DISTRIBUTION OF SEQUENCES RELATED TO L-FUNCTIONS

BY

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DISSERTATION

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Abstract

This thesis consists of three projects.

The first project focuses on the distribution of zeros of linear combinations of derivatives of L-functions. We consider a collection of such combinations and prove asymptotic formulas for the supremum of the real parts of their zeros. Moreover, an investigation of an inverse-type question related to the case of the Riemann zeta function is included.

In the second part of this thesis, we expand the class of Dirichlet series whose monotonicity properties are known. In particular, we describe a large class of Dirichlet series that are not logarithmically completely monotonic. Using similar techniques, an equivalent formulation of the Riemann Hypothesis for the Ramanujan-tau L-function is provided.

The last project is related to walks to infinity. Our main object is the subset P of the complex plane that includes all the primes of all rings of integers of all imaginary quadratic fields. One would want to know if it is possible to walk to infinity stepping only on points in P and such that the sequence of lengths of steps used in the process is bounded. However, the problem is surprisingly connected to some famous and notoriously difficult unsolved problems. We study more general walks on the set P, where the length of the steps is not forced to be bounded throughout the walk.

To my family, for all their love and support.

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Contents

List of Abbreviations

RH	Riemann Hypothesis.
CM	Completely Monotonic.
LCM	Logarithmically Completely Monotonic.

List of Symbols

$\zeta(s)$	The Riemann zeta function.
$\Gamma(s)$	The Gamma function.
$\phi(n)$	The Euler totient function.
$\mu(n)$	The Mobius function.
$\operatorname{sgn}(x)$	The signum function.
(m, n)	The greatest common divisor of m and n.
$[m, n]$	The least common multiple of m and n.
$\lceil x \rceil$	The smallest integer greater than or equal to x.
$\{x\}$	The fractional part or x.
$f(x) = \mathcal{O}(g(x))$	There exists a positive constant C such that $\lvert f(x)\rvert \leq Cg(x)$ for relevant values of x. The notation $f(x) = \mathcal{O}_\lambda(g(x))$ indicates that the implicit constants may depend on the parameter λ.
$f(x) \ll g(x)$	Same as $f(x) = \mathcal{O}(g(x))$. The notation $f(x) \ll_\lambda g(x)$ is the same as $f(x) = \mathcal{O}_\lambda(g(x))$.
$\lVert\vec{c}\rVert_\infty$	$\max\{\lvert c_1\rvert, \ldots, \lvert c_n\rvert\}$, where $\vec{c} = (c_1, \ldots, c_n)$.
$D(c, R)$	Disk centered at c with radius R.
$\langle\alpha\rangle$	The principal ideal generated by α.

Chapter 1

Linear combinations of ζ and its derivatives and vertical distribution of their zeros

1.1 Introduction

The distribution of zeros of the Riemann zeta-function and that of its derivatives appear to be closely related. One of the most well-known results around this observation was proved by Speiser [Spe35] who showed that the Riemann Hypothesis (RH) is equivalent to $\zeta'(s)$ having no zeros in $0 < \operatorname{Re} s < \frac{1}{2}$. Assuming RH, Levinson and Montgomery [LM74a] established that for $k \geq 1$, $\zeta^{(k)}(s)$ has at most a finite number of complex zeros for $\operatorname{Re} s < \frac{1}{2}$, and it was proved by Yildirim [Yıl00] that under RH, $\zeta''(s)$ and $\zeta'''(s)$ have no zeros in the strip $0 \leq \operatorname{Re} s < \frac{1}{2}$. Spira [Spi65a], [Spi70] studied the zero-free regions of the derivatives of $\zeta(s)$. In [Spi65a], he showed that if $k \geq 3$, then $\zeta^{(k)}(s) \neq 0$ for $\operatorname{Re} s \geq \frac{7}{4}k + 2$. In [Spi70] Spira proved that there is a half-plane free of complex zeros on the left as well, that is, there exists a number α_k such that $\zeta^{(k)}(s)$ has only real zeros for $\operatorname{Re} s \leq \alpha_k$. Verma and Kaur[VK82] revised Spira's first result showing that if $k \geq 3$ then $\zeta^{(k)}(s)$ has no zeros for $\operatorname{Re} s \geq (1.13588...)k + 2$. There are also some new developments about the location of the zeros of the derivatives of $\zeta(s)$ by Binder, Farr, Pauli, Saidak (see [FP13] and [BPS10]).

Another direction of study is finding asymptotic formulas for the number of zeros up to a certain height T. Let $N(T)$ and $N_k(T)$ denote respectively, the number of zeros of $\zeta(s)$ and $\zeta^{(k)}(s)$ with imaginary part between 0 and T. The first estimate of $N(T)$ goes back to Riemann. He stated in 1859 that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T),$$

which was proved by von-Mangoldt in 1905. Spira [Spi65a] made a strong conjecture about $N_k(T)$. The formula he suggested is a relation between $N(T)$ and $N_k(T)$ without any error terms:

$$N(T) = N_k(T) + \left[\frac{T\log 2}{2\pi}\right] \pm 1.$$

Berndt [Ber70] attacked this problem and obtained a relationship between $N(T)$ and $N_k(T)$ that includes an error term:

$$N(T) = N_k(T) + \frac{T\log 2}{2\pi} + \mathcal{O}(\log T).$$

In this chapter, we will study the distribution of zeros of a combination of the Riemann zeta-function and its derivatives. We denote $F(s) = c_0\zeta(s) + c_1\zeta'(s) + \cdots + c_k\zeta^{(k)}(s)$ with $c_0, c_1, \ldots, c_n$ real, $c_0 \neq 0$ and $c_k \neq 0$. Here $s = \sigma + it$ as usual and since we are interested in the zeros of this function, we take $c_0 = 1$ without loss of generality. We let $N_F(T)$ be the number of zeros of the function F such that $0 < t < T$. The main result of this chapter is an asymptotic formula for $N_F(T)$, given by the following theorem.

Theorem 1.1. *For any function F defined as above we have*

$$N_F(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \mathcal{O}_F(\log T).$$

1.2 Zero Free Regions

We examine the function $F(s) = \zeta(s) + c_1\zeta'(s) + \cdots + c_k\zeta^{(k)}(s)$ with $c_k \neq 0$. For $\sigma > 1$, the m-th derivative of the Riemann zeta-function is given by

$$\zeta^{(m)}(s) = (-1)^m \sum_{n=2}^{\infty} \frac{\log^m n}{n^s},$$

hence for such σ, $F(s)$ can be written as

$$F(s) = 1 + \sum_{n=2}^{\infty} \frac{1 - c_1 \log n + c_2 \log^2 n + \cdots + (-1)^k c_k \log^k n}{n^s}.$$

From the general theory of Dirichlet series it follows that $F(s)$ has a right half plane free of zeros, specifically there exists a number β_F such that $F(s) \neq 0$ for $\sigma > \beta_F$. In the next proposition we exhibit a rough value of such β_F.

Proposition 1.2 (Right zero-free regions)**.** *Let $c = \max_{j=0,\ldots,k} |c_j|$, where $c_0 = 1$. If $\sigma > \beta_F = k + 2 + \dfrac{\log(c(k+1))}{\log 2}$, then $F(s) \neq 0$.*

Proof. We rewrite $F(s)$ as

$$F(s) = 1 + \frac{1 - c_1 \log 2 + \cdots + (-1)^k c_k \log^k 2}{2^s} + \sum_{n=3}^{\infty} \frac{1 - c_1 \log n + \cdots + (-1)^k c_k \log^k n}{n^s}. \tag{1.1}$$

Since $\log^j 2 < 1$ for any j, and $\sigma > k + 2 + \dfrac{\log(c(k+1))}{\log 2}$, then

$$\left| \frac{1 - c_1 \log 2 + \cdots + (-1)^k c_k \log^k 2}{2^s} \right| \leq \frac{(k+1)c}{2^\sigma} < \frac{1}{2^{k+2}}. \tag{1.2}$$

On the other hand, $\log^j n > 1$ for $n \geq 3$, so

$$\begin{aligned} \left| \sum_{n=3}^{\infty} \frac{1 - c_1 \log n + \cdots + (-1)^k c_k \log^k n}{n^s} \right| &\leq \sum_{n=3}^{\infty} \frac{(k+1) c \log^k n}{n^\sigma} \\ &\leq (k+1)c \sum_{n=3}^{\infty} \frac{n^k}{n^\sigma} = (k+1)c \sum_{n=3}^{\infty} \frac{1}{n^{\sigma-k}} \leq (k+1)c \int_2^\infty \frac{1}{x^{\sigma-k}} dx \\ &= (k+1)c \frac{1}{(\sigma - k - 1) 2^{\sigma-k-1}} \leq \frac{(k+1)c}{2^{\sigma-k-1}} < \frac{1}{2}, \end{aligned} \tag{1.3}$$

where in the last step the inequality $\sigma > k + 1 + \dfrac{\log(2c(k+1))}{\log 2}$ is used. Finally, taking (1.1), (1.2) and (1.3) together we see that

$$\begin{aligned}
|F(s)| &= \left|1 + \frac{1 - c_1 \log 2 + \cdots + c_k(-1)^k \log^k 2}{2^s} + \sum_{n=3}^{\infty} \frac{1 - c_1 \log n + \cdots + (-1)^k c_k \log^k n}{n^s}\right| \\
&\geq 1 - \left|\frac{1 - c_1 \log 2 + \cdots + (-1)^k c_k \log^k 2}{2^s}\right| - \left|\sum_{n=3}^{\infty} \frac{1 - c_1 \log n + \cdots + (-1)^k c_k \log^k n}{n^s}\right| \\
&> 1 - \frac{1}{2^{k+2}} - \frac{1}{2} > 0,
\end{aligned}$$

which proves the proposition. □

In the next proposition we show that there is a left half plane where the function $F(s)$ has no non-real zeros under the condition that c_i's are real.

Proposition 1.3 (Left "zero-free" regions). *If all the coefficients c_i in the definition of $F(s)$ are real, then there exists a number α_F so that $F(s)$ has only real zeros for $\sigma < \alpha_F$, and has exactly one real zero in each open interval of the form $(-1-2n, 1-2n)$ for each positive integer n satisfying $1-2n \leq \alpha_F$.*

Remark. *Note that the proposition does not provide any information about the sign of α_F. It will be assumed to be negative in what follows.*

Proof. We start with the asymmetric functional equation for $\zeta(s)$,

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\frac{\pi s}{2} \Gamma(s)\zeta(s).$$

Differentiating this equality m times using Leibniz's rule we find

$$(-1)^m \zeta^{(m)}(1-s) = 2(2\pi)^{-s} \sum_{j=0}^{m} \Gamma^{(j)}(s) R_{jm}(s), \tag{1.4}$$

where

$$R_{jm}(s) = P_{jm}(s) \cos\frac{\pi s}{2} + Q_{jm}(s) \sin\frac{\pi s}{2}, \tag{1.5}$$

$$P_{jm}(s) = \sum_{n=0}^{m} a_{jmn}\zeta^{(n)}(s), \quad P_{mm}(s) = \zeta(s), \tag{1.6}$$

$$Q_{jm}(s) = \sum_{n=0}^{k} b_{jmn}\zeta^{(n)}(s), \quad Q_{mm}(s) = 0, \tag{1.7}$$

for some constants a_{jmn} and b_{jmn}[Spi65a].

For each summand in the combination $\zeta(1-s)+c_1\zeta'(1-s)+\cdots+c_k\zeta^{(k)}(1-s)$ we apply (1.4) and collect the terms involving the same derivatives of Γ to arrive at the following representation for $F(1-s)$:

$$\begin{aligned}
F(1-s) &= \sum_{m=0}^{k} c_m\zeta^{(m)}(1-s) = \sum_{m=0}^{k} c_m(-1)^m 2(2\pi)^{-s}\sum_{j=0}^{m}\Gamma^{(j)}(s)R_{jm}(s)\\
&= 2(2\pi)^{-s}\left[\sum_{m=0}^{k-1} c_m(-1)^m\sum_{j=0}^{m}\Gamma^{(j)}(s)R_{jm}(s) + c_k(-1)^k\sum_{j=0}^{k}\Gamma^{(j)}(s)R_{jm}(s)\right]\\
&= 2(2\pi)^{-s}\left[\sum_{m=0}^{k-1}\sum_{j=0}^{m} c_m(-1)^m\Gamma^{(j)}(s)R_{jm}(s)\right.\\
&\qquad \left. + c_k(-1)^k\sum_{j=0}^{k-1}\Gamma^{(j)}(s)R_{jk}(s) + c_k(-1)^k\Gamma^{(k)}(s)R_{kk}(s)\right]\\
&= 2(2\pi)^{-s}\sum_{j=0}^{k-1}\left[\sum_{m=j}^{k-1} c_m(-1)^m\Gamma^{(j)}(s)R_{jm}(s) + c_k(-1)^k\Gamma^{(j)}(s)R_{jk}(s)\right]\\
&\qquad + 2(2\pi)^{-s}c_k(-1)^k\Gamma^{(k)}(s)\zeta(s)\cos\frac{\pi s}{2}.
\end{aligned}$$

Therefore,

$$F(1-s) = 2(2\pi)^{-s}\sum_{j=0}^{k}\Gamma^{(j)}(s)\tilde{R}_{jk}(s), \tag{1.8}$$

where

$$\tilde{R}_{jk}(s) = c_k(-1)^k R_{jk}(s) + \sum_{m=j}^{k-1} c_m(-1)^m R_{jm}(s), \text{ for } j \le k-1, \tag{1.9}$$

$$\tilde{R}_{kk}(s) = c_k(-1)^k \zeta(s) \cos\frac{\pi s}{2}.$$

We next write

$$(-1)^k F(1-s)(2\pi)^s/2c_k = f(s) + g(s), \tag{1.10}$$

where

$$f(s) = \zeta(s)\Gamma^{(k)}(s)\cos\left(\frac{\pi s}{2}\right),$$

$$g(s) = \frac{(-1)^k}{c_k}\sum_{j=0}^{k-1}\Gamma^{(j)}(s)\tilde{R}_{jk}(s).$$

Next, we apply Rouché's theorem to the square with vertices $2n \pm i$, $2n+2 \pm i$ to deduce that the function $f(s)+g(s)$ has exactly the same number of zeros as $f(s)$ inside that square. We need to show that $|f(s)| > |g(s)|$ on the boundary and the proof of that is almost identical to the one done by Spira in [Spi70]. From [Spi65a] we have

$$\Gamma^{(j)}(s) = \Gamma(s)\left[\log^j s + \sum_{n=0}^{j-1} E_{nj}(s)\log^n s\right], \tag{1.11}$$

where each $E_{nj}(s)$ is $\mathcal{O}(1/s)$, so we can write

$$f(s) = \zeta(s)\cos\frac{\pi s}{2}\Gamma(s)\log^{k-1}(s)\left[\log s + \sum_{n=0}^{k-1} E_{nk}(s)\log^{n+1-k}(s)\right],$$

and by introducing $\tilde{R}^*_{jm}(s) = \dfrac{(-1)^k}{c_k\cos(\pi s/2)}\tilde{R}_{jm}(s)$, we have

$$g(s) = \zeta(s)\cos\frac{\pi s}{2}\Gamma(s)\log^{k-1}(s)\sum_{j=0}^{k-1}\frac{\tilde{R}^*_{jm}(s)}{\zeta(s)}\left[\frac{1}{\log^{k-1-j} s} + \sum_{n=0}^{j-1}\frac{E_{nj}(s)}{\log^{k-1-n}(s)}\right].$$

After applying the triangle inequality we see that we will have $|f(s)| > |g(s)|$ on the boundary provided

$$|\log s| > \sum_{n=0}^{k-1} \left| E_{nk}(s) \log^{n+1-k} s \right| + \left| \sum_{j=0}^{k-1} \frac{\tilde{R}^*_{jk}(s)}{\zeta(s)} \left[\frac{1}{\log^{k-1-j} s} + \sum_{n=0}^{j-1} \frac{E_{nj}(s)}{\log^{k-1-n}(s)} \right] \right|. \tag{1.12}$$

Using (1.9) we can write

$$\begin{aligned}
\tilde{R}^*_{jk}(s) &= \frac{R_{jk}(s)}{\cos(\pi s/2)} + \sum_{m=j}^{k-1} \frac{c_m}{c_k} (-1)^{m+k} \frac{R_{jm}(s)}{\cos(\pi s/2)} \\
&= \left[P_{jk}(s) + Q_{jk}(s) \tan \frac{\pi s}{2} \right] + \sum_{m=j}^{k-1} \frac{c_m}{c_k} (-1)^{m+k} \left[P_{jm}(s) + Q_{jm}(s) \tan \frac{\pi s}{2} \right] \\
&= \tilde{P}^*_{jk}(s) + \tilde{Q}^*_{jk}(s) \tan \frac{\pi s}{2},
\end{aligned}$$

where

$$\tilde{P}^*_{jk}(s) = P_{jk}(s) + \sum_{m=j}^{k-1} \frac{c_m}{c_k} (-1)^{m+k} P_{jm}(s),$$

and

$$\tilde{Q}^*_{jk}(s) = Q_{jk}(s) + \sum_{m=j}^{k-1} \frac{c_m}{c_k} (-1)^{m+k} Q_{jm}(s).$$

From (1.6) and (1.7) it follows that $\tilde{P}^*_{jk}(s)$ and $\tilde{Q}^*_{jk}(s)$ are linear combinations of $\zeta(s)$, $\zeta'(s)$,..., $\zeta^{(k)}(s)$, so their absolute values are bounded for large σ. The same holds for $\tilde{R}^*_{jk}(s)$ since $\tan \frac{\pi s}{2}$ is also bounded on the sides of the square. Lastly, $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(s) n^{-s}$ is also bounded and we get that the right side of (1.12) is bounded, whereas on the left side we have $|\log s| > \log |s|$ which is unbounded. Moreover, (1.11) implies that for σ sufficiently large, $\Gamma^{(j)}(s) \neq 0$, hence $f(s)$ has a single zero in the square due to $\cos(\pi s/2)$. Rouché's theorem then implies that $f(s) + g(s)$ (by (1.10), also $F(1-s)$) has exactly one zero inside the square. Since the coefficients c_i are all real, then the zeros of $F(s)$ occur in conjugate pairs, thus the unique zero must lie on the real axis and the proof is complete. □

1.3 Preliminary results on the zeros of $F(s)$

In this section, we first obtain an infinite product representation for the function $F(s)$, which will in turn lead to an expression for $F'(s)/F(s)$. In order to do this, we define the function $G(s) = (s-1)^{k+1}F(s)$ which is an entire function and, in fact, of order 1 [Dav00]. Therefore, there are constants A_F and B_F, depending only on the function F, such that

$$G(s) = s^m e^{A_F + sB_F} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$$

for all $s \in \mathbb{C}$, where $m \geq 0$ is the order of $s = 0$ as a zero of G and the product is over all the non-zero zeros of G. All the zeros of G are exactly the zeros of F and

$$F(s) = \frac{s^m}{(s-1)^{k+1}} e^{A_F + sB_F} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \tag{1.13}$$

for $s \in \mathbb{C}, s \neq 1$.

