

# A Primer on Energy Conditions

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**Abstract** An energy condition, in the context of a wide class of spacetime theories (including general relativity), is, crudely speaking, a relation one demands the stress-energy tensor of matter satisfy in order to try to capture the idea that “energy should be positive”. The remarkable fact I will discuss in this paper is that such simple, general, almost trivial seeming propositions have profound and far-reaching import for our understanding of the structure of relativistic spacetimes. It is therefore especially surprising when one also learns that we have no clear understanding of the nature of these conditions, what theoretical status they have with respect to fundamental physics, what epistemic status they may have, when we should and should not expect them to be satisfied, and even in many cases how they and their consequences should be interpreted physically. Or so I shall argue, by a detailed analysis of the technical and conceptual character of all the standard conditions used in physics today, including examination of their consequences and the circumstances in which they are believed to be violated.

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43

# 1 The Character of Energy Conditions

An energy condition, in the context of a wide class of spacetime theories (including general relativity), is, crudely speaking, a relation one demands the stress-energy tensor of matter satisfy in order to try to capture the idea that “energy should be positive”.<sup>1</sup> Perhaps the simplest example is the so-called weak energy condition: for any timelike vector  $\xi^a$  at any point of the spacetime manifold, the stress-energy tensor  $T_{ab}$  satisfies  $T_{mn}\xi^m\xi^n \geq 0$ . This has, *prima facie*, a simple physical interpretation: the (ordinary) energy density of the fields contributing to  $T_{ab}$ , as measured in a natural way by any observer (*e.g.*, using instruments at rest relative to that observer), is never negative. The remarkable fact I will discuss in this paper is that such simple, general, almost trivial seeming propositions have profound and far-reaching import for our understanding of the structure of relativistic spacetimes. It is therefore, especially, surprising when one also learns that we have no clear understanding of the nature of these conditions, what theoretical status they have *vis-à-vis* fundamental physics, what epistemic status they may have, when we should and should not expect them to be satisfied, and even in many cases how they and their consequences should be interpreted physically. Or so I shall argue.

Geroch and Horowitz [92, p. 260], in discussing the form of singularity theorems in general relativity, outline perhaps the most fundamental reason for the importance of energy conditions with the following pregnant observation:

One would of course have to impose *some* restriction on the stress-energy of matter in order to obtain any singularity theorems, for with no restrictions Einstein’s equation has no content. One might have thought, however, that only a detailed specification of the stress-energy at each point would suffice, *e.g.* that one might have to prove a separate theorem for each combination of the innumerable substances which could be introduced into spacetime. It is the energy condition which intervenes to make this subject simple. On the one hand it seems to be a physically reasonable condition on all types of classical matter, while on the other it is precisely the condition on the matter one needs for the singularity theorem.

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<sup>1</sup>From hereon until §5, unless explicitly stated otherwise, the discussion should be understood to be restricted to the context of general relativity. Almost everything I say until then will in fact hold in a very wide class of spacetime theories, but the fixed context will greatly simplify the exposition. In general relativity, the fundamental theoretical unit, so to speak, is a spacetime model consisting of an ordered pair  $(\mathcal{M}, g_{ab})$ , where  $\mathcal{M}$  is a four-dimensional, paracompact, Hausdorff, connected, differential manifold and  $g_{ab}$  is a pseudo-Riemannian metric on it of Lorentzian signature. ‘ $T_{ab}$ ’ will always refer to the stress-energy tensor picked out in a spacetime model by the Einstein field equation, ‘ $T$ ’ to the trace of  $T_{ab}$  ( $T^n_n$ ), ‘ $R_{ab}$ ’ to the Ricci tensor associated with the Riemann tensor  $R^a_{bcd}$  associated with the unique torsion-free derivative operator  $\nabla$  associated with  $g_{ab}$ , ‘ $R$ ’ to the trace of the Ricci tensor ( $R^n_n$ , the Gaussian scalar curvature), and ‘ $G_{ab}$ ’ to the Einstein tensor ( $R_{ab} - \frac{1}{2}Rg_{ab}$ ). For conventions about the metric signature and the exact definitions of these tensors, I follow Malament [133]. Unless otherwise explicitly noted, indicial lowercase Latin letters ( $a, b, \dots$ ) designate abstract tensor-indices, indicial lower-case Greek letters ( $\mu, \nu, \dots$ ) designate components with respect to a fixed coordinate system or tetrad of tangent vectors ( $\mu \in \{0, 1, 2, 3\}$ ), and hatted indicial lower-case Greek letters ( $\hat{\mu}, \hat{\nu}, \dots$ ) designate the spacelike components ( $\hat{\mu} \in \{1, 2, 3\}$ ) with respect to a fixed 1+3 tetrad system. (For an exposition of the abstract index notation, see Penrose and Rindler [155], Wald [203], or Malament [133].)

I will return to this quote later, in §5, but for now the salient point is that a generic condition one imposes on the stress-energy tensor, “generic” in the sense that it can be formulated independently of the details of the internal structure of the tensor, which is to say independently of any quantitative or structural feature or idiosyncrasy of any particular matter fields, suffices to prove theorems of great depth and scope. Indeed, as Geroch and Horowitz suggest, without the possibility of relying on conditions of such a generic character, we would not have the extraordinarily general and far-reaching singularity theorems we do have. And it is not only singularity theorems that rely for their scope and power on these energy conditions—it is no exaggeration to say that the great renaissance in the study of general relativity itself that started in the 1950s with the work of Synge, Wheeler, Misner, Sachs, Bondi, Pirani, *et al.*, and the blossoming of the investigation of the global structure of relativistic spacetimes at the hands of Penrose, Hawking, Geroch, *et al.*, in the 1960’s could not have happened without the formulation and use of such energy conditions.

What is perhaps even more remarkable is that many of the most profound results in the study of global structure—*e.g.*, the Hawking Area Theorem—do not depend on the Einstein field equation at all, but rather assume only a purely formal condition imposed on the Ricci tensor, which itself can be thought of as an “energy” condition if one invokes the Einstein field equation to provide a physical interpretation of the Ricci tensor. In a sense, therefore, energy conditions seem to reach down to and get a hold of a level of structure in our understanding of gravitation and relativistic spacetimes even more fundamental than the Einstein field equation itself. (I will discuss in §5 this idea of “levels of structure” in our understanding of general relativity in particular, and of gravitation and spacetime more generally.)

Now, most propositions of a fundamental character in general relativity admit of interpretation as either a postulate of the theory or as a derived consequence from some other propositions taken as postulates. That is to say, the theory allows one a great deal of freedom in what one will take as given and what one will demand a proof of. One can, for example, either assume the so-called Geodesic Principle from the start as a fundamental regulative principle of the theory, as, for example, in the exposition of Malament [133], or one can assume other propositions as fundamental, perhaps ones fixing the behavior of ideal clocks and rods, and derive the Geodesic Principle as a consequence of those propositions, as, for example, in the exposition of Eddington [64]. Which way one goes for any given proposition depends, in general, on the context one is working in, the aims of one’s investigation, one’s physical and philosophical intuitions and predilections, *etc.*<sup>2</sup>

This interpretive flexibility does not seem to hold, however, for energy conditions. I know of no substantive proposition that, starting from some set of other important “fundamental postulates”, has as its consequence an energy condition. One either imposes an energy condition by fiat, or one shows that it holds for stress-energy tensors associated with particular forms of matter fields. One never imposes general

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<sup>2</sup>See Weatherall [209, this volume] for an insightful discussion of a view of the foundations of spacetime theories, with particular regard to this issue, that I find sympathetic to my own views as I sketch them here.

conditions on other geometrical structures (*e.g.*, the Riemann tensor or the topology or the global causal structure) and derives therefrom the satisfaction of an energy condition (except in the trivial case where one imposes conditions directly on the Ricci or Einstein tensor, standing as a direct proxy for the stress-energy tensor by dint of the relation between them embodied by the Einstein field equation).<sup>3</sup> There are a plethora of results that show when various energy conditions may or must be violated both theoretically and according to observation, which I discuss in §3.2, but none that show nontrivially when one must hold. Indeed, this inability to prove them is an essential part of what seems to make them structure “at a deeper level” perhaps even than causality conditions (many of which can be derived from other fundamental assumptions), and so applicable across a *very* wide range of possible theories of spacetime.

In a similar vein, they occupy an odd methodological and theoretical niche quite generally. None is implied by any known general theory, though each can be formulated in the frameworks of a wide spectrum of different theories, and several can be shown to be inconsistent with a wide spectrum of theories (in the strong sense that one can derive their respective negations in the context of the theories). Indeed, they are among the very few physical propositions I know that can be used either to exclude as physically unreasonable individual solutions to the field equations of a particular theory (as for, *e.g.*, a wide class of FLRW spacetimes in general relativity that have strongly negative pressures<sup>4</sup>), or to exclude entire theories (such as the Hoyle Bondi steady-state theory of cosmology, as I discuss below in §3.2). Whether or not one should consider them as “part of” any given theory, therefore, seems a problematic question at best, and an ill-posed one at worst.

It is difficult to get a grip on their epistemic status as well. They seem in no sense to be laws, under any standard account in the literature, for none of them holds for all known “physically reasonable” types of matter, and each of them is in fact violated in what seem to be physically important circumstances. Neither do they appear to be empirical or inductive generalizations, for the same reason.<sup>5</sup> And yet we think that (at least) one of them—or something close to them—likely holds generically in

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<sup>3</sup>The one possible exception to this claim I know of is the attempt by Wall [206] to derive the so-called averaged null energy condition (ANEC) from the Generalized Second Law of thermodynamics. While I find his arguments of great interest, I also find them problematic at best. See Curiel [48] for discussion.

<sup>4</sup>See Curiel [47] for discussion.

<sup>5</sup>It should be noted, however, that, to the best of my knowledge, there has never been direct experimental observation of a violation of any of the standard energy conditions I discuss in §2. We do, however, have extremely good *indirect* experimental and observational evidence for violations of several of them, as I discuss in §3. See Curiel [47] for an extended discussion of evidence for their violation in cosmology, and Curiel [48] for one in the context of quantum field theory on curved spacetime. Even direct experimental verification of the Casimir effect does not yield direct measurement of negative energy densities, though the Casimir effect relies essentially on the existence of such; rather, the negative energy densities are inferred from measurement of the Casimir force itself [28].

the actual universe, at the level of classical (*i.e.*, non-quantum) physics at least, and even that one or more of them, appropriately reformulated, should hold generically at the quantum level as well.<sup>6</sup> Even more, as I have already indicated, there seem to be very good reasons for thinking that the sense in which they do obtain, whatever that may be, is grounded in structure at a level of our understanding even deeper than the Einstein field equation itself, which we surely do think of as a law, under any reasonable construal of the notion.

So what are they? The remainder of this paper consists of an attempt to come to grips with this question, by exploring their formulations, their consequences, their relations to other fundamental structures and principles, and their role in constraining the possible forms a viable theory of spacetime may take. Those who hope for a decisive answer to the question will leave disappointed. I feel I will have succeeded well enough if I am able only to survey the most important issues and questions, clarify and sharpen some of them, propose a few conjectures, and generally open the field up for other investigators to do more work in it.<sup>7</sup>

## 2 The Standard Energy Conditions

There are several different ways to formulate all the energy conditions standardly deployed in classical general relativity, both as a group and individually. I will focus here on three ways of formulating them as a group, what one may think of as the *geometric*, the *physical* and the *effective* ways, and will for a few of them discuss as well alternative individual formulations according to the geometric and physical ways, as they variously allow different insights into the character of the conditions.<sup>8</sup> The geometric and physical ways are easy to characterize: for the former, one writes down formal conditions expressed by use only of the value of a purely geometric tensor (such as the Ricci or Weyl tensor), perhaps as it is required to stand in relation to a fixed family of vectors or other tensors; for the latter, one writes down formal conditions expressed by use only of the value of the stress-energy tensor itself, perhaps as it is required to stand in relation to a fixed family of vectors or other tensors.<sup>9</sup> In every

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<sup>6</sup>See Curiel [48] for discussion.

<sup>7</sup>This paper, in other words, has as its goal a more modest version of that of Earman's wonderful book *A Primer on Determinism*, to which the name of this paper is an homage.

<sup>8</sup>In this section, aside from a few idiosyncracies, such as my classification of different types of formulation, I follow in part the exposition of [195, ch. 12] and in part that of [133, §2.5 and §2.8] for the formulations of the conditions themselves. See Curiel [47] for another formulation of them, based on the scale factor  $a(t)$  in generic cosmological models, and discussion thereof.

<sup>9</sup>Another interesting way to study the properties and behavior of  $T_{ab}$  is by the Segré algebraic classification of symmetric rank-two covariant tensors. (See, *e.g.*, Hall [94].) It is beyond the scope of the current paper to discuss that.

case, the physical formulation is logically equivalent to the geometric formulation if the Einstein field equation is assumed to hold.<sup>10</sup>

The effective way requires a bit of groundwork to explain. According to a useful classification of stress-energy tensors given by [107, p. 89], a stress-energy tensor is said to be of type I if at every point there is a  $1 + 3$  orthonormal frame with respect to which it is diagonal, *i.e.*, if its only nonzero components as computed in the given frame are on the diagonal in its matrix form. In this case, it is natural to interpret the timelike-timelike component as the ordinary (mass-)energy density  $\rho$  as represented in the given frame, and the three spacelike-spacelike components to be the three principal pressures  $p_{\hat{\mu}}$  ( $\hat{\mu} \in \{1, 2, 3\}$ ) as represented in the frame, to be understood by analogy with the case of a fluid or an elastic body. The effective formulation of an energy condition can then be stated as a quantitative relation among  $\rho$  and  $p_{\hat{\mu}}$ . Since all known “physically reasonable” classical fields (and indeed many unreasonable ones) have associated stress-energy tensors of type I, this is no serious restriction.<sup>11</sup> Thus, except for one special case to be discussed below, the effective formulation should be understood to be in all ways physically equivalent to the geometric and the physical formulations, under the assumption that the Einstein field equation holds, and matter is not too exotic. Under that assumption, the effective formulations become especially useful in cosmological investigations, since the matter fields in standard cosmological models, the FLRW spacetimes, can always be thought of as fluids.

It will be convenient to break the conditions up into two further classes, those (*pointilliste*) that constrain behavior at individual points and those (*impressionist*) that constrain average behavior over spacetime regions. I shall first list the definitions of all the former, then discuss the significance and interpretation of each as it will be useful to have them all in hand at once for the purposes of comparison, then do the same for the latter class.

## 2.1 Pointilliste Energy Conditions

### null energy condition (NEC)

- geometric** for any null vector  $k^a$ ,  $R_{mn}k^m k^n \geq 0$
- physical** for any null vector  $k^a$ ,  $T_{mn}k^m k^n \geq 0$
- effective** for each  $\hat{\mu}$ ,  $\rho + p_{\hat{\mu}} \geq 0$

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<sup>10</sup>This equivalence between the physical and the geometrical formulations does not hold in general if *and only if* the Einstein field equation holds. The biconditional holds in general relativity (for minimally coupled fields, at least). In other spacetime theories with field equations similar to but distinct from the Einstein field equation, the biconditional will not in general hold. I will discuss this further in §5.

<sup>11</sup>The one possible exception to this claim is a null fluid, which has a stress-energy tensor of the form  $T_{ab} = \rho k_a k_b + p_1 x_a x_b + p_2 y_a y_b$ , where  $k^a$  is null and  $x^a$  and  $y^a$  are unit spacelike vectors orthogonal to  $k^a$  and to each other.

**weak energy condition (WEC)**

- geometric** for any timelike vector  $\xi^a$ ,  $G_{mn}\xi^m\xi^n \geq 0$
- physical** for any timelike vector  $\xi^a$ ,  $T_{mn}\xi^m\xi^n \geq 0$
- effective**  $\rho \geq 0$ , and for each  $\hat{\mu}$ ,  $\rho + p_{\hat{\mu}} \geq 0$

**strong energy condition (SEC)**

- geometric** for any timelike vector  $\xi^a$ ,  $R_{mn}\xi^m\xi^n \geq 0$
- physical** for any timelike vector  $\xi^a$ ,  $(T_{mn} - \frac{1}{2}Tg_{mn})\xi^m\xi^n \geq 0$
- effective**  $\rho + \sum_{\hat{\mu}} p_{\hat{\mu}} \geq 0$ , and for each  $\hat{\mu}$ ,  $\rho + p_{\hat{\mu}} \geq 0$

**dominant energy condition (DEC)**

- geometric**
  1. for any timelike vector  $\xi^a$ ,  $G_{mn}\xi^m\xi^n \geq 0$ , and  $G^a_n\xi^n$  is causal
  2. for any two co-oriented timelike vectors  $\xi^a$  and  $\eta^a$ ,  $G_{mn}\xi^m\eta^n \geq 0$
- physical**
  1. for any timelike vector  $\xi^a$ ,  $T_{mn}\xi^m\xi^n \geq 0$ , and  $T^a_n\xi^n$  is causal
  2. for any two co-oriented timelike vectors  $\xi^a$  and  $\eta^a$ ,  $T_{mn}\xi^m\eta^n \geq 0$
- effective**  $\rho \geq 0$ , and for each  $\hat{\mu}$ ,  $|p_{\hat{\mu}}| \leq \rho$

**strengthened dominant energy condition (SDEC)**

- geometric**
  1. for any timelike vector  $\xi^a$ ,  $G_{mn}\xi^m\xi^n \geq 0$ , and, if  $R_{ab} \neq 0$ , then  $G^a_n\xi^n$  is timelike
  2. either  $G_{ab} = 0$ , or, given any two co-oriented causal vectors  $\xi^a$  and  $\eta^a$ ,  $G_{mn}\xi^m\eta^n > 0$
- physical**
  1. for any timelike vector  $\xi^a$ ,  $T_{mn}\xi^m\xi^n \geq 0$ , and, if  $T_{ab} \neq 0$ , then  $T^a_n\xi^n$  is timelike
  2. either  $T_{ab} = 0$ , or, given any two co-oriented causal vectors  $\xi^a$  and  $\eta^a$ ,  $T_{mn}\xi^m\eta^n > 0$
- effective**  $\rho \geq 0$ , and for each  $\hat{\mu}$ ,  $|p_{\hat{\mu}}| \leq \rho$

(It is not a typo that the given effective forms of the DEC and the SDEC are identical; this is the one special case, mentioned above, in which the effective form of the energy condition diverges from the geometrical and physical forms. Of course, it is the case that when one restricts attention to stress-energy tensors of type I, then the geometrical and physical forms of the DEC and SDEC also coincide.) I first sketch the most more or less straightforward interpretations of the conditions, before discussing problems with those interpretations.

