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Research Article

Convergence Properties and Fixed Points of Two General Iterative Schemes with Composed Maps in Banach Spaces with Applications to Guaranteed Global Stability

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This paper investigates the boundedness and convergence properties of two general iterative processes which involve sequences of self-mappings on either complete metric or Banach spaces. The sequences of self-mappings considered in the first iterative scheme are constructed by linear combinations of a set of self-mappings, each of them being a weighted version of a certain primary self-mapping on the same space. The sequences of self-mappings of the second iterative scheme are powers of an iteration-dependent scaled version of the primary self-mapping. Some applications are also given to the important problem of global stability of a class of extended nonlinear polytopic-type parameterizations of certain dynamic systems.

1. Introduction

The problems of boundedness and convergence of sequences of iterative schemes are very important in numerical analysis and the numerical implementation of discrete schemes; see [1–4] and references therein. In particular, [1] describes in detail and with rigor the associated problems linked to the theory of fixed points in various types of spaces like metric spaces, complete and compact metric spaces, and Banach spaces, while it also contains, discusses, and compares results of a number of relevant background references on the subject. In other papers, related problems of fixed point theory or stability are focused on approximations including, in some cases, issues from a computational point of view eventually involving modified numerical methods like, for instance, Aitken's delta-squared methods or Steffensen's method [4–11]. Also, a counterpart theory has been also formulated in the framework of common fixed points and coincidence points for several mappings and in the framework of multivalued functions. An important background on fixed, best proximity, and proximal points concerned with nonexpansive, contractive, weakly contractive, and strictly contractive mappings has been developed; see, for instance, [1–4, 8–25] and references therein. In particular, a relevant effort has been also focused on the formulations of extensions of the above problems to the study of existence and uniqueness of fixed and best proximity points in cyclic self-mappings as well to proximal contractions [12–14, 17–20, 24, 25] and to the characterization of approximate fixed and coincidence points [21, 22]. Direct applications of fixed point theory to the study of the stability of dynamic systems including the property of ultimate boundedness for the trajectory solutions having mixed nonexpansive and expansive properties through time or being subject to impulsive controls have been given in [21, 24, 25]. This paper is focused on the study of boundedness and convergence of sequences of distances and iterated points and the characterization of fixed points of a class of composite self-maps in metric spaces. Such maps are built with combinations of sets of elementary self-maps which can be expansive or nonexpansive and the last ones can be contractive (including the case of strict contractions). The composite maps are defined by switching rules which select some self-map (the "active" self-map) on a certain interval of definition of the running index of the sequence of iterates being built. The above-mentioned properties concerning the sequences of iterates being generated from given initial points are investigated under particular constraints for the switching rule. Note, on the other hand, that the properties of controllability, observability, and stability of differential or difference equations as well as the various kinds of dynamic systems are of a wide interest in theory and applications including the cases of presence of disturbances and/or unmodeled dynamics [23–45]; see, for instance, related problems associated with continuous-time, discrete-time, digital, and hybrid systems and those involving delayed dynamics [27, 30, 33, 37–39], hybrid [34–36, 41], and switched dynamic systems [31, 32, 38–43] and references therein. The above problems are often studied in an integrated or combined fashion in the sense that the presence of uncertainties of any nature (basically unmeasurable noise or unmodeled dynamics) is incorporated to the description of differential, difference, or hybrid systems with eventual external delays or delayed dynamics. The stability is studied with different tools as Lyapunov theory, matrix inequalities, or fixed point theory. Fractional calculus has also been widely used in the investigation of the solutions of differential, functional-differential, and dynamic systems; see, for instance, [44, 45] and some references therein.

This paper is firstly devoted to giving a framework for the contractive properties of two general iterative schemes which are constructed via combinations of elementary self-maps in appropriate metric or Banach spaces. The sequences of self-mappings of the first scheme are constructed by linear combinations of a set of self-mappings, each of them being a weighted version of a certain primary self-mapping on the same space. Such weights are nonnegative real sequences in general. The single parameterizations of the first iterative scheme include polytopic-type ones, where a set of real scalar sequences define both the sequence of self-mappings of interest and the individual parameterizations as a particular case. The second iterative scheme is a generalization of De Figueiredo scheme [8], where the sequences of self-mappings are integer powers of a scaled version of a primary elementary self-mapping. Such powers are iteration-dependent, while the scaling weights can be iteration-dependent. A second objective is to describe an application of the developed theoretical framework to study the stability properties of (in general) nonlinear switched dynamic systems under appropriate stabilizing switching rules. The obtained formal results can also be useful to investigate the stability of dynamic systems under combinations of single parameterizations.

1.1. Notation. $\{T_n\} \rightrightarrows T^*$ (i.e., $\limsup_{n\to\infty}\{\|T_nx - T^*x\| : x \in \operatorname{Dom} T_n\} = 0$; $\forall x \in \operatorname{Dom} T$) and $\{T_n\} \to T^*$ (i.e., $\lim_{n\to\infty} T_nx = T^*x$; $\forall x \in \operatorname{Dom} T_n$) for $T^*, T_n : X \to X$, $\forall n \in \mathbf{Z}_{0+}$; denote, respectively, uniform and point-wise convergence in X of $T_n : X \to X$ to $T^* : X \to X$ provided that all of them have the same domain.

$\operatorname{Fix}(T)$ denotes the set of fixed points of $T : X \to X$ and $\overline{k} = \{0, 1, \dots, k\}$.

2. Iterative Scheme 1

Consider the following iterative scheme under a sequence of self-mappings $T_n : X \to X$, $\forall n \in \mathbf{Z}_{0+}$, on a vector space X:

$$x_{n+1} = T_n x_n = \sum_{i=0}^{k} \alpha_i^{(n)} T^i x_n, \quad \forall n \in \mathbf{Z}_{0+}, \tag{1}$$

for any given $x_0 \in X$ with $T : X \to X$ and $T_n : X \to X$, $\forall n \in \mathbf{Z}_{0+}$, being defined by $T_nx = (\sum_{i=0}^{k} \alpha_i^{(n)} T^i)x$ for any $x \in X$ and the nonnegative real parameterization sequences being subject to $\sum_{i=0}^{k} \alpha_i^{(n)} > 0$, $\forall i \in \overline{k} = \{0, 1, \dots, k\}$, $\forall n \in \mathbf{Z}_{0+}$.

Theorem 1. *Consider the iterative scheme* (1) *on a vector space X, with $0 \in X$, under the following assumptions.*

(1) *Either $(X, \|\|)$ is a normed space endowed with a norm $\|\|$ or, respectively, (X, d) is a metric space endowed with a homogeneous translation-invariant metric $d : X \times X \to \mathbf{R}_{0+}$.*

(2) *$\sum_{i=0}^{k} \alpha_i^{(n)} > 0$ and $0 \le \alpha_i^{(n+1)} = (1 + \widetilde{\alpha}_i^{(n)})\alpha_i^{(n)}$, $\forall i \in \overline{k} = \{0, 1, \dots, k\}$, $\forall n \in \mathbf{Z}_{0+}$, and $\inf_{n\in\mathbf{Z}_{0+}} \max_{1\le i\le k} \alpha_i^{(n)} > 0$, with the nonnegative real sequences $\{\alpha_i^{(n)}\}$, $\forall i \in \overline{k}$, being subject to the constraints $|\widetilde{\alpha}_i^{(n)}| \le \widetilde{\alpha}_n \le m_n(d(x_n, x_{n+1})/d(x_{n+1}, 0))$ and $m_n = o[d(x_{n+1}, 0)]$, $\forall i \in \overline{k}$, $\forall n \in \mathbf{Z}_{0+}$, where the relative one-step increment parameterization sequences are $\widetilde{\alpha}_i^{(n)} = (\alpha_i^{(n+1)} - \alpha_i^{(n)})/\alpha_i^{(n)}$, $\forall i \in \overline{k}$, $\forall n \in \mathbf{Z}_{0+}$.*

(3) *$T : X \to X$ possesses the (nonnecessarily contractive) condition $d(Tx, Ty) \le K d(x, y)$, $\forall x, y \in X$, for some $K \in \mathbf{R}_+$.*

(4) *$(1 + m_n)(\sum_{i=0}^{k} \alpha_i^{(n)} K^i) \le \rho < 1$, $\forall n \in \mathbf{Z}_{0+}$.*

Then, the following properties hold.

(i) *There exists the limit $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ for any given initial point $x_0 \in X$ of the iterative scheme* (1) *and the sequence $\{x_n\}$ is bounded.*

(ii) *If, in addition, $\{T_n\} \rightrightarrows T^*$ for some limit $T^* : X \to X$ and if either $(X, \|\|)$ is a Banach space or (X, d) is complete, then $\{x_n\}$ is a Cauchy sequence and thus convergent to some z in X which is the unique fixed point of $T^* : X \to X$ and thus independent of the initial point $x_0 \in X$ of the iterative scheme* (1)*. All the self-mappings of the sequence $\{T_n\}$ as well as $T^* : X \to X$ are strict contractions.*

(iii) *If either $(X, \|\|)$ is a Banach space or (X, d) is complete and $\{\alpha_i^{(n)}\} \to \alpha_i$ as $n \to \infty$, $\forall i \in \overline{k}$, for some self-mapping T^* on X, then there is a unique $z^* \in \operatorname{Fix}(T^*)$ in X such that $z^* = z$ for any given initial point $x_0 \in X$ of the iterative scheme* (1)*. Also, $T^* : X \to X$ is a strict contraction and thus a strict Picard self-mapping with a unique fixed point $z^*(=z) \in X$ such that $T^{*n}x_0 \to z^*$,*

$\widehat{T}_n x_0 \to z^*$ as $n \to \infty$ for any given initial point $x_0 \in X$, where $\widehat{T}_n : \mathbf{Z}_{0+} \times X \to X$ is the composite mapping $\widehat{T}_n = T_n T_{n-1} \cdots T_0$, $\forall n \in \mathbf{Z}_{0+}$.

