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REMARK ON DYADIC POINTWISE DOMINATION AND MEDIAN OSCILLATION DECOMPOSITION

TIMO S. HÄNNINEN

ABSTRACT. In this note, we do the following:

a) By using Lacey’s recent technique, we give an alternative proof for Conde-Alonso and Rey’s domination theorem, which states that each positive dyadic operator of arbitrary complexity is pointwise dominated by a positive dyadic operator of zero complexity:

$$\sum_{S \in \mathcal{S}} \langle f \rangle_{S^{(k)}}^\mu 1_S \lesssim (k+1) \sum_{S' \in \mathcal{S}'} \langle f \rangle_{S'}^\mu 1_{S'}.$$

b) By following the analogue between median and mean oscillation, we extend Lerner’s local median oscillation decomposition to arbitrary (possibly non-doubling) measures:

$$|f - m(f, \hat{S}_0)| 1_{S_0} \lesssim \sum_{S \in \mathcal{S}} (\omega_\lambda(f; S) + |m(f, S) - m(f, \hat{S})|) 1_S.$$

This can be viewed as a median oscillation decomposition adapted to the dyadic (martingale) BMO. As an application of the decomposition, we give an alternative proof for the dyadic (martingale) John–Nirenberg inequality, and for Lacey’s domination theorem, which states that each martingale transform is pointwise dominated by a positive dyadic operator of complexity zero.

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NOTATION

μ	An arbitrary locally finite Borel measure on \mathbb{R}^d .
f	An arbitrary measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$.
k	An arbitrary non-negative integer.
$\langle f \rangle_Q$	The average of f on Q , $\langle f \rangle_Q := \langle f \rangle_Q^\mu := \frac{1}{\mu(Q)} \int_Q f \, d\mu$.
L^p	$L^p := L^p(\mu)$.
\mathcal{D}	The collection of dyadic cubes.
\hat{Q}	The dyadic parent of a dyadic cube Q
$Q^{(k)}$	The k th dyadic ancestor of a dyadic cube Q , defined recursively by $Q^{(k+1)} = \widehat{Q^{(k)}}$.
$\text{ch}_{\mathcal{D}}^{(k)}(Q)$	The k th dyadic descendants of a dyadic cube Q , defined by $\text{ch}_{\mathcal{D}}^{(k)}(Q) := \{Q' \in \mathcal{D} : Q'^{(k)} = Q\}$.
$\text{ch}_{\mathcal{F}}(F)$	The \mathcal{F} -children $\text{ch}_{\mathcal{F}}(F)$ of a dyadic cube F , defined by $\text{ch}_{\mathcal{F}}(F) := \{F' \in \mathcal{D} : F' \text{ maximal such that } F' \not\subseteq F\}$.
$E_{\mathcal{F}}(F)$	$E_{\mathcal{F}}(F) := F \setminus \bigcup_{F' \in \text{ch}_{\mathcal{F}}(F)} F'$.
$(g(x))_{x \in Q}$	The notation for the constant value of a function g on Q . It is implicitly understood that the function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is constant on Q .
$m(f, Q)$	Any median of f on Q , defined in Subsection 3.1.
$r_\lambda(f, Q)$	The relative median oscillation of f (about zero) on Q , defined in Subsection 3.1. It is implicitly understood that $\lambda \in (0, 1/2)$.
$\omega_\lambda(f, Q)$	The median oscillation of f on Q , defined in Subsection 3.1. It is implicitly understood that $\lambda \in (0, 1/2)$.

- A collection $\mathcal{F} \subseteq \mathcal{D}$ is *sparse* if there exists $\gamma \in (0, 1)$ such that $\sum_{F' \in \text{ch}_{\mathcal{F}}(F)} \mu(F') \leq \gamma \mu(F)$ for every $F \in \mathcal{F}$.

1. INTRODUCTION

In this note, by adapting Lacey's recent technique [5], we give an alternative proof for Conde-Alonso and Rey's domination theorem [1]. Furthermore, we extend Lerner's local median oscillation decomposition [6, 7] to arbitrary (possibly non-doubling) measures.

First, we consider the domination theorem. Conde-Alonso and Rey proved that:

Theorem 1.1 (Pointwise domination theorem for positive dyadic operators, Theorem A in [1]). *Let \mathcal{S} be a sparse collection that contains a maximal cube. Then there exists a sparse collection \mathcal{T} such that*

$$\sum_{S \in \mathcal{S}} \langle f \rangle_{S^{(k)}} 1_S \lesssim (k+1) \sum_{T \in \mathcal{T}} \langle f \rangle_T 1_T$$

μ -almost everywhere. The collection \mathcal{T} depends on the measure μ , the collection \mathcal{S} , the integer k , and the function f .

