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Nonlinear Torsional Wave Beams

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Abstract. An evolution equation with cubic nonlinearity is presented for a torsional wave beam in an isotropic elastic solid. Analytical solutions are presented for the fundamental and third harmonic in the far field of a uniform circular source. Numerical results are presented for harmonic beam patterns at an intermediate distance between the near and far fields, and for a torsional waveform with shocks.

Keywords: torsional waves, diffraction, cubic nonlinearity

PACS: 43.25.Dc

INTRODUCTION

Previous theoretical investigations of cubic nonlinearity in diffracting beams apply only to shear waves with linear polarization, [1, 2, 3] and only recently have elliptical and circular polarizations been considered for plane nonlinear shear waves. [4] At the 19th International Congress on Acoustics we reported on nonlinear shear wave beams with elliptical and circular polarizations. [5] In the present work we consider azimuthal polarization, whereby the beam is produced by rotational oscillation of a disk about its axis. Although wavefront curvature introduces quadratic nonlinearity in shear waves, [1] considered here are beams of quasi-plane shear waves affected only by cubic nonlinearity.

EVOLUTION EQUATIONS

We consider a shear wave beam that propagates along the z axis in an isotropic elastic solid. The pair of coupled evolution equations for the particle velocity components v_x and v_y , perpendicular to the propagation axis are found to be, in the parabolic approximation and with only cubic nonlinearity taken into account, [6]

$$\frac{\partial v_x}{\partial z} = \frac{\beta}{3c^3} \frac{\partial}{\partial \tau} [v_x(v_x^2 + v_y^2)] + \frac{\eta}{2\rho c^3} \frac{\partial^2 v_x}{\partial \tau^2} + \frac{c}{2} \int_{-\infty}^{\tau} \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) d\tau', \quad (1)$$

$$\frac{\partial v_y}{\partial z} = \frac{\beta}{3c^3} \frac{\partial}{\partial \tau} [v_y(v_x^2 + v_y^2)] + \frac{\eta}{2\rho c^3} \frac{\partial^2 v_y}{\partial \tau^2} + \frac{c}{2} \int_{-\infty}^{\tau} \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) d\tau', \quad (2)$$

where β is the coefficient of nonlinearity, $c = (\mu/\rho)^{1/2}$ the small-signal wave speed, μ the shear modulus, ρ the density of the material in its undeformed state, $\tau = t - z/c$ the retarded time, and η the coefficient of shear viscosity. For an isotropic elastic solid with its strain energy density expressed in the form [7]

$$\mathcal{E} = \mu I_2 + \left(\frac{1}{2}K - \frac{1}{3}\mu\right) I_1^2 + \frac{1}{3}A I_3 + B I_1 I_2 + \frac{1}{3}C I_1^3 + E I_1 I_3 + F I_1^2 I_2 + G I_2^2 + H I_1^4, \quad (3)$$

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where I_1, I_2 and I_3 are invariants of the Lagrangian strain tensor, K is the bulk modulus, A, B and C are the third-order elastic constants of Landau and Lifshitz, [8] and E, F, G and H are fourth-order elastic constants, the coefficient of nonlinearity is

$$\beta = \frac{3}{4\mu} \left[K + \frac{4}{3}\mu + A + 2B + 2G - \frac{(K + \frac{4}{3}\mu + \frac{1}{2}A + B)^2}{K + \frac{1}{3}\mu} \right]. \quad (4)$$

For soft elastic media characterized by $\mu \ll K$, the strain energy density may be expressed in the reduced form $\mathcal{E} = \mu I_2 + \frac{1}{3}AI_3 + DI_2^2$, [7] and the coefficient of nonlinearity becomes $\beta = \frac{3}{2}[1 + (\frac{1}{2}A + D)/\mu]$. [4]