From the theory of entire functions of order 1 we know that $\sum\limits_{\rho} |\rho|^{-1-\eta} < \infty$ for any $\eta > 0$, and in particular $\sum\limits_{\rho} |\rho|^{-2} < \infty$.

Logarithmic differentiation of (1.13) yields

$$\frac{F'(s)}{F(s)} = \frac{m}{s} - \frac{k+1}{s-1} + B_F + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{1.14}$$

valid for $s \in \mathbb{C}, s \neq 0, s \neq 1, s \neq \rho$.

On the other hand, the absolute convergence of the above sum allows us to rewrite this as

$$\frac{F'(s)}{F(s)} = \frac{m}{s} - \frac{k+1}{s-1} + B_F + \sum_{\rho \,\text{real}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{\substack{\gamma>0 \\ \rho=\beta+i\gamma}} \left(\frac{1}{s-\rho} + \frac{1}{\rho} + \frac{1}{s-\overline{\rho}} + \frac{1}{\overline{\rho}}\right),$$

while the absolute convergence of each of the series

$$\sum_{\substack{\gamma>0\\ \rho=\beta+i\gamma}} \left(\frac{1}{s-\rho}+\frac{1}{s-\overline{\rho}}\right)$$

and

$$\sum_{\substack{\gamma>0\\ \rho=\beta+i\gamma}} \left(\frac{1}{\rho}+\frac{1}{\overline{\rho}}\right)$$

give us

$$\frac{F'(s)}{F(s)} = \frac{m}{s} - \frac{k+1}{s-1} + \widetilde{B}_F + \sum_{\rho\,\mathrm{real}}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + \sum_{\substack{\gamma>0\\ \rho=\beta+i\gamma}} \left(\frac{1}{s-\rho}+\frac{1}{s-\overline{\rho}}\right) \tag{1.15}$$

for $s \in \mathbb{C}$, $s \neq 0, s \neq 1, s \neq \rho$, where

$$\widetilde{B}_F = B_F + \sum_{\substack{\gamma>0\\ \rho=\beta+i\gamma}} \left(\frac{1}{\rho}+\frac{1}{\overline{\rho}}\right)$$

is a constant independent of s.

Lemma 1.4. *For $s_0 = \beta^* + iT$, where $\beta^* \geq \alpha_F$ and $T \geq 2$, we have*

$$\sum_{\rho\ real}\left|\frac{1}{s_0-\rho}+\frac{1}{\rho}\right| = \mathcal{O}_F(\log T).$$

Proof. It is enough to prove the above estimate for the sum restricted over the zeros $\rho < \alpha_F$, since for $\alpha_F \leq \rho \leq \beta_F$, each summand is bounded by

$$\frac{1}{2} + \frac{1}{\min\{|\rho| : \alpha_F \leq \rho \leq \beta_F, F(\rho) = 0\}},$$

and the number of such ρ is finite.

Now, write

$$\sum_{\rho<\alpha_F}\left|\frac{1}{s_0-\rho}+\frac{1}{\rho}\right| = \sum_{\rho<\alpha_F}\frac{|s_0|}{|\rho||s_0-\rho|} =: S_1+S_2,$$

where S_1 and S_2 are the sums restricted over the zeros $\rho<-2T$ and $-2T\le\rho<\alpha_F$, respectively.

The result follows immediately from the following estimates:

$$S_1 = \sum_{\rho<-2T}\frac{|s_0|}{|\rho||s_0-\rho|} \ll_F \sum_{\rho<-2T}\frac{|s_0|}{|\rho|^2} \ll_F T\sum_{\rho<-2T}\frac{1}{|\rho|^2} \ll_F T\sum_{n>2T}\frac{1}{n^2} = \mathcal{O}_F(1),$$

and

$$S_2 \ll_F \sum_{-2T\le\rho<\alpha_F}\frac{1}{|\rho|} \ll_F \sum_{\frac{-\alpha_F}{2}<n\le T}\frac{1}{n} = \mathcal{O}_F(\log T).$$

□

Now combining (1.15) and Lemma 1.4 and using the fact that F'/F is bounded at $s_0=\beta^*+iT$, say by C, we get

$$\begin{aligned}\operatorname{Re}\left\{\sum_{\substack{\gamma>0\\ \rho=\beta+i\gamma}}\left(\frac{1}{s_0-\rho}+\frac{1}{s_0-\overline{\rho}}\right)\right\} &= \operatorname{Re}\left\{\frac{F'(s_0)}{F(s_0)}-\frac{m}{s_0}+\frac{k+1}{s_0-1}-\widetilde{B}_F-\sum_{\rho\,\mathrm{real}}\left(\frac{1}{s_0-\rho}+\frac{1}{\rho}\right)\right\}\\ &\le C+\frac{m+k+1}{T}+|\widetilde{B}_F|+\sum_{\rho\,\mathrm{real}}\left|\left\{\frac{1}{s_0-\rho}+\frac{1}{\rho}\right\}\right|\\ &= \mathcal{O}_F(\log T).\end{aligned}$$

Also,

$$\sum_{\substack{\gamma>0\\ \rho=\beta+i\gamma}}\operatorname{Re}\left\{\frac{1}{s_0-\rho}+\frac{1}{s_0-\overline{\rho}}\right\} \ge \sum_{\substack{|\gamma-T|\le1\\ \rho=\beta+i\gamma}}\operatorname{Re}\left\{\frac{1}{s_0-\rho}\right\},$$

since

$$\operatorname{Re}\left\{\frac{1}{s_0-\overline{\rho}}\right\} = \frac{\operatorname{Re}\{\overline{s_0}-\rho\}}{|s_0-\overline{\rho}|^2} \ge \frac{\beta^*-\beta_F}{|s_0-\overline{\rho}|^2} > 0,$$

and then

$$\sum_{\substack{|\gamma-T|\le 1\\ \rho=\beta+i\gamma}} \mathrm{Re}\left\{\frac{1}{s_0-\rho}\right\} = \mathcal{O}_F(\log T). \tag{1.16}$$

The following proposition is the first result on the number of zeros of F.

Proposition 1.5. *Let $T > 2$. The number of zeros $\rho = \beta + i\gamma$ of $F(s)$ with $T-1 \le \gamma \le T+1$ is $\mathcal{O}_F(\log T)$.*

Proof. For ρ with $|\gamma - T| \le 1$ and $s_0 = \beta^* + iT$, $\beta^* > \beta_F$, we have

$$\begin{aligned} |s_0-\rho| &= |\beta^* + iT - \mathrm{Re}\,\rho - i\gamma| \\ &\le |\beta^*| + |\mathrm{Re}\,\rho| + |T-\gamma| \\ &\le |\beta^*| + \max\{|\alpha_F|, |\beta_F|\} + 1, \end{aligned}$$

and

$$\mathrm{Re}\,\frac{1}{s_0-\rho} = \frac{\mathrm{Re}\{\overline{s_0}-\overline{\rho}\}}{|s_0-\rho|^2} \ge \frac{\beta^*-\beta_F}{(|\beta^*| + \max\{|\alpha_F|, |\beta_F|\}+1)^2} =: C_1(F).$$

Hence,

$$\sum_{\substack{|\gamma-T|\le 1\\ \rho=\beta+i\gamma}} \mathrm{Re}\left\{\frac{1}{s_0-\rho}\right\} \ge \sum_{\substack{|\gamma-T|\le 1\\ \rho=\beta+i\gamma}} C_1(F) \ge C_1(F)\cdot \#\{\rho=\beta+i\gamma : \gamma\in[T-1,T+1]\},$$

and the result follows from (1.16). $\square$

Before the next section and the proof of the main theorem we will need one more lemma.

Lemma 1.6. *If $s = \sigma + iT$, then for large T (not coinciding with the ordinate of a zero) and $\alpha_F \le \sigma \le \beta_F + 1$,*

$$\frac{F'(s)}{F(s)} = \sum_{\substack{|\gamma-T|\le 1\\ \rho=\beta+i\gamma}} \frac{1}{s-\rho} + \mathcal{O}_F(\log T).$$

Proof. By Lemma 1.4 and (1.14) applied at s and $\beta_F + 1 + iT$ we get

$$\frac{F'(s)}{F(s)} = \mathcal{O}_F(\log T) + \sum_{\substack{\gamma\neq 0\\ \rho=\beta+i\gamma}} \left(\frac{1}{s-\rho} - \frac{1}{\beta_F+1+iT-\rho}\right).$$

For the terms with $|\gamma - T| \geq 1$, we have

$$\left|\frac{1}{s-\rho} - \frac{1}{\beta_F+1+iT-\rho}\right| = \frac{\beta_F+1-\sigma}{\left|(s-\rho)(\beta_F+1+iT-\rho)\right|} \leq \frac{\beta_F+1-\alpha_F}{|\gamma-T|^2},$$

and the sum of these is $\mathcal{O}_F(\log T)$. For if we write

$$\sum_{|\gamma-T|^2\geq 1} \frac{1}{|\gamma-T|^2} = \sum_{m=1}^{\infty} \sum_{m\leq|\gamma-T|^2<m+1} \frac{1}{|\gamma-T|^2},$$

then we get

$$\begin{aligned}\sum_{|\gamma-T|^2\geq 1} \frac{\beta_F+1-\alpha_F}{|\gamma-T|^2} &\ll_F \sum_{m=1}^{\infty}\left\{\frac{1}{m^2}\sum_{m\leq|\gamma-T|^2<m+1} 1\right\}\\ &\ll_F \sum_{m=1}^{\infty}\left\{\frac{1}{m^2}\mathcal{O}_F(\log(T+m+1))\right\},\end{aligned}$$

where Proposition 1.5 was used. The last sum above is $\mathcal{O}_F(\log T)$.

As for the terms with $|\gamma - T| \leq 1$ we have $|\beta_F + 1 + iT - \rho| \geq 1$ and the number of these is $\mathcal{O}_F(\log T)$. $\square$

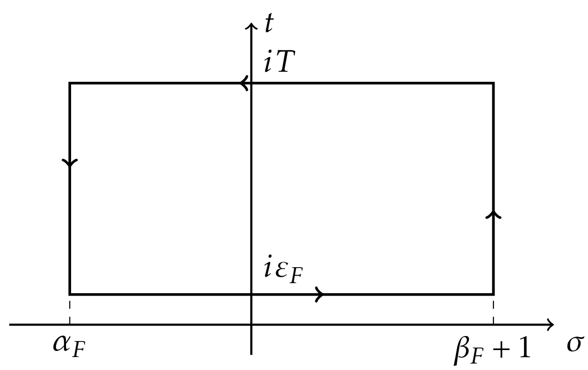
Now we are ready to proceed to the proof of Theorem 1.1 .

1.4 Proof of Theorem 1.1

Recall that $s = \sigma + it$ with σ and t both real and $N_F(T)$ is the number of zeros of the function F with $0 < t < T$.

Proof. From Section 1.2 we know that $F(s)$ has no zeros with $\sigma \geq \beta_F > 1$, and no non-real zeros for $\sigma \leq \alpha_F$. Choose $\varepsilon_F > 0$ so that $F(s)$ has no non-real zeros for $0 < t \leq \varepsilon_F$ and suppose that T does not coincide with the ordinate of a zero of $F(s)$. Let $\mathcal{R}$ be the rectangle with vertices

$$\alpha_F + i\varepsilon_F,\ \beta_F + 1 + i\varepsilon_F,\ \beta_F + 1 + iT,\ \alpha_F + iT.$$



By the argument principle, the number of zeros of F inside the rectangle is given by

$$\begin{aligned} N_F(T) &= \frac{1}{2\pi i} \int_{\mathcal{R}} \frac{F'(s)}{F(s)}\, ds \\ &= \frac{1}{2\pi} \{I_1 + I_2 + I_3 + I_4\}, \end{aligned}$$

where I_1, I_2, I_3 and I_4 are respectively the imaginary parts of the integrals of F'/F over the line segments

$$L_1 = \{\sigma + i\varepsilon_F,\ \alpha_F \leq \sigma \leq \beta_F + 1\},$$

$$L_2 = \{\beta_F + 1 + it,\ \varepsilon_F \leq t \leq T\},$$

$$L_3 = \{-\sigma + \alpha_F + \beta_F + 1 + iT,\, \alpha_F \le \sigma \le \beta_F + 1\}, \text{ and}$$

$$L_4 = \{\alpha_F + i(-t + \varepsilon_F + T),\, \varepsilon_F \le t \le T\}.$$

Firstly, $I_1 = \mathcal{O}_F(1)$, since it is independent of T.

Secondly,

$$\begin{aligned} I_2 &= \operatorname{Im}\left\{\int_{L_2} \frac{F'(s)}{F(s)}\,ds\right\} \\ &= \operatorname{Im}\left\{\int_{L_2} \frac{d}{ds}\left[\log F(s)\right] ds\right\} \\ &= \left[\arg F(s)\right]_{\beta_F+1+i\varepsilon_F}^{\beta_F+1+iT}. \end{aligned}$$

Note that for $s = \beta_F + 1 + it$

$$\begin{aligned} \operatorname{Re}\{1 + (F(s) - 1)\} &\ge 1 - |\operatorname{Re}\{F(s) - 1\}| \\ &\ge 1 - |F(s) - 1| \\ &\ge 1 - \frac{1}{2} - \frac{1}{2^{k+2}} > 0, \end{aligned}$$

where the last inequality uses (1.2) and (1.3). Hence, the argument of $F(s)$ is less than π, and so $I_2 = \mathcal{O}_F(1)$.

Next, to estimate I_3 we use Lemma 1.6. Thus,

$$I_3 = \int_{L_3} \operatorname{Im}\{F'(s)/F(s)\}\,ds$$

$$
\begin{aligned}
&= \int\limits_{L_3} \left(\sum_{\substack{|\gamma - T| \le 1 \\ \rho = \beta + i\gamma}} \mathrm{Im} \left\{ \frac{1}{s-\rho} \right\} + \mathcal{O}(\log T) \right) ds \\
&= \sum_{\substack{|\gamma - T| \le 1 \\ \rho = \beta + i\gamma}} \mathrm{Im} \left\{ \int\limits_{L_3} \frac{1}{s-\rho}\, ds \right\} + \mathcal{O}(\log T),
\end{aligned}
$$

and since $\frac{1}{s-\rho} = \frac{d}{ds}(\log(s-\rho))$, by Proposition 1.5,

$$
\begin{aligned}
I_3 &= \sum_{\substack{|\gamma - T| \le 1 \\ \rho = \beta + i\gamma}} [\arg(s-\rho)]_{\beta_F + iT - \rho}^{\alpha_F + iT - \rho} + \mathcal{O}(\log T) \\
&\le \sum_{\substack{|\gamma - T| \le 1 \\ \rho = \beta + i\gamma}} \pi + \mathcal{O}(\log T) = \mathcal{O}_F(\log T).
\end{aligned}
$$

Lastly, for estimating I_4 we use (1.8) where s is replaced with $1-s$, to get

$$
F(s) = \frac{1}{\pi}(2\pi)^s \sum_{j=0}^{k} \tilde{R}_{jk}(1-s)\Gamma^{(j)}(1-s),
$$

and rewrite this as

$$
F(s) = (2\pi)^s \Gamma(1-s) e^{-\frac{i\pi s}{2}} \left\{ \sum_{j=0}^{k-1} \frac{\tilde{R}_{jk}(1-s)}{\pi e^{\frac{-i\pi s}{2}}} \frac{\Gamma^{(j)}(1-s)}{\Gamma(1-s)} + \frac{\tilde{R}_{kk}(1-s)}{\pi e^{-\frac{i\pi s}{2}}} \frac{\Gamma^{(k)}(1-s)}{\Gamma(1-s)} \right\}. \tag{1.17}
$$

Equation (1.11) gives

$$
\frac{\Gamma^{(j)}(1-s)}{\Gamma(1-s)} = \log^j(1-s) + \mathcal{O}_j\left(\frac{\log^{j-1}(1-s)}{1-s} \right).
$$

Next, we write

$$
F(s) = (2\pi)^s e^{-i\frac{\pi s}{2}} \Gamma(1-s) \{R_1(s) + R_2(s)\}
$$

$$= (2\pi)^s e^{-i\frac{\pi s}{2}} \Gamma(1-s) R_1(s) \left\{ 1 + \frac{R_2(s)}{R_1(s)} \right\},$$

where

$$\begin{aligned} R_1(s) &= \log^k(1-s) \frac{\tilde{R}_{kk}(1-s)}{\pi e^{-\frac{i\pi s}{2}}} \\ &= \log^k(1-s) \frac{c_k(-1)^k \zeta(1-s) \sin(\frac{\pi s}{2})}{\pi e^{-\frac{i\pi s}{2}}} \end{aligned}$$

and

$$R_2(s) = \sum_{j=0}^{k-1} \frac{\tilde{R}_{jk}(1-s)}{\pi e^{\frac{-i\pi s}{2}}} \frac{\Gamma^{(j)}(1-s)}{\Gamma(1-s)} + \mathcal{O}_k \left(\frac{\tilde{R}_{kk}(1-s)}{e^{-\frac{i\pi s}{2}}} \frac{\log^{k-1}(1-s)}{1-s} \right).$$

For all $1 \le j, m \le k$, by (1.5)

$$\left| R_{jm}(1-s) e^{\frac{i\pi s}{2}} \right| = e^{-\pi T/2} \left| P_{jm}(1-s) \sin \frac{\pi s}{2} + Q_{jm}(1-s) \cos \frac{\pi s}{2} \right|.$$

On the other hand,

$$\left| \sin \frac{\pi s}{2} \right| = \left| \frac{e^{\frac{i\pi s}{2}} - e^{-\frac{i\pi s}{2}}}{2i} \right| = e^{\pi T/2} \left| \frac{e^{i\pi s} - 1}{2} \right| \le e^{\pi T/2},$$

and similarly we get $\left| \cos \frac{\pi s}{2} \right| \le e^{\pi T/2}$. Now, since the line of integration is away from $s = 1$, $P_{jm}(1-s)$ and $Q_{jm}(1-s)$ are bounded for each fixed $1 \le j, m \le k$. Combining this with (1.9) we finally get that, as $T \to \infty$,

$$e^{i\pi s/2} \tilde{R}_{jk}(1-s) = \mathcal{O}_F(1),$$

and consequently,

$$R_2(s) = \mathcal{O}_F(\log^{k-1}(1-s)).$$

We assume that α_F is sufficiently negative, so that

$$\left|\frac{R_2(s)}{R_1(s)}\right| < 1,$$

when s belongs to L_4. Hence the change of the argument of $1 + \dfrac{R_2(s)}{R_1(s)}$ is less than π as s travels along L_4. Moreover, $\frac{c_k(-1)^k \sin(\frac{\pi s}{2})}{\pi e^{-\frac{i\pi s}{2}}}$ approaches a nonzero complex number as t increases, while $\zeta(1-s)$ remains bounded and so the change of argument on L_4 is also $\mathcal{O}_F(1)$.