Remark. This result improves on Lerner's domination result [7, Proof of Theorem 1.1], which states the domination in any Banach function space norm (in particular, in the L^p norm).

In Section 2, we give an alternative proof for Theorem 1.1 by adapting Lacey's recent technique [5, Proof of Theorem 2.4].

Then, we consider the local median oscillation decomposition. Lerner proved that:

Theorem 1.2 (Median oscillation decomposition, Theorem 1.1 in [6] and Theorem 4.5 in [7]). *Let μ be a locally finite Borel measure. Assume that μ is doubling. Let F_0 be an initial cube. Then, there exists a sparse collection \mathcal{F} of dyadic subcubes of F_0 such that*

$$|f - m(f, F_0)|1_{F_0} \lesssim \sum_{F \in \mathcal{F}} \omega_\lambda(f; F)1_F$$

μ -almost everywhere. The collection \mathcal{F} depends on the initial cube F_0 and the function f , and the parameter λ depends on the doubling constant.

Remark. The original decomposition by Lerner in [6, Theorem 1.1] and [7, Theorem 4.5] contains an additional term (a median oscillation maximal function), which was removed by Hytönen in [3, Theorem 2.3]. Furthermore, the localization on an initial cube was removed by Lerner and Nazarov [8, Theorem 10.2].

In Section 3, we extend Theorem 1.3 as follows:

Theorem 1.3 (Median oscillation decomposition, adapted to the dyadic martingale BMO). *Let μ be an arbitrary (possibly non-doubling) locally finite Borel measure. Let F_0 be an initial cube. Then, there exists a sparse collection \mathcal{F} of dyadic subcubes of F_0 such that*

$$|f - m(f, \hat{F}_0)|1_{F_0} \lesssim \sum_{F \in \mathcal{F}} (\omega_\lambda(f; F) + |m(f, F) - m(f, \hat{F})|)1_F$$

μ -almost everywhere. The collection \mathcal{F} depends on the initial cube F_0 and the function f .

Remark. Because of the analogy between median oscillation and mean oscillation, this can be viewed as a median oscillation decomposition adapted to the dyadic (martingale) BMO, as explained in Subsection 3.5.

To keep this note as short as possible, only a tiny part of the story on the dyadic positive operators (story which revolves around the A_2 theorem) is told; For a bigger picture, see, for example, the introduction and the discussion in Lacey's paper [5], or Hytönen's survey on the A_2 theorem [3].

2. POINTWISE DOMINATION THEOREM FOR POSITIVE DYADIC OPERATORS

2.1. Alternative proof by adapting Lacey's recent technique.

Alternative proof for Theorem 1.1. To avoid writing the absolute value $|\cdot|$, we assume that the function f is non-negative.

We define

$$(2.1) \quad A_k f := \sum_{S \in \mathcal{S}} \langle f \rangle_{S^{(k)}} 1_S = \sum_{Q \in \mathcal{D}} \langle f \rangle_Q \sum_{\substack{S \in \mathcal{S}: \\ S^{(k)}=Q}} 1_S =: \sum_{Q \in \mathcal{D}} \langle f \rangle_Q \eta_Q.$$

We observe that each auxiliary function n_Q satisfies $\eta_Q \leq 1_Q$. Moreover, the auxiliary function η_Q is constant on each $Q' \in \text{ch}_{\mathcal{D}}^{(k)}(Q)$.

For each $F \in \mathcal{D}$, let $\text{ch}_{\mathcal{F}}(F)$ denote the collection of all the maximal $F' \in \{F' \in \mathcal{D} : F' \subseteq F\}$ such that

$$(2.2) \quad \left(\sum_{\substack{Q \in \mathcal{D}: \\ F'^{(k)} \subseteq Q \subseteq F}} \langle f \rangle_Q \eta_Q(x) \right)_{x \in F'} > 4 \|A_k\|_{L^1 \rightarrow L^{1,\infty}} \langle f \rangle_F,$$

or

$$(2.3) \quad \langle f \rangle_{F'} > 4 \langle f \rangle_F.$$

We observe that the weak- L^1 estimate implies that the cubes F' satisfying the first stopping condition satisfy the measure condition:

$$\begin{aligned} \sum_{F'} \mu(F') &= \sum_{F'} \mu(F' \cap \{ \sum_{\substack{Q \in \mathcal{D}: \\ F'^{(k)} \subseteq Q \subseteq F}} \langle f \rangle_Q \eta_Q > 4 \langle f \rangle_F \}) \\ &\leq \mu(\{A_k(1_F f) > 4 \|A_k\|_{L^1 \rightarrow L^{1,\infty}} \langle f \rangle_F\}) \leq \frac{1}{4} \mu(F). \end{aligned}$$

Similarly, the cubes F' satisfying the second stopping condition satisfy the measure condition $\sum_{F'} \mu(F') \leq \frac{1}{4} \mu(F)$. Altogether, $\sum_{F' \in \text{ch}_{\mathcal{F}}(F)} \mu(F') \leq \frac{1}{2} \mu(F)$.

