A comparison of Euclidean and Heisenberg Hausdorff measures

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Abstract

We prove some geometric properties of sets in the first Heisenberg group whose Heisenberg Hausdorff dimension is the minimal or maximal possible in relation to their Euclidean one and the corresponding Hausdorff measures are positive and finite. In the first case we show that these sets must be in a sense horizontal and in the second case vertical. We show the sharpness of our results with some examples.

1 Introduction

Let $\mathcal{H}^s_E$ denote the Euclidean Hausdorff measure in the first Heisenberg group $\mathbb{H}^1$ and let $\mathcal{H}^s_H$ denote the Hausdorff measure with respect to some homogeneous metric. Let $\dim_E$ and $\dim_H$ denote the corresponding Hausdorff dimensions. Generally, for a set $A \subset \mathbb{H}^1$, $\dim_E A$ and $\dim_H A$ can be different. For example, every line has Euclidean Hausdorff dimension 1 but there are lines that have Hausdorff dimension 2 with respect to any homogeneous metric, such as the vertical axis (corresponding to the t-axis when $\mathbb{H}^1$ is identified with $\mathbb{R}^3$ and points have coordinates (x, y, t), as we will see in Section 2). On the other hand, any so-called horizontal line has $\dim_H$ equal to 1. Horizontal lines, which are lines through the origin in the xy-plane or left translations of them with respect to the group operation, and horizontal planes (defined in (3)) play a special role in the Heisenberg group, as we will see also in our results.

Balogh, Rickly and Serra Cassano in [BRSC] compared $\dim_E$ and $\dim_H$ for general sets, proving what follows. For $0 \le s \le 3$ let

$$\beta_-(s) = \max\{s, 2s-2\}, \beta_+(s) = \min\{2s, s+1\}.$$

Then for any $A \subset \mathbb{H}^1$,

$$\beta_-(\dim_E A) \le \dim_H A \le \beta_+(\dim_E A).$$

Moreover, they also showed the sharpness of some of these inequalities, which was then completed by Balogh and Tyson in [BT]: for any $0 < s < 3$ they constructed compact subsets

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F_1 and F_2 of $\mathbb{H}^1$ such that $\mathcal{H}^s_E(F_1)$ and $\mathcal{H}^{\beta_-(s)}_H(F_1)$ are positive and finite and $\mathcal{H}^s_E(F_2)$ and $\mathcal{H}^{\beta_+(s)}_H(F_2)$ are positive and finite. The example F_1, for $0 < s \leq 2$, is in a sense horizontal and F_2 is in a sense vertical. In this paper we show that this must be so. We prove in Theorem 2 that for any set $A \subset \mathbb{H}^1$ and $0 < s \leq 2$ (recall that then $\beta_-(s) = s$), if both $\mathcal{H}^s_E(A)$ and $\mathcal{H}^s_H(A)$ are positive and finite, then in some arbitrarily small neighbourhoods around its typical points p, most of A lies close to the horizontal plane through p. We shall construct an example (see Example 5) to show that this need not hold for all small neighbourhoods and another example (see Example 7) to show that this does not hold when $s > 2$ and both $\mathcal{H}^s_E(A)$ and $\mathcal{H}^{2s-2}_H(A)$ are positive and finite. Note that $\beta_-(s) = 2s - 2$ when $2 < s \leq 3$. Corresponding to the second case we show in Theorems 3 and 4 that if both $\mathcal{H}^s_E(A)$ and $\mathcal{H}^{\beta_+(s)}_H(A)$ are positive and finite, then in some arbitrarily small neighbourhoods around its typical points p, a large part of A lies off the horizontal plane through p.

In [BTW] Balogh, Tyson and Warhurst solved the dimension comparison problem in general Carnot groups, but here we restrict to the first Heisenberg group.

2 Preliminaries

In a metric space X for $0 < s < \infty$ the s-dimensional Hausdorff measure of $A \subset X$ is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

where

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_{i=1}^{\infty} \operatorname{diam}(B_i)^s : A \subset \bigcup_{i=1}^{\infty} B_i, \operatorname{diam}(B_i) < \delta\}.$$

The Hausdorff dimension of A is

$$\dim A = \inf\{s : \mathcal{H}^s(A) = 0\}.$$

Let $B(p, r)$ be the closed ball with centre $p \in X$ and radius r. We have the basic upper density theorem for Hausdorff measures, see, e,g., [F], 2.10.19.

Theorem 1. *Let $A \subset X$ be $\mathcal{H}^s$ measurable with $\mathcal{H}^s(A) < \infty$. Then for $\mathcal{H}^s$ almost all $p \in A$,*

$$2^{-s} \leq \limsup_{r \to 0} \frac{\mathcal{H}^s(A \cap B(p, r))}{(2r)^s} \leq 1$$

and for $\mathcal{H}^s$ almost all $p \in X \setminus A$,

$$\lim_{r \to 0} \frac{\mathcal{H}^s(A \cap B(p, r))}{(2r)^s} = 0.$$

It is easy to construct examples, in Euclidean and many other metric spaces, where the lower density $\liminf_{r \to 0} \frac{\mathcal{H}^s(A \cap B(p,r))}{(2r)^s}$ is 0 for all $p \in X$, see, for example, [M], 4.12.

Let $\mathbb{H}^1$ be the first Heisenberg group. It can be identified as $\mathbb{R}^3$ with the non-Abelian group operation

$$p \cdot p' = (x + x', y + y', t + t' + 2(x'y - xy')),$$

for $p=(x,y,t), p'=(x',y',t')$, and with the metric

$$d_H(p,p') = \left(((x-x')^2+(y-y')^2)^2+(t-t'+2(x'y-xy'))^2\right)^{1/4}.$$

In addition to d_H we shall also use the Euclidean metric, which we denote by d_E. Then for $0<R<\infty$ there exists a constant $c_R>0$ such that for every $p,p'\in B_E(0,R)$,

$$\frac{1}{c_R} d_E(p,p') \leq d_H(p,p') \leq c_R d_E(p,p')^{1/2}. \tag{1}$$

The closed ball $B(p,r)$ is denoted by $B_H(p,r)$ when the metric is d_H and by $B_E(p,r)$ when the metric is d_E. The s-dimensional Hausdorff measures and dimensions with respect to d_H and d_E are denoted by $\mathcal{H}^s_H, \mathcal{H}^s_E, \dim_H$ and $\dim_E$. In place of d_H we could use any homogeneous metric on $\mathbb{H}^1$, that is, any left invariant metric d satisfying $d((\delta x,\delta y,\delta^2 t),(\delta x',\delta y',\delta^2 t')) = \delta d((x,y,t),(x',y',t'))$. By 5.1.5 in [BLU], these metrics are locally bi-Lipschitz equivalent.