The idea of average radial acceleration (explained in detail in the technical appendix §2.5 below) offers one seemingly promising route toward an interpretation of the geometric and physical forms of the NEC. Roughly speaking, the average radial acceleration of a geodesic  $\gamma$  at a point  $p$  is the averaged magnitude of the acceleration of neighboring geodesics relative to  $\gamma$  in directions orthogonal to  $\gamma$ . If the average

radial acceleration is negative, then this represents the fact that, again roughly speaking, neighboring geodesics tend to fall inwards towards  $\gamma$  at  $p$ . Thus, according to equation (2.4), the geometric form of the NEC requires that null geodesic congruences tend to be convergent in sufficiently small neighborhoods of every spacetime point (or at least not divergent). Assuming the Einstein field equation, the physical interpretation of negative average radial acceleration for causal geodesics is that, again roughly speaking, the “gravitational field” generated by the ambient stress-energy is “attractive”. Thus, according to equation (2.5), the interpretation of the physical form is that particles following null geodesics will observe that “gravity” tends locally to be “attractive” (or at least not repulsive) when acting on nearby particles also following null geodesics. Another possible interpretation of the physical form of the NEC is that an observer traversing a null curve will measure the ambient (ordinary) energy density to be positive.

The interpretation of the effective form of the NEC is that the natural measure either of mass–energy or of pressure in any given spacelike direction can be negative as determined by an observer traversing a null curve, but not both, and, if either is negative, it must be less so than the other is positive. In so far as one may think of pressure as a momentum flux, therefore, and so equivalent relativistically to a mass–energy flow, the effective form requires that ordinary mass–energy density at any point cannot be negatively dominated by momentum fluxes in any given spacelike direction as determined by an observer traversing a null curve: one cannot indefinitely “mine” energy from a system by subjecting it to negative momentum flux.

The interpretation of the physical form of the WEC is straightforward: the (ordinary) total energy density of all matter fields, as measured in a natural way by any observer traversing a timelike curve, is never negative. The interpretation of the geometric form is not straightforward. Indeed, I know of no simple, intuitive picture that captures the geometrical significance of the condition.<sup>12</sup> The interpretation of the effective form is similar to that for the NEC. Ordinary mass–energy density must be nonnegative as experienced by any observer traversing a timelike curve, and the

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<sup>12</sup>It has gone oddly unremarked in the physics and philosophy literatures, but is surely worth puzzling over, that the Einstein tensor itself, the fundamental constituent of the Einstein field equation, has no simple, natural geometrical interpretation, in the way, *e.g.*, that the Riemann tensor can naturally be thought of as a measure of geodesic deviation. Perhaps one could try to use the Bianchi identity to construct a geometric interpretation for  $G_{ab}$ , or the Lanczos tensor (see footnote 22), but it is not immediately obvious to me what such a thing would look like, if possible. One can give a geometrical interpretation of  $G_{ab}$  at a point by considering all unit timelike vectors at the point; the Einstein tensor can then be reconstructed by defining it to be the unique symmetric two-index covariant tensor at that point such that its double contraction with every unit timelike vector equals minus one-half the spatial scalar curvature of the spacelike hypersurface with vanishing extrinsic curvature orthogonal to the given vector. (See Malament [133, ch. 2, §7].) This may be only a matter of taste, but I find this interpretation obscure and Baroque, certainly not simple and natural, in large part because it relies on structure in a family of three-dimensional objects to fix the meaning of a four-dimensional object.



pressure in any given spacelike direction can never be so negative as to dominate that value.<sup>13</sup>

It is easy to see, by considerations of continuity, that the WEC implies the NEC. Tipler [189] proved two propositions that give some insight into the relation between the NEC and the WEC, and into the character of the WEC itself. He first showed that, in a natural sense, the WEC is the weakest local energy condition one can define. (“Local” here means something like: holding at a point, for all observers.) In particular, he proved the following: if  $T_{mn}\xi^m\xi^n$  is finitely bounded from below for all timelike  $\xi^a$ , *i.e.*, if there exists a  $b > 0$  such that  $T_{mn}\xi^m\xi^n \geq -b$  for all timelike  $\xi^a$ , then WEC holds (*i.e.*, the infimum of all such  $b$  is 0). He next proved that one cannot do better by imposing further natural constraints on the condition: if  $T_{mn}\xi^m\xi^n$  is finitely bounded from below for all unit timelike  $\xi^a$ , and  $T_{ab}$  is of type I, then the NEC holds. The effective form of the WEC, therefore, is in fact essentially equivalent to the NEC. Thus, though the WEC is not the weakest condition in a logical sense one can impose, it is the weakest in a loose, physical sense: one cannot do better by imposing further natural restrictions.

The interpretation of the geometric form of the SEC is similar to that of the NEC. According to equation (2.4), the geometric form of the SEC requires that timelike geodesic congruences tend to be convergent in sufficiently small neighborhoods of every spacetime point. This implies that congruences of null geodesics at that point are also convergent. Similarly, according to equation (2.5), the interpretation of the physical form is that observers following timelike geodesics will see that “gravity” tends locally to be “attractive” in its action on stuff following both timelike and null geodesics.<sup>14</sup> The effective form of the SEC has part of its interpretation the same as that of the WEC, *viz.*, ordinary mass–energy density at any point cannot be negatively dominated by momentum fluxes in any given spacelike direction as determined by an observer traversing a timelike curve. It also says, however, that ordinary mass–energy density cannot be negatively dominated by the sum of the individual pressures (momentum fluxes) at any point, as determined by an observer traversing a timelike curve. I know of no compelling elucidation of the physical content of that relation. The SEC does not imply the WEC, for the SEC can be satisfied even if the ordinary mass–density is negative. The SEC does, however, imply the NEC.

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<sup>13</sup>Classically, some fluids such as water are known to exhibit negative pressures in some regimes as measured by observers traversing timelike curves (*e.g.*, us), but these negative pressures are never large enough to dominate the fluid’s mass–energy. Indeed, when one considers how large the relativistic mass–energy of, say, 1 g of water is, and so correlatively how extraordinarily intense a momentum flux would have to be to achieve a mass–energy content comparable to that, one gets a good feel for just how “exotic” any stuff would be that violates the NEC.

<sup>14</sup>This explication of the physical form of the SEC clearly illustrates why it is problematic to try to think of general relativity as a theory of “gravity”, in the sense of a force exerted on a body: for bodies traversing non-geodesic curves, that is, for bodies experiencing nontrivial acceleration, one has no natural way to judge whether “the force of gravity” is acting attractively or repulsively, not even when one fixes a standard of rest (a fiducial body traversing a timelike geodesic). *Pace* particle physicists, general relativity simply cannot be comprehended as a theory describing a dynamical “force” at all.

As for the WEC, the interpretations of the geometrical forms of the DEC and the SDEC are not clear. The interpretations of their physical forms are apparent: every timelike observer will measure ordinary mass–energy density to be nonnegative, and will also measure total flux of energy–momentum to be causal, with the flow oriented in the same direction as the observer’s proper time. The SDEC, as the name suggests, is slightly stronger in that it requires energy–momentum flux as measured by any timelike observer to be strictly timelike for nontrivial stress-energy distributions. The DEC (and *a fortiori* the SDEC) are, therefore, standardly taken to rule out “superluminal propagation of stress-energy”. (See, *e.g.*, the exemplary remarks of Wald [203, p. 219].) As already noted, the effective forms of the DEC and SDEC are identical. Their interpretation, besides the now-familiar demand that locally measured energy density be nonnegative, is that pressures be strictly bounded both above and below by the energy density. This means that the effective fluid can be neither too “stiff” nor too “lax”, but must lie in a middling Goldilocks regime.<sup>15</sup> The second given geometric and physical forms of the SDEC make it manifest that the SDEC is in fact logically stronger than the DEC. Of course, any  $T_{ab}$  that satisfied the DEC but violated the SDEC would have to be not of Hawking-Ellis type I, for it is only in that case that the two come apart. Clearly, the SDEC implies the DEC, which implies the WEC.

Before turning to examine the so-called impressionist energy conditions, I briefly discuss a few problems with the interpretations I have sketched of the pointilliste conditions. The interpretations of the geometrical and physical forms of the NEC based on average radial acceleration is undermined by the fact that convergence of null geodesics at a point does not in general imply convergence of all timelike geodesics at that point. This is why I hedged the proposed interpretations with slippery terms like ‘tends to’: even if the NEC is satisfied at a point, an observer traversing a timelike geodesic may still see “gravity acting repulsively” in a small neighborhood. The existence of a positive cosmological constant is a case in which NEC is satisfied, but, by the failure of the SEC, there is still divergence of timelike geodesics: “gravity acts repulsively” on matter following timelike geodesics, even though it “acts attractively” on stuff following null geodesics.<sup>16</sup> The other proposed interpretation of the physical form of the NEC—that observers traversing null curves will measure nonnegative energy density—suffers from the fact that it is difficult to see

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<sup>15</sup>See Curiel [47] for a discussion of the consequences of allowing the effective fluid to be too lax, which is to say, allowing the barotropic index  $w$  to be less than  $-1$ , in the context of cosmology.  $w := \frac{p}{\rho}$ , and so is a useful measure of the “stiffness” of whatever (nearly) homogeneous, isotropic stuff fills spacetime in cosmological models.

<sup>16</sup>It should be kept in mind that the physical consequences of a “positive” versus a “negative” cosmological constant in this context depend on one’s conventions for writing the Einstein field equation and on one’s conventions for the metric signature. With the conventions I am using, a positive value of  $\Lambda$  itself leads to negative momentum flux in spacelike directions, and that is the condition that leads to accelerated expansion on the cosmological scale, as actually observed, and so the theoretical need for “dark energy”.

what physical sense can be made of the idea of an observer traveling at the speed of light making (ordinary) energy measurements. One cannot try to ameliorate this problem by positing that the condition means only that a physical system traversing a null curve will “experience” only nonnegative energy densities in its couplings with other systems, irrespective of whether it is an observer making measurements: ordinary energy density is not an observer-independent quantity, and so it can mediate no physical interaction in any way with intrinsic physical significance. No physical system will “experience” ordinary energy density at all.<sup>17</sup>

The interpretation of the effective form of the NEC suffers the same difficulty: what physical content does it have to compare the magnitude of ordinary energy density and that of momentum flux in a given spacelike direction, as determined by an observer traversing a *null* curve? There is an even more serious problem here, though, which the effective form makes particularly clear, showing the limitations of the physical significance of the NEC. Assuming a well behaved barotropic equation of state for the effective fluid described by the stress-energy tensor, *i.e.*, a fixed relation  $\rho(p)$  expressing  $\rho$  as an invertible function of the single isotropic pressure  $p$ , and assuming the medium is not too strongly dispersive, the speed of sound is  $c_s^2 = \frac{dp}{d\rho}$ . It should be clear that the NEC does *not* require that  $c_s \leq 1$ ; in other words, stuff can satisfy the NEC while still permitting superluminal propagation of physically significant structure. It is thus unclear in the end what real physical significance the requirement that mass–energy density not be negatively dominated by momentum fluxes has.

The problems with the effective interpretation of the WEC are much the same as for the NEC: it is not clear what physical significance the given relations among energy density and pressure can have when they permit superluminal propagation of physical structure. The fact that the WEC requires energy density always to be positive may make one at first glance think that it will be violated in the ergosphere of a Kerr black hole, where, as is well known, ordinary systems can have in a natural sense negative energy [150, 154]. In fact, though, there is an equivocation on ‘energy’ here that points to a subtle and important point. The energy that can be negative near a Kerr black hole is the energy defined by the stationary Killing field of the spacetime, not the ordinary energy density as measured by any observer using tools at rest with respect to herself. (Because the stationary Killing field is spacelike in the ergosphere, no observer can have any of its orbits as worldline.) Now, as I remarked in footnote 17, ordinary

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<sup>17</sup>We decompose  $T_{ab}$  into energy density, momentum flux and stress in our *representations* of our experiments, for various pragmatic and psychological reasons; the decomposition represents nothing of intrinsic physical significance about the world. This fact perhaps lies at the root of most if not all the difficulties and puzzles that plague the energy conditions, especially why they do not seem to be derivable from other fundamental principles. Of course, this fact also makes it even more puzzling that they should have such profound, physically significant consequences as they do. What is going on here?

energy density, not being an observer-independent quantity, is not a particularly natural concept in general relativity. The energy defined by a stationary Killing field, however, *is* observer-independent and so has *prima facie* physical significance, even more so given that it obeys both a local and an integral conservation law. Why is it not troubling that *this* quantity, a manifestly deep and important one, can be negative, whereas the negativity of the observer-dependent ordinary energy density throws us into fits? Why do we depend so strongly on conditions formulated using quantities that, under their standard physical interpretation, are not observer-independent, especially when proving results about quantities and structures such as event horizons that are observer-independent? I don't know. Perhaps the lesson here is that the geometric form of the energy conditions are the ones to be thought of as fundamental, in so far as they rely for their statement and interpretation only on invariant, geometrical structures and concepts. It would then be an interesting problem why in the context of some theories, such as general relativity, the physical interpretation of the conditions turns out to have questionable significance. Perhaps this is telling us to look for theories in which these important geometric conditions have physically significant interpretations. I will return to discuss this question in §5.

With regard to the SEC, because the convergence of all timelike geodesics at a point does imply the convergence of null geodesics there, the proposed interpretations of its geometric and physical forms, that “gravity tends to be attractive”, are on firmer ground than for the NEC. There is still a problem, though, even here. Averaged radial acceleration is, after all, only an average, factitious quantity. That it be negative does not say that individual freely falling ordinary bodies cannot in fact accelerate away from each other for no apparent reason, only that, on average, they do not do so. Thus, the idea that average geodesic convergence should be thought of as a representation of the attractiveness of gravity is dicey at best. And, again, there is the issue that this condition says nothing at all about the “effect of gravity” on bodies accelerating under the action of other forces.

The DEC (and *a fortiori* the SDEC) are standardly taken to rule out “superluminal propagation of stress-energy”. Once again, however, it is clear that the DEC does not preclude superluminal speeds of sound for fields, so it is not clear what work the prohibition on superluminal propagation of stress-energy is doing. Even if we put that point aside, though, there are other problems, as Earman [62] argues, claiming the DEC ought not be interpreted as prohibiting superluminal propagation of stress-energy. His argument goes in two steps. He first argues for the positive conclusion that the proper way to conceive of a prohibition on superluminal propagation is the existence of a well posed (in the sense of Hadamard) initial value formulation for all fields on spacetime. Then, based on Geroch [91], he shows that physical systems can have well posed initial value formulations even when the DEC is violated. Earman's arguments are buttressed by a recent argument due to Wong [212]. As Wong notes (along with Earman), the evidence almost always cited in support of the idea that DEC prohibits superluminal propagation of stress-energy is the theorem that states that, if a covariantly divergence-free  $T_{ab}$  is required to satisfy the DEC and it vanishes on a closed, achronal set, then it vanishes in the domain of dependence of that set

[102, 107].<sup>18</sup> Wong, I think rightly, points out that this theorem in fact shows only that DEC prohibits “the edge of a vacuum” (or vacuum fluctuations, in a quantum context) from propagating superluminally, not arbitrary stress-energy distributions. Given the nonlinearity of the Einstein field equation, I find it plausible that there may be problems in trying to naively generalize this result to arbitrary stress-energy tensors, whether they obey the DEC or not.

Comparing the strengths and weaknesses of the interpretations of the different forms of the conditions amongst themselves reveals some interesting questions. Consider the NEC: on the face of it, the geometric form has a relatively unproblematic interpretation, whereas the interpretations of the physical and effective forms are beset with more serious problems. The case is just the opposite for the WEC: the geometric form has no clear interpretation, whereas the physical form and at least part of the effective form (the positivity of energy density) are relatively unproblematic. The DEC occupies yet more treacherous ground, in so far as the geometric form has no clear interpretation, the physical interpretation (as Earman’s and Wong’s arguments show) is muddled at best, and the effective is only partially unproblematic. And yet these statements are, modulo the assumption of the Einstein field equation, logically equivalent. Ought unclarity of interpretation of one form push us to question the seeming clarity of interpretation of other forms? How can this happen, that the interpretation of one proposition can be problematic while the interpretation of a proposition logically equivalent is not (or, at least, is less so)? Can we lay all the blame on the assumption of the Einstein field equation? I don’t think so, for, if we could, then surely the forms that had interpretive problems would all be of the same type, but that is not the case here. Sometimes it is the geometric that is less problematic, and other times it is the more problematic.

This is not the place to try to address these questions. I will remark only that this topic would provide very rich fodder for an investigation into the relations between pure geometry and the physical systems that geometry purports to represent in a given theory, what must be in place in order to extract physically significant information from the geometry of those systems, and what the difference is between having an interpretation of a piece of pure mathematics and having a physical interpretation of it in the context of a theory. I have the sense that it is often a tacit assumption in philosophical discussions of the meaning of theoretical terms that, if a mathematical structure has a clear physical interpretation in a theory, then it itself must have a clear mathematical interpretation already. These examples show that this need not be so. They also provide interesting case studies of how theoretically equivalent statements can seemingly have very different physical meanings.