Now, by decomposing the summation and invoking the stopping conditions,

$$\begin{aligned} S_{\subseteq F} &:= \sum_{Q \in \mathcal{D}: Q \subseteq F} \langle f \rangle_Q \eta_Q \\ &= \sum_{Q \in \mathcal{D}: Q \subseteq F} \langle f \rangle_Q \eta_Q 1_{E_{\mathcal{F}}(F)} + \sum_{F' \in \text{ch}_{\mathcal{F}}(F)} \sum_{Q \in \mathcal{D}: Q \subseteq F} \langle f \rangle_Q \eta_Q 1_{F'} \\ &= \sum_{Q \in \mathcal{D}: Q \subseteq F} \langle f \rangle_Q \eta_Q 1_{E_{\mathcal{F}}(F)} + \sum_{F' \in \text{ch}_{\mathcal{F}}(F)} \sum_{\substack{Q \in \mathcal{D}: \\ F'^{(k+1)} \subseteq Q \subseteq F}} \langle f \rangle_Q \eta_Q 1_{F'} \\ (2.4) \quad &+ \sum_{F' \in \text{ch}_{\mathcal{F}}(F)} \sum_{\substack{Q \in \mathcal{D}: \\ F'^{(1)} \subseteq Q \subseteq \min\{F'^{(k)}, F\}}} \langle f \rangle_Q \eta_Q 1_{F'} + \sum_{F' \in \text{ch}_{\mathcal{F}}(F)} \sum_{Q \in \mathcal{D}: Q \subseteq F'} \langle f \rangle_Q \eta_Q \\ &\leq 4 \|A_k\|_{L^1 \rightarrow L^{1,\infty}} \langle f \rangle_F (1_{E_{\mathcal{F}}(F)} + \sum_{F' \in \text{ch}_{\mathcal{F}}(F)} 1_{F'}) + k 4 \langle f \rangle_F 1_F + \sum_{F'} S_{\subseteq F'}, \end{aligned}$$

where the last step follows from the following observations:

- For each $x \in E_{\mathcal{F}}(F)$, every $R \in \{R \in \mathcal{D} : R \subseteq F\}$ such that $R \ni x$ satisfies the opposite of the stopping condition (2.2). Therefore,

$$\sum_{Q \in \mathcal{D}: Q \subseteq F} \langle f \rangle_Q \eta_Q(x) = \lim_{Q \in \mathcal{D}: R \ni x, \ell(R) \rightarrow 0} \left(\sum_{Q \in \mathcal{D}: R^{(k)} \subseteq Q \subseteq F} \langle f \rangle_Q \eta_Q \right)_{x \in R} \leq 4 \|A_k\|_{L^1 \rightarrow L^{1,\infty}} \langle f \rangle_F.$$

- By maximality, the cube $F'^{(1)}$ satisfies the opposite of the stopping condition (2.2). Therefore,

$$\sum_{\substack{Q \in \mathcal{D}: \\ F'^{(k+1)} = (F'^{(1)})^{(k)} \subseteq Q \subseteq F}} \langle f \rangle_Q \eta_Q 1_{F'} \leq 4 \|A_k\|_{L^1 \rightarrow L^{1,\infty}} \langle f \rangle_F.$$

- By maximality, every cube $Q \in \mathcal{D}$ such that $F'^{(1)} \subseteq Q \subseteq \min\{F'^{(k)}, F\}$ satisfies the opposite of the stopping condition (2.3). Therefore, $\langle f \rangle_Q \leq 4 \langle f \rangle_F$ for all such cubes Q .