Finally, by (1.17)

$$\begin{aligned}
&[\log F(s)]_{\alpha_F+iT}^{\alpha_F+i\varepsilon_F} \\
&\qquad = \left[\log(2\pi)^s - \frac{1}{2}i\pi s + \log\Gamma(1-s) + \log R_1(s) + \log\left(1 + \frac{R_2(s)}{R_1(s)}\right)\right]_{\alpha_F+iT}^{\alpha_F+i\varepsilon_F} \\
&\qquad\qquad = -iT\log 2\pi + \mathcal{O}_F(1) - \frac{1}{2}\pi T + \mathcal{O}_F(1) \\
&\qquad\qquad + \left[-\left(s - \frac{1}{2}\right)\log(1-s) - (1-s) + \mathcal{O}_F(1)\right]_{\alpha_F+iT}^{\alpha_F+i\varepsilon_F} + \mathcal{O}_F(\log(\log^k(T))),
\end{aligned}$$

where in the last step we used Stirling's formula for $\log\Gamma(1-s)$, and the logarithm in $R_1(s)$ gives the error term. Now,

$$\begin{aligned}
\left(\alpha_F + iT - \frac{1}{2}\right)\log(1-\alpha_F - iT) &= \left(\alpha_F - \frac{1}{2} + iT\right)\log\left(-iT\left(1 - \frac{1-\alpha_F}{iT}\right)\right) \\
&= \left(\alpha_F + iT - \frac{1}{2}\right)\log(-iT) + \mathcal{O}_F(1) = iT\log T + \frac{\pi T}{2} + \mathcal{O}_F(\log T),
\end{aligned}$$

and putting all the estimates together we reach

$$I_4 = T(\log T - \log 2\pi - 1) + \mathcal{O}_F(\log T).$$

Hence, the main contribution in the number of zeros comes from I_4 and we have

$$N_F(T) = \frac{1}{2\pi}\sum_{j=1}^{4} \operatorname{Im} I_j = \frac{1}{2\pi} T\left(\log T - \log 2\pi - 1\right) + \mathcal{O}_F(\log T).$$

This completes the proof of the theorem. $\square$

Chapter 2

Linear combinations of ζ and its derivatives and horizontal distribution of their zeros

2.1 Introduction

In this chapter, we consider the number β_k which is defined as the supremum of the real parts of $\rho \in G_k$, where G_k is the set of all zeros of all combinations

$$F(s) = \zeta(s) + c_1\zeta'(s) + \cdots + c_k\zeta^{(k)}(s) \tag{2.1}$$

satisfying $|c_1|, |c_2|, \ldots, |c_k| \leq 1$. The next theorem provides a sharp asymptotic formula for β_k.

Theorem 2.1. *With β_k defined as above, we have, as $k \to \infty$,*

$$\beta_k = \frac{\log\log 15}{\log 15}k + C + \mathcal{O}_F(\eta^k). \tag{2.2}$$

Here η is a fixed positive number less than 1 and C is an absolute constant.

We prove this result throughout Sections 2.2 and 2.3 and also provide the exact values of the constants C and η.

In Section 2.4, we investigate an inverse-type problem. Given a real number $\beta > 1$, it is easy to show that there exists a linear combination of the form (2.1) which vanishes at β. However, can the degree of the largest derivative involved be small? In other words, if we define $k(\beta)$ to be the minimum integer k for which there exist real constants $c_1, \ldots, c_k$ of absolute value at most 1, with $c_k \neq 0$, such that $\zeta(\beta) + c_1\zeta'(\beta) + \cdots + c_k\zeta^{(k)}(\beta) = 0$, how does $k(\beta)$ relate to β?
This is the content of the following theorem.

Theorem 2.2. *1. For all real $\beta > 1$, with the exception of a set E of finite Lebesgue measure, and with $k(\beta)$ defined as above,*

$$k(\beta) = \left\lceil \frac{\beta - C}{L} \right\rceil,$$

where

$$C = \frac{\log\log 15 - \log(\log 15 - 1)}{\log 15} \qquad \textit{and} \qquad L = \frac{\log\log 15}{\log 15}.$$

2. For almost all integers $j > 1$,

$$k(j) = \left\lceil \frac{j - C}{L} \right\rceil, \tag{$*$}$$

with C and L as in part (1).

For example, one can show that $k(1,000,000) = 2,178,301$,which means that any linear combination, with coefficients satisfying the usual restrictions, that vanishes at $\beta = 1,000,000$ will involve a derivative of order at least $2,178,301$.

Extensive numerical evidence, as well as (2) of Theorem 2.2, lead to the following conjecture.

Conjecture 2.3. *For all but finitely many integers j, $(*)$ holds.*

2.2 β_k and β_k^*

In Chapter 1, we studied functions of the form $F(s) = \zeta(s) + c_1\zeta'(s) \cdots + c_k\zeta^{(k)}(s)$, where the non-negative integer k, as well as the real coefficients $c_1, c_2, \ldots, c_k$, were fixed. In the following sections, we investigate a family of such functions. To that end, we associate each vector $\vec{c} = (c_1, c_2, \ldots, c_k)$ in $\mathbb{R}^k$ with the function $F_{\vec{c}}(s) = \zeta(s) + c_1\zeta'(s) + \cdots + c_k\zeta^{(k)}(s)$ and define $\mathcal{F}_k$ to be the family of functions consisting of all $F_{\vec{c}}$ with $\vec{c} \in \mathbb{R}^k$ and $||\vec{c}||_\infty \leq 1$. Moreover, we define G_k to be the set of all complex numbers ρ for which there exists a function $F_{\vec{c}} \in \mathcal{F}_k$ such that $F_{\vec{c}}(\rho) = 0$. Our main goal is to prove an asymptotic formula for the number $\beta_k := sup\{\operatorname{Re}\rho : \rho \in G_k\}$. However, it is difficult to establish such an asymptotic directly from the definition of β_k. For this reason, we introduce the auxiliary number β_k^*, whose relation to β_k will be clear by the end of this section.

Lemma 2.4. *Let $F_k^*(s)$ be the function in $\mathcal{F}_k$ that corresponds to the vector $\vec{c} = (1, -1, \ldots, (-1)^{k-1})$, that is $F_k^*(s) = \zeta(s) + \zeta'(s) - \zeta''(s) + \cdots + (-1)^{k-1}\zeta^{(k)}(s)$. For $k \geq 2$, $F_k^*(s)$ has exactly one real zero β_k^* with $1 < \beta_k^* < \infty$, and in fact,*

$$\frac{\log\log 15}{\log 15}k < \beta_k^* < k + 2.$$

Remark. *The alternating signs in the definition of $F_k^*(s)$ start from the second term.*

Proof. Using the Dirichlet series of ζ and its derivatives we write

$$F_k^*(s) = 1 + \sum_{n=2}^{\infty} \frac{1 - \log n - \log^2 n - \cdots - \log^k n}{n^s} \tag{2.3}$$

for s with $\operatorname{Re} s > 1$.

For $k \geq 2$, F_k^* restricted to $1 < s < \infty$ is increasing, so that if it has a zero in that range it must be unique. The existence follows from the observation that $\lim\limits_{\substack{s \to 1^+ \\ s \in \mathbb{R}}} F_k^*(s) = -\infty$ and $\lim\limits_{\substack{s \to +\infty \\ s \in \mathbb{R}}} F_k^*(s) = 1$.

Now, to obtain the more precise bounds for β_k^* as stated in the lemma we make use of the fact that $F_k^*(\beta_k^*) = 0$, that is

$$1 = \sum_{n=2}^{\infty} \frac{-1 + \log n + \log^2 n + \cdots + \log^k n}{n^{\beta_k^*}}. \tag{2.4}$$

The lower bound follows easily, since by (2.3)

$$1 = \sum_{n=2}^{\infty} \frac{-1 + \log n + \log^2 n + \cdots + \log^k n}{n^{\beta_k^*}} > \frac{-1 + \log 15 + \log^2 15 + \cdots + \log^k 15}{15^{\beta_k^*}} > \frac{\log^k 15}{15^{\beta_k^*}},$$

and so

$$\beta_k^* > \frac{\log\log 15}{\log 15}k. \tag{2.5}$$

To get an upper bound, we use the inequalities $1 + x + x^2 + \cdots + x^k \le (1+x)^k$ and $1 + \log x \le x$ both valid for $x \ge 1$ and $k \ge 1$. If $\beta_k^* - k \le 1$ then $\beta_k^* \le k+1$ and the claim holds. Otherwise, $\beta_k^* - k > 1$ and we have

$$\begin{aligned}
1 &= \sum_{n=2}^{\infty} \frac{-1 + \log n + \log^2 n + \cdots + \log^k n}{n^{\beta_k^*}} \\
&\le \sum_{n=2}^{\infty} \frac{1 + \log n + \log^2 n + \cdots + \log^k n}{n^{\beta_k^*}} \\
&\le \sum_{n=2}^{\infty} \frac{(1 + \log n)^k}{n^{\beta_k^*}} \\
&\le \sum_{n=2}^{\infty} \frac{n^k}{n^{\beta_k^*}} = \zeta(\beta_k^* - k) - 1.
\end{aligned}$$

This shows that $\zeta(\beta_k^* - k) > \frac{\pi^2}{6} = \zeta(2)$, which leads to $\beta_k^* < k+2$ and the end of the proof. □

Remark. *In fact, we can get a better upper bound for β_k^* as follows. Since $\zeta(\beta_k^*) > 2$, then $\beta_k^* < k + \zeta^{-1}(2)$ (where ζ^{-1} means the inverse of the zeta function on the real line), so that $\beta_k^* < k + 1.73$.*

Now, we show the relation between β_k and β_k^*.

Lemma 2.5. *With β_k and β_k^* as before,*

$$\beta_k - \beta_k^* = \mathcal{O}\left(\frac{1}{2^{k/3}}\right)$$

as $k \to \infty$.

Proof. We assume that $\beta_k \ne \beta_k^*$, otherwise there is nothing to prove. By definition of β_k, for all $\epsilon > 0$ there exists a function $\widetilde{F} \in \mathcal{F}_k$ and an element ρ of G_k so that $\widetilde{F}(\rho) = 0$ and $\beta_k - \epsilon < \operatorname{Re}\rho \le \beta_k$. Suppose that $\vec{c} = (\tilde{c}_1, \ldots, \tilde{c}_k)$ is the vector that corresponds to $\widetilde{F}$.

Then,

$$1 = \sum_{n=2}^{\infty} \frac{-1 + \tilde{c}_1 \log n + \cdots + (-1)^{k+1} \tilde{c}_k \log^k n}{n^{\rho}} \le \sum_{n=2}^{\infty} \frac{1 + \log n + \cdots + \log^k n}{n^{\operatorname{Re}\rho}},$$

since $\|\vec{c}\|_\infty \le 1$. Let $\delta = \operatorname{Re}\rho - \beta_k^*$. Without loss of generality, we can assume that $\epsilon < \beta_k - \beta_k^*$ so that $\delta > 0$. The above inequalities together with (2.4) become

$$\sum_{n=2}^{\infty} \frac{-1+\log n+\log^2 n+\cdots+\log^k n}{n^{\beta_k^*}} = 1 \leq \sum_{n=2}^{\infty} \frac{1+\log n+\cdots+\log^k n}{n^{\delta+\beta_k^*}},$$

which gives

$$\sum_{n=2}^{\infty} \frac{\log n+\log^2 n+\cdots+\log^k n}{n^{\beta_k^*}}\left(1-\frac{1}{n^{\delta}}\right) \leq \sum_{n=2}^{\infty} \frac{1}{n^{\beta_k^*}}\left(1+\frac{1}{n^{\delta}}\right).$$

The left hand side is greater than $1-\frac{1}{2^{\delta}}$ by (2.4), while the right hand side is less than

$$2\sum_{n=2}^{\infty} \frac{1}{n^{\beta_k^*}} \leq 2\left(\frac{1}{2^{\beta_k^*}}+\int_2^{\infty} \frac{1}{x^{\beta_k^*}}\,dx\right) = 2\left(\frac{1}{2^{\beta_k^*}}+\frac{2^{1-\beta_k^*}}{\beta_k^*-1}\right) \leq \frac{3}{2^{\beta_k^*}},$$

where the last inequality holds if k is sufficiently large, by Lemma 2.4.

Hence,

$$1-\frac{1}{2^{\delta}} \leq \frac{3}{2^{\beta_k^*}},$$

and we get the following inequality for δ,

$$\delta \leq \frac{-\log\left(1-\frac{3}{2^{\beta_k^*}}\right)}{\log 2} \leq \frac{1}{2^{\beta_k^*}}\frac{3}{\log 2} \leq \frac{1}{2^{\beta_k^*-3}}.$$

Finally,

$$\beta_k-\beta_k^* = \beta_k-\operatorname{Re}\rho+\operatorname{Re}\rho-\beta_k^* \leq \epsilon+\delta \leq \epsilon+\frac{1}{2^{\beta_k^*-3}}$$

for any positive ϵ, and since

$$\beta_k^* \geq \frac{\log\log 15}{\log 15}k \geq \frac{k}{3},$$

we have

$$\beta_k-\beta_k^* = \mathcal{O}\left(\frac{1}{2^{k/3}}\right).$$

$\square$

2.3 Asymptotic formula for β_k^*

Recall that according to the definition of β_k^*,

$$1-\sum_{n=2}^{\infty}\frac{-1+\log n+\log^2 n+\cdots+\log^k n}{n^{\beta_k^*}}=0. \tag{2.6}$$

We define

$$H_k(x)=\frac{-1+\log x+\cdots+\log^k x}{x^{\beta_k^*}}=\frac{\frac{\log^{k+1}x-1}{\log x-1}-2}{x^{\beta_k^*}}, \tag{2.7}$$

and (2.6) becomes $\sum_{n=2}^{\infty}H_k(n)=1$.