(iv) *The "a priori" and "a posteriori" error estimates and the convergence rate are, respectively, given by the subsequent relations:*

$$d(x_n, z) \le \frac{\rho^n}{1-\rho} d(x_0, x_1), \tag{2}$$

$$d(x_n, z) \le \frac{\rho}{1-\rho} d(x_{n-1}, x_n), \tag{3}$$

$$d(x_n, z) \le \rho^n d(x_0, z). \tag{4}$$

Proof. Define the $(k+1)$ error sequences $\{\widetilde{\alpha}_i^{(n)}\}$ by $\widetilde{\alpha}_i^{(n)} = \alpha_i^{(n+1)} - \alpha_i^{(n)}$, $\forall i \in \overline{k} = \{0, 1, \ldots, k\}$, $\forall n \in \mathbf{Z}_{0+}$. If $(X, \|\|)$ is a normed space, then there is always a metric-induced norm $d(x,y) = \|x - y\|$, $\forall x, y \in X$. On the other hand, if (X, d) is a metric space endowed with a homogeneous translation-invariant metric $d : X \times X \to \mathbf{R}_{0+}$ then there is a metric-induced norm $\|x\| = d(x, 0)$, $\forall x, y \in X$. Both spaces $(X, \|\|)$ and (X, d) are formally identical and they can both deal with a metric-induced norm by using the standard metric properties and its homogeneous and translation-invariance properties. Thus, one gets via recursive calculations that

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &= d\left(\sum_{i=0}^{k} \left(\alpha_i^{(n)} T^i x_{n+1} + \widetilde{\alpha}_i^{(n)} T^i x_{n+1}\right), \sum_{i=0}^{k} \alpha_i^{(n)} T^i x_n\right) \\ &\le d\left(\alpha_0^{(n)} x_{n+1}, \alpha_0^{(n)} x_n\right) \\ &\quad + d\left(\alpha_1^{(n)} T x_{n+1}, \alpha_1^{(n)} T x_n\right) \\ &\quad + d\left(\sum_{i=2}^{k} \alpha_i^{(n)} T^i x_{n+1}, \sum_{i=2}^{k} \alpha_i^{(n)} T^i x_n\right) \\ &\quad + d\left(\widetilde{\alpha}_0^{(n)} x_{n+1}, 0\right) + d\left(\widetilde{\alpha}_1^{(n)} T x_{n+1}, 0\right) \\ &\quad + d\left(\sum_{i=2}^{k} \widetilde{\alpha}_i^{(n)} T^i x_{n+1}, 0\right) \\ &\le \cdots \le \sum_{i=0}^{k} d\left(\alpha_i^{(n)} T^i x_{n+1}, \alpha_i^{(n)} T^i x_n\right) \\ &\quad + \sum_{i=0}^{k} d\left(\widetilde{\alpha}_i^{(n)} T^i x_{n+1}, 0\right) \\ &\le (1 + m_n)\left(\sum_{i=0}^{k} \alpha_i^{(n)} K^i\right) d(x_n, x_{n+1}); \\ &\qquad \forall n \in \mathbf{Z}_{0+}. \end{aligned} \tag{5}$$

Thus,

$$\begin{aligned} d(x_{n+1}, x_n) &\le \left(\prod_{j=0}^{n-1} \left[\left(1 + m_j\right)\left(\sum_{i=0}^{k} \alpha_i^{(j)} K^i\right)\right]\right) d(T_0 x_0, x_0) \\ &\le \rho d(x_n, x_{n-1}) \le \rho^n d(x_1, x_0); \quad \forall n \in \mathbf{Z}_+, \end{aligned} \tag{6}$$

so that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ for any given $x_0 \in X$. It follows from (6) that, for any given initial $x_0 \in X$,

$$\begin{aligned} d(x_n, x_0) &\le \sum_{j=0}^{n-1} d\left(x_{j+1}, x_j\right) \le \frac{1-\rho^n}{1-\rho} d(T_0 x_0, x_0) < +\infty, \\ \limsup_{n\to\infty} d(x_n, x_0) &\le d(x_1, x_0)\left(\sum_{j=0}^{\infty} \rho^j\right) \\ &\le \frac{d(T_0 x_0, x_0)}{1-\rho} < +\infty, \end{aligned} \tag{7}$$

since $\rho < 1$ so that $\{x_n\}$ is bounded for any given $x_0 \in X$ and $\{d(x_{n+1}, x_n)\} \to 0$ from (6). All the self-mappings $T_n : X \to X$, $\forall n \in \mathbf{Z}_{0+}$, are strict contractions by construction from assumption 4. On the other hand, note that, since $\{T_n\} \rightrightarrows T^*$, one gets

$$\begin{aligned} d(T^* x, T^* y) &= \lim_{n\to\infty} d(T_n x, T_n y) \\ &= d\left(\lim_{n\to\infty}(T_n) x, \lim_{n\to\infty}(T_n) x\right) \\ &\le \rho d(x, y); \quad \forall x, y \in X, \end{aligned} \tag{8}$$

so that $T^* : X \to X$ is a strict contraction. Since $T_n : X \to X$, $\forall n \in \mathbf{Z}_{0+}$, are all strict ρ-contractions, $\{T_n\} \rightrightarrows T^*$, $\operatorname{Fix}(T^*) = \{z\}$, and $\operatorname{Fix}(T_n) = \{z_n\}$, $\forall n \in \mathbf{Z}_{0+}$, so that $d(T_n x, T^* x) < \varepsilon(1-\rho)$, $\forall x \in X$, so that

$$\begin{aligned} d(z_n, z) &= d(T_n z_n, T^* z) \\ &\le d(T_n z_n, T^* z_n) + d(T^* z_n, T^* z) \\ &< \varepsilon(1-\rho) + \rho d(z_n, z), \end{aligned} \tag{9}$$

and then $d(z_n, z) < \varepsilon$, $\forall n \ge n_0$, so that $\{z_n\} \to z$. Also, $\{d(T_n x_n, x_n)\} \to 0$ from (6) implies $\{d(T_n x_n, z_n)\} \to 0$ and $\{T_n z_n\} \to \{T_n z\}$ (since $\{T_n x_n\} \to z_n$ and $\{z_n\} \to z$ with $\operatorname{Fix}(T_n) = \{z_n\}$, $\forall n \in \mathbf{Z}_{0+}$), $\{d(T_n x_n, z)\} \to 0$, $\{T_n x_n\} \to z$ (since $\{z_n\} \to z$), and $\{T_n x_n\} \to \{T^* x_n\}$ (since $\{T_n\} \rightrightarrows T^*$). Thus, it follows that $\{T^* x_n\} \to T^* z$ $(= z)$ which implies that $\{x_n\} \to z$. Also, $\{x_n\}$ is a Cauchy sequence convergent to $z \in X$ if $(X, \|\|)$ is a Banach space and if (X, d) is a complete metric space, respectively.

On the other hand, $x_{n+1} = T_n x_n = \widehat{T}_n x_0 \to z$ $(\in X)$ as $n \to \infty$, $\forall x_0 \in X$, where $\widehat{T}_n : \mathbf{Z}_{0+} \times X \to X$ is the composite mapping $\widehat{T}_n = T_n T_{n-1} \cdots T_0$, $\forall n \in \mathbf{Z}_{0+}$. From (6), the self-mappings $T_n : X \to X$, $\forall n \in \mathbf{Z}_+$, are all strict contractions. Now, we prove that the limit point z is independent of the initial condition x_0 and thus unique. Assume two distinct initial values $x_0, y_0 \in X$ such that $\widehat{T}_n x_0 \to z(=z(x_0))$, $\widehat{T}_n y_0 \to \omega(=\omega(y_0))$ as $n \to \infty$ for some $z, \omega(\neq z) \in X$. Note from (6) that, since $\rho \in (0,1)$ is independent of the sequences $\{\widehat{T}_n x_0\}$ and $\{\widehat{T}_n y_0\}$, one gets

$$d\left(\widehat{T}_n x_0, \widehat{T}_n y_0\right) \leq \rho^n d\left(x_0, y_0\right); \quad \forall n \in \mathbf{Z}_+;$$
$$\lim_{n\to\infty} d\left(\widehat{T}_n x_0, \widehat{T}_n y_0\right) = 0. \tag{10}$$

Since $\omega \neq z$, one has the following from the triangle inequality:

$$0 < d(\omega, z) \leq d\left(\widehat{T}_n y_0, \widehat{T}_n x_0\right) \leq d\left(\omega, \widehat{T}_n y_0\right)$$
$$+ d\left(\widehat{T}_n x_0, \widehat{T}_n y_0\right) + d\left(\widehat{T}_n x_0, z\right); \quad \forall n \in \mathbf{Z}_+, \tag{11}$$

and then one gets the contradiction below to the assumption $\omega \neq z$:

$$0 < \lim_{n\to\infty} \left[d\left(\omega, \widehat{T}_n y_0\right) + d\left(\widehat{T}_n x_0, \widehat{T}_n y_0\right) + d\left(\widehat{T}_n x_0, z\right)\right] = 0, \tag{12}$$

so that $\omega = z$ and $\widehat{T}_n x_0 \to z$ as $n \to \infty$ with z being independent of the initial point x_0 of the iterative scheme (1). Hence, properties (i)-(ii) have been proven.