By Proposition 2.1, we have $\|A_k\|_{L^1(\mu) \rightarrow L^{1,\infty}(\mu)} \lesssim 1$. Note that the weak L^1 estimate for the operator A_k is independent of k , whereas the weak L^1 estimate for the adjoint

operator A_k^* depends linearly on k . The proof is completed by iteration, starting from the maximal cube (which exists, by assumption) of the collection \mathcal{S} . \square

2.2. Weak- L^1 estimate for positive dyadic operators.

Proposition 2.1 (Weak L^1 for positive dyadic operators). *Let μ be a locally finite Borel measure. Let A_k be defined as in (2.1). Then*

$$\|A_k f\|_{L^1 \rightarrow L^{1,\infty}} \lesssim 1.$$

Remark. The weak L^1 estimate for the operator A_k is proven using the Calderón–Zygmund decomposition: In the case of a doubling measure, this is proven as in [2, Proof of Proposition 5.1] or in [7, Proof of Lemma 5.4]; In the case of an arbitrary (possibly non-doubling) measure, the Calderón–Zygmund decomposition contains an additional term, for which the weak L^1 estimate is checked in what follows.

We prove the weak- L^1 boundedness by using the Calderón–Zygmund decomposition for general measures obtained by López–Sánchez, Martell, and Parcet:

Lemma 2.2 (Calderón–Zygmund decomposition for general measures, Theorem 2.1 in [9]). *Let μ be a locally finite Borel measure on \mathbb{R}^d . Assume that the measure of each d -dimensional quadrant is infinite. Then, for each $f \in L^1$ and $\lambda > 0$, there exists a decomposition*

$$f = g + b + \beta$$

such that the pieces satisfy the following properties:

- The function g satisfies

$$\|g\|_{L^p}^p \lesssim_p \lambda^{p-1} \|f\|_{L^1}$$

for every $1 \leq p < \infty$.

- The function b has the decomposition $b = \sum_{T \in \mathcal{T}} b_T$ such that

$$\text{supp}(b_T) \subseteq T, \quad \int b_T \, d\mu = 0, \quad \sum_{T \in \mathcal{T}} \|b_T\|_{L^1} \lesssim \|f\|_{L^1}.$$

- The function β has the decomposition $\beta = \sum_{\hat{T} \in \hat{\mathcal{T}}} \beta_{\hat{T}}$ such that

$$\text{supp}(\beta_{\hat{T}}) \subseteq \hat{T}, \quad \int \beta_{\hat{T}} \, d\mu = 0, \quad \sum_{\hat{T} \in \hat{\mathcal{T}}} \|\beta_{\hat{T}}\|_{L^1} \lesssim \|f\|_{L^1},$$

and $\beta_{\hat{T}}$ is constant on T and on $\hat{T} \setminus T$.

- The cubes T are the maximal (which exist because, by assumption, the measure of each d -dimensional quadrant is infinite) dyadic cubes such that $\langle |f| \rangle_T > \lambda$. Hence, they are pairwise disjoint, and their union $\Omega := \cup_T T$ satisfies $\mu(\Omega) \leq \frac{1}{\lambda} \int |f| \, d\mu$.

Proof of Proposition 2.1. We suppress the complexity k in the notation. By using the Calderón–Zygmund decomposition (Lemma 2.2), we decompose

$$\mu(\{|Af| > \lambda\}) \leq \mu(\{|Ag| > \frac{\lambda}{3}\}) + \mu(\{|Ab| > \frac{\lambda}{3}\} \cap \Omega^c) + \mu(\{|A\beta| > \frac{\lambda}{3}\} \cap \Omega^c) + \mu(\Omega).$$

By Chebyshev’s inequality together with the $L^p \rightarrow L^p$ boundedness of the operator A_k , we have

$$\mu(\{|Ag| > \frac{\lambda}{3}\}) \lesssim \frac{1}{\lambda^p} \|Ag\|_{L^p} \lesssim_p \frac{1}{\lambda^p} \|g\|_{L^p}^p \lesssim_p \frac{1}{\lambda^p} \lambda^{(p-1)} \|f\|_{L^1}.$$

We observe that $1_{T^c}A(h_T) = 0$ whenever h_T is such that $\text{supp}(h_T) \subseteq T$ and $\int h_T d\mu = 0$. This together with Chebyshev's inequality implies that

$$\mu(\{Ab > \frac{\lambda}{3}\} \cap \Omega^c) \lesssim \frac{1}{\lambda} \sum_T \int_{T^c} |Ab_T| d\mu = 0,$$

and

$$\mu(\{A\beta > \frac{\lambda}{3}\} \cap \Omega^c) \lesssim \frac{1}{\lambda} \sum_T \int_{T^c} |A\beta_{\hat{T}}| d\mu = \frac{1}{\lambda} \sum_T \int_{\hat{T} \setminus T} |A(\beta_{\hat{T}})| d\mu.$$

