Supersymmetry and Polytopes

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Abstract

We make an imaginative comparison between the Minimal Supersymmetric Standard Model and the 24-cell polytope in four dimensions, the Octacube.

1 Introduction

The Standard Model (SM) of particles and forces has a natural extension incorporating Supersymmetry, in particular there is the Minimal Supersymmetric Standard Model (MSSM): this scheme requires 128 boson and 128 fermion states in two diffferent sets, the ordinary particles and the Susy partners. In this report we shall refer mainly to the three forces of the microworld, excluding gravitation, the graviton, etc. However, some references to gravity would be unavoidable, mainly in relation to extant Supergravity models. No one of the Susy partners has been seen so far, and we still lack 5 Higgs bosons to complete the ordinary set of particles; these are hopefully to be found soon (at least one Higgs!) in the LHC accelerator, scheduled to start in late 2008.

The question arises whether there is any geometry behind this sharing of particles and partners, so the 2^8 states would reflect some hidden pattern of symmetry in Nature. In some proposed models this symmetry is already apparent. For example, in the old (1978) 11-dimensional SuperGravity theory of Cremmer, Julia and Scherk [1], the supermultiplet $(44, 128, 84,\ \text{ or } 44 - 128 + 84)$, explicitly

$$\text{Graviton } h(44) - \text{Gravitino } \Psi(128) + \text{ 3 - form C}(84), \tag{1}$$

has as underlying geometry the Moufang octonionic plane OP^2, in the following sense [2]: as a symmetric space, it is $OP^2 = F_4/B_4$ where F_4 represents the compact form of

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the second exceptional simple Lie group, and B_4 is represented by the spin group in nine dimensions, Spin(9); as the Euler number $\chi(OP^2) = 3$, there are several triplets of representations of Spin(9) with matching dimensions, one of them corresponding to the above splitting (1). We use Cartan´s notation in which B_n stands for the Spin$(2n+1)$ group as compact connected and simply connected representative. For the general philosophy of constructing these "Euler multiplets" and other examples see [3].

However, the connection of this 11d supermultiplet with the particles we see in Nature in our 4 dimension spacetime is very remote, to say the least; to begin with, it is a supergravity multiplet, whereas we aim first to understand the non-gravitation forces. But the number of required states ($128 = 2^7$) is not far from the observed helicity states of known particles (123), and this is the inspiration for the considerations which follow.

A few remarks on the geometry underlying (1) are in order. Pengpan and Ramond [4] found several of the triplets referred to above, but these really come from the SO(16) group, where $16 = \dim OP^2$ and dim Spin(16) $= 2^{16/2-1} = 128$, and it turns out that under the reduction Spin(9) $\subset$ SO(16), we have $128 - 128 = (44 + 84) - 128$, that is, our field content in (1). A further remark, anticipated in [1], is the orthosymplectic group OSp(1|32) as invariance supergroup of the action related to (1).

2 The MSSM multiplets

The simplest supersymmetry is perhaps the pattern $\{(8, 8) \equiv (8 - 8)\}$ of vector 8_v and (one of the two) spinor 8_s representations of $O(8)$, part of triality, corresponding (in the Ramond-Kostant language of [2]) to the sphere $S^8 = OP^1 =$ Spin(9)/Spin(8); indeed, by squaring this doublet $|8_v - 8_s|^2$ we get the 11d triplet of above, once we ascend to 11 dimensions and M-Theory (the 8 dimensions here are just the transverse dimensions of the 10d superstring theory, of course). It is remarkable that squaring the square, say $|8_v - 8_s|^4$, and ascending to $12 = (10, 2)$ dimensions, it gives a putative matter content for F-Theory [5].

