partial p_n(X)} \quad n > 0, \tag{4.21}$$

with the central element α_0 acting as a constant. An alternative basis to that consisting of monomials in the creation operators α_{-n}, which corresponds to the power sum basis $p_\lambda(X)$, is the basis consisting of all Schur functions $s_\lambda(X)$. The symmetric-function basis has proven convenient for carrying out calculations in bosonic Fock spaces, using the realization (4.21). In the case of two commuting copies $\{\alpha_{-n}, \tilde{\alpha}_{-m}\}$ of the Heisenberg algebra realized on the space $\Lambda(X)\times\Lambda(Y)$ for a state $|u\rangle = \alpha_{-n_1}\cdots\alpha_{-n_p}\tilde{\alpha}_{-m_1}\cdots\tilde{\alpha}_{-m_r}|0\rangle$, we obtain $||u||^2 = ||\alpha_{-n_1}\cdots\alpha_{-n_p}|0\rangle||^2 \cdot ||\tilde{\alpha}_{-m_1}\cdots\tilde{\alpha}_{-m_r}|0\rangle||^2$. Therefore in the language of symmetric functions, this corresponds to using the inner product $\langle\cdots, \cdots\rangle_{\Lambda(X)\times\Lambda(Y)}$ on the space $\Lambda(X)\times\Lambda(Y)$.

B. Replicated argument and spectral functions of hyperbolic geometry

The Schur-Hall scalar product may be used to define skew Schur functions $s_{\lambda/\mu}$ through the identities $c_{\mu,\nu}^\lambda = \langle s_\mu\, s_\nu | s_\lambda\rangle = \langle s_\nu\, | s_\mu^\perp(s_\lambda)\rangle = \langle s_\nu | s_{\lambda/\mu}\rangle$ so that $s_{\lambda/\mu} = \sum_\nu\, c_{\mu,\nu}^\lambda\, s_\nu$. In what follows, we shall make considerable use of several infinite series of Schur functions. The most important of these are the mutually inverse pair defined by

$$\mathcal{F}(t;X) = \prod_{i\geq 1}(1-t\,x_i)^{-1} = \sum_{m\geq 0} h_m(X)t^m, \tag{4.22}$$

$$\mathcal{G}(t;X) = \prod_{i\geq 1}(1-t\,x_i) \quad = \sum_{m\geq 0}(-1)^m e_m(X)t^m, \tag{4.23}$$

where Schur functions $h_m(X) = s_{(m)}\,(X)$ and $e_m(X) = s_{(1^m)}(X)$.

Remark 4.1. There are some expansions which are different from power series expansions that are useful in empirical studies (for a detailed description see Refs. 8 and 14). Indeed the following result holds:

$$\prod_{n=\ell}^{\infty}(1-q^{an+\varepsilon})^{bn} = \mathcal{R}(s=(a\ell+\varepsilon)(1-i\varrho(\vartheta))+1-a)^{b\ell}$$

$$\times\prod_{n=\ell+1}^{\infty}\mathcal{R}(s=(an+\varepsilon)(1-i\varrho(\vartheta))+1-a)^{b}, \tag{4.24}$$

$$\mathcal{F}(X)^{a_n} = \prod_{n=1}^{\infty}(1-q^n)^{-a_n} = 1 + \sum_{n=1}^{\infty}\mathfrak{B}_n q^n, \tag{4.25}$$

$$n\mathfrak{B}_n = \sum_{j=1}^{n} D_j \mathfrak{B}_{n-j} q^n, \quad D_j = \sum_{d|j} d a_d. \tag{4.26}$$

Here a_n and $\mathfrak{B}_n$ are integers, function $\mathcal{R}(s)$ is an alternating product of more complicate factors—Patterson-Selberg spectral zeta-functions. Note that if either sequence a_n or $\mathfrak{B}_n$ is given, the other is uniquely determined by (4.26).