Lemma 2.6. *As $k\to\infty$ we have*

$$\sum_{\substack{n\geq 2\\ n\neq 15}}H_k(n)=\mathcal{O}(\eta^k), \tag{2.8}$$

where

$$\eta=2^{\frac{\log\log 16}{\log 16}-\frac{\log\log 15}{\log 15}}=0.99995258\ldots$$

Proof. Using (2.7) we write

$$\begin{aligned}
\sum_{\substack{n\geq 2\\ n\neq 15}}H_k(n)&=\sum_{\substack{n\geq 2\\ n\neq 15}}\frac{\frac{\log^{k+1}n-1}{\log n-1}-2}{n^{\beta_k^*}}\leq\sum_{\substack{n\geq 2\\ n\neq 15}}\frac{\log^{k+1}n-1}{(\log n-1)n^{\beta_k^*}}\\
&\leq\sum_{\substack{n\geq 2\\ n\neq 15}}\frac{\log^{k+1}n}{(\log n-1)n^{\beta_k^*}}\leq 2\sum_{\substack{n\geq 2\\ n\neq 15}}\frac{\log^k n}{n^{\beta_k^*}}\\
&=2\sum_{\substack{n\geq 2\\ n\neq 15}}\frac{1}{n^{\beta_k^*-k\frac{\log\log n}{\log n}}}\leq 2\sum_{\substack{n\geq 2\\ n\neq 15}}\frac{1}{n^{k\left(\frac{\log\log 15}{\log 15}-\frac{\log\log 16}{\log 16}\right)}},
\end{aligned}$$

where in the last step we used the lower bound (2.5) for β_k^* and the fact that, for $n\neq 15$,

$$\frac{\log\log n}{\log n}\leq\frac{\log\log 16}{\log 16}.$$

This means that

$$
\begin{aligned}
\sum_{\substack{n\geq 2\\ n\neq 15}} H_k(n) &\leq \frac{2}{2^{k\left(\frac{\log\log 15}{\log 15}-\frac{\log\log 16}{\log 16}\right)}} + 2\int_2^\infty \frac{dx}{x^{k\left(\frac{\log\log 15}{\log 15}-\frac{\log\log 16}{\log 16}\right)}} \\
&= \frac{2}{2^{k\left(\frac{\log\log 15}{\log 15}-\frac{\log\log 16}{\log 16}\right)}} + \frac{2}{2^{k\left(\frac{\log\log 15}{\log 15}-\frac{\log\log 16}{\log 16}\right)-1}\left(k\left(\frac{\log\log 15}{\log 15}-\frac{\log\log 16}{\log 16}\right)-1\right)} \\
&\ll \left(2^{\frac{\log\log 16}{\log 16}-\frac{\log\log 15}{\log 15}}\right)^k \ll \eta^k,
\end{aligned}
$$

and the lemma is proven. $\square$

Proposition 2.7. *The following asymptotic formula for β_k^* holds:*

$$\beta_k^* = \frac{\log\log 15}{\log 15}k + C + \mathcal{O}(\eta^k),$$

where

$$C = \frac{\log\log 15 - \log(\log 15 - 1)}{\log 15}$$

is a constant and η is as in the previous lemma.

Proof. Recall that we have

$$1 = \sum_{n=2}^{\infty} H_k(n) = H_k(15) + \sum_{\substack{n\geq 2\\ n\neq 15}} H_k(n).$$

So (2.8) gives us $H_k(15) = 1 + \mathcal{O}(\eta^k)$, that is

$$\frac{\frac{\log^{k+1} 15-1}{\log 15-1} - 2}{15^{\beta_k^*}} = 1 + \mathcal{O}(\eta^k),$$

which implies

$$\frac{\log^{k+1} 15}{15^{\beta_k^*}(\log 15 - 1)} = 1 + \mathcal{O}(\eta^k).$$

Taking logarithm of both sides we get

$$\beta_k^* \log 15 + \log(\log 15 - 1) - (k+1) \log\log 15 = \mathcal{O}(\eta^k),$$

which implies

$$\beta_k^* = \frac{\log\log 15}{\log 15} k + \frac{\log\log 15 - \log(\log 15 - 1)}{\log 15} + \mathcal{O}(\eta^k),$$

and the proposition is proven. □

We finally combine Proposition 2.7 and Lemma 2.5 to get the statement of Theorem 2.1.

2.4 Inverse-type problem

For $t > 1$, let

$$F_k^*(t) = \zeta(t) + \zeta'(t) + \cdots + (-1)^{k-1} \zeta^{(k)}(t),$$

and recall that β_k^* is its unique zero. Notice that $F_k^*(t)$ is increasing as a function of t. Moreover, the sequence $\{\beta_k^*\}_{k\in\mathbb{N}}$ is increasing and as $k \to \infty$, we have

$$\beta_k^* = \gamma_k + \mathcal{O}(\eta^k),$$

where $\gamma_k = L \cdot k + C$, with the constants C and L as in the statement of Theorem 2.2 and

$$\eta = 2^{\frac{\log\log 16}{\log 16} - \frac{\log\log 15}{\log 15}} = 0.99995258\ldots$$

Furthermore, we assume that for large enough k, say $k \geq k_0$,

$$|\beta_k^* - \gamma_k| < 1000 \cdot \eta^k. \tag{2.9}$$

In order to proceed to the proof of Theorem 2.2, we first prove some preliminary results.

Proposition 2.8. *For $\beta > 1$, we have $k(\beta) \leq m$ if and only if $\beta \leq \beta_m^*$.*

Proof. First, assume that $k(\beta) = l \leq m$. By definition, there exist constants $c_1, \dots, c_l$ of absolute value at most 1, with $c_l \neq 0$, such that

$$\zeta(\beta) + \sum_{i=1}^{l} c_i \zeta^{(i)}(\beta) = 0.$$

Therefore,

$$F_l^*(\beta_l^*) = 0 = \zeta(\beta) + \sum_{i=1}^{l} c_i \zeta^{(i)}(\beta) \geq \zeta(\beta) + \sum_{i=1}^{l} (-1)^{i-1} \zeta^{(i)}(\beta) = F_l^*(\beta),$$

and since F_l^* is increasing, we get

$$\beta \leq \beta_l^* \leq \beta_m^*.$$

To prove the converse direction, assume that $\beta \leq \beta_m^*$. More specifically, there exists an integer $l \leq m$, such that $\beta_{l-1}^* \leq \beta \leq \beta_l^*$. Let $g_\beta(\theta) := F_{l-1}^*(\beta) + \theta(-1)^{l-1}\zeta^{(l)}(\beta)$. Notice that $g_\beta(\theta)$ is a continuous function of θ and that

$$g_\beta(0) = F_{l-1}^*(\beta) \geq F_{l-1}^*(\beta_{l-1}^*) = 0,$$

$$g_\beta(1) = F_l^*(\beta) \leq F_l^*(\beta_l^*) = 0,$$

so we must have $g_\beta(\theta) = 0$, for some $\theta \in [0, 1]$. In other words, there exists a linear combination whose higher derivative is at most of order l that vanishes at β. Hence, $k(\beta) \leq l \leq m$. □

The following is an immediate corollary to the above proposition that makes clear the fact that $k(\beta)$ is a step function.

Corollary 2.9. *For $\beta > 1$, we have $k(\beta) = m$ if and only if $\beta_{m-1}^* < \beta \leq \beta_m^*$.*

Let us now prove our main theorem.

Proof of Theorem 2.2. *(1)* The set $E := \left\{t > 1 : k(t) \neq \left\lceil \frac{t-C}{L} \right\rceil\right\}$ is measurable, since it consists of points for which two step functions attain different values. We will prove that E has finite Lebesgue measure, by showing that

$$\mu\left(\bigcup_{k>k_0} E \cap (\beta^*_{k-1}, \beta^*_k]\right) < \infty.$$

Let $\alpha_k = \min\{\gamma_k, \beta^*_k\}$ and $\lambda_k = \max\{\gamma_k, \beta^*_k\}$. Given $\beta \in (\beta^*_{k-1}, \beta^*_k]$, for $k > k_0$, exactly one of the following is true:

<u>Case 1</u>: If $\beta \in (\lambda_{k-1}, \alpha_k]$, then $\gamma_{k-1} < \beta \leq \gamma_k$, which implies $k-1 < \frac{\beta-C}{L} \leq k$. This gives $k(\beta) = k = \left\lceil \frac{\beta-C}{L} \right\rceil$ and so β does not belong to E.

<u>Case 2</u>: If $\beta \in \left(\beta^*_{k-1}, \lambda_{k-1}\right]$, then the interval is not empty, so $\beta^*_{k-1} < \beta \leq \gamma_{k-1}$. Moreover, $\beta > \gamma_{k-2}$ and we get $k-2 < \frac{\beta-C}{L} \leq k-1$. In this case, $\beta \in E$, since $k(\beta) = k = \left\lceil \frac{\beta-C}{L} \right\rceil + 1$.

<u>Case 3</u>: This case is similar to the previous one with the only difference being that for $\beta \in \left(\alpha_k, \beta^*_k\right]$, we have $k(\beta) = k = \left\lceil \dfrac{\beta - C}{L} \right\rceil - 1$.

In conclusion,

$$\bigcup_{k>k_0} E \cap (\beta^*_{k-1}, \beta^*_k] = \bigcup_{k>k_0} \left(\beta^*_{k-1}, \lambda_{k-1}\right] \cup \left(\alpha_k, \beta^*_k\right],$$

and therefore,

$$\begin{aligned}\mu\left(\bigcup_{k>k_0} E \cap (\beta^*_{k-1}, \beta^*_k]\right) &= \mu\left(\bigcup_{k>k_0} \left(\beta^*_{k-1}, \lambda_{k-1}\right] \cup \left(\alpha_k, \beta^*_k\right]\right) \\ &= \sum_{k>k_0} (\lambda_{k-1} - \beta^*_{k-1} + \beta^*_k - \alpha_k)\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k>k_0} (|\gamma_{k-1} - \beta^*_{k-1}| + |\beta^*_k - \gamma_k|) \\
&\leq \sum_{k>k_0} 2000 \cdot \eta^{k-1} = \mathcal{O}(1),
\end{aligned}
$$

where the bound (2.9) is used in the last inequality.

(2) We first show that the number $L = \frac{\log\log 15}{\log 15}$ is irrational. Suppose to the contrary that $\frac{\log\log 15}{\log 15}$ is rational, which would imply that $15^{\frac{\log\log 15}{\log 15}}$ is a non-zero algebraic number. By the Lindemann-Weierstrass Theorem (see for example [Bak75, Thm 1.4]), it would follow that $e^{15^{\frac{\log\log 15}{\log 15}}} = 15$ is transcendental, which leaves us with a contradiction. Now, by a theorem of Weyl [KN74, Thm 2.4], the sequence $\left\{\dfrac{k-C}{L}\right\}_{k=1}^{\infty}$ is uniformly distributed modulo 1. Consequently, almost all positive integers k will fall outside the exceptional set E. □

Observe that large enough integers in the exceptional set should be "exponentially close" to one of the β^*_k's. Finding approximate values of these numbers is numerically expensive for large k because of the high order derivatives of ζ. Therefore to show that a certain integer is non-exceptional we check it being outside the $1000\eta^k$-neighborhoods of the γ_k's. We numerically calculated these neighborhoods for the first $1,000,000$ values of k and it turns out that all integers larger than $100,000$ fall outside these intervals. In fact, only one integer after $30,000$, namely $k = 32,810$ is in the mentioned neighborhood. Currently we do not know if this is an exceptional number, because our bound (2.9) is not the best possible.

Chapter 3

Linear combinations of L-functions and their derivatives

3.1 Introduction

Theorem 2.1 can be extended to a larger class of Dirichlet series. Consider the complex linear combinations

$$F_{k,G}(s) = G(s) + c_1 G'(s) + \cdots + c_k G^{(k)}(s), \qquad |c_1|, \ldots, |c_k| \leq 1, \tag{3.1}$$

where $G(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent for $\operatorname{Re} s > 1$, and for any fixed $\epsilon > 0$, its Dirichlet coefficients satisfy $a_1 = 1$ and $a_n \ll_\epsilon n^\epsilon$ for all integers n.

Stating the more general theorem requires the following definition.

Definition 3.1. *The relevant number to the Dirichlet series $G(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, denoted simply by n^*, is defined to be the unique integer n with $a_n \neq 0$, that maximizes the quantity $\frac{\log\log n}{\log n}$. Moreover, denote by $n^\#$ the integer n with $a_n \neq 0$, that maximizes the quantity $\frac{\log\log n}{\log n}$ for $n \neq n^*$.*

Remark. *Note that n=2 will never be relevant as $\frac{\log\log 2}{\log 2} < 0$ and $a_n \neq 0$ for some $n \neq 2$.*

The proof of the following result is similar in spirit to that of Theorem 2.1 and is provided in the next section.

Theorem 3.2. *Define*

$$\beta_{k,G} = \sup\{\operatorname{Re}\rho > 1 : F_{k,G}(\rho) = 0 \text{ for some } F_{k,G} \text{ as in (3.1)}\}.$$

Then, as $k \to \infty$,

$$\beta_{k,G} = \frac{\log\log n^*}{\log n^*} k + \frac{\log|a_{n^*}| + \log\log n^* - \log(\log n^* - 1)}{\log n^*} + \mathcal{O}_G(\eta_G^k), \tag{3.2}$$

where n^* is the relevant number to the Dirichlet series $G(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, and η_G is a positive number less than 1 that depends only on G.

Referring to Table 3.1 in Section 3.3, one can immediately see that, in the case of the Riemann-zeta function, $n^* = 15$ and the asymptotic formula (3.2) becomes in fact (2.2). However, there are Dirichlet series whose fifteenth coefficient is zero and then the asymptotic formula for $\beta_{k,G}$ looks slightly different.

In Section 3.3, we restrict our attention to those Dirichlet series associated to a Dirichlet character χ. The multiplicative properties of Dirichlet characters prevent some integers from being relevant at all.

The following theorem describes the set of all relevant numbers to Dirichlet L-functions.

Theorem 3.3. *The set of relevant numbers to Dirichlet L-functions consists of all primes $p \geq 11$ and the composite integers* $15, 16, 21, 25, 27$ *and* 49.

The remainder of the section is devoted to computing the frequency of each relevant number.

3.2 Proof of Theorem 3.2

The proof of Theorem 3.2 is similar in spirit to that of Theorem 2.1.

First, we choose the coefficients $c_j = (-1)^{j-1} \dfrac{\overline{a_{n^*}}}{|a_{n^*}|}$, for $1 \leq j \leq k$, and write

$$F^*_{k,G}(s) = 1 - \sum_{n=2}^{\infty} H_n(s), \tag{3.3}$$

where

$$H_n(s) = H_{n,k,G}(s) := \frac{a_n}{n^s}\left(-1 + \frac{\overline{a_{n^*}}}{|a_{n^*}|}\frac{\log^{k+1} n - \log n}{\log n - 1}\right).$$

Our goal is to show that the function $F^*_{k,G}(s)$ has a zero $\beta^*_{k,G}$ which is exponentially close to the number $\beta_{k,G}$ and satisfies the desired asymptotic formula. To this end, we do the following things:

1. Show that the sum of the terms in (3.3) that correspond to $n \neq n^*$ is exponentially small.
2. Prove that a zero $\beta^*_{k,G}$ exists and that its real part satisfies the asymptotic (3.2).
3. Bound the difference between $\beta_{k,G}$ and $\operatorname{Re}\beta^*_{k,G}$.

3.2.1 Exponentially small tail

Let

$$s^* = k\frac{\log\log n^*}{\log n^*} + \frac{\log|a_{n^*}| + \log\log n^* - \log(\log n^* - 1)}{\log n^*}.$$

For $\operatorname{Re} s \geq s^* - R$, where R can be anything that satisfies $R = \mathcal{O}(1)$, as $k \to \infty$,

$$\begin{aligned}
\sum_{\substack{n=3\\ n\neq n^*}}^{\infty} |H_n(s)| &= \sum_{\substack{n=3\\ n\neq n^*}}^{\infty} \left|\frac{a_n}{n^s}\left(-1 + \frac{\overline{a_{n^*}}}{|a_{n^*}|}\frac{\log^{k+1} n - \log n}{\log n - 1}\right)\right| \\
&\leq \sum_{\substack{n=3\\ n\neq n^*}}^{\infty} \frac{|a_n|}{n^{\operatorname{Re} s}}\left(1 + \frac{\log^{k+1} n - \log n}{\log n - 1}\right) \\
&\ll_\epsilon \sum_{\substack{n=3\\ n\neq n^*}}^{\infty} \frac{n^\epsilon}{n^{s^*-R}}\log^k n \\
&= \sum_{\substack{n=3\\ n\neq n^*}}^{\infty} \frac{1}{n^{s^*-R-\epsilon-k\frac{\log\log n}{\log n}}} \\
&\leq \sum_{\substack{n=3\\ n\neq n^*}}^{\infty} \frac{1}{n^{k\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right) + \frac{\log|a_{n^*}|}{\log n^*} - R}}
\end{aligned}$$

where we used the fact that $\frac{\log\log n}{\log n} \le \frac{\log\log n^\#}{\log n^\#}$, for $n \ne n^*$ and made the choice $\epsilon = \epsilon(G) = \frac{\log\log n^* - \log(\log n^* - 1)}{\log n^*}$.

For large enough k, we have

$$k\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right) + \frac{\log|a_{n^*}|}{\log n^*} - R > \frac{1}{2}k\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right) > 2,$$

and so

$$\begin{aligned}\sum_{\substack{n=3\\ n\ne n^*}}^{\infty} |H_n(s)| &\ll_G \int_2^\infty \frac{1}{x^{\frac{1}{2}k\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right)}}\, dx \\ &= \frac{1}{\frac{1}{2}k\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right) - 1} \cdot \frac{2}{2^{\frac{1}{2}k\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right)}} \\ &\le 4\left(\frac{1}{2^{\frac{1}{2}\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right)}}\right)^k \end{aligned} \tag{3.4}$$

Similarly, for $n = 2$, and large k,

$$|H_2(s)| \le \lambda_G \left(\frac{1}{2^{\frac{1}{2}\frac{\log\log n^*}{\log n^*}}}\right)^k, \tag{3.5}$$

where the constant λ_G depends only on the function G.

In conclusion,

$$\sum_{\substack{n=2\\ n\ne n^*}}^{\infty} |H_n(s)| \ll_G \left(\frac{1}{2^{\frac{1}{2}\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right)}}\right)^k,$$

for all s with $\operatorname{Re} s \ge s^* - R$.

3.2.2 Existence of $\beta^*_{k,G}$

We apply Rouché's theorem with the second holomorphic function being $1 - H_{n^*}(s)$. It is easy to check that the equation $H_{n^*}(s) = 1$ has a complex zero s_0, with

$$|s_0 - s^*| \le \frac{9}{2}\left(\frac{1}{\log n^*}\right)^k =: r.$$

The region we are working with is a disk centered at s^* with radius R that will be determined later.