The particles of the Standard Model are: spin-1 "gaugeons", carriers of the three microscopic forces, spin-1/2 fermions (divided into quarks and leptons), and putative spin-0 Higgses, only 3 "seen" at the moment in the form of longitudinal degrees of the carriers of the weak force. Do we see any symmetry pattern in the masses and groupings of the corresponding helicity states? Yes, perhaps. We have, for *gaugeons* 24 degrees of freedom:

$$\sharp\text{spin - 1 states} = 2{\cdot}(\sharp SU(3) + \sharp SU(2) + \sharp U(1)) = 2{\cdot}(8 + 3 + 1) = 24 \qquad (2)$$

Of course, 24 is a dear number to mathematicians (if only for Dedekind's η-function or the Leech lattice), and to string theorist (if only for the bosonic string). This numbering counts the three massive gaugeons as massless (and hence counting 4, not 1; or 8, not 5 Higgses, later). One can also entertain 27 helicity states for gaugeons and one minimal

Higgs; indeed, 27 is also a distinguished number (fundamental representation of E_6, and some curve intersections in algebraic geometry).

Of course, superstrings include gravitation and live in $10 = 8+2 = (9,1)$ dimensions; so we cite here more reasons why the number 24 is favoured, besides the bosonic string. (a): in R^4, the maximum number of identical spheres touching each other (the *kissing* number) is 24 [6]. (b) The Leech lattice, just mentioned, optimizes the sphere packing in dimension 24, and links up also with another important mathematical construct, the Monster group (or largest sporadic finite simple group). The relation is this: the kissing number for the Leech lattice is 196 560. Now there are *four* classes of sporadic (i.e. non-generic) finite simple groups; besides de " pariah" class, there are three consecutive levels: the Mathieu-group(s) level M_{24}, related to the exceptional automorphism of the symmetric group S_6; the Conway-group(s)level, associated to the automorphism group of the Leech lattice, and the Monster-group level. The Monster group itself, of order 8×10^{53} approximately, is constructed (Lepowski; Borcherds) with vertex operators from string theory; it first faithful irreducible representation (*irrep*) has dimension 196 883, number tantalizing close to the kissing number of the Leech lattice. It presents also the "Moonshine phenomenon": simple combinations of the dimensions of the *irreps* give the coeficients of a very important modular function, the so-called $J(\tau)$ function. See [7].

For *fermions* we have leptons and quarks in three generations:

$$\sharp\text{spin - }1/2\text{ states} = \text{leptons} + \text{quarks} = 4{\cdot}(2\times 3) + 3{\cdot}4{\cdot}(2\times 3) = 24 + 72 = 96 \tag{3}$$

where (3) is for generations, (2) for isospin, 4$\cdot$ for massive Dirac states and 3$\cdot$ for color. Again, this supposes three sets of massive Dirac neutrinos, what still is to be confirmed experimentally (as the absence of neutrinoless double beta decay, for example), which is for the moment uncertain.

We shall see below what to do with the big number, $96 = 24{\cdot}(1+3)$.

In the SM it is enough a single complex doublet of Higgses, but we need *two* doublets in the MSSM, even in the "ordinary" sector, because (among other reasons) up- and down-type quarks couple differently. Accepting this, we have

$$\sharp\text{spin - }0\text{ states} = \text{Higgses in the MSSM} = 2{\cdot}2{\cdot}2 = 8 \tag{4}$$

where two complex doublets make up the 8. Again, 8 is a special number (if only because the dimension of the last composition algebras!).

Now, in the Susy partners sector, separated from the ordinary one by the so-called $R-$symmetry, and with Susy broken in order to prevent unseen mass coincidences between ordinary and Susy particles, we have to have

Spin-1/2 gauginos (24), Spin-1/2 Higgsinos (8); Spin-0 squarks (72) and sleptons (24)

So 32 fermions and (72+24 = 96) bosons with R-number, to match the opposites in the ordinary sector. Notice $32 = 2^5$ appears in strings again (the gauge group $O(32)$ in heterotic and open strings); note also the satisfactory feature that the Susy partners do *not* introduce new forces, because there are no more expected spin-1 fields.