1. Q-symmetric functions

Let us introduce some more symmetric functions, which are called Q-functions. The original definition for $Q_{(\lambda_1,\ldots,\lambda_p)}(x_1,\ldots,x_n)$ for a finite number of arguments is[15]

$$Q_{(\lambda_1,\ldots,\lambda_p)}(x_1,\ldots,x_n) \stackrel{\rm def}{=} 2^p \sum_{j_i,\ldots,j_p=1}^{n} \frac{x_{j_1}^{\lambda_1}\cdots x_{j_p}^{\lambda_p}}{u_{j_1}\cdots u_{j_p}} \mathcal{A}(x_{j_p},\ldots,x_{j_2},x_{j_1}), \tag{4.27}$$

where

$$\mathcal{A}(y_1,\ldots,y_p) = \prod_{1\leq i<j\leq p} \frac{y_i - y_j}{y_i + y_j}, \quad u_j = \prod_{1\leq i\leq n, i\neq j} \frac{x_j - x_i}{x_j + x_i}. \tag{4.28}$$

2. The Hall-Littlewood functions

Note the generalization of the idea of symmetric functions, the Hall-Littlewood function[16] in the variables $x_1, x_2, \ldots, x_n$ defined for a partition of length $\ell(\lambda) \leq n$ is as follows:

$$Q_\lambda(x_1,\ldots,x_n;t) \stackrel{\rm def}{=} (1-t)^{\ell(\lambda)} \sum_{\sigma\in S_n} \sigma\left(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{1\leq i<j\leq n} \frac{x_i - t x_j}{x_i - x_j}\right), \tag{4.29}$$

where σ acts as $\sigma(x_1^{\lambda_1}\cdots x_n^{\lambda_n}) = x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(n)}^{\lambda_n}$ and t is some parameter. When $t = 0$, Q_λ reduces to the S-function s_λ. Given a field F, let $\Lambda_F = \Lambda \otimes_{\mathbb{Z}} F$ be the ring of symmetric functions over F. In the case of the Hall-Littlewood functions (4.29) in an infinite number of indeterminates, it is known[9] that they form a basis for $F = \mathbb{Q}(t)$, the field of rational functions in t. Functions $Q_\lambda(X)$ obey a Cauchy identity

$$\sum_\lambda 2^{-\ell(\lambda)} Q_\lambda(X) Q_\lambda(Y) = \prod_{i,j=1}^{n} \left(\frac{1 + x_i y_j}{1 - x_i y_j}\right). \tag{4.30}$$

3. The Jack symmetric functions

Another symmetric functions are the Jack symmetric functions $P_\lambda^{(\alpha)}(X)$, which are defined as a particular limit of a Macdonald function

$$P_\lambda^{(\alpha)}(X) = \lim_{t\to 1} P_\lambda(X; t^\alpha, t). \tag{4.31}$$

It has been pointed out that the Jack symmetric functions $P_n^{(\alpha)}(X)$ can be expressed in the form $P_n^{(1/\alpha)}(X) = (n!/\alpha^n) s_n(\alpha X)$, which specialize to zonal symmetric functions for $\alpha = 2$. For these functions, there is an inner product $\langle\cdot,\cdot\rangle_\alpha$ on the ring Λ_G of symmetric functions with coefficients in $G = \mathbb{Q}(\alpha)$ which is defined by

$$\langle p_\lambda(X), p_\mu(X)\rangle_\alpha = \delta_{\lambda,\mu}\, \mathfrak{z}_\lambda \alpha^{\ell(\lambda)}, \tag{4.32}$$

under which the Jack symmetric functions obey the orthogonality relation

$$\langle P_\lambda^{(\alpha)}, P_\mu^{(\alpha)}\rangle_\alpha = \delta_{\lambda,\mu} j_\lambda, \tag{4.33}$$

where the calculation of the numerical factor j_λ can be found in Ref. 17, Theorem 5.8. Let $g_n^{(\alpha)}(X) = P_{(n)}^{(\alpha)}(X)/j_{(n)}$ denote the elementary Jack function, which has the generating function $\sum_{n=0}^{\infty} g_n^{(\alpha)}(X)z^n = \prod_j (1-zx_j)^{1/\alpha}$. The Gram-Schmidt orthogonalization procedure gives a unique orthogonal basis for Λ_F.