It follows from (3.4) and (3.5) that for $|s - s^*| \le R$,

$$\begin{aligned}\left|F^*_{k,G}(s) - (1 - H_{n^*}(s))\right| &\le \sum_{\substack{n=2 \\ n\ne n^*}}^{\infty} |H_n(s)| \\ &\ll_G \left(\frac{1}{2^{\frac{1}{2}\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right)}}\right)^k.\end{aligned}$$

On the other hand, for s on the boundary of the disk $D(s^*, R)$,

$$\begin{aligned}|1 - H_{n^*}(s)| &= |1 - H_{n^*}(s_0 - s_0 + s)| = |1 - n^{s_0 - s}| \\ &= |1 - e^{(s_0 - s)\log n^*}| \ge \frac{|s_0 - s|\log n^*}{2} \ge \frac{(R - r)\log n^*}{2}.\end{aligned}$$

Therefore, it is enough to choose R so that

$$\left(\frac{1}{2^{\frac{1}{2}\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right)}}\right)^k \ll_G \frac{(R - r)\log n^*}{2},$$

or just,

$$R = C_G\left(\frac{1}{2^{\frac{1}{2}\left(\frac{\log\log n^*}{\log n^*} - \frac{\log\log n^\#}{\log n^\#}\right)}}\right)^k.$$

With this choice of R, we conclude that $F^*_{k,G}(s)$ has a zero $\beta^*_{k,G}$ that satisfies

$$|\beta^*_{k,G} - s^*| = \mathcal{O}_G(\eta_G^k),$$

and in particular

$$\operatorname{Re}\beta^*_{k,G} = k\frac{\log\log n^*}{\log n^*} + \frac{\log|a_{n^*}| + \log\log n^* - \log(\log n^* - 1)}{\log n^*} + \mathcal{O}_G(\eta_G^k). \tag{3.6}$$

3.2.3 $\beta_{k,G}$ and $\operatorname{Re}\beta^*_{k,G}$ are exponentially close

By definition of $\beta_{k,G}$, for all $\epsilon > 0$ there exists a function

$$\widetilde{F}_{k,G}(s) = \sum_{n=2}^{\infty} \frac{-1 + \tilde{c}_1 \log n + \cdots + (-1)^{k+1}\tilde{c}_k \log^k n}{n^\rho}$$

and a $\rho \in \mathbb{C}$ so that $\widetilde{F}_{k,G}(\rho) = 0$ and $\beta_{k,G} - \epsilon < \operatorname{Re}\rho \le \beta_{k,G}$. Suppose that $\vec{c} = (\tilde{c}_1, \dots, \tilde{c}_k)$ is the vector that corresponds to $\widetilde{F}$. Moreover, let $\delta = \operatorname{Re}\rho - \beta^*_{k,G}$. Without loss of generality, we can assume that $\epsilon < \beta_{k,G} - \beta^*_{k,G}$ so that $\delta > 0$. Then,

$$\begin{aligned}
1 = 1 - \widetilde{F}_{k,G}(\rho) &\le \sum_{n=2}^{\infty} \frac{|a_n|}{n^{\operatorname{Re}\rho}} \left(1 + \log n + \cdots + \log^k n\right) \\
&= \frac{|a_{n^*}|}{(n^*)^{\operatorname{Re}\rho}} \left(1 + \log n^* + \cdots + \log^k n^*\right) + \mathcal{O}_G(\eta_G^k) \\
&= \frac{|a_{n^*}|}{(n^*)^{\operatorname{Re}\rho}} \frac{\log^{k+1} n^* - 1}{\log n^* - 1} + \mathcal{O}_G(\eta_G^k) \\
&\le 2\frac{|a_{n^*}|}{(n^*)^{\operatorname{Re}\rho}} \log^k n^* + \mathcal{O}_G(\eta_G^k),
\end{aligned}$$

since $\operatorname{Re}\rho > \beta^*_{k,G} = s^* + \mathcal{O}_G(\eta_G^k)$.

On the other hand,

$$\begin{aligned} 1 = \left|H_{n^*}(\beta^*_{k,G})\right| + \mathcal{O}_G(\eta_G^k) &\geq \frac{|a_{n^*}|}{(n^*)^{\operatorname{Re}\beta^*_{k,G}}}\left(\log n^* + \cdots + \log^k n^* - 1\right) + \mathcal{O}_G(\eta_G^k) \\ &\geq \frac{|a_{n^*}|}{(n^*)^{\operatorname{Re}\beta^*_{k,G}}}\log^k n^* + \mathcal{O}_G(\eta_G^k). \end{aligned}$$

We put together the above inequalities

$$\frac{|a_{n^*}|}{(n^*)^{\operatorname{Re}\beta^*_{k,G}}}\log^k n^* + \mathcal{O}_G(\eta_G^k) \leq 1 \leq 2\,\frac{|a_{n^*}|}{(n^*)^{\operatorname{Re}\rho}}\log^k n^* + \mathcal{O}_G(\eta_G^k),$$

and get

$$\frac{\log^k n^*}{(n^*)^{\beta^*_{k,G}}}\left(1 - \frac{2}{n^\delta}\right) \leq \mathcal{O}_G(\eta_G^k).$$

Now, since

$$\frac{(n^*)^{\beta^*_{k,G}}}{\log^k n^*} = (n^*)^{\beta^*_{k,G} - k\frac{\log\log n^*}{\log n^*}} = (n^*)^{\mathcal{O}_G(1)} = \mathcal{O}_G(1),$$

we have

$$\frac{1}{2^\delta} \geq 1 + \mathcal{O}_G(\eta_G^k),$$

and we get that $\delta = \mathcal{O}_G(\eta_G^k)$.

3.3 Relevant numbers

In this section, we work with the Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \quad \operatorname{Re}s > 1,$$

associated to a Dirichlet character χ modulo q.

For a vector $\vec{c} = (c_1, c_2, \ldots, c_k) \in \mathbb{C}^k$ with $\|\vec{c}\|_\infty \le 1$, define

$$F_\chi(s) = L(s,\chi) + c_1 L'(s,\chi) + \cdots + c_k L^{(k)}(s,\chi). \tag{3.7}$$

For each fixed Dirichlet character χ, denote by $\beta_{k,\chi}$ the supremum of real parts of all zeros of all combinations as in (3.7). Theorem 3.2 gives the asymptotic formula for $\beta_{k,\chi}$ when k is large enough, where the relevant number n^* is determined by the unique integer n with $\chi(n) \neq 0$, which maximizes $\frac{\log\log n}{\log n}$. In other words, the relevant number is the first number n in Table 3.1 below, for which $\chi(n) \neq 0$.

n	$\frac{\log\log n}{\log n}$	n	$\frac{\log\log n}{\log n}$	n	$\frac{\log\log n}{\log n}$	n	$\frac{\log\log n}{\log n}$
15	0.367877...	21	0.365687...	28	0.361213...	45	0.351161...
16	0.367808...	22	0.36509...	$\vdots$	$\cdots$	46	0.350649...
14	0.367715...	11	0.364733...	32	0.358632...	$\vdots$	$\cdots$
17	0.367573...	23	0.364468...	9	0.358268...	65	0.342318...
13	0.367235...	24	0.363829...	33	0.358004...	7	0.342117...
18	0.367214...	25	0.363180...	$\vdots$	$\cdots$	66	0.341942...
19	0.366765...	26	0.362526...	43	0.352212...	67	0.341571...
12	0.366306...	10	0.362216...	8	0.352065...	$\vdots$	$\cdots$
20	0.366251...	27	0.361869...	44	0.351682...		

Table 3.1: List of values of $\frac{\log\log n}{\log n}$ in decreasing order.

We begin by proving Theorem 3.3.

Proof of Theorem 3.3. First observe that the relevant number to an L-function depends only on the modulus of the corresponding character. We separately check which multiples of small primes can be relevant. One can immediately notice that the only even relevant number is 16. Indeed, if a number $2m$ is relevant to some character χ, then $\chi(2m) \neq 0$ implying $\chi(2) \neq 0$, hence $\chi(16) \neq 0$. However, 16 appears earlier in Table 3.1 than any other even number, so 16 must be the relevant number to χ. The existence of characters whose relevant number is 16 is clear when one takes a character of modulus q with $(q,2) = 1$ and $(q,15) \neq 1$. Next, any odd multiple of 3 being relevant means $\chi(3) \neq 0$, hence $\chi(27) \neq 0$. Since any odd multiple of 3 other than 15 and 21 appear in Table 3.1 later than 27, the only possible relevant numbers divisible by 3 are $15, 21$ and 27. Again, all three can appear as relevant numbers if one chooses the modulus q appropriately: 15 is relevant to any character modulo q with $(q, 15) = 1$; 21 is relevant to those characters modulo q with $(q, 21) = 1$ and $m|q$ where $m = 2, 5, 13, 17, 19$; 27 is relevant to those with $(q, 3) = 1$ and $m|q$, where $m = 2, 5, 7, 11, 13, 17, 19, 23$. The reader can similarly verify that the only relevant numbers that are a multiple of 5 are 25 and 15, those that are a multiple of 7 are 21 and 49, and all the primes $p \geq 11$. □

To understand how often an integer n appears as a relevant number to some Dirichlet L-function we introduce the following notation. First, write $R(\chi) := R(q)$ for the relevant number corresponding to the character χ modulo q. Next, we define

$$h(r,x) = \frac{\sum\limits_{q \leq x} \sum\limits_{\substack{\chi \pmod q \\ R(\chi)=n}} 1}{\sum\limits_{q \leq x} \sum\limits_{\chi \pmod q} 1} = \frac{\sum\limits_{\substack{q \leq x \\ R(\chi)=n}} \phi(q)}{\sum\limits_{q \leq x} \phi(q)}. \tag{3.8}$$

and set $h(n) := \lim\limits_{x \to \infty} h(n,x)$. The number $h(n)$ represents the probability of n being relevant. In the limit, as $x \to \infty$, the denominator has the asymptotic formula (see for instance [HW54], Theorem 330, p. 268)

$$\sum_{q \leq x} \phi(q) = \frac{3}{\pi^2} x^2 + \mathcal{O}(x \log x). \tag{3.9}$$

The following two lemmas will provide us with an asymptotic formula for the numerator.

Lemma 3.4. *For any fixed square-free integer D, and any $x \geq 2$,*

$$\sum_{\substack{q \leq x \\ (q,D)=1}} \phi(q) = \frac{3}{\pi^2} x^2 \prod_{\substack{p|D \\ p \text{ prime}}} \frac{p}{p+1} + \mathcal{O}_D(x \log x).$$

Proof. Using the fact that $\phi(q) = q \sum_{d|q} \frac{\mu(d)}{d}$ and changing the order of summation as necessary, we find that

$$\begin{aligned}
\sum_{\substack{q\le x\\(q,D)=1}} \phi(q) &= \sum_{\substack{q\le x\\(q,D)=1}} q \sum_{d|q} \frac{\mu(d)}{d} \\
&= \sum_{1\le d\le x} \frac{\mu(d)}{d} \sum_{\substack{q\le x\\ d|q\\(q,D)=1}} q \\
&= \sum_{1\le d\le x} \frac{\mu(d)}{d} \sum_{\substack{q\le x\\ d|q}} q \sum_{\substack{\tilde{d}|D\\ \tilde{d}|q}} \mu(\tilde{d}) \\
&= \sum_{\tilde{d}|D} \mu(\tilde{d}) \sum_{1\le d\le x} \frac{\mu(d)}{d} \sum_{\substack{q\le x\\ d|q,\, \tilde{d}|q}} q. \qquad (3.10)
\end{aligned}$$

Moreover, it is easy to see that

$$\sum_{\substack{q\le x\\ d|q,\, \tilde{d}|q}} q = \frac{x^2}{2\cdot[d,\tilde{d}]} + \mathcal{O}(x). \qquad (3.11)$$

Therefore, using (3.11) in (3.10), we have

$$\begin{aligned}
\sum_{\substack{q\le x\\(q,D)=1}} \phi(q) &= \frac{x^2}{2} \sum_{\tilde{d}|D} \mu(\tilde{d}) \sum_{1\le d\le x} \frac{\mu(d)}{d\cdot[d,\tilde{d}]} + \mathcal{O}\left(x \cdot \sum_{\tilde{d}|D} \sum_{1\le d\le x} \frac{1}{d}\right) \\
&= \frac{x^2}{2} \sum_{\tilde{d}|D} \mu(\tilde{d}) \sum_{1\le d\le x} \frac{\mu(d)\cdot(d,\tilde{d})}{d^2\cdot\tilde{d}} + \mathcal{O}_D\left(x\log x\right) \\
&= \frac{x^2}{2} \sum_{\tilde{d}|D} \frac{\mu(\tilde{d})}{\tilde{d}} \sum_{d\ge 1} \frac{\mu(d)\cdot(d,\tilde{d})}{d^2} + \mathcal{O}\left(x^2 \cdot \sum_{\tilde{d}|D} \sum_{d\ge x} \frac{1}{d^2}\right) + \mathcal{O}_D\left(x\log x\right) \\
&= C_D \cdot x^2 + \mathcal{O}_D(x\log x),
\end{aligned}$$

where

$$C_D = \frac{1}{2}\sum_{\tilde{d}|D}\frac{\mu(\tilde{d})}{\tilde{d}}\sum_{d\geq 1}\frac{\mu(d)\cdot(d,\tilde{d})}{d^2} = \frac{1}{2}\sum_{d\geq 1}\frac{\mu(d)}{d^2}\sum_{\tilde{d}|D}\frac{\mu(\tilde{d})\cdot(d,\tilde{d})}{\tilde{d}}.$$

Let

$$A_{D,d} := \sum_{\tilde{d}|D}\frac{\mu(\tilde{d})\cdot(d,\tilde{d})}{\tilde{d}},$$

and suppose that the square-free integers D and d have the prime factorizations $D = p_1\cdots p_k\cdot q_1\cdots q_l$ and $d = p_1\cdots p_k\cdot r_1\cdots r_t$, where $p_1,\dots,p_k, q_1,\dots,q_l, r_1,\dots,r_t$ are all distinct primes. Then,

$$\begin{aligned}
A_{D,d} &= \sum_{\substack{a|p_1\cdots p_k\\ b|q_1\cdots q_l}}\frac{\mu(ab)\cdot(d,ab)}{ab}\\
&= \sum_{\substack{a|p_1\cdots p_k\\ b|q_1\cdots q_l}}\frac{\mu(a)\mu(b)}{b}\\
&= \sum_{a|p_1\cdots p_k}\mu(a)\sum_{b|q_1\cdots q_l}\frac{\mu(b)}{b}.
\end{aligned}$$

It is clear that $A_{D,d} = 0$ when $p_1\cdots p_k > 1$, or in other words, when $(D,d) > 1$. On the other hand, if $(D,d) = 1$, then

$$A_{D,d} = \sum_{b|q_1\cdots q_l}\frac{\mu(b)}{b} = \frac{\phi(q_1\cdots q_l)}{q_1\cdots q_l} = \frac{\phi(D)}{D},$$

and so

$$\begin{aligned}
C_D &= \frac{\phi(D)}{2D}\frac{1}{2}\sum_{d\geq 1}\frac{\mu(d)}{d^2}\\
&= \frac{\phi(D)}{2D}\prod_{\substack{p\nmid D\\ p\text{ prime}}}\left(1-\frac{1}{p^2}\right)\\
&= \frac{1}{2\zeta(2)}\frac{\phi(D)}{D}\prod_{\substack{p|D\\ p\text{ prime}}}\left(1-\frac{1}{p^2}\right)^{-1}
\end{aligned}$$

$$= \frac{3}{\pi^2} \prod_{\substack{p|D \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right)^{-1}$$

$$= \frac{3}{\pi^2} \prod_{\substack{p|D \\ p \text{ prime}}} \frac{p}{p+1}.$$

□

The next lemma follows from the previous one by induction.

Lemma 3.5. *For any square-free integers d and D, with $(D,d) = 1$, and any $x \geq 2$,*

$$\sum_{\substack{q \leq x \\ (q,D)=1 \\ d|q}} \phi(q) = \frac{3}{\pi^2} x^2 \prod_{\substack{p|D \\ p \text{ prime}}} \frac{p}{p+1} \prod_{\substack{r|d \\ r \text{ prime}}} \frac{1}{q+1} + \mathcal{O}_{D,d}(x \log x). \tag{3.12}$$

Proof. We proceed by induction on the number of prime divisors of d. For $d = 1$ the formula reduces to Lemma 3.4. Assume that the inductive hypothesis is true when d has m prime divisors. In that case,

$$\begin{aligned}
\sum_{\substack{q \leq x \\ (q,D)=1 \\ r_1 \cdots r_{m+1}|q}} \phi(q) &= \sum_{\substack{q \leq x \\ (q,D)=1 \\ r_1 \cdots r_m|q}} \phi(q) - \sum_{\substack{q \leq x \\ (q,D)=1 \\ (q,r_{m+1})=1 \\ r_1 \cdots r_m|q}} \phi(q) \\
&= \frac{3}{\pi^2} x^2 \prod_{\substack{p|D \\ p \text{ prime}}} \frac{p}{p+1} \prod_{i=1}^{m} \frac{1}{r_i+1} - \\
&\qquad - \frac{3}{\pi^2} x^2 \frac{r_{m+1}}{r_{m+1}+1} \prod_{\substack{p|D \\ p \text{ prime}}} \frac{p}{p+1} \prod_{i=1}^{m} \frac{1}{r_i+1} + \mathcal{O}_{D,d}(x \log x) \\
&= \frac{3}{\pi^2} x^2 \prod_{\substack{p|D \\ p \text{ prime}}} \frac{p}{p+1} \prod_{i=1}^{m} \frac{1}{r_i+1} \left(1 - \frac{r_{m+1}}{r_{m+1}+1}\right) + \mathcal{O}_{D,d}(x \log x)
\end{aligned}$$

$$= \frac{3}{\pi^2}x^2 \prod_{\substack{p|D \\ p \text{ prime}}} \frac{p}{p+1} \prod_{i=1}^{m+1} \frac{1}{r_i+1} + \mathcal{O}_{D,d}(x \log x),$$

and the proof is complete. □

We now have all the necessary tools to prove the following theorem, which gives the frequency for each relevant number.