I conclude this section with an observation of what is *not* here: there are no standard energy conditions based on the Weyl conformal tensor  $C^a{}_{bcd}$  or on the Bel–Robinson

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<sup>18</sup>A region of spacetime is achronal if no two of its points stand in timelike relation to each other. The domain of dependence  $D(\Sigma)$  of a closed achronal set  $\Sigma$  is the collection of all points  $p$  in spacetime such that every inextendible causal curve passing through  $p$  intersects  $\Sigma$ .

tensor  $T_{abcd}$ .<sup>19</sup> I find this odd. Because there is no object in general relativity that one can reasonably interpret as the stress-energy tensor of the “gravitational field”, the standard pointilliste energy conditions do not directly constrain the behavior of anything one may want to think of as gravitational stress-energy, and yet one may still want to try to do so.<sup>20</sup> The possible need for trying to do so becomes clear when one considers how strange, even pathological, purely vacuum spacetimes can be, such as Taub-NUT spacetime and some gravitational plane wave spacetimes.<sup>21</sup> Because the Weyl tensor is not directly constrained by the stress-energy tensor of matter, in the sense that it may be nonzero even when  $T_{ab}$  is zero, it is often thought to represent “purely gravitational” degrees of freedom.<sup>22</sup> The Bel–Robinson tensor, moreover, may usefully be thought of as a measure of a kind of “super-energy” associated with purely gravitational phenomena, and directly measures in a precise sense the intensity of gravitational radiation in infinitesimal regions. These two tensors, therefore, would seem perfect candidates to serve as the basis for conditions that would constrain the behavior of purely gravitational phenomena and, more particularly, of vacuum spacetimes. I think it would be of great interest to investigate whether there are natural conditions based on these two tensors that would constrain behavior in vacuum spacetimes so as to rule out such pathologies. I conjecture that there are indeed such conditions.<sup>23</sup> One potentially promising place to start a search for such

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<sup>19</sup>For characterization and discussion of the Bel–Robinson tensor and its properties, see Penrose and Rindler [155], Senovilla [177, 178], Garecki [85] and García-Parrado Gómez-Lobo [84].

<sup>20</sup>There does not exist in general relativity a satisfactory definition for a “gravitational” stress-energy tensor, one that represents localized stress-energy of purely “gravitational” systems. (See Curiel [51].) One may want to think of this as a limitation on the possible physical content of the standard pointilliste energy conditions, as I discuss at the end of §2.1.

<sup>21</sup>See, *e.g.*, Misner [137] and Ellis and Schmidt [71], respectively, and Curiel [45] for further discussion.

<sup>22</sup>Still,  $C^a{}_{bcd}$  and  $T_{ab}$  are not entirely independent of each other. If we define the so-called Lanczos tensor

$$\begin{aligned} J_{abc} &:= \frac{1}{2} \nabla_{[b} R_{a]c} + \frac{1}{6} g_{c[a} \nabla_{b]} R \\ &= 4\pi \nabla_{[b} T_{a]c} - \frac{1}{12} g_{c[b} \nabla_{a]} T \end{aligned} \quad (2.1)$$

then the Bianchi identities may be rewritten

$$\nabla_n C^n{}_{abc} = J_{abc}$$

The similarity of this equation to the sourced Maxwell equation suggests regarding the Bianchi identities as field equations for the Weyl tensor, specifying how at a point it depends on the distribution of matter at nearby points. (This approach is especially useful in the analysis of gravitational radiation; see, for example, Newman and Penrose [142], Newman and Unti [143], and Hawking [99].) Thus, conditions imposed on the Weyl tensor might still be plausibly interpretable as energy conditions in spacetimes with nontrivial  $T_{ab}$ .

<sup>23</sup>It is well known that the Bel–Robinson tensor automatically satisfies the so-called “dominant super-energy condition”, *viz.*,  $T_{mnr{s}} \xi^m \xi^n \xi^r \xi^s \geq 0$ , for all causal vectors  $\xi^a$  in all spacetimes. Because of the complete universality of the condition, however, it cannot rule out pathologies.

conditions might be the Weyl Curvature Hypothesis of Penrose [152], and recent work attempting to formulate expressions for gravitational entropy based on these two tensors.<sup>24</sup>

## 2.2 Impressionist Energy Conditions

Before exhibiting the impressionist energy conditions, a little technical background is in order. If  $\gamma$  is a timelike curve, then it is natural to parameterize the line integral of a quantity along  $\gamma$  by proper time. If  $\gamma$  is a null curve, however, one does not have a natural parameterization of it available. In this case, it is convenient to use a generalized affine parameter.<sup>25</sup> The generalized affine parameter is especially useful in that it does not depend on the tetrad basis chosen in one crucial respect: whether or not the generalized affine parameter of the curve increases without bound.

In order to express the impressionist conditions in effective form, it will be convenient to define direction cosines for causal tangent vectors. Fix a  $1 + 3$  orthonormal frame with respect to which the stress-energy tensor (assumed, recall, for the effective form, to be of Hawking-Ellis type I) is diagonal. Let  $k^\mu$  be the components of the null vector  $k^a$  with respect to the fixed frame. Then define the normalization function  $\nu_n$  and the direction cosines  $\cos \alpha_\mu$  so that  $\cos \alpha_0 = 1$  and  $k^\mu = \nu_n(k^a) \cos \alpha_\mu$ . Let  $\xi^\mu$  be the components of the timelike vector  $\xi^a$  with respect to the fixed frame. Then define the normalization function  $\nu_t$ , the real number  $\beta$ , and the direction cosines  $\cos \alpha_\mu$  so that  $\cos \alpha_0 = 1$ ,  $\xi^0 = \nu_t(\xi^a) \cos \alpha_0$  and  $\xi^{\hat{\mu}} = \nu_t(\xi^a) \beta \cos \alpha_{\hat{\mu}}$ .

Although in principle one could define impressionist energy conditions based on spacetime regions of any dimension or topology, in practice, at least in the classical regime, they have all been defined using curves of various types. In my exposition of them here, I will give what is in effect only a template for the ones actually used to prove theorems, which often qualify the basic template in some way. I will explain or at least mention some of those qualifications in my discussion below in this section, and also in §3. All the impressionist energy conditions based on curves have this in common: the characteristic property that is postulated is required to hold on every curve in some fixed class  $\Gamma$  of curves on spacetime.

### averaged null energy condition (ANEC)

**geometric** for every  $\gamma$  in the fixed class of null curves  $\Gamma$ ,

$$\int_{\gamma} R_{mn} k^m k^n d\theta \geq 0$$

where  $\gamma$  has tangent vector  $k^a$  and  $\theta$  is a generalized affine parameter along  $\gamma$

**physical** for every  $\gamma$  in the fixed class of null curves  $\Gamma$ ,

<sup>24</sup>See, e.g., Cotsakis and Klaoudatou [44] and Clifton [42].

<sup>25</sup>See, e.g., Schmidt [171] for a definition and discussion.

$$\int_{\gamma} T_{mn} k^m k^n d\theta \geq 0$$

where  $\gamma$  has tangent vector  $k^a$  and  $\theta$  is a generalized affine parameter along  $\gamma$   
**effective** for every  $\gamma$  in the fixed class of null curves  $\Gamma$ ,

$$\int_{\gamma} \left( \rho + \sum_{\hat{\mu}} p_{\hat{\mu}} \cos^2 \alpha_{\hat{\mu}} \right) v_n^2(k^a) d\theta \geq 0$$

where  $\gamma$  has tangent vector  $k^a$  and  $\theta$  is a generalized affine parameter along  $\gamma$   
**averaged weak energy condition (AWEC)**

**geometric** for every  $\gamma$  in the fixed class of timelike curves  $\Gamma$ ,

$$\int_{\gamma} G_{mn} \xi^m \xi^n ds \geq 0$$

where  $\gamma$  has tangent vector  $\xi^a$  and  $s$  is proper time  
**physical** for every  $\gamma$  in the fixed class of timelike curves  $\Gamma$ ,

$$\int_{\gamma} T_{mn} \xi^m \xi^n ds \geq 0$$

where  $\gamma$  has tangent vector  $\xi^a$  and  $s$  is proper time  
**effective** for every  $\gamma$  in the fixed class of timelike curves  $\Gamma$ ,

$$\int_{\gamma} \left( \rho + \beta^2 \sum_{\hat{\mu}} p_{\hat{\mu}} \cos^2 \alpha_{\hat{\mu}} \right) v_t^2(\xi^a) ds \geq 0$$

where  $\gamma$  has tangent vector  $\xi^a$  and  $s$  is proper time  
**averaged strong energy condition (ASEC)**

**geometric** for every  $\gamma$  in the fixed class of timelike curves  $\Gamma$ ,

$$\int_{\gamma} R_{mn} \xi^m \xi^n ds \geq 0$$

where  $\gamma$  has tangent vector  $\xi^a$  and  $s$  is proper time  
**physical** for every  $\gamma$  in the fixed class of timelike curves  $\Gamma$ ,

$$\int_{\gamma} \left( T_{mn} - \frac{1}{2} T g_{mn} \right) \xi^m \xi^n ds \geq 0$$



where  $\gamma$  has tangent vector  $\xi^a$  and  $s$  is proper time  
**effective** for every  $\gamma$  in the fixed class of timelike curves  $\Gamma$ ,

$$\int_{\gamma} \left\{ \left( \rho + \beta^2 \sum_{\hat{\mu}} p_{\hat{\mu}} \cos^2 \alpha_{\hat{\mu}} \right) v_r^2(\xi^a) - \frac{1}{2} \xi^n \xi_n \left( \rho - \sum_{\hat{\mu}} p_{\hat{\mu}} \right) \right\} ds \geq 0$$

where  $\gamma$  has tangent vector  $\xi^a$  and  $s$  is proper time

Before discussing their respective interpretations, a few remarks are in order. No reasonable impressionist analogue of either of the pointilliste dominant conditions are known.<sup>26</sup> In practice, one generally requires that  $\Gamma$  consist of a suitably large family of inextendible geodesics of the appropriate type. For the ANEC, if  $\Gamma$  consists of null geodesics, then one can replace the generalized affine parameter with the ordinary affine parameter. In no case can one allow arbitrary parameterizations for null curves in the defining integral, as that would simply reduce the ANEC to the NEC. If one further requires for the ANEC that the curves in  $\Gamma$  be achronal, then the condition is often called the ‘averaged achronal null energy condition’ (AANEC). For the AWEC, if  $\Gamma$  contains enough timelike geodesics and the spacetime is well behaved, then there may be null geodesics that are limit curves of subfamilies of  $\Gamma$ ; in this case, the relevant characteristic integral will be nonnegative for those null geodesics, and the AWEC with the fixed  $\Gamma$  can be said to imply the ANEC for the family of limiting null geodesics. Even in well-behaved spacetimes, however, there may be null geodesics that are not the limit of any family of timelike geodesics, so in general the AWEC does not imply the ANEC. The ASEC does not imply either the AWEC or the ANEC. Clearly, the NEC, WEC and SEC respectively imply the ANEC, AWEC, and ASEC.

I am sorry to say the discussion of the possible interpretations of, or even just motivations for, the standard impressionist energy conditions is a simple one to have: there are no compelling geometrical, physical or effective interpretations of these conditions, not even hand-waving, rough or approximate ones, and no compelling physical or philosophical motivations for them.

I should perhaps clarify what I mean in claiming that there are no compelling interpretations or motivations of these conditions. One can certainly describe in simple, clear, physical language the sorts of spacetimes in which they will be satisfied—geodesics experience more positive than negative energy, the regions in which the pointilliste conditions are violated are bounded in various ways, *etc.*—but it is difficult, at best, to understand these classes of spacetimes as being related in any but accidental ways. There is nothing principled or lawlike that makes these spacetimes similar or the same in any deep sense. It is not easy to imagine principled conditions one could impose on theories of matter or fields—say, a form for the Lagrangian, or manifestation of a symmetry, *etc.*—that would ensure the sort of behavior captured

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<sup>26</sup>One could in flat spacetimes, and possibly in stationary spacetimes, circumvent the obvious problems with formulating a dominant-like impressionist energy condition, but, being confined to flat (and possibly stationary) spacetimes, such a condition would have little import or relevance.

by the averaged conditions. This somewhat vague qualm is substantiated by the ease with which violations of the averaged conditions can be found, in both the classical and the quantum cases, as I discuss in §3.2.

More to the point, there is at least one interesting way of making this vague qualm more precise, that at the same time shows clearly the artificiality of the impressionist conditions as compared to the pointilliste conditions: none of the quantities constrained by the impressionist conditions enter the equations of motion or the field equations of any known kinds of physical system, and, correlatively, no couplings between any known kinds of physical system are mediated by those quantities; the opposite is true for the pointilliste conditions, whose constrained quantities promiscuously appear in equations of motion, field equations and couplings for many if not most known kinds of physical system. Finally, the restriction to geodesics has no compelling physical or philosophical basis that I can see, but appears to be dictated by pragmatic considerations about the technical tractability of required calculations.

Still, there is more to say about them, even though none has a clear, principled interpretation or motivation. These conditions were all constructed by reverse engineering—an investigator looked for the weakest condition she could impose on the averaged behavior of some quantity depending on curvature or stress-energy in order to derive the result of interest to her. (Indeed, I think it is not going too far to say that many of them represent a case of outright gerrymandering by the relativity community.<sup>27</sup>) Other researchers were impressed by the weakness of the condition used to derive the important result, and so picked it up and used it themselves. And so the impressionist conditions have been passed down through the generations of relativists, hand to hand from teacher to student, powerful, talismanic runes to be brought out and invoked with precise ceremony on formal occasions, but whose inner significance is beyond our ken, though their very familiarity often obscures that fact.<sup>28</sup>

This is not to say the impressionist energy conditions have no foundational or physical interest at all. It is often important to find the weakest conditions one can to prove theorems whose conclusions have great weight or significance, such as the positive energy theorems or the singularity theorems, if only, for example, to get as clear as one can on what those conclusions really depend on. If one wants to try to extend or modify one's global theory while ensuring that certain results remain true, for example, it behooves one to find the weakest conditions from which one

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<sup>27</sup>The only physicists I know of to express similar concerns are Visser and Barceló [6, 199]; indeed they seem to be of the opinion that it is difficult to think of all energy conditions, not just the impressionist ones, as anything more than pragmatically convenient tools whose formulation is driven by the technical needs in proving desired theorems.

<sup>28</sup> und Das und Den,  
 die man schon nicht mehr sah  
 (so täglich waren sie und so gewöhnlich),  
 auf einmal anzuschauen: sanft, versöhnlich  
 und wie an einem Anfang und von nah  
 — Rainer Maria Rilke, “Der Auszug des verlorenen Sohnes”

can derive those results. For we who are interested in the foundations of the theory in and of itself, however, these impressionist conditions have little to offer. Still, because they have been used to prove deep results of great interest in themselves, it is important to understand what sorts of system violate and what sorts satisfy the conditions (which I will discuss in §3).

Before moving on, it will be edifying to examine in a little detail two of the most important technical qualifications made to the templates I gave of the averaged conditions. Tipler [189], which if my history is not mistaken was the first use of an averaged condition to prove results of any depth, required the additional constraint that the characteristic integral of the averaged condition at issue can equal zero for any curve only if its integrand (*e.g.*,  $T_{mn}\xi^m\xi^n$  for the physical AWEC) equals zero along the entire curve. As Borde [21] points out, this constraint raises problems for the physical plausibility, or at least possible scope, of the conditions.<sup>29</sup> To see the problem, let us for the sake of definiteness focus attention for the moment on the physical AWEC. Then Tipler's constraint rules out cases where the integral equals zero because the relevant curve passes endlessly in and out of regions of positive and negative energy density. This may not sound so bad at first, until one realizes it means that, for a spacetime to satisfy the constrained condition, every curve in the fixed class must eventually traverse only regions of nonnegative energy density, both to the past and the future: violations of the WEC are to be allowed only in bounded regions in the interior of spacetime, so to speak. There seems even less physical justification for demanding this than for the bare AWEC in the first place.

To try to address this problem, Borde proposed modifications to the averaged conditions. The technical details of his proposals, while ingenious, are not worth working through for my purposes, as they are complicated and shed little light on the issues I am discussing. The gist of his proposed modifications is this: rather than requiring that the salient integral equal zero only when its integrand equal zero everywhere along the curve, we require only that, if the integral equal zero, then the integrand must be suitably periodic along the entire curve, *i.e.*, roughly speaking, that the integrand visit a neighborhood of zero frequently and that the lengths of the intervals it spends visiting those neighborhoods not approach zero as one heads along the curve in either direction. This allows application of the averaged condition to situations in which the total integral may essentially be zero even though there are large and long violations of the relevant pointilliste condition, such as may occur for the SEC during inflationary periods of a spacetime. In this sense, Borde's modifications do seem an improvement on Tipler's original version. One cannot help but feel though, given the intricacy and physical opacity of the mathematical machinery required to formulate Borde's condition, that the problems of physical interpretation in the sense I sketched above—not having in hand a principled justification for the condition founded on general, fundamental principles, but rather only reverse engineering the weakest suitable condition one can manage to prove the results one

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<sup>29</sup>Chicone and Ehrlich [38] also pointed out that there were lacunæ in Tipler's proofs, unrelated to Borde's problems, but that is by the by for our purposes, as they also showed how to fix the problems.

wants for the particular class of spacetimes one is interested in—become perhaps even more severe than before.

### 2.3 Appendix: A Failed Attempt to Derive the NEC and SEC

It is sometimes claimed (*e.g.*, Liu and Rebouças [130]) that one can derive the NEC and the SEC from the Raychaudhuri equation. Even though I think the argument fails, it is of interest to try to pinpoint exactly why it fails, as it sheds light on why it appears to be difficult to derive the energy conditions from other fundamental principles (the difficulty strongly suggested by the lack of convincing derivations). I will sketch the argument only for the SEC, as that for the NEC is essentially the same, with only a few inessential technical differences.