We shall be interested in the cases

$$\xi_n = \alpha\left(\frac{q^{\kappa n} - q^{-\kappa n}}{q^{2n} - q^{-2n}}\right) = \alpha[\kappa/2]_q = \alpha\left(\frac{\sin(2\pi\kappa\vartheta n)}{\sin(4\pi\vartheta n)}\right), \tag{4.34}$$

where $\alpha \in \mathbb{R}$ and $\kappa \in \mathbb{Z}$. Recall the following:

1. The Hall-Littlewood symmetric functions correspond to the case when $\xi_n = (1-t^n)^{-1}$.
2. The Maconald functions correspond to the case $\xi_n = (1-q^n)/(1-t^n)$.

Let us discuss the set of symmetric functions over the field $F = \mathbb{Q}(q,t)$, which are generalizations of the Hall-Littlewood functions. Define an inner product on the power sum symmetric functions by

$$\langle p_\lambda(X), p_\mu(X)\rangle_{(q,t)} = \mathfrak{z}_\lambda(q,t)\delta_{\lambda,\mu}, \qquad \mathfrak{z}_\lambda(q,t) = \mathfrak{z}_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}. \tag{4.35}$$

Letting $b_\lambda^{-1}(q,t) \equiv \langle P_\lambda(q,t), P_\lambda(q,t)\rangle_{(q,t)}$, define $Q_\lambda(q,t) = b_\lambda(q,t)P_\lambda(q,t)$ and use the following condition: $\langle P_\lambda(X;q,t), P_\mu(X;q,t)\rangle_{(q,t)} = 0$, for $\lambda \neq \mu$. Then we have

$$\langle P_\lambda(q,t), Q_\mu(q,t)\rangle_{(q,t)} = \delta_{\lambda,\mu}. \tag{4.36}$$

Define

$$(a;q)_n \overset{\text{def}}{=} (1-a)(1-aq)\cdots(1-aq^{n-1}); \quad (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n); \quad (a;q)_0 = 1, \tag{4.37}$$

$$\Omega(tx_iy_j;\vartheta) := \log(tx_iy_j)/2\pi i\vartheta\,. \tag{4.38}$$

From Eq. (4.36), the functions P_λ and Q_λ are dual basis for Λ_F; it follows that the Macdonald functions $P_\lambda(X;q,t)$ obey the identity[9]

$$\begin{aligned}\sum_\lambda \mathfrak{z}_\lambda^{-1}(q,t)p_\lambda(X)p_\lambda(Y) &= \sum_\lambda P_\lambda(X;q,t)Q_\lambda(Y;q,t)\\ &= \prod_{i,j}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} = \prod_{i,j}\prod_{n=0}^{\infty}\frac{(1-tx_iy_jq^n)}{(1-x_iy_jq^n)}\\ &= \prod_{i,j}\prod_{n=0}^{\infty}\frac{(1-q^{n+\Omega(tx_iy_j;\vartheta)})}{(1-q^{n+\Omega(x_iy_j;\vartheta)})}\\ &= \prod_{i,j}\frac{\mathcal{R}\left(s=\Omega(tx_iy_j;\vartheta)(1-i\varrho(\vartheta))\right)}{\mathcal{R}\left(s=\Omega(x_iy_j;\vartheta)(1-i\varrho(\vartheta))\right)},\end{aligned} \tag{4.39}$$

with the last equality in (4.39) obtained by using the relation (2.18). Let us define the dual functions $Q_\lambda(X;q,\kappa,\alpha) \overset{\text{def}}{=} b_\lambda(q,\kappa,\alpha)P_\lambda(X;q,\kappa,\alpha)$ and $b_\lambda(q,\kappa,\alpha) = ||P_\lambda(X;q,\kappa,\alpha)||^{-2}$ such that $\langle P_\lambda(X;q,\kappa,\alpha), Q_\mu(X;q,\kappa,\alpha)\rangle = \delta_{\lambda\mu}$. From definition (4.34), we see that

1. $\lim_{q\to 1^-} P_\lambda(X;q,\kappa,\alpha) = P_\lambda^{(\kappa\alpha/2)}(X)$.
2. When $\kappa = 2$, the symmetric functions are Jack symmetric functions for all values of q.
3. When $\alpha = 1$, $P_\lambda(X;q,\kappa,\alpha)$ are identical to the Macdonald's function $P_\lambda(q^{2\kappa}, q^4)$.

