MODEL THEORY OF PARTIALLY RANDOM STRUCTURES

BY

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DISSERTATION

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Abstract

Many interactions between mathematical objects, e.g. the interaction between the set of primes and the additive structure of $\mathbb{N}$, can be usefully thought of as random modulo some obvious obstructions. In the first part of this thesis, we document several such situations, show that the randomness in these interactions can be captured using first-order logic, and deduce in consequence many model-theoretic properties of the corresponding structures. The second part of this thesis develops a framework to study the aforementioned situations uniformly, shows that many examples of interest in model theory fit into this framework, and recovers many known model-theoretic results about these examples from our theory.

To my family and friends.

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Notation and conventions

We include here notation and conventions which will be in force throughout the entire thesis.

Uniformity conventions. Whenever we declare that a particular letter denotes a certain type of object in some portion of this thesis, that letter with decorations is also assumed to be the same type of object in that portion of the thesis. For example, k is used throughout for integers, so k', k_1, k^*, etc. also denote integers whenever they appear.

Numbers conventions. Throughout, h, k, and l range over the set $\mathbb{Z}$ of integers, m and n range over the set $\mathbb{N}$ of natural numbers (which contains 0), p ranges over the set of (positive) prime numbers, and q ranges over the set $\{p^m : m \geqslant 1\}$ of positive powers of primes.

Set theory conventions. We assume the reader is familiar with basic concepts and definitions from set theory. The letter κ will be reserved for a cardinal.

Logic and model theory conventions. We work in multi-sorted first-order logic. Our semantics allows empty sorts and empty structures. Our syntax includes logical constants $\top$ and $\bot$ interpreted as true and false, respectively. We view constant symbols as 0-ary function symbols. The equality symbol is considered a logical symbol.

We let L denote a (possibly multi-sorted) first-order language, and let $\mathcal{M}$ be a structure in some language. Let x, y, and z be (possibly infinite) tuples of variables; strictly speaking, we should also specify the languages, but these are always obvious from the context. Suppose L has S its set of sort and $\mathcal{M}$ is an L-structure. We use the corresponding capital letter M to denote the S-indexed family $(M_s)_{s\in S}$ of underlying sets of the sorts of $\mathcal{M}$. By $A \subseteq M$, we mean $A = (A_s)_{s\in S}$ with $A_s \subseteq M_s$ for each $s \in S$. If $A \subseteq M$, then a tuple of elements (possibly infinite) in A is a tuple each of whose components is in A_s for some $s \in S$. If $x = (x_j)_{j\in J}$ is a tuple of variables, we let $A^x = \prod_{j\in J} A_{s(x_j)}$ where $s(x_j)$ is the sort of the variable x_j.

Suppose $\mathcal{M}$ is an L-structure. For $B \subseteq M$ and $b \in B^y$, let $L(B)$ and $L(b)$ be the extensions of L obtained by adding constant symbols for the elements of B and for the components of b, and view $\mathcal{M}$ in the obvious way as an $L(B)$-structure and an $L(b)$-structure. For an L-formula $\varphi(x, y)$ and $b \in M^y$, we let $\varphi(\mathcal{M}, b)$ be the set defined in the structure $\mathcal{M}$ by the $L(b)$-formula $\varphi(x, b)$.

CHAPTER 1

Introduction

This thesis is a fusion of four papers [**7, 80, 54, 79**] and some material from a paper under preparation [**53**]. They share a common theme dealing with the model theory of partially random structures, that is, structures that contain both predictable/algebraic features and random/generic features. These preprints consist of my stand-alone work as well as joint works with Neer Bhardwaj, Alex Kruckman, and Erik Walsberg.

In this introduction, I would like to give some justification for the current endeavor and offer a bird's-eye view of the whole thesis. I will start by colloquially explaining what it means to study the model theory of a structure or a class of structures and why it might be interesting to do so. Then I provide reasons for studying the model theory of partially random structures. The primary target audience of this part are fellow graduate students from outside model theory, but I hope it will also amuse/annoy some experts. Afterward, I will go into a more detailed description of the structure of this thesis and the main results of the chapters; this part is essentially a fusion of the introductions of the aforementioned papers. For it, I will assume more familiarity with model theory.

1.1. Why model theory and partially random structures?

Viewing a mathematical problem in a geometric light is often desirable and sometimes the key to its solution. This is the case even for very discrete problems like solving systems of polynomial equations over finite fields or counting the number of solutions of such systems. Model theory can be described as "geometry from a logical perspective": the subject allows us to put even more exotic problems under the lens of geometry, albeit in a weaker sense.

Let us make more sense of the above. Model theory is a subject that belongs to the "modern wing" of mathematical logic. The focus is no longer on using mathematical methods to investigate the way humans reason or to provide a foundation of mathematics. Instead, we want to study mathematical structures or classes of structures with a perspective informed by logic. A structure here consists of an ambient space M and relations on M, i.e., subsets of M^n for varying n thought of as relations between n elements of M; most mathematical objects can be seen as a structure in this way. Studying structures in the logic way means considering sets, functions, and groups that "can be described or constructed" from the basic relations of the structure (or class of structures) and then studying various mathematical phenomena that arise from these settings. The meaning of "can be described or constructed" varies as we move across the areas of logic or even the areas of model theory. It should be noted that similar ideas are also native to other fields of mathematics. For instance, algebraic geometry studies affine algebraic varieties over a field, which are sets admitting a particular description, namely, as the solution set of a system of polynomial equation. Stretching the meaning further, one can think of a manifold as "can be described or constructed" compared to a totally arbitrary subset of the ambient space. Hence, one should not be too surprised that some aspects of classical geometric theories have analogues in favorable logical settings.

Model theory studies the above analogues, which we call notions of tameness, and the above favorable settings, which we call tame structures. Understanding these notions and structures is desirable as it makes available new geometric tools beyond the reach of classical geometric theories. The machinery has been applied to solve many problems outside logic which make model theory connected to virtually every major area of mathematics. We arrived at three main goals of model theory: Isolating and studying various notions of tameness, finding interesting structures and either showing that they satisfy some tameness notions or showing the opposite (often referred to as establishing the model-theoretic properties of the structure), and looking for opportunities elsewhere in mathematics to apply our understanding. These three tasks roughly corresponds to pure model theory, "middle-of-the-road" model theory, and applied model theory. They are intricately connected, as many notions of tameness arising out of purely logical consideration turned out to be keys to application.

Let us clarify the meaning of "partial randomness" through an example before saying why it ought to be studied. Consider $(\mathbb{N}; +, \mathrm{Pr})$ where Pr is the set of prime numbers. The interaction between Pr and $+$ is not fully random, it is quite different from how a set Rd given by coin-flipping interacts with $+$. For instance, $4a$ is not in Pr for all $a \in \mathbb{N}$, but there is $b \in \mathbb{N}$ with $4b \in \mathrm{Rd}$ (with probability 1). Nonetheless, it is possible to think of Pr as interacting with $+$ randomly modulo such obstructions. It is also useful to do so as many conjectures in analytic number theory depends on such intuition. Strictly speaking, this situation should be called "partially pseudo-random" as Pr is completely predictable, but we will blur this distinction.

There are many other structures in mathematics that can be viewed in the same way as above. Moreover, a respectable strategy to deal with a structure is to decompose it into a predictable (algebraic) part and a random (generic) part and then try to handle them separately. Developing model theory (establishing tameness notions/generalized geometrical principles) for these structure might therefore give us new tools to solve problems.

This thesis observes that many natural partially random structures indeed satisfy certain known notions of tameness/generalized geometrical principles. We also build a general theory to study all these structures uniformly. Our tools need probably to be sharpened much further before they can find applications outside model theory, so what we have done so far is very "middle of the road". In the mean time, we have found that our framework is quite powerful for the purpose of organizing examples in model theory. We can view many important examples in model theory as an instance of partially random structures, and recover known model-theoretic results for them from our theory. On the other hand, our theory suggests new notions of tameness that ought to be studied, so there is hope that interesting pure model theory can come out of it as well.

1.2. What is in this thesis?

This thesis has two parts. The first part, **concrete partially random structures**, consists of the next three chapters. These correspond to three papers: my stand-alone paper [**79**], my joint paper [**7**] with Bhardwaj, and my joint paper with Erik [**80**]. The second part, **abstract partially random structures**, consists of the last five chapters. They comes from the two joint papers [**54**] and [**53**] with Kruckman and Walsberg.

As most of the above papers are joint, I am only partially to credit for many of the results presented here. In fact, a few items came close to being solely the work of my collaborators: The idea and proof of Section 7.7 on structures and fields with automorphism are mostly by to Walsberg. The conjecture behind Section 9.7 was made by me with input from Walsberg, and I obtained some partial results. However, the proof in its current final form is entirely due to Kruckman, who brought in many new ideas and came to the project with his own perspective. I will present these results here anyway as they make the story more complete.

Part 1. Concrete partially random structures. We consider several situations where randomness plays an important role in understanding the model-theoretic properties of structures. Chapter 2 looks at structures of the form $(\mathbb{F}, \triangleleft\,)$ where $\mathbb{F}$ is an algebraic closure of a finite field, and $\triangleleft$ is a circular ordering on the multiplicative group $\mathbb{F}^\times$ which respects multiplication. Chapter 3 is about the structure $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ with $\mathbb{Z}$ the additive group of integers $\mathrm{SF}^{\mathbb{Z}}$ the set of square-free integers and several other structures in the same vein. In both chapters, establishing the model-theoretic properties of the structures under consideration requires observing that they are built up from two components interacting in a random way modulo obvious obstructions. Chapter 4 studies structures of the form $(\mathbb{Z}, \triangleleft\,)$ where $\triangleleft$ is a circular ordering on the additive group $\mathbb{Z}$ which respects addition. Here, randomness plays a role in classifying such structures up to interdefinability. We now give more detailed summaries of the chapters, together with the neccessary background to consider them.

Chapter 2. Tame structures via character sums over finite fields. Throughout the current summary and the corresponding chapter, $\mathbb{F}$ is an algebraic closure of a finite field. We are interested in the following question:

Are there natural expansions of $\mathbb{F}$ *by order-type relations which are also model-theoretically tame?*

There is no known order-type relation on $\mathbb{F}$ which interacts in a sensible way with both addition and multiplication. This is in stark contrast to the situation with the field $\mathbb{C}$ where addition and multiplication are compatible with the Euclidean metric induced by the natural order on its subfield $\mathbb{R}$. It is not hard to see the reason: the additive group of $\mathbb{F}$ is an infinite torsion group of finite exponent, so even finding an additively compatible order-type relation seems unlikely. On the other hand, the multiplicative group $\mathbb{F}^\times$ is a union of cyclic groups, so it is fairly natural to consider circular orders $\lhd$ on $\mathbb{F}^\times$ which are compatible with the multiplicative structure. In this paper, we will show that the resulting structures $(\mathbb{F}, \lhd)$ give a positive answer to some aspects of the above question.

We will take a step back to be more precise and to study the above structures as members of a natural class. A **circular order** on a group G is a ternary relation $\lhd$ on G which is invariant under multiplication by elements in G and satisfies the following conditions for all $a, b, c \in G$:

(1) if $\lhd(a,b,c)$, then $\lhd(b,c,a)$;
(2) if $\lhd(a,b,c)$, then not $\lhd(c,b,a)$;
(3) if $\lhd(a,b,c)$ and $\lhd(a,c,d)$, then $\lhd(a,b,d)$;
(4) if a,b,c are distinct, then either $\lhd(a,b,c)$ or $\lhd(c,b,a)$.

A canonical example, also used later on, is $(\mathbb{T}, \lhd)$ where $\mathbb{T}$ is the multiplicative group of complex numbers with norm 1, and $\lhd$ is the clockwise circular order (i.e., $\lhd(a,b,c)$ if b lies in the clockwise open arc from a to c viewing $\mathbb{T}$ as the unit circle).

A **multiplicative circular order** on a field F is a circular order on the multiplicative group $F^\times$, viewed as a ternary relation on F. If $\lhd$ is a multiplicative circular order on F, then $(F, \lhd)$ is a structure in the total language L_{t} extending the language $L_{\mathrm{f}} = \{0, 1, +, -, \times, \square^{-1}\}$ of fields by a ternary predicate symbol $\lhd$. Let ACFO^- be the L_{t}-theory whose models are such $(F, \lhd)$ where F is algebraically closed. Section 2.1 and Section 2.2 establish our first main result:

Theorem 1.1. *The theory* ACFO^- *has a model companion* ACFO.

Underlying the proof of Theorem 1.1 is the following heuristic: the existential closed models of ACFO^- are $(F, \lhd) \models \mathrm{ACFO}^-$ where $(F^\times, \lhd)$ is "sufficiently rich", and $+$ interacts in a "random fashion" with $\lhd$ modulo their compatibility with $\times$. The challenges involve making sense of "sufficiently rich" and "random fashion", justifying this heuristic, and showing that these properties are first-order axiomatizable.

In section 2.3, we return to the structures described in the first paragraph:

Theorem 1.2. *If $\lhd$ is a multiplicative circular order on $\mathbb{F}$, then $(\mathbb{F}, \lhd) \models \mathrm{ACFO}$.*

Every injective group homomorphism $\chi : \mathbb{F}^\times \to \mathbb{T}$ induces a multiplicative circular order on $\mathbb{F}$, namely, the pullback $\lhd_\chi$ of the clockwise circular order $\lhd$ on $\mathbb{T}$ by the map χ. It turns out that every multiplicative circular order on $\mathbb{F}$ is of this form; see Corollary 2.6. The main idea of the proof of Theorem 1.2 is to exploit this connection and results on character sums over finite fields. These results are useful here as they reflect "number-theoretic randomness" [**50**]. This is precisely what we want for the interaction between $+$ and $\lhd$.

This work is a response to the question below by van den Dries and Hrushovski; Kowalski also asked a related question in [**51**].

Do results on exponential sums and character sums over finite fields yield any model-theoretically tame structures?

Behind this question is the hope to find analogies of Ax's results in [**4**]. There, the model-theoretic tameness of ultraproducts of finite fields essentially follows from results on counting points over finite fields. The theory ACFO is our proposed counterpart of the theory of pseudo-finite fields, and the above two theorems correspond to the fact that the theory of finite-fields is almost model complete and the fact that nonprincipal ultraproducts of finite fields are pseudofinite fields (in the definition given by Ax). There are also reasons to believe that there are deeper connections between ACFO and the theory of pseudo-finite fields. Both theories include certain "random features" and can be put under the framework of interpolative fusions discussed in Part 2; see Sections 7.4 and 7.7 for details.

Chapter 2 is essentially the updated version of [**79**]. The earlier versions of [**79**] contained several other results. Some of these are now generalized into results about interpolative fusions; see Chapter 9. We do not include them here to minimize overlapping.

The structures $(\mathbb{F}, \lhd)$ in Theorem 1.2 are not simple (in the sense of model theory) as they define dense linear orders. They also have IP by a result of Shelah and Simon [**71**]. It turns out that these structure do not even have TP_2 (see Proposition 2.7). This brings $(\mathbb{F}, \lhd)$ outside the current known boundary of the combinatorially tame universe. We hope these structures provide some motivation to push the boundary further and include them as well.

Chapter 3. Additive groups of $\mathbb{Z}$ and $\mathbb{Q}$ and predicates for being square-free. In this the current summary and the corresponding chapter, $\mathbb{Z}$ is the additive group of integers implicitly assumed to contain the element 1 as a distinguished constant. Likewise, $\mathbb{Q}$ is the additive group of rational numbers with 1 as a distinguished constant.

In [**45**], Kaplan and Shelah showed under the assumption of Dickson's conjecture that if Pr is the set of $a \in \mathbb{Z}$ such that either a or $-a$ is prime, then the theory of $(\mathbb{Z}, \mathrm{Pr})$ is model complete, decidable, and super-simple of U-rank 1. This result can be interpreted as an example of the central theme of this thesis where we can often capture aspects of randomness inside a structure using first-order logic and deduce in consequence several model-theoretic properties of that structure. In $(\mathbb{Z}, \mathrm{Pr})$, the conjectural randomness is that of the set of primes with respect to addition. Dickson's conjecture is useful here as it reflects this randomness in a fashion which can be made first-order.

This viewpoint in particular predicts that there are analogues of Kaplan and Shelah's results with Pr replaced by other random subsets of $\mathbb{Z}$. We confirm the above prediction here without the assumption of any conjecture when Pr is replaced with the set

$$\mathrm{SF}^{\mathbb{Z}} = \{a \in \mathbb{Z} : v_p(a) \leqslant 1 \text{ for all } p\}$$

where v_p is the p-adic valuation associated to the prime p. As the reader can guess, "SF" stands for "square-free". We will introduce a first-order notion of "genericity" which encapsulates the partial randomness in the interaction between $\mathrm{SF}^{\mathbb{Z}}$ and the additive structure on $\mathbb{Z}$. Using an approach with the same underlying principle as that in [**45**], we obtain:

Theorem 1.3. *The theory of $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ is model complete, decidable, supersimple of U-rank 1, and is k-independent for all $k \in \mathbb{N}^{\geqslant 1}$.*

The theorem above gives us without assuming any conjecture the first natural example of a simple unstable expansion of $\mathbb{Z}$. From the same notion of "genericity", we deduce entirely different consequences for a related structure:

Theorem 1.4. *The theory of $(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$ defines multiplication.*

The proof adapts the strategy Bateman, Jockusch, and Woods used in [**6**] to show that $\mathrm{Th}(\mathbb{N}; +, <, \mathrm{Pr})$ with Pr the set of primes interprets arithmetic. The above two theorems are in stark contrast with one another in view of the fact that $(\mathbb{Z}, <)$ is a minimal proper expansion of $\mathbb{Z}$; indeed, Conant proved in [**19**] that adding any new definable set from $(\mathbb{Z}, <)$ to $\mathbb{Z}$ results in defining $<$. On the other hand, Dolich and Goodrick showed in [**26**] that there is no strong expansion of the theory of Presburger arithmetic, so the second theorem is perhaps not completely unexpected.

It is also natural to consider the structures $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ and $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ where $\mathrm{SF}^{\mathbb{Q}}$ is the set $\{a \in \mathbb{Q} : v_p(a) \leqslant 1 \text{ for all primes } p\}$, and the relation $<$ on $\mathbb{Q}$ is the natural ordering. (We do not study $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Z}})$ as the set $\mathbb{Z}$ is definable in $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Z}})$. Indeed, it follows from Lemma 3.2 that every integer is a sum of two elements in $\mathrm{SF}^{\mathbb{Z}}$.) The main new technical aspect here lies in getting other suitable notions of "genericity" and using this to prove:

Theorem 1.5. *The theory of* $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ *is model complete, decidable, simple but not supersimple, and is* k*-independent for all* $k \in \mathbb{N}^{\geqslant 1}$.

From the above, $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ is "less tame" than $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$. The reader might therefore suspect that $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ is wild. However, this is not the case:

Theorem 1.6. *The theory* $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ *is model complete, decidable, has* NTP_2 *but is not strong, and is* k*-independent for all* $k \in \mathbb{N}^{\geqslant 1}$.

Above we presented the material of Chapter 2 structure by structure. However, the chapter actually proceeds by considering all the four structures in parallel fashion, and prove related results for them consecutively. More precisely, the first theorem is Theorem 3.3 and Theorem 3.6 put together, the second theorem is Theorem 3.5, the third theorem is a consequence of Theorem 3.4 and Theorem 3.7, and the fourth theorem is a consequence of Theorem 3.4 and Theorem 3.8. Having the same principle running through four structures hints that randomness can be indeed used as a framework to explain model-theoretic properties of multiple structures uniformly.

Chapter 4. A family of dp-minimal expansions of the additive group $\mathbb{Z}$. In this chapter and summary, $\mathbb{Z}$ is the additive group of integers. We are interested in the following classification-type question:

What are the dp-minimal expansions of $\mathbb{Z}$?

For a definition of dp-minimality, see [**72**, Chapter 4]. The terms expansion and reduct here are as used in the sense of definability: If $\mathcal{M}_1$ and $\mathcal{M}_2$ are structures with underlying set M and every $\mathcal{M}_1$-definable set is also definable in $\mathcal{M}_2$, we say that $\mathcal{M}_1$ is a **reduct** of $\mathcal{M}_2$ and that $\mathcal{M}_2$ is an **expansion** of $\mathcal{M}_1$. Two structures are **definably equivalent** if each is a reduct of the other.