To prove property (iii), note that the assumption of uniform convergence $\{T_n\} \rightrightarrows T^*$ in X is weakened to point-wise convergence $\{T_n\} \to T^*$ in X since $\{\alpha_i^{(n)}\} \to \alpha_i$ and then $\{\widetilde{\alpha}_i^{(n)}\} \to 0$; $\forall i \in \overline{k}$ and $T^* : X \to X$ is a ρ-contraction from assumption 4. Thus, $\{d(x_{n+1}, x_n)\} \to 0$ implies $\{d(T_n x_n, z)\} \to 0$ and $\{x_n\} \to z$ implies $\{T_n x_n\} \to T^* z(=z)$. Since (X,d) is complete and $T_n : X \to X$ is a strict contraction then T^* is also a strict contraction and thus a strict Picard self-mapping on X and there is a unique $z^* \in \mathrm{Fix}(T^*)$ in X. Assume that $\widehat{T}_n x_0 \to z$ as $n \to \infty$ for any given $x_0 \in X$ and $z^* \neq z$. Take the sequence $\{T_n x_n\} \equiv \{\widehat{T}_n x_0\}$. Define $\delta T_n = \widehat{T}_n - T^{*n+1}$ by $(\delta T_n)x = (\widehat{T}_n - T^{*n+1})x$ for $x \in X$. Then, note that

$$\begin{aligned} d\left(\widehat{T}_{m+n} x_{n+1}, T^{*n+m+1} z^*\right) &= d\left(\widehat{T}_m x_{n+1}, T^{*m+1} z^*\right) \\ &= d\left(\left[T^{*m+1} + \delta T_m\right] x_{n+1}, T^{*m+1} z^*\right) \\ &\leq d\left(T^{*m+1} x_{n+1}, T^{*m+1} z^*\right) + \left\|\left(\delta T_m\right) x_{n+1}\right\|, \end{aligned} \tag{13}$$

and since $\{x_n\}$ is bounded, $T^{*m} z^* = z^*$, $\delta T_m \to 0$ as $m \to \infty$, $x_m \to z$, $T^{*m} x_n \to z^*$, and $T^{*m} z \to z^*$ as $m \to \infty$ then the following contradiction holds if $z^* \neq z$:

$$\begin{aligned} &\lim_{n,m\to\infty} d\left(T^{*m+1} x_{n+1}, T^{*m+1} z^*\right) \\ &\quad + \lim_{n,m\to\infty} \left\|\left(\delta T_m\right) x_{n+1}\right\| = 0 \\ &\geq \lim_{n,m\to\infty} d\left(\widehat{T}_{m+n} x_{n+1}, T^{*n+m+1} z^*\right) \\ &= d\left(\lim_{n,m\to\infty} \widehat{T}_{m+n} x_{n+1}, z^*\right) = d\left(z, z^*\right) > 0, \end{aligned} \tag{14}$$

and then $z^* = z$. As a result, $T^* : X \to X$ is a strict contraction and thus a strict Picard self-mapping with a unique fixed point $z^* \in X$ such that $T^{*n} x_0 \to z^*$ and $\widehat{T}_n x_0 \to z^*$ as $n \to \infty$ for any given initial point $x_0 \in X$, where $\widehat{T}_n : \mathbf{Z}_{0+} \times X \to X$ is the composite mapping $\widehat{T}_n = T_n T_{n-1} \cdots T_0$, $\forall n \in \mathbf{Z}_{0+}$.

Property (iv) is well known for Picard iterations. □

Remark 2. Note that the parameterization sequences $0 \leq \alpha_i^{(n)} \leq 1$, $\forall i \in \overline{k} = \{0, 1, \ldots, k\}$, $\forall n \in \mathbf{Z}_{0+}$, are not necessarily constant in Theorem 1 and $\alpha_1^{(n)}$ can be zero for some $n \in \mathbf{Z}_{0+}$ and the positive amount $\sum_{i=0}^{n} \alpha_i^{(n)}$ is not necessarily identically equal to one. Furthermore, the constant K can be equal to or greater than unity in assumption 3 of Theorem 1. Thus, the iterative scheme generalizes that proposed and analyzed by Cho et al. [1].

Remark 3. Note also that if $\{T_n\} \to T^*$ (or if the stronger condition $\{T_n\} \rightrightarrows T^*$ holds) then $T^{*n} x_0 \to z^*(\in \mathrm{Fix}(T^*) = \{z^*\})$ and $\widehat{T}_n x_0 \to z^*$ as $n \to \infty$ irrespective of the given $x_0 \in X$. However, if the property $(T_n - T^*) \to 0$ as $n \to \infty$ does not hold then $T_m^n x_0 \to z_m(\in \mathrm{Fix}(T_m^n) = \{z_m\})$ as $n \to \infty$, $\forall m \in \mathbf{Z}_{0+}$, for the given $x_0 \in X$ since all the self-mappings T_m on X are strict contractions but z_m can be distinct of z^*.

The following result relaxes condition (3) of strict contraction mappings in the sequence $\{T_n\}$ of Theorem 1 to weaker condition in terms of those mappings to be contractive in compact metric spaces.

Theorem 4. *Consider the iterative scheme* (1) *on a compact metric space* (X,d) *endowed with a homogeneous translation-invariant metric* $d : X \times X \to \mathbf{R}_{0+}$, *where* X *is a vector space, with* $0 \in X$, *under the following assumptions:*

(1) $\sum_{i=0}^{k} \alpha_i^{(n)} > 0$, $\alpha_i^{(n)} \geq 0$, $\forall i \in \overline{k} = \{0, 1, \ldots, k\}$, $\forall n \in \mathbf{Z}_{0+}$, *and* $\inf_{n\in\mathbf{Z}_{0+}} \max_{1\leq i\leq k} \alpha_i^{(n)} > 0$, *with the nonnegative real sequences* $\{\alpha_i^{(n)}\}$, $\forall i \in \overline{k}$, *and* $\{m_n\}$ *being subject to the constraints* $\alpha_i^{(n+1)} = (1 + \widetilde{\alpha}_i^{(n)})\alpha_i^{(n)}$, $|\widetilde{\alpha}_i^{(n)}| \leq \widetilde{\alpha}_n \leq m_n(d(x_n, x_{n+1})/d(x_{n+1}, 0))$ *and* $m_n = o[d(x_{n+1}, 0)]$, $\forall i \in \overline{k}$, $\forall n \in \mathbf{Z}_{0+}$, *where* $\widetilde{\alpha}_i^{(n)} = (\alpha_i^{(n+1)} - \alpha_i^{(n)})/\alpha_i^{(n)}$, $\forall i \in \overline{k}$, $\forall n \in \mathbf{Z}_{0+}$.

(2) $T : X \to X$ *possesses the weak contractive condition* $d(Tx, Ty) < d(x, y)$, $\forall x, y(\neq x) \in X$.

(3) $(1 + m_n)(\sum_{i=0}^{k} \alpha_i^{(n)}) < 1$, $\forall n \in \mathbf{Z}_{0+}$.

Then, the following properties hold.

(i) *There exists* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ *for any given initial point* $x_0 \in X$ *of the iterative scheme* (1).

(ii) *If, in addition,* $\{T_n\} \rightrightarrows T^*$ *for some limit* $T^* : X \to X$ *and* $\mathrm{Fix}(T_n) = \{z_n^*\}$, $\forall n \in \mathbf{Z}_{0+}$, *then the iterated sequence* $\{x_n\}$ *is a Cauchy sequence and thus convergent to some* z *in* X. *All the self-mappings of the sequence* $\{T_n\}$ *as well as* $T^* : X \to X$ *are contractive.*

(iii) *If* $\{T_n\} \to T^*$ *for some point-wise limit self-mapping* T^* *on* X*, then there is a unique* $z^* \in \operatorname{Fix}(T^*)$ *in* X *such that* $z^* = z$ *to which any sequence* $\{x_n\}$ *of the iterative scheme* (1) *converges for any given initial point* $x_0 \in X$*. Also,* $T^*: X \to X$ *is a contractive and thus a Picard self-mapping with a unique fixed point* $z^*(=z) \in X$ *such that* $T^{*n}x_0 \to z^*$, $\widehat{T}_n x_0 \to z^*$ *as* $n \to \infty$ *for any given initial point* $x_0 \in X$*, where* $\widehat{T}_n: \mathbf{Z}_{0+} \times X \to X$ *is the composite mapping* $\widehat{T}_n = T_n T_{n-1} \cdots T_0$, $\forall n \in \mathbf{Z}_{0+}$.

Proof. Note that a metric space is compact if and only if it is complete and totally bounded. Note also that $(X, \|\|)$ is a Banach space formally identical to the compact (and then complete) metric space (X, d) when endowed with a homogeneous and translation-invariant metric $d: X \times X \to \mathbf{R}_{0+}$ if $\|\|$ is the norm-induced metric. Thus, one concludes that

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}) < \cdots < d(x_1, x_0); \quad \forall n \in \mathbf{Z}_+, \tag{15}$$

which implies that $\{d(x_{n+1}, x_n)\}$ is a convergent sequence with $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ for any given $x_0 \in X$. Hence, property (i) follows. On the other hand, since the metric space (X, d) is a compact metric space (and thus complete) then the iterated sequence $\{x_n\}$, with $x_{n+1} = T_n x_n$ and the point-wise convergence of $\{T_n\}$ to $T^*: X \to X$, is a Cauchy sequence $\{T_n\} \rightrightarrows T^*$ and $\operatorname{Fix}(T_n) = \{z_n^*\}$, $\forall n \in \mathbf{Z}_{0+}$. Assume that $\{z_n^*\} \to z^*$ is untrue. Then,

$$\begin{aligned} d(z_n^*, z) &= d(T_n z_n^*, T^* z) \\ &\leq d(T_n z_n^*, T^* z_n^*) + d(T^* z_n^*, T^* z) \\ &= d(T_n z_n^* - T^* z_n^*, 0) + d(T^* z_n^*, T^* z) \\ &< d(T_n z_n^* - T^* z_n^*, 0) + d(z_n^*, z) \end{aligned} \tag{16}$$

so that the contradiction $0 = \liminf_{n\to\infty} d(T_n z_n^* - T^* z_n^*, 0) > 0$ since the metric is homogeneous and translation-invariant, $\{T_n\} \rightrightarrows T^*$, so that $T_n z_n^* \to T^* z_n^*$ as $n \to \infty$ since $z_n^* \in X$, and $T^*: X \to X$ is contractive. Hence, $\{z_n^*\} \to z$ for some z in X and any given $x_0 \in X$, all the self-mappings $T_n : X \to X$; $\forall n \in \mathbf{Z}_{0+}$ in the sequence $\{T_n\}$ are contractive, and then Picard mappings (since (X, d) is a compact metric space) so that the composite mapping $\widehat{T}_n: X \to X$ is also a Picard mapping. As a result, $\widehat{T}_n x_0 \to z^*$ as $n \to \infty$ for any given initial point $x_0 \in X$ and $T_n^m x_0 \to z_n^*$ as $m \to \infty$, $\forall n \in \mathbf{Z}_{0+}$, $\forall x_0 \in X$ with $\operatorname{Fix}(T_n) = \{z_n\}$, $\forall n \in \mathbf{Z}_{0+}$, for any $T_n(:X \to X) \in \{T_n\}$. If $\{T_n\} \to T^*$ then $T^{*n}x_0 \to z^*$, $\widehat{T}_n x_0 \to z^*$ as $n \to \infty$, and $\operatorname{Fix}(T^*) = \{z^*\}$. Hence, properties (ii)-(iii) have been proven. □

Remark 5. Note that a metric space is compact if and only if it is complete and totally bounded. Equivalently, a metric space is compact if and only if every family of closed subsets of X with the finite intersection property (i.e., the intersection of any finite collection of sets in the family is nonempty) has a nonempty intersection.

