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by

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Abstract

In this thesis, we consider the standard voting model with a finite set of alternatives A and 2 voters, and address the following question: besides Domains $\mathfrak{D}$ satisfying the Property T (Chatterji & Sen (2011)), what are other characteristics of domains that induce every strategy-proof and unanimous social choice function $f : \mathfrak{D}^n \to A$ to satisfy the tops-only property? We impose a minimal richness condition which ensures that for every alternative $a \in A$, there exists a preference ordering where a is maximal. We identify a more general condition on domains that is sufficient for strategy-proofness and unanimity to imply tops-onlyness in the case of 2 voters. This condition is shown to apply to Linked Domains (Aswal, Chatterji & Sen (2003)).

Contents

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Chapter 1

Introduction

A social choice function (SCF) is a mapping from profiles of ordinal preference orderings over a fixed set of alternatives to the same set of alternatives. Our first requirement is that the SCF should be strategy-proof, which implies that for every voter, true preference revelation is a dominant strategy. Next, we impose the requirement of unanimity on SCFs, which says that when an alternative is the most-preferred one for all voters in a profile of preferences, it must be the social outcome. In this thesis, along with these two properties mentioned above, we are going to discuss another important property of SCFs: the tops-only property. The tops-only property requires that given any profile of preferences, the social choice be determined by the peaks of all preferences in that profile. A tops-only SCF is attractive from a design perspective. A Domain $\mathcal{D}$ is a subset of the set containing all possible linear orders over the alternative set A. Tops-only domains are those domains on which every strategy-proof and unanimous SCF satisfies the tops-only property. The objective of this thesis is to investigate the connection between tops-onlyness and strategy-proofness with unanimity.

Given a SCF, a dictator is a voter whose most-preferred alternative is always the social choice for every profile of preferences. Correspondingly, dictatorial domains are those domains on which every strategy-proof and unanimous SCF has a dictator. For instance, the Universal Domain (considered in the Gibbard-Satterthwaite Theorem), Linked Domains (Aswal *et al.*, 2003) and Circular Domains (Sato (2010)) are all dictatorial domains. Since a dictatorial SCF always picks the top-ranked alternative of a dictator, dictatorial domains are necessarily tops-only domains. Even in the case of infinite alternatives, strategy-proof SCFs on domains of continuous preferences with a unique most-preferred alternatives on the range also satisfies the tops-only property (Weymark (2008)). Besides dictatorial domains, some other prominent domains are also tops-only domains, say the Domain of Separable Preferences and the Domain of Single-Peaked Preferences.

In this thesis, we focus on the case where the set of alternatives is finite. We assume the domain is the same for every voter. Our analysis is concerned with domains satisfying the minimal richness property, which says that for each alternative in A, there exists a preference ordering that sets that alternative as the top-ranked one.

This thesis is essentially close to Chatterji and Sen (2011). They provide a sufficient condition, named Property T, for a domain of preferences to be a tops-only domain in the case of 2 voters. This condition is satisfied by many restricted domains, like Adjacency Rich Single-Peaked Domains (Chatterji & Sen (2011)). However, it is not applicable to Linked Domains which are dictatorial and hence necessarily tops-only. In this thesis, we provide a more general version of Property T, named Modified Property T, which ensures that a minimally rich domain is a tops-only domain in the case of 2 voters. Linked Domains satisfy Modified Property T.

The thesis is organized as follows. Section 2 introduces definitions and the basic model. Section 3 describes the sufficient condition for tops-onlyness. Section 4 provides three related lemmas, while section 5 gives the proof of the theorem.

Chapter 2

Preliminaries

Let $I = \{ 1, 2, \ldots, N \}$ denote the set of voters, $|I| = n$ and $n \geq 2$. Let A denote the finite set of alternatives, $|A| = m$ and $m \geq 3$. Let $\mathbb{P}$ denote the set of all strict linear orders over A. A subset $\mathcal{D} \subset \mathbb{P}$ denotes an admissible domain. A typical preference ordering for voter i is denoted by P_i. Let $P = (P_1, P_2, \ldots, P_N) \in \mathcal{D}^n$ denote a profile of preferences. We will write $P = (P_i, P_{-i})$ in the usual way. Let a sub-domain $\mathcal{D}^S = \{P_i \in \mathcal{D} \mid \tau(P_i) \in S\}$, where $S \subseteq A$. When S is a singleton set, say $S = \{a\}$, we write $\mathcal{D}^S$ as $\mathcal{D}^a$. For 2 alternatives: a, $b \in A$, $a \, P_i \, b$ represents that a is strictly preferred to b in P_i. Let $r_k(P_i)$ denote the k_{th} ranked alternative in P_i, while $\tau(P_i)$ denotes the top-ranked alternative in P_i.