A torsional wave beam may be produced by rotational oscillation of a disk about the z axis. For simplicity the source is assumed to be axisymmetric, such that in the cylindrical coordinates (r, ϕ, z) one obtains for the components of the particle velocity vector in the plane perpendicular to the z axis $v_r = 0$ and $v_\phi = v_\phi(r, z, \tau)$. In this case $v_x = -v_\phi \sin \phi$, $v_y = v_\phi \cos \phi$, and $v_x^2 + v_y^2 = v_\phi^2$, such that Eqs. (1) and (2) yield

$$\frac{\partial v_\phi}{\partial z} = \frac{\beta}{c^3} v_\phi^2 \frac{\partial v_\phi}{\partial \tau} + \frac{\eta}{2\rho c^3} \frac{\partial^2 v_\phi}{\partial \tau^2} + \frac{c}{2} \int_{-\infty}^{\tau} \left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2} \right) d\tau'. \quad (5)$$

Due to the term v_ϕ/r^2 , the quantity under the integral is no longer a Laplacian of the field variable. Torsional waves are often more conveniently expressed in terms of the angular velocity in the medium, $\Omega = \partial \phi / \partial \tau$, for which $v_\phi = r\Omega$ and Eq. (5) becomes

$$\frac{\partial \Omega}{\partial z} = r^2 \frac{\beta}{c^3} \Omega^2 \frac{\partial \Omega}{\partial \tau} + \frac{\eta}{2\rho c^3} \frac{\partial^2 \Omega}{\partial \tau^2} + \frac{c}{2} \int_{-\infty}^{\tau} \left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{3}{r} \frac{\partial \Omega}{\partial r} \right) d\tau'. \quad (6)$$

From this equation it is evident that nonlinearity vanishes along the axis of the beam.

ANALYTICAL RESULTS

The source condition for radiation at angular frequency ω is expressed in the form $\Omega(r, 0, t) = \Omega_0 \text{Re}[f(r)e^{i\omega t}]$, and the solutions for the resulting harmonics are expressed as $\Omega_n(r, z, \tau) = \text{Re}[W_n(r, z)e^{in\omega\tau}]$. For weak nonlinearity, one may obtain solutions of Eq. (6) by successive approximations using the Green's function for the n th harmonic,

$$g_n(r, z|r', z') = -\frac{nkrr'/r}{2\pi(z-z')} J_1\left(\frac{nkrr'}{z-z'}\right) \exp\left[-n^2\alpha(z-z') - \frac{ink(r^2+r'^2)}{2(z-z')}\right], \quad (7)$$

where $k = \omega/c$ is the wavenumber and $\alpha = \eta\omega^2/2\rho c^3$ the absorption coefficient at the source frequency, and J_n is the n th order Bessel function of the first kind. The surface integral $W_1(r, z) = \Omega_0 \int_S f(r') g_1(r, z|r', 0) dS'$ for the linear solution can be evaluated for the Gaussian source function $f(r) = \exp(-r^2/a^2)$ to obtain

$$W_1(r, z) = \frac{\Omega_0}{(1-iz/z_0)^2} \exp\left(-\alpha z - \frac{r^2/a^2}{1-iz/z_0}\right), \quad (8)$$

where $z_0 = ka^2/2$ is the Rayleigh distance. In the far field, the decay rate associated with spherical spreading is z^{-2} rather than z^{-1} . This is a consequence of the dipole-like nature of the source, and therefore the entire field, because $v_{x,y}(r, \phi, z, \tau) = -v_{x,y}(r, \phi + \pi, z, \tau)$. For any pair of points that are symmetric with respect to the z axis, the particle velocity vectors in the (x, y) plane are equal in magnitude and opposite in direction.