An extension of Theorem 1 follows below by admitting the failure of the contractive condition of assumption 4 of Theorem 1 within connected subsets of finite length of $\mathbf{Z}_{0+}$ which are adjacent to connected subsets where the contractive condition holds.

Theorem 6. *Consider the iterative scheme* (1) *on a vector space* X *under the assumptions (1)–(3) and (5) of Theorem 1 and, furthermore,*

$$\prod_{n=0}^{m_j} \left[(1 + m_n) \left(\sum_{i=0}^{k} \alpha_i^{(n)} K^i \right) \right] \leq \rho < 1; \qquad \forall p_j \in S \subseteq \mathbf{Z}_{0+}, \quad \forall j \in \mathbf{Z}_{0+}, \tag{17}$$

where $S = \{p_k : k \in \mathbf{Z}_{0+}\}$ *is a strictly increasing sequence of nonnegative integer numbers subject to* $p_0 \leq p^* < +\infty$ *and* $p_{k+1} - p_k \leq p < +\infty$, $\forall k \in \mathbf{Z}_{0+}$.

(i) *There exists the limit* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ *for any given initial point* $x_0 \in X$ *of the iterative scheme* (1) *and the sequence* $\{x_n\}$ *is bounded.*

(ii) *If, in addition,* $\{T_n\} \rightrightarrows T^*$ *for some limit self-mapping* $T^*: X \to X$ *and if either* $(X, \|\|)$ *is a Banach space or* (X, d) *is complete, then* $\{x_n\}$ *is a Cauchy sequence and thus convergent to some* z *in* X *which is unique and thus independent of the initial point* $x_0 \in X$ *of the iterative scheme* (1)*. Also, all the self-mappings in the sequence* $\{T_n\}$ *and* $T^*: X \to X$ *are strict contractions.*

(iii) *If either* $(X, \|\|)$ *is a Banach space or* (X, d) *is complete and* $\{T_n\} \to T^*$ *as* $n \to \infty$ *for some self-mapping* T^* *on* X*, then there is a unique* $z^* \in \operatorname{Fix}(T^*)$ *in* X *such that* $z^* = z$ *for any given initial point* $x_0 \in X$ *of the iterative scheme* (1)*. Also,* $T^*: X \to X$ *is a strict contraction and thus a strict Picard self-mapping with a unique fixed point* $z^*(=z) \in X$ *such that* $T^{*n}x_0 \to z^*$, $\widehat{T}_n x_0 \to z^*$ *as* $n \to \infty$ *for any given initial point* $x_0 \in X$*, where* $\widehat{T}_n: \mathbf{Z}_{0+} \times X \to X$ *is the composite mapping* $\widehat{T}_n = T_n T_{n-1} \cdots T_0$, $\forall n \in \mathbf{Z}_{0+}$.

(iv) *The "a priori" and "a posteriori" error estimates and the convergence rate are, respectively, given by the subsequent relations:*

$$\begin{aligned} &d(x_{p_n}, z) \leq \frac{\rho^n}{1-\rho} d(x_{p_0}, x_{p_1}) \leq \frac{M\rho^n}{1-\rho} d(x_0, x_1); \\ &d(x_{p_n+j}, z) \leq \frac{M^2\rho^n}{1-\rho} d(x_0, x_1), \\ &d(x_{p_n}, z) \leq \frac{\rho}{1-\rho} d(x_{p_{n-1}}, x_{p_n}) \leq \frac{M\rho}{1-\rho} d(x_0, x_1); \\ &d(x_{p_n+j}, z) \leq \frac{M^2\rho}{1-\rho} d(x_0, x_1), \\ &d(x_{p_n}, z) \leq \rho^n d(p_{n_0}, z) \leq M\rho^n d(p_{n_0}, z); \\ &d(x_{p_n+j}, z) \leq M^2\rho^n d(x_0, x_1), \end{aligned} \tag{18}$$

for some $M \in \mathbf{R}_+$ *and any integer* $j \in (1, p_{n+1} - p_n)$, $\forall n \in \mathbf{Z}_{0+}$.

Proof. Note from (17) and (6) that since $p_k \to \infty$ as $k \to \infty$, one gets

$$d\left(x_{p_{k+1}+1}, x_{p_{k+1}}\right) \le \rho d\left(x_{p_{k+1}}, x_{p_k}\right); \quad \forall p_k \in S, \tag{19a}$$

$$\begin{aligned} d\left(x_{p_{k+2}}, x_{p_{k+1}}\right) &\le \left(1 + L + \cdots + L^{p-1}\right) \rho d\left(x_{p_{k+1}}, x_{p_k}\right) \\ &\le \rho_0^{k+1} d\left(x_{p_0}, x_{p_1}\right), \end{aligned} \tag{19b}$$

and, provided that $\rho \in (0,1)$ is small enough for the given $p \in \mathbf{Z}_+$ so that $\rho_0 = \rho(1+L+\cdots+L^{p-1}) < 1$, $d(x_{p_{k+2}}, x_{p_{k+1}}) \to 0$ as $k \to \infty$ for any given $x_0 \in X$. Then, $\{d(x_{p_{k+1}}, x_{p_k})\}$ is a convergent sequence with $\lim_{k\to\infty} d(x_{p_{k+1}}, x_{p_k}) = \lim_{p_k\to\infty} d(x_{p_{k+1}}, x_{p_k}) = 0$ for any given $x_0 \in X$. It follows from (19a), (19b), since $p_1 \le \overline{p} + p < +\infty$ and $p_{k+1} - p_k \le p < +\infty$, that for any given initial $x_0 \in X$,

$$d\left(x_{p_k}, x_{p_0}\right) \le \frac{1-\rho^k}{1-\rho} d\left(x_{p_1}, x_{p_0}\right) < +\infty; \quad \forall k \in \mathbf{Z}_+, \tag{20}$$

$$\begin{aligned} \limsup_{k\to\infty} d\left(x_{p_k}, x_{p_0}\right) &\le d\left(x_{p_1}, x_{p_0}\right)\left(\sum_{j=0}^{\infty} \rho^j\right) \\ &\le \frac{d\left(x_{p_1}, x_{p_0}\right)}{1-\rho} < +\infty, \end{aligned} \tag{21}$$

$$\begin{aligned} &d\left(x_{p_k+j}, x_{p_k+j-1}\right) \\ &\quad \le \prod_{n=0}^{j-1}\left[\left(1 + m_{p_k+n}\right)\left(\sum_{i=0}^{k} \alpha_i^{(p_k+n)} K^i\right)\right] d\left(x_{p_k}, x_{p_{k-1}}\right) \\ &\quad < +\infty, \end{aligned} \tag{22}$$

$$\begin{aligned} &d\left(x_{p_k+j}, x_{p_k}\right) \\ &\quad \le \rho^k\left(\sum_{i=0}^{j-1}\prod_{n=0}^{i-1}\left[\left(1 + m_{p_k+n}\right)\right.\right. \\ &\qquad \left.\left.\times\left(\sum_{\ell=0}^{k} \alpha_\ell^{(p_k+n)} K^\ell\right)\right]\right) d\left(x_{p_0}, x_{p_1}\right) \\ &\quad < +\infty, \end{aligned} \tag{23}$$

for $j = 0, 1, \ldots, p_{k+1} - p_k$, $\forall k \in \mathbf{Z}_+$, and then, since $\{p_k\}$ is strictly increasing with k, $\rho \in [0,1)$ and $p_1 \le p_0 + p \le p + \overline{p} < +\infty$, one gets $\lim_{k\to\infty} d(x_{p_k+j}, x_{p_k}) = 0$ for $j = 0, 1, \ldots, p_{k+1} - p_k$, $\forall k \in \mathbf{Z}_+$. Then, $\{d(x_{n+1}, x_n)\} \to 0$ as $n \to \infty$ from (23) and $\{x_n\}$ is bounded for any initial $x_0 \in X$. However, $\{x_n\}$ is not a Cauchy sequence, in general, since the constraint $d(x_{n+2}, x_{n+1}) < d(x_{n+1}, x_n)$ does not necessarily hold for all $n \in \mathbf{Z}_{0+}$.