A very remarkable common feature of the known dp-minimal expansions of $\mathbb{Z}$ is their "rigidity". In [**21**], Conant and Pillay showed that all proper stable expansions of $\mathbb{Z}$ have infinite weight, hence infinite dp-rank, and so in particular are not dp-minimal. The expansion $(\mathbb{Z}, <)$, well-known to be dp-minimal, does not have any proper dp-minimal expansion (a result in [**3**] by Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko), or any proper expansion of finite dp-rank, or even any proper strong expansion (a resut in [**26**] by Dolich and Goodrick). Moreover, Conant showed that any reduct $(\mathbb{Z},<)$ expanding $\mathbb{Z}$ is definably equivalent to $\mathbb{Z}$ or $(\mathbb{Z},<)$ [**20**]. Recently, Alouf and d'Elbée showed in [**24**] that $(\mathbb{Z},\prec_p)$ is dp-minimal for all p where $\prec_p$ is the partial order on $\mathbb{Z}$ given by declaring $k \prec_p l$ if and only if $v_p(k) < v_p(l)$ with v_p the p-adic valuation on $\mathbb{Z}$. In the same paper, they showed that any reduct of $(\mathbb{Z},\prec_p)$ expanding $\mathbb{Z}$ is definably equivalent to either $\mathbb{Z}$ or $(\mathbb{Z},\prec_p)$.

The above "rigidity" gives hope for a classification of dp-minimal expansions of $\mathbb{Z}$ analogous to Johnson's classification of dp-minimal fields [**42**]. In [**2**] (also by by Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko), the authors asked whether every dp-minimal expansion of $\mathbb{Z}$ is a reduct of $(\mathbb{Z},<)$. In view of results in [**24**], the natural modified question is whether every dp-minimal expansion of $\mathbb{Z}$ is a reduct of $(\mathbb{Z},<)$ or $(\mathbb{Z},\prec_p)$ for some p.

With the notion of circularly ordered abelian groups defined in the summary of Chapter 2, we show that:

Theorem 1.7. *Every circularly ordered abelian group* $(\mathbb{Z},\lhd)$ *is dp-minimal.*

In Section 4.1, we characterize unary definable sets in these expansions of $\mathbb{Z}$, classify these structures up to definable equivalence, and show that there are continuumm many up to definable equivalence. Hence, we get a strong negative answer to the aforementioned question. The proof of many of the above results notably makes use of Kronecker's approximation theorem, which can be seen as reflecting randomness.

Part 2. Abstract partially random structures. We aim to develop a general framework to study structures with partial randomness. Chapter 5 is a preliminary chapter, but with several original results. Chapter 6 introduces the notion of an interpolative structure, which makes precise what it means to say that a structure is built up from multiple components interacting randomly over a common part. Chapter 7 shows that many examples with model theoretic interest fit into this framework. Chapter 8 provides several sufficient conditions under which randomness can be captured using first-order logic. Chapter 9 develops a general theory which allows us to understand definable sets in an interpolative structure in terms of definable sets in the components. Below we describe the chapters in more details omitting chapter 5 as it is a necessary supplement but not part of the storyline.

Chapter 6. Interpolative structures and interpolative fusions. For expository purpose, we only consider here a special case of the setting introduced in Chapter 6. In this summary, L_1 and L_2 are first-order languages with the same sorts, $L_\cap = L_1 \cap L_2$, and $L_\cup = L_1 \cup L_2$. We let T_1 and T_2 be L_1 and L_2-theories, respectively, with a common set $T_\cap$ of $L_\cap$-consequences, and $T_\cup = T_1 \cup T_2$. Finally, $\mathcal{M}_\cup$ is an $L_\cup$-structure, $\mathcal{M}_\square$ is the $L_\square$-reduct of $\mathcal{M}_\cup$, and $X_\square$ ranges over $\mathcal{M}_\square$-definable sets for $\square \in \{1,2,\cap\}$.

We say that $\mathcal{M}_\cup$ is **interpolative** if for all $X_1 \subseteq X_2$, there is an $X_\cap$ such that

$$X_1 \subseteq X_\cap \text{ and } X_\cap \subseteq X_2$$

(more symmetrically: for all disjoint X_1 and X_2, there are $\mathcal{M}_\cap$-definable sets $X^1_\cap$ and $X^2_\cap$ such that $X_1 \subseteq X^1_\cap$, $X_2 \subseteq X^2_\cap$, and $X^1_\cap \cap X^2_\cap = \varnothing$). This notion is an attempt to capture the idea that $\mathcal{M}_1$ and $\mathcal{M}_2$ interact, with respect to definability, in a generic, independent, or random fashion over the reduct $\mathcal{M}_\cap$. Informally, the above definition says that the only information $\mathcal{M}_1$ has about $\mathcal{M}_2$ comes from $\mathcal{M}_\cap$. If the class of interpolative models of $T_\cup$ is elementary with theory $T^*_\cup$, then we say that $T^*_\cup$ is the **interpolative fusion** (of T_1 and T_2 over $T_\cap$). We also say that "$T^*_\cup$ exists" if the class of interpolative $T_\cup$-models is elementary.

The reader may notice similarities with the Craig interpolation theorem: for every L_1-sentence φ_1 and L_2-sentence φ_2 for which $\models \varphi_1 \to \varphi_2$, there is an $L_\cap$-formula $\varphi_\cap$ such that $\models \varphi_1 \to \varphi_\cap$ and $\models \varphi_\cap \to \varphi_2$. The resemblance is consequential. It allows us to prove:

Theorem 1.8. *Suppose T_1 and T_2 are model-complete. Then $\mathcal{M}_\cup \models T_\cup$ is interpolative if and only if $\mathcal{M}_\cup$ is existentially closed in the class of $T_\cup$-models. Hence, $T^*_\cup$ exists if and only if $T_\cup$ has a model companion, in which case $T^*_\cup$ is a model companion of $T_\cup$.*

In the case that T_1 and T_2 are not model-complete, we can still think of $T^*_\cup$ as a relative model companion of $T_\cup$, see Proposition 6.2.

Chapter 7. Examples of interpolative fusions. We adopt here the notational conventions of the summary of Chapter 6. We show that many theories of model-theoretic interest can be construed as interpolative fusions.

We show in Section 7.1 that if P is an infinite and co-infinite unary predicate on a single-sorted structure $\mathcal{M}$ with underlying set M, then P is a generic predicate as defined by Chatzidakis and Pillay [**11**] if and only if $(\mathcal{M}; P)$ is a model of the interpolative fusion of the theories of $\mathcal{M}$ and (M, P) over the theory of the pure set M. Another source of examples with the same flavor is the expansion of a structure by a generic predicate for a reduct, recently described by d'Elbée [**23**]. We will discuss the latter examples and others in [**53**].

Certain notions of independence in mathematics give us interpolative fusions. Let K be an algebraically closed field and v_1, v_2 be non-trivial valuations which induce distinct topologies on K. It follows from results in [**43**, Chapter 11] by Johnson that (K, v_1, v_2) is a model of the interpolative fusion of the theories of (K, v_1) and (K, v_2) over the theory of K (see Section 7.2). Now write $k \preccurlyeq_p l$ if $v_p(k) \leqslant v_p(l)$ with v_p the p-adic valuation. Following results in [**24**], if p and p' are distinct, and $\mathbb{Z}$ is the additive group of integers, then $(\mathbb{Z}, \preccurlyeq_p, \preccurlyeq_{p'})$ is a model of the interpolative fusion of the theories of $(\mathbb{Z}, \preccurlyeq_p)$ and $(\mathbb{Z}, \preccurlyeq_{p'})$ over the theory of $\mathbb{Z}$.

Consider the structures in the summary of Chapter 2: $(\mathbb{F}; +, \times)$ is an algebraic closure of a finite field with the underlying set $\mathbb{F}$, and $\triangleleft$ is a multiplicative circular order on $(\mathbb{F}; +, \times)$. It follows rather easily from the main theorems of Chapter 2 that $(\mathbb{F}; +, \times, \triangleleft)$ is a model of the interpolative fusion of the theories of $(\mathbb{F}; +, \times)$ and $(\mathbb{F}; \times, \triangleleft)$ over the theory of $(\mathbb{F}; \times)$. The initial motivation to introduce the notion of interpolative fusions was to find a common generalization of this example and the first example in the preceding paragraph.

Many interesting theories are not themselves interpolative fusion, but bi-interpretable with one. Let σ be an automorphism of a model-complete L-structure $\mathcal{M}$, $\mathcal{N}$ another L-structure, and τ an isomorphism from $\mathcal{M}$ to $\mathcal{N}$. Let T be the theory of $\mathcal{M}$ and T_{Aut} be the theory of a T-model expanded by an L-automorphism. We show in Section 7.7 that $(\mathcal{M}, \mathcal{N}; \tau)$ and $(\mathcal{M}, \mathcal{N}; \tau \circ \sigma)$ are both canonically bi-interpretable with $\mathcal{M}$ and $(\mathcal{M}, \mathcal{N}; \tau, \tau \circ \sigma)$ is canonically bi-interpretable with $(\mathcal{M}, \sigma)$. Further, $(\mathcal{M}, \sigma)$ is existentially closed in the collection of T_{Aut}-models if and only if $(\mathcal{M}, \mathcal{N}; \tau, \tau \circ \sigma)$ is an interpolative structure. It follows that if T_{Aut} has a model companion T^*_{Aut}, then T^*_{Aut} is bi-interpretable with the interpolative fusion of two theories, each of which is bi-interpretable with T.

As a special case of the remarks in the preceding paragraph, we see that the model companion ACFA of the theory of difference fields is bi-interpretable with an interpolative fusion of two theories, each of which is bi-interpretable with the theory of algebraically closed fields. We also show that the analogous statement holds for the theory DCF of differentially closed fields. The general algebraic framework of $\mathcal{D}$-fields, developed by Moosa and Scanlon [**60**], gives a way of uniformly handling both ACFA and DCF. We show in Section 7.8 that the model companion of the theory of $\mathcal{D}$-fields of characteristic 0 is always bi-interpretable with an interpolative fusion of two theories, each of which is bi-interpretable with ACF_0.

Chapter 8. Existence results. We adopt here the notational conventions of the summary of Chapter 6. In general, $T^*_\cup$ need not exist, and the existence of $T^*_\cup$ may even involve classification-theoretic issues. For example, it is conjectured that if T is unstable, then T_{Aut} does not have a model companion. In Chapter 8 we give general "pseudo-topological" conditions on T_1, T_2, and $T_\cap$ which ensure the existence of $T^*_\cup$. These conditions are highly nontrivial, but they are satisfied in many interesting examples. We also give a natural set of pseudo-topological axioms for $T^*_\cup$ when the pseudo-topological conditions are satisfied.

Suppose we can assign to each $\mathcal{M}_\cap$-definable set $X_\cap$ in $\mathcal{M}_\cap \models T_\cap$ an ordinal dimension $\dim(X_\cap)$, and dim satisfies some minimal conditions given in Section 8.1. Most tame theories come with a canonical dimension. We say that an arbitrary set A is **pseudo-dense** in $X_\cap$ if A intersects every $\mathcal{M}_\cap$-definable $Y_\cap \subseteq X_\cap$ such that $\dim Y_\cap = \dim X_\cap$. We say that $X_\cap$ is a pseudo-closure of A if A is pseudo-dense in $X_\cap$ and $A \subseteq X_\cap$.

For $i \in \{1,2\}$, we say that $\mathcal{M}_i$ is **approximable over** $\mathcal{M}_\cap$ if every $\mathcal{M}_i$-definable set has a pseudo-closure, and we say that T_i is **approximable over** $T_\cap$ if the same situation holds for every T_i-model. Then T_i satisfies the **pseudo-topological** conditions if T_i is approximable over $T_\cap$ and T_i defines pseudo-denseness (see Section 8.1 for a precise definition of the latter). If T_1 and T_2 satisfy the pseudo-topological conditions, then $\mathcal{M}_\cup$ is interpolative if and only if $X_1 \cap X_2 \neq \varnothing$ whenever X_1 and X_2 are both pseudo-dense in some $X_\cap$. The **definability of pseudo-denseness** ensures this property is axiomatizable. In many settings of interest, the notions of approximability and definability of pseudo-denseness turn out to be equivalent to very natural notions in those settings.

The use of the term "pseudo-topological" is motivated by consideration of the case, treated in Section 8.3, when $T_\cap$ is o-minimal and dim is the canonical o-minimal dimension. In this case, any theory extending $T_\cap$ defines pseudo-denseness. Furthermore T_i is approximable over $T_\cap$ if and only if $T_\cap$ is an *open core* of T_i, i.e. the closure of any $\mathcal{M}_i$-definable set in any T_i-model $\mathcal{M}_i$ is already $\mathcal{M}_\cap$-definable. This leads to the following:

Theorem 1.9. *Suppose $T_\cap$ is o-minimal. If $T_\cap$ is an open core of both T_1 and T_2 then $T^*_\cup$ exists.*

In the case when $L_\cap = \varnothing$ and $T_\cap$ is the theory of an infinite set, the notion of interpolative fusion is essentially known and was studied by Winkler in his thesis [**89**]. Winkler shows that $T^*_\cup$ exists if only if T_1 and T_2 both eliminate $\exists^\infty$. In Section 8.4, we show that if $T_\cap$ is $\aleph_0$-stable, and dim is Morley rank, then any theory extending $T_\cap$ is approximable over $T_\cap$ (e.g. if $T_\cap$ is the theory of algebraically closed fields, then this follows from the fact that every Zariski closed set is definable). In Section 8.5, we show that if $T_\cap$ is $\aleph_0$-stable, $\aleph_0$-categorical, and weakly eliminates imaginaries, then T_i defines pseudo-denseness if and only if T_i eliminates $\exists^\infty$. This yields a generalization of Winkler's theorem:

Theorem 1.10. *Suppose that $T_\cap$ is $\aleph_0$-stable, $\aleph_0$-categorical, and weakly eliminates imaginaries. If T_1 and T_2 both eliminate $\exists^\infty$, then $T^*_\cup$ exists.*

The preceding theorem can also be used to prove another main result of [**89**]: the existence of generic Skolemizations of model-complete theories eliminating $\exists^\infty$. We explain this in Section 7.5.

In [**81**, Chapter 3], van den Dries notes a similarity between his main result and Winkler's theorem and remarks that this similarity *"... suggests a common generalization of Winkler's and my results"*. This chapter can be seen as a moral answer to this suggestion, but not yet the final one, as our result only covers a special case of the main result in [**81**, Chapter 3].

Chapter 9. Preservation results. We adopt here the notational convention of the summary of Chapter 6. Suppose that the interpolative fusion $T^*_\cup$ exists. The examples described above motivate the following question:

How are the model-theoretic properties of $T^*_\cup$ determined by T_1, T_2, and $T_\cap$?

Model-theoretic properties of $T^*_\cup$ should be largely determined by how T_i relates to $T_\cap$ for $i \in \{1,2\}$, and not by any relationship between T_1 and T_2. We describe a general framework for strengthenings of model-completeness in Section 5.2 and prove syntactic preservation results in Chapter 9. The most important is the following, see Proposition 9.2.

Theorem 1.11. *Suppose $T_\cap$ is stable with weak elimination of imaginaries. Suppose $T^*_\cup$ exists. Then every $L_\cup$-formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas of the form*

$$\exists y\,(\varphi_1(x,y) \wedge \varphi_2(x,y))$$

where $\varphi_i(x,y)$ is an L_i-formula for $i \in \{1,2\}$ and $(\varphi_1(x,y) \wedge \varphi_2(x,y))$ is bounded in y, i.e. there exists k such that $T^*_\cup \models \forall x\, \exists^{\leqslant k} y\,(\varphi_1(x,y) \wedge \varphi_2(x,y))$.

This result is close to optimal, as $L_\cup$-formulas are in general not $T^*_\cup$-equivalent to boolean combinations of L_1 and L_2-formulas. However, in Proposition 9.4, we show that certain restrictive conditions on algebraic closure in T_1 and T_2 do imply that every $L_\cup$-formula is $T^*_\cup$-equivalent to a boolean combination of L_1 and L_2-formulas. If this special situation holds, and if T_1 and T_2 are both stable (NIP), then $T^*_\cup$ must also be stable (NIP), see Section 9.5.

These syntactic preservation results can be applied to obtain classification-theoretic preservation results which relate the (neo)stability-theoretic properties of $T^*_\cup$ to those of T_1, T_2, and $T_\cap$. The most notable result we have obtained so far in this direction is the following:

Theorem 1.12. *If both T_1 and T_2 have* NSOP_1 *and $T_\cap$ is stable with* 3*-uniqueness, then $T^*_\cup$ has* NSOP_1.

On the other hand, NTP_2 is not preserved even in very natural situations, which brings us to the hope that the boundary of the tame universe can be extended to include these examples as well.

Part 1

Concrete partially random structures

CHAPTER 2

Tame structures via character sums over finite fields

To minimize repetition, we treat this chapter as the continuation of the corresponding summary in Section 1.2, and keep the definitions and statements of theorems given there. Throughout this chapter, we also assume that $x = (x_1, \dots, x_m)$ is an m-tuple of variables, $y = (y_1, \dots, y_n)$ is an n-tuple of variables, G is a multiplicative abelian group, F is a field, and $F^\times$ is the multiplicative group of F. Again, $\mathbb{T}$ is the multiplicative group of complex numbers of norm 1, and $\mathbb{F}$ is an algebraic closure of a finite field.

2.1. Almost model companion of GMO^-

For understanding ACFO^- and finding its model companion, we need to first understand $F^\times$ and $(F^\times, \triangleleft)$ as $(F, \triangleleft)$ ranges over the models of ACFO^-. Two phenomena turn out to be important later on:

(1) the "reduct" of ACFO^- to the language of multiplicative groups is very simple: it "almost" admits quantifier elimination and has a natural notion of dimension;
(2) the "reduct" of ACFO^- to the language of circularly ordered multiplicative groups "almost" has a model companion.

2.1.1. Multiplicative groups of algebraically closed fields. We will consider the theory of multiplicative groups of algebraically closed fields (i.e., the set of statements which hold in all such structures) in a suitable language, show that this theory "almost" admits quantifier elimination and coincides with the theory of multiplicative groups of ACFO^--models, and obtain an axiomatization along the way as usual.

Throughout Section 2.1.1, G is a multiplicative abelian group. If a and b in G are such that $b^n = a$, we call b an n**th root** of a. If $n \geqslant 1$, an nth root of the identity element 1_G of G is **trivial** if it is 1_G. An nth root of $1_\mathbb{T}$ in the multiplicative group $\mathbb{T}$ for some $n \geqslant 1$ is called a **root of unity**. Let $\mathbb{U} \subseteq \mathbb{T}$ be the multiplicative group of roots of unity. For a given p, let $\mathbb{U}_{(p)} \subseteq \mathbb{T}$ be the multiplicative group of roots of unity whose order is coprime to p. So $\mathbb{U}_{(p)}$ is isomorphic to $\mathbb{F}^\times$ as a group when $\mathrm{char}(\mathbb{F}) = p$.

Let $L_\mathrm{m} = \{1, \times, \square^{-1}\}$ be the language of multiplicative groups. It is easy to obtain an L_m-theory GM such that $G \vDash \mathrm{GM}$ if and only if the following conditions hold:

($0^\times$) every finite subgroup of G is cyclic;
($1^\times$) the group G is divisible;
($2^\times$) for any two distinct prime numbers p and l, either 1_G has a nontrivial pth root, or 1_G has a nontrivial lth root.

The theory GM is our candidate for axiomatizing the theory of multiplicative groups of algebraically closed fields. It has several natural extensions. For a given p, let GM_p be the L_{m}-theory whose models are $G \models \mathrm{GM}$ which satisfy the following extra property:

($\mathrm{c}_p^\times$) every pth root of 1_G is trivial.

It is easy to see that if $G \models \mathrm{GM}_p$, then 1_G has a nontrivial lth root for any prime number $l \neq p$. Let GM_0 be the L_{m}-theory whose models are $G \models \mathrm{GM}$ which satisfy the following extra property:

($\mathrm{c}_0^\times$) for all prime numbers l, 1_G has a nontrivial lth root.

Hence, a model of GM is either a model of GM_p for some p or a model of GM_0.

Remark 2.1. Suppose G satisfies conditions ($0^\times$) and ($1^\times$), and l is a prime number. The condition that 1_G has a nontrivial l-root is also equivalent to several other conditions:

(1) 1_G has exactly l many lth roots;
(2) for all $k \geqslant 1$, 1_G has exactly l^k many l^kth roots;
(3) for all $k \geqslant 1$, every $a \in G$ has exactly l^kth many l^kth roots.

Likewise, the condition that every pth root of 1_G is trivial is also equivalent to two other conditions:

(1) for all k, every p^kth root of 1_G is trivial;
(2) for all $k \geqslant 1$, every $a \in G$ has exactly one p^kth root.

From Remark 2.1, we easily deduce the following:

Remark 2.2. For every p, $\mathbb{U}_{(p)}$ is a model of GM_p, and so is $\mathbb{F}^\times$ when $\mathrm{char}(\mathbb{F}) = p$. Moreover, if G is a model of GM_p, then the group of torsion elements of G is isomorphic to $\mathbb{U}_{(p)}$. The group $\mathbb{U}$ is a model of GM_0 and is isomorphic to the group of torsion elements of any GM_0-model.