Since $\beta_{\hat{T}}$ is constant on $\hat{T} \setminus T$, we have

$$1_{\hat{T} \setminus T} A(\beta_{\hat{T}}) = \sum_{Q: Q \subseteq \hat{T} \setminus T} \langle \beta_{\hat{T}} \rangle_Q \eta_Q = \langle \beta_{\hat{T}} \rangle_{\hat{T} \setminus T} \sum_{Q: Q \subseteq \hat{T} \setminus T} \eta_Q.$$

Recall that, by definition, $\eta_Q := \sum_{S \in \mathcal{S}: S^{(k)} = Q} 1_S$, where \mathcal{S} is a sparse collection. Therefore, by sparseness,

$$\begin{aligned} \int_{\hat{T} \setminus T} |A(\beta_{\hat{T}})| d\mu &\leq |\langle \beta_{\hat{T}} \rangle_{\hat{T} \setminus T}| \sum_{S \in \mathcal{S}: S \subseteq \hat{T} \setminus T} \mu(S) \\ &\lesssim |\langle \beta_{\hat{T}} \rangle_{\hat{T} \setminus T}| \sum_{S \in \mathcal{S}: S \subseteq \hat{T} \setminus T} \mu(E_{\mathcal{S}}(S)) \leq |\langle \beta_{\hat{T}} \rangle_{\hat{T} \setminus T}| \mu(\hat{T} \setminus T) \leq \|\beta_{\hat{T}}\|_{L^1}. \end{aligned}$$

The proof is completed by the property $\sum_{T \in \mathcal{T}} \|b_{\hat{T}}\|_{L^1} \lesssim \|f\|_{L^1}$. □

3. MEDIAN OSCILLATION DECOMPOSITION

Convention. Throughout this section, the parameter λ is an arbitrary real number such that $0 < \lambda < 1/2$.

3.1. Definition of median and median oscillation.

- The *median* $m(f; Q)$ of a function f on a cube Q is defined as any real number such that

$$\frac{\mu(Q \cap \{f > m(f; Q)\})}{\mu(Q)} \leq \frac{1}{2} \quad \text{and} \quad \frac{\mu(Q \cap \{f < m(f; Q)\})}{\mu(Q)} \leq \frac{1}{2}.$$

- The *relative median oscillation* $r_{\lambda}(f; Q)$ of a function f (*about zero*) on a cube Q is defined by

$$r_{\lambda}(f; Q) := \min\{r \geq 0 : \mu(Q \cap \{|f| > r\}) \leq \lambda \mu(Q)\}.$$

Note that, by means of decreasing rearrangement, the relative median oscillation is written as $r_{\lambda}(f; Q) = (1_Q f)^*(\lambda \mu(Q))$. The quantity $r_{\lambda}(f - c; Q)$ is the *relative median oscillation* of a function f *about a real number* c on a cube Q .

- The *median oscillation* $\omega_{\lambda}(f; Q)$ of a function f on a cube Q is defined by

$$\omega_{\lambda}(f; Q) := \inf_{c \in \mathbb{R}} r_{\lambda}(f - c; Q).$$

3.2. Properties of median and median oscillation. For reader's convenience, we summarize the properties of median that we need. The properties are all well-known. For proofs, see, for example, the lecture notes [4, Section 5].

Lemma 3.1 (Every median quasiminimizes the median oscillation). We have

$$r_\lambda(f - m(f; Q); Q) \leq 2\omega_\lambda(f; Q).$$

Lemma 3.2 (Median is linear). We have

$$m(f + c; Q) = m(f; Q) + c.$$

Since median is not unique, this slight abuse of notation is understood as an identity for the set of all medians: $\{m : m \text{ is a median of } (f + c) \text{ on } Q\} = \{m' : m' \text{ is a median of } f \text{ on } Q\} + c$.

Lemma 3.3 (Median is controlled by the relative median oscillation). We have

$$|m(f; Q) - c| \leq 3r_\lambda(f - c; Q).$$

Proof. Using the fact that every median quasiminimizes the median oscillation (Lemma 3.1), and the definition of median oscillation, we have

$$r_\lambda(f - m(f; Q)) \leq 2\omega_\lambda(f; Q) \leq 2r_\lambda(f - c; Q).$$

This, by the definition of relative median oscillation, implies that

$$\mu(\{|f - m(f; Q)| > 2r_\lambda(f - c; Q)\}) \leq \lambda\mu(Q), \text{ and } \mu(\{|f - c| > r_\lambda(f - c; Q)\}) \leq \lambda\mu(Q).$$

From this together with the implicit assumption $0 < \lambda < 1/2$, it follows that there exists $x \in Q$ such that $|f(x) - m(f; Q)| \leq 2r_\lambda(f - c; Q)$ and $|f(x) - c| \leq r_\lambda(f - c; Q)$. The proof is completed by the triangle inequality. \square

Lemma 3.4 (Fujii's Lemma). We have

$$\lim_{\substack{Q \in \mathcal{D}: \\ Q \ni x, \ell(Q) \rightarrow 0}} m(f; Q) = f(x)$$

for μ -almost every $x \in \mathbb{R}^d$.