Let us now develop a Cauchy formula for the functions $P_\lambda(X; q, \kappa, \alpha)$. First, we have the result

$$\begin{aligned}\sum_\lambda \mathfrak{z}_\lambda^{-1}(q,\kappa,\alpha)p_\lambda(X)p_\lambda(Y) &= \exp\left(\frac{1}{\alpha}\sum_{n>0}\frac{\sin(2\pi\kappa\vartheta n)}{\sin(4\pi\vartheta n)}p_n(X)p_n(Y)\right)\\ &= \prod_{i,j}\frac{(x_iy_jq^{\kappa+2};q^{2\kappa})_\infty^{1/\alpha}}{(x_iy_jq^{\kappa-2};q^{2\kappa})_\infty^{1/\alpha}}\\ &\overset{\text{by (2.18)}}{=\!=} \prod_{i,j}\left(\frac{\mathcal{R}(s=(\Omega(x_iy_j;\vartheta)+2)(1-i\varrho(\vartheta))-2)}{\mathcal{R}(s=(\Omega(x_iy_j;\vartheta)-2)(1-i\varrho(\vartheta)))}\right)^{1/\alpha}, \end{aligned} \tag{4.40}$$

where we have denoted $\mathfrak{z}_\lambda(q,\kappa,\alpha) = \mathfrak{z}_\lambda\xi_\lambda$ for the particular choice (4.34) of ξ_λ and $(x;q)_\infty^a = \prod_{j=0}^\infty (1-xq^j)^a$. Equation (4.40) is proved by a standard calculation (see Ref. 9, for example). From this, we obtain the following Cauchy identity:

$$\begin{aligned}\sum_\lambda P_\lambda(X;q,\kappa,\alpha), Q_\lambda(X;q,\kappa,\alpha) &= \prod_{i,j}\frac{(x_iy_jq^{\kappa+2};q^{2\kappa})_\infty^{1/\alpha}}{(x_iy_jq^{\kappa-2};q^{2\kappa})_\infty^{1/\alpha}}\\ &\overset{\text{by (2.18)}}{=\!=} \prod_{i,j}\left(\frac{\mathcal{R}(s=(\Omega(x_iy_j;\vartheta)+2)(1-i\varrho(\vartheta))-2)}{\mathcal{R}(s=(\Omega(x_iy_j;\vartheta)-2)(1-i\varrho(\vartheta)))}\right)^{1/\alpha}. \end{aligned} \tag{4.41}$$

The functions $P_\lambda(X; q, \kappa, \alpha)$ form a basis for the ring Λ_F, so there exist structure constants $f_{\mu\nu}^\lambda \equiv f_{\mu\nu}^\lambda(q\kappa,\alpha)$ (actually rational functions of the indeterminates q and α) such that

$$P_\mu(X;q,\kappa,\alpha)P_\lambda(X;q,\kappa,\alpha) = \sum_\lambda f_{\mu\nu}^\lambda P_\lambda(X;q,\kappa,\alpha) \quad \text{or} \quad \text{equivalently} \tag{4.42}$$

$$Q_\mu(X;q,\kappa,\alpha)Q_\nu(X;q,\kappa,\alpha) = \sum_\lambda \overline{f}_{\mu\nu}^\lambda Q_\lambda(X;q,\kappa,\alpha), \tag{4.43}$$