Raychaudhuri’s equation expresses the rate of change of the scalar expansion of a congruence of geodesics, as one sweeps along the congruence, as a function of the expansion itself, of the congruence’s shear and twist tensors, and of the Ricci tensor. For a congruence of timelike geodesics with tangent vector  $\xi^a$ , it takes the form

$$\xi^n \nabla_n \theta = -\frac{1}{3}\theta^2 - \sigma_{mn}\sigma^{mn} + \omega_{mn}\omega^{mn} - R_{mn}\xi^m\xi^n \quad (2.2)$$

where  $\theta$  is the expansion of the congruence,  $\sigma_{ab}$  its shear and  $\omega_{ab}$  its twist.<sup>30</sup> If the total sum on the right-hand side is negative, then the expansion of the congruence is decreasing with proper time, *i.e.*, the geodesics in the congruence are everywhere converging on each other. The first term on the right-hand side is manifestly negative, as is the second, since  $\sigma_{ab}$  is spacelike in both indices, and so  $\sigma_{mn}\sigma^{mn} \geq 0$ . For a hypersurface orthogonal congruence, it follows directly from Frobenius’s Theorem that  $\omega_{ab} = 0$ . Thus, if we assume that “gravity is everywhere attractive”, and we interpret this to mean that congruences of timelike geodesics which have vanishing twist should always converge, then, in order to ensure that the total right-hand side of equation (2.2) is always negative, we require that  $R_{mn}\xi^m\xi^n \geq 0$ , which is just the geometrical form of the SEC.

It should be clear why I fail to find the argument compelling. In fact, all one can conclude from the demand that the righthand side of equation (2.2) be nonpositive (when  $\omega_{ab} = 0$ ) is that

$$R_{mn}\xi^m\xi^n \geq -\frac{1}{3}\theta^2 - \sigma_{mn}\sigma^{mn} \quad (2.3)$$

everywhere. Of course, this is not the SEC, but only a weaker form of the geometric formulation, one that sets a nonconstant lower bound on “how negative” mass–energy

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<sup>30</sup>See, *e.g.*, Wald [203, ch. 9, §2] for a derivation and explanation of the Raychaudhuri equation for both timelike and null congruences. There is a generalization of the Raychaudhuri equation that treats congruences of accelerated curves, but nothing would be gained for our purposes by discussing it.

and momentum-energy flux can get (invoking the physical form of the condition).<sup>31</sup> When one considers that one can, in every spacetime, find at every point a congruence of timelike geodesics that has divergent expansion as one approaches that point, one realizes that the inequality (2.3) is vacuous, for the right-hand side of the inequality can be made as negative as one likes. (Proof: in any spacetime, at any point  $p$ , consider the family of timelike geodesics defined by the family of unit, past-directed, timelike vectors at  $p$ , parametrized by proper time so that each geodesic's parameter has the value 0 at  $p$ ; there will be some real number  $\epsilon$  such that the class of geodesic-segments defined by considering all geodesics in the family for proper time values in the open interval  $(-\epsilon, 0)$  defines a proper congruence; that congruence will have divergent expansion along all its members as one approaches proper time 0, *i.e.*, the point  $p$ , as can be seen by the fact that any spacelike volume swept along the flow of the congruence toward  $p$  will converge to 0.)

The heart of the problem should now be clear. Geodesic congruences are a dime a dozen. You can't throw a rock in a relativistic spacetime without hitting a zillion of them, most of them having no intrinsic physical significance. Because the pointilliste energy conditions, moreover, constrain the behavior of curvature terms only at individual points, and that by reference to all timelike or null (or both) vectors at those points, one can always find geodesic congruences that are as badly behaved as one wants, in just about any way one wants to make that idea precise, with respect to how various measures of curvature evolve along the congruences. Nonetheless, geodesic congruences seem to be about the only structure one has naturally available to work with, if one wants to try to constrain the behavior of curvature as measured by the contractions of curvature tensors with causal vectors. So long as one wants to work with geodesic congruences, therefore, it seems one must find some way to restrict the class one allows as relevant to those that are "physically significant" in some important and clear way. I know of no way to try to address that problem in any generality. Of course, one could always try to work with structures other than geodesic congruences, but, again, I know of no other natural candidates to try to use to constrain the behavior of measures of curvature, given the typical form of the energy conditions.

Even if one could find natural, compelling ways to restrict attention to a privileged class of congruences in such a way as to resolve the technical problems I raised for this kind of argument, there would still be interpretative problems with this kind of argument. As I discussed at the end of §2.1 above, I do not find it convincing to interpret the fact that causal congruences are convergent as a representation of the idea that "gravity is attractive". Without that interpretation, however, one has little motivation for invoking Raychaudhuri's equation in the first place without ancillary physical justification.

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<sup>31</sup>Because the lower bound is variable, the propositions of Tipler [189] I discussed in §2.1 do not allow one to infer that this weaker condition is in fact equivalent to the WEC.

## 2.4 Appendix: Very Recent Work

Recently, Abreu et al. [1] introduced a new classical energy condition

### flux energy condition (FEC)

#### geometric

1. for any timelike vector  $\xi^a$ ,  $G^a_n \xi^n$  is causal
2. for any timelike vector  $\xi^a$ ,  $G^m_r G_{ms} \xi^r \xi^s \geq 0$

#### physical

1. for any timelike vector  $\xi^a$ ,  $T^a_n \xi^n$  is causal
2. for any timelike vector  $\xi^a$ ,  $T^m_r T_{ms} \xi^r \xi^s \geq 0$

#### effective for each $\hat{\mu}$ , $\rho^2 \geq p_{\hat{\mu}}^2$

There is, as is to be expected, no simple interpretation of its geometric form. The simplest interpretation of its physical form is that the total flux of energy–momentum as measured by any timelike observer is always causal, albeit the temporal direction of the flux is not restricted. Because isotropic tachyonic gases always satisfy  $\rho < \frac{1}{3}p$ , with weaker bounds for anisotropic tachyonic material, the effective form may be interpreted as ruling out the possibility of tachyonic matter. Otherwise, I know of no compelling interpretation of it, as it allows energy density to be unboundedly negative, so long as the absolute value of pressure is not too great.

Abreu et al. [1] argue that the FEC gives better support to the claim that the cosmological equation-of-state parameter  $w$  (the so-called barotropic index—see footnote 15) must be  $\leq 1$ , and so better substantiates arguments in favor of entropy bounds they give based on that assumption. Martín-Moruno and Visser [134, 135] investigated its properties and proposed a quantum analogue of it, which, they claim, works in several respects better than the standard quantum energy conditions.<sup>32</sup> The FEC, therefore, shows *prima facie* promise as being of real physical interest. It is, moreover, manifestly weaker than all the other standard energy conditions, as its characteristic nonlinearity (most easily seen in the second given articulations of its geometric and physical forms, and in its effective form) ensures that essentially no limit is placed on the possible negativity of the ordinary mass–energy of matter. If, therefore, it bears out its promise for leading to, or at least supporting, results of interest, it would be a great improvement on the standard energy conditions. Because, however, its properties and consequences are virtually unknown as compared to the standard conditions, I shall not discuss it further.

Even more recently, Martín-Moruno and Visser [135] proposed two more energy conditions, the determinant energy condition (DETEC) and the trace-of-square energy condition (TOSEC), and also proposed quantum analogues for them. Again, these energy conditions seem *prima facie* interesting, but even less work has been done on and with them than the FEC, so I shall not discuss them here either.

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<sup>32</sup>See Curiel [48] for extended discussion of energy conditions in quantum field theory on curved spacetime.

## 2.5 Technical Appendix: Average Radial Acceleration

To characterize the idea of the average radial acceleration of a causal geodesic,<sup>33</sup> let  $\xi^a$  be a future-directed causal vector field whose integral curves  $\gamma$  are affinely parametrized geodesics. If  $\gamma$  is timelike, then assume  $\xi^a$  to be unit. Let  $\lambda^a$  be a vector field on  $\gamma$  such that at one point  $\lambda^n \xi_n = 0$  and  $\xi_\xi \lambda^a = 0$ . (Note that if  $\xi^a$  is null, then  $\lambda^a$  may be proportional to  $\xi^a$ ; otherwise it must be spacelike.) Then automatically  $\lambda^n \xi_n = 0$  at all points of  $\gamma$ .  $\lambda^a$  is usefully thought of as a “connecting field” that joins the image of  $\gamma$  to the image of another, “infinitesimally close” integral curve of  $\xi^a$ . Then  $\xi^m \nabla_m (\xi^n \nabla_n \lambda^a)$  represents the acceleration of that neighboring geodesic relative to  $\gamma$ . According to the equation of geodesic deviation,

$$\xi^m \nabla_m (\xi^n \nabla_n \lambda^a) = R^a{}_{mnr} \xi^m \lambda^n \xi^r$$

Now, fix an orthonormal triad-field  $\{\lambda^a\}_{\mu \in \{1, 2, 3\}}$  along  $\gamma$  such that each  $\lambda^a$  forms a connecting (relative acceleration) field along  $\gamma$ . The magnitude of the radial component of the relative acceleration in the  $\mu^{\text{th}}$  direction then is  $-\lambda_r \xi^m \nabla_m (\xi^n \nabla_n \lambda^r)$ . Fix a point  $p \in \gamma$ . The *average radial acceleration*  $A_r$  of  $\gamma$  at  $p$  is defined to be

$$A_r := -\frac{1}{k} \sum_{\mu}^{\mu} \lambda_r \xi^m \nabla_m (\xi^n \nabla_n \lambda^r)$$

where  $k$  is 3 if  $\xi^a$  is timelike and 2 if null. It is straightforward to verify that the average radial acceleration is independent of the choice of orthonormal triad, so it encodes a quantity of intrinsic geometric (and physical) significance accruing to  $\xi^a$ . A simple calculation using the equation of geodesic deviation then shows that

$$A_r = -\frac{1}{k} R_{mn} \xi^m \xi^n \tag{2.4}$$

If the Einstein field equation is assumed to hold, it follows that

$$A_r = -\frac{8\pi}{k} (T_{mn} - \frac{1}{2} T g_{mn}) \xi^m \xi^n \tag{2.5}$$

which reduces in the case of null vectors to

$$A_r = -4\pi T_{mn} \xi^m \xi^n \tag{2.6}$$

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<sup>33</sup>I follow the exposition of Malament [133, §2.7], with a few emendations.

### 3 Consequences and Violations

To study the role of energy conditions in spacetime theories, I will look at results that do not depend on the imposition of any field equations (*e.g.*, the Einstein field equation) and yet directly constrain spacetime geometry. One often hears the claim that such-and-such result (*e.g.*, various singularity theorems, various versions of the geodesic postulate, the Zeroth Law of black hole mechanics, *etc.*) that assumes an energy condition does require the Einstein field equation for its proof, but one must be careful of such claims. It is almost always the case, in fact, that the Einstein field equation is logically independent of the result (in the strong sense that one can assume the negation of the Einstein field equation and still derive the result); the Einstein field equation is used in such cases *only* to provide a physical interpretation of the assumed energy condition; mathematically, one in general needs only the geometric form of the condition, which is why I distinguish the geometric from the physical form.<sup>34</sup> In this section, every consequence of the energy conditions I discuss is of this type: it is logically independent of the Einstein field equation, and relies on the Einstein field equation only for the physical interpretation of the assumed geometric energy condition.<sup>35</sup> Many of the violations of the energy conditions I list here, however, do rely on assuming the Einstein field equation for their derivation, in so far as they use the Lagrangian formulation of the relevant forms of matter to derive the violation, or in so far as they rely on the effective form of the energy conditions in conjunction with, *e.g.*, the Friedmann equations to derive the violation.

I will begin with a list of the consequences of the energy conditions, *i.e.*, the results each energy condition is used to derive, and then discuss the roles the conditions play in the derivations of those results. I then list the classical cases in which each energy condition is known to fail, then discuss how the known failures may or may not undermine our confidence in the consequences.<sup>36</sup> In several of the references I give in the list of consequences, no explicit mention is made of energy conditions, but, if one works through their arguments, one will see that the relevant energy condition is indeed being implicitly assumed. In other works I cite, an energy condition is explicitly assumed, but in fact, according to the arguments of those works, either a weaker one is sufficient or a stronger one is required; in such cases, I cite the result under the sufficient or required condition. For almost none of the statements in the list

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<sup>34</sup>There is perhaps room for debate over this claim, at least in a few cases. Some elements of the black hole uniqueness theorems, *e.g.*, “use” the Einstein field equation to show that certain distinguished spacelike hypersurfaces must be spatially conformally flat when the entire spacetime is assumed to be vacuum; in such a case, the spatial conformal flatness follows from the vanishing of the Ricci tensor, which follows from the vanishing of the stress-energy tensor by the Einstein field equation. I would still argue in such cases that the Einstein field equation is not necessary for the proof of the theorem—only  $R_{ab} = 0$  is—and, again, the Einstein field equation is used only to provide the necessary condition a physical interpretation.

<sup>35</sup>In cosmology, several of the most interesting results do require assumption of the Einstein field equation. For this reason, and also because it is such a large and rich field on its own, I explore the role and character of energy conditions in the context of cosmology at some length in Curiel [47].

<sup>36</sup>See Curiel [48] for examination of the cases of failure in the quantum regime.



of consequences is it the case that the energy condition alone is necessary or sufficient; it is rather that the energy condition is one assumption among others in the only known way (or ways) to prove the result. When I list the same proposition as a consequence of more than one energy condition (*e.g.*, “prohibition on spatial topology change” under both WEC and ANEC), it means that there are different proofs of the statement using different ancillary assumptions. When I qualify a spacetime as “spatially open” or “spatially closed”, it should be understood that the spacetime is globally hyperbolic and the openness or closedness refers to the topology of spacelike Cauchy surfaces in a natural slicing of the spacetime.

### 3.1 Consequences

#### NEC

1. formation of singularities after gravitational collapse in spatially open spacetimes [148]
2. formation of singularities in asymptotically flat spacetimes with non-simply connected Cauchy surface [82, 128]
3. formation of an event horizon after gravitational collapse [148–150]
4. trapped and marginally trapped surfaces and apparent horizons must be inside asymptotically flat black holes [203]
5. Hawking’s Area Theorem for asymptotically flat black holes (Second Law of black hole mechanics) [103]
6. the area of a generalized black hole always increases<sup>37</sup> (Second Law of generalized black hole mechanics) [111]
7. asymptotically predictable black holes cannot bifurcate<sup>38</sup> [203]
8. the domain of outer communication of a stationary, asymptotically flat, causally well behaved spacetime is simply connected<sup>39</sup> [40, 79, 81]
9. a stationary, asymptotically flat black hole has topology  $S^2$ , if the domain of outer communication is globally hyperbolic and the closure of the black hole is compact<sup>40</sup> [40, 78]

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<sup>37</sup>Hayward [111] defines a generalized notion of black hole, one applicable to spacetimes that are not asymptotically flat, by the use of what he calls “trapping horizons”. In the same paper, he shows that generalized black holes obey laws analogous to the standard Laws of black hole mechanics.

<sup>38</sup>A spacetime is asymptotically predictable if it is asymptotically flat, and there is a partial Cauchy surface whose boundary is the event horizon, such that future null infinity is contained in its future domain of dependence.

<sup>39</sup>The domain of outer communication of an asymptotically flat spacetime is, roughly speaking, the exterior of the black hole region. See Chruściel et al. [41, §2.4] for a precise definition. This theorem is similar to, but stronger than, the original Topological Censorship Theorem of Friedman et al. [77]; see footnote 59. The theorem due to Galloway and Woolgar [81] in fact requires only the ANEC.

<sup>40</sup>This is also a constituent of the proof of the full No-Hair Theorem, but is important enough a result to warrant its own entry in the list; see footnote 41.

10. almost all the constituents of the black hole No Hair Theorem for asymptotically flat black holes<sup>41</sup> [12, 31, 119, 120, 136, 141, 164, 165, 185, 186, 201, 202]
11. generalized black holes are regions of “no escape” [110]
12. limits on energy extraction by gravitational radiation from colliding asymptotically flat black holes [103]
13. positivity of ADM mass<sup>42</sup> [4, 153]
14. the Generalized Second Law of Thermodynamics<sup>43</sup> [73]
15. Bousso’s covariant universal entropy bound<sup>44</sup> [73]
16. the Shapiro “time-delay” is always a delay, never an advance<sup>45</sup> [200]
17. chronology implies causality<sup>46</sup> [107]
18. standard formulations of the classical Chronology Protection Conjecture<sup>47</sup> [105]

## WEC

1. asymptotically flat spacetimes without naked singularities are asymptotically predictable [104]
2. asymptotically flat black holes cannot bifurcate [104]
3. the event horizon of a stationary black hole is a Killing horizon<sup>48</sup> [104, 107]

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<sup>41</sup>The No Hair Theorem states that an asymptotically flat, stationary black hole is completely characterized by three parameters, *viz.*, its mass, angular momentum and electric charge. The proof of this theorem logically comprises many steps, each of interest in its own right, and historically stretched from the original papers of Israel [119, 120] to the final results of Mazur [136]. There are too many constituents of the proof to list each individually. A few remaining constituents require the DEC; see that list for details. Heusler [113] provides an excellent, relatively up-to-date overview of all the known results. There is an analogous No Hair Theorems for the generalized black holes of Hayward [111], but I will not discuss them.

<sup>42</sup>Earlier proofs relied on the DEC; see that list for details.

<sup>43</sup>The total entropy of the world, *i.e.*, the entropy of ordinary matter plus the entropy of a black hole as measured by its surface area, never decreases.

<sup>44</sup>Bousso [24, 25], clarifying and improving on earlier work by Bekenstein [13, 15–17], ‘t Hooft [184], Smolin [182, 183], Corley and Jacobson [43], and Fischler and Susskind [72], conjectured that in any spacetime satisfying the DEC the total entropy flux  $S_L$  through any null hypersurface  $L$  satisfying some natural geometrical conditions must be such that  $S_L \leq A/4$ , where  $A$  is a spatial area canonically associated with  $L$ . Flanagan et al. [73] managed to prove the bound using the weaker NEC.

<sup>45</sup>One can understand this result physically as a prohibition on a certain form of “hyper-fast” travel or communication. Roughly speaking, this is travel in spacetime in which the traveler is measured by external observers, in a natural way, to travel faster than the speed of light, even though the traveler’s worldline is everywhere timelike. It is closely related, though not equivalent, to the idea of traversable wormholes.