The variation in the proof development of the concerns derived from the assumption $\{T_n\} \rightrightarrows T^*$ of Theorem 1 (ii) is addressed as follows. Since $\{T_n\} \rightrightarrows T^*$ and $\mathrm{Fix}(T_n) = \{z_n^*\}$, $\forall n \in \mathbf{Z}_{0+}$, then (17) necessarily leads to $T^* : X \to X$ being a strict contraction, $\{z_n^*\} \to z^*$ with $z^* \in \mathrm{Fix}(T^*)$ $(=\{z^*\})$, and $\lim_{k\to\infty}(p_{k+1} - p_k) = 1$. Therefore, the remaining proofs of properties (i)–(iii) follow in a very close way as their counterparts of Theorem 1. Also, note that

$$\begin{aligned} &d\left(x_{p_0}, x_{p_0-1}\right) \\ &\quad \le \sum_{i=0}^{p_0-1} d\left(x_{i+1}, x_i\right) \\ &\quad \le \sum_{i=0}^{p_0-1}\prod_{n=0}^{i-1}\left[\left(1 + m_n\right)\left(\sum_{\ell=0}^{k} \alpha_\ell^{(n)} K^\ell\right)\right] d\left(x_0, x_1\right), \end{aligned} \tag{24}$$

and then define

$$\begin{aligned} M = \sup_{k\in\mathbf{Z}_{0+}} \max_{p_k \le n < p_{k+1}} &\left(\sum_{i=0}^{j-1}\prod_{n=0}^{i-1}\left[\left(1 + m_{p_k+n}\right)\right.\right. \\ &\left.\left.\times\left(\sum_{\ell=0}^{k} \alpha_\ell^{(p_k+n)} K^\ell\right)\right]\right), \end{aligned} \tag{25}$$

so that property (iv) follows from (23) and Theorem 1 (iv). □

Remark 7. Note that assumption 4 of Theorem 1 is relaxed to the constraint (17) which holds for a set of connected finite intervals within a strictly increasing sequence of points with the difference between any two consecutive ones being upper-bounded by a prescribed bound.

Remark 8. Note that Theorems 1 (i), 4 (ii), and 6 (iii) hold irrespective of the convergence of the sequence of self-mappings to a limit.

3. Iterative Scheme 2 and Some Generalizations

Now, consider the iterative scheme

$$x_{n+1} = T_n^{f_n} x_n, \quad T_n x = g_n T x; \quad \forall n \in \mathbf{Z}_{0+}, \tag{26}$$

for any given $x_0 \in X$ which is a further generalization of the De Figueiredo iteration [8]. The following result holds.

Theorem 9. *Let the iterative scheme* (1) *with the nonexpansive self-mapping $T : X \to X$ on a vector space X, with $0 \in X$, under the following additional assumptions.*

(1) *Either $(X, \|\|)$ is a Banach space endowed with a norm $\|\|$ or, respectively, (X, d) is a complete metric space endowed with a homogeneous translation-invariant metric $d : X \times X \to \mathbf{R}_{0+}$.*

(2) *$\{g_n\} \subset (0,1) \cap \mathbf{R}_{0+}$ is a real parameterization sequence with $0 < g_n < 1$, $\forall n \in \mathbf{Z}_{0+}$, and $\{f_n\} \subseteq \mathbf{Z}_{0+}$ is an integer sequence with $f_n > 0$; $\forall n \in \mathbf{Z}_+$.*

(3) *There exist the following limits: $\lim_{n\to\infty} g_n = 1$, $\lim_{n\to\infty}(f_n/n) = +\infty$, and either $\lim_{n\to\infty}(f_{n+1} \log g_{n+1}/f_n \log g_n) < 1$ or $|\lim_{n\to\infty} n(1 - (f_{n+1} \log g_{n+1}/f_n \log g_n))| < +\infty$.*

Then, the subsequent properties hold.

(i) $\{x_n\}$ *converges to a fixed point of* $T: X \to X$.

(ii) *If* $T: X \to X$ *is a strict contraction then* $\{x_n\}$ *converges to the unique fixed point of* $T: X \to X$.

Proof. As in the proof of Theorem 1, the following considerations are applicable for the proof.

(1) If $(X, \|\|)$ is a normed space then there is always a metric-induced norm $d(x, y) = \|x - y\|$; $\forall x, y \in X$.

(2) If (X, d) is a metric space endowed with a homogeneous translation-invariant metric $d: X \times X \to \mathbf{R}_{0+}$ then there is a norm-induced metric $\|x\| = d(x, 0)$; $\forall x, y \in X$.

Both spaces $(X, \|\|)$ and (X, d) are formally identical under assumption 1 and both possess either a metric-induced norm by using the standard metric properties and its homogeneous and translation-invariance properties or a norm-induced metric, respectively. Now, note from (26) that

$$x_{n+k} = \left(\prod_{i=0}^{k-1}\left[g_{n+i}^{f_{n+i}}\right] T^{\sum_{i=0}^{k-1} f_{n+i}} x_n\right),$$

$$\begin{aligned}
&x_{n+k} - x_{n+k-1} \\
&\quad = \left(\prod_{i=0}^{k-1}\left[g_{n+i}^{f_{n+i}}\right] T^{\sum_{i=0}^{k-1} f_{n+i}} - \prod_{i=0}^{k-2}\left[g_{n+i}^{f_{n+i}}\right] T^{\sum_{i=0}^{k-2} f_{n+i}}\right) x_n \\
&\quad = \prod_{i=0}^{k-1}\left[g_{n+i}^{f_{n+i}}\right] \left(T^{\sum_{i=0}^{k-1} f_{n+i}} - T^{\sum_{i=0}^{k-2} f_{n+i}}\right. \\
&\qquad \left. + \left(1 - \frac{1}{\left[g_{n+k-1}^{f_{n+k-1}}\right]}\right) T^{\sum_{i=0}^{k-2} f_{n+i}}\right) x_n; \\
&\qquad \forall n \in \mathbf{Z}_{0+}, \quad \forall k \in \mathbf{Z}_+, \\
&\left\|x_{n+k} - x_{n+k-1}\right\| \\
&\quad \le \left\|\prod_{i=0}^{k-1}\left[g_{n+i}^{f_{n+i}}\right]\left(I - T^{f_{n+k-1}}\right)\right\| \times \left\|T^{\sum_{i=0}^{k-2} f_{n+i}} x_n\right\| \\
&\qquad + \left\|\prod_{i=0}^{k-1}\left[g_{n+i}^{f_{n+i}}\right]\left(\left|1 - \frac{1}{\left(g_{n+k-1}^{f_{n+k-1}}\right)}\right| T^{\sum_{i=0}^{k-2} f_{n+i}}\right) x_n\right\|; \\
&\qquad \forall n \in \mathbf{Z}_{0+}, \quad \forall k \in \mathbf{Z}_+, \\
&\limsup_{n \to \infty} \left\|x_{n+k} - x_{n+k-1}\right\| \\
&\quad \le 2\left(\prod_{i=0}^{k-1}\left[g_{n+i}^{f_{n+i}}\right]\right) \limsup_{n \to \infty} \left\|T^{\sum_{i=0}^{k-2} f_{n+i}} x_n\right\| \\
&\qquad + \left(\prod_{i=0}^{k-1}\left[g_{n+i}^{f_{n+i}}\right]\right) \\
&\qquad \times \limsup_{n \to \infty}\left(\left\|\left(\left|1 - \frac{1}{\left(g_{n+k-1}^{f_{n+k-1}}\right)}\right| T^{\sum_{i=0}^{k-2} f_{n+i}}\right) x_n\right\|\right) \\
&\quad = 2\left(\limsup_{n \to \infty} L_n(k)\right) \limsup_{n \to \infty} \left\|T^{\sum_{i=0}^{k-2} f_{n+i}} x_n\right\| < +\infty; \\
&\qquad \forall k \in \mathbf{Z}_+,
\end{aligned} \tag{27}$$

since $\|I - T^{f_{n+k-1}}\| \le 2$, $\forall n \in \mathbf{Z}_{0+}$, $\forall k \in \mathbf{Z}_+$ because $T: X \to X$ is nonexpansive, $\lim_{n \to \infty} g_{n+k-1}^{f_{n+k-1}} = 1$; $\forall k \in \mathbf{Z}_+$ and $\prod_{i=0}^{k-1}[g_{n+i}^{f_{n+i}}] = L_n(k) \le L_n(\infty) < +\infty$; $\forall k \in \mathbf{Z}_+$, $\forall n \in \mathbf{Z}_{0+}$ because $\{g_n\} \subset (0, 1) \cap \mathbf{R}_{0+}$ and either $\lim_{n \to \infty}(f_{n+1} \log g_{n+1}/f_n \log g_n) < 1$ or $|\lim_{n \to \infty} n(1 - (f_{n+1} \log g_{n+1}/f_n \log g_n))| < +\infty$. Then $\prod_{i=0}^{\infty}[g_{n+i}^{f_{n+i}}]$ is convergent since the corresponding logarithmic series of positive numbers converges according to either d'Alembert or Raabe convergence criteria of series of nonnegative real numbers. Then, the sequence $\{x_{n+1} - x_n\}$ is bounded. In the same way, we get

$$\begin{aligned}
&\limsup_{k \to \infty} \left\|x_k - x_{k-1}\right\| \\
&\quad \le 2\left(\prod_{i=0}^{k-1}\left[g_i^{f_i}\right]\right) \limsup_{k \to \infty} \left\|T^{\sum_{i=0}^{k-2} f_i} x_0\right\| \\
&\qquad + \left(\prod_{i=0}^{k-1}\left[g_i^{f_i}\right]\right) \\
&\qquad \times \limsup_{k \to \infty}\left(\left\|\left(\left|1 - \frac{1}{\left(g_{k-1}^{f_{k-1}}\right)}\right| T^{\sum_{i=0}^{k-2} f_i}\right) x_0\right\|\right) \\
&\quad = 2\left(\limsup_{k \to \infty} L_0(k)\right) \limsup_{k \to \infty} \left\|T^{\sum_{i=0}^{k-2} f_i} x_0\right\| \\
&\quad = 2\left(\limsup_{k \to \infty} L_0(k)\right) x^* = 0,
\end{aligned} \tag{28}$$

since $\exists \lim_{k \to \infty} L_0(k) = \prod_{i=0}^{k-1}[g_i^{f_i}] = 0$ since $g_n < 1$, $\forall n \in \mathbf{Z}_{0+}$, and $\lim_{n \to \infty}(f_n/n) = +\infty$ so that $\exists \lim_{n \to \infty}\|T^{\sum_{i=0}^{k-2} f_{n+i}} x_n\| = x^* (\in \mathrm{Fix}(T))$. Thus, $\{d(x_{n+1}, x_n)\}$ converges to zero for any given $x_0 \in X$ and

$$d(x_{n+1}, x^*) \le d(x_{n+1}, x_n) + d(x_n, x^*) \tag{29}$$

so that, since $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$, $\lim_{n \to \infty}(d(x_{n+1}, x^*) - d(x_n, x^*)) = 0$. Then, $\{d(x_n, x^*)\}$ converges and $\{x_n\}$ converges as well to some point of X since (X, d) is complete so that $x_n \to (x^* + q)$ as $n \to \infty$ for some $q \in X$ and since $T : X \to X$ is nonexpansive so that it is K-Lipschitz-continuous (i.e., continuous with a Lipschitz constant $K \le 1$), one gets