Recall the definitions of $\beta_-(s)$ and $\beta_+(s)$ from the introduction. Then by Proposition 3.1 in [BTW], for any positive number R there exists a constant C_R such that for $A\subset B_E(0,R)$ and for $0<s<3$,

$$\mathcal{H}^{\beta_+(s)}_H(A)/C_R \leq \mathcal{H}^s_E(A) \leq C_R \mathcal{H}^{\beta_-(s)}_H(A). \tag{2}$$

Let $V(p)$ denote the horizontal plane passing through $p=(x_0,y_0,t_0)\in\mathbb{H}^1$. This is the set of points $q=(x,y,t)$ such that

$$t-t_0-2(xy_0-yx_0)=0. \tag{3}$$

The Euclidean distance of a point $q=(x,y,t)$ to the plane $V(p)$ is

$$d_E(q,V(p)) = \frac{|t-t_0-2(xy_0-yx_0)|}{\sqrt{1+4(x_0^2+y_0^2)}}. \tag{4}$$

We let $A(\delta)$ denote the closed δ neighbourhood of $A\subset\mathbb{H}^1$ in the Euclidean metric. Observe that $B_H(p,r)$ looks like $V(p)(r^2)\cap B_E(p,r)$, more precisely, for p as above with $x_0^2+y_0^2\leq R^2$ and $0<r<1$,

$$V(p)\left(\frac{r^2}{\sqrt{2(1+4R^2)}}\right)\cap B_E\left(p,\frac{r}{2}\right) \subset B_H(p,r) \subset V(p)(r^2)\cap B_E(p,c_R r), \tag{5}$$

where c_R is the constant in (1).

The restriction of a measure μ to a set $A\subset X$ is denoted by $\mu\llcorner A$; $\mu\llcorner A(B)=\mu(A\cap B)$.

3 The theorems

Theorem 2. *Let $0<s\leq 2$ and let $A\subset\mathbb{H}^1$ be such that $\mathcal{H}^s_H(A)<\infty$. Then for $\mathcal{H}^s_E$ almost every $p\in A$ there exists $0<\epsilon<1$ such that*

$$\liminf_{r\to 0}\frac{\mathcal{H}^s_E(A\cap B_E(p,r)\setminus V(p)(r^{1+\epsilon}))}{r^s}=0.$$

Proof. By the Borel regularity of Hausdorff measures we may assume that A is a Borel set. Changing ϵ a bit it suffices to prove for $\mathcal{H}^s_E$ almost every $p \in A$ that

$$\liminf_{r\to 0} \frac{\mathcal{H}^s_E(A \cap B_E(p,r) \setminus V(p)(7r^{1+\epsilon}))}{r^s} = 0. \tag{6}$$

Writing $A_R = A \cap B_E(0,R)$, we have $A = \cup_{R=1}^{\infty} A_R$ and we could run the argument for any R such that $\mathcal{H}^s_H(A_R) > 0$. Hence we may assume that for some positive number R,

$$A \subset B_E(0,R).$$

First, let us see that we can reduce to the case when there is a positive number C such that

$$\frac{1}{C}\mathcal{H}^s_E(B) \leq \mathcal{H}^s_H(B) \leq C\mathcal{H}^s_E(B) \tag{7}$$

for every $B \subset A$. The left-hand side inequality holds because of (1). We can decompose A as

$$A = C \cup D,$$

where

$$\mathcal{H}^s_E(C) = 0 \quad \text{and} \quad \mathcal{H}^s_E(B) = 0 \Leftrightarrow \mathcal{H}^s_H(B) = 0 \quad \forall B \subset D. \tag{8}$$

This can be done as follows. Let $\mu_E = \mathcal{H}^s_E \llcorner A$ and $\mu_H = \mathcal{H}^s_H \llcorner A$. Since $\mu_E << \mu_H$, we have that for every Borel set $B \subset \mathbb{H}^1$,

$$\mu_E(B) = \int_B D(\mu_E, \mu_H, x) d\mu_H x,$$

where $D(\mu_E, \mu_H, x)$ is the Radon-Nikodym derivative of μ_E with respect to μ_H. Thus if we let

$$C = \{x \in A : D(\mu_E, \mu_H, x) = 0\}, \quad D = \{x \in A : D(\mu_E, \mu_H, x) > 0\},$$

then C and D satisfy (8). Moreover, we can write

$$D = \bigcup_{j=1}^{\infty} D_j, \quad \text{where} \quad \mathcal{H}^s_E(B) \geq \frac{1}{j}\mathcal{H}^s_H(B) \quad \forall B \subset D_j$$

by taking

$$D_j = \{x \in D : D(\mu_E, \mu_H, x) > \frac{1}{j}\}.$$

Thus (7) holds for every D_j in place of A. If we can prove (6) under the assumption (7), and so for every D_j, it follows that (6) holds for A by the the second part of the upper density theorem 1. More precisely, $\mathcal{H}^s_E$ almost every $p \in A$ belongs to some D_j and for $\mathcal{H}^s_E$ almost every $p \in D_j$ we have

$$\lim_{r\to 0} \frac{\mathcal{H}^s_E((A \setminus D_j) \cap B(p,r))}{(2r)^s} = 0,$$

so

$$\liminf_{r\to 0} \frac{\mathcal{H}^s_E(A \cap B_E(p,r) \setminus V(p)(r^{1+\epsilon}))}{r^s} = \liminf_{r\to 0} \frac{\mathcal{H}^s_E(D_j \cap B_E(p,r) \setminus V(p)(r^{1+\epsilon}))}{r^s}.$$

Hence we can assume (7).

Let $\epsilon > 0$. Supposing that (6) is false, we will reach a contradiction at the end of the proof if ϵ is small enough depending only on s, R and the constant C appearing in (7). By Theorem 1 there exist $c > 0$, $0 < r_0 < 1$ and $A' \subset A$, $\mathcal{H}^s_E(A') > 0$, such that

$$\mathcal{H}^s_E(A \cap B_E(p,r)) \leq 3^s r^s \tag{9}$$

and

$$\mathcal{H}^s_E(A \cap B_E(p,r) \setminus V(p)(7r^{1+\epsilon})) > cr^s \tag{10}$$

for every $p \in A'$ and $0 < r < r_0$. Let $0 < r < r_0/5$ and $p \in A'$ be such that $r^2 << r^{1+\epsilon}$ and

$$\mathcal{H}^s_E(B_H(p,r) \cap A') \geq \frac{1}{C}\mathcal{H}^s_H(B_H(p,r) \cap A') > c'r^s \tag{11}$$

with $c' = 1/(2C)$ (we can find these by Theorem 1). Let $k \in \mathbb{N}$ be such that $r^2 < r^{(1+\epsilon)^k}$ and $r^2 \geq r^{(1+\epsilon)^{k+1}}$, whence $k \geq \log 2/(2\log(1+\epsilon))$. By the $5r$ covering theorem, see, e.g., Theorem 2.1 in [M], for $j = 1, \ldots, k$ we can find $p_{j,i} \in A' \cap B_H(p,r)$ such that