3 The Polytopes

From the above counting the reader should keep mainly the numbers 24 and $96 = 24\cdot(1+3)$ in mind. We seek for some *discrete exceptional* mathematical objects where these numbers would appear. Discreteness is obvious, and exceptional because we subscribe to the philosophy that the mathematical model of Nature is likely to be exceptional, not generic, as WE are unique(!).

Polytopes are generalization of 2d polygons and 3d polyhedra to higher dimensions; they were systematically investigated first by Schläfli around 1850, and are thoroughly studied in the book of Coxeter [8]. Starting with the triangle T_2 and the square H_2 in the plane R^2, there is a straightforward generalization in arbitrary dimension n to the *regular generic* n-polytopes lying in R^n: hyper-tetrahedron T_n, which is self-dual (=palindromic in the arrangement of vertices, edges, ..., cells), and hypercube H_n, with dual hyper-octahedron H_n^*. In *even* dimensions $2n$, their simplices (vertices V, edges A, faces F,..., cells, etc.) make up a "supersymmetric" *alternate* sum (to zero) because the Euler number of odd spheres S^{2n-1} is zero; for example

$$\text{in } 2d: \; H_2 \text{ is } (4-4); \quad \text{in } 6d: \; T_6 \text{ is } (7-21+35-35+21-7) \tag{5}$$

If one includes "1" for the vacuum (with dimension -1!) and another "1" for wholeness (the solid body), there is also "supersymmetry" for n odd, of course, as (+2) is the Euler number for even spheres; for instance

$$3d: H_3^* \; (1-6+12-8+1); \quad 5d: T_5 \; (1-6+15-20+15-6+1). \tag{6}$$

This counting works because projecting the simplices to the circumsphere of the polytope tessellates the sphere.

Besides these *generic* regular polytopes there a few *exceptional* regular polytopes: everybody knows of polygons of p sides, with p any integer > 2, with "Susy" of type ($p-p$ or $1-p+p-1$). The Greeks constructed the icosahedron Y $(12, 30, 20)$ and -with more effort- its dual dodecahedron Y^* $(20, 30, 12)$; and Schläfli determined that there are *only* a few more exceptions, all in dimension four; see also [9]. Today we understand all these exceptions as related to the complex numbers (dimension 2) and to the quaternion numbers (dim 4, descending to 3). For example, p-sided regular polygons lying in R^2 are related to the group $SO(2) = U(1)$ being abelian and divisible (injective in the category of abelian groups); see [10], [12].

For example, one can inscribe a regular polygon in a circle $S^1 = U(1)$, with rotation symmetry Z_n; now $U(1)/Z_n \approx U(1)$, as $U(1)$ is *injective* in the category of abelian groups (or Z-modules), in the same way that Z is *projective*; for these elementary notions of homological algebra consult [11]. For an alternative viewpoint, in which the division (composition) algebras come first, see [12].

Clearly the regular polytopes Π_n have a center, and two related groups: a rotation symmetry group $\mathrm{Rot}(\Pi_n) \subset SO(n)$ and an isometry group $\mathrm{Iso}(\Pi_n) \subset O(n)$, where the index $[O : SO] = 2$; for example $\mathrm{Iso}(T_n) = S_{n+1}$, and $\mathrm{Rot}(T_n) = A_{n+1}$, the alternating subgroup of the symmetric group. Examples for other polytopes are (where $\prec$ means semidirect product)

$$\mathrm{Rot}(H_3) = (Z_2 \times Z_2) \prec S_3. \;\; \mathrm{Rot}(Y = Y_3) = A_5. \;\; \sharp Iso(H_n) = 2^n \times n! \qquad (7)$$

Coxeter explains wonderfully the concept of *truncation*, which produces some intermediate, quasiregular polytopes by "cutting corners"; for example, the ordinary cube H_3 and the ordinary octahedron H_3^* are dual of each other; so starting e.g. from the cube we get an hybrid, named *cubeoctahedron* H^*H_3, with counting $(12, 24, 14)$, which is a quasiregular polyhedron with 6 squares and 8 triangles as faces, known from antiquity. The process generalizes to arbitrary dimensions and polytopes.