The integral for the third harmonic does not appear to admit a closed-form solution for a Gaussian beam. However, an analytic solution for the primary wave in the far field is sufficient to determine the corresponding asymptotic solution for the third harmonic. We now consider radiation from a uniform circular disk of radius a with $f(r) = H(a - r)$, where $H(\cdot)$ is the Heaviside unit step function. The solutions for the fundamental and third harmonic in the far field are found to be

$$W_1(\theta, z) \sim -\frac{1}{2}\Omega_0 z_0^2 \frac{e^{-\alpha z}}{z^2} D(\theta) \exp\left(-\frac{1}{2}ikz \tan^2 \theta\right), \quad (9)$$

$$W_3(\theta, z) \sim -\frac{i\beta\Omega_0^3 z_0^5}{384\alpha c^2} \frac{e^{-3\alpha z}}{z^4} (k a \tan \theta)^2 D^3(\theta) \exp\left(-\frac{3}{2}ikz \tan^2 \theta\right), \quad (10)$$

where the directivity function for the primary wave,

$$D(\theta) = \frac{8J_2(k a \tan \theta)}{(k a \tan \theta)^2}, \quad (11)$$

is normalized to be unity on axis.

Like the linear solution for a Gaussian beam, Eq. (9) exhibits a $z^{-2}e^{-\alpha z}$ decay rate. The factor $(k a \tan \theta)^2$ multiplying $D^3(\theta)$ in Eq. (10) indicates that the third harmonic vanishes along the axis of the beam in the far field. The maximum amplitude of the third harmonic in the far field is located at the angle θ_0 where $k a \tan \theta_0 = 1.92$, for which $(k a \tan \theta_0)^2 D^3(\theta_0) = 1.41$.

NUMERICAL RESULTS

Equation (6) was solved with an algorithm developed for the KZK equation [9] but without coordinate transformation, and with minor changes in the nonlinearity and diffraction subroutines. The source is taken to be a uniform circular disk of radius a , with $f(r) = H(a - r)$. Diffraction is characterized by the Rayleigh distance z_0 , nonlinearity by the characteristic shock formation distance $z_{sh} = c^2/\beta k a^2 \Omega_0^2$, and absorption by the length $z_{abs} = 1/\alpha$. In terms of these length scales, the two dimensionless parameters used to characterize solutions of Eq. (6) relative to diffraction are $A = z_0/z_{abs}$ for absorption and $N = z_0/z_{sh}$ for nonlinearity.

Results are presented in Fig. 1 for $A = 0.01$ and $N = 3$. Figure 1(a) shows harmonic beam patterns at $z = z_0$ for the source frequency and its third and fifth harmonics. Note that the levels of the third and fifth harmonics on the beam axis are maxima, not minima. The discrepancy with Eq. (10), which predicts zero for the amplitude of the third harmonic along the beam axis, is because the wave is not yet in the far field. Even though the nonlinear term in Eq. (6) is zero everywhere along the beam axis, harmonics generated nonlinearly off axis in the near field diffract into the axial region. Figure 1(b) shows the axial waveform at $z = z_0$, which is seen to be symmetric.

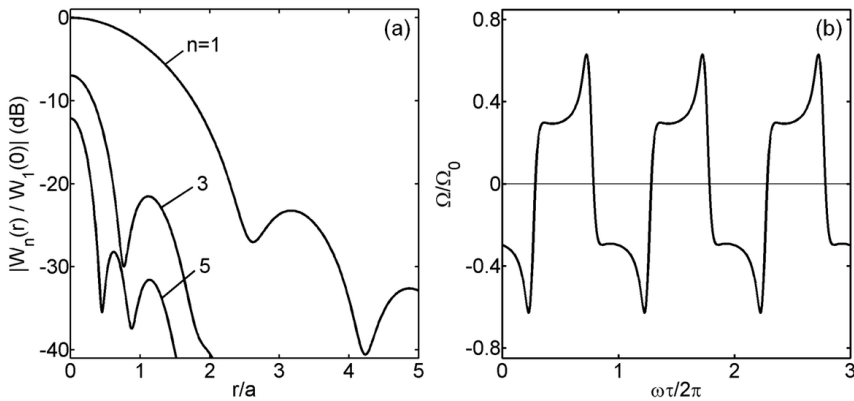


FIGURE 1. (a) Harmonic beam patterns and (b) axial time waveform at $z = z_0$ in a torsional wave beam.

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