Definition 1 *A Social Choice Function (SCF) is a mapping* $f : \mathcal{D}^n \rightarrow A$.

Definition 2 *The Domain* $\mathcal{D}$ *satisfies minimal richness, if for all* $a \in A$*, there exists* $P_i \in \mathcal{D}$*, such that* $\tau(P_i) = a$.

Remark 1 *Minimal richness implies* $|\mathcal{D}| \geq m$.

Definition 3 *A SCF* $f : \mathcal{D}^n \to A$ *satisfies unanimity, if for all* $a \in A$ *and* $P \in \mathcal{D}^n$*, such that* $\tau(P_i) = a$*, for all* $i \in I$*, then* $f(P) = a$*.*

Unanimity property plays an important role in our following analysis. Unanimity implies that the SCF is an onto function.

Definition 4 *A SCF* $f : \mathcal{D}^n \to A$ *is strategy-proof, if for all* $i \in I$*;* $P_i, P_i' \in \mathcal{D}$ *and* $P_{-i} \in \mathcal{D}^{n-1}$*, we have either* $f(P_i, P_{-i})\, P_i\, f(P_i', P_{-i})$*, or* $f(P_i, P_{-i}) = f(P_i', P_{-i})$*.*

Definition 5 *The preference profiles* P *and* P' *are tops-equivalent, if* $\tau(P_i) = \tau(P_i')$*, for all* $i \in I$*. A SCF* $f : \mathcal{D}^n \to A$ *satisfies the tops-only property, if* $f(P) = f(P')$*, whenever* P *and* P' *are tops-equivalent.*

Definition 6 *A Domain* $\mathcal{D}$ *satisfies the tops-only property, if every strategy-proof and unanimous SCF* $f : \mathcal{D}^n \to A$ *satisfies the tops-only property.*

A domain that satisfies the tops-only property will be referred to as a tops-only domain.

Remark 2 *Dictatorial domains are tops-only domains.*

The objective of this thesis is to provide a sufficient condition that ensures that a domain is a tops-only domain. Recent work on tops-only domains (Weymark (2008) and Chatterji & Sen (2011)) uses the notion of option set[1].

Definition 7 *For a strategy-proof and unanimous SCF* $f : \mathcal{D}^n \to A$*, given* $P_{-i} \in \mathcal{D}^{n-1}$*, the option set for voter* i *is the set* $O_i(P_{-i}) = \{a \in A \mid a = f(P_i, P_{-i}), P_i \in \mathcal{D}\}$*.*

[1] Introduced originally by Barberà & Peleg (1990).

Chatterji & Sen (2011) (Proposition 1) shows that a SCF satisfies the tops-only property if and only if the option sets for every voter satisfy the tops-only property, that is, a SCF $f : \mathcal{D}^n \to A$ satisfies the tops-only property, if and only if $O_i(P_{-i}) = O_i(P'_{-i})$, for all $i \in I$ and $P_{-i}, P'_{-i} \in \mathcal{D}^{n-1}$ which are tops-equivalent. Thus, the subsequent analysis focuses on the tops-onlyness of option sets.

2.1 The case of two voters

We henceforth specialize the discussion to the case of 2 voters: the set of voters is $I = \{i, j\}$, and the SCF is $f : \mathcal{D}^2 \to A$. We now describe Linked Domains (Aswal *et al.*, 2003).

Definition 8 *A pair of alternatives: $a, b \in A$, are connected, denoted $a \sim b$, if $\exists P_i, P'_i \in \mathcal{D}$, such that $\tau(P_i) = a$, $r_2(P_i) = b$, $\tau(P'_i) = b$ and $r_2(P'_i) = a$.*

Definition 9 *Let $B \subset A$, and $a \in A - B$. Then a is linked to B, if $\exists b, c \in B$, such that $a \sim b$ and $a \sim c$.*

Definition 10 *The Domain $\mathcal{D}$ is linked if there exists a one to one function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$, such that*

1. $a_{\sigma(1)} \sim a_{\sigma(2)}$.
2. $a_{\sigma(j)}$ *is linked to* $\{a_{\sigma(1)}, \ldots a_{\sigma(j-1)}\}$, $j = 3, \ldots, m$.