<sup>46</sup>Chronology holds if there are no closed timelike curves; causality holds if there are no closed causal curves.

<sup>47</sup>This states, roughly, that the formation of closed timelike curves always requires either the presence of singularities or else pathological behavior “at infinity”.

<sup>48</sup>A Killing horizon is a null hypersurface generated by the orbits of a non-degenerate null Killing field.

4. Third Law of black hole mechanics<sup>49</sup> [121]
5. limits on energy extraction by gravitational radiation from asymptotically flat colliding black holes [104]
6. formation of singularities after gravitational collapse in spatially open spacetimes [89, 189]
7. cosmological singularities in spatially open or flat spacetimes [89, 96]
8. cosmological singularities in globally hyperbolic spacetimes that are non-compactly regular near infinity<sup>50</sup> [83]
9. prohibition on spatial topology change [87, 187]
10. geodesic theorems for “point-particles” [64, 70]
11. mass limits for stability of hydrostatic spheres against gravitational collapse [19]
12. some standard forms of the Cosmic Censorship Hypothesis [123]

## SEC

1. cosmological singularities in spatially closed spacetimes [86, 89, 98, 101, 106, 109]
2. cosmological singularities in spatially open spacetimes [89, 97, 100, 106, 109]
3. cosmological singularities in spacetimes with partial Cauchy surfaces [89, 97, 100, 101, 109]
4. formation of singularities after gravitational collapse in spatially closed spacetimes [89, 101, 109]
5. formation of singularities after gravitational collapse in spatially open spacetimes [89, 109]
6. Lorentzian splitting theorem<sup>51</sup> [80, 214]
7. a given globally hyperbolic extension of a spacetime is the maximal such extension [163]
8. existence and uniqueness of constant-mean-curvature foliations for spacetimes with compact Cauchy surfaces [10, 26, 161, 162]

## DEC

1. formation of a closed trapped surface after gravitational collapse of arbitrary (*i.e.*, not necessarily close to spherical) matter distribution [174]

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<sup>49</sup>No physical process can reduce the surface gravity of an asymptotically flat black hole to zero in a finite amount of time.

<sup>50</sup>Roughly speaking, a globally hyperbolic spacetime is noncompactly regular near infinity if it has a (partial) Cauchy surface that is the union of well behaved nested sets, each having compact boundary, that are themselves noncompact near infinity.

<sup>51</sup>I will give two versions of the theorem; see Galloway and Horta [80] for proofs of both. In order to state the first version of the theorem, define a *timelike line* to be an inextendible timelike geodesic that realizes the supremal Lorentzian distance between every two of its points [68]. Then the theorem, as first conjectured by Yau [214], is as follows: let  $(\mathcal{M}, g_{ab})$  be a timelike geodesically complete spacetime satisfying the SEC; if it contains a timelike line, then it is isometric

2. a stationary, asymptotically flat black hole is topologically  $\mathbb{S}^{252}$  [104]
3. a generalized black hole is topologically  $\mathbb{S}^{253}$  [111]
4. constituents of the black hole No Hair Theorems for asymptotically flat black holes<sup>54</sup> [12, 32, 107]
5. Zeroth Law of black hole mechanics<sup>55</sup> [7]
6. Zeroth Law of generalized black hole mechanics<sup>56</sup> [111]
7. every past timelike geodesic in spatially open, nonrotating spacetimes with nonzero spatially averaged energy densities is incomplete<sup>57</sup> [179, 180]
8. positivity of ADM energy [172, 211]
9. positivity of Bondi energy [112, 115, 131, 173]
10. asymptotic energy-area inequality in the spherically symmetric case<sup>58</sup> [112]
11. if a covariantly divergence-free  $T_{ab}$  vanishes on a closed, achronal set, it vanishes in the domain of dependence of that set [102, 107]

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(Footnote 51 continued)

to  $(\mathbb{R} \times \Sigma, t_a t_b - h_{ab})$ , where  $(\Sigma, h_{ab})$  is a complete Riemannian manifold and  $t^a$  is a timelike vector field in  $\mathcal{M}$ . (In particular,  $(\mathcal{M}, g_{ab})$  must be globally hyperbolic and static.)

In order to state the second, we need two more definitions. First, the *edge* of an achronal, closed set  $\Sigma$  is the set of points  $p \in \Sigma$  such that every open neighborhood of  $p$  contains a point  $q \in I^-(p)$ , a point  $r \in I^+(p)$  and a timelike curve from  $q$  to  $r$  that does not intersect  $\Sigma$ . Second, let  $\Sigma$  be a nonempty subset of spacetime; then a future inextendible causal curve is a *future  $\Sigma$ -ray* if it realizes the supremal Lorentzian distance between  $\Sigma$  and any of its points lying to the future of  $\Sigma$  [80]; *mutatis mutandis* for a *past  $\Sigma$ -ray*. (If  $\gamma$  is a  $\Sigma$ -ray, it necessarily intersects  $\Sigma$ .) The second version of the theorem is as follows: let  $(\mathcal{M}, g_{ab})$  be a spacetime that contains a compact, acausal spacelike hypersurface  $\Sigma$  without edge and obeys the SEC; if it is timelike geodesically complete and contains a future  $\Sigma$ -ray  $\gamma$  and a past  $\Sigma$ -ray  $\eta$  such that  $I^-(\gamma) \cap I^+(\eta) \neq \emptyset$ , then it is isometric to  $(\mathbb{R} \times \Sigma, t_a t_b - h_{ab})$ , where  $(\Sigma, h_{ab})$  is a compact Riemannian manifold and  $t^a$  is a timelike vector field in  $\mathcal{M}$ . (In particular,  $(\mathcal{M}, g_{ab})$  must be globally hyperbolic and static.)

I discuss the physical meaning of the splitting theorems below.

<sup>52</sup>This is also a constituent of the proof of the full No Hair theorem, but is important enough a result to warrant its own entry in the list; see footnote 41. Hawking's original proof was not rigorous; in particular, it did not completely rule out a toroidal topology. See Gannon [83] for a rigorous proof of the theorem in electrovac spacetimes, and Galloway [78] and Chruściel and Wald [40] for a rigorous proof using the NEC for otherwise arbitrary stress-energy tensors but more stringent constraints on the global topology of the spacetime.

<sup>53</sup>See footnote 37.

<sup>54</sup>See footnote 41.

<sup>55</sup>The surface gravity is constant on the event horizon of a stationary, asymptotically flat black hole.

<sup>56</sup>The total trapping gravity of a generalized black hole is bounded from above, and achieves its maximal value if and only if the trapping gravity is constant on the trapping horizon, which happens when the horizon is stationary. (See footnote 37.)

<sup>57</sup>This theorem is particularly strong: it implies that any singularity-free spacetime satisfying the other conditions must have everywhere vanishing averaged spatial energies, making them highly non-generic.

<sup>58</sup>This inequality, first conjectured by Penrose [151], states that if a spacelike hypersurface in a spherically symmetric, asymptotically flat spacetime contains an outermost marginally trapped sphere of radius  $R$  (in coordinates respecting the spherical symmetry), then the ADM energy  $\geq \frac{1}{2}R$ . The DEC need hold only on the spacelike hypersurface, not in the whole spacetime.

12. standard statements of the initial value formulation of the Einstein field equation with nontrivial  $T_{ab}$  is well posed (in the sense of Hadamard) [107, 203]
13. natural definition of the center of mass, multipole moments and equations of motion for an extended body [55–58, 65, 67, 169, 170]
14. some standard forms of the Cosmic Censorship Hypothesis [92, 123, 152, 203]

### SDEC

1. geodesic theorem for “arbitrarily small” bodies, neglecting self-gravitational effects [93, 133, 208]
2. geodesic theorem for “arbitrarily small” bodies, including self-gravitational effects [66]

### ANEC

1. a stationary, asymptotically flat black hole is topologically  $\mathbb{S}^2$  [122]
2. focusing theorems for congruences of causal geodesics [21]
3. formation of singularities after gravitational collapse in spatially open spacetimes [166, 176]
4. Topological Censorship Theorem<sup>59</sup> [77]
5. prohibition on traversable wormholes [140]
6. prohibition on spatial topology change [22]
7. positivity of ADM energy [156]

### AWEC $\emptyset$

### ASEC

1. cosmological singularities in spatially closed spacetimes<sup>60</sup> [176, 189]
2. cosmological singularities in spatially open spacetimes<sup>61</sup> [176, 189]

There is a striking absentee from the list of consequences: strictly speaking, the First Law of black hole mechanics (for asymptotically flat black holes)—conservation of mass–energy—does not require for its validity the assumption of

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<sup>59</sup>The theorem states: fix an asymptotically flat, globally hyperbolic spacetime satisfying the ANEC; let  $\gamma$  be a causal curve with endpoints on past and future null infinity that lies in a simply connected neighborhood of null infinity; then every causal curve with endpoints on past and future null infinity is smoothly deformable to  $\gamma$ . Roughly speaking, this theorem says that no observer remaining outside a black hole can ever have enough time to probe the spatial topology of spacetime: isolated, nontrivial topological structure with positive energy will collapse into black holes too quickly for light to cross it. Loosely speaking, the region outside black holes is topologically trivial.

<sup>60</sup>Strictly speaking, Tipler’s proof requires the ASEC with the additional constraint that its characteristic integral can equal 0 for any geodesic only if its integrand ( $R_{mn}\xi^m\xi^n$ ) equals 0 along the entire geodesic. Senovilla’s proof does not require these extra assumptions, though it does require the existence of a Cauchy surface with vanishing second fundamental form.

<sup>61</sup>Strictly speaking, Tipler’s proof of this theorem requires the WEC as well as the ASEC, and also requires the same further constraint on the ASEC as described in footnote 60. Senovilla’s proof also requires what is described in footnote 60.

any energy condition (unlike the other three Laws).<sup>62</sup> The issue is somewhat delicate in the details, however. The delicacy arises from the fact that all the most rigorous and the most physically compelling derivations of the Law I know [7, 205] assume that the surface gravity of the black hole is constant on the event horizon. This, of course, is the Zeroth Law of black hole mechanics, and all known proofs of the most general form of the Zeroth Law rely on the DEC. The qualification “most general” is required because there are weaker forms of the Zeroth Law that require no energy condition for their proof: any sufficiently regular Killing horizon must be bifurcate, and the appropriate generalization of surface gravity for a bifurcate Killing horizon must be constant on the entire horizon, without the need to impose any energy condition [113, 125, 159, 160, 204].<sup>63</sup> This is a weaker form of the Zeroth Law, in so far as it is not known whether the event horizons of all “physically reasonable” black holes are sufficiently regular in any of the senses required, though in fact the event horizons of all known exact black hole solutions are, and the condition of sufficient regularity has strong physical plausibility on its own, at least if one accepts any version of Cosmic Censorship—it almost necessarily follows that any non-sufficiently regular horizon will eventuate in a naked singularity.

Whether one considers the First Law a consequence of the DEC, therefore, depends on whether one thinks it suffices simply to assume the Zeroth Law in its most general form, whether one thinks one should include a derivation of the most general form of the Zeroth Law in a derivation of the First Law, or whether one thinks that the weaker form of the Zeroth Law, which requires no energy condition, suffices for the purposes of the First Law. The delicacy is exacerbated by the fact that (at least) two conceptually distinct formulations of the First Law appear in the literature, what (following Wald [204, ch. 6, §2]) I will call the physical-process version and the equilibrium version. The former fixes the relations among the changes in an initially stationary black hole’s mass, surface gravity, area, angular velocity, angular momentum, electric potential and electric charge when the black hole is perturbed by throwing in an “infinitesimally small” bit of matter, after the black hole settles back down to stationarity. The latter considers the relation among all those quantities for two black holes in “infinitesimally close” stationary states, or, more precisely, for two “infinitesimally close” black hole spacetimes.

The roles the assumption of the Zeroth Law plays in the proofs of the two versions of the First Law differ significantly, moreover, so it is not clear one could give a single principled answer to the question of whether or not the First Law is a consequence

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<sup>62</sup>Hayward [111] does give a proof of what he calls the First Law for generalized black holes (footnote 37), and that does explicitly require the NEC, but the physical interpretation of Hayward’s result is vexed (as he himself admits), so I did not list it among the consequences of the NEC. The physical interpretation of that result would be an interesting problem to resolve, as it would likely shed light on the already vexed problem of understanding energy in general relativity.

<sup>63</sup>Roughly speaking, a Killing horizon is sufficiently regular in the relevant sense if: it is (locally) bifurcate; or the null geodesic congruence constituting it is geodesically complete; or the twist of the null geodesic congruence has vanishing exterior derivative; or the domain of exterior communication is static; or the domain of exterior communication is stationary, axisymmetric, and the 2-surfaces orthogonal to the two Killing fields are hypersurface orthogonal (and so integrable themselves).

of the DEC that covered both versions at once. For example, in the physical-process version, but not in the equilibrium version, one must assume that the black hole settles back down to a stationary state after one throws in the small bit of matter, and so, *a fortiori*, that the event horizon is not destroyed when one does so, resulting in a naked singularity. I know of no rigorous proofs of the stability of an event horizon under generic small perturbations. All the most compelling arguments in favor of a reasonably broad kind of stability I know, however, do assume constraints on the form of the matter causing the perturbation, constraints that usually look a lot like energy conditions.<sup>64</sup>

Why is there this problem with understanding the relation of the First Law to the energy conditions? The difficulty seems especially surprising in light of the fact that it is the only one of the Laws that constrains mass–energy! Is it, perhaps, that mere conservation doesn’t care whether mass–energy is negative or positive?

As striking as the difficulty in that case is, however, I still find more striking the number, variety and depth of what are indubitably consequences that the energy conditions *do* have, especially without input from the Einstein field equation. The two most numerous types of theorems in the list of consequences are those pertaining to singularities and those to black holes (including horizons), respectively. Indeed, it was the epoch-making result of Penrose [148] showing that a singularity would inevitably result from gravitational collapse in an open universe that first demonstrated the power that the qualitative abstraction of energy conditions gives in proving far-reaching results of great physical importance. I will first discuss some interesting features of the singularity theorems and the role that energy conditions play in their proofs, then do the same for theorems about black holes, positive energy, geodesic theorems and entropy bounds.<sup>65</sup> In §3.2, I will then review the violations of the energy conditions and discuss whether they give us grounds for doubting the physical relevance of the positive consequences.

The weakest condition, the NEC, already has remarkably strong consequences. Among the singularity theorems it supports, to my mind the most astonishing is the one due to Gannon [82] and Lee [128].<sup>66</sup> In any asymptotically flat spacetime with a non-simply connected Cauchy surface, a singularity is bound to form. Topological complexity by itself, with the only constraint on metrical structure being the mild one of the NEC, suffices for the formation of singularities (in the guise of the incompleteness of a causal geodesic). The theorem gives one no information about the singularity, whether it will be a timelike or null geodesic that is incomplete, or whether it will be associated with pathology in the curvature, or something that looks like collapse of a material body, or will be cosmological in character (such as a big bang or big crunch), but the simple fact that nontrivial topology plus the weak-

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<sup>64</sup>See, *e.g.*, Press and Teukolsky [124, 158], Kay and Wald [124], Carter [33, 34], and Kokkotas and Schmidt [126].

<sup>65</sup>I will not discuss the role of energy conditions in ensuring that the initial value formulation of general relativity is well posed, as the relation between the two is complex and very little is known about it. Although I will make a few remarks on the subject in §5, it is work for a future project.

<sup>66</sup>Gannon and Lee discovered it independently, roughly simultaneously.

est energy condition, irrespective of dynamics, suffices for geodesic incompleteness already shows the profound power of these conditions. It is tempting to relate Gannon's and Lee's singularity theorem to Topological Censorship, especially in so far as the latter requires only the ANEC, which the NEC implies. If one assumes that the singularity predicted by the theorem will be hidden behind an event horizon, then the theorem gives some insight into why nontrivial spatial topological structure will always (quickly come to be) hidden inside a black hole. (See footnote 59.) It also suggests that, in some rough sense, nontrivial topological structure may have mass–energy associated with it (perhaps of an ADM-type). It would be of some interest to see whether that idea can be made precise; one possible approach would be to see whether one could attribute some physically reasonable, nonzero ADM-like mass to flat, topologically nontrivial spacetimes. If so, I think this would give insight into the vexed question of the meaning of “mass” and “energy” in general relativity. If such a definition were to be had, I conjecture that nontrivial topological structure could have either positive or negative mass–energy, depending on the form of the structure; otherwise, it would not seem necessary to assume an energy condition in order to derive the Topological Censorship Theorem.<sup>67</sup>

Another striking feature of the list is that the only important consequences of the SEC (and the ASEC) are singularity theorems,<sup>68</sup> and among them the most physically salient ones, whereas the DEC, contrarily, is used in only one type of singularity theorem (Senovilla [179, 180]), and that of a character completely different from the other singularity theorems. The singularity theorems following from the SEC are the most physically salient both because they tend to have the weakest ancillary assumptions, and because they apply to physically important situations, both for collapsing bodies and for cosmology. I have no compelling explanation for why the SEC should have no important consequences other than singularity theorems. Perhaps it has to do with the fact that the SEC has a relatively clear geometrical interpretation (convergence of timelike geodesics) that is manifestly relevant to the formation of singularities, whereas its physical and effective interpretations are obscure at best. If so, then one may want to consider the SEC a case of gerrymandering, the relativity community simply having posited the weakest formal condition it could find to prove the results it wants. This line of thought becomes especially attractive when one contemplates the many possible violations of the SEC and even more the strong preponderance of indirect observational evidence that the SEC has been widely violated on cosmological scales at many different epochs in the actual universe, and is likely being violated

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<sup>67</sup>A good place to start might be the investigation of asymptotically flat spacetimes with nontrivial second Stiefel-Whitney class, as it is known that such spacetimes cannot support a global spinor structure [88, 90]. That shows already that there is something physically *outré* about those spacetimes.