$$A' \cap B_H(p,r) = \bigcup_{i=1}^{m_j} A' \cap B_H(p,r) \cap B_E(p_{j,i}, 5r^{(1+\epsilon)^j}), \tag{12}$$

where the balls $B_E(p_{j,i}, r^{(1+\epsilon)^j})$, $i = 1, \ldots, m_j$, are disjoint. Since by (11), (12) and (9),

$$\begin{aligned} c'r^s &< \mathcal{H}^s_E(B_H(p,r) \cap A') \\ &\leq \sum_{i=1}^{m_j} \mathcal{H}^s_E(A' \cap B_H(p,r) \cap B_E(p_{j,i}, 5r^{(1+\epsilon)^j})) \\ &\leq m_j 3^s 5^s r^{s(1+\epsilon)^j}, \end{aligned}$$

we obtain

$$m_j \geq c_1 r^{s(1-(1+\epsilon)^j)}, \tag{13}$$

with $c_1 = c'/15^s$ depending only on s and C.

In what follows we shall show that the sets

$$\bigcup_{i=1}^{m_j} B_E(p_{j,i}, r^{(1+\epsilon)^j}) \setminus V(p_{j,i})(7r^{(1+\epsilon)^{j+1}}),\ j = 1, \ldots, k, \tag{14}$$

are disjoint. Let $j \in \{1, \ldots, k-1\}$ and let $B_E(p_{j,i}, r^{(1+\epsilon)^j})$ and $B_E(p_{n,l}, r^{(1+\epsilon)^n})$, $j+1 \leq n \leq k$, $i \in \{1, \ldots, m_j\}, l \in \{1, \ldots, m_n\}$, be such that $B_E(p_{j,i}, r^{(1+\epsilon)^j}) \cap B_E(p_{n,l}, r^{(1+\epsilon)^n}) \neq \emptyset$. We want to show that

$$(B_E(p_{j,i}, r^{(1+\epsilon)^j}) \setminus V(p_{j,i})(7r^{(1+\epsilon)^{j+1}})) \cap B_E(p_{n,l}, r^{(1+\epsilon)^n}) = \emptyset. \tag{15}$$

Let us denote $p = (\bar{x}, \bar{y}, \bar{t})$, $p_{j,i} = (x_i, y_i, t_i)$, $p_{n,l} = (x_l, y_l, t_l)$. Since $p_{j,i}, p_{n,l} \in B_H(p,r)$, we have

$$((\bar{x} - x_i)^2 + (\bar{y} - y_i)^2)^2 + (\bar{t} - t_i + 2(x_i\bar{y} - y_i\bar{x}))^2 \leq r^4, \tag{16}$$

and

$$((\bar{x}-x_l)^2+(\bar{y}-y_l)^2)^2+(\bar{t}-t_l+2(x_l\bar{y}-y_l\bar{x}))^2\le r^4. \tag{17}$$

Moreover,

$$d_E(p_{j,i},p_{n,l})\le r^{(1+\epsilon)^j}+r^{(1+\epsilon)^n}\le 2r^{(1+\epsilon)^j}. \tag{18}$$

We now want to show that $d_E(p_{n,l},V(p_{j,i}))\le 6r^{(1+\epsilon)^{j+1}}$. Indeed by (4), (16), (17) and (18) we have

$$\begin{aligned}
d_E(p_{n,l},V(p_{j,i}))=&\frac{|t_l-t_i-2(x_ly_i-y_lx_i)|}{\sqrt{1+4(y_i^2+x_i^2)}}\\
\le&|t_l-t_i-2(x_ly_i-y_lx_i)|\\
\le&|t_l-\bar{t}-2(x_l\bar{y}-y_l\bar{x})|+|2(y_l\bar{x}-x_l\bar{y})+2(x_i\bar{y}-y_i\bar{x})\\
&+2(x_ly_i-y_lx_i)|+|\bar{t}-t_i+2(x_i\bar{y}-y_i\bar{x})|\\
\le&2r^2+2|(x_l-x_i)(y_i-\bar{y})-(y_l-y_i)(x_i-\bar{x})|\\
=&2r^2+2|\langle(x_l-x_i,y_i-y_l),(y_i-\bar{y},x_i-\bar{x})\rangle|\\
\le&2r^2+4r^{(1+\epsilon)^j}r\le 6r^2,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product and we used Cauchy-Schwarz inequality. Since $r^2<r^{(1+\epsilon)^k}\le r^{(1+\epsilon)^{j+1}}$, we have $d_E(p_{n,l},V(p_{j,i}))\le 6r^{(1+\epsilon)^{j+1}}$. Thus

$$B_E(p_{n,l},r^{(1+\epsilon)^n})\subset V(p_{j,i})(7r^{(1+\epsilon)^{j+1}}),$$

which implies (15). Hence the sets in (14) are disjoint.

We have $\mathcal{H}_E^s(A\cap B_E(p_{j,i},r^{(1+\epsilon)^j})\setminus V(p_{j,i})(7r^{(1+\epsilon)^{j+1}}))>cr^{s(1+\epsilon)^j}$ by (10) hence using (13) and the fact that $B_E(p_{j,i},r^{(1+\epsilon)^j})\subset B_E(p,3r)$ we get

$$\begin{aligned}
\mathcal{H}_E^s(A\cap B_E(p,3r))\ge&\sum_{j=1}^k\sum_{i=1}^{m_j}\mathcal{H}_E^s(A\cap B_E(p_{j,i},r^{(1+\epsilon)^j})\setminus V(p_{j,i})(7r^{(1+\epsilon)^{j+1}}))\\
\ge& c\sum_{j=1}^k m_jr^{s(1+\epsilon)^j}\ge cc_1\sum_{j=1}^k r^{s(1-(1+\epsilon)^j)}r^{s(1+\epsilon)^j}\\
=&cc_1kr^s\ge cc_1\log 2/(2\log(1+\epsilon))r^s.
\end{aligned}$$

When ϵ is small enough, the last term is greater than 7^sr^s. This yields a contradiction with Theorem 1. □

Remark. The above proof shows that if $A\subset B_E(0,R)$ satisfies (7), then we can choose ϵ depending only on s,R and C.

Theorem 3. *Let $s\ge 1$ and $A\subset\mathbb{H}^1$ be such that $\mathcal{H}_E^s(A)<\infty$. Then for $\mathcal{H}_H^{s+1}$ almost every $p\in A$ there exists $\delta>0$ such that*

$$\limsup_{r\to 0}\frac{\mathcal{H}_E^s(A\cap B_E(p,r)\setminus V(p)(\delta r))}{(2r)^s}>\frac{1}{2^{s+1}}. \tag{19}$$

Proof. We may assume that A is a Borel set and $A \subset B_E(0,R)$ for some $R > 0$. We can again, using Theorem 1, reduce the proof to the case where there exists a constant $C > 0$ such that for every $B \subset A$ we have

$$\frac{1}{C}\mathcal{H}^s_E(B) \le \mathcal{H}^{s+1}_H(B) \le C\mathcal{H}^s_E(B). \tag{20}$$

Indeed, this follows from a similar reasoning as was used to prove the right-hand side inequality in (7) since $\mathcal{H}^{s+1}_H << \mathcal{H}^s_E$ holds always by (2) (when $s \ge 1$, $\beta_+(s) = s+1$).