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = d\left(\lim_{n\to\infty} x_{n+1}, \lim_{n\to\infty} x_n\right) = d(x^* + q, x^* + q) = 0. \tag{30}$$

Since $\{x_n\}$ converges and (X, d) is a metric space then $\{x_n\}$ is a Cauchy sequence (and a bounded sequence) and there is $a \in \mathbf{R}_{0+}$ such that

$$\begin{aligned} a &= \lim_{n\to\infty} d(x_n, x^*) = d(x^* + q, x^*) \\ &= \lim_{n\to\infty} d\left((g_n T)^{f_n} x_n, x^*\right) \\ &= d\left(\lim_{n\to\infty} T^{f_n}(x^* + q), x^*\right) = d(y^*, x^*), \end{aligned} \tag{31}$$

since the metric is translation-invariant, $g_n \to 1$ as $n \to \infty$, and since $T: X \to X$ is nonexpansive, it is $K(\le 1)$-Lipschitz-continuous and $T^{f_n}(x^* + q) \to y^*$ with $y^*, x^* \in \mathrm{Fix}(T)$. If $a = 0$ then $y^* = x^*$ $(\in \mathrm{Fix}(T))$ and we have proven that $\{x_n\}$ converges to the fixed point x^* of $T: X \to X$. Now, assume that $y^*(\neq x^*) \in \mathrm{Fix}(T)$. The result is again proven since $\{x_n\}$ converges to a fixed point of $T : X \to X$ which is distinct of x^*. Finally, assume that $y^* \notin \mathrm{Fix}(T)$ and proceed by contradiction to prove that this assertion is false. Since $T : X \to X$ is nonexpansive, one gets $d(T^n y^*, T^n x^*) \le \cdots \le d(y^*, x^*) = a$ so that $d(T^n y^*, x^*) \to a_1(\le a)$ as $n \to \infty$; then by everywhere Lipschitz continuity of the nonexpansive self-mapping $T: X \to X$,

$$\begin{aligned} d\left(\lim_{n\to\infty} T^n(Ty^*), x^*\right) &= d\left(T\left(\lim_{n\to\infty} T^n y^*\right), x^*\right) \\ &= d\left(\lim_{n\to\infty} T^n y^*, x^*\right) = d(y_1^*, x^*) \\ &= a_1 \le a, \end{aligned} \tag{32}$$

and $T^n y^* \to y_1^* (\in \mathrm{Fix}(T))$ and $T^{f_n} y^* \to y_1^*$. Since y^* is a limit point of $T^{f_n} x_n$, $a_1 = a$ and then $y^*(= y_1^*) \in \mathrm{Fix}(T)$, a contradiction to $y^* \notin \mathrm{Fix}(T)$. Thus, $\{T^{f_n} x_n\}$ converges to a fixed point of $T : X \to X$. Property (i) has been proven. Also,

$$\begin{aligned} d\left(T_n^{f_n} x_n, T^{f_n} x_n\right) &= d\left(T_n^{f_n} x_n - T^{f_n} x_n, 0\right) \\ &\le \left\| T_n^{f_n} - T^{f_n} \right\| \left\| x_n \right\|; \quad \forall n \in \mathbf{Z}_{0+}. \end{aligned} \tag{33}$$

Since $\|T_n^{f_n} - T^{f_n}\| \le (1 - g_n^{f_n})\|T^{f_n}\|$, $\forall n \in \mathbf{Z}_{0+}$ with $(1 - g_n^{f_n}) \to 0$ as $n \to \infty$, it follows that $\|T_n^{f_n} - T^{f_n}\| \to 0$ as $n \to \infty$ and since $\{T^{f_n} x_n\}$ converges to a fixed point of $T: X \to X$ then $\{T_n^{f_n} x_n\}$ also converges to the same fixed point of $T: X \to X$. Hence, property (i) follows.

On the other hand, if $T: X \to X$ is a strict contraction then $\mathrm{Fix}(T) = \{x^*\}$, since (X, d) is complete so that $a = 0$ and $y^* = x^*$ in (31) and, hence, property (ii) follows as well. □

Theorem 9 has the following derived result.

Corollary 10. *Let the iterative scheme* (1) *be under the nonexpansive self-mapping $T : C \to C$, where C is a nonempty closed and convex subset of a Hilbert space $(X, \|\|)$, with $0 \in C$, subject to all the assumptions of Theorem 9. Then, the subsequent properties hold.*

(i) *$\{x_n\}$ converges strongly to a fixed point of $T : C \to C$.*

(ii) *If $T : C \to C$ is a strict contraction then $\{x_n\}$ converges to the unique fixed point of $T : C \to C$.*

Proof. Property (i) follows from Theorem 1 since $(X, \|\|)$ is uniformly convex since it is a Hilbert space; $T : C \to C$ is nonexpansive and contains a bounded sequence (since C is nonempty, closed, and convex) and then it has at least a fixed point. Property (ii) follows since such a fixed point is unique if $T: C \to C$ is a strict contraction. □

The iterative scheme (26) is now generalized by using some ideas of Section 2 as follows:

$$x_{n+1} = T_n^{f_n} x_n, \quad T_n x = g_n\left(\sum_{i=0}^{k} \alpha_i T^i\right) x_n; \quad \forall n \in \mathbf{Z}_{0+}, \tag{34}$$

for any given $x_0 \in X$.

Theorem 11. *Let the iterative scheme* (34) *generated by the self-mapping $T : X \to X$ on a vector space X, with $0 \in X$, and assumptions 1–3 of Theorem 9 hold as well as the following additional assumptions:*

(1) *$\sum_{i=0}^{k} \alpha_i > 0$ for nonnegative real scalars α_i, $\forall i \in \overline{k}$, and $\max_{1\le i\le k} \alpha_i > 0$;*

(2) *$T : X \to X$ satisfies the condition $d(Tx, Ty) \le K d(x, y)$, $\forall x, y \in X$, for some $K \in \mathbf{R}_+$;*

(3) *$\sum_{i=0}^{k} \alpha_i K^i \le 1$; $\forall n \in \mathbf{Z}_{0+}$.*

Then, the subsequent properties hold.

(i) *$\{x_n\}$ converges to a fixed point of the nonexpansive self-mapping $\overline{T} : X \to X$ defined by $\overline{T}x = (\sum_{i=0}^{k} \alpha_i T^i)x$, $\forall x \in X$.*

(ii) *If $\overline{T}: X \to X$ is a strict contraction fulfilling $\sum_{i=0}^{k} \alpha_i K^i \le \rho < 1$ then $\{x_n\}$ converges to the unique fixed point of $\overline{T} : X \to X$.*

Proof. As in Theorem 1 and Theorem 9, both spaces $(X, \|\|)$ and (X, d) are formally identical under assumption 1 of Theorem 9 and both possess either a metric-induced norm by using the standard metric properties and its homogeneous and translation-invariance properties or a norm-induced metric, respectively. Now, define the mapping $\overline{T} : X \to X$ by $\overline{T}x = (\sum_{i=0}^{k} \alpha_i T^i)x$, $\forall x \in X$. Thus, (27) in the proof of Theorem 9 still holds with the replacement $T \to \overline{T}$. Note that $\|I - \overline{T}^{f_{n+k-1}}\| \le 2$, $\forall n \in \mathbf{Z}_{0+}$, $\forall k \in \mathbf{Z}_+$, since $\overline{T}: X \to X$ is nonexpansive (even if $T : X \to X$ is expansive with $K > 1$ in the assumption 2), from assumptions 2-3, and $\overline{K}$-Lipschitz-continuous from the assumption 3 with $\overline{K} = \sum_{i=0}^{k} \alpha_i K^i \le 1$.

Note also that $\lim_{n\to\infty} g_{n+k-1}^{f_{n+k-1}} = 1, \forall k \in \mathbf{Z}_+$, and $\prod_{i=0}^{k-1}[g_{n+i}^{f_{n+i}}] \le L_n(k) \le L_n(\infty) < +\infty, \forall k \in \mathbf{Z}_+, \forall n \in \mathbf{Z}_{0+}$, and then the sequence $\{x_{n+1} - x_n\}$ obtained from the iterative scheme (34) is bounded for any $x_0 \in X$. In the same way, (28) holds from assumptions 2-3 of Theorem 9, since $g_n < 1, \forall n \in \mathbf{Z}_{0+}$, and $\lim_{n\to\infty}(f_n/n) = +\infty$ so that $\exists \lim_{n\to\infty}\|\overline{T}^{\sum_{i=0}^{k-2} f_{n+i}} x_n\| = x^*$ ($\in\mathrm{Fix}(\overline{T})$). Thus, $\{d(x_{n+1}, x_n)\}$ converges to zero for any given $x_0 \in X$ since (X, d) is complete. Then, it follows (as it is deduced from (30) in the proof of Theorem 9) that $\{x_n\}$ converges so that it is a Cauchy and bounded sequence. Finally, it can be proven in a similar way as in Theorem 9 that $\{\overline{T}^{f_n} x_n\}$ converges to some fixed point $\overline{x}^*$ of the nonexpansive self-mapping $\overline{T}: X \to X$ for each given initial point $x_0 \in X$ of the iteration (34). Such a fixed point is unique if $\overline{T}: X \to X$ is a strict contraction. □

In a similar way as Corollary 10 is got from Theorem 9, one gets the following.