Lemma 3.5 (Relative median oscillation is controlled by the weak L^1 norm). We have

$$r_\lambda(f; Q) \leq \frac{1}{\lambda} \frac{\|f\|_{L^{1,\infty}}}{\mu(Q)}.$$

3.3. Proof of the decomposition adapted to the dyadic BMO.

Proof of Theorem 1.3. From Lemma 3.3 and Lemma 3.1, it follows that

$$(3.1) \quad \begin{aligned} \omega_\lambda(f; F) + |m(f, F) - m(f, \hat{F})| &\approx r_\lambda(f - m(f, F)) + |m(f, F) - m(f, \hat{F})| \\ &\approx r_\lambda(f - m(f; \hat{F}); F). \end{aligned}$$

For each $F \in \mathcal{D}$, let $\text{ch}_{\mathcal{F}}(F)$ denote the collection of all the maximal $F' \in \{F' \in \mathcal{D} : F' \subseteq F\}$ such that

$$(3.2) \quad |m(f; F') - m(f; \hat{F})| > 3r_\lambda(f - m(f; \hat{F}); F)$$

By decomposing and using the stopping condition,

$$\begin{aligned} |f - m(f, \hat{F})|1_F &\leq |f - m(f; \hat{F})|1_{E_{\mathcal{F}}(F)} + \sum_{F' \in \text{ch}_{\mathcal{F}}(F)} |m(f; \hat{F}') - m(f; \hat{F})|1_{F'} + \\ &\quad \sum_{F' \in \text{ch}_{\mathcal{F}}(F)} |f - m(f, \hat{F}')|1_{F'} \\ &\leq 3r_{\lambda}(f - m(f, \hat{F}); F)1_F + \sum_{F' \in \text{ch}_{\mathcal{F}}(F)} |f - m(f, \hat{F}')|1_{F'}, \end{aligned}$$

where the last step follows from the following observations:

- For each $x \in E_{\mathcal{F}}(F)$, every cube $Q \in \{Q \in \mathcal{D} : Q \subseteq F\}$ such that $Q \ni x$ satisfies the opposite of the stopping inequality (3.2). Therefore, by Fujii's Lemma (Lemma 3.4),

$$|f(x) - m(f; \hat{F})| = \lim_{\substack{Q \in \mathcal{D}: \\ Q \ni x, \ell(Q) \rightarrow 0}} |m(f; Q) - m(f; \hat{F})| \leq 3r_{\lambda}(f - m(f, \hat{F}); F)$$

μ -almost every $x \in E_{\mathcal{F}}(F)$.

- By maximality, the cube \hat{F}' satisfies the opposite of the stopping inequality (3.2). Therefore,

$$|m(f; \hat{F}') - m(f; \hat{F})| \leq 3r_{\lambda}(f - m(f, \hat{F}); F).$$

Finally, we check that $\sum_{F' \in \text{ch}_{\mathcal{F}}(F)} \mu(F') \leq 2\lambda\mu(F)$. Let $\kappa \in (0, 1/2)$ be an auxiliary parameter. We note the following assertion:

$$\text{If } \mu(Q \cap \{|f - c| > r\}) \leq \kappa\mu(Q), \text{ then } |m(f; Q) - c| \leq 3r;$$

This is because $\mu(Q \cap \{|f - c| > r\}) \leq \kappa\mu(Q)$ implies, by definition, that $r_{\kappa}(f - c; Q) \leq r$, from which, by Lemma 3.3, it follows that $|m(f; Q) - c| \leq 3r$. The contrapositive of this assertion applied to the stopping inequality (3.2) (where we have $Q := F'$, $c := m(f, \hat{F})$ and $r := r_{\lambda}(f - m(f; \hat{F}); F)$) implies that

$$(3.3) \quad \mu(F' \cap \{|f - m(f, \hat{F})| > r_{\lambda}(f - m(f; \hat{F}); F)\}) > \kappa\mu(F').$$