By Theorem 1 for $\mathcal{H}^s_E$ almost all $p \in A$ there exists $0 < r_p < 1$ such that for every $0 < r < r_p$

$$\mathcal{H}^s_E(A \cap B_E(p,r)) \le 3^s r^s \tag{21}$$

and

$$\mathcal{H}^{s+1}_H(A \cap B_H(p,r)) \le 3^{s+1} r^{s+1}. \tag{22}$$

For $j = 1, 2, \dots,$ let

$$A_j = \{p \in A : 2^{-j} \le r_p < 2^{-j+1}\}.$$

Then $\mathcal{H}^s_E(A \setminus \cup_{j=1}^\infty A_j) = 0$.

Let $p \in A_l$ and $\mathcal{H}^s_E(A_l) > 0$ for some l and let

$$0 < \delta < \frac{1}{2^{s+2}3^{s+2}C\sqrt{1+4R^2}}, \tag{23}$$

where C is as in (20). Hence, as in the previous proof, δ depends only on s, R and C. For every $0 < r < 2^{-l}, r < \delta$, we want to show that there exist $p_1, \dots, p_k$, $k \approx (\delta/r)\sqrt{1+4R^2}$, such that

$$B_E(p,r) \cap V(p)(\delta r) \subset \bigcup_{i=1}^k B_H(p_i, 2r). \tag{24}$$

Let $p = (\bar{x}, \bar{y}, \bar{t})$. By (3) the horizontal plane $V(p)$ is the set of points $(x,y,t) \in \mathbb{H}^1$ such that

$$\bar{t} - t - 2(x\bar{y} - y\bar{x}) = 0.$$

Let $L(p)$ be the vertical line passing through p, that is $L(p) = \{(\bar{x}, \bar{y}, t) : t \in \mathbb{R}\}$. If $q = (\bar{x}, \bar{y}, t) \in L(p)$ and $d_E(q, V(p)) \le \delta r$ then $|t - \bar{t}| \le \delta r\sqrt{1+4R^2}$. Indeed, by (4) we have

$$\begin{aligned} \delta r \ge d_E(q, V(p)) &= \frac{|t - \bar{t} - 2(\bar{x}\bar{y} - \bar{y}\bar{x})|}{\sqrt{1+4(\bar{x}^2+\bar{y}^2)}} \\ &= \frac{|t-\bar{t}|}{\sqrt{1+4(\bar{x}^2+\bar{y}^2)}} \ge \frac{|t-\bar{t}|}{\sqrt{1+4R^2}}. \end{aligned} \tag{25}$$

Cover the interval $[\bar{t} - \delta r\sqrt{1+4R^2}, \bar{t} + \delta r\sqrt{1+4R^2}]$ with intervals $[t_i, t_{i+1}]$, $i = 1, \dots, k$, with $t_{i+1} - t_i = r^2$ and

$$k \le 3(\delta/r)\sqrt{1+4R^2}. \tag{26}$$

Let

$$p_i = (\bar{x}, \bar{y}, t_i) \in L(p).$$

If $u \in L(p) \cap V(p)(\delta r)$ then there exists i such that $d_E(u, p_i) \leq r^2$ by (25). To see that (24) holds, let $q = (x, y, t) \in B_E(p, r) \cap V(p)(\delta r)$ and let q' be the point of intersection between the plane passing through q parallel to $V(p)$ and the line $L(p)$. This means that

$$q' = (\bar{x}, \bar{y}, t') \quad \text{and} \quad t' - t - 2((\bar{x} - x)\bar{y} - (\bar{y} - y)\bar{x}) = 0,$$

hence

$$q' = (\bar{x}, \bar{y}, t + 2(y\bar{x} - x\bar{y})).$$

Since $q' \in L(p) \cap V(p)(\delta r)$, there exists $j \in \{1, \ldots, k\}$ such that

$$d_E(q', p_j) = |t + 2(y\bar{x} - x\bar{y}) - t_j| \leq r^2. \tag{27}$$

Let us now see that $q \in B_H(p_j, 2r)$, that is $d_H(q, p_j) \leq 2r$. Indeed,

$$d_H(q, p_j)^4 = ((x - \bar{x})^2 + (y - \bar{y})^2)^2 + (t - t_j + 2(\bar{x}y - \bar{y}x))^2. \tag{28}$$

Since $q \in B_E(p, r)$, we have

$$(x - \bar{x})^2 + (y - \bar{y})^2 \leq r^2,$$

and by (27) the second term in (28) is $\leq r^4$. It follows that $d_H(q, p_j) \leq 2r$, which proves (24).

Hence by (24), (20), (22), (26) and (23) we have that for every $0 < r < 2^{-l}$,

$$\begin{aligned}
\mathcal{H}^s_E(A \cap B_E(p, r) \cap V(p)(\delta r)) &\leq \sum_{i=1}^{k} \mathcal{H}^s_E(A \cap B_H(p_i, 2r)) \\
&\leq C \sum_{i=1}^{k} \mathcal{H}^{s+1}_H(A \cap B_H(p_i, 2r)) \\
&\leq Ck3^{s+1}2^{s+1}r^{s+1} \\
&\leq C3^{s+2}2^{s+1}\frac{\delta}{r}\sqrt{1 + 4R^2}r^{s+1} < \frac{1}{2}r^s.
\end{aligned}$$

Thus for $\mathcal{H}^s_E$ almost every $p \in A$ there exists $\delta > 0$ such that

$$\limsup_{r \to 0} \frac{\mathcal{H}^s_E(A \cap B_E(p, r) \cap V(p)(\delta r))}{(2r)^s} < \frac{1}{2^{s+1}},$$

which proves (19) by Theorem 1. □

Remark. It is easy to give examples where the lower limit of the expression in (19), and in (29) in Theorem 4, is 0 everywhere. For example, this is so for any set of lower density zero, recall the comment after Theorem 1. On the other hand, for many sets the lower limit can be positive, for example, for classical Cantor sets C in the vertical axis for which $0 < \mathcal{H}^s_E(C) < \infty$ and $0 < \mathcal{H}^{2s}_H(C) < \infty$ for some $0 < s < 1$

Theorem 4. *Let $0 < s < 1$ and $A \subset \mathbb{H}^1$ be such that $\mathcal{H}^s_E(A) < \infty$. Then for $\mathcal{H}^{2s}_H$ almost every $p \in A$ there exists $\delta > 0$ such that*

$$\limsup_{r \to 0} \frac{\mathcal{H}^s_E(A \cap B_E(p, r) \setminus V(p)(\delta r))}{(2r)^s} > 0. \tag{29}$$

We do not know if one can replace 0 with a positive constant as in the previous theorem, our proof would give only a constant depending on the point p.