Lemma 2.1 confirms that our candidate GM at least meets the basic requirements:

Lemma 2.1. *If G is the multiplicative group of an* ACF*-model, then $G \models \mathrm{GM}$. Similar statements hold for* ACF_p *together with* GM_p *for an arbitrary p and for* ACF_0 *together with* GM_0.

Proof. It is easy to see that if G is the multiplicative group of a prime model of ACF, then conditions $(0^\times)$, $(1^\times)$, and $(2^\times)$ are satisfied. Hence, the first statement follows from the fact that ACF is model complete. The proof of the second statement is similar. $\square$

Suppose B is a subset of G, and $t(x)$ and $t'(x)$ are $L_{\mathrm{m}}(B)$-terms. Then we call the atomic formula $t(x) = t'(x)$ a **multiplicative equation over** B. A multiplicative equation over B is **trivial** if it defines in every abelian group G' extending $\langle B\rangle$ the set $(G')^m$. If $a \in G^m$ does not satisfy any nontrivial multiplicative equation over B, we say that a is **multiplicatively independent** over B.

Proposition 2.1 below is the "almost" quantifier-elimination result we promised. This can be seen as folklore and can be obtained as a consequence of the characterization of elementary embeddings of abelian groups [**30**] and the quantifier-reduction for abelian groups [**65**, page 46]. Since the situation is relatively simple, we briefly indicate a direct proof:

Proposition 2.1. *For each p, the theory GM_p is complete and admits quantifier elimination. A similar statement holds for GM_0. However, GM is not model complete.*

Proof. We will only prove the first statement for GM_p with p fixed as the proof for GM_0 is very similar. By Remark 2.2, $\mathbb{U}_{(p)}$ is an L_{m}-substructure of every model of GM_p, so completeness will follow from quantifier elimination. Using a standard test for quantifier elimination, we need to show the following: if G and G' are models of GM_p such that G' is $|G|^+$-saturated, and f is a partial L_{m}-isomorphism from G to G' (i.e., f is an L_{m}-isomorphism from an L_{m}-substructure of G to an L_{m}-substructure of G') such that $\mathrm{Domain}(f) \neq G$, then there is a partial L_{m}-isomorphism from G to G' which properly extends f.

In each of the following cases, we will obtain a in $G \smallsetminus \mathrm{Domain}(f)$ and a' in $G' \smallsetminus \mathrm{Image}(f)$. A proper extension of f can then be defined by

$$a^k b \mapsto (a')^k f(b) \quad \text{for } k \in \mathbb{Z} \text{ and } b \in \mathrm{Domain}(f).$$

We will leave the reader to check that the function is well-defined and is a partial L_{m}-isomorphism from G to G'.

Suppose l is a prime number, and $a \in G \smallsetminus \mathrm{Domain}(f)$ is a nontrivial lth root of 1_G. As G satisfies $(\mathrm{c}_p^\times)$, $l \neq p$. Since G and G' both satisfy $(0^\times)$, $\mathrm{Domain}(f)$ and $\mathrm{Image}(f)$ contain no nontrivial lth roots of 1_G and $1_{G'}$ respectively. We can then choose $a' \in G' \smallsetminus \mathrm{Image}(f)$ to be an lth root of $1_{G'}$, which must exist because G' satisfies $(\mathrm{c}_p^\times)$ and $(2^\times)$.

Now suppose $\mathrm{Domain}(f)$ contains all roots of 1_G with prime order, l is a prime and $a \in G \smallsetminus \mathrm{Domain}(f)$ is such that $a^l \in \mathrm{Domain}(f)$. If b is another lth root of a^l, then ab^{-1} is an lth root of 1_G. Hence, $\mathrm{Domain}(f)$ contains no lth root of a^l, and $\mathrm{Image}(f)$ contains no

lth root of $f(a^l)$. We then choose a' to be an lth root of $f(a^l)$ which must exist because G' satisfies $(1^\times)$.

The last case is when $\mathrm{Domain}(f)$ is divisibly closed in G, and $a \in G \smallsetminus \mathrm{Domain}(f)$. Using the fact that G' is $|G|^+$-saturated, we obtain $a' \in G'$ which is multiplicatively independent over $\mathrm{Image}(f)$.

For the last statement, note that both $\mathbb{U}_{(p)}$ and $\mathbb{U}$ are models of GM, $\mathbb{U}_{(p)}$ is a substructure of $\mathbb{U}$, but $\mathbb{U}_p$ is not not an elementary substructure of $\mathbb{U}$. $\square$

Fact 2.1 is an easy consequence of Želeva's characterization of circularly orderable groups [**85**] and Levi's characterization of linearly orderable abelian group [**55**]:

Fact 2.1. *An abelian group is circularly orderable if and only if it satisfies* $(0^\times)$.

Combining Proposition 2.1 and Fact 2.1 confirms the validity of our candidate GM:

Corollary 2.1. *Every model of* GM *is elementarily equivalent to both the multiplicative group of an algebraically closed field and the multiplicative group of a model of* ACFO^-.

Many other model-theoretic properties of the theory GM are also immediate:

Corollary 2.2. *The theory* GM *is strongly minimal.*

Hence, definable sets, types, and elements in a model of GM can be given a canonical dimension mdim which coincides with Morley rank and the $\mathrm{acl}_{\mathrm{m}}$-dimension; see [**58**] for details. Proposition 2.1 also yields:

Corollary 2.3. *Suppose G is a model of* GM*, B is a subset of G, and a is in G^m. Then* $\mathrm{mdim}(a|B) < m$ *if and only if a is multiplicatively dependent over B.*

2.1.2. Circularly ordered multiplicative groups of ACFO^--models. We next consider the theory of circularly ordered multiplicative groups of models of ACFO^-. We want to show that that this theory "almost" has a model companion and obtain an axiomatization for this model companion along the way.

Throughout Section 2.1.2, we adopt the notational conventions of Section 2.1.1. Moreover, G is assumed to be circularly orderable, and $(G, \lhd)$ ranges over the circularly ordered multiplicative abelian groups. For each $(G, \lhd)$, we define the linear order $<$ on $(G, \lhd)$ by setting $1_G < a$ for all $a \in G \smallsetminus \{1_G\}$ and

$$a < b \text{ if and only if } \lhd(1_G, a, b) \quad \text{for } a, b \in G \smallsetminus \{1_G\}.$$

When G is $\mathbb{T}$, $\mathbb{U}$, or $\mathbb{U}_{(p)}$, we let $\lhd$ denotes the clockwise circular orders on the respective sets. From Fact 2.1, we can easily deduce the following:

Remark 2.3. For given $(G, \lhd)$ and finite subgroup A of G, if $a \in A \setminus \{1_G\}$ is minimal with respect to $\lessdot$, then $A = \langle a \rangle$.

Let $L_{\mathrm{mc}} = L_{\mathrm{m}} \cup \{\lhd\}$ be the language of circularly ordered abelian groups. Let GMO^- be the theory whose models are $(G, \lhd)$ such that $G \models \mathrm{GM}$, or equivalently, G satisfies $(1^\times)$ and $(2^\times)$ (as $(0^\times)$ is automatic by Fact 2.1). Let $\mathrm{GMO}_p^- = \mathrm{GMO}^- \cup \mathrm{GM}_p$ for all p, and let $\mathrm{GMO}_0^- = \mathrm{GMO}^- \cup \mathrm{GM}_0$. We show below that GMO^- is an axiomatization of the theory of circularly ordered multiplicative groups of algebraically closed fields:

Lemma 2.2. *An L_{mc}-structure $(G, \lhd)$ is a model of GMO^- if and only if $(G, \lhd)$ is elementarily equivalent to the circularly ordered group of an ACF-model. Similar statements hold for GMO_p^- together with ACF_p for an arbitrary p and GMO_0^- together with ACF_0.*

Proof. The backward implication of the first statement follows immediately from Lemma 2.1. For the forward implication of the first statement, suppose $(G, \lhd)$ is a model of GMO^-. We assume further that $(G, \lhd)$ is a model of GMO_p^- and omit the proof of the similar case where $(G, \lhd)$ is a model of GMO_0^-. Replacing $(G, \lhd)$ by an elementary extension if necessary, we can arrange that $|G| = \kappa > \aleph_0$. By Corollary 2.2, GMO_p^- is κ-categorical. Hence, G is is isomorphic to the multiplicative group G' of a model of ACF_p of size κ. Pushing forward $\lhd$ by the isomorphism we get a circular orderding $\lhd'$ on G' such that $(G', \lhd')$ is L_{mc}-isomorphic to $(G, \lhd)$. This also proved the second statement. $\square$

A rather awkward aspect dealing with $(G, \lhd)$ comes from the fact that $\lessdot$ is not invariant under translation. We will consider here a partial rectification. The **winding number** $W(a_1, \ldots, a_n)$ of $(a_1, \ldots, a_n) \in G^n$ is defined to be the cardinality of the set

$$\Big\{k : 1 \leqslant k \leqslant n-1, \prod_{i=1}^{k+1} a_i \lessdot \prod_{i=1}^{k} a_i\Big\}.$$

It is intuitively the number of times the sequence $a_1, a_1a_2, \ldots, \prod_{i=1}^{n-1} a_i, \prod_{i=1}^{n} a_i$ "winds around the circle". If $a_1 = \ldots = a_n = a$, we also denote $W(a_1, \ldots, a_n)$ as $W_n(a)$.

Remark 2.4. Suppose a and b in G satisfy $a \lessdot b$. Then for all $c \in G$, either $ac \lessdot bc$ or $W(a, c) < W(b, c)$. So in this sense the notion of winding number accounts for the non-invariant of $\lessdot$.

For $a \in G$, we say that a is **n-divisible with winding number** r if a has an nth root b with $W_n(b) = r$.

Remark 2.5. Consider $(G, \lhd)$ and $a \in G$. From Remark 2.4, it is easy to see that every $a \in G$ has at most one nth root b such that $W_n(b) = r$. So if there are distinct $b_1, \ldots, b_n$ such that $b_i^n = a$ for all $i \in \{1, \ldots, n\}$, then for each $r \in \{1, \ldots, n\}$ there is exactly one $i \in \{1, \ldots, n\}$ such that $W_n(b_i) = r$.

Let GMO be the L_{mc}-theory such that its models are $(G, \lhd)$ with $G \vDash \mathrm{GM}$ and the following density condition is satisfied:

($\mathrm{d}^{\times}$) for any given n, $r \in \{0, \ldots, n-1\}$, and c and d in G, there is $a \in G$ such that $\lhd(c, a, d)$ and a is n-divisible with winding number r.

The theory GMO is our candidate for the "almost" model companion of GMO^{-}. Also set $\mathrm{GMO}_p = \mathrm{GMO} \cup \mathrm{GMO}_p^{-}$ for an arbitrary p, and set $\mathrm{GMO}_0 = \mathrm{GMO} \cup \mathrm{GMO}_0^{-}$.

To handle circularly ordered groups, it is convenient to "linearize" them; see also [**34**] and [**31**] for related material. Let $(H, <)$ be a linearly ordered additive group with identity element 0_H, and let $\omega \in H$ be a distinguished positive element such that $(n\omega)_{n>0}$ is cofinal in $(H, <)$ (i.e., for every $\alpha \in H$, $\alpha < n\omega$ for sufficiently large n). For every k, set

$$[k, k+1)_H = \{\alpha \in H : k\omega \leqslant \alpha < (k+1)\omega\}.$$

A surjective group homomorphism $e : H \to G$ with kernel $\langle \omega \rangle$ is a **covering map** from $(H, \omega, <)$ to $(G, \lhd)$ if for all n and all $\alpha, \beta, \gamma \in [n, n+1)_H$ $\lhd(e(\alpha), e(\beta), e(\gamma))$ is equivalent to

$$\alpha < \beta < \gamma \quad \text{or} \quad \beta < \gamma < \alpha \quad \text{or} \quad \gamma < \alpha < \beta.$$

If there is a covering map from $(H, \omega, <)$ to $(G, \lhd)$, we call $(H, \omega, <)$ a **universal cover** of $(G, \lhd)$.

Remark 2.6. Suppose $(H, \omega, <)$ is as described in the preceding paragraph. Then the above definition also allow us to construct $(G, \lhd)$ such that $(H, \omega, <)$ is a universal cover of $(G, \lhd)$.

The examples in the following remark will hopefully make this notion concrete:

Remark 2.7. Let the additive groups $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}_{(p)}$ be equipped with their natural orders $<$. With $\alpha \mapsto e^{2\pi i \alpha}$ the covering map, we have the following:

(1) $(\mathbb{R}, 1, <)$ is a universal cover of $(\mathbb{T}, \lhd)$;
(2) $(\mathbb{Q}, 1, <)$ is a universal cover of $(\mathbb{U}, \lhd)$;
(3) $(\mathbb{Z}_{(p)}, 1, <)$ is a universal cover of $(\mathbb{U}_{(p)}, \lhd)$.

The lemma below illustrates the advantage of having a universal cover.

Lemma 2.3. *Suppose $(H, \omega, <)$ is a universal cover of $(G, \lhd)$ with e the covering map, $\alpha_1, \ldots, \alpha_n$ are in $[0, 1)_H$, and $a_i = e(\alpha_i)$ for $i \in \{1, \ldots, n\}$. Then*

$$W(a_1, \ldots, a_n) = r \text{ if and only if } \alpha_1 + \ldots + \alpha_n \in [r, r+1)_H.$$

Proof. It follows from the definition of a universal cover that $\sum_{i=1}^{k} \alpha_i \in [l, l+1)_H$ and $\sum_{i=1}^{k+1} \alpha_i \in [l+1, l+2)_H$ if and only if $\prod_{i=1}^{k+1} a_i \lessdot \prod_{i=1}^{k} a_i$. The desired conclusion follows. □

Applying Lemma 2.3 into the setting where $a_1 = \ldots = a_n = a$, we get:

Corollary 2.4. *Suppose $(H, \omega, <)$ is a universal cover of $(G, \lhd)$ with e the covering map, a is in G, $n \geqslant 1$, r is in $\{0, \ldots, n-1\}$, and $\alpha \in [r, r+1)_H$ is such that $e(\alpha) = a$. Then the following are equivalent:*

(i) *a is n-divisible with winding number r;*
(ii) *α is n-divisible.*

We can view such $(H, \omega, <)$ as a structure in a language L_{al} consisting of function symbols for 0, ω, and $+$ and a relation symbol for $<$. It turns out that the convenience of a universal cover is something we can always afford. Moreover, we get it partially definably:

Lemma 2.4. *Every $(G, \lhd)$ has a universal cover $(H, \omega, <)$. Moreover, there is an L_{mc}-isomorphic copy $(\tilde{G}, \tilde{\lhd})$ of $(G, \lhd)$ such that the underlying set of $\tilde{G}$ is $[0,1)_H$, and the multiplication on $\tilde{G}$ and $\tilde{\lhd}$ can be defined by L_{al}-formulas whose choice is independent of the choice of $(G, \lhd)$ and the choice of $(H, \omega, <)$.*

Proof. Set $H = \mathbb{Z} \times G$, and define

$$(k, a) + (k', a') = (k + k' + W(a, a'), aa')$$

for (k, a) and (k', a') in H. Let $<$ be the lexicographic product of the usual order on $\mathbb{Z}$ and the linear order $\lessdot$ on G. Set $0_H = (0_{\mathbb{Z}}, 1_G)$ and $\omega = (1_{\mathbb{Z}}, 1_G)$. We can easily check that $(H, \omega, +, <)$ is a universal cover of G. For $a, a' \in [0_H, \omega)_H$, set

$$a \mathbin{\tilde{\times}} a' = \begin{cases} a + a' & \text{if } a + a' \in [0,1)_H, \\ a + a' - \omega & \text{otherwise.} \end{cases}$$

Define $\tilde{\lhd}$ by setting $\tilde{\lhd}(a, b, c)$ for any $a, b, c \in [0,1)_H$ such that $a < b < c$ or $b < c < a$ or $c < a < b$. It is easy to see the quotient map $H \to G$ induces an isomorphism from $([0,1)_H, \tilde{\times}, \tilde{\lhd})$ to $(G, \lhd)$. $\square$

The universal cover notion is functorial in the following sense:

Lemma 2.5. *Suppose $(H, \omega, <)$ is a universal cover of $(G, \lhd)$ with convering map e, and $(H', \omega', <')$ is a universal cover of $(G', \lhd')$ with convering map e'. Then we have the following:*

(i) *if g is an L_{mc}-embedding from $(G, \lhd)$ to $(G', \lhd')$, then there is a unique L_{al}-embedding h from $(H, \omega, <)$ to $(H', \omega', <')$ such that the diagram below commutes:*

$$\begin{array}{ccc} H' & \xrightarrow{e'} & G' \\ \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle g} \\ H & \xrightarrow{e} & G \end{array} ;$$

in particular, e is the unique covering map from $(H, \omega, <)$ to $(G, \lhd)$, and any two universal coverings of $(G, \lhd)$ are isomorphic as L_{al}-structures;

(ii) *if h is an L_{al}-embedding from $(H, \omega, <)$ to $(H', \omega', <')$, then there is a unique L_{mc}-embedding g from $(G, \lhd)$ to $(G', \lhd')$ such that the same diagram above commutes.*

Proof. For (i), let $h : H \to H'$ be such that $\alpha \in [k, k+1)_H$ is mapped to the unique $\beta \in [k, k+1)_{H'}$ with $g \circ e(\alpha) = e'(\beta)$. For (ii), let $g : G \to G'$ be such that $e(\alpha)$ is mapped to $e' \circ h(\alpha)$ for $\alpha \in H$. It is easy to check that h and g are as desired. □

We extend the "linearization" procedure to theories GMO^- and GMO. Let HAO^- be an L_{al}-theory such that an L_{al}-structure $(H, \omega, <)$ is a model of HAO^- if and only if $(H, <)$ is a linearly ordered additive abelian group, ω is a positive element in H, and the the following additional two properties are satisfied:

(1^+) for each n and $\alpha \in H$, there is at least one $r \in \{0, \dots, n-1\}$ such that $\alpha + r\omega$ is n-divisible;

(2^+) for any prime numbers p and l, ω is either p-divisible or l-divisible.

Note that (1^+) and (2^+) correspond to ($1^\times$) and ($2^\times$). There is no (0^+) because ($0^\times$) is trivial in our current setting. The condition that ω is cofinal in H cannot be included here as it is not first-order. For a given p, let HAO_p^- be the L_{al}-theory whose models are the $(H, \omega, <) \models \mathrm{HAO}^-$ which satisfy the addition condition:

(c_p^+) ω is not p-divisible.

We also let let HAO_0^- be the L_{al}-theory whose models are the $(H, \omega, <) \models \mathrm{HAO}^-$ which satisfy the additional condition:

(c_0^+) for all prime numbers l, ω is l-divisible.

Let HAO be an L_{al}-theory whose models are the $(H, \omega, <) \models \mathrm{HAO}^-$ which also satisfy the additional condition:

(d^+) for any given n and $\beta, \gamma \in H$ with $\beta < \gamma$, there is $\alpha \in H$ such that α is n-divisible and $\beta < \alpha < \gamma$.