where $\overline{f}_{\mu\nu}^\lambda = (b_\mu(q,\kappa,\alpha)b_\nu(q,\kappa,\alpha)/(b_\lambda(q,\kappa,\alpha))f_{\mu\nu}^\lambda$. It then follows from (4.41) the following indeterminates:

$$P_\lambda(X,y;q,\kappa,\alpha) = \sum_\sigma P_{\lambda/\sigma}(X;q,\kappa,\alpha)P_\sigma(X;q,\kappa,\alpha), \tag{4.44}$$

$$Q_\lambda(X,y;q,\kappa,\alpha) = \sum_\sigma Q_{\lambda/\sigma}(X;q,\kappa,\alpha)Q_\sigma(X;q,\kappa,\alpha). \tag{4.45}$$

Introduce the symmetric function of a replicated argument. In order to define the function $P_\lambda(X^{(\tau)}; q, \kappa, \alpha)$, $\tau = m$, an integer, we put

$$P_\lambda(X^{(\tau)};q,\kappa,\alpha) := P_\lambda(\overbrace{x_1,\ldots,x_1}^{m},\overbrace{x_2,\ldots,x_2}^{m},\ldots;q,\kappa,\alpha). \tag{4.46}$$

We introduce also the transition matrix $Y_\lambda^\mu \equiv Y_\lambda^\mu(q,\kappa,\alpha)$ between the power sums and the functions P_λ,

$$p_\lambda(X) = \sum_\mu Y_\lambda^\mu P_\mu(X;q,\kappa,\alpha)\,. \tag{4.47}$$

The functions Y_λ^μ have been studied in Ref. 18 in the case $\alpha = 1$ (the Macdonald case). From the Cauchy identities (4.40) and (4.41), it follows that we have orthogonality relations of the form

$$\sum_\rho \mathfrak{z}_\rho^{-1}(q,\kappa,\alpha)Y_\rho^\lambda Y_\rho^\mu = b_\lambda(q,\kappa,\alpha)\delta_{\lambda\mu}, \tag{4.48}$$

$$\sum_\lambda b_\rho^{-1}(q,\kappa,\alpha)Y_\rho^\lambda Y_\sigma^\mu = \mathfrak{z}_\rho(q,\kappa,\alpha)\delta_{\rho\sigma}. \tag{4.49}$$

$Y_{\mu}^{(n)} = 1$ for all partitions $\mu \vdash n$. The Cauchy identity is

$$\sum_{\lambda} P_{\lambda}(X^{(\tau)}; q, \kappa, \alpha) Q_{\lambda}(Y^{(\eta)}; q, \kappa, \alpha) = \prod_{i,j} \left(\frac{(x_i y_j q^{\kappa+2}; q^{2\kappa})_{\infty}^{1/\alpha}}{(x_i, y_j, q^{\kappa-2}; q^{2\kappa})_{\infty}^{1/\alpha}} \right)^{\tau\eta/\alpha} \tag{4.50}$$

$$\underset{\overline{\overline{}}}{\text{by (2.18)}} \prod_{i,j} \left(\frac{\mathcal{R}(s = (\Omega(x_i y_j; \vartheta) + 2)(1 - i\varrho(\vartheta)) - 2)^{1/\alpha}}{\mathcal{R}(s = (\Omega(x_i y_j; \vartheta) - 2)(1 - i\varrho(\vartheta)) - 2)^{1/\alpha}} \right)^{\tau\eta/\alpha}, \tag{4.51}$$

and therefore

$$\begin{aligned} \sum_{n=0}^{\infty} Q_{(n)}(X; q, \kappa, \alpha) z^n &= \exp\left(\frac{1}{\alpha} \sum_{n>0} \frac{q^{\kappa n} - q^{-\kappa n}}{q^{2n} - q^{-2n}} p_n(X) p_n(Y) \right) \\ &= \prod_i \left(\frac{(x_i z q^{\kappa+2}; q^{2\kappa})_{\infty}}{(x_i z q^{\kappa-2}; q^{2\kappa})_{\infty}} \right)^{1/\alpha} \\ &\underset{\overline{\overline{}}}{\text{by (2.18)}} \prod_i \left(\frac{\mathcal{R}(s = (\Omega(x_i z; \vartheta) + 2)(1 - i\varrho(\vartheta)) - 2)}{\mathcal{R}(s = (\Omega(x_i z; \vartheta) - 2)(1 - i\varrho(\vartheta)) - 2)} \right)^{1/\alpha}. \end{aligned} \tag{4.52}$$