Proof. We may assume that A is a Borel set and $A \subset B_E(0,R)$ for some $R > 0$. Since $\mathcal{H}_H^{2s} << \mathcal{H}_E^s$ always holds (here $\beta_+(s) = 2s$ because $s < 1$), we can assume, as in the proof of Theorem 3, that there exists $C > 0$ such that

$$\frac{1}{C}\mathcal{H}_E^s(B) \leq \mathcal{H}_H^{2s}(B) \leq C\mathcal{H}_E^s(B) \tag{30}$$

for every $B \subset A$.

Suppose that (29) does not hold. Let $A_0 \subset A$ be a Borel set such that $\mathcal{H}_E^s(A_0) > 0$ and that (29) fails for $p \in A_0$ for every $\delta > 0$. Fix $\delta > 0$ and $\epsilon > 0$, to be chosen sufficiently small at the end of the proof. Then there exist a Borel set $A' \subset A_0$ and $r_0 > 0$ such that $\mathcal{H}_E^s(A') > \mathcal{H}_E^s(A_0)/2$ and for every $p \in A'$ and for every $0 < r < r_0$,

$$\mathcal{H}_E^s(A \cap B_E(p,r) \setminus V(p)(\delta r)) < \epsilon r^s. \tag{31}$$

Let $0 < \eta < \min\{r_0, \delta\}$. Let $c = 3^s$. Then by Theorem 1 for $\mathcal{H}_E^s$ almost all $p \in A'$ there is $r_p < \eta$ such that

$$\frac{r_p^s}{c} \leq \mathcal{H}_E^s(A' \cap B_E(p,r_p)) \leq c r_p^s. \tag{32}$$

Applying Vitali's covering theorem (see Theorem 2.8 in [M]) to the family of balls

$$\{B_E(p,r_p) : p \in A' \text{ such that } r_p \text{ satisfying (32) exists}\},$$

we find a subfamily of disjoint balls, $\{B_E(p_i,r_i)\}_{i=1}^{\infty}$, such that

$$\mathcal{H}_E^s\left(A' \setminus \bigcup_{i=1}^{\infty} B_E(p_i,r_i)\right) = 0. \tag{33}$$

Hence we have by (32)

$$\begin{aligned}\mathcal{H}_E^s(A') &= \mathcal{H}_E^s\left(A' \cap \bigcup_{i=1}^{\infty} B_E(p_i,r_i)\right) \\ &= \sum_{i=1}^{\infty} \mathcal{H}_E^s(A' \cap B_E(p_i,r_i)) \geq \frac{1}{c}\sum_{i=1}^{\infty} r_i^s. \end{aligned} \tag{34}$$

Since $p_i \in B_E(0,R)$, we have by (1) that

$$\mathrm{diam}_H(B_E(p_i,r_i)) \leq c_R \mathrm{diam}_E(B_E(p_i,r_i))^{1/2} \leq c_R\sqrt{2\eta}.$$

Let $\eta' = c_R\sqrt{2\eta}$. Then we have

$$\begin{aligned}\mathcal{H}_{H,\eta'}^{2s}(A') &\leq \sum_{i=1}^{\infty} \mathcal{H}_{H,\eta'}^{2s}(A' \cap B_E(p_i,r_i)) \\ &\leq \sum_{i=1}^{\infty} \mathcal{H}_{H,\eta'}^{2s}(A' \cap B_E(p_i,r_i) \cap V(p_i)(\delta r_i)) \\ &\quad + \sum_{i=1}^{\infty} \mathcal{H}_{H,\eta'}^{2s}(A' \cap B_E(p_i,r_i) \setminus V(p_i)(\delta r_i)). \end{aligned} \tag{35}$$

Moreover, we have

$$\begin{aligned}\mathcal{H}^{2s}_{H,\eta'}(A' \cap B_E(p_i, r_i) \cap V(p_i)(\delta r_i)) &\leq \operatorname{diam}_H(B_E(p_i, r_i) \cap V(p_i)(\delta r_i))^{2s} \\ &\leq (2(c_R + 1)\sqrt{\delta r_i})^{2s} = C''(\delta r_i)^s. \qquad (36)\end{aligned}$$

To see this, let $q, q' \in B_E(p_i, r_i) \cap V(p_i)(\delta r_i)$ and let $\bar{q}, \bar{q}' \in B_E(p_i, r_i) \cap V(p_i)$ be such that $d_E(q, \bar{q}) \leq \delta r_i$ and $d_E(q', \bar{q}') \leq \delta r_i$. Then

$$d_H(q, q') \leq d_H(q, \bar{q}) + d_H(\bar{q}, \bar{q}') + d_H(\bar{q}', q').$$

We have $d_H(q, \bar{q}) \leq c_R d_E(q, \bar{q})^{1/2} \leq c_R(\delta r_i)^{1/2}$, $d_H(q', \bar{q}') \leq c_R d_E(q', \bar{q}')^{1/2} \leq c_R(\delta r_i)^{1/2}$ by (1) and

$$d_H(\bar{q}, \bar{q}') \leq d_H(\bar{q}, p_i) + d_H(p_i, \bar{q}') = d_E(\bar{q}, p_i) + d_E(p_i, \bar{q}') \leq 2r_i,$$

where we used the fact that $d_H(u, p_i) = d_E(u, p_i)$ if $u \in V(p_i)$. Since $r_i < \eta < \delta$, it follows that $r_i \leq (\delta r_i)^{1/2}$, hence

$$d_H(q, q') \leq 2c_R(\delta r_i)^{1/2} + 2r_i \leq 2(c_R + 1)(\delta r_i)^{1/2},$$

which proves (36). On the other hand, by (30) and (31) we have

$$\mathcal{H}^{2s}_H(A' \cap B_E(p_i, r_i) \setminus V(p_i)(\delta r_i)) \leq C\mathcal{H}^s_E(A' \cap B_E(p_i, r_i) \setminus V(p_i)(\delta r_i)) < C\epsilon r_i^s,$$

thus also

$$\mathcal{H}^{2s}_{H,\eta'}(A' \cap B_E(p_i, r_i) \setminus V(p_i)(\delta r_i)) < C\epsilon r_i^s. \qquad (37)$$

Hence we have by (35), (36), (37) and (34)

$$\begin{aligned}\mathcal{H}^{2s}_{H,\eta'}(A') &\leq (C''\delta^s + C\epsilon) \sum_{i=1}^{\infty} r_i^s \\ &\leq c(C''\delta^s + C\epsilon)\mathcal{H}^s_E(A') \\ &\leq 2c(C''\delta^s + C\epsilon)\mathcal{H}^s_E(A_0).\end{aligned}$$

whence letting η and η' tend to 0,

$$0 < \mathcal{H}^{2s}_H(A_0) < 2\mathcal{H}^{2s}_H(A') \leq 2c(C''\delta^s + C\epsilon)\mathcal{H}^s_E(A_0).$$