<sup>68</sup>Although the proposition that a given globally hyperbolic extension of a spacetime is the maximal such extension depends for its only known proof on the assumption of the SEC, this is not really a counter-example to my claim: roughly speaking, the proof works by showing that the given globally hyperbolic extension cannot be extended (and so is maximal) because to do so would result “immediately” in singularities, contradicting the assumption of extendibility.



right now.<sup>69</sup> The result of Ansoldi [3], however, that black holes with singularity-free interiors necessarily violate the SEC, may push one towards the opposite view, in so far as it comes close to making the SEC both necessary and sufficient for the occurrence of certain types of singularities. (The construction of singularity-free FLRW spacetimes violating the SEC, in Bekenstein [14], buttresses this line of thought; I discuss this further below.)

I have no explanation for why the DEC should be used in almost no singularity theorems, except for the simple observation that the only real addition the DEC makes to the NEC and the WEC, that energy–momentum flux be causal, has no obvious connection to the convergence of geodesics. The one type of singularity theorem (Senovilla [179, 180]) it is used in, moreover, is the only one to make substantive, explicit assumptions (over and above the energy conditions themselves) about the distribution of stress-energy, in this case in the demand for nonzero averaged spatial energy density. Perhaps that is why the DEC comes into play in this theorem, though I have no real insight into how or why the DEC may bear on averaged spatial energy density and its relation to the convergence of geodesic congruences.

The Lorentzian splitting theorems may be thought of as rigidity theorems for singularity theorems invoking the SEC, for the splitting theorems show that, under certain other assumptions, there will be no singularities only when the spacetime is static and globally hyperbolic.<sup>70</sup> Static and globally hyperbolic spacetimes, however, are “of measure zero” in the space of all spacetimes, and so being free of singularities is, under the ancillary conditions, unstable under arbitrarily small perturbations.<sup>71</sup> Thus, they go some way towards proving the conjecture of Geroch [86] that essentially all spatially closed spacetimes either have singularities or do not satisfy the SEC.<sup>72</sup>

As a group, the singularity theorems are perhaps the most striking example of the importance of ascertaining the status and nature of the energy conditions, because all the assumptions used in proving essentially all of them have strong observational or theoretical support *except* the energy conditions, as Sciama [175] emphasized even before there were serious observational grounds for doubting any of the energy

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<sup>69</sup>See §3.2 for discussion, and Curiel [47] for a more extensive and thorough analysis.

<sup>70</sup>See footnote 51 for a statement and explication of the splitting theorems. See Beem et al. [11, ch. 14] for a beautiful discussion of the rationale behind and intent of rigidity theorems, as well as an exposition of many of the most important ones.

<sup>71</sup>One should bear in mind that this argument is hand-waving at best. First, there is no known natural measure on the space of spacetimes; second, even if there were, being a measure on an infinite-dimensional space, it is possible that every open set (in some natural topology, of which there is also not one known) would have zero measure or infinite measure. (There is no Borel measure on an infinite-dimensional Fréchet manifold; thus measure and topology tend to come apart.) In that case, in a natural sense “arbitrarily small” perturbations of a static, globally hyperbolic spacetime could in fact yield another static, globally hyperbolic spacetime. This problem is not unique to this argument but plagues all hand-waving arguments invoking “measure zero” sets in the space of all spacetimes, which are a dime a dozen, especially in the cosmology literature. See Curiel [49] for detailed discussion of all these issues.

<sup>72</sup>If this conjecture were to be precisely formulated and proven, perhaps one could view it as providing something like an *a posteriori* partial physical interpretation of the SEC.

conditions. This raises the question of the necessity of the energy conditions for the singularity theorems. That some of the impressionist energy conditions can be used to prove essentially identical theorems already shows that satisfaction of the pointilliste conditions is not necessary for validity of at least some of the theorems. The original singularity theorem, the demonstration by Penrose [148] that singularities should form after gravitational collapse in spatially open universes, holds under the weaker assumption of the ANEC [166, 176]. Likewise, the existence of cosmological (*i.e.*, non-collapse) singularities in both spatially open and closed universes can be shown under the assumption of the ASEC [176, 189], without the full SEC. So far as I know, there is no proof that gravitational collapse will lead to singularities in the case of spatially closed spacetimes under the weaker assumption of an impressionist energy condition. I conjecture that there are such theorems; it would be of some interest to formulate and prove one or to construct a counter-example.

With the possible exception of the First Law of black hole mechanics (for asymptotically flat black holes), every fundamental result about black holes requires an energy condition for its proof, with the majority relying either on the NEC or the DEC. Roughly speaking, the results pertaining to black holes fall into three categories: those constraining the topological and Killing structure of horizons; those constraining the kinds of property black holes can possess; and those constraining the relations among the horizon and the properties. Almost all of the first category invokes the NEC for their proof. One can perhaps see why the NEC is relevant for the results about the topological and Killing structure of horizons associated with asymptotically flat black holes: such a black hole is defined as an event horizon, which is the boundary of the causal past of future null infinity, and the boundary of the causal past or future of any closed set is a null surface, *i.e.*, is generated by null geodesics and so may be thought of as a null geodesic congruence. The proofs of many of those results, moreover, tend to have the same structure: very broadly speaking, one assumes the result is not true and then derives a contradiction with the fact that null geodesic congruences, by dint of the NEC, must be convergent (or at least not divergent). This suggests that the NEC is necessary for these theorems, a suspicion strengthened by the facts that, first, there is no weaker energy condition that one could attempt to replace it with (except perhaps the FEC, if it turns out to be viable—see §2.4), and, second, no such results are known to follow from any of the impressionist energy conditions. Again, it would be of interest to see whether the impressionist energy conditions could be used to prove theorems about the topological and Killing structure of black hole horizons, or else to construct counter-examples to the results in spacetimes in which the impressionist but not the pointilliste conditions hold. The NEC is also used to prove many results about the kinds of properties required to characterize black holes (the constituents of the No Hair Theorems), *viz.*, that stationary black holes can be entirely characterized by three parameters, mass, angular momentum and electric charge. I have no physically compelling story to tell about why the NEC relates intimately to these kinds of result. Again, the lack of such results depending on impressionist conditions suggests that the pointilliste conditions are necessary, and, again, it would be of some interest either to prove analogous results using the impressionist conditions or to find counter-examples.

Every consequence of the DEC pertaining to black holes is of the kind that constrains topological or Killing structure of the horizons. There is, however, no common thread to the role the DEC plays in the proofs of those various results analogous to the way that the NEC plays essentially the same role in the proofs of many of its consequences. It is thus difficult even to hazard a guess about the necessity of the DEC for these consequences. It would be of great interest to work through the various results to see whether counter-examples to them satisfying or violating the DEC could be found, or whether proofs using weaker energy conditions can be found. That there is no impressionist analogue to the DEC may suggest that the DEC is necessary for these results.

Roughly speaking, the idea of the Cosmic Censorship Hypothesis is that “naked singularities” should not be allowed to occur in nature, where, continuing in the same rough vein, a naked singularity is one that is visible from future null infinity. Now, the relation of the energy conditions to the status of the Cosmic Censorship Hypothesis is complicated, first and foremost, by the fact that there are a multitude of different formulations of the Hypothesis (thus calling into question the common practice of honoring the thing with the definite article and the capitalization of its name). Because the presence of naked singularities would seem to herald a spectacular breakdown in predictability and even determinism associated with dynamical evolution in general relativity (such as it is),<sup>73</sup> many attempts to make the Hypothesis precise focus on the initial value formulation of general relativity. The most common formulations invoke either the WEC or the DEC [123] as a constraint on the matter fields permissible for the initial value formulation. As initially plausible as are such attempts at formulating a precise version of the Hypothesis that would admit of rigorous proof, there are in fact cases where satisfaction of an energy condition actually seems to aid the development of a naked singularity after gravitational collapse, *e.g.*, the WEC in the case of the self-similar collapse of a perfect fluid [123]. In such cases, one can show that the focusing effects the energy condition induces in geodesic congruences actually contribute directly to the lack of an event horizon. It is thus parlous to attempt to draw any concrete conclusions regarding the relation of the energy conditions to the Cosmic Censorship Hypothesis in our current state of knowledge.

With regard to results about positivity of global mass, because the NEC does not require the convergence of timelike geodesics (as I discussed in §2.1), and so does not entail that “gravity be attractive” for bodies traversing such curves, it is particularly striking that Penrose [153] and Ashtekar and Penrose [4] were able to prove positivity of ADM mass using only it, and that Penrose et al. [156] were able to prove it using the even weaker ANEC, and not the significantly stronger DEC, as all other known proofs require. All known proofs of the positivity of the Bondi mass do require the DEC, which is perhaps not surprising, in light of the fact that the Bondi energy essentially tracks mass–energy radiated away along null curves to future null infinity. If the DEC were to fail, then it seems plausible that the Bondi energy could become negative, if negative mass–energy radiated to null infinity. It

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<sup>73</sup>See, *e.g.*, Earman [61] for a thorough discussion, and Curiel [45] for arguments arriving at somewhat contrary conclusions.

would be of some interest to try to find a spacetime model with negative Bondi mass in which the DEC is not violated. Perhaps matter fields with “superluminal acoustic modes” that still satisfied the DEC (§2.1) might provide such examples.

The most precise, rigorous and strongest geodesic theorems [66, 93] both assume the SDEC.<sup>74</sup> Under the assumptions used to prove the theorem of Geroch and Jang [93], Malament [132] showed that the SDEC is necessary for the body to follow a geodesic, and not just any timelike curve. Weatherall [208] strengthened the result by showing that the SDEC is necessary for the geodesic to be timelike, not spacelike. He showed as well that the SDEC is not strong enough to ensure that the curve not be null: there is a spacetime with a null geodesic satisfying all the conditions of the Geroch-Jang Theorem. It is perhaps important that the example Weatherall [208] produces to show that a null curve can satisfy all of the theorem’s conditions rely on a stress-energy tensor not of Hawking-Ellis type I. Since stress-energy tensors not of type I are generally considered “unphysical”, it would be of interest to determine whether there are counter-examples to the Geroch and Jang [93] and Ehlers and Geroch [66] theorems that rely on stress-energy tensors of type I. Because of the character of the proofs of the theorems and of the counter-examples that Weatherall [208] produces, I conjecture that there are no such counter-examples, and thus that null curves satisfying the conditions of the theorem require nonstandard stress-energy tensors.<sup>75</sup>

Whether or not my conjecture is correct, I think the necessity of the strongest energy condition for the validity of the theorems poses a problem for many attempts to analyze and clarify the conceptual foundations of general relativity. Many attempts to provide interpretations of the formalism of general relativity, for instance, place fundamental weight on the so-called Geodesic Principle, that “small bodies”, when acted on by no external forces, traverse timelike geodesics. The “fact” that the Geodesic Principle is a consequence of the Einstein field equation is often cited as justification for the validity of the Principle (*e.g.*, Brown [27]). The work of Malament [132] and Weatherall [208], however, show that, at best, such approaches to the foundations of general relativity must be more subtle where the Geodesic Principle is concerned, and, at worst, that the Principle may in fact not be suitable at all for playing a fundamental role in giving an interpretation of the theory.<sup>76</sup>

With regard to entropy bounds such as that of Bousso [24, 25], if in fact the NEC or DEC were necessary for their validity, this could spell serious trouble for

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<sup>74</sup>The statement of the theorems in each of those papers in fact uses the DEC, but an examination of the proof shows that they both actually use the SDEC, in both cases in order to ensure that a constructed scalar quantity that can be thought of as the mass of an “arbitrarily small” body is strictly greater than zero.

<sup>75</sup>I have not had the opportunity to work through the arguments of Dixon [55–58], Ehlers and Rudolph [67] and Schattner [169, 170] to determine whether their results on the definability of the center of mass of an extended body and the formulation of equations of motion for that center of mass in fact rely on the SDEC rather than, as they explicitly assumed, the DEC. Because of the intimate connection of these relations with the geodesic theorems, this would be of some interest to determine.

<sup>76</sup>See Weatherall [209, this volume] for extended discussion of these issues.

many programs in quantum gravity, or at least for the ways that research in such programs are currently being carried out, in so far as many programs place enormous motivational, argumentative and interpretational weight on such entropy bounds, and we already know that essentially all energy conditions are promiscuously violated when quantum effects are taken into account.<sup>77</sup>

### 3.2 Violations

In some of the cases of violations I list, the circumstance or condition possibly leading to a violation of the germane energy condition (*e.g.*, for some subset of possible values for relevant parameters); in other cases, it necessarily does so. I will indicate which is which. When I list the same type of system as violating different energy conditions (*e.g.*, “big bang” singularities for both NEC and SEC), it means that different instances of that type of system (having different parameters) violate the different conditions.

#### NEC

1. conformally coupled massless and massive scalar fields [**possibly**] [6, 199]
2. generically non-minimally coupled massless and massive scalar fields [**possibly**] [6, 59, 74, 199]
3. “big bang” and “big crunch” singularities<sup>78</sup> [**possibly**] [35, 37]
4. “big rip” singularities<sup>79</sup> [**necessarily**] [35, 37]
5. sudden future singularities<sup>80</sup> [**possibly**] [8, 9, 35, 37]
6. naked singularities [**possibly**] [5, 123, 152]
7. closed timelike curves [**possibly**] [195]
8. Tolman wormholes and Einstein–Rosen bridges [**necessarily**] [5]
9. any fluid with a barotropic index  $w < -1$ <sup>81</sup> (such as those postulated in so-called phantom cosmologies) [**necessarily**] [53, 195]

<sup>77</sup>See Curiel [48] for more detailed discussion of all these issues.

<sup>78</sup>A big bang or a big crunch is a singularity in a standard cosmological model where the expansion factor  $a(t) \rightarrow 0$  in a finite period of time to the past or future, respectively. See, *e.g.*, Weinberg [210] or Wald [203]. In the specific context of FLRW spacetimes, this condition implies that a singularity is “strong” in the sense of Tipler [188].

<sup>79</sup>A big rip is a singularity in a standard cosmological model where the expansion factor  $a(t) \rightarrow \infty$  in a finite period of time. If, as is currently believed, the universe is expanding at an accelerated rate, and it continues to do so, it is possible that such a big rip will occur. See, *e.g.*, Caldwell [29], Caldwell et al. [30] and Chimento and Lazkoz [39].

<sup>80</sup>These are singularities in standard cosmological models in which the pressure of the effective fluid or some higher derivative of the expansion factor  $a(t)$  diverges, even though the energy density and curvature remain well behaved. They are very strange, not least because they do not necessarily lead to curve incompleteness of any kind. See Curiel [50] for further discussion.

<sup>81</sup>See footnote 15.

10. “hyper-fast” travel<sup>82</sup> **[possibly]** [200]

## WEC

1. naked singularities **[possibly]** [76]
2. closed timelike curves **[possibly]** [195]
3. physically traversable wormholes **[necessarily]** [139, 193, 194]
4. cosmological steady-state theories of Bondi and Gold [20] and Hoyle [116]<sup>83</sup> **[necessarily]**
5. classical Dirac fields **[possibly]** [203]
6. a negative cosmological constant (*e.g.*, anti-de Sitter Space) **[necessarily]** [107, 195]
7. future eternal inflationary cosmologies **[possibly]** [23]
8. “hyper-fast” travel<sup>84</sup> **[necessarily]** [2, 127, 144]

## SEC

1. “big bang” and “big crunch” singularities<sup>85</sup> **[possibly]** [35, 37]
2. sudden future singularities<sup>86</sup> **[possibly]** [8, 9, 35, 37]
3. cosmological “bounces”<sup>87</sup> **[necessarily]** [35, 37]
4. just before or just after a cosmological “inflexion”<sup>88</sup> **[possibly]** [35, 37]
5. spatially closed, expanding, singularity-free spacetimes **[necessarily]** [176]
6. cosmological inflation **[necessarily]** [195]
7. a positive cosmological constant, as in de Sitter spacetime, and the “dark energy” postulated to drive the observed accelerated expansion of the universe **[necessarily]** [29, 30, 54, 107]
8. asymptotically flat black holes with regular (nonsingular) interiors **[necessarily]** [3]
9. closed timelike curves **[possibly]** [195]
10. physically traversable wormholes **[necessarily]** [114, 138]
11. minimally coupled massless and massive scalar fields **[possibly]** [6, 199]
12. massive Klein-Gordon fields **[possibly]** [195]
13. typical gauge theories with spontaneously broken symmetries **[possibly]** [189]
14. conformal scalar fields coupled with dust **[possibly]** [14]

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<sup>82</sup>See footnote 45.

<sup>83</sup>See also Pirani [157], Hoyle and Narlikar [117], and Hawking and Ellis [107, §4.3, pp. 90–91; §5.2, p. 126].

<sup>84</sup>See footnote 45.

<sup>85</sup>See footnote 78.

<sup>86</sup>See footnote 80.

<sup>87</sup>A bounce, in the context of a standard cosmological model, is a local minimum of the expansion factor  $a(t)$ . See, *e.g.*, Bekenstein [14] and Molina-Paris and Visser [138].

<sup>88</sup>An inflexion, in the context of a standard cosmological model, is a saddle point of the expansion factor  $a(t)$ . See, *e.g.*, Sahni et al. [167] and Sahni and Shtanov [168].

15. “hyper-fast” travel<sup>89</sup> [**necessarily**] [2, 127, 144]

### DEC

1. “big bang” and “big crunch” singularities<sup>90</sup> [**possibly**] [35, 37]
2. sudden future singularities<sup>91</sup> [**possibly**] [8, 9, 35, 37]
3. classical Dirac fields [**necessarily**] [155]

### ANEC

1. massless conformally coupled scalar fields<sup>92</sup> [**possibly**] [6, 199]
2. massless and massive non-minimally coupled scalar fields [**possibly**] [59, 74]
3. closed timelike curves [**possibly**] [195]
4. traversable wormholes [**possibly**] [140]

### AWEC

1. cosmological steady-state theories of Bondi and Gold [20] and Hoyle [116] [**necessarily**] (my calculation)
2. a negative cosmological constant (*e.g.*, anti-de Sitter Space) [**necessarily**] (my calculation)
3. classical Dirac fields [**possibly**] (my calculation)
4. closed timelike curves [**possibly**] [195]
5. physically traversable wormholes [**possibly**] (my calculation)
6. “hyper-fast” travel<sup>93</sup> [**possibly**] (my calculation)

### ASEC

1. a positive cosmological constant, as in de Sitter spacetime, and the “dark energy” postulated to drive the observed accelerated expansion of the universe [**necessarily**] (my calculation)
2. cosmological inflation [**possibly**] (my calculation)
3. massive Klein-Gordon fields [**possibly**] (my calculation)
4. typical gauge theories with spontaneously broken symmetries [**possibly**] (my calculation)
5. conformal scalar fields coupled with dust [**possibly**] (my calculation)

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<sup>89</sup>See footnote 45.