C. Vertex operator traces

Vertex operators play a fruitful role in string theory, quantum field theory, mathematical constructions of group representations, and combinatorial constructions. We cite their applications to affine Lie algebras,[19,20] quantum affine algebras. Variations on the theme of symmetric functions are applications, for example, to Q-functions, Hall-Littlewood functions, Macdonald functions, Jack functions (see in particular, Sec. IV B), and Kerov's symmetric functions (and a specialization of S-functions introduced by Kerov[21]). By considering different specializations of Kerov's symmetric functions, the trace calculations in representations of the levels quantum affine algebra $U_q(\mathfrak{gl}_N)$ (see Ref. 22 for appropriate results) can be feasible. The extension of these mathematical tools to other (quantum) affine algebras and superalgebras is also practicable and provides the relevant vertex operator realizations of those algebras.

Note that any irreducible highest weight representation of a Kac-Moody algebra can be constructed as the quotient of a Verma module by its maximal proper submodule. This construction suffices for some purposes, but in some cases, other constructions are known which give a connection with physics.[23,24] In some cases, this construction is known as the *vertex*.[19,20,25] Any Kac-Moody algebra has a root system, a Weyl group, simple roots, and highest representations. In the affine case, this gives the famous Macdonald identities for powers of the Dedekind η-function.[26]

We consider here a general vertex operator that describes the currents of this realization and which is able to connect with the symmetric and spectral functions. We specially note that realizations of (homogeneous) vertex operators are important in the high level representation theory of quantum affine algebras. Define generalized vertex operators as

$$\begin{aligned} V(\overrightarrow{\tau} * Z; \overrightarrow{\eta} * W; \xi) = \exp&\left(\sum_{m>0} \frac{1}{m\xi_m} p_m(\tau_1 z_1^m + \cdots + \tau_n z_n^m) \right) \\ &\times \exp\left(\sum_{m>0} \frac{1}{m\xi_m} D(p_m)(\eta_1 w_1^m + \cdots + \eta_n w_n^m) \right), \end{aligned} \tag{4.53}$$

where D is the adjoint operator with respect to the inner product (4.35), that is, $D(p_m) = m\xi_m \partial/\partial p_m$.

As an approach to generalising the vertex operators, the observations made in Secs. IV B 1–IV B 3 allow us to write down expressions for replicated or parameterized vertex operators. In the simplest case, this is exemplified by

$$V(\alpha z; -\alpha z^{-1}; 1) := V_\alpha(z) = \mathcal{F}(\alpha z; X)\, \mathcal{G}(\alpha z^{-1}; X)$$
$$= \exp\left(\alpha \sum_{k\geq 1} \frac{z^k}{k}\, p_k\right) \exp\left(-\alpha \sum_{k\geq 1} z^{-k}\, \frac{\partial}{\partial p_k}\right), . \tag{4.54}$$

for any α, integer, rational, real, or complex. In its simplest form (4.54), the vertex construction gives a representation $\hat{\mathfrak{g}}$ for $\mathfrak{g}$ of type A, D, or E from the even integral root lattice Λ of $\mathfrak{g}$.[27] Then we have

$$\mathcal{F}(\alpha z; X) = \mathcal{F}(z; X)^\alpha = \prod_{i\geq 1}(1 - z\, x_i)^{-\alpha} = \sum_{\sigma} s_\sigma(\alpha z)\, s_\sigma(X)$$
$$= \sum_{\sigma} z^{|\sigma|}\, \dim_\sigma(\alpha)\, s_\sigma(X), \tag{4.55}$$
$$\mathcal{G}(\alpha z^{-1}; X) = \mathcal{G}(z^{-1}; X)^\alpha = \prod_{i\geq 1}(1 - z^{-1}\, x_i)^{\alpha} = \sum_{\tau} (-1)^{|\tau|}\, s_\tau(\alpha z^{-1})\, s_{\tau'}(X)$$
$$= \sum_{\tau} (-z)^{-|\tau|}\, \dim_\tau(\alpha)\, s_{\tau'}(X), \tag{4.56}$$