Theorem 3.6. *For all primes $p \geq 53$,*

$$h(p) = \frac{p}{p+1} \prod_{\substack{r \text{ prime} \\ r<p}} \frac{1}{r+1}.$$

The values of $h(n)$ for the other relevant numbers n are given in Table 3.2 below.

Proof. Let d_p be the product of all the primes less than p. One can easily check that for primes $p \geq 53$, the condition $R(\chi) = p$ is equivalent to $(p,q) = 1$ and $d_p|q$, where q is the modulus of χ. By Lemma 3.5,

$$\sum_{\substack{q \leq x \\ R(\chi)=p}} \phi(q) = \sum_{\substack{q \leq x \\ d_p|q \\ (p,q)=1}} \phi(q) = \frac{3}{\pi^2}x^2 \frac{p}{p+1} \prod_{\substack{r \text{ prime} \\ r<p}} \frac{1}{r+1} + \mathcal{O}_{p,d_p}(x \log x).$$

Taking the limit as $x \to \infty$ in (3.8) we get the desired result for $p \geq 53$.

To find $h(n)$ for the remaining relevant numbers, one should apply (3.12) with appropriate D and d. For example, 15 is the relevant number to a character modulo q if and only if $(q,15) = 1$, so we use (3.12) with $D = 15$. Taking the limit in (3.8) we get $h(15) = \frac{3}{3+1} \cdot \frac{5}{5+1} = \frac{5}{8}$. Similarly, 13 is relevant to any character modulo q with $(q,13) = 1$, and $(q,15) \neq 1, (q,2) \neq 1$ and $(q,17) \neq 1$.

In this case, $D = 13$ and $d = 2 \cdot 15 \cdot 17$ and the numerator of (3.8) becomes

$$\begin{aligned}
\sum_{\substack{q \le x \\ R(\chi)=13}} \phi(q) = \sum_{\substack{q \le x \\ (q,13)=1, \\ 2\cdot 15\cdot 17 | q}} \phi(q) &= \sum_{\substack{q \le x \\ (q,13)=1 \\ 2\cdot 17 | q}} \phi(q) - \sum_{\substack{q \le x \\ (q,13\cdot 3\cdot 5)=1 \\ 2\cdot 17 | q}} \phi(q) \\
&= \frac{3}{\pi^2} x^2 \frac{13}{14} \cdot \frac{1}{3} \cdot \frac{1}{18} - \frac{3}{\pi^2} x^2 \frac{13}{14} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{3} \cdot \frac{1}{18} + \mathcal{O}(x \log x) \\
&= \frac{3}{\pi^2} x^2 \cdot \frac{13}{2016} + \mathcal{O}(x \log x),
\end{aligned}$$

and therefore $h(13) = \dfrac{13}{2016}$.

The verification of the remainder of Table 3.2 is straightforward, and so we skip the computations.

$\square$

n	$h(n)$	n	$h(n)$	n	$h(n)$
11	$\frac{187}{11612160}$	21	$\frac{1}{138240}$	37	$\frac{37}{3812504371200}$
13	$\frac{13}{2016}$	23	$\frac{391}{278691840}$	41	$\frac{41}{160125183590400}$
15	$\frac{5}{8}$	25	$\frac{1}{20901888}$	43	$\frac{43}{7045508077977600}$
16	$\frac{1}{4}$	27	$\frac{1}{278691840}$	47	$\frac{47}{338184387742924800}$
17	$\frac{17}{144}$	29	$\frac{29}{3135283200}$	49	$\frac{1}{386496443134771200}$
19	$\frac{19}{40320}$	31	$\frac{31}{100329062400}$		

Table 3.2: List of values of $h(n)$ for relevant $n < 53$.

Remark. *It can be checked that the sum of values of* $h(n)$ *from the table along with the sum of* $h(p)$ *for* $p \geq 53$ *is indeed* 1.

Chapter 4

Monotonicity properties of L-functions

4.1 Introduction

A number of interesting results on the horizontal monotonicity of various Dirichlet series, for instance, for the Riemann zeta function and related L-functions, have been known, some of which can be attributed to Spira [Spi65b], Saidak and Zvengrowski [SZ03], Matiyasevich, Saidak and Zvengrowski [MSZ14], Zhang [Zha14], and the references therein. Concerning horizontal monotonicity along the real line, Alzer [AB02] proved the monotonicity properties of a function related to the Riemann zeta function given by $\left(1 - \frac{1}{\zeta(s)}\right)^{1/(s-a)}$. A finer property than monotonicity is that of being completely monotonic and logarithmically completely monotonic. A function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic (CM for short) if it has derivatives of all orders and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for all $x > 0, n = 0, 1, 2, \ldots$. The notion of a logarithmically completely monotonic function was first used by Attanassov and Tsoukrovski [AT88] in their investigation of properties of completely monotonic functions. A function $f : (0, \infty) \to (0, \infty)$ is logarithmically completely monotonic (LCM for short), if $(\log f(x))^{(k)}$ exists for $k \geq 1$ and

$$(-1)^k [\log f(x)]^{(k)} \geq 0,$$

for all $x > 0$ and $k \geq 1$. Attanassov and Tsoukrovski also showed that the class of LCM functions are contained in the class of CM functions. Considerable work has been done by various authors in providing new classes of CM and LCM functions and proving properties of these functions, see for example [QGC06], [Che07], [QG09], [CE15], and the references therein. Most of the classes of functions studied above involve Euler's gamma function, digamma, polygamma or certain combinations of gamma function. When it comes to other classes of functions, a good example is the Riemann zeta function, whose LCM property follows by simply looking at the Dirichlet series of its logarithmic derivative. For the Dirichlet L-functions, it was proved in [DRZ14] that the Dirichlet L-functions associated to real primitive characters are not LCM. One is naturally interested in examining LCM properties for other Dirichlet series not restricted to Dirichlet L-functions or $\zeta(s)$. The purpose here is to expand the class of Dirichlet series whose LCM property is known. We describe a large class of Dirichlet series that are not LCM, including the Dirichlet L-functions, the derivatives of the Riemann zeta function, and the Ramanujan-tau L-function among others. This class is defined as follows:

Let $\mathcal{A}$ be the class of those real Dirichlet series $G(s) = \sum\limits_{n=1}^{\infty} a_n/n^s$, which satisfy the following four properties:

1. $G(s)$ has finite abscissa of convergence σ_α.
2. $G(\sigma) > 0$ for $\sigma > \sigma_\alpha$.
3. $G(s)$ has a meromorphic continuation to the whole complex plane with finitely many poles $\beta_1, \beta_2, \dots, \beta_m$ of orders $\mu_1, \mu_2, \dots, \mu_m$, respectively, and the function $(s-\beta_1)^{\mu_1} \dots (s-\beta_m)^{\mu_m} G(s)$ is entire of order 1.
4. If we define $\mathcal{E} = \{\text{real zeros}\} \cup \{\text{poles}\}$ and $\mu = \max\{\operatorname{Re} z : z \in \mathcal{E}\}$, then G(s) has a complex zero ρ with $\operatorname{Re}\rho > \mu$.

We have the following theorem whose proof is the content of Section 4.2.

Theorem 4.1. *For any real Dirichlet series $G(s) = \sum\limits_{n=1}^{\infty} a_n/n^s$ in the class $\mathcal{A}$, there exists a real λ such that $G(s)$ is not logarithmically completely monotonic in any subinterval of $[\lambda, \infty)$.*

Section 4.3 is devoted to examples of functions in $\mathcal{A}$ with two notable examples being $\zeta'(s)$ and the L-function associated to the Ramanujan-tau function.

In the same spirit of studying signs of derivatives of logarithms of Dirichlet series, we are tempted to know, for a fixed k, the sign changes of the kth derivative of the logarithm of a Dirichlet series at any two distinct points. In Section 4.4, we investigate this question for the Ramanujan-tau L-function and the answer to which yields another formulation of the Riemann Hypothesis for this L-function. The Ramanujan-tau function $\tau(n)$ is defined as the coefficient of x^n in the power series expansion of the product $x\prod_{n=1}^{\infty}(1-x^n)^{24}$. That is,

$$\sum_{n=1}^{\infty}\tau(n)x^n = x\prod_{n=1}^{\infty}(1-x^n)^{24}.$$

Mordell proved that τ is a multiplicative function and that

$$\tau(p^{r+1}) = \tau(p)\tau(p)^r - p^{11}\tau(p^{r-1}),$$

where p is a prime and r a positive integer. We consider the Dirichlet series associated to τ

$$L_\Delta(s) = \sum_{n=1}^{\infty}\frac{\tau(n)}{n^s},$$

which is absolutely convergent for $\sigma = \mathrm{Re}(s) > 13/2$, but we will instead work with the normalized or shifted L-function given by

$$L(s) = \sum_{n=1}^{\infty}\frac{\tau(n)}{n^{s+11/2}},$$

for $\sigma > 1$. $L(s)$ is entire and satisfies a functional equation given by

$$(2\pi)^{-s}\Gamma\left(s+\frac{11}{2}\right)L(s) = (2\pi)^{s-1}\Gamma\left(\frac{13}{2}-s\right)L(1-s),$$

from which one finds that the trivial zeros of $L(s)$ are at the poles of $\Gamma\left(s+\frac{11}{2}\right)$, i.e. the half integers $-m-11/2$, $m \geq 0$. All the non-trivial zeros lie in the critical strip $0 < \sigma < 1$ and the Riemann Hypothesis for $L(s)$ asserts that they lie on the critical line $\sigma = 1/2$. For any $k \geq 1$, denote by

$\mathcal{F}^{(k)}(s)$ the k-th derivative of $\log(L(s))$. Define a function

$$\mathcal{G} : [1/2, \infty) \to B, \tag{4.1}$$

where the set B consists of functions $f : \mathbb{N} \to \{-1, 0, 1\}$ with the equivalence relation $f \sim g$, if $f(n) = g(n)$ for all n large enough. For every $s \in [1/2, \infty)$, $\mathcal{G}(s) \in B$ where $\mathcal{G}(s) : \mathbb{N} \to \{-1, 0, 1\}$ defined by $\mathcal{G}(s)(k) := \operatorname{sgn}(\mathcal{F}^{(k)}(s))$. Using this function, we have the following formulation of the Riemann hypothesis for $L(s)$.

Theorem 4.2. *Let $\rho_0 = \frac{1}{2} + i\gamma_0$ be the non-trivial non-real zero of $L(s)$ with the least imaginary part γ_0. Then the Riemann Hypothesis is true for $L(s)$ if and only if $\mathcal{G}$ is injective in the interval $(\frac{\gamma_0^2}{12} - \frac{5}{2}, \infty)$.*

4.2 Proof of Theorem 5.1

We begin this section by introducing some notation necessary to prove an auxiliary result. Unless otherwise mentioned, here and in what follows s will denote a real variable. For $s > \mu$, denote by $l(s)$, the distance between s and the set of zeros of G, that is,

$$l(s) := \min\{|s - \rho| : G(\rho) = 0\},$$

and also define

$$\lambda := \inf\{c > \mu : s > c \Rightarrow |s - \mu| > l(s)\}.$$

Let $\rho_0 = \beta_0 + i\gamma_0$ be a zero of $G(s)$ such that $\gamma_0 = \min\{\operatorname{Im}\rho : G(\rho) = 0, \ \operatorname{Re}\rho > \mu\}$. For $s > \frac{1}{2}\left(\mu + \beta_0 + \frac{{\gamma_0}^2}{\beta_0 - \mu}\right)$, we have

$$(s - \mu)^2 > (s - \beta_0)^2 + {\gamma_0}^2 \geq (l(s))^2,$$

and therefore $\lambda \leq \frac{1}{2}\left(\mu + \beta_0 + \frac{{\gamma_0}^2}{\beta_0 - \mu}\right)$. Moreover, notice that for $s > \lambda$ the minimum $l(s)$ must be attained at a zero ρ of G to the right of the line $\operatorname{Re} s = \mu$.

Lemma 4.3. *Assume that the Dirichlet series $G(s) = \sum_{n=1}^{\infty} a_n/n^s$, where the coefficients a_n are real, is in $\mathcal{A}$. Then for any fixed $t > \max\{\lambda, \sigma_\alpha\}$, where λ is as above and σ_α is the abscissa of convergence of $G(s)$, there exist an $\epsilon > 0$, a unique zero $\rho_{1,t}$ of G with $\operatorname{Re}(\rho_{1,t}) > \mu$, and a zero $\rho_{2,t}$ such that for all $s \in (t, t+\epsilon)$, we have*

$$|s - \rho_{1,t}| < |s - \rho_{2,t}| \leq |s - \rho|,$$

for all zeros $\rho \neq \rho_{1,t}, \overline{\rho_{1,t}}$.

Proof. Fix $t > \max\{\lambda, \sigma_\alpha\}$, and let $\rho_{1,t}$ be the zero of G with maximum real part among those zeros for which the minimum $l(t)$ is attained. Take any $s^* > t$ and denote by $R(t, s^*)$ the closed region that lies inside the circle centered at s^* with radius $|s^* - \rho_{1,t}|$, outside the circle centered at t with radius $l(t)$, and above the real line (see shaded region in Figure 4.1). The perpendicular bisector of the line segment joining $\rho_{1,t}$ and any other zero ρ in $R(t, s^*)$, crosses the x-axis between t and s^*. The collection of all such points of intersection with the x-axis is a finite set, and therefore there exists a $\delta_1 > 0$ for which the interval $(t, t + \delta_1)$ is free of such crossings. We claim that for any $s \in (t, t + \delta_1)$, the minimum distance from s to a zero of G is attained uniquely at $\rho_{1,t}$. On the contrary, suppose that there is an s in the above-mentioned interval and a zero ρ_s of G such that $|s - \rho_s| < |s - \rho_{1,t}|$. It follows that ρ_s is in $R(t, s^*)$ and that the perpendicular bisector of the line segment joining ρ_s and $\rho_{1,t}$ divides the plane into two half-planes leaving s and ρ_s on one side, and t and $\rho_{1,t}$ on the other. In other words, the perpendicular bisector crosses the real axis between s and t, contradicting the fact that the interval $(t, t + \delta_1)$ doesn't include such points.

Next, we consider $\rho_{2,t}$ to be the zero of G with maximum real part among those zeros for which the minimum $\tilde{l}(t) := \min\{|t - \rho| : G(\rho) = 0,\ \rho \neq \rho_{1,t}\}$ is attained. Repeating the above argument where we replace $l(t)$ by $\tilde{l}(t)$ and $\rho_{1,t}$ by $\rho_{2,t}$, produces a $\delta_2 > 0$ with the following property: for any $s \in (t, t+\delta_2)$, the minimum distance from s to a zero of G other than $\rho_{1,t}$ is attained uniquely at $\rho_{2,t}$.

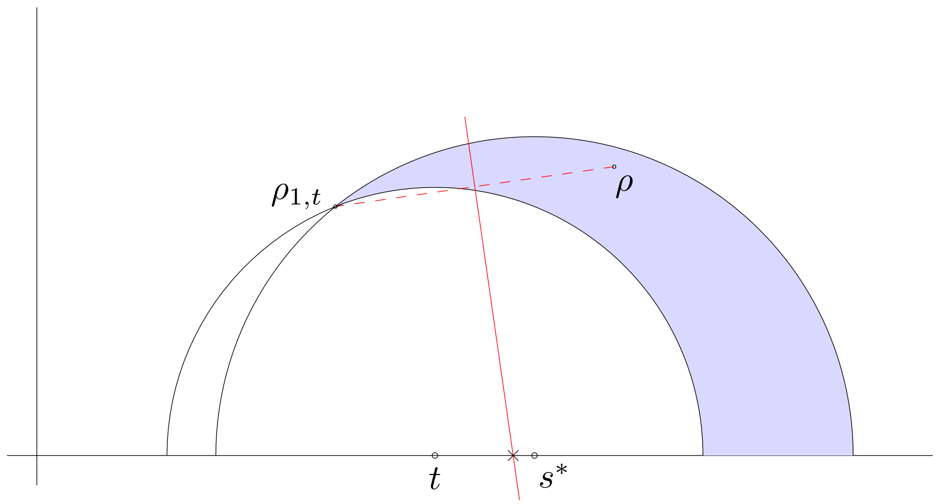


Figure 4.1: Region $R(t, s^*)$ for typical t.

Taking $\epsilon = \min\{\delta_1, \delta_2\}$ completes the proof. $\square$

By Hadamard's theory of integral functions, the entire function of order one, $g(s) = (s-\beta_1)^{\mu_1}\cdots(s-\beta_m)^{\mu_m}G(s)$, can be written as

$$g(s) = s^l\, e^{(As+B)} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \tag{4.2}$$

where A and B are constants, $l \geq 0$ is possibly the order of $s = 0$ as a zero and the product is over the nonzero roots of $G(s)$. Taking the logarithmic derivative of (4.2) gives

$$F'(s) = \frac{G'(s)}{G(s)} = A + \frac{l}{s} + \sum_{i=1}^{m} \frac{-\mu_i}{s-\beta_i} + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{4.3}$$

Differentiating the above expression an additional $k-1$ times, yields the following representation

$$F^{(k)}(s) = (-1)^{k-1}(k-1)!\left(\frac{l}{s^k} - \sum_{i=1}^{m} \frac{\mu_i}{(s-\beta_i)^k} + \sum_\rho \frac{1}{(s-\rho)^k}\right).$$

We now proceed to the proof of Theorem 5.1 where the above representation is used to prove infinitely many sign changes of $F^{(k)}(s)$.