Finally, set $\mathrm{HAO}_p = \mathrm{HAO} \cup \mathrm{HAO}_p^-$ for each p, and $\mathrm{HAO}_0 = \mathrm{HAO} \cup \mathrm{HAO}_0^-$; in fact, HAO_0 is just the theory of divisible ordered abelian groups. The next Lemma explains precisely what it means by saying that these are "linearization" of GMO^- and GMO:

Lemma 2.6. *Suppose $(H, \omega, <)$ is a universal cover of $(G, \lhd)$. Then we have:*

(i) *for all p, $(H, \omega, <) \models \mathrm{HAO}^-$ if and only if $(G, \lhd) \models \mathrm{GMO}^-$. Similar statements hold for HAO_p^- together with GMO_p^- and HAO_0^- together with GMO_0^-;*

(ii) *for all p, $(H, \omega, <) \models \mathrm{HAO}$ if and only if $(G, \lhd) \models \mathrm{GMO}$. Similar statements hold for HAO_p together with GMO_p and HAO_0 together with GMO_0.*

Proof. All these statements are immediate consequences of Corollary 2.4. □

Lemma 2.6 allows us to deduce results for GMO$^-$-models and GMO-models from generally much easier results for HAO$^-$-models and HAO-models. Below is the first demonstration of its usefulness:

Lemma 2.7. *Let* $(\mathbb{Z}_{(p)}, 1, <)$ *and* $(\mathbb{Q}, 1, <)$ *be as in Remark 2.7. Then we have* $(\mathbb{Z}_{(p)}, 1, <) \models \mathrm{HAO}_p$ *and* $(\mathbb{Q}, 1, <) \models \mathrm{HAO}_0$*. Moreover, there is a unique* L_{al}*-embedding of* $(\mathbb{Z}_{(p)}, 1, <)$ *into every* HAO^-_p*-model and a unique* L_{al}*-embedding of* $(\mathbb{Q}, 1, <)$ *into every* HAO^-_0*-model.*

Proof. It is easy to verify that $(\mathbb{Z}_{(p)}, 1, <)$ is a model of HAO^-_p. Since $\mathbb{Z}_{(p)}$ is dense in $\mathbb{R}$ with respect to the natural order, it follows that $(\mathbb{Z}_{(p)}, 1, <)$ is a model of HAO_p. Suppose $(H, \omega, <)$ is a model of HAO^-_p. Then the subgroup of H generated by ω is an isomorphic copy of $\mathbb{Z}_{(p)}$. This gives us an L_{al}-embedding of $(\mathbb{Z}_{(p)}, 1, <)$ into $(H, \omega, <)$. This L_{al}-embedding is unique as any such L_{al}-embedding must send 1 to ω. The statements for $(\mathbb{Q}, 1, <)$ can be proven similarly. □

Combining with Lemma 2.5 and Lemma 2.6, we get:

Proposition 2.2. *We have* $(\mathbb{U}_{(p)}, \lhd) \models \mathrm{GMO}_p$ *and* $(\mathbb{U}, \lhd) \models \mathrm{GMO}_0$*. Moreover, there is a unique* L_{mc}*-embedding of* $(\mathbb{U}_{(p)}, \lhd)$ *into every* GMO^-_p*-model and a unique* L_{mc}*-embedding of* $(\mathbb{U}, \lhd)$ *into every* GMO^-_0*-model.*

Suppose σ is an L_{m}-automorphism of $\mathbb{U}_{(p)}$. Define $\lhd_\sigma$ to be the image of the clockwise circular order $\lhd$ under σ. From Lemma 2.2, we get the following:

Corollary 2.5. *Every circular order on* $\mathbb{U}_{(p)}$ *is equal to* $\lhd_\sigma$ *for a unique* L_{m}*-automorphism* σ *of* $\mathbb{U}_{(p)}$*.*

For an injective group homomorphism $\chi : \mathbb{F}^\times \to \mathbb{T}$, define the circular order $\lhd_\chi$ to be the pullback of $\lhd$ via χ. Note that $\chi(\mathbb{F}^\times) = \mathbb{U}_{(p)}$ as a subgroup of $\mathbb{T}$. So applying Corrollary 2.5, we get:

Corollary 2.6. *Every multiplicative circular order on* $\mathbb{F}$ *is equal to* $\lhd_\chi$ *for a unique injective group homomorphism* $\chi : \mathbb{F}^\times \to \mathbb{T}$*.*

Toward showing that GMO is "almost" the model companion of GMO$^-$, we first show the "linearized" version of the result. This is also folklore [**88**], but not everything we want is written down, so we briefly indicate a proof.

Lemma 2.8. *For each* p*, the theory* HAO_p *is complete and is the model companion of* HAO^-_p*. A similar statement holds for* HAO_0 *and* HAO^-_0*. The theory* HAO *is not model complete.*

Proof. To show the first statement, we require some preparation. For each $(H, \omega, <) \models \mathrm{HAO}_p^-$, define the family $D = (D_n)_{n>0}$ of unary relations on H by setting

$$D_n = \{\alpha \in H : \text{ there is } \beta \in H \text{ such that } n\beta = \alpha.\}$$

Then such $(H, \omega, <, D)$ is naturally a structure in a language $L_{\mathrm{al}}^{\diamond}$ extending L_{al} by adding a family of unary relation symbols for D. The theory HAO_p^- and HAO_p can be naturally expanded to $L_{\mathrm{al}}^{\diamond}$-theories by adding the obvious axioms defining such D. Note that when $(H, \omega, <) \models \mathrm{HAO}_p^-$ and $D = (D_n)_{n>0}$ are as above, we also have

$$D_n = \{\alpha \in H : \text{ for all } \beta \in H, \bigwedge_{r=1}^{n-1} n\beta \neq \alpha + r\omega.\}$$

It follows that such D_n is both universally and existentially definable. Moreover, we can choose the formula defining such D_n independent of the choice of $(H, \omega, <)$. Thus, the problem is reduced to showing that the natural $L_{\mathrm{al}}^{\diamond}$-expansion of HAO_p is complete, admits quantifier elimination, and is the model companion of the natural $L_{\mathrm{al}}^{\diamond}$ expansion of HAO_p^-.

It follows from Lemma 2.7 that $(\mathbb{Z}_{(p)}, 1, <, D)$ can be canonically viewed as a $L_{\mathrm{al}}^{\diamond}$-substructure of any model of the natural $L_{\mathrm{al}}^{\diamond}$-expansion of HAO_p. Hence,it suffices to show the following: if $(H, \omega, <, D)$ is the natural $L_{\mathrm{al}}^{\diamond}$-expansion of a model of HAO_p^-, $(H', \omega', <', D')$ is the natural $L_{\mathrm{al}}^{\diamond}$-expansion of a model of HAO_p and is moreover $|H|^+$-saturated, and $f : H \to H'$ is a partial $L_{\mathrm{al}}^{\diamond}$-isomorphism from $(H, \omega, <, D)$ to $(H', \omega', <', D')$ with $\mathrm{Domain}(f) \neq H$, then we can find a partial $L_{\mathrm{al}}^{\diamond}$-embedding which properly extends f.

It is easy to reduce to the case where $\mathrm{Domain}(f)$ is a divisibly closed subgroup of H. Let $\alpha \in H \setminus \mathrm{Domain}(f)$. If $\alpha - r\omega$ is p^k-divisible and $\beta < \alpha < \beta'$ for β and β' in $\mathrm{Domain}(f)$, then we can find α' in $H' \setminus \mathrm{Image}(f)$ such that $\alpha' - r\omega'$ is p^k-divisible and $f(\beta) < \alpha' < f(\beta')$ using the fact that $(H', \omega', <')$ satisfies (d^+). As H' is $|H|^+$-saturated, we can arrange that α' satisfies all such conditions simultaneously. Let g be the obvious extension of f sending α to α'. It is easy to check that g is as desired.

The proof of the second statement is similar to the proof of the first statement. Note that $(\mathbb{Z}_{(p)}, 1, <)$ is an L_{al}-substructure of $(\mathbb{Q}, 1, <)$, and both are models of HAO, but the former is not an elementary substructure of the latter. So HAO is not model complete. □

Proposition 2.3. *For each p, the theory GMO_p is complete and is the model companion of GMO_p^-. A similar statement holds for GMO_0 and GMO_0^-. However,* GMO *is not model complete.*

Proof. By Proposition 2.2, $(\mathbb{U}_{(p)}, \lhd)$ is a model of GMO_p and is an L_{mc}-substructure of any model of GMO_p. Hence to get the completeness of GMO_p, it suffices to show that GMO_p is model complete. Suppose $(G, \lhd)$ and $(G', \lhd)$ are models of GMO_p and that the former is a substructure of the latter. By Lemma 2.6, the universal covers $(H, \omega, <)$ and $(H', \omega, <)$ of $(G, \lhd)$ and $(G', \lhd)$ are models of HAO_p. It follows from Lemma 2.5 and Lemma 2.8 that $(H, \omega, <)$ can be viewed as an elementary substructure of $(H', \omega, <)$. Combining this with the second part of Lemma 2.4, we get that $(G, \lhd)$ is an elementary substructure of $(G', \lhd)$.

Next we show that every model of GMO_p^- can be embedded into a model of GMO_p. Suppose $(G, \lhd)$ is a model of GMO_p^-. By Lemma 2.6, the universal cover $(H, \omega, <)$ of $(G, \lhd)$ is a model of HAO_p^-. Hence, it follows from Lemma 2.8 that $(H, \omega, <)$ has an extension $(H', \omega, <)$ which is a model of HAO_p. Construct $(G', \lhd)$ as mentioned in Remark 2.6. Then $(G', \lhd)$ is a model of GMO_p by Lemma 2.6 and $(G', \lhd)$ can be considered a substructure of $(G', \lhd)$ by Lemma 2.5.

The second statement can be proved similarly. The third statement can be deduced from Lemma 2.8 using similar ideas. It can also be observed directly by looking at $(\mathbb{U}_{(p)}, \lhd)$ and $(\mathbb{Q}, \lhd)$. ☐

Remark 2.8. The theory HAO_0 is just the theory of divisible ordered abelian groups, so HAO_0 has quantifier elimination. For the second statement of Proposition 2.3, we can also get that GMO_0 admits quantifier elimination. Since we will not use this later on, we leave it to the interested reader.

2.2. Model companion of ACFO^-

We will establish that ACFO^- has a model companion in two steps:

(1) obtaining a characterization of the existentially closed models of ACFO^- following the ideas in [**11**];
(2) showing that the class of ACFO^--models satisfying the characterization in (1) is elementary by using model-theoretic/geometric properties of the reducts of ACFO^- to the language of rings and the language of circularly ordered multiplicative groups.

2.2.1. Geometric characterization of the existentially closed models. Intuitively, in an existentially closed model $(F, \lhd)$ of ACFO^-, the field F interacts "randomly" with the circularly ordered abelian group $(F^\times, \lhd) \models \mathrm{GMO}$ over their "common reduct" $F^\times$. In this section, we will make precise this intuition through a "geometric characterization" and then verify its correctness.

We keep the the notational conventions of Section 2.1.1 and Section 2.1.2. Suppose $(G, \square)$ is an L-structure expanding G. For convenience, we call a set $X \subseteq G^m$ which is defined in $(G, \square)$ by a quantifier-free $L(G)$-formula a **qf-set** in $(G, \square)$. For $X \subseteq G^m$ definable in $(G, \square)$ and an elementary extension $(G', \square)$ of $(G, \square)$, let $X(G') \subseteq (G')^m$ be the set defined in $(G', \square)$ by any $L(G)$-formula formula $\varphi(x)$ that defines X.

We first correct a minor issue: the group $F^\times$ is, strictly speaking, not a reduct of F, as 0 is not an element of $F^\times$. Set

$$\Sigma_{n+1} = \{(a_1, \dots, a_{n+1}) \in (F^\times)^{n+1} : a_1 + \dots + a_n = a_{n+1}\}$$

and let $\Sigma = (\Sigma_{n+1})$. We call $(F^\times, \Sigma)$ the **punctured field** associated to F. Then $(F^\times, \Sigma)$ is naturally a structure in the language $L_\mathrm{f}^\times = L_\mathrm{m} \cup \{\Sigma_{n+1}\}$. The group $F^\times$ is now an honest reduct of $(F^\times, \Sigma)$.

We will see in the proof of Lemma 2.14 a more substantial advantage working with $L_\mathrm{f}^\times$ instead of L_f, namely, $L_\mathrm{f}^\times$ expands L_m only by relation symbols and not by function symbols.

Remark 2.9. The following "adding 0" procedure allows us to recover an isomorphic copy of a field from its associated punctured field, but the procedure is applicable to any $L_\mathrm{f}^\times$-structure expanding a multiplicative abelian group. Starting with an $L_\mathrm{f}^\times$-structure (G, Σ), set $F = G \cup \{0\}$, define $+$ on F^2 by pretending that G is $F^\times$ (i.e. $0 + 0 = 0$, $a + 0 = 0 + a = a$ for $a \in G$, $a + b = c$ for a and b in G if c is the unique element of G satisfying $\Sigma_3(a, b, c)$, and $a + b = 0$ for the remaining cases), and define $\times$ on F^2 similarly.

As immediate consequence of Remark 2.9, we get:

Remark 2.10. Suppose F is a field and $(F^\times, \Sigma)$ is its associated punctured field. Then $X \subseteq (F^\times)^m$ is definable in F if and only if X is definable in $(F^\times, \Sigma)$.

From Remark 2.9, it is also easy to find an $L_\mathrm{f}^\times$-theory whose models are precisely the punctured fields. Likewise, we get $L_\mathrm{f}^\times$-theories $\mathrm{ACF}^\times$, $\mathrm{ACF}_p^\times$ for every p, and $\mathrm{ACF}_0^\times$ whose models are punctured models of $\mathrm{ACF}^\times$, $\mathrm{ACF}_p^\times$, and $\mathrm{ACF}_0^\times$ respectively. The basic model theory $\mathrm{ACF}^\times$ can be obtained:

Lemma 2.9. *The theory* $\mathrm{ACF}^\times$ *admits quantifier elimination and is the model companion of the theory of punctured fields. The theories* $\mathrm{ACF}_p^\times$ *for various* p *and* $\mathrm{ACF}_0^\times$ *are the only completions of* $\mathrm{ACF}^\times$*.*

Proof. These statements are easy consequences of Remark 2.9, Remark 2.10, the quantifier elimination of ACF, and the fact that ACF_p for various p and ACF_0 are the only completions of ACF. $\square$

Let $(F, \lhd)$ be an ACFO^--model and $(F^\times, \Sigma)$ the punctured field associated to F. We call $(F^\times, \Sigma, \lhd)$ the **punctured** ACFO^-**-model** associated to $(F, \lhd)$. Then $(F^\times, \Sigma, \lhd)$ is a structure in the language $L_\mathrm{t}^\times = L_\mathrm{f}^\times \cup L_\mathrm{mc}$. We define punctured ACFO_p^--models for various p and punctured ACFO_0^--models likewise.

By the discussion on "adding 0" in Remark 2.9, it is easy to see that there is an $L_{\mathrm{t}}^{\times}$-theory whose models are precisely the punctured ACFO^{-}-models. We say that a punctured ACFO^{-}-model is **existentially closed** if it is an existentially closed model of this theory.

The following Lemma allows us to trade existentially closed ACFO^{-}-models with existentially closed punctured ACFO^{-}-models:

Lemma 2.10. *An* ACFO^{-}*-model is existentially closed if and only if its associated punctured* ACFO^{-}*-model is existentially closed.*

Proof. Let $(F^{\times}, \Sigma, \lhd)$ be the punctured ACFO^{-}-model associated to a model $(F, \lhd)$ of ACFO^{-}, and suppose $(F^{\times}, \Sigma, \lhd)$ is existentially closed. We assume further that $(F, \lhd) \vDash$ ACFO$_p^{-}$ for a fixed p and omit the proof of the similar case where $(F, \lhd)$ is a model of ACFO$_0^{-}$. Note that an ACFO^{-}-model extending $(F, \lhd)$ is then automatically an ACFO$_p^{-}$-model. Let $\varphi(x)$ be a quantifier-free L_{t}-formula which defines a nonempty set in some ACFO$_p^{-}$-model extending $(F, \lhd)$. To get the backward implication, we need to show that $\varphi(x)$ already defines a nonempty set in $(F, \lhd)$.

Note that $\varphi(x)$ is logically equivalent to $(\varphi(x) \wedge x_i = 0) \vee (\varphi(x) \wedge x_i \neq 0)$ with $i \in \{1, \ldots, m\}$, and $\varphi(x) \wedge x_i = 0$ is equivalent over ACFO$_p^{-}$ to a quantifier-free formular with fewer variables. So we reduced to the case where $\varphi(x)$ is logically equivalent to $\varphi(x) \wedge \bigwedge_{i=1}^{m} x_i \neq 0$. Consider the special case where the only atomic formulas of $\varphi(x)$ in which $+$ appears are of the form

$$t_1(x) + \ldots + t_n(x) = t_{n+1}(x)$$

where $t_i(x)$ does not further contain $+$ for $i \in \{1, \ldots, n+1\}$. Such a formula $t_1(x) + \ldots + t_n(x) = t_{n+1}(x)$ defines in an arbitary ACFO$_p^{-}$-model the same set that $\Sigma_{n+1}(t_1(x), \ldots, t_n(x), t_{n+1}(x))$ defines in the associated punctured ACFO$_p^{-}$-model. So we get an $L_{\mathrm{t}}^{\times}$ formula $\varphi^{\times}(x)$ such that $\varphi(x)$ defines in an ACFO$_p^{-}$-model the same set that $\varphi^{\times}(x)$ defines in its associated punctured ACFO$_p^{-}$-model. In particular, $\varphi^{\times}(x)$ defines a nonempty set in a punctured ACFO$_p^{-}$-model extending $(F^{\times}, \Sigma, \lhd)$. As $(F^{\times}, \Sigma, \lhd)$ is existentially closed, $\varphi^{\times}(x)$ defines a nonempty set in $(F^{\times}, \Sigma, \lhd)$. Thus, in this special case, $\varphi(x)$ defines a nonempty set in $(F, \lhd)$.

We next reduce to the special case in the preceding paragraph. Arrange that every term $t(x)$ appearing in $\varphi(x)$ is of the form $t_1(x) + \ldots + t_k(x)$ where $t_i(x)$ does not contain $+$ for $i \in \{1, \ldots, k\}$. Introduce a new variable y_t for each such t, and set y to be the tuple of variables built up from such y_t. Replace the appearance of each aforementioned $t(x)$ in $\varphi(x)$ with y_t to get a formula $\psi(x, y)$. Then $\varphi(x)$ is equivalent across ACFO$_p^{-}$-models to

$$\exists y \Big(\psi(x, y) \wedge \bigwedge_t t(x) = y_t \Big).$$

Note that $\psi(x,y) \wedge \bigwedge_t y_t = t(x)$ is of the form in the preceding paragraph and defines a nonempty set in an ACFO_p-models extending $(F, \lhd)$. So $\psi(x,y) \wedge \bigwedge_t y_t = t(x)$ defines a nonempty set in $(F, \lhd)$. Hence, $\varphi(x)$ also defines a nonempty set in $(F, \lhd)$, which concludes the proof of the backward implication.

The forward implication is much easier. The main difficulty with the backward implication comes from the fact that $+$ has no corresponding function symbol in $L^{\times}_{\mathrm{t}}$. On the other hand, basic functions of $(F^{\times}, \Sigma, \lhd)$ are restrictions of basic functions of $(F, \lhd)$ to 0-definable sets of $(F, \lhd)$, and basic relations of $(F^{\times}, \Sigma, \lhd)$ are 0-definable sets of $(F, \lhd)$. So the analogous argument can be carried out without worrying about the aforementioned difficulty. $\square$

In order to "geometrically" characterize the existentially closed models of the expansion of a theory by a unary predicate, Chatzidakis and Pillay implicitly introduced a "largeness property" for definable sets **[11]**; the name largeness here is taken from the paper **[71]** by Shelah and Simon. We will provide an analogous notion in this setting.

Suppose $(G, \square)$ is an expansion of G, $X \subseteq G^m$ is definable in $(G, \square)$, and $(\boldsymbol{G}, \square)$ is a monster elementary extension of $(G, \square)$. We say that X is **multiplicatively large** if there is $a \in X(\boldsymbol{G})$ which is not a solution of any nontrivial multiplicative equations over G.

The above definition in particular applies to definable sets in a circularly ordered abelian group $(G, \lhd)$ and definable sets in a punctured field (G, Σ). We can extend it in an obvious way to cover definable sets in a field F. A definable subset $X \subseteq F^m$ is **multiplicative large** if $X \cap (F^{\times})^m$ is multiplicatively large as a definable subset of the punctured field $(F^{\times}, \Sigma)$ associated to F.

Remark 2.11. Suppose $(G, \square)$ is an expansion of G, and X, X_1, and X_2 are definable in $(G, \square)$ with $X = X_1 \cup X_2$. Then X is multiplicatively large if and only if either X_1 is multiplicatively large or X_2 is multiplicatively large.

Even though multiplicative largeness can be defined very generally, it only behaves well under stronger assumptions:

Lemma 2.11. *Suppose $(G, \square)$ is an expansion of $G \models \mathrm{GM}$, and $X \subseteq G^m$ is a multiplicatively large definable set in $(G, \square)$. If $(G', \square)$ is an elementary extension of $(G, \square)$, then $X(G')$ is a multiplicatively large definable set in $(G', \square)$.*

Proof. Suppose $(G, \square)$, $(G', \square)$, and X are as above. It follows from Lemma 2.3 that $\mathrm{mdim}(X) = m$. As mdim coincides with Morley rank which is preserved under taking elementary extension, $\mathrm{mdim}(X(G')) = m$. Applying Lemma 2.3 again, we get that $X(G')$ is also multiplicatively large. $\square$

We now move on to give the geometric characterization we promised. Suppose $(G, \Sigma, \lhd)$ is a punctured ACFO_p^--model. We say that $(G, \Sigma, \lhd)$ satisfies the **geometric characterization** if the following two conditions are satisfied:

(1) $(G, \lhd) \models \mathrm{GMO}$;

(2) if $X_1 \subseteq G^m$ is a multiplicatively large qf-set in (G, Σ) and $X_2 \subseteq G^m$ is a multiplicative large qf-set in $(G, \lhd)$, then $X_1 \cap X_2 \neq \varnothing$.