as given first in Ref. 28. The Cauchy kernel $\mathcal{F}(XZ)$ serves as a generating function for characters of $GL(n)$ in the sense that

$$\mathcal{F}(XY) = \prod_{i,j}(1 - x_i y_j) = \sum_{\lambda} s_\lambda(X) s_\lambda(Y), \tag{4.57}$$

where $s_\lambda(X)$ is the character of the irreducible representation $V^\lambda_{GL(n)}$ of highest weight λ evaluated at group elements whose eigenvalues are the element of X. We summarize some useful formulas,

$$\mathcal{F}(q; XY) = \prod_{i,j}(1 - q x_i y_j)^{-1} = \sum_{\alpha} q^\alpha s_\alpha(X) s_\alpha(Y),$$
$$\mathcal{G}(q; XY) = \prod_{i,j}(1 - q x_i y_j) \quad = \quad \sum_{\alpha}(-q)^{|\alpha|} s_\alpha(X) s_{\alpha'}(Y). \tag{4.58}$$

Following the standard calculation,[28] remark that the matrix elements of the above vertex operator in a basis of Kerov symmetric functions take the form

$$\langle P_\mu(X;\xi), VQ_\nu(X;\xi)\rangle = \sum_{\zeta} P_{\mu/\zeta}(\overrightarrow{\tau} * Z;\, \xi)\, Q_{\nu/\zeta}(\overrightarrow{\eta} * W;\, \xi)\,. \tag{4.59}$$

Suppose we want to calculate the regularized trace of the vertex operator V over the space Λ_FF. Define[28]

$$S_{p/r} = \sum_{\mu\nu} p^{|\mu|} r^{|\nu|} P_{\mu/\nu}(\overrightarrow{\tau} * Z;\, \xi)\, Q_{\mu/\nu}(\overrightarrow{\eta} * W;\, \xi), \tag{4.60}$$
$$A_{\lambda\mu} = \sum_{\zeta} p^{|\zeta|} P_{\zeta/\lambda}(\overrightarrow{\tau} * Z;\, \xi)\, Q_{\zeta/\mu}(\overrightarrow{\eta} * W;\, \xi)\,. \tag{4.61}$$

Suppose that the Kerov functions with replicated arguments obey a very general Cauchy identity

$$\sum_{\lambda} r^{|\lambda|} P_\lambda(X^{(\tau)};\xi)\; Q_\lambda(Y^{(\eta)};\xi) = J^{\tau\eta}_r(X, Y;\xi) \tag{4.62}$$

so that for the functions $P_\lambda(X;\xi)$ with ξ_λ defined by (4.34), for example, the expression on the right has the form

$$J^{\tau\eta}_r(X, Y;\xi) \quad = \quad \prod_{i,j}\left(\frac{(x_i y_j q^{\kappa+2} r;\, q^{2\kappa})_\infty}{(x_i y_j q^{\kappa-2} r;\, q^{2\kappa})_\infty}\right)^{\tau\eta/\alpha}$$
$$\overset{\text{by (2.18)}}{=\!=\!=} \prod_{i,j}\left(\frac{\mathcal{R}(s = (\Omega(x_i y_j r;\vartheta) + 2)(1 - i\varrho(\vartheta)) - 2)}{\mathcal{R}(s = (\Omega(x_i y_j r;\vartheta) - 2)(1 - i\varrho(\vartheta)) - 2)}\right)^{\tau\eta/\alpha}. \tag{4.63}$$