Proof of Theorem 5.1. Suppose I is a fixed subinterval of $[\lambda, \infty)$ and let t be an interior point of I. If ϵ, $\rho_{1,t}$ and $\rho_{2,t}$ are as in Lemma 4.3, then for any $s \in [a,b] \subset I \cap (t, t+\epsilon)$, we have

$$F^{(k)}(s) = (-1)^{k-1}(k-1)!\Bigg(\frac{1}{(s-\rho_1^t)^k} + \frac{1}{(s-\overline{\rho_1^t})^k} + \frac{l}{s^k} - \\ - \sum_{i=1}^{m} \frac{\mu_i}{(s-\beta_i)^k} + \sum_{\substack{\rho \\ \rho \neq \rho_1^t, \overline{\rho_1^t}}} \frac{1}{(s-\rho)^k} \Bigg).$$

If we let $s - \rho_{1,t} = r_s e^{i\theta_s}$, then the above becomes

$$F^{(k)}(s) = \frac{(-1)^{k-1}(k-1)!}{r_s^k}\Bigg(2\cos(k\theta_s) + l\left(\frac{r_s}{s}\right)^k - \\ - \sum_{i=1}^{m} \mu_i \left(\frac{r_s}{s-\beta_i}\right)^k + \sum_{\substack{\rho \\ \rho \neq \rho_{1,t}, \overline{\rho_{1,t}}}} \left(\frac{r_s}{s-\rho}\right)^k \Bigg). \tag{4.4}$$

It is not hard to see that

$$\left| \sum_{i=1}^{m} \mu_i \left(\frac{r_s}{s-\beta_i}\right)^k \right| \leq \sum_{i=1}^{m} |\mu_i| \left| \frac{s-\rho_1^t}{s-\beta_i} \right|^k \leq \left(\sum_{i=1}^{m} |\mu_i| \right) \cdot L^k$$

and

$$\left| l\left(\frac{r_s}{s}\right)^k \right| = l \cdot \left| \frac{s-\rho_1^t}{s} \right|^k \leq l \cdot M^k,$$

where $L := \sup_{1\leq i\leq m} \sup_{s\in[a,b]} \left|\frac{s-\rho_{1,t}}{s-\beta_i}\right| < 1$ and $M := \sup_{s\in[a,b]} \left|\frac{s-\rho_{1,t}}{s}\right| < 1$. Note that when $s = 0$ is not a zero of $g(s)$, we have $l = 0$, and the last estimate becomes redundant.

Moreover,

$$\begin{aligned}
\left| \sum_{\substack{\rho \\ \rho \neq \rho_{1,t}, \overline{\rho_{1,t}}}} \left(\frac{r_s}{s-\rho} \right)^k \right| &\leq \sum_{\substack{\rho \\ \rho \neq \rho_{1,t}, \overline{\rho_{1,t}}}} \left| \frac{s-\rho_{1,t}}{s-\rho} \right|^k \\
&= |s-\rho_{1,t}|^2 \sum_{\substack{\rho \\ \rho \neq \rho_{1,t}, \overline{\rho_{1,t}}}} \frac{1}{|s-\rho|^2} \left| \frac{s-\rho_{1,t}}{s-\rho} \right|^{k-2} \\
&\leq |d-\rho_{1,t}|^2 \left| \frac{s-\rho_{1,t}}{s-\rho_{2,t}} \right|^{k-2} \sum_{\substack{\rho \\ \rho \neq \rho_{1,t}, \overline{\rho_{1,t}}}} \frac{1}{|c-\rho|^2} \leq C_{a,b,G} \cdot K^{k-2},
\end{aligned}$$

where $K := \sup\limits_{s \in [a,b]} \left| \frac{s-\rho_{1,t}}{s-\rho_{2,t}} \right| < 1$ and the constant $C_{a,b,G}$ depends only on a, b and G.

The above estimates turn (4.4) into

$$F^{(k)}(s) = \frac{(-1)^{k-1}(k-1)!}{r_s^k} \Big(2\cos(k\theta_s) + \mathcal{O}_{a,b,G}(\eta^k) \Big), \tag{4.5}$$

with $\eta = \eta_{a,b,G} := \max\{K, L, M\} < 1$.

For $s \in [a,b]$ we have that θ_s is between θ_a and θ_b, and since for all large enough k, $|k\theta_b - k\theta_a|$ is at least 2π, $\cos(k\theta_s)$ attains all the values in the interval $[-1,1]$. Also, for large k, the error term in (4.5) is negligible. Therefore, for all large k, $F^{(k)}(s)$ changes sign in $[a,b]$ implying that $G(s)$ is not LCM in these intervals. □

4.3 Examples of functions in class $\mathcal{A}$

Let $s = \sigma + it$. Recall,

$$\zeta'(s) = -\sum_{n \geq 2} \frac{\log n}{n^s} \; ; \; \sigma > 1.$$

By analytic continuation, $\zeta'(s)$ can be extended to the whole complex plane with an exception of a pole of order two at $s = 1$. Moreover from [LM74b], for $n \geq 2$, there is exactly one trivial zero of $\zeta'(s)$ in the interval $(-2n, -2n+2)$ and there are no other zeros for $\mathrm{Re}(s) \leq 0$. Some of the non-trivial zeros of $\zeta'(s)$ have been computed numerically, see Figure 1 in [DnFF$^+$10], and one finds the existence of imaginary zeros of $\zeta'(s)$ with real part strictly greater than 1. All these properties together imply that $\zeta'(s)$ is in $\mathcal{A}$. This critical implication combined with Theorem 5.1 also presents us with an important and first example of an arithmetic function whose associated Dirichlet series is CM, but not LCM.

The Dirichlet L-functions satisfy the first three conditions to belong to $\mathcal{A}$. Moreover, it is proved in [CS02] that there exist infinitely many (at least 20% in a suitable sense) of all primitive quadratic Dirichlet characters whose Dirichlet L-functions do not vanish on the line segment $[0, 1]$. And since these have infinitely many zeros on the critical line $\sigma = 1/2$, it follows that these functions also satisfy condition (4) of the definition of the class $\mathcal{A}$.

In the context of more general L-functions, in practice, if a concrete L-function is given, one can first check whether or not this L-function has any real zeros (or poles, if it is the case) in the critical strip. If it does not, then the given L-function will satisfy condition (4), since all L-functions have infinitely many nontrivial zeros inside their critical strip. This is for example the case with the L-function associated to the Ramanujan tau function. Conjecturally, all self-dual modular form L-functions which do not vanish at the central point should similarly satisfy property (4). On GRH, if an L-function has a zero at the central point, then it is not in $\mathcal{A}$. But if it has no zero at the central point, then it is (at least on GRH) in $\mathcal{A}$. Recently, the notion of super-positivity has been introduced into the theory of L-functions by Yun and Zhang [YZ17]. An automorphic self dual L-function has the super-positivity property if all derivatives of the completed L-function at the central point $s = 1/2$ are nonnegative and all derivatives at a real point $s > 1/2$ are positive. Goldfeld and Huang [GH16], [GH18] have shown that at least 12% of L-functions associated to Hecke basis cusp forms of weight 2 and large prime level q have the super-positivity property. They also proved that at least 49% of such L-functions have no real zeros for $\mathrm{Re}\, s > 0$, except possibly at $s = 1/2$. As a consequence of their results, for each of these more than 49% of such L-

functions, whether condition (4) holds true or fails is completely determined by the value at $s = 1$ and by the Riemann Hypothesis being true or false for that particular L-function. To be precise, if the L-function in question vanishes at $s = 1/2$ then the L-function satisfies (4) if and only if RH fails for the given L-function; if the L-function does not vanish at $s = 1/2$, then condition (4) is satisfied regardless of whether or not RH holds for that L-function.

4.4 Monotonicity of Ramanujan tau L-function

In the case of the L-function associated to the Ramanujan tau function $L(s)$, one can explicitly find an interval in which $L(s)$ is not logarithmically completely monotonic. Such an interval can be obtained since the location of the zeros of $L(s)$ is well-known ([Spi73]). Although we know that $L(s)$ is not LCM in certain intervals, it is not known if $L(s)$ is CM or not and we pose this question to the interested reader. The non-trivial zero with the smallest imaginary part lies on the critical line $\sigma = \frac{1}{2}$ and has imaginary part $\gamma_0 \approx 9.222$. From the discussion at the beginning of Section 4.2, and since $\mu = -\frac{11}{2}$, we conclude that $L(s)$ is not logarithmically completely monotonic in (λ_τ, ∞), where $\lambda_\tau = \frac{\gamma_0^2}{12} - \frac{5}{2}$.

We first state and prove a coloring lemma which will help us prove Theorem 5.2.

Lemma 4.4 (Coloring Lemma)**.** *Let α, β, δ be real numbers such that $0 < \alpha < \frac{1}{4}$, $0 < \beta < \frac{1}{4}$, $0 < \delta < \min\{\frac{\alpha}{2}, \frac{\beta}{2}, \frac{1}{4} - \alpha, \frac{1}{4} - \beta\} \leq \frac{1}{12}$. Color the real line with three colors such that for any $x \in \mathbb{R}$, x is green if $\{x\} \in [0, \frac{1}{4} - \delta) \cap (\frac{3}{4} + \delta, 1)$, x is yellow if $\{x\} \in [\frac{1}{4} - \delta, \frac{1}{4} + \delta) \cap [\frac{3}{4} - \delta, \frac{3}{4} + \delta]$, and finally x is red if $\{x\} \in (\frac{1}{4} + \delta, \frac{3}{4} - \delta)$. If the sets*

$$\{k \in \mathbb{N} : k\alpha \text{ is green and } k\beta \text{ is red}\} \text{ and } \{k \in \mathbb{N} : k\alpha \text{ is red and } k\beta \text{ is green}\}$$

are finite then $\alpha = \beta$.

Proof. This scheme of coloring yields the following relations between the colors of pairs of real numbers.

1. There are no integers $m \in \mathbb{Z}$ such that both $m\alpha, (m+1)\alpha$ are yellow.

2. If $n \in \mathbb{Z}$ is such that $n\alpha$ is green then at least one of $(n+1)\alpha$ or $(n-1)\alpha$ is green.

3. If $n \in \mathbb{Z}$ is such that $n\alpha$ is red then at least one of $(n+1)\alpha$ or $(n-1)\alpha$ is red.

4. If $n \in \mathbb{Z}$ is such that $n\alpha$ is yellow then either $(n+1)\alpha$ is red and $(n-1)\alpha$ is green or $(n+1)\alpha$ is green and $(n-1)\alpha$ is red.

The above statements remain valid if we replace α by β.

Let k^* be an integer that is strictly larger than any element of the two finite sets provided in the statement of the lemma. For any $k \geq k^*$, if $k\alpha$ is green then $k\beta$ is either green or yellow, and if $k\alpha$ is red then $k\beta$ is either red or yellow. Now, fix $m_0 > k^*$, such that $m_0\alpha$ is green. Since $m_0 > k^*$, the color of $m_0\beta$ is either green or yellow. If $m_0\beta$ is green, then set $k_0 := m_0$. If $m_0\beta$ is yellow, then by statement (4) above, one of $(m_0-1)\beta$ and $(m_0+1)\beta$ is green and the other is red, and for simplicity, let us assume that $(m_0+1)\beta$ is green and $(m_0-1)\beta$ is red. Meanwhile, since $m_0\alpha$ is green, statement (2) implies that at least one of $(m_0+1)\alpha$ or $(m_0-1)\alpha$ is green. However, $m_0 - 1 \geq k^*$ and $(m_0-1)\beta$ is red, so $(m_0-1)\alpha$ cannot be green. Therefore $(m_0+1)\alpha$ is green, and in this case, we set $k_0 := m_0 + 1$.

By a similar argument, if $(m_0-1)\beta$ is green, one can deduce that $(m_0-1)\alpha$ is also green and we set $k_0 := m_0 - 1$. In any case, k_0 denotes an integer for which both $k_0\alpha$ and $k_0\beta$ are green.

Denote by I_0 the green interval that contains $k_0\alpha$, and by J_0 the green interval that contains $k_0\beta$. These intervals will each be of the form $\left(r - \frac{1}{4} + \delta, r + \frac{1}{4} - \delta\right)$ for some $r \in \mathbb{Z}$. Next, denote the green intervals following I_0 by $I_1, I_2, \ldots, I_m, \ldots$, and the green intervals following J_0 by $J_1, J_2, \ldots, J_m, \ldots$.

Claim 4.5. *For any integer $j \geq 0$, there exists an integer $K_j \geq k_0$ such that $K_j\alpha \in I_j$ and $K_j\beta \in J_j$.*

Proof of Claim. We prove this by induction on j. For $j = 0$, we take $K_0 = k_0$. Assume the statement holds for $j = l$, i.e. there exists an integer $K_l \geq k_0$, such that $K_l\alpha \in I_l$ and $K_l\beta \in J_l$. Let m_1 be the smallest integer m for which $(K_l + m)\alpha \in I_{l+1}$ and $(K_l + m + 1)\alpha \in I_{l+1}$. The existence of such an integer follows from the observation that the interval I_{l+1} has length at least 2α. On the other hand, neither $(K_l + m)\beta$, nor $(K_l + m + 1)\beta$ are red, because $K_l + m > k_0$. Moreover, they cannot both be yellow, since $(K_l + m + 1)\beta - (K_l + m)\beta = \beta$ and $2\delta < \beta < \frac{1}{4} + \delta \leq \frac{1}{2} - 2\delta$, and the intervals colored with yellow have length 2δ and are at distance at least $\frac{1}{2} - 2\delta$ from each other. We choose $K_{l+1} = K_l + m$, if $(K_l + m)\beta$ is green, otherwise we take $K_{l+1} = K_l + m + 1$. This completes the proof of the claim.

Therefore, for any $j \geq 0$,

$$K_j\alpha \in I_j = \left(r_I + j - \frac{1}{4} + \delta, r_I + j + \frac{1}{4} + \delta\right),$$

and

$$K_j\beta \in I_j = \left(r_J + j - \frac{1}{4} + \delta, r_J + j + \frac{1}{4} + \delta\right).$$

Hence, for j large enough, $K_j\alpha = j + \mathcal{O}(1)$ and $K_j\beta = j + \mathcal{O}(1)$. This implies that $\frac{K_j}{j} = \frac{1}{\alpha} + \mathcal{O}(\frac{1}{j})$, and $\frac{K_j}{j} = \frac{1}{\beta} + \mathcal{O}(\frac{1}{j})$. Taking $j \to \infty$, yields $\alpha = \beta$. □

Let us first prove the necessary implication in Theorem 5.2. Assume $L(s)$ satisfies the Riemann Hypothesis. We show that the function $\mathcal{G}$ from (4.1), is injective in the interval (c, ∞).

In order to show this, we show that for any $c < s_1 < s_2$, there are infinitely many integers k such that $\operatorname{sgn}\left(\mathcal{F}^{(k)}(s_1)\right) \neq \operatorname{sgn}\left(\mathcal{F}^{(k)}(s_2)\right)$. We now prove part (1) of Theorem 5.2.

Proof of Part (1) *of Theorem 5.2.* Under the Riemann hypothesis for $L(s)$, for all $s > \lambda_\tau$, the minimum $l(s)$ is attained at the non-trivial zero $\rho_0 = \frac{1}{2} + i\gamma_0$. Fix $s_1, s_2 > \lambda_\tau$, with $s_2 > s_1$. Following the proof of Theorem 5.1, for all $s \in [s_1, s_2]$, we have

$$\mathcal{F}^{(k)}(s) = \frac{(-1)^{k-1}(k-1)!}{|s - \rho_0|^k}\left(2\cos(k\theta_s) + \mathcal{O}_{s_1,s_2,L}(\eta^k)\right),$$

with $\eta = \sup_{s\in[s_1,s_2]} \left|\frac{s-\rho_0}{s-\rho_1}\right| < 1$, where ρ_2 is the second zero of $L(s)$ on the critical line.

If θ_{s_1} and θ_{s_2} are the arguments of $s_1 - \rho_0$ and $s_2 - \rho_0$ respectively, then $0 < \theta_{s_2} < \theta_{s_1} < \frac{\pi}{2}$. Let $\alpha = \frac{\theta_{s_1}}{2\pi}$, $\beta = \frac{\theta_{s_2}}{2\pi}$ and $0 < \delta < \beta/2 = \min\{\alpha/2, \beta/2, 1-\alpha/4, 1-\beta/4\}$. This choice of α, β and δ satisfies the requirements of the Coloring Lemma, except that $\alpha \neq \beta$. Therefore, there are infinitely many integers k such that $k\alpha = \frac{k\theta_{s_1}}{2\pi}$ is green and $k\beta = \frac{k\theta_{s_2}}{2\pi}$ is red (or there are infinitely many integers k for which $k\alpha$ is red and $k\beta$ is green, but one can similarly reach the same results). This implies that infinitely many $\left\{\frac{k\theta_{s_1}}{2\pi}\right\}$ lie in $\left(0, \frac{1}{4}-\delta\right)\cup\left(\frac{3}{4}+\delta, 1\right)$ and infinitely many $\left\{\frac{k\theta_{s_2}}{2\pi}\right\}$ lie in the interval $\left(\frac{1}{4}+\delta, \frac{3}{4}-\delta\right)$. Therefore, we get infinitely many $k \in \mathbb{N}$ for which $\operatorname{sgn}(\cos(k\theta_{s_1})) \neq \operatorname{sgn}(\cos(k\theta_{s_2}))$, and $\operatorname{sgn}(\mathcal{F}^{(k)}(s_1)) = \operatorname{sgn}(\cos(k\theta_{s_1}))$ and $\operatorname{sgn}(\mathcal{F}^{(k)}(s_2)) = \operatorname{sgn}(\cos(k\theta_{s_2}))$. $\square$

Next we prove part (2) of Theorem 5.2. We show that if the Riemann hypothesis for $L(s)$ fails then $\mathcal{G}$ is not injective in the interval $[c, \infty)$.