Linked Domains are dictatorial domains. In the case of 2 voters, we can find a fixed voter, for instance, voter i, such that the option set for voter i is equal to the set of alternatives, given any preference of voter j in the domain: $O_i(P_j) = A$, for all $P_j \in \mathcal{D}$. This indicates that voter i is the dictator.

We now turn to domains that satisfy the tops-only property. Chatterji & Sen (2011) provides a sufficient condition, named Property T, which ensures that a minimally rich domain is a tops-only domain in the case of 2 voters.

Definition 11 *The Domain $\mathcal{D}$ satisfies Property T, if for all $P_i \in \mathcal{D}$ and $a \in A - \{\tau(P_i)\}$, $\exists b \in A - \{a\}$, such that $b\, P_i\, a$ and $b \sim a$.*

Many restricted domains meet the requirement of Property T. For instance, Circular Domains, Adjacency Rich Single-Peaked Domains (Chatterji & Sen (2011)) and Domains of Separable Preferences all satisfy the Property T. However, Property T may be inapplicable for some restricted domains. We now provide an example that is neither linked, nor satisfies Property T.

Example 1 *$A = \{a_1, a_2, a_3, a_4, a_5\}$, the Domain $\mathcal{D}_{usp}$ is constructed by the following 12 preference orderings:*

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
a_1	a_2	a_1	a_3	a_2	a_3	a_2	a_4	a_3	a_4	a_4	a_5
a_2	a_1	a_3	a_1	a_3	a_2	a_4	a_2	a_4	a_3	a_5	a_4
a_3	a_3	a_2	a_2	a_1	a_1	·	·	·	·	a_2	a_1
a_4	a_4	a_4	a_4	a_4	a_4	·	·	·	·	a_3	a_2
a_5	a_5	a_5	a_5	a_5	a_5	·	·	·	·	a_1	a_3

Table 2.1: Domain $\mathcal{D}_{usp}$

Here, dots on P_k, $k = 7, 8, 9, 10$, signify that alternatives ranked 3 and beyond are arbitrarily assigned. The corresponding connectivity graph of $\mathcal{D}_{usp}$ is following:

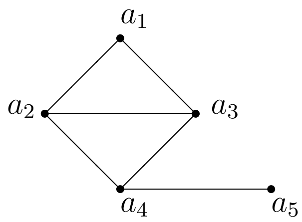

Figure 2.1: Connectivity Graph of Domain $\mathcal{D}_{usp}$

According to Figure 2.1, it is evident that $\mathcal{D}_{usp}$ is not linked. Since for all $P_k \in \mathcal{D}^{a_5}_{usp}$, $a_4 = r_2(P_k)$, the unique seconds property[1] is satisfied by $\mathcal{D}_{usp}$. Applying Theorem 5.1 in Aswal *et al.* (1999), it follows that $\mathcal{D}_{usp}$ is non-dictatorial. Next, though the first 11 preferences in $\mathcal{D}_{usp}$ all satisfy the requirement of Property T, the preference P_{12} does not satisfy Property T, as neither a_5, nor a_4 is connected to a_1. Existing results in the literature do not allow us to decide whether or not $\mathcal{D}_{usp}$ is a tops-only domain. This thesis provides a more general sufficient condition that applies to such domains.

[1]Introduced originally by Aswal *et al.* (1999). A Domain $\mathcal{D}$ satisfies the unique seconds property, if $\exists a, b \in A$, such that for all $P_i \in \mathcal{D}^a$, $b = r_2(P_i)$.

Chapter 3

Modified Property T

We first introduce the notions of a dictatorial sub-domain and a linked sub-domain.