Since δ and ϵ are allowed to depend on A_0 and they can be chosen arbitrarily small, we have a contradiction which completes the proof. □

4 Examples

We show the sharpness of Theorem 2 with three examples. Example 5 shows that we cannot replace lim inf by lim sup, Example 6 shows that we cannot replace the $r^{1+\epsilon}$-neighbourhood by Mr^2-neighbourhood for any positive number M, in particular we cannot replace it with the Heisenberg ball $B_H(p, r)$. We shall construct these two examples only for $s = 1$, but very likely similar examples can be given for any $0 < s < 2$. Example 7 shows that when $s > 2$ then in arbitrarily small neighbourhoods around a point p the set cannot lie too close to the horizontal plane through p, in the sense that we cannot obtain the same conclusion as in Theorem 2.

Example 5. *There exists a compact set $F \subset \mathbb{H}^1$ such that for some positive constant C, $\mathcal{H}^1_H(F) > 0$ and $\mathcal{H}^1_H(A) \leq C\mathcal{H}^1_E(A) < \infty$ for $A \subset F$, and for $p \in F$,*

$$\limsup_{r\to 0} \frac{\mathcal{H}^1_E(F \cap B_E(p,r) \setminus V(p)(r/8))}{2r} \geq \frac{1}{8}. \tag{38}$$

Example 6. *For any $M, 1 < M < \infty$, there exists a compact set $F \subset \mathbb{H}^1$ such that for some positive constant C, $\mathcal{H}^1_H(F) > 0$ and $\mathcal{H}^1_H(A) \leq C\mathcal{H}^1_E(A) < \infty$ for $A \subset F$, and for $p \in F$,*

$$\liminf_{r\to 0} \frac{\mathcal{H}^1_E(F \cap B_E(p,r) \setminus V(p)(Mr^2))}{2r} \geq \frac{1}{16}. \tag{39}$$

Both examples will follow from the same construction which we now describe. In both cases F will be a subset of the vertical plane $V = \{(x,y,t) : y = 0\}$, whose points will now be written as (x,t). The metric d_H restricted to this plane is given by

$$d_H((x_1,t_1),(x_2,t_2)) = \left((x_1-x_2)^4 + (t_1-t_2)^2\right)^{1/4}.$$

For $p = (x,t) \in V$, the horizontal plane $V(p)$ intersects V along the line $\{(u,t) : u \in \mathbb{R}\}$.

For $p, q \in V$ we have $d_E(p,q) \leq d_H(p,q)$ if $d_E(p,q) \leq 1/2$. Thus

$$H^1_E(B) \leq \mathcal{H}^1_H(B) \text{ for } B \subset V.$$

Let n be an integer, $n \geq 1$, and λ a positive number, $0 < \lambda \leq 1/2$. For a rectangle $R = [a,b] \times [c,d] \subset V$ such that $\lambda(b-a) < d-c$ we let $\mathcal{R}(R,n,\lambda)$ be the collection of the following $2n$ subrectangles:

$$[a + 2i\frac{b-a}{2n}, a + 2i\frac{b-a}{2n} + \frac{b-a}{2n}] \times [c, c+\lambda(b-a)],$$
$$[a + (2i+1)\frac{b-a}{2n}, a + (2i+1)\frac{b-a}{2n} + \frac{b-a}{2n}] \times [d-\lambda(b-a), d],$$

for $i = 0, \ldots, n-1$.

Let (n_k) be a sequence of integers, $n_k \geq 1$, and (λ_k) a sequence of positive numbers, $\lambda_k \leq 1/2$. We define for $k \geq 1$,

$$\mathcal{R}_0 = \mathcal{R}([0,1]^2, 1, 1/2),$$
$$\mathcal{R}_k = \bigcup_{R\in\mathcal{R}_{k-1}} \mathcal{R}(R, n_k, \lambda_k),$$

and

$$F = \bigcap_{k=0}^{\infty} \bigcup_{R\in\mathcal{R}_k} R.$$

Then $F \subset V$ is compact and the projection of F on the x-axis is $[0,1]$. Thus both $\mathcal{H}^1_E(F)$ and $\mathcal{H}^1_H(F)$ are at least 1. Using the natural coverings with the rectangles of $\mathcal{R}_k$, one easily checks that $\mathcal{H}^1_E(F)$ and $\mathcal{H}^1_H(F)$ are also finite provided λ_k goes to 0 sufficiently fast. More precisely, let h_k be the length of the horizontal sides of the rectangles of $\mathcal{R}_k$ and let v_k be the length of their vertical sides. Then the Euclidean diameter of each $R \in \mathcal{R}_k$ is $(h_k^2 + v_k^2)^{1/2}$ and the Heisenberg diameter is $(h_k^4 + v_k^2)^{1/4}$. If v_k/h_k tends to zero as $k \to \infty$, then

$$\mathcal{H}_E^1(F \cap R) = h_k \text{ for } R \in \mathcal{R}_k, \tag{40}$$

in particular, $\mathcal{H}_E^1(F) = 1$. If moreover, $v_k \leq C h_k^2$ for all large enough k, then

$$\mathcal{H}_H^1(A) \leq (1 + C^2)^{1/4} \mathcal{H}_E^1(A) \text{ for } A \subset F. \tag{41}$$

These conditions on h_k and v_k will be satisfied in both examples below; in Example 5 $v_k = h_k^2$ and in Example 6 $v_k = 34M h_k^2$ for large k.

For Example 5 we choose $n_k = 2^{2^{k-1}-1}$ and $\lambda_k = 2^{-3 \cdot 2^{k-1}}$. As a consequence, the rectangles in $\mathcal{R}_k$ have horizontal sides of length $h_k = 2^{-2^k}$ and the vertical sides of length $2^{-2^{k+1}}$. For $R \in \mathcal{R}_k$, the horizontal sides of each rectangle R' of $\mathcal{R}_{k+1}$ inside R thus has the same length as the vertical sides of R (see Figure 1). This implies that for $p \in R'$ and $r_k = 4h_{k+1}$, $B_E(p, r_k) \setminus V(p)(r_k/8)$ contains another rectangle R'' of $\mathcal{R}_{k+1}$. Hence

$$\frac{\mathcal{H}_E^1(F \cap B_E(p, r_k) \setminus V(p)(r_k/8))}{2r_k} \geq \frac{\mathcal{H}_E^1(F \cap R'')}{8h_{k+1}} = \frac{1}{8},$$

from which, recalling also (41), the asserted properties follow.