Corollary 12. *Consider the iterative scheme* (34) *under the nonexpansive self-mapping* $\overline{T} : C \to C$, *where* C *is a nonempty closed and convex subset of a Hilbert space* $(X, \|\|)$, *with* $0 \in C$, *subject to all the assumptions of Theorem 9. Then, the subsequent properties hold.*

(i) $\{x_n\}$ *converges strongly to a fixed point of* $\overline{T}: C \to C$.

(ii) *If* $\overline{T}: C \to C$ *is a strict contraction fulfilling* $\sum_{i=0}^{k} \alpha_i K^i \le \rho < 1$ *then* $\{x_n\}$ *converges to the unique fixed point of* $\overline{T}: C \to C$.

Note that in Theorem 9 (i) and Corollary 10 (i), $T: X \to X$ and $T_n^{f_n} : X \to X$ for $n \in \mathbf{Z}_+$ are not necessarily Picard mappings since the limiting points can be dependent on the initial condition of the iterative schemes. The same conclusion arises for $\overline{T} : X \to X$ and $\overline{T}_n^{f_n} : X \to X$ for $n \in \mathbf{Z}_{0+}$ in Theorem 11 (i) and Corollary 12 (i). However, the above self-mappings are Picard iterations in the corresponding parts (ii) of such results since the relevant mappings are strict contractions.

Note also that Theorem 11 and Corollary 12 still hold by replacing $\alpha_i \to \alpha_i^{(n)}$ for $i \in \overline{k}$ and the replacement of the constraint $\max_{1\le i\le k}\alpha_i > 0$ with $\inf_{n\in\mathbf{Z}_{0+}}\max_{1\le i\le k}\alpha_i^{(n)} > 0$.

4. Simulation Examples towards an Application Perspective on Discrete Nonlinear Dynamic Systems

This section contains two numerical examples. The first one is related to the Iterative Scheme 1 introduced in Section 2 while the second one concerns the Iterative Scheme 2 discussed in Section 3.

4.1. Iterative Scheme 1. Consider the iterative scheme defined by (1) with $T(x) = x/2(1 + x)$ on $[0, +\infty)$ and

$$x_{n+1} = \left(\alpha_3^{(n)}T^3 + \alpha_2^{(n)}T^2 + \alpha_1^{(n)}T^1 + \alpha_0^{(n)}I\right)x_n. \tag{35}$$

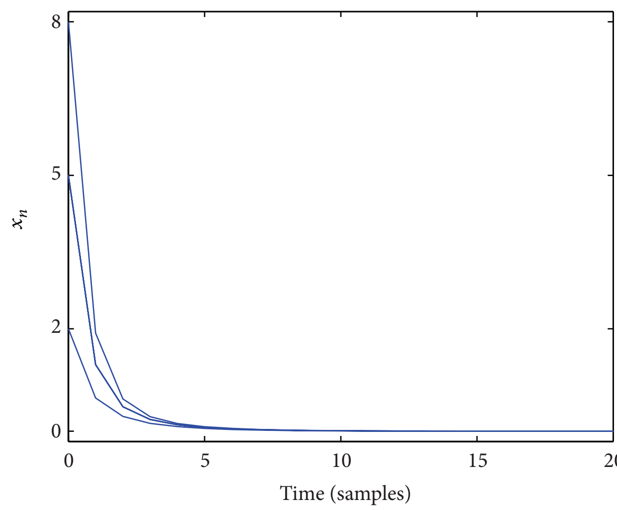

FIGURE 1: Evolution of the iterates for different initial conditions.

T is a strict contraction satisfying the condition $d(T(x), T(y)) \le Kd(x, y)$ with $K = 1/2$ (for the Euclidean distance) and, hence, it possesses a unique fixed point at $x = 0$. Note that the above description can also be considered as that of a nonlinear discrete time-varying dynamic system where the state evolves from initial conditions according to the sequence $\{x_n\}$ with initial condition x_0 while the output is defined by the real map $x \to Tx$. Note that the fixed point $x = 0$ is also an equilibrium point of the dynamic system which is suited to be globally asymptotically stable. Consider, firstly, the sequence of constant weights $\alpha^{(n)} = [0.2 \;\; 0.3 \;\; 0.8 \;\; 0.9]$ for all $n \ge 0$. We are now in conditions of applying Theorem 1 since $\sum_{i=0}^{3}\alpha_i^{(n)} = 2.2 > 0$, $\tilde{\alpha}_i^{(n)} = m_n = 0$, and $\sum_{i=0}^{3}\alpha_i^{(n)}K^i = 0.6625 < 1$ for all integers $n \ge 0$. In this case, the system parameterization is close to, but more general than, a polytopic-type time-invariant one but, in particular, the usual constraint $\sum_{i=0}^{3}\alpha_i^{(n)} = 1$ is not needed. Accordingly, the sequence of iterates $\{x_n\}$ is bounded for all $n \ge 0$ and converges to the unique fixed point of T, $x = 0$. Moreover, the iterates converge to the unique fixed point regardless of the initial value x_0. These claims are verified through a numerical simulation in Figure 1.

Furthermore, Theorem 1 (iv) also provides an upper-bound for the rate of convergence of the sequence of iterates to the fixed point. Therefore, one gets from (4) $d(x_n, 0) \le \rho^n d(x_0, 0) = 0.6625^n d(x_0, 0)$. Figure 2 displays the evolution of iterates along with the calculated upper-bound for the case $x_0 = 8$.

Consider the time-varying parameterization under the time-varying weights given by

$$\alpha_i^{(n+1)} = \begin{cases} \lambda_i\alpha_i^{(n)}, & \alpha_i^{(n+1)} \ge 0.1, \\ 0.1, & \text{otherwise}, \end{cases} \tag{36}$$

for all $n \ge 0$ and $0 \le i \le 3$ with $\alpha^{(0)} = [0.2 \;\; 0.3 \;\; 0.8 \;\; 0.9]$, $\lambda_0 = 0.95$, $\lambda_1 = 0.9$, $\lambda_2 = 0.85$, and $\lambda_3 = 0.8$. The 0.1 lower bound has been included in (36) so as to satisfy the condition $\inf_{n\in\mathbf{Z}_{0+}}\max_{1\le i\le k}\alpha_i^{(n)} > 0$.

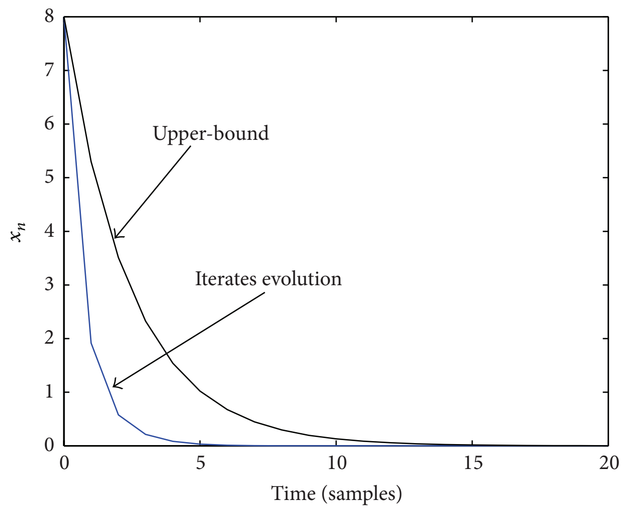

FIGURE 2: Upper-bounding of the convergence rate of iteration.

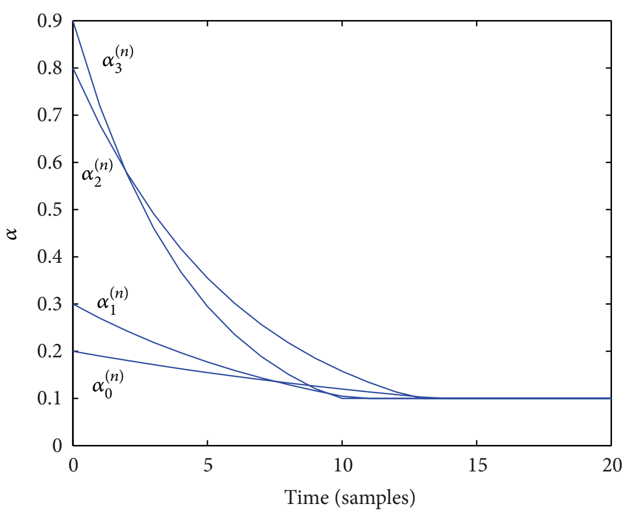

FIGURE 3: Evolution of the time-varying weights $\alpha_0^{(n)}$, $\alpha_1^{(n)}$, $\alpha_2^{(n)}$, and $\alpha_3^{(n)}$ through (36).

As it can be appreciated in Figure 3, the weights are decreasing with time until they reach the constant lower bound of 0.1 where they stop decreasing and become time-invariant. In fact, Figure 3 shows that this happens for $n \geq 14$. Thus, we are in conditions of applying the results stated in Theorem 6 for the case when the stability condition only holds on a subset of the nonnegative integer numbers. In this way, we have $\tilde{\alpha}_i^{(n)} = m_n = 0$ for all $n \geq 14$ and (17) of Theorem 6 is satisfied since

$$\sum_{i=0}^{3} \alpha_i^{(n)} K^i = \sum_{i=0}^{3} 0.1 K^i = 0.1875 < 1, \tag{37}$$

for all $n \geq 14$. Thus, Theorem 6 guarantees the convergence of iterates to the unique fixed point irrespectively of the initial condition. This fact is shown in Figure 4.