Definition 12 *Let $S \subseteq A$, $|S| = k \geq 3$. A sub-domain $\mathcal{D}^S$ is a dictatorial sub-domain, if for all strategy-proof and unanimous SCFs, there exists a fixed voter, for instance, voter i, such that $O_i(P_j) \bigcap S = S$, for all $P_j \in \mathcal{D}^S$. Then for voter j, $O_j(P_i) \bigcap S = \{\tau(P_i)\}$, for all $P_i \in \mathcal{D}^S$.*

Definition 13 *Let $S \subseteq A$, $|S| = k \geq 3$. A sub-domain $\mathcal{D}^S$ is a linked sub-domain, if for all elements in S, there exists a one to one function $\sigma : \{1, 2, \cdots, k\} \rightarrow \{1, 2, \cdots, k\}$, such that*

1. $a_{\sigma(1)} \sim a_{\sigma(2)}$.
2. $a_{\sigma(i)}$ *is linked to* $\{\, a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(i-1)} \,\}$, $i = 3, \cdots, k$.

Remark 3 *The Domain $\mathcal{D}$ is linked, if and only if $k = m$, where $|A| = m$.*

Remark 4 *A linked sub-domain is a dictatorial sub-domain. This is implied in the proof of lemma 3.4 - 3.9 in Aswal et al. (2003).*

We illustrate with the example in Table 2.1. Though $\mathcal{D}_{usp}$ is not linked, it contains a linked sub-domain: $\mathcal{D}^S_{usp}$, where $S = \{a_1, a_2, a_3, a_4\}$. Next, we provide the definition of weak path connectivity in a sub-domain[1].

Definition 14 *A sub-domain $\mathcal{D}^S$, $|S| \geq 2$, is weakly path-connected, if for all $a, b \in S$, there exists a sequence $\{a_k\}_{k=1}^t \subseteq S$, $t \geq 2$, such that $a_1 = a$, $a_t = b$ and $a_{k-1} \sim a_k$, for $k = 2, \ldots, t$.*

Remark 5 *A linked sub-domain is weakly path-connected. However, a dictatorial sub-domain may not be necessarily weakly path-connected. Weak path connectivity provides a path for the transitivity of a property, which is crucial in what follows.*

We now turn to the description of Modified Property T, which is the central notion of this thesis. We begin with some notation.

Notation 1 *Given $P_i \in \mathcal{D}$, $X(P_i) = \{x \in A - \{\tau(P_i)\} \mid \nexists y \in A,$ such that $y \, P_i \, x$ and $y \sim x\}$.*

Notation 2 *Given $P_i \in \mathcal{D}$ and $a \in A$, The upper contour set and lower contour set of a in P_i are sets: $B(P_i, a) = \{x \in A \,|\, x \, P_i \, a\}$ and $W(P_i, a) = \{x \in A \,|\, a \, P_i \, x\}$.*

[1] The definition of weakly path-connected domain is originally introduced by Chatterji, Sanver & Sen (2010). We apply their definition to the sub-domain.

Definition 15 *A Domain* $\mathcal{D}$ *satisfies Modified Property T, if for all* $P_i \in \mathcal{D}$ *and* $a \in A - \{\tau(P_i)\}$*, either* $a \notin X(P_i)$*, or if* $a \in X(P_i)$*, then*

(i) $\exists S \subseteq A$*, such that* $a \in S$*.*

(ii) *The sub-domain* $\mathcal{D}^S$ *is a weakly path-connected dictatorial sub-domain.*

(iii) $B(P_i, a) \bigcap S \neq \emptyset$*.*

Remark 6 *A Domain satisfying Property T satisfies Modified Property T. Indeed, if Domain* $\mathcal{D}$ *satisfies Property T, then* $X(P_i) = \emptyset$*, for all* $P_i \in \mathcal{D}$*.*

Remark 7 *Linked Domains satisfy Modified Property T. A Linked Domain* $\mathcal{D}_L$ *itself is also a linked sub-domain. Then for all* $P_i \in \mathcal{D}_L$ *and* $a \in A - \{\tau(P_i)\}$*,* $\exists S = A$*, such that* $a \in S$ *and* $B(P_i, a) \bigcap S \neq \emptyset$*.*

Returning to the example in Table 2.1, recall that $\mathcal{D}_{usp}$ contains a linked sub-domain. Observe that the preference $P_{12} \in \mathcal{D}_{usp}$ satisfies all three requirements of Modified Property T. It follows that $\mathcal{D}_{usp}$ satisfies Modified Property T. Our theorem will prove that domains satisfying Modified Property T are tops-only.