Proof of Part (2) of Theorem 5.2. Assume for the moment that there is only one zero ρ_1 of $L(s)$ lying above $y = \gamma_0$ and to the right of the critical line. Denote by s_0 and s_1, the points of intersection of the x-axis with the lines perpendicular to the line segment joining ρ_0 and ρ_1 that pass through ρ_0 and ρ_1, respectively. Note that $s_1 > s_0 > \lambda_\tau$ and if we let $s_0 - \rho_0 = r_0 e^{i\theta}$ and $s_1 - \rho_1 = r_1 e^{i\theta}$, then $l(s_0) = r_0$ and $l(s_1) = r_1$. By Lemma 4.3, we can find an $\epsilon > 0$, such that for all $s \in (s_0, s_0+\epsilon)$, the minimum distance $l(s)$ is attained at ρ_0, and for all $s \in (s_1, s_1+\epsilon)$, the minimum distance $l(s)$ is attained at ρ_1. Using the fact that the sequence $\{n\sqrt{m}\}$ with m any fixed positive integer greater than 1, is dense, one can find integers α and β, such that

$$\frac{\theta}{2\pi} < \alpha + \beta\sqrt{m} < \min\left(\frac{\arg(s_0+\epsilon-\rho_0)}{2\pi}, \frac{\arg(s_1+\epsilon-\rho_1)}{2\pi}\right).$$

Therefore, there exist $s' \in (s_0, s_0+\epsilon)$ and $s'' \in (s_1, s_1+\epsilon)$ for which $s' - \rho_0 = l(s')e^{i2\pi(\alpha+\beta\sqrt{m})}$ and $s'' - \rho_0 = l(s'')e^{i2\pi(\alpha+\beta\sqrt{m})}$. Following the proof of Theorem 5.1, we have

$$\mathcal{F}^{(k)}(s') = (-1)^{k-1}(k-1)!\left(\frac{2}{l(s')^k}\cos(2\pi k(\alpha+\beta\sqrt{m})) + S(s')\right),$$

and

$$\mathcal{F}^{(k)}(s'') = (-1)^{k-1}(k-1)!\left(\frac{2}{l(s'')^k}\cos(2\pi k(\alpha+\beta\sqrt{m})) + S(s'')\right),$$

where $|S(s')|$, $|S(s'')| < \tilde{C}\cdot\tilde{\eta}^k$, with the constants $\tilde{C}$ and $\tilde{\eta}$ depending only on s', s'' and L.

Using arguments similar to those in [DRZ14], one can show that there exists a positive constant $C_{\alpha,\beta}$, such that for k large enough and any integer r, $|4k(\alpha+\beta\sqrt{m})+r| > \frac{C_{\alpha,\beta}}{k}$. Now for l such that either $|4k(\alpha+\beta\sqrt{m})+1+4l| < 1$ or $|4k(\alpha+\beta\sqrt{m})-1+4l| < 1$, we can write,

$$\begin{aligned}
|\cos(2\pi k(a+b\sqrt{m}))| &= |\sin(2\pi k(a+b\sqrt{m}))+\pi/2)| \\
&= |\sin(\frac{\pi}{2}(4k(a+b\sqrt{m})\pm 1+4l))| \\
&\geq |4k(a+b\sqrt{m})\pm 1+4l| \\
&> \frac{C_{\alpha,\beta}}{k} > \tilde{C}\cdot\tilde{\eta}^k,
\end{aligned}$$

for all large k. This implies that for distinct points s' and s'', $\mathcal{F}^{(k)}(s'')$ and $\mathcal{F}^{(k)}(s'')$ have the same sign for all large enough k and so $\mathcal{G}$ is not injective on $[\lambda_\tau, \infty)$.

Recall that so far we worked with the assumption that ρ_1 is the only zero to the right of the critical line. If there are more such zeros, we choose ρ_1 to be the one with the smallest imaginary part among the zeros with the property that the line joining ρ_0 and ρ_1 has slope smaller than or equal to that of the line joining ρ_0 and any other ρ to the right of the critical line. □

Chapter 5

Walks to infinity

5.1 Introduction

In his well known book on unsolved problems in number theory ([Guy04]), Richard Guy mentions the following problem: is it possible to walk to infinity stepping only on Gaussian primes and taking steps of bounded length? The problem proved to be exceedingly difficult, and very little is known at present. Taking the difficulty of this problem into account, it may be worth studying the following related question, which in theory should be easier. Suppose that besides Gaussian primes we collect in a subset $\mathcal{P}$ of the complex plane all the primes of all rings of integers of all imaginary quadratic fields. Then ask a similar question to the one above: is it possible to walk to infinity stepping only on points in $\mathcal{P}$ by taking steps of bounded lentgh?

This problem has both an algebraic flavor and an analytic flavor. In connection with the algebraic aspect, one knows that there are only nine imaginary quadratic number fields with class number 1. For all the other imaginary quadratic number fields the corresponding ring of integers is not factorial. One has unique factorization in prime ideals, not in prime elements of the ring of integers. Therefore, by choosing a prime in such a ring, we mean choosing a prime ideal. But then such an ideal will not correspond to a point in the complex plane, unless the ideal is a principal ideal. In that case, the ideal will be generated by an element in the ring of integers, and that element will then be an acceptable element of $\mathcal{P}$. Thus, in order to create a walk stepping only on points from $\mathcal{P}$, one needs at each step to find a suitable imaginary quadratic number field, and in its ring of integers find a suitable prime ideal which is principal, and generated by an element that lies in a prescribed neighborhood where one intends to step in.

In connection with the analytic aspect, the problem is surprisingly connected to some famous and notoriously difficult unsolved problems. The connection is as follows. On one hand, the norm of a prime ideal in a quadratic number field is either a prime number or the square of a prime number. The squares of primes form a sequence that is much more sparse than the sequence of primes. Therefore, if one is to succeed to construct a walk on $\mathcal{P}$ with the desired properties, one would very likely need to use the sequence of primes rather than squares of primes. On the other hand, the norm of a prime ideal which in addition is principal can also be obtained by computing the norm of its generator. Since this generator belongs to an imaginary quadratic number field, its norm also equals the square of the distance from this generator to the origin. In conclusion, each element of $\mathcal{P}$ lies on a circle around the origin at distance a prime number or the square root of a prime number. Now, if there are large gaps between two consecutive such circles, then of course there is no way to jump from points in $\mathcal{P}$ which belong to one of these circles to points in $\mathcal{P}$ which belong to the second circle. Thus, if the sequence of gaps between consecutive circles is unbounded, then the above problem will have a negative answer. This, however is not what one expects. In fact, a well known conjecture in number theory states that between any two consecutive squares of integer numbers there exists at least one prime number. This is not known in general, not even assuming the powerful Riemann Hypothesis. If we do assume this conjecture about squares and primes, then, taking square roots, we deduce that between any two consecutive positive integer numbers there exists at least one square root of a prime. In other words, at least one of the above circles of radius square root of a prime passes between those two consecutive integers. In this scenario the distance between any two consecutive circles is less than 2. This means that one can jump from one circle to the next by using steps of length less than 2, but it does not mean that one can use a step of length less than 2 to jump from a point on the first circle which belongs to $\mathcal{P}$ to a point on the second circle which also belongs to $\mathcal{P}$. Here we remark that there are only finitely many points in P which belong to a given circle. This follows from the fact that for each imaginary quadratic number field, an integral basis involves square root of the discriminant, and if the discriminant is large enough the ring of integers will not produce any new points in $\mathcal{P}$ on the given circle.

One direction would be to study the distribution of those points from $\mathcal{P}$ which belong to the same circle, the goal being to estimate the size of the arcs which join such points along the given circle, and see if the arcs are small enough to allow jumping from one point in $\mathcal{P}$ to the next point in $\mathcal{P}$ along the circle. A second part of this project would be to study more general walks on the same set $\mathcal{P}$ as above, where the length of the steps is not forced to be bounded throughout the walk. In this more general context, a natural choice would be to consider walks where the length of the steps is allowed to increase logarithmically as a function of the distance to the origin. The choice of a logarithmic increase is reasonable for at least two different reasons. One reason is that in such scenario the problem cannot be solved by only using walks along the real axis, in other words by only using walks on prime numbers, in view of the occasional larger than logarithmic gaps between consecutive primes. Therefore, for walks using steps of logarithmic size, walks on rational primes are not sufficient and one needs to make use of intrinsic properties of primes in rings of algebraic integers of imaginary quadratic number fields. A second reason for choosing steps of logarithmic size is that in this context one does not need to assume the conjecture on the existence of prime numbers between consecutive squares. Instead, in order to control the distance between consecutive circles of radii square root of primes one may assume the Riemann Hypothesis, a hypothesis which is frequently assumed by researchers in analytic number theory. Under the same hypothesis, an additional part of the project would be to establish whether there exist walks as above which also have the property that the trajectory passes through almost all the points in the given set $\mathcal{P}$.

5.2 Main results

5.2.1 Existence of a path to infinity

Let $\mathcal{C}$ denote the collection of all imaginary quadratic number fields, that is $\mathcal{C} = \left\{\mathbb{Q}[\sqrt{-d}\,] : d > 0 \text{ and } d \text{ squarefree}\right\}$, and let $\mathcal{O}_K$ represent the ring of integers of the imaginary quadratic number field K. Define the set of "stepping stones" as $\mathcal{P} = \{\alpha : \alpha \in \mathcal{O}_K \text{ for some } K \in \mathcal{C} \text{ and } \langle\alpha\rangle \text{ is prime}\}$. Note that if $\alpha = a + b\sqrt{-d} \in \mathcal{P}$, then $|\alpha|$ must be a prime or the square root of a prime, and that a and b will be integers of half-integers, depending on d.

In the following theorems, condition 1 says that the z_n's form a walk in our set $\mathcal{P}$ of primes in quadratic imaginary number fields, condition 2 means that these are "walks to infinity", and conditions 3 and 3' show that there are such walks for which the length $|z_{n+1} - z_n|$ of the steps is not too large in terms of $|z_n|$.

Theorem 5.1. *There exists a sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers satisfying the following properties:*

1. $z_n \in \mathcal{P}$ *for all* $n \in \mathbb{N}$,
2. $|z_n| \to \infty$ *as* $n \to \infty$, *and*
3. $|z_{n+1} - z_n| < |z_n|^{1/20}$ *for all* n *large enough.*

Proof. Let $z_n = \sqrt{-p_n}$, where p_n is the n-th prime number in $\mathbb{Z}$. For each fixed n, the ideal $\langle z_n\rangle$ is prime in the ring of integers of $\mathbb{Q}[\sqrt{-p_n}\,]$. We use the unconditional upper bound on the n-th prime gap obtained in [BHP01] and valid for large n, to get

$$p_{n+1} < p_n + p_n^{21/40} < (\sqrt{p_n})^2 + 2\sqrt{p_n}p_n^{1/40} < (\sqrt{p_n} + p_n^{1/40})^2.$$

The third property follows immediately since for such n

$$|z_{n+1} - z_n| = \sqrt{p_{n+1}} - \sqrt{p_n} < p_n^{1/40} = |z_n|^{1/20}.$$

$\square$

The next theorem provides a path with steps of logarithmic size, but is conditional on the Riemann Hypothesis.

Theorem 5.2. *Assume the Riemann Hypothesis. Then there exists a sequence* $(z_n)_{n\in\mathbb{N}}$ *of complex numbers satisfying the following properties:*

1. $z_n \in \mathcal{P}$ *for all* $n \in \mathbb{N}$,

2. $|z_n| \to \infty$ *as* $n \to \infty$, *and*

3'. $|z_{n+1} - z_n| < \frac{22}{25} log|z_n|$ *for all n large enough.*

Proof. Using the same sequence of z_n as in Theorem 5.1, and a conditional upper bound on the gaps between primes [CMS], we get

$$p_{n+1} < p_n + \frac{22}{25}\sqrt{p_n}\log p_n < (\sqrt{p_n} + \frac{11}{25}\log p_n)^2,$$

and so,

$$|z_{n+1} - z_n| < \sqrt{p_{n+1}} - \sqrt{p_n} < \frac{11}{25}\log p_n = \frac{22}{25} log|z_n|.$$

$\square$

Remark. *If one walks along the real axis, that is, if one walks on prime numbers, one can simply take* $z_n = p_n$, *the n-th prime number. But then using known bounds on gaps between consecutive primes* $p_{n+1} - p_n = \mathcal{O}((p_n)^{0.525})$ *due to Baker, Harman and Pintz ([BHP01]), one obtains a much worse result than in the statement of Theorem 5.1.*

5.2.2 Ideals with prime norm

Theorem 5.1 provides a way to move from one circle with radius the square root of a prime p to the circle with radius the square root of the next prime with steps of size that do not exceed

$p^{1/40}$. We now focus on elements in $\mathcal{P}$ that lie on a single such circle.

For a fixed prime p and any integer $0 \le a < \sqrt{p}$, one can always find b and squarefree $d > 0$, such that $p - a^2 = b^2 d$ or $|a + b\sqrt{-d}|^2 = p$. In that case, the ideal generated by α in $\mathbb{Q}[\sqrt{-d}]$ is prime and therefore $\alpha \in \mathcal{P}$.

Let $\alpha = a + b\sqrt{-d}$ and $\alpha' = a + 1 + b'\sqrt{-d'}$ be two elements of $\mathcal{P}$ such that $|\alpha| = |\alpha'| = \sqrt{p}$.

The square of the gap between α and α' is given by

$$|\alpha - \alpha'|^2 = 1 + \left(b\sqrt{d} - b'\sqrt{d'}\right)^2 = 1 + \left(\sqrt{p - a^2} - \sqrt{p - (a+1)^2}\right)^2 =$$

since $p = a^2 + b^2 d = (a+1)^2 + (b')^2 d'$ which gives $b\sqrt{d} = \sqrt{p - a^2}$ and $b'\sqrt{d'} = \sqrt{p - (a+1)^2}$.

The following inequality

$$\frac{2a+1}{2\sqrt{p - a^2}} \le \sqrt{p - a^2} - \sqrt{p - (a+1)^2} = \frac{2a+1}{\sqrt{p - a^2} + \sqrt{p - (a+1)^2}} \le \frac{2a+1}{\sqrt{p - a^2}}$$

leads to

$$1 + \frac{(2a+1)^2}{4(p - a^2)} \le |\alpha - \alpha'|^2 \le 1 + \frac{(2a+1)^2}{(p - a^2)}.$$

If $a \le (1-\epsilon)\sqrt{p}$, then $a^2 \le (1-\epsilon)^2 p$, so $p - a^2 \ge (1 - (1-\epsilon)^2)p = (2\epsilon - \epsilon^2)p$.

This gives the bound

$$|\alpha - \alpha'|^2 \le 1 + \frac{(2a+1)^2}{(p - a^2)} \le 1 + \frac{9a^2}{(2\epsilon - \epsilon^2)p} \le 1 + \frac{9(1-\epsilon)^2}{2\epsilon - \epsilon^2} = 1 + \frac{9}{2}\left(\frac{1}{2\epsilon - \epsilon^2} - 1\right).$$

Choosing $\epsilon = p^{-1/40}$ gives

$$|\alpha - \alpha'|^2 \ll p^{1/40}.$$

This means that on the circle with radius $\sqrt{p}$, all elements $z = a + b\sqrt{-d}$ that belong to $\mathcal{P}$ can be accessed with a step of size $p^{1/40}$, except possibly those with $(1 - p^{-1/40})\sqrt{p} \leq a \leq \sqrt{p}$.

5.2.3 A path that covers almost all elements of $\mathcal{P}$

Let $P_x = P \cap D(0; x)$, where $D(0; x)$ is the disk centered at the origin with radius x, and let $E_x(s)$ denote the set of those elements in P_x that cannot be accessed using steps of size s.

We will show that we can cover *almost all* elements of $\mathcal{P}$ with walks of step size $|z|^{1/20}$, in the sense that

$$\lim_{x\to\infty} \frac{E_x\left(|z|^{1/20}\right)}{P_x} = 0.$$

For each prime $p \leq x$, the set P_x contains all $z = a + b\sqrt{-d}$, with $a, b \in \mathbb{Z}$ and $0 \leq a \leq p$ that lie on a circle of radius a prime or the square root of a prime, and so

$$\begin{aligned}
|P_x| &\geq \sum_{\substack{p\leq x\\ p \text{ prime}}} p + \sum_{\substack{p\leq x^2\\ p \text{ prime}}} \sqrt{p} \\
&\geq \sum_{\substack{x/2\leq p\leq x\\ p \text{ prime}}} p + \sum_{\substack{x^2/4\leq p\leq x^2\\ p \text{ prime}}} \sqrt{p} \\
&\geq \frac{x}{2}\left(\pi(x) - \pi(x/2)\right) + \frac{x}{2}\left(\pi(x^2) - \pi(x^2/4)\right) \gg \frac{x^3}{\log x}.
\end{aligned} \tag{5.1}$$

For an upper bound on the number of elements of E_x, we count all the elements with integer x-coordinate that lie on a circle of radius a prime number, as well as those points discussed in the previous subsection, and we have

$$
\begin{aligned}
|E_x| &\le 2 \cdot \sum_{\substack{p \le x \\ p \text{ prime}}} (2p+1) + 4 \cdot \sum_{\substack{p \le x^2 \\ p \text{ prime}}} p^{\frac{1}{2} - \frac{1}{40}} \\
&\ll x \cdot \pi(x) + x^{1-\frac{1}{20}} \cdot \pi(x^2) \\
&\ll \frac{x^{3-\frac{1}{20}}}{\log x}. \qquad (5.2)
\end{aligned}
$$

When we combine (5.1) and (5.2) we get that

$$
\lim_{x \to \infty} \frac{E_x\left(|z|^{1/20}\right)}{P_x} = 0.
$$

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