4 The MSSM and the "Octacube"

We focus here in the ONLY case in which the cube-octahedron mixing of above produces a *regular* polytope, the so-called *24-cell* or (3,4,3) in Schläfli (p, q, r) notation; its Coxeter diagram is that of the Lie group F_4; A. Ocneanu [13] calls it Octacube, living in 4 dimensions, which we shall write H^*H_4. The reason why is a regular polytope is related (besides its origin in the quaternions) to the fact that the distance from the centre of the hypercube H_4 to the vertices is the length of the edges, as $\sqrt{1+1+1+1} = 2$. The 4-cube H_4 is (16, 32, 24, 8) and the 4-octahedron H_4^* is the dual; the hybrid H^*H_4 becomes (24, 96, 96, 24): is regular and selfdual! It is the 24-cell, and beautiful projections of it to 3 and 2 dimensions are drawn in [8]; see also [13].

The automorphism (isometry) group of the 24-cell is the Weyl group of the Dynkin diagram for the exceptional Lie group F_4 (again!), of order $1152 = 384 \cdot 3$, where $384 = 2^4 \cdot 4! =$ order of Automorphism group of H_4; notice $\sharp\mathrm{Rot}(24\text{ - cell}) = 1152/2 = 576 = 24^2$, where $24 = \sharp\mathrm{Rot}(H_3)$, as $\mathrm{Rot}(H_3) = S_4$; this is a reminder that Spin(4) =Spin(3) $\times$ Spin(3), [10]. This enhanced symmetry, the factor of 3, in the H^*H_4 with respect to H_4, does *not* occur in the other cubeoctahedra, and it will be nice if it could be related, through triality, to the number of generations in particle physics!

Now 96 and 24 are the same numbers as fermions (96) and gaugeons (24) in Nature. Even repeated, as required for the R-sector! Is this coincidental? Perhaps, but let us play the game:

If "24" correspond to the gaugeons and "96"$= 24 + 24 \cdot 3$ to leptons and quarks, what about the Higgses? We venture to associate them to the two "1" missing in the whole Susy pattern of the 24-cell $(1 - 24 + 96 - 96 + 24 - 1)$, *except* that the "1" must be "8": we have 8 Higgs and 8 Higgsinos; perhaps the mismatch 1 vs. 8 has something to do with octonions, but granted, this is a point we lack understanding. Accepting that suggestion, however, the pattern of the 256 expected helicity states in the MSSM would look like (with $\pm$ for Bose/Fermi)

-1(8)	+24	-96	+96	-24	+1(8)
$\tilde{\mathrm{H}}$	g	$\mathrm{q} + \ell$	$\tilde{\mathrm{q}} + \tilde{\ell}$	$\tilde{\mathrm{g}}$	H
Higgsinos Spin 1/2	Gaugeons Spin 1	Quarks & Leptons Spin 1/2	sQuarks & sLeptons Spin 0	Gauginos Spin 1/2	Higgs Spin 0

Table 1: Supposed correspondence of the MSSM with the 24-cell.

That is our correspondence; it goes without saying this pure speculative scheme is meant only to stimulate further thoughts and works; in particular, it does not help at all to understand the pattern of masses we see. The idea is only that the same type of "palindromic" symmetry of the 24-cell polytope is present in the spectrum of elementary particles in the MSSM; we are not yet aiming at the deeper reasons for that.

We stress finally the four features we find fairly unique in our suggestion: first, the main numbers 24 and 96 are present naturally; second, the repeated pattern (twice of each) approaches duplication between particles and their Susy partners. Third, the 24-cell polytope is absolutely unique, rather than only exceptional. Fourth, there is a hint for family triplication, as the group F_4, with 3-Torsion, is related to octonion triality; for example, in the "Mercedes" Dynkin diagram for $O(8)$ triality is manifest; adding to Spin(8) the three equivalent representations, we reach F_4; [14].