We observe that if we add a new preference P_{13}: $\tau(P_{13}) = a_1$ and $r_2(P_{13}) = a_5$ to $\mathcal{D}_{usp}$, the new Domain $\mathcal{D}'_{usp}$ violates Modified Property T. By the unique seconds property, we propose a strategy-proof and unanimous SCF for $\mathcal{D}'_{usp}$:

$$f(P_i, P_j) = \begin{cases} \tau(P_i) & \text{, if } \quad \tau(P_i) \neq a_5 \\ \max(P_j, \{a_4, a_5\}) & \text{, if } \quad \tau(P_i) = a_5 \end{cases}$$

Given three preferences: $P_i = P_{12}$, $P_j = P_1$ and $P'_j = P_{13}$, the SCF gives two different outcomes: $f(P_i, P_j) = a_4$ and $f(P_i, P'_j) = a_5$, which shows that $\mathcal{D}'_{usp}$ doesn't satisfy the tops-only property.

Chapter 4

Lemmas

Lemma 1 *Given a preference* P_i*, if* $x \in O_j(P_i)$*,* $z\,P_i\,x$ *and* $z \sim x$*, then* $x \in O_j(\bar{P}_i)$*, for all* $\bar{P}_i \in \mathcal{D}$*, such that* $z \in O_j(\bar{P}_i)$*.*[1]

Proof. Suppose not, then $\exists P_i' \in \mathcal{D}$, such that $z \in O_j(P_i')$ and $x \notin O_j(P_i')$. Given a preference P_j: $\tau(P_j) = x$ and $r_2(P_j) = z$, then $f(P_i', P_j) = Max(P_j, O_j(P_i'))$ $= z$; and $f(P_i, P_j) = Max(P_j, O_j(P_i)) = x$. Since $z\,P_i\,x$, voter i will manipulate at (P_i, P_j) via P_i'. Therefore, $x \in O_j(\bar{P}_i)$, whenever $O_j(\bar{P}_i)$ contains z. ■

Lemma 2 *Given a preference* P_i*, if* $x \in O_j(P_i)$*,* $z\,P_i\,x$ *and* $z \sim x$*, then* $z \in O_j(P_i)$*.*[2]

Proof. Suppose not, then $z \notin O_j(P_i)$. Given P_j: $\tau(P_j) = z$ and $r_2(P_j) = x$, then $f(P_i, P_j) = Max(P_j, O_j(P_i)) = x$; and $f(P_i', P_j) = z$, for all $P_i' \in \mathcal{D}^z$. Since $z\,P_i\,x$, voter i will manipulate at (P_i, P_j) via P_i'. Hence, $z \in O_j(P_i)$. ■

[1] See the appendix of Aswal *et al.* (1999).

[2] See the appendix of Aswal *et al.* (1999).

Lemma 3 *Suppose a sub-domain $\mathcal{D}^S$, where $S \subset A$, is a weakly path-connected dictatorial sub-domain with $O_j(P_i) \bigcap S = S$, for all $P_i \in \mathcal{D}^S$. Given a preference $P_i^* \notin \mathcal{D}^S$, if $\exists b \in S$, such that $b \in O_j(P_i^*)$, then $S \subset O_j(P_i^*)$.*

Proof. The sub-domain $\mathcal{D}^S$ is weakly path-connected, then for all $c \in S - \{b\}$, there exists a sequence $\{x_k\}_{k=1}^t \subseteq S$, such that $x_1 = c$, $x_t = b$ and $x_{k-1} \sim x_k$, for $k = 2, \ldots, t$. Since $O_j(P_i) \bigcap S = S$, for all $P_i \in \mathcal{D}^S$, there exists $P_i' \in \mathcal{D}^b$, such that $x_{t-1} \in O_j(P_i')$. Next, by lemma 1, we have $x_{t-1} \in O_j(P_i^*)$. Following the sequence, repeatedly applying lemma 1, we conclude that $c \in O_j(P_i^*)$. Hence, $S \subset O_j(P_i^*)$. ■

Chapter 5

Theorem

Theorem 1 *Assume $n = 2$. If a minimally rich Domain $\mathcal{D}$ satisfies Modified Property T, then it satisfies the tops-only property.*

Proof. Consider a preference ordering P_i^*: $a \succ \cdots \succ x_1 \succ \cdots \succ x_k \succ \cdots$, where $X(P_i^*) = \{x_1, \ldots, x_k\}$ satisfies the definition of notation 1.