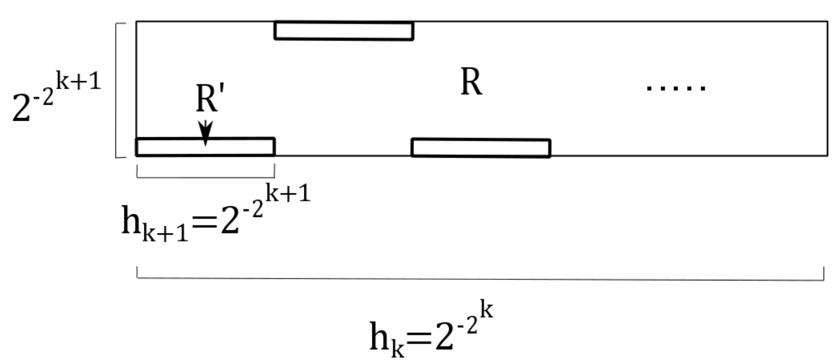

Figure 1: A rectangle $R \in \mathcal{R}_k$ and a rectangle $R' \in \mathcal{R}_{k+1}$ inside R in Example 5

For Example 6 we choose $n_k = 1$ and we let $\lambda_k = 1/2$, when $34M4^{-k} \geq 2^{-k}$, that is, $2^k \leq 34M$, and $\lambda_k = 34M2^{-k-1}$ for all larger k. As a consequence, for large enough k, the rectangles in $\mathcal{R}_k$ have horizontal sides of length $h_k = 2^{-k}$ and vertical sides of length $34M4^{-k}$. For $R \in \mathcal{R}_k$, we have two rectangles R_1 and R_2 of $\mathcal{R}_{k+1}$ inside R, one along the lower side of R and one along the upper. The distance between these rectangles is $34M4^{-k} - 2 \cdot 34M4^{-k-1} = 17M4^{-k}$ (see Figure 2).

Let $0 < r < 1$ and let k be such that $2^{1-k} \leq r < 2^{2-k}$. We assume that r is small enough so that $2^k > 68M$. Let R, R_1 and R_2 be as above and $p \in R_2$. Then $Mr^2 \leq M4^{2-k} < 17M4^{-k}$, whence R_1 lies outside $V(p)(Mr^2)$. On the other hand, as $2^{1-k} \leq r$, R_1 lies inside $B_E(p,r)$. This implies that $B_E(p,r) \setminus V(p)(Mr^2)$ contains R_1, from which the asserted properties follow as in the case of Example 5.

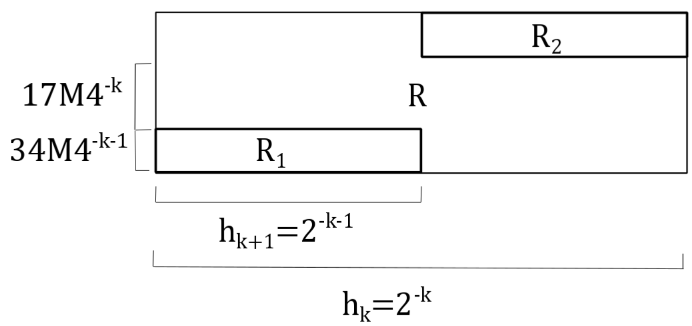

Figure 2: A rectangle $R \in \mathcal{R}_k$ and the rectangles $R_1, R_2 \in \mathcal{R}_{k+1}$ inside R in Example 6

The next example shows that the conclusion of Theorem 2 fails when $2 < s < 3$.

Example 7. *For any $2 < s < 3$ there exist constants $c_s, \delta_s > 0$ and a set $F_s \subset \mathbb{H}^1$ such that $\mathcal{H}^s_E(F_s) > 0$, $\mathcal{H}^{2s-2}_H(F_s) < \infty$ and for $\mathcal{H}^s_E$ almost every $p \in F_s$,*

$$\liminf_{r \to 0} \frac{\mathcal{H}^s_E(F_s \cap B_E(p,r) \setminus V(p)(\delta_s r))}{r^s} \geq c_s. \tag{42}$$

This example is taken from Theorem 4.1 in [BT] (see also [S]), where it is used to show the sharpness of some of the dimension inequalities. We will consider the Heisenberg square Q_H and a certain Cantor set above each point of Q_H. The Heisenberg square is the invariant set of the affine iterated function system F_1, F_2, F_3, F_4, that is $Q_H = \cup_{j=1}^4 F_j(Q_H)$. The maps $F_i : \mathbb{H}^1 \to \mathbb{H}^1$, $i = 1,2,3,4$, are similarities with respect to d_H with contraction ratio $1/2$ and they are horizontal lifts of f_j, $j = 1,2,3,4$, which are maps in the plane. This means that $\pi \circ F_j = f_j \circ \pi$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection $\pi(x,y,t) = (x,y)$. These maps have the form $f_j(x,y) = \frac{1}{2}((x,y) + v_j)$, where $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (0,1)$ and $v_4 = (1,1)$. Then we have $\pi(Q_H) = Q = [0,1]^2$, where Q is the invariant set of the iterated function system f_1, f_2, f_3, f_4. See [BHT] and [BT] for more details. We will use the symbolic dynamics notation: for $m \geq 1$ and $w = w_1 w_2 \dots w_m \in W_m = \{1,2,3,4\}^m$ we let $F_w = F_{w_1} \circ \dots \circ F_{w_m}$. Then $Q_H = \cup_{w \in W_m} F_w(Q_H)$ for every m.

Given $2 < s < 3$, let $d = s - 2$ and let C_d be a standard symmetric Cantor set in the t-axis such that $0 < \mathcal{H}^d_E(C_d) < \infty$. Then $0 < \mathcal{H}^{2d}_H(C_d) < \infty$. Moreover, C_d is d-Ahlfors regular, which implies that for $0 < r < 1$ and $(0,0,t') \in C_d$,

$$\mathcal{H}_E^d(\{(0,0,t) \in C_d : c_0 r \leq |t - t'| \leq \frac{r}{4}\}) \geq c_d r^d \tag{43}$$

for some constants c_0 and c_d. The set C_d is the invariant set associated to two maps G_1, G_2, which are $2^{-1/2d}$-Lipschitz with respect to d_H. Let

$$F_s = \{(x, y, t + t') : (x, y, t) \in Q_H, (0, 0, t') \in C_d\}.$$

It is shown in Theorem 4.1 in [BT] that

$$\mathcal{H}_E^s(F_s) > 0 \quad \text{and} \quad \mathcal{H}_H^{2s-2}(F_s) < \infty.$$