We say that an ACFO^--model $(F, \lhd)$ satisfies the **geometric characterization** if its associated punctured ACFO^--model does, or equivalently, if it satisfies a definition as above but with $(G, \lhd)$ replaced by $(F^\times, \lhd)$ and (G, Σ) replaced by F. Note that $\mathrm{ACF}^\times$ admits quantifier elimination, so the assumption that X_1 is a qf-set is in fact unnecessary.

In the rest of the section, we will show that an ACFO^--model is existentially closed if and only if it satisfies the geometric characterization. If T is an L-theory, let $T(\forall)$ denote the set of L-consequences of T.

Lemma 2.12. *Suppose $(G, \Sigma, \lhd)$ is an $L_{\mathrm{t}}^\times$-structure with $(G, \Sigma) \models \mathrm{ACF}_p^\times(\forall)$ and $(G, \lhd) \models \mathrm{GMO}_p(\forall)$. Then $(G, \Sigma, \lhd)$ can be $L_{\mathrm{t}}^\times$-embedded into a punctured ACFO_p^--model $(G', \Sigma, \lhd)$ such that $(G', \lhd)$ is a model of GMO. A similar statement holds for $L_{\mathrm{t}}^\times$-structure with $(G, \Sigma) \models \mathrm{ACF}_0^\times(\forall)$ and $(G, \Sigma) \models \mathrm{GMO}_0(\forall)$.*

Proof. We will only prove the first statement as the proof of the second statement is similar. Let $(G, \Sigma, \lhd)$ be as above. We will construct a sequence $(G_n, \Sigma, \lhd)_n$ of $L_{\mathrm{t}}^\times$-structure such that

(1) $(G_0, \Sigma, \lhd)$ extends $(G, \Sigma, \lhd)$ as an $L_{\mathrm{t}}^\times$-structure;

(2) $(G_{n+1}, \Sigma, \lhd)$ is an $L_{\mathrm{t}}^\times$-extension of $(G_n, \Sigma, \lhd)$;

(3) (G_{2n}, Σ) is a punctured ACFO_p^--model;

(4) $(G_{2n+1}, \lhd) \models \mathrm{GMO}_p$ and (G_{2n+1}, Σ) is a model of $\mathrm{ACF}_p^\times(\forall)$.

Then we can take $(G', \Sigma, \lhd)$ to be the union of $(G_n, \Sigma, \lhd)_n$. Note that $\mathrm{ACF}_p^\times$ and GMO_p are both inductive theories, as they are model complete, so it is easy to see that $(G', \Sigma, \lhd)$ satisfied the desired conclusion.

As $(G, \Sigma) \models \mathrm{ACF}_p^\times(\forall)$, we can get $(G_0, \Sigma) \models \mathrm{ACF}_p^\times$ extending (G, Σ) as an $L_{\mathrm{f}}^\times$-structure. On the other hand $(G, \lhd) \models \mathrm{GMO}_p(\forall)$, so we can get a monster model $(\boldsymbol{G}, \lhd)$ of GMO_p extending $(G, \lhd)$ as an L_{mc}-structure. Now both G_0 and $\boldsymbol{G}$ are models of GM_p, and recall that the theory GM_p admits quantifier elimination by Proposition 2.1. Hence, there is an embedding of G_0 into $\boldsymbol{G}$. We can then define $\lhd$ on G_0 by pull-back via f. Clearly, $(G_0, \Sigma, \lhd)$ is a model of ACFO_p^-.

Suppose we have constructed $(G_{2n}, \Sigma, \lhd)$ satisfying both (2) and (3). Then $(G_{2n}, \lhd)$ is a model of GMO_p^- which has GMO_p as a model companion, so we can get $(G_{2n+1}, \lhd) \models \mathrm{GMO}_p$

extending $(G_{2n}, \lhd)$ as an L_{mc}-structure. Choose a monster model $(\boldsymbol{G}, \Sigma)$ of $\mathrm{ACF}_p^{\times}$ extending (G_{2n}, Σ). Note that G_{2n}, G_{2n+1}, and $\boldsymbol{G}$ are models of GM_p, which is a model complete L_{m}-theory. So there is an L_{m}-embedding $f : G_{2n+1} \to \boldsymbol{G}$ which extends the identity map on G_{2n}. Define the family of relations Σ on $(G_{2n+1}, \lhd)$ as the pull-back via f of the family Σ on $\boldsymbol{G}$. Then $G_{2n+1} \models \mathrm{ACF}_p^{\times}(\forall)$ by construction, and so $(G_{2n+1}, \lhd) \models \mathrm{GMO}_p$ satisfies (4).

Finally, suppose we have constructed $(G_{2n+1}, \Sigma, \lhd)$ satisfying both (2) and (4). We note that the only thing used in the preceding paragraph is that GMO_p is the model companion of GMO_p^- and that GM_p is model complete. Hence, we can carry out exactly the same strategy to get the desired conclusion by replacing the former with the fact that $\mathrm{ACF}_p^{\times}$ is the model companion of $\mathrm{ACF}_p(\forall)$ and reusing the latter. □

Remark 2.12. Using the fact that GM_p is strongly minimal, one can produce a quicker proof of Lemma 2.12 by constructing $(G', \Sigma, \lhd)$ directly. We still choose to present the longer proof here to make the neccesary ingredients transparent.

We need another embedding lemma:

Lemma 2.13. *Suppose $(G, \Sigma, \lhd)$ is a punctured ACFO_p^--model with $(G, \lhd) \models \mathrm{GMO}_p$. Then $(G, \Sigma, \lhd)$ can be $L_{\mathrm{t}}^{\times}$-embedded into a punctured ACFO_p^--model which satisfies the geometric characterization. A similar statement holds for a punctured ACFO_0^--model $(G, \Sigma, \lhd)$ with $(G, \lhd) \models \mathrm{GMO}_0$.*

Proof. We will only prove the first statement, as the proof for the second statement is similar. Suppose $(G, \Sigma, \lhd)$ is as stated. Let $X_1 \subseteq (F^{\times})^m$ is a multiplicative large qf-set in (G, Σ) and $X_2 \subseteq (F^{\times})^m$ is a multiplicative large qf-set in $(G, \lhd)$. Our problem can be reduced to finding $(G', \Sigma, \lhd) \models \mathrm{ACFO}_p^-$ extending $(G, \Sigma, \lhd)$ with $(G', \lhd) \models \mathrm{GMO}$ such that $X_1(G') \cap X_2(G') \neq \varnothing$. Indeed, we can simply iterate this construction and take the union.

We now construct the aforementioned $(G', \Sigma, \lhd)$. Take a monster elementary extension $(\boldsymbol{G_1}, \Sigma)$ of (G, Σ) and a monster elementary extension $(\boldsymbol{G_2}, \lhd)$ of $(G, \lhd)$. As X_1 is multiplicatively large, we get $a' \in X_1(\boldsymbol{G}_1)$ whose components are multiplicatively independent over G. Likewise, we get $b' \in X_1(\boldsymbol{G}_2)$ whose components are multiplicatively independent over G. Let $f : \langle G, a' \rangle \to \boldsymbol{G}_2$ be the unique map which extends the identity map on G and maps a' to b'. Define the family Σ on $\langle G, a' \rangle$ by restricting the family with the same name on $\boldsymbol{G_1}$, and define the relation $\lhd$ on $\langle G, a' \rangle$ by pulling back via f the relation with the same name on $\boldsymbol{G_2}$. Then $(\langle G, a' \rangle, \Sigma, \lhd)$ is an $L_{\mathrm{t}}^{\times}$-structure with $(\langle G, a' \rangle, \Sigma) \models \mathrm{ACF}^{\times}(\forall)$ and $(\langle G, a' \rangle, \Sigma) \models \mathrm{GMO}(\forall)$. Applying Lemma 2.12, we get the desired conclusion. □

Lemma 2.13 essentially gives us that the characterization works in one direction:

Corollary 2.7. *Suppose* $(G, \Sigma, \lhd)$ *is an existentially closed punctured* ACFO$^-$*-model. Then* $(G, \lhd)$ *satisfies the geometric characterization.*

Proof. We consider only the case where $(G, \Sigma, \lhd)$ is a punctured ACFO$_p^-$-model; the other case with $(G, \Sigma, \lhd)$ a punctured ACFO$_0^-$-model is very similar. Using Lemma 2.13, we obtain a punctured ACFO$_p^-$-model $(G', \Sigma, \lhd)$ extending $(G, \Sigma, \lhd)$ as an $L_{\mathrm{t}}^{\times}$-structure such that $(G', \Sigma, \lhd)$ satisfies the geometric characterization.

The structure $(G, \Sigma, \lhd)$ is existentially closed in $(G', \Sigma, \lhd)$, so $(G, \lhd)$ is existentially closed in $(G', \lhd)$. As $(G', \Sigma, \lhd)$ satisfies the geometric characterization, $(G', \lhd)$ is a model of GMO$_p$. The theory GMO$_p$ is model complete, so we can assume that it only consists of $\forall\exists$ statements. It follows that $(G, \lhd)$ is also a model of GMO$_p$.

Suppose $X_1 \subseteq G^m$ is a multiplicatively large qf-set in (G, Σ) and $X_2 \subseteq G^m$ is multiplicatively large qf-set in $(G, \lhd)$. Then $X_1(G')$ is multiplicatively large in (G', Σ), and $X_2(G')$ is multiplicatively large in $(G', \lhd)$ by Lemma 2.11 and the fact that both ACF$_p^{\times}$ and GMO$_p$ are model complete. As $(G', \Sigma, \lhd)$ satisfies the geometric characterization, $X_1(G') \cap X_2(G') \neq \varnothing$. So $X_1 \cap X_2 \neq \varnothing$ as well by the fact that $(G, \Sigma, \lhd)$ is existentially closed. The desired conclusion follows. ☐

We next verify that the characterization works in the other direction:

Lemma 2.14. *Suppose* $(G, \Sigma, \lhd)$ *is a punctured model of* ACFO$^-$ *that satisfies the geometric characterization. Then* $(G, \Sigma, \lhd)$ *is existentially closed.*

Proof. It suffices to prove the corresponding statements for punctured ACFO$_p^-$-models and punctured ACFO$_0^-$-models. We will only prove the former as the latter is very similar. Suppose $(G, \Sigma, \lhd)$ is a generic punctured model of ACFO$_p^-$, and $\varphi(x)$ is a quantifier-free $L_{\mathrm{t}}^{\times}(G)$-formula which defines a nonempty set in a punctured ACFO$_p^-$-model $(G', \Sigma, \lhd)$ extending $(G, \Sigma, \lhd)$. Our job is to show that $\varphi(x)$ defines a nonempty set in $(G, \Sigma, \lhd)$.

As the only function symbols in both $L_{\mathrm{f}}^{\times}$ and L_{mc} already appear in L_{m}, we can reduce to the case where $\varphi(x) = \psi(x) \wedge \theta(x)$ with $\psi(x)$ a quantifier-free $L_{\mathrm{f}}^{\times}(G)$-formula and $\theta(x)$ a quantifier-free $L_{\mathrm{mc}}(G)$-formula. Let $a' \in (G')^m$ be such that $(G', \Sigma, \lhd) \models \varphi(a')$. Suppose a' is multiplicatively independent. Then $\psi(x)$ and $\theta(x)$ define multiplicatively large sets in (G, Σ) and $(G, \lhd)$. So $\varphi(x)$ defines a nonempty set in $(G, \Sigma, \lhd)$ by the assumption that $(G, \Sigma, \lhd)$ satisfies the geometric characterization.

Now consider the general case where a' might not be multiplicatively independent. Then we can choose a tuple $b' \in (G')^n$ which is multiplicatively independent such that $a' = t(b')$ with $t = (t_1, \dots, t_m)$ and $t_i(y)$ is an L_{m}-term for $i \in \{1, \dots, m\}$. Applying the earlier case for the formula $\varphi(t(y)) = \psi(t(y)) \wedge \theta(t(y))$, we get $b \in G^y$ such that $(G, \Sigma, \lhd) \models \varphi(t(b))$. Thus, $\varphi(x)$ defines in $(G, \Sigma, \lhd)$ a nonempty set, which is our desired conclusion. ☐

Finally, we put everything together:

Proposition 2.4. *An* ACFO^{-}*-model* $(F, \lhd)$ *is existentially closed if and only if* $(F, \lhd)$ *satisfies the geometric characterization.*

PROOF. This follows easily from Lemma 2.10, Corollary 2.7, and Lemma 2.14. ☐

2.2.2. Axiomatization. Just as in [**11**], we want to establish that the class of models of ACFO^{-} satisfying the geometric characterization is elementary. In order to do so, the key is to show that ACF and GMO each "defines multiplicative largeness".

Throughout Section 2.2.2, F is an algebraically closed field, and $V \subseteq (F^{\times})^m$ is a quasi-affine variety (i.e, a Zariski-open subset of an irreducible Zariski-closed subset of $(F^{\times})^m$). We equip $(F^{\times})^m$ with the group structure given by coordinate-wise multiplication, and let $1^{(m)}$ be the identity element of $(F^{\times})^m$.

Suppose T is an L-theory, $\varphi(x,y)$ is an L-formula, and $\mathcal{P}$ is a property of $\mathcal{M}$-definable subsets of M^x with $\mathcal{M} \models T$. We say that T **defines** $\mathcal{P}$ **for** $\varphi(x,y)$ if there is an L-formula $\delta(y)$ such that for all $\mathcal{M} \models T$ and $b \in M^y$, we have

$$\varphi(\mathcal{M}, b) \text{ satisfies } \mathcal{P} \quad \text{if and only if} \quad \mathcal{M} \models \delta(b).$$

A formula $\delta(y)$ as above is called a $\mathcal{P}$**-defining formula** over T for $\varphi(x,y)$. The fact that this is key to axiomatization can be seen through the following lemma:

Lemma 2.15. *Assume* ACF *defines multiplicative largeness for all* L_{f}*-formulas* $\varphi(x,y)$ *and* GMO *defines multiplicative largeness for all quantifier-free* L_{mc}*-formulas* $\psi(x,z)$*. Then the class of* ACFO^{-}*-models satisfying the geometric characterization is elementary.*

PROOF. Let ACFO$^{-1/2}$ be the L_{t}-theory whose models are $(F, \lhd) \models$ ACFO^{-} with $(F^{\times}, \lhd) \models$ GMO. Obtain ACFO from ACFO$^{-1/2}$ by adding for each L_{f}-formula $\varphi(x,y)$ and L_{mc}-formula $\psi(x,z)$ the formula

$$\forall y \forall z \big(\delta(y) \wedge \hat{\theta}(z) \to \exists x(\varphi(x,y) \wedge \hat{\psi}(x,z))\big),$$

where $\delta(y)$ is a multiplicative largeness-defining formula over ACF for $\varphi(x,y)$, $\theta(z)$ is a multiplicative largeness-defining formula over GMO for $\psi(x,z)$, and $\hat{\theta}(z)$ and $\hat{\psi}(x,z)$ are the obvious modifications of $\theta(z)$ and $\psi(x,z)$ such that $\hat{\theta}(z)$ and $\hat{\psi}(x,z)$ apply to all tuples with components in F. It is easy to see that ACFO axiomatizes the class of ACFO^{-}-models satisfying the geometric characterization. ☐

We next obtain various characterizations of multiplicative largeness in models of ACF and deduce from them the first condition of Lemma 2.15.

Lemma 2.16. *A quasi-affine variety $V \subseteq (F^{\times})^m$ is multiplicatively large if and only if no nontrivial multiplicative equation vanishes on V.*

Proof. The forward implication is immediate. Suppose no nontrivial multiplicative equation vanishes on V. As V is irreducible, it is not a subset of a finite union of solution sets of nontrivial multiplicative equations. The desired conclusion then follows a compactness argument. □

If $B \subseteq F^{\times}$, a **multiplicative system** over B is simply a conjunction of multiplicative equations over B. Fact 2.2 below about definable subgroups of $(F^{\times})^m$ is a consequence of the fact that definable subgroups of $(F^{\times})^m$ are closed and of the characterization of algebraic subgroups of $(F^{\times})^m$. For the former, see for instance [**58**, Lemma 7.4.9]. For the latter, see for instance [**8**, Corollary 3.2.15]; the proof there is for characteristic 0 but goes through in positive characteristics.

Fact 2.2. *Every connected definable subgroup of $(F^{\times})^m$ is defined by a multiplicative system over $\varnothing$.*

Suppose $X_1, \ldots, X_n$ are subsets of G. Set $X_1 \cdots X_n = \{a_1 \cdots a_n : a_i \in X_i \text{ for } 1 \leqslant i \leqslant n\}$. Moreover, if $X_1 = \cdots = X_n = X$, then we denote this as $\Pi_m(X)$. The following fact is a special case of Zilber's Indecomposability theorem for structures of finite Morley rank but was also known much earlier for algebraically closed fields; see [**58**, Theorem 7.3.2].

Fact 2.3. *Suppose $1^{(m)}$ is in V. Then $\Pi_{2m}(V)$ is (the underlying set of) a connected definable subgroup of $(F^{\times})^m$. Hence, $\Pi_{2m}(V)$ is a subgroup of every definable subgroup of $(F^{\times})^m$ containing V as a subset.*

We now have a simple criterion for multiplicative largeness:

Lemma 2.17. *If $1^{(m)}$ is in V, then V is multiplicatively large if and only if $\Pi_{2m}(V) = (F^{\times})^m$.*

Proof. For the forward implication, suppose V is multiplicatively large. Then by Lemma 2.16 and Fact 2.2, no proper definable subgroup of $(F^{\times})^m$ contains V as a subset. It then follows from Fact 2.3 that $\Pi_{2m}(V) = (F^{\times})^m$. The backward implication is immediate from Lemma 2.16. □

To get from quasi-affine varieties over F to general sets definable in F, we need a result related to defining irreducibility. This and other related results are included in Fact 2.4 as we will also need them later on; see [**43**, Chapter 10] for details.

Fact 2.4. *Supose $\varphi(x, y)$ is an L_{f}-formula, d is in $\mathbb{N}$, and r is in $\mathbb{N}^{\geqslant 1}$. Then there are formulas $\delta_d(y)$, $\mu_r(y)$, $\iota(y)$, and $\psi(x, z)$ such that if the families $(X_b)_{b \in Y}$ and $(X_c)_{c \in Z}$ are defined in F by $\varphi(x, y)$ and $\psi(x, z)$, we have the following:*

(i) $F \vDash \delta_d(b)$ *for* $b \in Y$ *if and only if* $\dim(X_b) = d$*;*
(ii) $F \vDash \mu_r(b)$ *for* $b \in Y$ *if and only if the Morley degree of* X_b *is* r*;*
(ii) $F \vDash \iota(b)$ *for* $b \in Y$ *if and only if* X_b *is a quasi-affine subvariety of* F^m*;*
(iv) $(X_c)_{c \in Z}$ *is a family of quasi-affine varieties which contains all irreducible components of members of* $(X_b)_{b \in Y}$.

We now put together Fact 2.4 and Lemma 2.17:

Proposition 2.5. *The theory* ACF *defines multiplicative largeness for every* L_{f}*-formula.*

PROOF. Suppose $\varphi(x,y)$ is an arbitrary $L_{\mathrm{f}}^{\times}$-formula, $\psi(x,z)$ is as in Fact 2.4, and $\varphi(x,y)$ and $\psi(x,z)$ define in F the families $(X_b)_{b \in Y}$ and $(X_c)_{c \in Z}$. Observe that X_b with $b \in Y$ is multiplicatively large if and only if there is $c \in Z$ with $X_c \subseteq X_b$ and $a \in X_c \cap (F^{\times})^m$ such that $\Pi_{2m}\big(a^{-1}(X_c \cap (F^{\times})^m)\big) = (F^{\times})^m$. It is easy to see from here that ACF defines multiplicative largeness for $\varphi(x,y)$. $\square$

Every multiplicative equation over $\varnothing$ is equivalent over GM to a multiplicative equation $t(x) = t'(x)$ of the simplified form where the power of every variable is nonnegative and each variable appears only on at most one side of the equation. The **degree** of a multiplicative equation is the highest power of x_i which appears in the above simplified form as i ranges over $\{1, \dots, m\}$. A simple application of the compactness theorem yields the following corollary:

Corollary 2.8. *Suppose* $(V_b)_{b \in Y}$ *is a family of quasi-affine subvarieties of* $(F^{\times})^m$ *passing through* $1^{(m)}$*. Then there is* $N > 0$ *such that for all* $b \in Y$*, either* V_b *is multiplicatively large or a nontrivial multiplicative equation over* $\varnothing$ *with degree at most* N *vanishes on* V_b*.*

We will next obtain a characterization of multiplicative largeness in models of GMO and from there obtain the second condition of Lemma 2.15.