<sup>90</sup>See footnote 78.

<sup>91</sup>See footnote 80.

<sup>92</sup>Urban and Olum [192] also show that AANEC can be violated by conformally coupled scalar fields in conformally flat spacetimes, such as the standard FLRW cosmological models.

<sup>93</sup>See footnote 45.

The most compelling empirical evidence for violations of energy conditions comes from cosmology. For instance, strongly substantiated cosmographic arguments comparing best estimates for the age of the oldest stars to the epoch of galaxy formation show that the SEC must have been violated in the relatively recent cosmological past (redshift  $z < 7$ ) [196–198]. Visser’s arguments, particularly as presented in the 1997 papers, are an especially striking example of the power of the energy conditions: years before there was any hard observational evidence for the acceleration of the current expansion of the universe, and so hard, direct support for the existence of a positive cosmological constant, Visser predicted on purely theoretical grounds that the most likely culprit for violation of SEC in the recent cosmological past must be a positive cosmological constant. In fact, if the current consensus that the expansion of the universe is accelerating is correct, and so some form of “dark energy” exists, then we know that the SEC is currently being violated on cosmological scales, entirely independently of any assumptions about the nature of the fields entering into the stress-energy tensor or cosmological constant [6, 35–37, 198, 199]. Finally, if any model of inflationary cosmology is correct, then we know that the SEC was necessarily violated at least during the inflationary period and possibly, depending on the particulars of the model, the ASEC as well. One glimmer of hope among the gloom, however, is that the presence of a positive cosmological constant does not yield violations of the NEC, so no matter how exotic so-called dark energy is, and whatever fundamental mechanism may underlie it, at the classical level at least it will still satisfy that condition.

Far and away the simplest theoretical mechanisms presently known for yielding violations of energy conditions, and in many ways the most plausible, come from models including scalar fields. Indeed, using classical scalar fields alone, without even having to resort to quantum weirdness, it is relatively easy to engineer violations of even the weakest conditions, the NEC and the ANEC, as the list of violations shows. We do not yet have indubitable evidence for the existence of a fundamental scalar field in nature. (The recently discovered Higgs field is without question phenomenologically a scalar field, but the jury is still out on whether or not it is a composite, bound state of underlying non-scalar entities.) The importance of scalar fields in fundamental theoretical physics, however, is indubitable.<sup>94</sup> For many theoretical and pragmatic reasons, the so-called inflaton field that drives cosmological inflation is most commonly modeled as a classical scalar field, and cosmological inflation necessarily violates SEC and, depending on particulars of the model, possibly ASEC. Many meson fields in the Standard Model (pions, kaons and many other mesons, including their “charmed”, “truth” and “beauty” correlates), moreover, are modeled to an extraordinarily high degree of accuracy as scalar fields, even though we believe they in fact consist of bound states of (non-scalar) quark–antiquark pairs. It is also widely believed that the so-called “strong CP problem”, the fact that no CP-violation in strong nuclear interactions has ever been observed, is best solved

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<sup>94</sup>It would be an interesting project to try to determine why theoretical physicists are firmly wedded to scalar fields as fundamental constituents of reality in the face of an almost complete lack of evidence for them, and whether their reasons for the marriage are really sound.



by the postulation of a scalar field called the axion [147], though to the best of my knowledge it is not known whether any classical models of the axion violate any of the energy conditions (any more than those of other quantum fields do, at any rate).

Now, violations of the NEC are disturbing for at least two important reasons. First and perhaps foremost, they imply violations of all other pointilliste energy conditions. Second, they already would seem to allow not only violations of the ordinary Second Law of thermodynamics [52, 75], but of the Generalized Second Law as well: send lots of negative energy (with positive entropy) through the event horizon into a black hole, and *voilà!*—the area of the black hole shrinks, even though arbitrary amounts of entropy have disappeared from outside the event horizon. Perhaps the most troubling violation of the NEC from the above list is the case of a conformally coupled scalar field, given the naturalness of “conformal coupling” for scalar fields in quantum field theory [6, 199], which is why in the list of violations I singled it out from the class of generically non-minimally coupled scalar fields.

The particular example of a massive conformal scalar field coupled with dust given by Bekenstein [14] in an example of how to construct a nonsingular FLRW model, exploiting the fact that the system can be made to violate the SEC, has interesting possible physical significance, which is why I singled it out in the list of systems for which energy conditions can fail: the pions that mediate the strong nuclear force can to a very high degree of approximation be represented by just such scalar fields. Thus, Bekenstein argues, nuclear matter in the very early, dense stages of the actual universe may not have satisfied the SEC, which may suggest that the initial singularity in standard Big Bang models may be avoidable. This may give reason to doubt the stability of at least some of the singularity theorems in regimes where the energy conditions fail. Because the SEC would have been necessarily violated during an epoch of inflationary expansion, moreover, and because inflationary theories have such strong support among many cosmologists, such doubts should perhaps cause further concern for advocates of an initial Big Bang singularity. In light of the fact that the strongest theorems for big bang singularities rely on the SEC, and that the Lorentzian Splitting Theorems (in conjunction with the results of Senovilla [176] to the effect that spatially closed, expanding, singularity-free spacetimes necessarily violate the SEC) come close to showing that the SEC is necessary for those theorems, then I think it becomes quite reasonable to question the current confidence in the so-called Standard Model of cosmology, which rests on the idea that the universe “started with” a big bang. That, moreover, both a cosmological “bounce” and a Tolman wormhole (perhaps the two most natural possible replacements for an initial big bang singularity) require violation *only* of the SEC [114, 138], not any of the other energy conditions, only exacerbates the problem.

Tipler [189], in a line of argument intended to mitigate such doubts, has pointed out an amusing poignancy in the role that homogeneity (high symmetry) plays in Bekenstein’s construction of nonsingular FLRW spacetimes that violate the SEC. It follows from a theorem Tipler proves that, if a black hole (marginally trapped surface) develops in one of Bekenstein’s spacetimes, then, because they do satisfy the WEC, a singularity would necessarily develop. Of course, a marginally trapped surface would form only if there were deviations from homogeneity. We would expect, however,

on physical grounds, that even slight deviations from homogeneity could lead to the development of marginally trapped surfaces. Thus, it is only the strict symmetry of the Bekenstein models that precludes singularities. This, of course, turns the standard (mistaken) pre-Penrose [148] argument on its head: that the singularities of the FLRW, Schwarzschild, and Oppenheimer and Snyder [145] spacetimes were simply an artifact of their unrealistic perfect symmetry. In the case of Bekenstein's spacetimes, it is only their unrealistic perfect symmetries that *precludes* singularities. Theorem 1 of Tipler [189], moreover, gives him even stronger grounds for thinking that violations of SEC will not necessarily block formation of singularities, at least for closed universes, so long as the period and extent of the failure is limited with respect to its satisfaction in the rest of spacetime, *i.e.*, so long as the ASEC holds.

The theorems predicting big bang and big crunch singularities face one more problem peculiar to them alone: all such theorems invoke energy conditions of various kinds, mostly the SEC, and yet one can show that, depending on the characteristics of a given big bang or big crunch singularity, the presence of the singularity itself implies a violation of the relevant energy condition. Roughly speaking, whether a big bang or big crunch implies a violation of a given energy condition depends on how “violent” the singularity is, which idea can be made precise by analysis of the nature of the matter fields present (*e.g.*, the value of the barotropic index of the ambient homogeneous cosmological fluid), or by the behavior of geodesic congruences in the immediate neighborhood of the singularity (*e.g.*, whether such singularities are strong in the sense of Tipler [189], and, if so, how quickly they squeeze spatial volumes to zero). What is one to say in such cases? Clearly, the known theorems do not apply to such singularities, but also clearly the exact spacetimes in which such singularities occur have been shown to exist. The only safe conclusion seems to be that, at least in the case of these kinds of singularity, violations of salient energy conditions need not preclude their existence. But then one must question the importance of the theorems themselves, especially in light of the growing body of observational evidence that, if there is a big bang or big crunch, it may well be of a type that violates energy conditions.

What about the remainder of the singularity theorems? Should any of the violations drive us to doubt their validity or physical relevancy? In order to try to answer this question with some generality, it will be useful to draw two distinctions, the first between types of violations, and the second between types of theorems.<sup>95</sup> First, roughly speaking, the violations fall into one of two classes, being associated either with a type of physical system (*e.g.*, conformally coupled scalar field, classical Dirac field), or with a type of “event” (very loosely construed, *e.g.*, traversable wormhole, closed timelike curve, or big rip singularity). Generally speaking, for the latter, the regions where the energy conditions are violated can be “localized” to a neighborhood of the “event”. The scare-quotes are to remind us of the fact that some such events—*e.g.*, many types of singularities—are not localizable in any reasonable sense

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<sup>95</sup>I do not think the classifications I sketch here are of relevance beyond the context of such discussions as this. I certainly do not think they capture anything of fundamental significance about the nature of violations of energy conditions or about singularities.

of the term.<sup>96</sup> The qualification “generally speaking” hedges against cases such as the traversable wormholes of Visser [194], for which travelers moving through the wormholes never experience a violation of any energy condition. Generally speaking, for violations of the former class (*viz.*, associated with a type of physical system), one cannot “localize” the regions of violation in any way, unless one can localize the system itself, or at least those spacetime regions in which the system is known to violate the energy conditions and one can also determine that the system violates them nowhere else.

As for the singularity theorems, they also fall roughly into two classes, which for lack of better terms I will refer to as pinpointing and not. Roughly speaking, pinpointing theorems, as the name suggests, in certain ways allow one to say where in spacetime the singularities occur, and so in a sense one can “localize” the singularities.<sup>97</sup> Such theorems demonstrate the existence of singularities associated with closed, trapped surfaces (for singularities contained in asymptotically flat black holes: Penrose [148], Hawking and Ellis [107]), or with trapping surfaces (for singularities contained in generalized black holes: Hayward [110, 111]), or with the “boundaries” of spacetime (such as big bang and big crunch singularities), or they place the defining incomplete, inextendible geodesic entirely in a compact subset of the spacetime (*e.g.*, Hawking and Ellis [107, pp. 290–292]). Singularity theorems that are not pinpointing, such as those of Gannon [82, 83], merely demonstrate the existence of incomplete, inextendible geodesics without giving one any information about “where the incompleteness of the geodesic” is in spacetime.

Now, the impact of possible violations will differ from theorem to theorem depending on whether the theorem at issue pinpoints or not, and on whether the violation can be localized in an appropriate sense to that region of spacetime in which the theorem locates the predicted singularity. For theorems that do not pinpoint, I think there is no principled reason to believe that any salient violations may or may not vitiate the theorem. For theorems that do pinpoint, there may be hope of showing that at least some salient violations may or likely will not vitiate the theorems, but one must work through them on a case by case basis to make the determination. If one has some reason to believe, for example, that a given type of salient violation can be segregated entirely from the region of spacetime in which a closed, trapped surface forms and evolves (because, *e.g.*, of the type of collapsing matter that eventuates in the trapped surface), then one also has some reason to believe that any theorem that both invokes the violated condition and places the singularity in such a closed, trapped surface may still hold despite the violation. It would take us too long to go through all the singularity theorems and all the types of violations to determine which violations can and cannot be relevantly segregated from the regions where the predicted singularities form or reside. I leave this as an exercise for the reader.

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<sup>96</sup>See Curiel [45] for discussion.

<sup>97</sup>Again, see Curiel [45] for discussion of why the scare-quotes are called for.

Similar considerations about pinpointing, type of violation, and the possibility of segregation come into play when trying to determine whether a given violation should give us reason to doubt the soundness of any other type of given consequence of an energy condition. I see no way to draw clean, general conclusions.

In sum, it seems difficult to escape the conclusion that we are faced with the horns of an important dilemma: either we must learn to live with the “exotic” physics that violations of energy conditions lead to (wormholes, closed timelike curves, sudden future singularities, spatial topology change, naked singularities, *et al.*), and so become much more skeptical of the plethora of seemingly important results that rely on the conditions; or else we must reconstruct fundamental physical theory root and branch, *e.g.*, by prohibiting the use of essentially all scalar fields, in order to rule out the possibility of such violations. I personally find it more realistic, if not more palatable, to grasp the first horn. An investigation of the consequences of this conclusion for projects that purport to provide fundamental explication and interpretation of the conceptual and physical structure of general relativity is beyond the scope of this paper, but is, I think, urgently called for.

### 3.3 Appendix: The Principle of Equivalence

There is an interesting, though not obvious, possible connection between the principle of equivalence (in at least some of its guises) and energy conditions. (See Wallace [207, this volume] for discussion of the principle of equivalence.) Postulating the lack of a preferred flat affine connection is, to my mind, one of the most promising ways of trying to formulate the principle of equivalence in a way that one can make somewhat precise [190, 191], even if one cannot show that such a principle must be true in the context of the theory. Could one derive an energy condition, or the violation of one, from the existence of a preferred flat affine structure? One way to determine such a privileged flat affine connection would be by use of the existence of a distinguished family of particles possessing what, for lack of a better term, I will call “anti-inertial charge”, which would couple with the “active gravitational mass” of ordinary matter in such a way as to result in the anti-inertial systems traversing curves whose images form the projective structure of a flat affine connection. For a force that picks out such a connection, one can assign to it a stress-energy tensor by solving the equation of geodesic deviation using it as a force that exactly cancels out the curvature terms due to the ordinary affine connection, and deriving an expression for an “effective” stress-energy tensor associated with the force.

One possible mechanism for producing anti-inertial charge is strongly suggested by the arguments of Bondi [18] showing that active and passive gravitational mass are not necessarily equal in general relativity, at least when negative mass is allowed.<sup>98</sup>

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<sup>98</sup>I put aside the problem that “mass” is, in general, not a well defined concept in general relativity. If one likes, one can consider the following discussion to be restricted to test particles in static spacetimes.

In particular, negative masses uniformly repel all other mass, irrespective of the sign of the other masses, and likewise positive masses uniformly attract all other masses, and so, most strikingly, a system consisting of one positive and one negative mass will spontaneously accelerate, even when no forces other than “gravitational” are present. a clear violation of the weak equivalence principle, that, roughly speaking, all small enough freely falling bodies traverse the same worldlines, *viz.*, geodesics. (Arguably, the inequality of passive and active gravitational mass already constitutes a violation of the weak principle of equivalence, at least in one of its guises.) In this case, negative mass plays the role of an anti-inertial charge. In the case that Bondi describes, therefore, the projective structure of the flat affine connection could possibly be determined by the acceleration curves of systems having equal parts positive and negative active gravitational mass.

This line of thought suggests the following.

**Conjecture 1** *If one were able to demonstrate the existence of a privileged flat affine connection, by the existence of a family of particles with anti-inertial charge, then one or more of the standard pointilliste energy conditions would be generically violated.*

### 3.4 Coda: The Trace Energy Condition

The history of what may be called the Trace Energy Condition (TEC) should give one pause before rejecting possible violations of the standard energy conditions on the grounds that the circumstances or types of matter involved in the violations seem to us today “too exotic”. The TEC states that the trace of the stress-energy tensor can never be negative ( $T = T^n_n \geq 0$ —or, depending on one’s metrical conventions, that it can never be positive). In its effective formulation, therefore, the condition requires that  $p \leq \frac{1}{3}\rho$  in a medium with isotropic pressure. Before 1961, it seemed to have been more or less universally believed in the general relativity community that this condition would always be satisfied, even under the most extreme physical conditions. It is, for instance, assumed without argument, or even remark, in the seminal papers of Oppenheimer and Volkoff [146] and Harrison et al. [95] on possible equations of state for neutron stars. It was not seriously questioned until the work of Zel’dovich in the early 1960s, in which he showed that a natural solution for a quantum field theory relevant to modeling the matter in neutron stars leads to macroscopic equations of state of the form  $p = \rho$ .<sup>99</sup> In fact, it is widely believed today that matter at densities above 10 times that of atomic nuclei, as we expect to find in the interior of neutron stars, behaves in exactly that manner [181, ch. 8].<sup>100</sup>

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<sup>99</sup>See Zel’dovich and Novikov [215] (especially p. 197) for a discussion.

<sup>100</sup>This coda was inspired by the discussion in Morris and Thorne [139].

## 4 Temporal Reversibility

For the purposes of the discussion in §5, and because it is of some interest in its own right, I will briefly discuss the relation of the energy conditions to the idea of temporal reversibility.

A spacetime is temporally orientable if one can consistently designate one lobe of the null cone at every point as the “future” lobe. A temporal orientation then is logically equivalent to the existence of a continuous timelike vector field  $\xi^a$ ; by convention, the future lobe of the null cone at each point is that into which  $\xi^a$  points, and a causal vector is itself future-directed if it points into or lies tangent to the future lobe. To reverse the temporal orientation is to take  $-\xi^a$  to point everywhere in the “future” direction. If  $T_{ab}$  is the stress-energy tensor in the original spacetime, then we want the time-reversed spacetime to have the stress-energy tensor  $T'_{ab}$  such that: the four-momentum of any particle as determined relative to any observer will be reversed in the time-reversed case; and the energy density of any particle as determined relative to any observer will stay the same. Formally

1.  $T'_{an}(-\xi^n) = -T_{an}\xi^n$
2.  $T'_{mn}(-\xi^m)(-\xi^n) = T_{mn}\xi^m\xi^n$

Clearly, then,  $T'_{ab} = T_{ab}$ . So, in sum, I claim the rule for constructing the time reverse of a (temporally orientable) relativistic spacetime is to leave everything the same except for the sense of parameterization of timelike (and null) curves, which should be reversed. (No problem arises with parameterization of spacelike curves: there is no natural or preferred sense for their parameterization in the first place.)