After the first draft of the paper was sent off, we became aware of the (now famous) preprint of G. Lisi [15]. In fact, we subscribe unconsciously to the Pati-Salam " leptons as the fourth color" philosophy, as Lisi does; he also uses polytopes and the F_4 group, and hints to a relation between triality and generations. He is more ambitious, though, as he considers gravitation as well, but does not adhere to supersymmetry.

5 Concluding Remarks

We are fully aware of the incompleteness of our approach. As said, gravitation has been deliberately left over in this essay. Also, as we believe octonions should play a role in the "final theory" (see e.g. [16]), the preliminary and provisional aspects of our considerations should be evident: the largest exceptional group E_8 appears conspicuously in theoretical constructions (e.g. in the heterotic string by duplicate, as the gauge group in M-Theory [17], etc.); of course, it plays a major role in Lisi theory [15]. In fact, octonions spring from the triality of the Spin(8) group, although this phenomenon does not lead to new regular polytopes but to some very special lattices (Gosset, 1897). So we feel the two omissions (gravitation and octonions, in particular E_8) should go together. As $E_6 \subset E_8$ naturally, is worth to recall that F_4 is the subgroup of E_6 fixed by the involutory automorphism of it, a kind of complex conjugation. In fact, there is a whole chain of groups/subgroups from E_8 to $SU(2)$ which includes triality, duality, automorphisms, etc. [18].

As a final trait of our incompleteness, we mention that the primes 2 and 3 enter through duality and triality in e.g. $SU(3)$, F_4 or E_6 as center, conjugation or torsion. The prime 5 appears only in E_8 as torsion, and the possible relevance of this "5" for physics escapes totally from us. In this context, see [19].

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[9] R. C. Lyndon, *Groups and Geometry.* Cambridge U.P. 1985, Ch.5.

[10] L. J. Boya, in preparation. The real peculiarity is that $SO(2)$ is abelian, that $SO(4)$ has a split Lie algebra, and that the covering of $SO(8)$ exhibits triality, that is, the symmetry group of the center lifts to an outer automorphism group of the whole Spin(8) group. From this one concludes several facts, in particular the existence of the division algebras $C, H$and O, and also the exceptional regular polytopes and some others not regular (in dim 8, descending to $7, 6$). For an alternative point of view, see [12].

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[12] J. H. Conway and D. Smith, *On Quaternions and Octonions.* A.K. Peters 2003.

[13] A. Ocneanu designed in 2005 a "3d projection" of the Octacube for a statue in Pennsylvania State University as a reminder of the fundamentalists attack on Sept. 11, 2001. He is a renowed mathematician, publishing few of his results. This information is from the Maths News Archive in InterNet.

[14] It is curious that the Coxeter-Dynkin diagram for F_4 is not simply laced, whereas one associates the "nicest" Dynkin diagrams to the simple-laced ADE series; but, as we say in Spain: "Lo mejor es enemigo de lo bueno".

[15] G. Lisi, *An Exceptionally Simple Theory of Everything*, arXiv 0711.0770, 6-XI-2007. For a careful criticism of this paper see the blog pages of J. Distler, "Musings" (Austin, TX). I thank Jacques for explaining his arguments to me (July, 2008).

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[17] D. Freed *et al., The M-Theory 3-form and the E_8 gauge theory.* arXiv: hep-th/03 012 069.

[18] In fact, there is a full chain of groups from E_8 to $A_1 = SU(2)$: e.g. from F_4 we pass to $O(8)$ as $3 \cdot 8 = 28 - 4$ and F_4 has $2 \cdot 24 + 4$ parameters, with 4 the rank. Now G_2 (dim 14) comes from Spin(8)/Aut, and again dim $G_2 = 2 \cdot 6 + 2$ with 2 the rank, so $SU(3)$ appears. Finally $SU(3)/\mathrm{Aut} = SO(3) = A_1$. I thank A. King (Bath, U.K.) for some of the above correspondences.

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