Suppose $(G, \lhd) \vDash$ GMO. The $\lhd$**-topology** on G^m is defined as the topology which has a basis consisting of sets of the form $U = U_1 \times \cdots \times U_m$ where U_i is the "interval"

$$\{a \in G : \lhd(d_i, a, d_i')\}$$

with d_i and d_i' in G for $i \in \{1, \dots, m\}$. It is also easy to see that the $\lhd$-topology on G^m is simply the product of the $\lhd$-topologies on the m copies of G.

Lemma 2.18. *Suppose* $(G, \lhd)$ *is a model of* GMO*. Then we have the following:*
(i) $\lhd$ *as a subset of* G^3 *is* $\lhd$*-open;*
(ii) *the multiplication map is continous.*

PROOF. It is well known that $\mathbb{U}_{(p)}$ for an arbitrary p and $\mathbb{U}$ are dense in $\mathbb{T}$ with respect to the Euclidean topology. Hence, when $(G, \lhd)$ is $(\mathbb{U}_{(p)}, \lhd)$ for some given p or $(\mathbb{U}, \lhd)$, the

$\triangleleft$-topology is just the subspace topology with respect to the usual Euclidean topology on $\mathbb{T}$. Hence, (i) and (ii) are automatic in these cases. Note that properties (i) and (ii) can be expressed as L_{mc}-statements, so the desired conclusion follows from Proposition 2.2 and Proposition 2.3. $\square$

Proposition 2.6. *Suppose* $(G, \triangleleft)$ *is a model of* GMO *and* $X \subseteq G^m$ *is defined in* $(G, \triangleleft)$ *by a quantifer-free* $L_{\mathrm{mc}}(G)$*-formula. Then* X *is multiplicatively large if and only if* X *contains a nonempty subset which is* $\triangleleft$*-open in* G^m*.*

Proof. Let $(G, \triangleleft)$ and X be as stated above. For the forward implication, suppose X is multiplicative large. Note that $\neg \triangleleft(x, y, z)$ is equivalent over GMO to

$$\triangleleft(z, y, x) \vee (x = y) \vee (y = z) \vee (z = x).$$

So quantifier free L_{mc}-formulas are equivalent over GMO to positive quantifier-free formulas. Applying also Remark 2.11, we can assume that X is defined by a formula of the form

$$\bigwedge_{i \in I} \triangleleft\big(t_i(x), t'_i(x), t''_i(x)\big) \wedge \bigwedge_{j \in J} \big(t_j(x) = t'_j(x)\big).$$

where $t_i(x), t'_i(x), t''_i(x), t_j(x), t'_j(x)$ are $L_{\mathrm{m}}(G)$-terms for all $i \in I$ and $j \in J$. As X is multiplicatively large, one must have that $t_j(x) = t'_j(x)$ is trivial for all $j \in J$. It then follows from Lemma 2.18 that X is open, which gives us the desired conclusion.

For the backward implication, suppose X contains a a nonempty subset which is $\triangleleft$-open in G^m. We may assume that $X = \prod_{i=1}^m X_i$ where

$$X_i = \{a_i \in G : \triangleleft(d_i, a_i, d'_i)\},$$

with $d_i, d'_i \in G$ and $d_i \neq d'_i$ for $i \in \{1, \dots, m\}$. Let $(\boldsymbol{G}, \triangleleft)$ be a monster elementary extension of $(G, \triangleleft)$. Using Proposition 2.3, it is easy to show that X_i is infinite by reducing to the special cases where $G = \mathbb{U}$ or $G = \mathbb{U}_{(p)}$ for some p. Hence, $|X_i(\boldsymbol{G})| > |G|$ for $i \in I$. Hence, we can choose the desired $a' = (a'_1, \dots, a'_m)$ in $X(\boldsymbol{G})$ by ensuring that $a'_{i+1} \in X_i(\boldsymbol{G})$ is multiplicatively independent over $a'_1, \dots, a'_i$ for $i \in \{1, \dots, m-1\}$. $\square$

From the definition of $\triangleleft$-topology, we immediately get:

Corollary 2.9. *The theory* GMO *defines multiplicative largeness for all quantifier free* L_{mc}*-formulas.*

Combining Proposition 2.6, Remark 2.11, and the well-known fact that definable sets in ACF-models are finite unions of quasi-affine varieties, we get:

Corollary 2.10. *Suppose* $(F, \triangleleft)$ *is a model of* ACFO^- *with* $(F^\times, \triangleleft) \models$ GMO*. Then* $(F, \triangleleft)$ *satisfies the geometric characterization if and only if all multiplicatively large* $V \subseteq (F^\times)^m$ *are dense with respect to the* $\triangleleft$*-topology.*

We now put together the results of this section and the preceding section to get the existence of the model companion ACFO of ACFO^{-}.

Proof of Theorem 1.1. The desired conclusion follows immediately from Proposition 2.4, Lemma 2.15, Proposition 2.5, and Corollary 2.9. ☐

As a side note, we will show that every model of ACFO has TP$_2$. The notion was defined in [**68**] by Shelah and systematically studied in [**16**] by Chernikov. We use here a finitary version of the definition given in [**16**]. Let $\mathcal{M}$ be an L-structure. An L-formula $\varphi(x,y)$ witnesses that $\mathcal{M}$ has TP$_2$ if for each finite set I, there is a family $(b_{ij})_{(i,j)\in I^2}$ of elements of M^n such that the following conditions hold:

(1) $\mathcal{M} \models \neg\exists x\big(\varphi(x,b_{ij}) \wedge \varphi(x,b_{ij'})\big)$ for every $i \in I$ and distinct j and j' in I;
(2) $\mathcal{M} \models \exists x \bigwedge_{i\in I} \varphi(x, b_{if(i)})$ for any $f : I \to I$.

We say that $\mathcal{M}$ has TP$_2$ if there is a formula $\varphi(x,y)$ which witnesses that $\mathcal{M}$ has TP$_2$ and say that $\mathcal{M}$ has NTP$_2$ otherwise.

Proposition 2.7. *Every model of* ACFO *has* TP$_2$.

Proof. Suppose $(F, \lhd)$ is a model of ACFO. Let x be a single variable, $y = (z,t,t')$ with z, t, and t' single variables, and $\varphi(x,y)$ the formula

$$\lhd(x+z,t,t').$$

Let I be an arbitrary finite set. Get a family $(c_i)_{i\in I}$ of distinct elements in F. Obtain a family $(d_j,d'_j)_{j\in I}$ of pairs of elements in $F^\times$ with $d_j \neq d'_j$ for all $j \in J$ and

$$(F,\lhd) \models \neg\exists x\big(\lhd(d_j,x,d'_j) \wedge \lhd(d_{j'},x,d'_{j'})\big) \quad \text{for distinct } j,j' \in I.$$

Set $b_{ij} = (c_i, d_j, d'_j)$. It is easy to see that $\varphi(x,y)$ together with $(b_{ij})_{(i,j)\in I^2}$ satisfy (1) in the definition of a TP$_2$-witness. We assume without loss of generality that $I = \{1,\dots,k\}$. Set

$$V = \{(a+c_1,\dots,a+c_k) : a \in F\}.$$

It suffices to show that V is multiplicatively large as it will follow that $\varphi(x,y)$ together with $(b_{ij})_{(i,j)\in I^2}$ satisfies (2) in the definition of a TP$_2$-formula as well. We can reduce further to showing triviality for an arbitrarily chosen multiplicative equation $x_1^{n_1}\cdots x_m^{n_m} = c x_1^{n'_1}\cdots x_k^{n'_m}$ vanishing on V where c is in $F^\times$, and n_i and n'_i are in $\mathbb{N}$ with either $n_i = 0$ or $n'_i = 0$ for $i \in \{1,\dots,m\}$. In this case, we have

$$(a+c_1)^{n_1}\cdots(a+c_m)^{n_m} = c(a+c_1)^{n'_1}\cdots(a+c_m)^{m'_m} \quad \text{for all } a \in F.$$

For $i \in \{1,\dots,m\}$, we substitute $a = -c_i$ and deduce $n_i = n'_i = 0$. Hence, we also get $c = 1$, and the desired conclusion follows. ☐

Remark 2.13. Proposition 2.7 is surprising: following [**11**], one would expect that models of ACFO have NTP_2. It suggests that NTP_2 is not quite "stable + order + random". In the same direction, recent evidence seems to suggest that "stable + random" is NSOP_1 instead of simple as earlier thought [**52, 53**]. We hope a new candidate for "stable + order + random" will be introduced in the near future.

2.3. Standard models are existentially closed

We will finally show that if $\lhd$ is a multiplicative circular order on $\mathbb{F}$, then $(\mathbb{F}, \lhd)$ is a model of ACFO, or in other words, $(\mathbb{F}, \lhd)$ satisfies the geometric characterization. This will require two steps:

(1) simplifying the characterization of ACFO-models given by Corollary 2.10 into a characterization that only concerns curves.
(2) using number-theoretic results on character sums over finite fields and counting points over finite fields in combination with Weyl's criterion for equidistribution to show that $(\mathbb{F}, \lhd)$ satisfies the characterization specified in (1).

2.3.1. Geometric characterization with curves. We will show that every multiplicatively large variety contains as a subset a curve which is multiplicatively large. This curve will be obtained by intersecting the original variety with suitably chosen hyperplanes of the ambient space. Combining this with Corollary 2.10, we will get the desired simplified characterization of ACFO-models.

Throughout Section 2.3.1, we work with a fixed algebraically closed fied F, and definable means definable in F. The notions of open, closed, irreducible, and dense are with respect to the Zariski topology, which is natural in this context. Let dim be the canonical dimension for algebraically closed fields, so dim coincides with Morley rank, topological dimension, acl-dimension, etc. Let mult be the Morley degree. If $X \subseteq F^m$ is definable with $\mathrm{mult} X = 1$, we say that X is **generically irreducible** and let the **maximal component** of X be the unique quasi-affine variety with maximal dimension in the decomposition of X into irreducible components. We let V range over the quasi-affine subvarieties of $(F^\times)^m$ and C range over the one-dimension quasi-affine subvarieties of $(F^\times)^m$.

Let S be $F^m \smallsetminus \{0^{(n)}\}$. If b is an element of S, let H_b be the hyperplane defined by the equation $b \cdot x = 1$ where $b \cdot x$ is the usual vector dot product between b and x. So S is essentially the space parametrizing the affine hyperplanes of F^m. For each definable set $X \subseteq F^m$ and $b_1, \ldots, b_n \in S$, set

$$X(b_1, \ldots, b_n) = X \cap H_{b_1} \cap \ldots \cap H_{b_n}.$$

Fact 2.5 below is a well-known consequence of Fact 2.4, Bezout's theorem [**67**, Section 4.1] and Bertini's theorem [**67**, Theorem 2.26].

Fact 2.5. *Suppose $W \subseteq F^m$ is generically irreducible, and* $\dim W = n+1$. *Then the set of $(b_1, \ldots, b_n) \in S^n$ satisfying the following conditions (i) and (ii) is definable and dense in S^n:*

(i) *$W(b_1, \ldots, b_i)$ is generically irreducible for $i \in \{1, \ldots, n\}$;*
(ii) $\dim W(b_1, \ldots, b_i) = \dim W(b_1, \ldots, b_{i-1}) - 1$ *for all $i \in \{1, \ldots, n\}$.*

Hence, for such $(b_1, \ldots, b_n) \in S^n$, the maximal component of $W(b_1, \ldots, b_i)$ is a subset of the maximal component of $W(b_1, \ldots, b_{i-1})$ for $i \in \{1, \ldots, n\}$.

For each $V \subseteq F^m$, define S_V to be the set of $b \in S$ such that V is a subset of H_b.

Remark 2.14. If V is a single point c, then S_c is an irreducible quasi-affine variety and $\dim S_c = m-1$. If $\dim(V) \geqslant 1$, then $\dim S_V \leqslant m-2$.

We need a variation of Fact 2.5:

Lemma 2.19. *Suppose $W \subseteq F^m$ is generically irreducible, and* $\dim W = n+1$. *Then there is c in the maximal component of W such that the set Y_c of $(b_1, \ldots, b_n) \in S_c^n$ satisfying conditions (i)-(iii) below is dense in S_c^n.*

(i) *$W(b_1, \ldots, b_i)$ is generically irreducible for $i \in \{1, \ldots, n\}$;*
(ii) $\dim W(b_1, \ldots, b_i) = \dim W(b_1, \ldots, b_{i-1}) - 1$ *for all* $i \in \{1, \ldots, n\}$;
(iii) *c is in $W(b_1, \ldots, b_n)$.*

Moreover, with the above c, if $X \subseteq W$ is definable and satisfies $\dim X < \dim W$, *then the set of $(b_1, \ldots, b_n) \in Y_c$ such that* $\dim X(b_1, \ldots, b_n) \leqslant 0$ *is also dense in S_c^n.*

Proof. Suppose W and n are as stated above, and $Y \subseteq S^n$ is the set obtained in Fact 2.5. We show the first statement of the lemma. Let Γ be

$$\{(b_1, \ldots, b_n, c) \in Y \times W : c \text{ is in the maximal component of } W(b_1, \ldots, b_n)\}.$$

We note that Γ is definable by Fact 2.4. Let $\pi_1 : \Gamma \to Y$ and $\pi_2 : \Gamma \to W$ be the projection maps. For each $c \in W$, we have $\pi_2^{-1}(c) \subseteq S_c^n \times \{c\}$ and $\pi_1(\pi_2^{-1}(c)) \subseteq S_c^n$. We want to find c such that $\pi_1(\pi_2^{-1}(c))$ is dense in S_c^n. As S_c^n is irreducible of dimension $n(m-1)$, for the current purpose, it suffices to find $c \in W$ such that $\dim \pi_2^{-1}(c) = n(m-1)$.

For each $(b_1, \ldots, b_n) \in Y$, we have $\dim \pi_1^{-1}(b_1, \ldots, b_n) = 1$. As $\dim Y = \dim S^n = mn$, it follows that $\dim \Gamma = mn + 1$. As $\dim W = n+1$, the set

$$\{c \in W : \dim \pi_2^{-1}(c) = n(m-1)\}$$

must be dense in W. So we obtain c such that the set Y_c of $(b_1, \ldots, b_n) \in S_c^n$ satisfying conditions (i)-(iii) is dense in S_c^n.

Now suppose X is as in the second part of the statement. Note that $\dim X(b_1,\dots,b_n)$ is at most $\dim W(b_1,\dots,b_n) = 1$. By Fact 2.4 and the irreducibility of S_c^n, exactly one of the following two possibilities happens:

(1) $\{(b_1,\dots,b_n) \in Y_c : \dim X(b_1,\dots,b_n) \leqslant 0\}$ is dense in S_c^n
(2) $\{(b_1,\dots,b_n) \in Y_c : \dim X(b_1,\dots,b_n) = 1\}$ is dense in S_c^n.

We need to show that (2) cannot happen. Suppose to the contrary that it does. Then using Fact 2.4, we get a definable dense subset R_c of S_c^n and $i \in \{1,\dots,n\}$ such that

$$\dim X(b_1,\dots,b_i) = \dim X(b_1,\dots,b_{i-1}) = \dim W(b_1,\dots,b_i) < \dim W(b_1,\dots,b_{i-1})$$

for all $(b_1,\dots,b_n)$ in R_c. Shrinking R_c further if necessary, we can arrange that $R_c = U_c^n$ with U_c a definable dense subset of S_c. Fix $(b_1,\dots,b_{i-1}) \in U_c^{i-1}$. Then for all b_i in U_c, the hyperplane H_{b_i} must contain an irreducible component of $X(b_1,\dots,b_{i-1})$ with dimension $\geqslant 1$. Remark 2.14 then gives us that $\dim U_c \leqslant m-2$ which contradicts the fact that U_c is dense in S_c and $\dim S_c = m-1$. The desired conclusion follows. □

Proposition 2.8. *Suppose V is multiplicatively large and $\dim V = n+1$. Then there is $(b_1,\dots,b_n) \in S^n$ such that $V(b_1,\dots,b_n)$ is generically irreducible with multiplicatively large maximal component of dimension one.*

Proof. Obtain c in V and Y_c as in the first part of Lemma 2.19. Then for all $(b_1,\dots,b_n) \in Y_c$, $V(b_1,\dots,b_n)$ is generically irreducible with maximal component $C(b_1,\dots,b_n)$ of dimension one. It suffices to show that the set

$$\{(b_1,\dots,b_n) \in Y_c : C(b_1,\dots,b_n) \text{ is multiplicatively large}\}$$

is dense in Y_c. Replacing V with $c^{-1}V$ and c with $1^{(m)}$ if neccesary, we arrange that $c = 1^{(m)}$. Hence, $(C(b_1,\dots,b_m))_{(b_1,\dots,b_m)\in Y_c}$ is a family of subvarieties of $(F^\times)^m$ passing through $1^{(m)}$. Obtain N as in Corollary 2.8 for this family. Let $(X_i)_{i=1}^k$ list the intersections of V with the solution sets of the multiplicative equations of the form $M(x) = 1$ with $\deg M < N$, and set $X = \bigcup_{i=1}^k X_i$. As V is multiplicatively large, $\dim X < \dim V$. It then follows from the second part of Lemma 2.19 that

$$\{(b_1,\dots,b_n) \in Y_c : \dim X(b_1,\dots,b_n) \leqslant 0\} \text{ is dense in } Y_c.$$

Suppose $(b_1,\dots,b_n)$ is in the above set. Then $C(b_1,\dots b_n)$ is not a subset of $X(b_1,\dots,b_n)$. So $C(b_1,\dots b_n)$ is not a subset of $X = \bigcup_{i=1}^k X_i$. By the property of N, $C(b_1,\dots,b_n)$ is multiplicatively large, which gives us the desired conclusion. □

Combining with Corollary 2.10, we get:

Corollary 2.11. *Suppose $(F,\lhd)$ is a model of* ACFO^- *and* $(F^\times,\lhd)\vDash \mathrm{GMO}$. *Then $(F,\lhd)$ satisfies the geometric characterization if and only if all multiplicatively large $C\subseteq (F^\times)^m$ are dense with respect to the $\lhd$-topology.*

2.3.2. Standard models and number-theoretic randomness. We now use the results in the preceding section to prove Theorem 1.2. Other ingredients include a variant of Weyl's criterion for equidistribution and results on counting points and character sums over finite fields, which are consequences of the Weil conjectures for curves over finite fields.

In Section 2.3.2, let $\lhd$ be the clockwise circular order on $\mathbb{T}$. The multiplicative group $\mathbb{T}^m$ is a compact topological group. So $\mathbb{T}^m$ is equipped with a unique normalized Haar measure μ. A sequence (X_n) of finite subsets of $\mathbb{T}^m$ becomes **equidistributed** in $\mathbb{T}^m$ if

$$\lim_{n\to\infty}\frac{|X_n\cap U|}{|X_n|}=\mu(U)\quad\text{for all }\lhd\text{-open } U\subseteq\mathbb{T}^m.$$

The following result is a variant of Weyl's criterion for this setting; a proof can be obtained by adapting that of [**77**, Theorem 2.1].

Fact 2.6. *A sequence (X_n) of finite subsets of $\mathbb{T}^m$ becomes equidistributed if and only if*

$$\lim_{n\to\infty}\left(\frac{1}{|X_n|}\sum_{a\in X_n}a_1^{l_1}\cdots a_m^{l_m}\right)=0\quad\textit{for all } (l_1,\dots,l_m)\in\mathbb{Z}^m\setminus\{0^{(m)}\}.$$

Below are the consequences of Weil conjectures for curves that we need; see [**87**] for Fact 2.7(i) and [**63**, Proposition 4.5] for a stronger version of Fact 2.7(ii).

Fact 2.7. *Suppose $C\subseteq\mathbb{F}^m$ is a one-dimensional quasi-affine variety over $\mathbb{F}$, $f\in\mathbb{F}[C]$ has image in $\mathbb{F}^\times$, char$(\mathbb{F})=p$, C and f are definable over $\mathbb{F}_q$ (in the model-theory sense, or equivalently for perfect fields like $\mathbb{F}_q$, in the field sense), and $\chi:\mathbb{F}^\times\to\mathbb{C}^\times$ is an injective group homomorphism. Then there is a constant $N\in\mathbb{N}^{\geqslant 1}$ such that for all $n\geqslant 1$,*

(i) $|C(\mathbb{F}_{q^n})|<q^n+N\sqrt{q^n}$;
(ii) $\left|\sum_{a\in C(\mathbb{F}_{q^n})}\chi(f(a))\right|<N\sqrt{q^n}$.

Here, $C(\mathbb{F}_{q^n})$ is the set of $\mathbb{F}_{q^n}$-points of C.