Define $T_l(P_i^*) = B(P_i^*, x_l) \bigcap W(P_i^*, x_{l-1})$, $1 \le l \le k$ and $x_0 = a$. Any alternative in $T_l(P_i^*)$ must be connected to an alternative from its upper contour set.

Let a sub-domain $\mathcal{D}^{S_l}$ denote a weakly path-connected dictatorial sub-domain, such that $x_l \in S_l$ and $\exists b_l \in B(P_i^*, x_l) \bigcap S_l$, where $l = 1, 2, \ldots, k$.

Let $x \in O_j(P_i^*)$, where $x \in B(P_i^*, x_l) \bigcup \{x_l\}$, $0 \le l \le k$, $x_0 = a$, $B(P_i^*, x_0) = \emptyset$; or $x \in W(P_i^*, x_k)$.[1] We will show by induction on l that $x \in O_j(P_i')$, for all $P_i' \in \mathcal{D}^a$.

Claim 5.1 *If $x = a$, then $x \in O_j(P_i')$.*

[1]The worst preferred alternative $r_m(P_i^*) \notin X(P_i^*)$, then $W(P_i^*, x_k) \neq \emptyset$. So, we must also consider the set $W(P_i^*, x_k)$.

The claim above follows from unanimity of the SCF.

Claim 5.2 *If $x \in T_1(P_i^*)$, then $x \in O_j(P_i')$.*

This claim follows from Theorem 1 of Chatterji & Sen (2011). For completeness, we verify it as follows. Since any alternative in $T_1(P_i^*)$ is connected to an alternative from its upper contour set, there exists a sequence $\{y_j\}_{j=1}^t$, $t \geq 2$, such that $y_1 = x$, $\{y_j\}_{j=2}^t \subseteq B(P_i^*, x)$, $y_t = a$, $y_{j-1} \sim y_j$ and $y_j \, P_i^* \, y_{j-1}$, for $j = 2, \ldots, t$. According to lemma 2, following the sequence, we have $\{y_j\}_{j=1}^t \subseteq O_j(P_i^*)$. Next, by unanimity, $y_t = a \in O_j(P_i')$, then according to lemma 1, $y_{t-1} \in O_j(P_i')$. Thus, repeated application of lemma 1 gives $x \in O_j(P_i')$.

Claim 5.3 *If $x = x_1$, then $x \in O_j(P_i')$.*

Firstly, we specify whether $O_j(P_i) \bigcap S_1 = S_1$, for all $P_i \in \mathfrak{D}^{S_1}$; or $O_i(P_j) \bigcap S_1 = S_1$, for all $P_j \in \mathfrak{D}^{S_1}$. Suppose $O_i(P_j) \bigcap S_1 = S_1$, for all $P_j \in \mathfrak{D}^{S_1}$. Given a preference $\bar{P}_j \in \mathfrak{D}^{x_1}$, then $f(P_i^*, \bar{P}_j) = Max(\bar{P}_j, O_j(P_i^*)) = x_1$, because $x_1 = x \in O_j(P_i^*)$. Given another preference $\bar{P}_i \in \mathfrak{D}^{b_1}$, then $f(\bar{P}_i, \bar{P}_j) = Max(\bar{P}_i, O_i(\bar{P}_j)) = b_1$, because $O_i(\bar{P}_j) \bigcap S_1 = S_1$ and $b_1 \in S_1$. Since $b_1 \, P_i^* \, x_1$, voter i will manipulate at $(P_i^*, \bar{P}_j)$ via $\bar{P}_i$. Therefore, $O_j(P_i) \bigcap S_1 = S_1$, for all $P_i \in \mathfrak{D}^{S_1}$.

Secondly, if $a \in S_1$, then by the definition of a dictatorial sub-domain, we have $x_1 \in O_j(P_i')$. Next, we complete the proof with the analysis of the case: $a \notin S_1$. Thus, $b_1 \in T_1(P_i^*)$. Since $b_1 \in S_1$ and $x_1 \in O_j(P_i^*)$, by lemma 3, $b_1 \in O_j(P_i^*)$. Next, claim 5.2 implies that $b_1 \in O_j(P_i')$. Applying lemma 3 again, we conclude that $x \in O_j(P_i')$.