On the other hand, by definition,

$$(3.4) \quad \lambda\mu(F) \geq \mu(F \cap \{|f - m(f; \hat{F})| > r_{\lambda}(f - m(f; \hat{F}); F)\}).$$

Summing over the cubes F' (which are pairwise disjoint and satisfy $F' \subseteq F$) in the inequality (3.3), combining this with the inequality (3.4), and taking $\kappa \rightarrow 1/2$ yields

$$\sum_{F' \in \text{ch}_{\mathcal{F}}(F)} \mu(F') \leq 2\lambda\mu(F).$$

The proof is completed by iteration. □

3.4. Corollaries. The dyadic (martingale) BMO norm is defined by

$$\|f\|_{BMO(\mu)} := \sup_{Q \in \mathcal{D}} \frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_Q| d\mu.$$

Note that, whenever the measure μ is doubling, the dyadic (martingale) BMO norm is comparable to the usual BMO norm: $\|f\|_{BMO(\mu)} \approx_{\mu} \sup_{Q \in \mathcal{D}} \frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_Q| d\mu$.

Proposition 3.6 (John–Nirenberg). *Let μ be a locally finite Borel measure. Then, there exist positive constants c and C such that*

$$\frac{1}{\mu(Q)} \int_Q \exp(c|f - \langle f \rangle_{\hat{Q}}|/\|f\|_{BMO}) d\mu \leq C$$

for every $f \in BMO$.

Proof by the dyadic median oscillation decomposition. By using the inequalities

$$r_\lambda(f; Q) \lesssim_\lambda \frac{1}{\mu(Q)} \int_Q |f| d\mu \quad \text{and} \quad |m(f; Q)| \leq 3r_\lambda(f; Q),$$

of which the first follows from Chebyshev's inequality and the second is stated in Lemma 3.3, and by using the linearity of median, we obtain

$$r_\lambda(f - m(f; \hat{Q}); Q) \lesssim \|f\|_{BMO}.$$

By the median oscillation decomposition (Theorem 1.3), there exists a sparse collection \mathcal{S} of dyadic subcubes of Q such that

$$|f - m(f, \hat{Q})|_{1_Q} \lesssim \sum_{S \in \mathcal{S}} r_\lambda(f - m(f; \hat{S}); S) 1_S.$$

Altogether,

$$|f - \langle f \rangle_{\hat{Q}}|_{1_Q} \leq |f - m(f, \hat{Q})|_{1_Q} + |m(f; \hat{Q}) - \langle f \rangle_{\hat{Q}}|_{1_Q} \lesssim \|f\|_{BMO} \sum_{S \in \mathcal{S}} 1_S.$$

By sparseness, $\mu(\{\sum_{S \in \mathcal{S}} 1_S = k\}) \lesssim 2^{-k} \mu(Q)$, from which the exponential integrability follows by splitting the integration as $\int_Q = \sum_{k=0}^{\infty} \int_{\{\sum_{S \in \mathcal{S}} 1_S = k\}}$. \square

The *martingale transform* T associated with the (constant) coefficients ϵ_Q satisfying $|\epsilon_Q| \leq 1$ is defined by

$$Tf := \sum_{Q \in \mathcal{D}} \epsilon_Q D_Q f := \sum_{Q \in \mathcal{D}} \epsilon_Q \left(\sum_{Q' \in \text{ch}_{\mathcal{D}}(Q)} \langle f \rangle_{Q'} 1_{Q'} - \langle f \rangle_Q \right).$$

Lacey [5, Theorem 2.4] proves that each martingale transform is pointwise dominated by a positive dyadic operator of zero complexity. Alternative proof for this is as follows: First, use the median oscillation decomposition (Theorem 1.3) to yield the domination by positive dyadic operators of complexity zero and one. Then, apply the domination for positive dyadic operators (Theorem 1.1) to reduce the complexity to zero.

Proposition 3.7 (A pointwise domination theorem for martingale transforms, see Lacey's Theorem 2.4 in [5] for a stronger version). *Let F_0 be an initial cube. Assume that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a locally integrable function that is supported on the cube F_0 . Then, there exists a sparse collection \mathcal{F} of dyadic subcubes of F_0 such that*

$$|Tf|_{1_{F_0}} \lesssim (\|T\|_{L^1 \rightarrow L^{1,\infty}} + 1) \left(\sum_{F \in \mathcal{F}} \langle |f| \rangle_F 1_F + \sum_{\hat{F} \in \mathcal{F}} \langle |f| \rangle_{\hat{F}} 1_{\hat{F}} \right).$$

Proof by the median oscillation decomposition. The theorem follows from the median oscillation decomposition (Theorem 1.3) together with an estimate for the oscillation quantities (Lemma 3.8). \square