We then form the generating function $\mathcal{J} = \sum_{\lambda\mu} A_{\lambda\mu} P_\lambda(\mathfrak{A})\, Q_\mu(\mathfrak{B})$, obtaining

$$\begin{aligned}\mathcal{J} &= \sum_{\zeta} p^{|\zeta|} P_\zeta(\vec{\tau} * Z,\, \mathcal{A};\, \xi)\; Q_\zeta(\vec{\eta} * W,\, \mathcal{B};\, \xi)\\ &= \prod_{i,j=1}^{n} J_p^{\tau_i\eta_j}(z_i, w_j; \xi) \prod_{k=1}^{n} J_p^{\tau_k,1}(z_k, \mathfrak{B}; \xi) J_p^{1,\eta_k}(\mathfrak{A}, w_k; \xi) J_p^{1,1}(\mathfrak{A}, \mathfrak{B}; \xi)\\ &= \prod_{i,j=1}^{n} J_p^{\tau_i\eta_j}(z_i, w_j; \xi) \sum_{\sigma,\sigma_1,\sigma_2,\lambda,\mu} p^{|\sigma|+|\sigma_1|+|\sigma_2|} P_{\sigma_1}(\vec{\tau} * Z;\, \xi)\\ &\times Q_{\sigma_2}(\vec{\eta} * W;\, \xi) f^{\lambda}_{\sigma_2\sigma} P_\lambda(\mathfrak{A}; \xi) \overline{f}^{\mu}_{\sigma_1\sigma} Q_\mu(\mathfrak{B}; \xi)\,. \end{aligned} \tag{4.64}$$

Finally we get (see also Ref. 29)

$$A_{\lambda\mu} = \prod_{i,j=1}^{n} J_p^{\tau_i\eta_j}(z_i, w_j; \xi) \sum_{\sigma} p^{|\lambda|+|\mu|-|\sigma|} P_{\mu/\sigma}(\vec{\tau} * Z;\, \xi)\; Q_{\lambda/\sigma}(\vec{\eta} * W;\, \xi), \tag{4.65}$$

$$S_{p/r} = \sum_{\nu} r^{|\nu|} A_{\nu\nu} = \prod_{i,j=1}^{n} J_p^{\tau_i\eta_j}(z_i, w_j; \xi) S_{rp^2/p^{-1}}\,. \tag{4.66}$$

In the Hall-Littlewood case, the above trace calculation leads to the particular identities

$$\begin{aligned}&\sum_{\mu\nu} q^{|\mu|} P_{\mu/\nu}(X^{(\alpha)}, Y^{(\beta)}; q)\; Q_{\mu/\nu}(W^{(\tau)}, Z^{(\eta)}; q) = \prod_{n=1}^{\infty}(1-q^n)^{-1} \prod_{i,j}(1 - qx_iw_j)^{-\alpha\tau}\\ &\times \prod_{k,l}(1 - qx_kz_l)^{-\alpha\eta} \prod_{m,n}(1 - qy_mw_n)^{-\beta\tau} \prod_{r,p}(1 - qy_rz_s)^{-\beta\eta} \overset{\text{by (2.18)}}{=\!=} \mathcal{R}(s = 1 - i\varrho(\vartheta))^{-1}\\ &\times \mathcal{F}(-\alpha\tau q; XW) \mathcal{F}(-\alpha\eta q; XZ) \mathcal{F}(-\beta\tau q; YW) \mathcal{F}(-\beta\eta q; YZ)\,. \end{aligned} \tag{4.67}$$

V. CONCLUSION

The symmetric functions with replicated argument and vertex operator traces have played a fruitful role in quantum field theory, string theory, and related subjects. During the last few years, the mathematical aspects of those fields have been actively studied and enriched by ideas coming from physics and vice versa. The 2N-piecewise string theory demonstrates such ideas well. The motivation for our work is based on the fact that the infinite-dimensional Heisenberg algebras (4.20) play a central role in applying the symmetric function techniques to various problems in string theories. In this connection, we have presented the vertex operator traces, realized for a replicated argument, with its connection to spectral functions of hyperbolic geometry.

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