Now assume that $x \in B(P_i^*, x_{l'}) \bigcup \{x_{l'}\}$ and $x \in O_j(P_i^*)$ imply $x \in O_j(P_i')$, for all $l' < l$. We will show that $x \in B(P_i^*, x_l) \bigcup \{x_l\}$ and $x \in O_j(P_i^*)$ imply

$x \in O_j(P_i')$. Thus, we only need to focus on the set $T_l(P_i^*) \bigcup \{x_l\}$.

Claim 5.4 *If $x \in T_l(P_i^*)$, then $x \in O_j(P_i')$.*

Similar to claim 5.2, there exists a sequence $\{y_j\}_{j=1}^s$, $s \geq 2$, such that $y_1 = x$, $\{y_j\}_{j=2}^{s-1} \subseteq B(P_i^*, x) \bigcap T_l(P_i^*)$, $y_s \in B(P_i^*, x_{l-1}) \bigcup \{x_{l-1}\}$, $y_{j-1} \sim y_j$ and $y_j \, P_i^* \, y_{j-1}$, for $j = 2, \ldots, s$.

By lemma 2, following the sequence, we have $\{y_j\}_{j=1}^s \subseteq O_j(P_i^*)$. Next, by induction hypothesis, $y_s \in O_j(P_i')$. Then, applying lemma 1 iteratively, we conclude that $x \in O_j(P_i')$.

Claim 5.5 *If $x = x_l$, then $x_l \in O_j(P_i')$.*

Firstly, similar to claim 5.3, we have $O_j(P_i) \bigcap S_l = S_l$, for all $P_i \in \mathcal{D}^{S_l}$.

Secondly, if $a \in S_l$, then the definition of a dictatorial sub-domain implies that $x_l \in O_j(P_i')$. Next, we complete the proof with the analysis of the case: $a \notin S_l$. Similar to claim 5.3, lemma 3 implies that $b_l \in O_j(P_i^*)$. Since $b_l \in B(P_i^*, x_l) = B(P_i^*, x_{l-1}) \bigcup \{x_{l-1}\} \bigcup T_l(P_i^*)$, according to induction hypothesis or claim 5.4, we have $b_l \in O_j(P_i')$. Apply lemma 3 again, we conclude that $x \in O_j(P_i')$.

Claim 5.6 *For all $x \in B(P_i^*, x_k) \bigcup \{x_k\}$, if $x \in O_j(P_i^*)$, then $x \in O_j(P_i')$, for all $P_i' \in \mathcal{D}^a$.*

This claim follows from claims 5.1 to 5.5.

Claim 5.7 *If $x \in W(P_i^*, x_k)$, then $x \in O_j(P_i')$.*

Similar to claim 5.4, since any alternative in $W(P_i^*, x_k)$ is connected to an alternative from its upper contour set, there exists a sequence $\{y_j\}_{j=1}^s$, $s \geq 2$, such that $y_1 = x$, $\{y_j\}_{j=2}^{s-1} \subseteq B(P_i^*, x) \bigcap W(P_i^*, x_k)$, $y_s \in B(P_i^*, x_k) \bigcup \{x_k\}$, $y_{j-1} \sim y_j$ and $y_j \, P_i^* \, y_{j-1}$, for $j = 2, \ldots, s$.

By lemma 2, following the sequence, we have $\{y_j\}_{j=1}^{s} \subseteq O_j(P_i^*)$. Next, by claim 5.6, $y_s \in O_j(P_i')$. Then, applying lemma 1 iteratively, we conclude that $x \in O_j(P_i')$.

In conclusion, for all $x \in A$, if $x \in O_j(P_i^*)$, then $x \in O_j(P_i')$, for all $P_i' \in \mathcal{D}^a$. This implies that $O_j(P_i^*) \subseteq O_j(P_i')$, for all $P_i' \in \mathcal{D}^a$. The proof for $O_j(P_i') \subseteq O_j(P_i^*)$ is identical. Therefore, $O_j(P_i^*) = O_j(P_i')$, for all $P_i^*, P_i' \in \mathcal{D}^a$. Hence, we conclude that $O_j(P_i) = O_j(P_i')$ whenever P_i and P_i' are tops-equivalent.

An identical argument with subscripts i and j interchanged gives a symmetric conclusion: $O_i(P_j) = O_i(P_j')$, whenever P_j and P_j' are tops-equivalent. Therefore, in the case of 2 voters, Modified Property T implies the tops-only property. ■

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