Let $p = (\bar{x}, \bar{y}, \bar{t} + \bar{t}') \in F_s$ and let $0 < r < \min\{1/20, c_0/6\}$. Let m be the integer such that $2^{-m+2} \operatorname{diam}_H(Q_H) \leq r < 2^{-m+3} \operatorname{diam}_H(Q_H)$, then

$$2^{-m+2} \operatorname{diam}_H(Q_H) < \min\{1/20, c_0/6\}.$$

Let n be the smallest integer such that $n \geq 2dm$. For $w \in \{1,2,3,4\}^m$ and $v \in \{1,2\}^n$, let

$$F_s^{vw} = \{(x, y, t + t') : (x, y, t) \in F_w(Q_H), (0, 0, t') \in G_v(C_d)\} \subset F_s.$$

Then

$$\operatorname{diam}_H(F_s^{vw}) \leq 2^{-m+2} \operatorname{diam}_H(Q_H) \leq r.$$

Indeed, if $\operatorname{diam}_H(F_s^{vw}) = d_H((x, y, t + t'), (\tilde{x}, \tilde{y}, \tilde{t} + \tilde{t}'))$, then we have, as shown in the proof of Theorem 4.1 in [BT],

$$\begin{aligned}
\operatorname{diam}_H(F_s^{vw})^4 &= ((x - \tilde{x})^2 + (y - \tilde{y})^2)^2 + (t + t' - \tilde{t} - \tilde{t}' + 2(\tilde{x}y - \tilde{y}x))^2 \\
&\leq 2(((x - \tilde{x})^2 + (y - \tilde{y})^2)^2 + (t - \tilde{t} + 2(\tilde{x}y - \tilde{y}x))^2 + (t' - \tilde{t}')^2) \\
&\leq 2(2^{-4m} \operatorname{diam}_H(Q_H)^4 + 2^{-2n/d}) \leq 2^{-4m+2} \operatorname{diam}_H(Q_H)^4.
\end{aligned}$$

Let w and v be such that $p \in F_s^{vw}$, so we have $F_s^{vw} \subset F_s \cap B_H(p, r)$. Let now $q = (x, y, t + t') \in F_s^{vw}$ be such that $|x - \bar{x}|^2 + |y - \bar{y}|^2 \leq r^2/400$. Then $q \in B_H(p, r)$ and we have

$$\begin{aligned}
|t + t' - \bar{t} - \bar{t}'| &\leq |t + t' - \bar{t} - \bar{t}' + 2(\bar{x}y - \bar{y}x)| + 2|\bar{x}y - \bar{y}x| \\
&\leq d_H(p, q)^2 + 2|\bar{x}(y - \bar{y}) + (\bar{x} - x)\bar{y}| \\
&\leq r^2 + \frac{4r}{20} \leq \frac{r}{20} + \frac{4r}{20} = \frac{r}{4}.
\end{aligned} \tag{44}$$

Let

$$C_d^q = \{(0, 0, t'') \in C_d : c_0 r \leq |t'' - t'| \leq \frac{r}{4}\}. \tag{45}$$

We want to show that the set

$$L_q = \{(x, y, t + t'') : (0, 0, t'') \in C_d^q\}$$

is contained in

$$D_r = F_s \cap B_E(p, r) \cap \{(x, y, t) : |x - \bar{x}|^2 + |y - \bar{y}|^2 \leq r^2/400\} \setminus V(p)(c_0 r/6).$$

Let $q'' = (x, y, t + t'') \in L_q$. Then $q'' \in F_s$ since $(x, y, t) \in Q_H$ and $(0, 0, t'') \in C_d$. Moreover, by (44) and (45) we have

$$|t + t'' - \bar{t} - \bar{t}'| \leq |t + t' - \bar{t} - \bar{t}'| + |t'' - t'| \leq \frac{r}{4} + \frac{r}{4} = \frac{r}{2},$$

thus

$$|q'' - p|^2 = |x - \bar{x}|^2 + |y - \bar{y}|^2 + |t + t'' - \bar{t} - \bar{t}'|^2 \leq \frac{r^2}{400} + \frac{r^2}{4} < r^2.$$

This implies that $q'' \in B_E(p, r)$. It remains to show that $d_E(q'', V(p)) \geq c_0 r/6$. Using (4), (45) and the facts that $\bar{x}^2 + \bar{y}^2 \leq 2$ and $d_H(p, q) \leq r < c_0/6$, we have

$$\begin{aligned} d_E(q'', V(p)) &= \frac{|t' + t'' - \bar{t} - \bar{t}' - 2(x\bar{y} - y\bar{x})|}{\sqrt{1 + 4(\bar{x}^2 + \bar{y}^2)}} \\ &\geq \frac{|t'' - t'|}{\sqrt{1 + 4(\bar{x}^2 + \bar{y}^2)}} - \frac{|t + t' - \bar{t} - \bar{t}' - 2(x\bar{y} - y\bar{x})|}{\sqrt{1 + 4(\bar{x}^2 + \bar{y}^2)}} \\ &\geq |t' - t''|/3 - d_H(q, p)^2 \geq c_0 r/3 - r^2 \geq c_0 r/6. \end{aligned}$$

Hence we have

$$L_q \subset D_r \subset F_s \cap B_E(p, r) \setminus V(p)(c_0 r/6). \tag{46}$$

In particular, for every point $(x, y, t) \in F_w(Q_H)$ such that $|x - \bar{x}|^2 + |y - \bar{y}|^2 \leq r^2/400$ there are points $(x, y, t + t'') \in D_r$. Thus

$$\begin{aligned} &\pi(F_s \cap B_E(p, r) \setminus V(p)(c_0 r/6)) \supset \pi(D_r) \\ &\supset \pi(F_w(Q_H)) \cap \{(x, y) : |x - \bar{x}|^2 + |y - \bar{y}|^2 \leq r^2/400\} \\ &= f_w(Q) \cap \{(x, y) : |x - \bar{x}|^2 + |y - \bar{y}|^2 \leq r^2/400\}, \end{aligned}$$

which implies

$$\mathcal{H}^2_E(\pi(D_r)) \geq \mathcal{H}^2_E(f_w(Q) \cap \{(x, y) : |x - \bar{x}|^2 + |y - \bar{y}|^2 \leq r^2/400\}) \geq c r^2 \tag{47}$$

for some constant c. Then by (46), (43), (47) and Theorem 7.7 in [M] we have for some constant $c' > 0$,

$$\begin{aligned} &\mathcal{H}^s{}_E(F_s \cap B_E(p, r) \setminus V(p)(c_0 r/6)) \geq \mathcal{H}^s_E(D_r) \\ &\geq c' \int_{\pi(D_r)} \mathcal{H}^d_E(\{(0, 0, t_q + t'') : q = (x_q, y_q, t_q + t'') \in D_r\}) d\mathcal{H}^2_E(x_q, y_q) \\ &\geq c' \int_{\pi(D_r)} \mathcal{H}^d_E(C^q_d) d\mathcal{H}^2_E(x_q, y_q) \\ &\geq c' c_d r^d \mathcal{H}^2_E(\pi(D_r)) \geq c' c_d c r^{2+d} = c_s r^s, \end{aligned}$$

which implies (42).

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