One advantage of the results in Theorem 6 for the sake of generality is that an arbitrary variation in the weights is

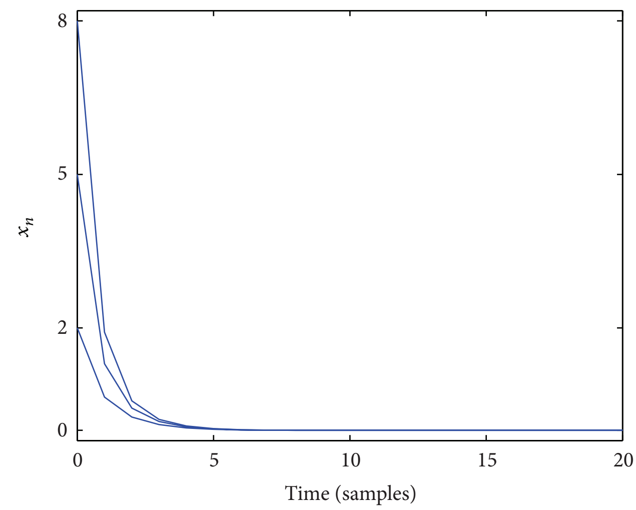

FIGURE 4: Iterates for different initial conditions and weights defined by (36).

admitted on certain subsets of the natural numbers. Thus, the family of admissible time variations for which the stability of the iteration scheme is guaranteed enlarges with respect to other approaches. Consider a time-varying set of weights defined with $m_n = d(x_{n+1}, 0)$ in condition (2) of Theorem 1 so that $\tilde{\alpha}_n \leq d(x_n, x_{n+1}) \leq K d(x_{n-1}, x_n) = 1/2 d(x_{n-1}, x_n)$ while condition (4) becomes

$$\begin{aligned}(1 + m_n) \sum_{i=0}^{3} \alpha_i^{(n)} K^i &\\ = (1 + m_n) \left(\alpha_0^{(n)} + \frac{1}{2} \alpha_1^{(n)} + \frac{1}{2^2} \alpha_2^{(n)} + \frac{1}{2^3} \alpha_3^{(n)} \right) < 1.\end{aligned} \tag{38}$$

Such constraints are satisfied, for instance, if we take $\alpha^{(0)} = [0.05 \quad 0.1 \quad 0.15 \quad 0.2]$ and

$$\tilde{\alpha}_n = 0.05 \sin(2\pi 0.05 n)\, d(x_{n-1}, x_n). \tag{39}$$

The weights evolution is displayed in Figure 5. Note that the weight variation defined by (39) satisfies the condition $\tilde{\alpha}_n \leq d(x_n, x_{n+1}) \leq 1/2 d(x_{n-1}, x_n)$ since $0.05 \sin(2\pi 0.05 n) \leq 0.05$, $\forall n \in \mathbf{Z}_{0+}$.

Figure 6 displays the sample-by-sample stability condition evaluation, in terms of the left-hand side of (38), showing that it remains smaller than unity. Therefore, according to Theorem 1, the iterates converge to zero as Figure 7 depicts.

Also, $d(x_{n-1}, x_n) \to 0$ while the weights converge to a real constant according to (39). Thus, the given theoretical results are useful to conclude the convergence of iteration schemes of the form (1).

4.2. Iterative Scheme 2. This second example is concerned with the iterative scheme defined by (26). Note that the first equation can describe, in particular, the state and output of

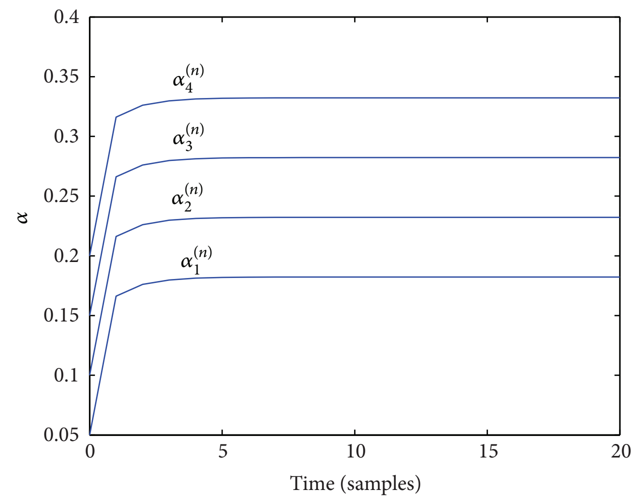

FIGURE 5: Evolution of the weights under (39).

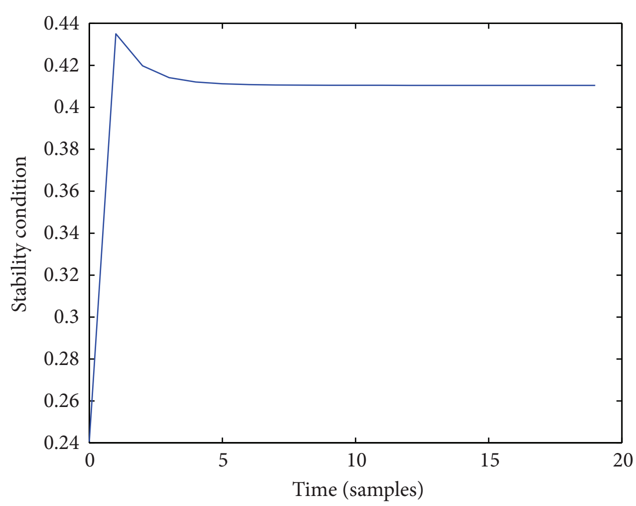

FIGURE 6: Numerical verification of the stability condition (38).

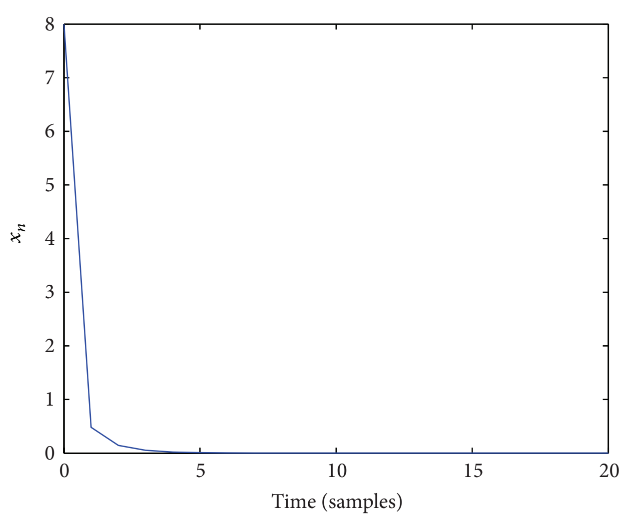

FIGURE 7: Convergence of the iterates to zero for time-varying weights given by (39).

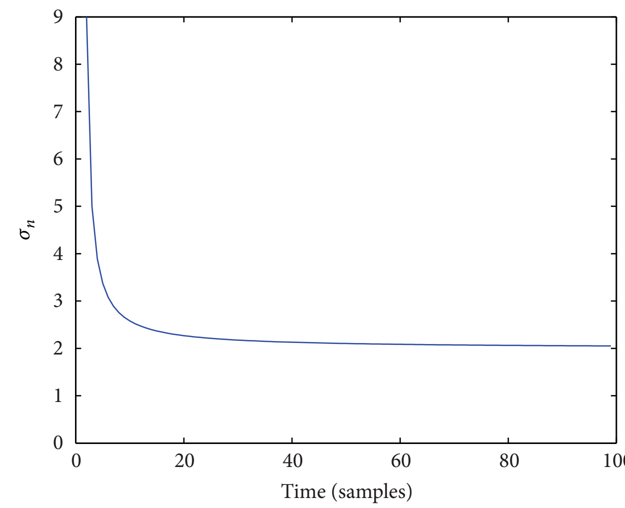

FIGURE 8: Evolution of σ_n with time.

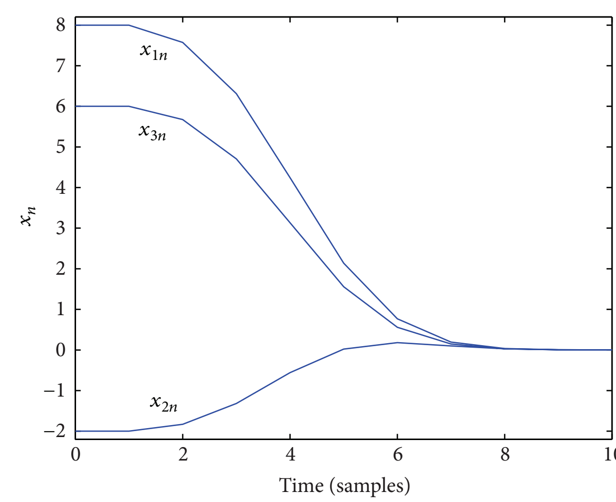

FIGURE 9: Evolution of the iterates through time.

a certain nonlinear discrete dynamic system. Thus, consider the linear discrete-time system given by

$$Tx = Ax = \begin{pmatrix} 0.955 & 0.01 & 0.005 \\ 0.005 & 0.96 & 0.005 \\ -0.005 & 0.01 & 0.965 \end{pmatrix} x, \tag{40}$$

with sequences $g_n = 1 - 0.1^{n+1}$ and $f_n = n^2$ for all $n \geq 0$. T is a strict contraction with eigenvalues $\{0.95, 0.96, 0.97\}$, $\|A\|_\infty = 0.98$, and $\|A\|_2 = 0.9717$. Thus, it has a unique fixed point at $x = 0$. These sequences satisfy conditions (2) and (3) stated in Theorem 9 since $\lim_{n\to\infty} g_n = 1$, $0 < g_n < 1$ for all $n \geq 0$, $\lim_{n\to\infty}(f_n/n) = \lim_{n\to\infty} n = +\infty$, and $\lim_{n\to\infty}\sigma_n < +\infty$ with $\sigma_n = |n(1 - (f_{n+1}\log g_{n+1}/f_n \log g_n))|$ and $\{\sigma_n\} \to 2$ (see Figure 8).

From Theorem 9, the sequence of iterates converges to the unique fixed point of T, $x = 0$, as it is confirmed in the numerical simulation displayed in Figure 9.

5. Conclusion

This paper has investigated the boundedness and convergence properties of two general iterative processes built with sequences of self-mappings in either complete metric or Banach spaces. The self-mappings of the first iterative scheme are built with linear combinations of a set of self-mappings each of them being a weighted version of a self-mapping on the same space. Those of the second scheme are powers of an iteration-dependent scaled version of the primary self-mapping. Some applications are given for global stability of a class of nonlinear polytopic-type parameterizations of dynamic systems.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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