This makes physical sense. The best way to see this is to ask what should happen to the metric under time reversal. I claim the answer is: nothing at all. The metric stays the same. Temporal orientation is not a metrical concept. It is a concept at the level of differential topology and conformal structure. The temporal orientation is determined by how one parameterizes temporal curves (which in turn, of course, depends on whether one can do so in a way that consistently singles out a choice of “future lobe of null cone” at every point of the manifold in the first place). It also makes geometrical sense. If one fixes a  $1 + 3$  tetrad  $\{\xi^\mu\}_{\mu \in \{0, 1, 2, 3\}}$  (not necessarily orthonormal) such that the metric at a point can be expressed as  $\sum_{\mu} \alpha_{\mu} \xi^{\mu}_a \xi^{\mu}_b$ , for some real coefficients  $\alpha_{\mu}$ , then reversing the sign of  $\xi^0$  clearly does not change the metric.<sup>101</sup> (One can always find such a tetrad at a single point, though it may not be extendible to a tetrad-field with the same property.)

It is a simple matter to verify that a spacetime satisfies any one of the standard energy conditions listed in §2 if and only if the time reverse of the spacetime does as

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<sup>101</sup> Another way to see this is to note that the only reasonable choice for “changing the metric” under time reversal would be to multiply it by  $-1$ ; that however, does not change the Einstein tensor, and so *a fortiori* cannot change the stress-energy tensor.

well. (The same holds as well for all the more recently proposed energy conditions discussed in §2.4.) On the face of it, this is somewhat surprising. A white hole, for instance, is the time reverse of a black hole, and surely that should violate some energy condition. But in fact, no, it shouldn't, as a perusal of the relevant Penrose diagram will show: a white hole will violate an energy condition if and only if its time-reversed black hole does so.

## 5 Constraints on the Character of Spacetime Theories

General relativity assumes the existence of a single object, the stress-energy tensor  $T_{ab}$  that encodes, for all fields of matter, all properties relevant to determining the relationship of the matter to the geometrical structure of spacetime. This relationship is governed by the Einstein field equation,

$$G_{ab} = 8\pi T_{ab}$$

This equation, conjoined with the definition of a spacetime model  $(\mathcal{M}, g_{ab})$ , constitutes the entirety of general relativity as a formal theory.

In order to do physics, however, we must give physical significance to the formal terms in the Einstein field equation, and this is where the idea of stress-energy enters. As its name suggests, the stress-energy tensor encodes for matter only information about what we normally think of as its energy, momentum and stress content. General relativity, then, assumes that what we normally think of as stress-energy content completely determines the relation of spacetime structure to matter—no other property of matter “couples” with spacetime structure at all, except in so far as it may have a part in determining the stress-energy of the matter. It is exactly this feature of general relativity that affords the energy conditions their power. Nonetheless, we fully expect, or at least fervently hope, that general relativity will one day give way to a deeper theory of gravity, one that will attend to the presumably quantum nature of phenomena in regions of extreme curvature.<sup>102</sup> It thus makes sense to explore alternative theories of spacetime even in the strictly classical regime, if only to get ideas about how to try to modify general relativity in the search for that deeper theory. Surely not everything is up for grabs, though. Even in the attempt to formulate alternative theories in the spirit of free exploration, some core structure or set of structures must be retained in order for the explorations to take place in the province of “spacetime theories”. What is that core? Is there a single one?

In particular, for our purposes, the most important question is: what must be true about the relation of stress-energy to the local and global structures of spacetime, in a

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<sup>102</sup>I will not discuss the relation of energy conditions to any programs in quantum gravity, as I do not feel any of them are mature enough as proposals for a physical theory to support serious analysis of this sort. See Curiel [46] for why I hold this view. See Wüthrich [213, this volume], among others, for arguments to the contrary.

candidate spacetime theory, for one to be able to formulate energy conditions and use them to derive results? What, we are thus led to ask, must a spacetime theory itself be like in order for it to be able to exploit the possibility that deep and extensive features of global structure depend only on purely qualitative properties of stress-energy? Any field equations it imposes must be “loose” enough to respect this fact. In particular, no global feature of the geometry, as constrained by a theory’s field equations, should depend on anything but purely qualitative properties of stress-energy; *a fortiori*, no global feature of the geometry should depend on the species of matter present, so long as that species manifests a relevant qualitative property. It is otherwise difficult to see how generic, purely qualitative conditions could determine specific, concrete features of spacetime geometry.

A useful way to begin to try to address these questions, and at the same time to begin to figure out the place of energy conditions in relation to potentially viable alternative spacetime theories, is to ask oneself, following a line of questioning introduced early in Geroch and Horowitz [92], what one can envisage needing to hold onto in future developments of physical theory, come what may. Not the Einstein field equation itself, most likely. Very likely causal structure of some sort. What else?

What follows is my attempt at such a list of structures, roughly ordered by “fundamentality”—where I mean by this only something like: what we would or should be willing to give up before what else, what we have more and less confidence will survive in future theories (not anything having to do with recent debates in the metaphysics literature). Such an ordering should respect, at a minimum, the fact that one needs in place already some structure in order to be able to define other structure—one could not countenance giving up the former before the latter.<sup>103</sup>

In constructing the list, I have been guided by the tenet that any physically reasonable spacetime theory should “look enough like” general relativity so as to make all the elements of the list sensible in its context. Not all the elements in the list, however, should be understood to be restricted to the form they take in standard accounts of general relativity. For instance, “causal structure” need not mean Lorentzian light cone structure; it may signify, for example, only some relation among events required by some feature of ambient matter fields, such as respecting the characteristic cones of matter obeying symmetric, quasilinear, hyperbolic equations of motion, whether or not those cones conform to the standard Lorentzian metric of spacetime.<sup>104</sup> Any such list, moreover, will ineluctably be shaped in part by the biases, prejudices and aesthetic and practical predilections of the one constructing it, so the following attempt should be taken with a healthy dose of salt.<sup>105</sup>

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<sup>103</sup>For a similar list, albeit constructed for a somewhat different purpose, and with a very different ordering than mine, see Isham [118, p. 10].

<sup>104</sup>See, *e.g.*, Geroch [91] and Earman [62].

<sup>105</sup>One could sharpen this list by distinguishing between local and global varieties of structure, *e.g.*, by allowing for the possibility that it makes sense to determine a local causal structure without necessarily requiring the existence of a global one. (In such a case, presumably something like transitivity of causal connectability would fail.) While I think such distinctions could have interest for some projects, they are too *recherché* for my purposes here.



1. event structure: primitive set of “events” constituting the fundamental building blocks of spacetime<sup>106</sup>
2. causal structure: primitive relation of “causal connectability” among events (not necessarily distinguishing between null and timelike connectability)
3. topology: spacetime dimension; notion of continuous curves and fields (maps to and from event structure); relative notions of “proximity” among events; global notions of “connectedness” and “hole-freeness” on event structure
4. projective structure, conformal structure, temporal orientability: notion of a set of events forming a “straight line”, and so physically a distinguished family of curves (but not yet a distinction between accelerated and non-accelerated motion); distinction among spacelike, null and timelike curves; preferred orientation for parameterization of causal curves; null geodesics (but not timelike or spacelike); asymptotic flatness; singularities (incomplete, inextendible causal curves); horizons (event, apparent, particle, *etc.*, and so asymptotically flat black holes)
5. differential structure: notion of smooth (or at least finitely differentiable) curves and fields; and so of tangent vectors, tensors, Lie derivatives and exterior derivatives; and so of field equations and equations of motion; spinor structure
6. affine structure: notion of accelerated versus non-accelerated motion, and so timelike geodesics; spacelike geodesics, and so characterization of “rigid bodies”; “hyperlocal” conservation laws (covariant divergence), at least for quantities “represented by” contravariant indices on tensors; comparison (ratios) of lengths of curve-segments, and so integrals along curves
7. metric structure: principled distinction between Ricci and Weyl curvature (“matter” versus “vacuum”); “hyperlocal” conservation laws (covariant divergence) for any quantity; volume element, and so integrals, and so integral conservation laws (in the presence of symmetries) for spacetime regions of any dimension; variational principles; convergence and divergence of geodesic congruences (Raychaudhuri equation), and so trapping surfaces (generalized black holes)
8. Einstein field equation: fixed relation between properties of ponderable matter and spacetime geometry; initial value formulation and dynamics

Now, granting the interest of the list for the sake of argument, where, if at all, should one place energy conditions on it? No matter what else is the case, so long as definitional dependence (what one needs in place already to define or characterize structure of a particular sort) is one criterion used in ordering such a list, it seems that energy conditions, in their standard forms, must be not so fundamental as differential structure: one needs differential structure in order to write down any tensor, and so *a fortiori* to write down a stress-energy tensor. Because all the standard energy conditions (and pretty much all the nonstandard ones), rely on the distinction between causal and noncausal vectors in general, and often on the distinction between null and timelike, it seems likely that energy conditions will be less fundamental than conformal structure as well. Energy conditions, however, do not seem to require a notion of temporal orientability, as the discussion of §4 strongly suggests, and, except

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<sup>106</sup>This does not presuppose that an “event” is a purely local entity, in any relevant sense of “local”.

for the impressionist conditions, neither do they require a projective structure. They also seem to be largely independent of topological structure (except in so far as it is required to define differential structure). The impressionist energy conditions do require an affine structure (for the definition of a line-integral along a geodesic), but since they have much murkier physical significance and far fewer important applications than the pointilliste ones, I would almost certainly prefer to forego them before foregoing an affine structure.

Now, if one accepts my ordering, or anything close to it, energy conditions do not seem to fit anywhere neatly in it. So what can we conclude? One possibility is that energy conditions are not clearly a part of any broad conception of what a spacetime theory is, and thus, perhaps, are not themselves of fundamental importance in the study of the foundations of spacetime theories. Alternatively, one could choose to take the fact that energy conditions seem to fit nowhere neatly in the list as a reason to change my groupings of structure into levels or to change my proposed order of levels. All of these considerations are complicated by the fact that the geometrical and physical forms of the energy conditions are equivalent if one assumes the Einstein field equation, but, if one does not, as in most if not all alternative theories of spacetime, all bets are off. In such cases, should one hold on to the geometrical formulations, to try to ensure that one will still be able to derive the consequences listed in §3.1? Or should one hold on to the physical formulations, so as to be able to investigate possible violations of the sort listed in §3.2?

One reason to think they should form part of any broad conception of what constitutes a spacetime theory, irrespective of which formulation one wants to hold on to, rests on the remark of Geroch and Horowitz [92] I quoted on page 49, that without energy conditions the Einstein field equation “has no content.” The conditions one needs to impose to make the initial value problem of general relativity merely consistent—the so-called Gauss-Codazzi constraints—look very much like conditions on the allowed forms of types of matter. So does the fact that the standard proofs showing existence and uniqueness of solutions to the initial value problem of general relativity require matter fields that yield quasilinear, hyperbolic equations of motion satisfying the DEC throughout all of spacetime (Hawking and Ellis [107, ch. 7, §7, pp. 254–5]; Wald [203, ch. 10, especially pp. 250 and 266–7]). This fact seems to place a constraint on spacetime theories—only theories that require nontrivial input about the nature of matter in order for the distribution of matter to constrain the geometry of spacetime ought to be counted as physically reasonable, at least if we want to try to hold on to the idea that a viable spacetime theory ought to support a cogent notion of dynamical evolution, and thus (at a minimum) ought to admit a well set initial value formulation.<sup>107</sup>

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<sup>107</sup>One ought to keep in mind, of course, that we already know the DEC is not necessary for a well defined initial value formulation, as the arguments of Geroch [91] show. What is at issue here is whether the DEC is necessary for the initial value formulation to be well set in the sense of Hadamard—whether, that is, we can show not only existence and uniqueness of solutions, but also stability under small perturbations. There is some evidence that solutions to the initial value problem for some particular types of matter fields violating the NEC will be unstable [59, 200], but the arguments are murky and often hand-waving.

One can try to make this idea precise, and at the same time to capture the kernel of Geroch and Horowitz’s remark, in the following way. First, note that globally hyperbolic spacetimes represent in a natural way possible solutions to the initial value problem of general relativity as it is normally posed.<sup>108</sup> Now, it is a trivial matter to find globally hyperbolic spacetimes that violate any energy condition. Proof: pick your favorite globally hyperbolic spacetime and some open set in it; from the formulæ in Wald [203, Appendix D], it follows that one can always find a conformal transformation of the metric that is the identity outside the open set and nontrivial inside such that at some point in the set the transformed stress-energy tensor will yield whatever one wants on contraction with a timelike or null vector; since conformal transformations preserve causal structure, the transformed spacetime is still globally hyperbolic.

Now, this fact poses a serious problem for any attempt to formulate a notion of dynamical evolution that would support any minimal notion of predictability or determinism. Fix a Cauchy surface in the original spacetime to the past of the open set one conformally jiggered in the proof I sketched. Take that Cauchy surface as initial data for the initial value problem of general relativity. Which spacetime will the Cauchy development off that Cauchy surface (the solution to the initial value problem with that initial data) yield? The original one? One of the conformally jiggered ones? Another one entirely? If one cannot give principled reasons for why exactly one of those spacetimes and no other is the natural result of dynamical evolution off the Cauchy surface according to the Einstein field equation, then one has captured one sense in which the Einstein field equation may “have no content.”<sup>109</sup> The fact that the only known proof of the theorem that a given globally hyperbolic extension of a spacetime is the maximal such extension requires the WEC [163], in conjunction with the fact that the assumptions of standard proofs of the well posedness of the initial value formulation for general relativity imply the DEC throughout the entirety of the derived spacetime [203, ch. 11], suggest that it may be the energy conditions that intervene to ensure a cogent notion of dynamical evolution that supports some minimal notion of predictability or determinism.<sup>110</sup>

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<sup>108</sup>But see, *e.g.*, Ringström [163] for a discussion of the formidable subtleties and complexities involved in trying to make even this seemingly simple idea precise.

<sup>109</sup>This is *not* the infamous Hole Argument [63], nor is it in any way related to it, as conformal transformations are not in general associated with diffeomorphisms.

<sup>110</sup>One may want to object that, inside the region where the conformal transformation is nontrivial, one has actually changed the stress-energy tensor in such a way that what type of matter now is there is not the same as it was before, and so must obey different field equations; thus, the requirement that the same field equations apply throughout the evolution suffices to guarantee uniqueness. I, however, find the notion of “same field equations” to be, in our current state of understanding, hopelessly ambiguous. It is a highly nontrivial matter to ascertain whether or not some matter field obeys the “same field equations” in different spacetimes, or even in different regions of the same spacetime. Say that the field couples to the scalar curvature, but it so happens that in the spacetime at issue the scalar curvature vanishes everywhere. After the conformal transformation, the scalar curvature may no longer vanish in the region where the conformal transformation is nontrivial, and so the field equations will look as though they have “changed form” when passing from one region to the other.

Holding on to everything in my list except for the Einstein field equation, so long as whatever field equations do hold depend only on something like the stress-energy tensor that does not depend on idiosyncratic features of particular kinds of matter, I strongly suspect that one will likely face the same problem. Thus, once again, we seem pushed toward the view that energy conditions play some fundamental role or other in any reasonably broad conception of spacetime theories or at least any such conception that would include a cogent notion of dynamical evolution.

If one does think energy conditions belong as a part of any reasonably broad conception of what constitutes spacetime theory, one tempting way to try to capture the sense in which they may hold at a level of structure deeper than the Einstein Field Equation invokes the thermodynamical character of stress-energy: all stress-energy is fungible, is interchangeable, in the strong sense that the form it takes (electromagnetic, viscoelastic, thermal, *etc.*), and so *a fortiori* any property or quality it may have idiosyncratic to that form, is irrelevant to its gravitational effects, both locally and globally. This is not a conclusion that follows by logical consequence from the observation that qualitative energy conditions suffice to prove theorems of great depth and strength about global structure. It is only one that is strongly suggested by what thermodynamics tells us about the nature of energy. I will not be able to discuss this idea further in this paper, however, as it would take us too far afield.<sup>111</sup>

The inability to derive the energy conditions from other propositions of a fundamental character constitutes an essential part of what pushes one to conceive of them as structure “at a deeper level” than many other elements on the list, perhaps even deeper than causality conditions (many of which can be derived from other fundamental assumptions), and so applicable across a *very* wide range of possible theories of spacetime. If, in the end, one does hold the view that they ought to be thought of as a fundamental part of a reasonably broad conception of what constitutes a spacetime theory, then perhaps, as I suggested in §2.1, the final lesson here is that the geometric form of the energy conditions are the ones to be thought of as fundamental, in so far as they rely for their statement and interpretation only on invariant, geometrical structures and concepts, whose consequences will hold irrespective of the exact field equation assumed by the given spacetime theory. If that is so, then perhaps one

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<sup>111</sup>In one of the first papers in which he tried to provide a fundamental derivation of the field equation bearing his name, Einstein [69, pp. 148–9] explicitly used a similar line of thought to motivate the idea that all gravitationally relevant mass-energetic quantities associated with matter of any kind is exhaustively captured by the stress-energy tensor:

The special theory of relativity has led to the conclusion that inert mass is nothing more or less than energy, which finds its complete mathematical expression in a symmetrical tensor of second rank, the energy tensor. Thus in the general theory of relativity we must introduce a corresponding energy tensor of matter  $T^\alpha_\sigma \dots$ . It must be admitted that this introduction of the energy tensor of matter is not justified by the relativity postulate alone. For this reason we have here deduced it from the requirement that the energy of the gravitational field shall act gravitatively in the same way as any other kind of energy.

potentially fruitful way to use the (poorly named?) energy conditions as a constraint on the construction of spacetime theories is to search for theories in which these important geometric conditions have unproblematic, physically significant interpretations.

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