Proof of Theorem 1.2. Applying Corollary 2.6, it suffices to verify for fixed $\mathbb{F}$, injective group homomorphism $\chi:\mathbb{F}^\times\to\mathbb{T}$, and multiplicative circular order $\lhd_\chi$ on $\mathbb{F}$ (as defined in the paragraph preceding the same corollary), that $(\mathbb{F},\lhd_\chi)$ is a model of ACFO. By Proposition 2.4, it suffices to show that $(\mathbb{F},\lhd_\chi)$ satisfies the geometric characterization. Using Proposition 2.2 and Corollary 2.11, we reduce the problem further to showing for a fixed multiplicatively large one-dimensional quasi-affine variety $C\subseteq(\mathbb{F}^\times)^m$ that C is dense in $(\mathbb{F}^\times)^m$ with respect to the $\lhd_\chi$-topology. This is equivalent to showing that $\chi(C)$ is dense in $\mathbb{T}^m$ with respect to the $\lhd$-topology.

Assume, without loss of generality, that $\mathrm{char}(\mathbb{F}) = p$ and C is definable over $\mathbb{F}_q$. Let $C(\mathbb{F}_{q^n})$ be the set of $\mathbb{F}_{q^n}$-points of C. Note that $C = \bigcup_n C(\mathbb{F}_{q^n})$. Hence, the denseness of $\chi(C)$ in $\mathbb{T}^m$ with respect to the $\triangleleft$-topology follows from a stronger result: if X_n is the image of $C(\mathbb{F}_{q^n})$ under χ, then the sequence (X_n) becomes equidistributed. Using Fact 2.6, we reduce the problem to verifying that

$$\lim_{n\to\infty}\left(\frac{1}{|C(\mathbb{F}_{q^n})|}\sum_{a\in C(\mathbb{F}_{q^n})}\chi(a_1^{l_1}\cdots a_m^{l_m})\right) = 0 \quad \text{for all } (l_1,\dots,l_m)\in\mathbb{Z}^m\setminus\{0^{(m)}\}.$$

It is easy to check that C together with $f(x) = x_1^{l_1}\cdots x_m^{l_m}$ satisfies all the conditions described in Fact 2.7, so we arrive at the desired conclusion. $\square$

Remark 2.15. All approaches to prove Theorem 1.2 so far require the use of character sums over finite fields, counting points over finite fields, and Weyl's criterion for equidistribution. However, slightly different paths could have been taken.

The original approach in our earlier write-up [**79**] did not go through Section 2.3.1, but directly used Corollary 2.10 and appealed to the much deeper results on character sums over varieties and counting points over varieties [**25**]. It is possible to avoid these results in the appendix through a rather lengthy proof of "Lang-Weil Theorem for character sums" provided in the appendix of the same manuscript.

We also proved there that ACFO is $\mathrm{acl}_{\mathrm{f}}$-complete (i.e., every complete type over a field-theoretic algebraically closed set is determined by the quantifier-free part of that type); this is a refinement of the fact that ACFO is model complete as every ACFO-model expands an algebraically closed field. Hrushovski pointed out a shorter path to $\mathrm{acl}_{\mathrm{f}}$-completeness which only uses character sums and counting points over curves: using a similar aproach as in our proof of Theorem 1.1, one can get a similar axiomatization correspoding to the simplified geometric characterization in Corollary 2.3.1; then one can show directly that the resulting theory is $\mathrm{acl}_{\mathrm{f}}$-complete by a back-and-forth argument. Only results for curves are necessary in this approach as in the back-and-forth argument, one can extend a field-theoretic algebraically closed set each time by an element with transcendence degree $\leqslant 1$.

The current approach is in between. It preserves some of the original intuitive ideas in [**79**] while not appealing to deep number-theoretic results. The current geometric characterization of existentially closed models of ACFO^- is closer to the notion of an interpolative structure in Chapter 6 than the simplified geometric characterization that one would get following the approach suggested by Hrushovski. Proposition 2.8 is interesting in its own right, and the technology might be useful elsewhere. The $\mathrm{acl}_{\mathrm{f}}$-completeness of ACFO can also be obtained now from the general machinery of interpolative fusions [**54**].

CHAPTER 3

Additive groups of $\mathbb{Z}$ and $\mathbb{Q}$ and predicates for being square-free

We also treat this chapter as the continuation of the corresponding summary in the introduction and keep the notational conventions, definitions, and statements of theorems given there. In Section 3.1, we define the appropriate notions of randomness for the structures under consideration. The model completeness and decidability results are proven in Section 3.2 and the combinatorial tameness results are proven in Section 3.3.

Throughout the chapter, let x be a single variable, y a finite tuple of variables of unspecified length, z the tuple $(z_1, \dots, z_n)$ of variables, and z' the tuple $(z'_1, \dots, z'_{n'})$ of variables. For an n-tuple a of elements from a certain set, we let a_i denote the i-th component of a for $i \in \{1, \dots, n\}$. For an abelian group G and $a \in G$, we define ka in the obvious way and write k for $k1$.

3.1. Genericity of the examples

We can view $\mathbb{Z}$ and $\mathbb{Q}$ as structures in a language L_{b} consisting of function symbols for 0, 1, $+$, $a \mapsto -a$; we will refer to L_{b} as the base language. So $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ and $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ are structures in the language L_{u} extending L_{b} by a unary predicate symbol for $\mathrm{SF}^{\mathbb{Z}}$, and $(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$ and $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ as structures in the language L_{ou} extending L_{u} by a binary predicate symbol for the natural orderings $<$.

We study the structure $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ indirectly by looking at its definable expansion to a richer language. For given p and l, set

$$U^{\mathbb{Z}}_{p,l} = \{a \in \mathbb{Z} : v_p(a) \geqslant l\}.$$

Let $\mathcal{U}^{\mathbb{Z}} = (U^{\mathbb{Z}}_{p,l})$. The definition for $l \leqslant 0$ is not too useful as $U^{\mathbb{Z}}_{p,l} = \mathbb{Z}$ in this case. However, we still keep this for the sake of uniformity as we treat $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ later. For $m > 0$, set

$$P^{\mathbb{Z}}_m = \{a \in \mathbb{Z} : v^{\mathbb{Z}}_p(a) < 2 + v_p(m) \text{ for all } p\}.$$

In particular, $P^{\mathbb{Z}}_1 = \mathrm{SF}^{\mathbb{Z}}$. Let $\mathcal{P}^{\mathbb{Z}} = (P^{\mathbb{Z}}_m)_{m>0}$. We have that $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$ is a structure in the language L^*_{u} extending L_{u} by families of unary predicate symbols for $\mathcal{U}^{\mathbb{Z}}$ and $(P^{\mathbb{Z}}_m)_{m>1}$. Note that

$$U^{\mathbb{Z}}_{p,l} = \mathbb{Z} \text{ for } l \leqslant 0, \quad U^{\mathbb{Z}}_{p,l} = p^l\mathbb{Z} \text{ for } l > 0, \text{ and } \quad P^{\mathbb{Z}}_m = \bigcup_{d|m} d\mathrm{SF}^{\mathbb{Z}} \text{ for } m > 0.$$

Hence, $U^{\mathbb{Z}}_{p,l}$ and $P^{\mathbb{Z}}_m$ are definable in $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$, and so a subset of $\mathbb{Z}$ is definable in $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$ if and only if it is definable in $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$.

Let $(G, \mathcal{P}^G, \mathcal{U}^G)$ be an L^*_{u}-structure. Then $\mathcal{U}^G$ is a family indexed by pairs (p,l), and $\mathcal{P}^G$ is a family indexed by m. For p, l, and m, define $U^G_{p,l} \subseteq G$ to be the member of $\mathcal{U}^G$ with index (p,l) and $P^G_m \subseteq G$ to be the member of the family $\mathcal{P}^G$ with index m. In particular, we have $\mathcal{U}^G = (U^G_{p,l})$ and $P^G = (P^G_m)_{m>0}$. Clearly, this generalizes the previous definition for $\mathbb{Z}$.

We isolate the basic first-order properties of $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$. Let $\mathrm{Sf}^*_{\mathbb{Z}}$ be a recursive set of L^*_{u}-sentences such that an L^*_{u}-structure $(G, \mathcal{U}^G, \mathcal{P}^G)$ is a model of $\mathrm{Sf}^*_{\mathbb{Z}}$ if and only if $(G, \mathcal{U}^G, \mathcal{P}^G)$ satisfies the following properties:

(Z1) G is elementarily equivalent to $\mathbb{Z}$;
(Z2) $U^G_{p,l} = G$ for $l \leqslant 0$, and $U^G_{p,l} = p^l G$ for $l > 0$;
(Z3) 0 and 1 are in P^G_1;
(Z4) for any given p, we have that $pa \in P^G_1$ if and only if $a \in P^G_1$ and $a \notin U^G_{p,1}$;
(Z5) $P^G_m = \bigcup_{d|m} dP^G_1$ for all $m > 0$.

The fact that we could choose $\mathrm{Sf}^*_{\mathbb{Z}}$ to be recursive follows from the well-known decidability of $\mathbb{Z}$. Clearly, $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$ is a model of $\mathrm{Sf}^*_{\mathbb{Z}}$. Several properties which hold in $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$ also hold in an arbitrary model of $\mathrm{Sf}^*_{\mathbb{Z}}$:

Lemma 3.1. *Let $(G, \mathcal{U}^G, \mathcal{P}^G)$ be a model of $\mathrm{Sf}^*_{\mathbb{Z}}$. Then we have the following:*
(i) *$(G, \mathcal{U}^G)$ is elementarily equivalent to $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}})$;*
(ii) *for all k, p, l, and $m > 0$, we have that*

$$k \in U^G_{p,l} \text{ if and only if } k \in U^{\mathbb{Z}}_{p,l} \quad \text{and} \quad k \in P^G_m \text{ if and only if } k \in P^{\mathbb{Z}}_m;$$

(iii) *for all $h \neq 0$, p, and l, we have that $ha \in U^G_{p,l}$ if and only if $a \in U^G_{p,l-v_p(h)}$;*
(iv) *if $a \in G$ is in $U^G_{p,2+v_p(m)}$ for some p, then $a \notin P^G_m$;*
(v) *for all $h \neq 0$ and $m > 0$, $ha \in P^G_m$ if and only if we have*

$$a \in P^G_m \quad \text{and} \quad a \notin U^G_{p,2+v_p(m)-v_p(h)} \text{ for all } p \text{ which divides } h;$$

(vi) *for all $h > 0$ and $m > 0$, $a \in P^G_m$ if and only if $ha \in P^G_{mh}$.*

Proof. Fix a model $(G, \mathcal{U}^G, \mathcal{P}^G)$ of $\mathrm{Sf}^*_{\mathbb{Z}}$. It follows from (Z2) that the same first-order formula defines both $U^G_{p,l}$ in G and $U^{\mathbb{Z}}_{p,l}$ in $\mathbb{Z}$. Then using (Z1), we get (i). The first assertion of (ii) is immediate from (i). Using this, (Z3), and (Z4), we get the second assertion of (ii) for the case $m = 1$. For $m \neq 1$, we reduce to the case $m = 1$ using property (Z5). Statement (iii) is an immediate consequence of (i). We only prove below the cases $m = 1$ of (iv – vi) as the remaining cases of the corresponding statements can be reduced to these using (Z5). Statement (iv) is immediate for the case $m = 1$ using (Z2) and (Z4). The case $m = 1$ of (v) follows from (iii), (iv),and repeated application of (Z4). The case $m = 1$ of (vi) follows from (iv), (v) and (Z5). $\square$

We next consider the structures $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ and $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$. For given p, l, and $m > 0$, in the same fashion as above, we set

$$U^{\mathbb{Q}}_{p,l} = \{a \in \mathbb{Q} : v_p(a) \geqslant l\} \quad \text{and} \quad P^{\mathbb{Q}}_m = \{a \in \mathbb{Q} : v_p(a) < 2 + v_p(m) \text{ for all } p\},$$

and let

$$\mathcal{U}^{\mathbb{Q}} = (U^{\mathbb{Q}}_{p,l}) \quad \text{and} \quad \mathcal{P}^{\mathbb{Q}} = (P^{\mathbb{Q}}_m)_{m>0}.$$

Then $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ is a structure in the language L^*_{u}. Clearly, every subset of $\mathbb{Q}^n$ definable in $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ is also definable in $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$. A similar statement holds for $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ and $(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$. We will show that the reverse implications are also true.

Lemma 3.2. *Every integer is a sum of two elements from* $\mathrm{SF}^{\mathbb{Z}}$.

Proof. We prove the statement for a given integer k. As $\mathrm{SF}^{\mathbb{Z}} = -\mathrm{SF}^{\mathbb{Z}}$ and the cases where $k = 0$ or $k = 1$ are immediate, we assume that $k > 1$. It follows from [**66**] that the number of square-free positive integers lesser than k is at least $\frac{53k}{88}$. Since $\frac{53}{88} > \frac{1}{2}$, this implies k can be written as a sum of two positive square-free integers which is the desired conclusion. $\square$

Lemma 3.3. *For all p and l, $U^{\mathbb{Q}}_{p,l}$ is existentially* 0*-definable in* $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$.

Proof. As $U^{\mathbb{Q}}_{p,l+n} = p^n U^{\mathbb{Q}}_{p,l}$ for all l and n, it suffices to show the statement for $l = 0$. Fix a prime p. We have for all $a \in \mathrm{SF}^{\mathbb{Q}}$ that

$$v_p(a) \geqslant 0 \ \text{ if and only if } p^2 a \notin \mathrm{SF}^{\mathbb{Q}}.$$

Using Lemma 3.2, for all $a \in \mathbb{Q}$, we have that $v_p(a) \geqslant 0$ if and only if there are $a_1, a_2 \in \mathbb{Q}$ such that

$$\big(a_1 \in \mathrm{SF}^{\mathbb{Q}} \wedge v_p(a_1) \geqslant 0\big) \wedge \big(a_2 \in \mathrm{SF}^{\mathbb{Q}} \wedge v_p(a_2) \geqslant 0\big) \ \text{ and } \ a = a_1 + a_2.$$

Hence, the set $U^{\mathbb{Q}}_{p,0} = \{a \in \mathbb{Q} : v_p(a) \geqslant 0\}$ is existentially definable in $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$. The desired conclusion follows. $\square$

It is also easy to see that for all m, $P^{\mathbb{Q}}_m = m\mathrm{SF}^{\mathbb{Q}}$ for all $m > 0$, and so $P^{\mathbb{Q}}_m$ is existentially 0-definable in $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$. Combining with Lemma 3.3, we get:

Proposition 3.1. *Every subset of $\mathbb{Q}^n$ definable in $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ is also definable in $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$. The corresponding statement for $(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ and $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ holds.*

In view of the first part of Proposition 3.1, we can analyze $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ via $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ in the same way we analyze $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ via $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$. Let $\mathrm{Sf}^*_{\mathbb{Q}}$ be a recursive set of L^*_{u}-sentences such that an L^*_{u}-structure $(G, \mathcal{U}^G, \mathcal{P}^G)$ is a model of $\mathrm{Sf}^*_{\mathbb{Q}}$ if and only if $(G, \mathcal{U}^G, \mathcal{P}^G)$ satisfies the following properties:

(Q1) G is elementarily equivalent to $\mathbb{Q}$;
(Q2) for any given p, $U^G_{p,0}$ is an n-divisible subgroup of G for all n coprime with p;
(Q3) $1 \in U^G_{p,0}$ and $1 \notin U^G_{p,1}$;
(Q4) for any given p, $p^{-l}U^G_{p,l} = U^G_{p,0}$ if $l < 0$ and $U_{p,l} = p^l U_{p,0}$ if $l > 0$;
(Q5) $U^G_{p,0}/U^G_{p,1}$ is isomorphic as a group to $\mathbb{Z}/p\mathbb{Z}$;
(Q6) $1 \in P^G_1$;
(Q7) for any given p, we have that $pa \in P^G_1$ if and only if $a \in P^G_1$ and $a \notin U^G_{p,1}$;
(Q8) $P^G_m = mP^G_1$ for $m > 0$;

The fact that we could choose $\mathrm{Sf}^*_{\mathbb{Q}}$ to be recursive follows from the well-known decidability of $\mathbb{Q}$. Obviously, $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ is a model of $\mathrm{Sf}^*_{\mathbb{Q}}$. Several properties which hold in $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ also hold in an arbitrary model of $\mathrm{Sf}^*_{\mathbb{Q}}$:

Lemma 3.4. *Let $(G, \mathcal{U}^G, \mathcal{P}^G)$ be a model of $\mathrm{Sf}^*_{\mathbb{Q}}$. Then we have the following:*

(i) *For all p and all $l, l' \in \mathbb{Z}$ with $l \leqslant l'$, we have $U^G_{p,l}$ is a subgroup of G, $U^G_{p,l'} \subseteq U^G_{p,l}$, and*
$$U^G_{p,l}/U^G_{p,l'} \cong_{L_\perp} \mathbb{Z}/(p^{l'-l}\mathbb{Z});$$
(ii) *for all h, $k \neq 0$, p, l, and $m > 0$, we have that*
$$\frac{h}{k} \in U^G_{p,l} \text{ if and only if } \frac{h}{k} \in U^{\mathbb{Q}}_{p,l} \quad \text{and} \quad \frac{h}{k} \in P^G_m \text{ if and only if } \frac{h}{k} \in P^{\mathbb{Q}}_m$$
where hk^{-1} is the obvious element in $\mathbb{Q}$ and in G;
(iii) *the replica of* (iii – vi) *of Lemma 3.1 holds.*

Proof. Fix a model $(G, \mathcal{U}^G, \mathcal{P}^G)$ of $\mathrm{Sf}^*_{\mathbb{Q}}$. From (Q2) we have that $U^G_{p,0}$ is a subgroup of G for all p. It follows from (Q4) that $U^G_{p,l'} \subseteq U^G_{p,l}$ are subgroups of G for all p and $l \leqslant l'$. We get an $L_\perp$-embedding of $\mathbb{Z}/(p^{l'-l}\mathbb{Z})$ into $U^G_{p,l}/U^G_{p,l'}$ and $|U^G_{p,l}/U^G_{p,l'}| = p^{(l'-l)}$ using (Q2)-(Q5) and induction on $l' - l$. So, the aforementioned embedding must be an isomorphism and we get (i). The first assertion of (ii) follows easily from (Q2)-Q(4). The second assertion for the case $m = 1$ follows from the first assertion, (Q6), and (Q7). Finally, the case with $m \neq 1$ follows from the case $m = 1$ using (Q8). The proof for (iii) is similar to the proofs for (iii – vi) of Lemma 3.1. ☐

As the reader may expect by now, we will study $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ via $(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$. Let L^*_{ou} be $L_{\mathrm{ou}} \cup L^*_{\mathrm{u}}$. Then $(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ can be construed as an L^*_{ou}-structure in the obvious way. Let $\mathrm{OSf}^*_{\mathbb{Q}}$ be a recursive set of L^*_{ou}-sentences such that an L^*_{ou}-structure $(G, \mathcal{U}^G, \mathcal{P}^G)$ is a model of $\mathrm{OSf}^*_{\mathbb{Q}}$ if and only if $(G, \mathcal{U}^G, \mathcal{P}^G)$ satisfies the following properties:

(1) $(G, <)$ is elementarily equivalent to $(\mathbb{Q}, <)$;
(2) $(G, \mathcal{U}^G, \mathcal{P}^G)$ is a model of $\mathrm{Sf}^*_{\mathbb{Q}}$.

As $\mathrm{Th}(\mathbb{Q}, <)$ is decidable, we could choose $\mathrm{OSf}^*_{\mathbb{Q}}$ to be recursive.

Returning to the theory $\mathrm{Sf}^*_{\mathbb{Z}}$, we see that it does not fully capture all the first-order properties of $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$. For instance, we will show later in Corollary 3.1 that for all $c \in \mathbb{Z}$, there is $a \in \mathbb{Z}$ such that

$$a + c \in \mathrm{SF}^{\mathbb{Z}} \quad \text{and} \quad a + c + 1 \in \mathrm{SF}^{\mathbb{Z}},$$

while the interested reader can construct models of $\mathrm{Sf}^*_{\mathbb{Z}}$ where the corresponding statement is not true. Likewise, the theories $\mathrm{Sf}^*_{\mathbb{Q}}$ and $\mathrm{OSf}^*_{\mathbb{Q}}$ do not fully capture all the first-order properties of $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ and $(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$.