Lemma 3.8 (Oscillations of a martingale transform). Let T be a martingale transform. Let R be a dyadic cube. Then

$$r_\lambda(Tf - m(Tf; \hat{R}); R) \leq (\|T\|_{L^1 \rightarrow L^{1,\infty}} + 1)(\langle |f| \rangle_R + \langle |f| \rangle_{\hat{R}}),$$

and

$$m(Tf; R) \lesssim \|T\|_{L^1 \rightarrow L^{1,\infty}} \frac{1}{\mu(Q)} \int_{\mathbb{R}^d} |f| d\mu$$

Proof. Let R be a dyadic cube. We split $1_R Tf = 1_RT(1_R f) + 1_RT(1_{R^c} f)$. We observe that $1_RT(1_{R^c} f) = 1_R \sum_{Q: Q \supseteq \hat{R}} \epsilon_Q D_Q f$ is constant on R , and denote this constant value by $c_R := \left(\sum_{Q \supseteq \hat{R}} \epsilon_Q D_Q f \right)_{x \in R}$. By using the linearity of median, we write

$$\begin{aligned} 1_RT(f) - 1_R m(Tf; \hat{R}) &= 1_RT(1_R f) + 1_R(c_R - c_{\hat{R}}) - 1_R m(T(1_{\hat{R}} f); \hat{R}) \\ &= 1_RT(1_R f) + 1_R \epsilon_{\hat{R}} D_{\hat{R}} f - 1_R m(T(1_{\hat{R}} f); \hat{R}) \\ &= 1_RT(1_R f) + 1_R \epsilon_{\hat{R}} \langle f \rangle_R - 1_R \epsilon_{\hat{R}} \langle f \rangle_{\hat{R}} + m(T(1_{\hat{R}} f); \hat{R}). \end{aligned}$$

Therefore,

$$r_\lambda(Tf - m(Tf; \hat{R}); R) \leq r_\lambda(T(1_R f); R) + |m(T(1_{\hat{R}} f); \hat{R})| + \langle |f| \rangle_R + \langle |f| \rangle_{\hat{R}}.$$

By using the estimate $|m(f; Q)| \leq 3r_\lambda(f; Q)$ (Lemma 3.3), and by dominating the median oscillation $r_\lambda(f; Q)$ by the weak L^1 estimate (Lemma 3.5), we obtain

$$\begin{aligned} r_\lambda(Tf - m(Tf; \hat{R}); R) &\lesssim_\lambda \frac{\|T(1_R f)\|_{L^{1,\infty}}}{\mu(R)} + \frac{\|T(1_{\hat{R}} f)\|_{L^{1,\infty}}}{\mu(\hat{R})} + \langle |f| \rangle_R + \langle |f| \rangle_{\hat{R}} \\ &\leq (\|T\|_{L^1 \rightarrow L^{1,\infty}} + 1)(\langle |f| \rangle_R + \langle |f| \rangle_{\hat{R}}). \quad \square \end{aligned}$$

3.5. Median oscillation decomposition adapted to the RBMO?. In the light of the analogue between median and mean, and median oscillation and mean oscillation,

$$m(f; Q) \leftrightarrow \langle f \rangle_Q, \quad r_\lambda(f - c, Q) \leftrightarrow \frac{1}{\mu(Q)} \int_Q |f - c| d\mu,$$

the passage from the usual BMO norm to the dyadic (martingale) BMO norm,

$$\frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_Q| \rightarrow \frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_Q| d\mu + |\langle f \rangle_Q - \langle f \rangle_{\hat{Q}}|,$$

is analogous to the passage

$$r_\lambda(f - m(f, Q); Q) \rightarrow r_\lambda(f - m(f, Q); Q) + |m(f; Q) - m(f; \hat{Q})|,$$

which we use to extend Lerner's local oscillation decomposition. Thus, our extension can be viewed as a local oscillation decomposition adapted to the dyadic (martingale) BMO.

The author believes that, in the same spirit, Lerner's local oscillation decomposition can be adapted to the RBMO space, and that this adapted decomposition can be used to pointwise dominate non-homogeneous Calderón–Zygmund operators by suitable positive averaging operators. (For the RBMO space, see [11], and, for non-homogeneous Calderón–Zygmund operators, see [10].)

We remark that a pointwise domination for non-homogeneous Calderón–Zygmund operators by positive averaging operators was obtained by Treil and Volberg, by adapting Lacey's technique [5, Proof of Theorem 5.2]. This result is announced by Lacey [5, Section 6].

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