To give a precise formulation of the missing first-order properties of $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$, $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$, and $(\mathbb{Q}, < \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$, we need more terminologies. Let $t(z)$ be an L^*_{u}-term (or equivalently an L^*_{ou}-term) with variables in z. If $(G, \mathcal{U}^G, \mathcal{P}^G)$ is either an L^*_{u}-structure or an L^*_{ou}-structure, and $c \in G^n$, define $t^G(c)$ to be the $\mathbb{Z}$-linear combination of the components of c given by $t(z)$. Define in the obvious way the formulas

$$t(z) = 0,\ t(z) \neq 0,\ t(z) < 0,\ t(z) > 0,\ t(z) \leqslant 0 \ \text{ and } \ t(z) \geqslant 0.$$

An L^*_{u}-formula (or an L^*_{ou}-formula) which is a boolean combination of formulas having the form $t(z) = 0$ where we allow t to vary is called an **equational condition**. Similarly, an L^*_{ou}-formula which is a boolean combination of formulas having the form $t(z) < 0$ where t is allowed to vary is called an **order-condition**. For any given p, l define $t(z) \in U_{p,l}$ to be the obvious formula in $L^*_{\mathrm{u}}(z)$ which defines in an arbitrary L^*_{u}-structure $(G, \mathcal{U}^G, \mathcal{P}^G)$ the set

$$\{c \in G^n : t^G(c) \in U^G_{p,l}\}.$$

Define the quantifier-free formulas $t(z) \notin U_{p,l}$, $t(z) \in P_m$, and $t(z) \notin P_m$ in $L^*_{\mathrm{u}}(z)$ for p, l, and for $m > 0$ likewise. For each prime p, An L^*_{u}-formula (or an L^*_{ou}-formula) which is a boolean combination of formulas of the form $t(z) \notin U_{p,l}$ where t and l are allowed to vary is called a p-**condition**. We call a p-condition as in the previous statement **trivial** if the boolean combination is the empty conjunction.

A **parameter choice** of variable type (x, z, z') is a triple (k, m, Θ) such that k is in $\mathbb{Z} \setminus \{0\}$, m is in $\mathbb{N}^{\geqslant 1}$, and $\Theta = \big(\theta_p(x, z, z')\big)$ where $\theta_p(x, z, z')$ is a p-condition for each prime p and is trivial for all but finitely many p. We say that an L^*_{u}-formula $\psi(x, z, z')$ is **special** if it has the form

$$\bigwedge_p \theta_p(x, z, z') \wedge \bigwedge_{i=1}^{n} (kx + z_i \in P_m) \wedge \bigwedge_{i'=1}^{n'} (kx + z'_i \notin P_m)$$

where k, m and $\theta_p(x, z, z')$ are taken from a parameter choice of variable type (x, z, z'). Every special formula corresponds to a unique parameter choice and vice versa. Special formulas are special enough that we have a "local to global" phenomenon in the structures of interest but general enough to represent quantifier free formula. We will explain the former point in the remaining part of the section and make the latter point precise with Theorem 3.10.

Let $\psi(x,z,z')$ be a special formula with parameter choice (k,m,Θ) and $\theta_p(x,z,z')$ is the p-condition in Θ for each p. We define the **associated equational condition** of $\varphi(x,z,z')$ to be the formula

$$\bigwedge_{i=1}^{n}\bigwedge_{i'=1}^{n'}(z_i \neq z'_{i'})$$

and the **associated p-condition** of $\varphi(x,z,z')$ to be the formula

$$\theta_p(x,z,z') \wedge \bigwedge_{i=1}^{n}(kx+z_i \notin U_{p,2+v_p(m)}).$$

It is easy to see for an arbitrary special formula that its associated equational condition and its associated p-condition for any prime p are its logical consequences.

Suppose $(G,\mathcal{U}^G,\mathcal{P}^G)$ and $(H,\mathcal{U}^H,\mathcal{P}^H)$ are L^*_{u}-structures such that the former is an L^*_{u}-substructure of the latter. Let $\psi(x,z,z')$ be a special formula, $\psi_=(z,z')$ the associated equational condition, and $\psi_p(x,z,z')$ the associated p-condition for any given prime p. For $c \in G^n$ and $c' \in G^{n'}$, we call the quantifier-free $L^*_{\mathrm{u}}(G)$-formula $\psi(x,c,c')$ a **G-system**. An element $a \in H$ such that $\psi(a,c,c')$ holds is called a **solution** of $\psi(x,c,c')$ in H. We say that $\psi(x,c,c')$ is **satisfiable** in H if it has a solution in H and **infinitely satisfiable** in H if it has infinitely many solutions in H. We say that $\psi(x,c,c')$ is **nontrivial** if $\psi_=(c,c')$ holds or more explicitly if c and c' have no common components. For a given p, we say that $\psi(x,c,c')$ is **p-satisfiable** in H if there is $a_p \in H$ such that $\psi_p(a_p,c,c')$ holds. A G-system is **locally satisfiable** in H if it is p-satisfiable in H for all p.

Suppose $(G,<,\mathcal{U}^G,\mathcal{P}^G)$ and $(H,<,\mathcal{U}^H,\mathcal{P}^H)$ are L^*_{ou}-structures such that the former is an L^*_{ou}-substructure of the latter. All the definitions in the previous paragraph have obvious adaptations to this new setting as $(G,\mathcal{U}^G,\mathcal{P}^G)$ and $(H,\mathcal{U}^H,\mathcal{P}^H)$ are L^*_{u}-structures. For b and b' in H such that $b<b'$, define

$$(b,b')^H = \{a \in H : b < a < b'\}.$$

A G-system $\psi(x,c,c')$ is **satisfiable in every H-interval** if it has a solution in the interval $(b,b')^H$ for all b and b' in H such that $b<b'$. The following observation is immediate:

Lemma 3.5. *Suppose $(G,\mathcal{U}^G,\mathcal{P}^G)$ is a model of either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$. Then every G-system which is satisfiable in G is nontrivial and locally satisfiable in G.*

It turns out that the converse and more are also true for the structures of interest. We say that a model $(G,\mathcal{U}^G,\mathcal{P}^G)$ of either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$ is **generic** if every nontrivial locally satisfiable G-system is infinitely satisfiable in G. A $\mathrm{OSf}^*_{\mathbb{Q}}$ model $(G,<,\mathcal{U}^G,\mathcal{P}^G)$ is **generic** if every nontrivial nontrivial locally satisfiable G-system is satisfiable in every G-interval. We will later show that $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$, $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$, and $(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ are generic.

Before that we will show that the above notions of *genericity* are first-order. Let $\psi(x, z, z')$ be the special formula corresponding to a parameter choice (k, m, Θ) with $\Theta = \big(\theta_p(x, z, z')\big)$. A **boundary** of $\psi(x, z, z')$ is a number $B \in \mathbb{N}^{>0}$ such that $B > \max\{|k|, n\}$ and $\theta_p(x, z, z')$ is trivial for all $p > B$.

Lemma 3.6. *Let $\psi(x, z, z')$ be a special formula, B a boundary of $\psi(x, z, z')$, and $(G, \mathcal{U}^G, \mathcal{P}^G)$ a model of either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$. Then every G-system $\psi(x, c, c')$ is p-satisfiable for $p > B$.*

Proof. Let $\psi(x, z, z')$ be the special formula corresponding to a parameter choice (k, m, Θ), and B, $(G, \mathcal{U}^G, \mathcal{P}^G)$ as in the statement of the lemma. Suppose $\psi(x, c, c')$ is a G-system, $p > B$, and $\psi_p(x, z, z')$ is the associated p-condition of $\psi(x, z, z')$. Then $\psi_p(x, c, c')$ is equivalent to

$$\bigwedge_{i=1}^{n} (kx + c_i \notin U_{p, 2+v_p(m)}) \text{ in } (G, \mathcal{U}^G, \mathcal{P}^G).$$

We will show a stronger statement that there is a $a_p \in \mathbb{Z}$ satisfying the latter. Note that for all $d \notin U^G_{p,0}$, we have that $(ka + d \notin U_{p,0})$ for all $a \in \mathbb{Z}$. From Lemma 3.4, we have that $U^G_{p,l} \subseteq U^G_{p,k}$ whenever $k < l$, so we can assume that $c_i \in U^G_{p,0}$ for $i \in \{1, \dots, n\}$. In light of Lemma 3.1 (i) and Lemma 3.4 (i), we have that

$$U^G_{p, 2+v_p(m)} / U^G_{p,0} \cong_{L_\perp} \mathbb{Z}/(p^{2+v_p(m)}\mathbb{Z}).$$

It is easy to see that k is invertible mod $p^{2+v_p(m)}$ and that $p^{2+v_p(m)} > n$. Choose a_p in $\{0, \dots, p^{2+v_p(m)} - 1\}$ such that the images of $ka_p + c_1, \dots, ka_p + c_n$ in $\mathbb{Z}/(p^{2+v_p(m)}\mathbb{Z})$ are not 0. We check that a_p is as desired. □

Corollary 3.1. *There is an L^*_{u}-theory $\mathrm{SF}^*_{\mathbb{Z}}$ such that the models of $\mathrm{SF}^*_{\mathbb{Z}}$ are the generic models of $\mathrm{Sf}^*_{\mathbb{Z}}$. Similarly, there is an L^*_{u}-theory $\mathrm{SF}^*_{\mathbb{Q}}$ and an L^*_{ou}-theory $\mathrm{OSF}^*_{\mathbb{Q}}$ satisfying the corresponding condition for $\mathrm{Sf}^*_{\mathbb{Q}}$ and $\mathrm{OSf}^*_{\mathbb{Q}}$.*

In the rest of the chapter, we fix $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ to be as in the previous lemma. We can moreover arrange them to be recursive. In the remaining part of this section, we will show that $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$, $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ and $(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Z}})$ are models of $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ respectively. The proof that the latter are in fact the full axiomatizations of the theories of the former needs to wait until next section. Further we fix $\mathrm{SF}_{\mathbb{Z}}$ and $\mathrm{SF}_{\mathbb{Q}}$ to the theories whose models are precisely the L_{u}-reducts of models of $\mathrm{SF}^*_{\mathbb{Z}}$ and $\mathrm{SF}^*_{\mathbb{Q}}$ respectively, and $\mathrm{OSF}_{\mathbb{Q}}$ to be the theory whose models are precisely L_{ou} reducts of models of $\mathrm{OSF}^*_{\mathbb{Q}}$. For the reader's reference, the following table lists all the languages, the corresponding theories and primary structures under consideration:

Languages	Theories	Primary structures
L_{u}	$\mathrm{SF}_{\mathbb{Z}}$, $\mathrm{SF}_{\mathbb{Q}}$	$(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$, $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$
L_{ou}	$\mathrm{OSF}_{\mathbb{Q}}$	$(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$, $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$
L^*_{u}	$\mathrm{Sf}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{Sf}^*_{\mathbb{Q}}$, $\mathrm{SF}^*_{\mathbb{Q}}$	$(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$, $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$
L^*_{ou}	$\mathrm{OSf}^*_{\mathbb{Q}}$, $\mathrm{OSF}^*_{\mathbb{Q}}$	$(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$

TABLE 3.1. Languages, theories, and primary structures

Suppose $h \neq 0$ and $\varphi(z)$ is a boolean combination of atomic formulas of the form $t(z) \in U_{p,l}$ or $t(z) \in P_m$ where $t(z)$ is an L^*_{u}-term. Define $\varphi^h(z)$ to be the formula obtained by replacing $t(z) \in U_{p,l}$ and $t(z) \in P_m$ in φ with $t(z) \in U_{p,l+v_p(h)}$ and $t(z) \in P_{mh}$ for every choice of p, l, m and L^*_{u}-term t. By construction and linearity of terms, across models of $\mathrm{Sf}^*_{\mathbb{Z}}$ and $\mathrm{Sf}^*_{\mathbb{Q}}$, using Lemma 3.1 (iii), (vi) and Lemma 3.4 (iii), we have that

$$\varphi^h(hz) \text{ is equivalent to } \varphi(z).$$

Moreover, if $\theta(z)$ is a p-condition, then $\theta^h(z)$ is also p-condition. If $\psi(x, z, z')$ is the special formula corresponding to a parameter choice (k, m, Θ) with $\Theta = \big(\theta_p(x, z, z')\big)$, then $\psi^h(x, z, z')$ is the special formula corresponding to the parameter choice (k, hm, Θ^h) with $\Theta^h = \big(\theta^h_p(x, z, z')\big)$. It is easy to see from here that:

Lemma 3.7. *Any boundary of a special formula $\psi(x, z, z')$ is also a boundary of $\psi^h(x, z, z')$ and vice versa.*

Let $\psi(x, z, z')$ be a special formula, $(G, \mathcal{U}^G, \mathcal{P}^G)$ a model of either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$, and $\psi(x, c, c')$ a G-system. Then $\psi^h(x, hc, hc')$ is also a G-system which we refer to as the h-**conjugate** of $\psi(x, c, c')$. This has the property that $\psi^h(ha, hc, hc')$ if and only if $\psi(a, c, c')$ for all $a \in G$.

For a and b in $\mathbb{Z}$, we write $a \equiv_n b$ if a and b have the same remainder when divided by n. We need the following version of Chinese remainder theorem:

Lemma 3.8. *Suppose B is in $\mathbb{N}^{>0}$, Θ is a family $\big(\theta_p(x, z)\big)_{p \leqslant B}$ where $\theta_p(x, z)$ is a p-condition for all $p \leqslant B$, and $c \in \mathbb{Z}^n$ is such that $\theta_p(x, c)$ defines a nonempty set in $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$ for all $p \leqslant B$. Then we can find $D \in \mathbb{N}^{>0}$ such that for all $h \neq 0$ with $\gcd(h, B!) = 1$, for some $r_h \in \{0, \dots, D-1\}$ we have that*

$$a \equiv_D r_h \text{ implies } \bigwedge_{p \leqslant B} \theta^h_p(a, hc) \quad \text{for all } a \in \mathbb{Z}.$$

PROOF. Let B, Θ, and c be as stated. Fix $h \neq 0$ such that $\gcd(h, B!) = 1$ and so we have that $\theta_p(x, z) = \theta^h_p(x, z)$ for all $p \leqslant B$. For each $p \leqslant B$, the p-condition $\theta^h_p(x, z)$ is a boolean combination of atomic formulas of the form $kx + t(z) \in U_{p,l}$ where $t(z)$ is an L_{u}^*-term. Now for $p \leqslant B$, let l_p be the largest value of l occurring in an atomic formula in $\theta_p(x,z)$. Set

$$D = \prod_{p \leqslant B} p^{l_p}.$$

Obtain a_p such that $\theta_p(a_p, c)$ holds. By the Chinese remainder theorem, we get r in $\{0, \ldots, D-1\}$ such that

$$r \equiv_{p^{l_p}} a_p \quad \text{for all } p \leqslant B.$$

Suppose $a \in \mathbb{Z}$ is such that $a \equiv_D hr$. By construction, if $p \leqslant B$ and $kx + t(z) \in U_{p,l}$ is any atomic formula, then $ka + t(hc) \in U_{p,l}^{\mathbb{Z}}$ if and only if $k(ha_p) + t(hc) \in U_{p,l}^{\mathbb{Z}}$. It follows that $\theta_p^h(a, hc)$ holds for all $p \leqslant B$. The desired conclusion follows with any $r_h \equiv_D hr$. ☐

Towards showing that the structures of interest are generic, the key number-theoretic ingredient we need is the following result:

Lemma 3.9. *Let $\psi(x,z,z')$ be a special formula and $\psi(x,c,c')$ a nontrivial $\mathbb{Z}$-system which is locally satisfiable in $\mathbb{Z}$. For $h > 0$, and $s, t \in \mathbb{Q}$ with $s < t$, set*

$$\Psi^h(hs, ht) = \{a \in \mathbb{Z} : \psi^h(a, hc, hc') \textit{ holds and } hs < a < ht\}.$$

Then there exists $N \in \mathbb{N}^{>0}$, $\varepsilon \in (0,1)$, and $C \in \mathbb{R}$ such that for all $h > 0$ with $\gcd(h, N!) = 1$ *and $s, t \in \mathbb{Q}$ with $s < t$, we have that*

$$|\Psi^h(hs, ht)| \geqslant \varepsilon h(t-s) - \left(\sum_{i=1}^{n} \sqrt{|hks + hc_i|} + \sqrt{|hkt + hc_i|}\right) + C.$$

Proof. Throughout this proof, let $\psi(x,z,z')$, $\psi(x,c,c')$, and $\Psi^h(hs,ht)$ be as stated. We first make a number of observations. Suppose $\psi(x,z,z')$ corresponds to the parameter choice (k, m, Θ) and has a boundary B, and $\psi_p(x,z,z')$ is the associated p-condition of $\psi(x,z,z')$. Then $\psi^h(x,z,z')$ corresponds to the parameter choice (k, hm, Θ^h), and B is also a boundary of $\psi^h(x,z,z')$ by Corollary 3.7. Moreover $\psi_p^h(x,z,z')$ is the associated p-condition of $\psi^h(x,z,z')$. Using Lemma 3.8, we fix $D \in \mathbb{N}^{>0}$ and obtain for each $h > 0$ with $\gcd(h, B!) = 1$ an $r_h \in \{0, \ldots, D-1\}$ such that

$$a \equiv_D r_h \text{ implies } \bigwedge_{p \leqslant B} \psi_p^h(a, hc, hc') \quad \text{for all } a \in \mathbb{Z}.$$

We note that D here is independent of the choice of h for all h with $\gcd(h, B!) = 1$.

We introduce a variant of $\Psi^h(hs,ht)$ which is needed in our estimation of $|\Psi^h(hs,ht)|$. Until the end of the proof, set $l_p = 2 + v_p(m)$. Fix primes $p_1, \ldots, p_{n'}$ such that $p_1 > c_i$ for all $i \in \{1, \ldots, n\}$, $p_1 > c'_{i'}$ for all $i' \in \{1, \ldots, n'\}$ and

$$B < p_1 < \ldots < p_{n'}.$$

For $M > p_{n'}$, $h > 0$ with $\gcd(h, B!) = 1$, and corresponding r_h, define $\Psi^h_M(hs, ht)$ to be the set of $a \in \mathbb{Z}$ such that $hs < a < ht$ and

$$(a \equiv_D r_h) \wedge \bigwedge_{B<p\leqslant M} \Big(\bigwedge_{i=1}^{n} (ka + hc_i \not\equiv_{p^{l_p+v_p(h)}} 0) \Big) \wedge \bigwedge_{i'=1}^{n'} (ka + hc'_{i'} \notin P^{\mathbb{Z}}_{hm}).$$

It is not hard to see that $\Psi^h(hs, ht) \cap \{a \in \mathbb{Z} : a \equiv_D r_h\} \subseteq \Psi^h_M(hs, ht)$, and the latter is intended to be and upper approximation of the former. The desired lower bound for $|\Psi^h(hs, ht)|$ will be obtained via a lower bound for $|\Psi^h_M(hs, ht)|$ and an upper bound for $|\Psi^h_M(hs, ht) \smallsetminus \Psi^h(hs, ht)|$.

Now we work towards establishing a lower bound on $|\Psi^h_M(hs, ht))|$ in the case where $M > p_{n'}$, $h > 0$, and $\gcd(h, M!) = 1$. The latter assumption implies in particular that $p^{l_p+v_p(h)} = p^{l_p}$ for all $p \leqslant M$. For $p > B$, we have that $p > |k|$ and so k is invertible mod p^{l_p}. Set

$$\Delta = \{p : B < p \leqslant M\} \smallsetminus \{p_{i'} : 1 \leqslant i' \leqslant n'\}.$$

For $p \in \Delta$, as k is invertible mod p^{l_p}, there are at least $p^{l_p} - n$ (note we have $p > B > n$) choices of r_p in $\{0, \dots, p^{l_p} - 1\}$ such that if $a \equiv_{p^{l_p}} r_p$, then

$$\bigwedge_{i=1}^{n} (ka + hc_i \not\equiv_{p^{l_p}} 0).$$

Suppose $p = p_{i'}$ for some $i' \in \{1, \dots, n'\}$. By the assumption that $\psi(x, c, c')$ is nontrivial, c has no common components with c'. Since $\gcd(h, M!) = 1$, h and p are coprime, and so the components of hc and hc' are pairwise distinct mod p^{l_p}. As k is invertible mod p^{l_p}, there is exactly one r_p in $\{0, \dots, p^{l_p} - 1\}$ such that if $a \equiv_{p^{l_p}} r_p$, then

$$\bigwedge_{i=1}^{n} (ka + hc_i \not\equiv_{p^{l_p}} 0) \wedge (ka + hc'_{i'} \equiv_{p^{l_p}} 0) \;\text{ and consequently }\; ka + hc'_{i'} \notin P^{\mathbb{Z}}_{hm}.$$

Now it follows by the Chinese remainder theorem that,

$$|\Psi^h_M(hs, ht)| \geqslant \left\lfloor \frac{ht - hs}{D \prod_{B<p\leqslant M} p^{l_p}} \right\rfloor \prod_{p \in \Delta} \left(p^{l_p} - n\right).$$

Then it follows that,

$$|\Psi^h_M(hs, ht)| \geqslant \frac{ht - hs}{D} \prod_{p \leqslant p_{n'}} \frac{1}{p^{l_p}} \prod_{p > p_{n'}}^{\leqslant M} \left(1 - \frac{n}{p^{l_p}}\right) - \prod_{p \leqslant M} p^{l_p}.$$

Set

$$\varepsilon = \frac{1}{2D} \prod_{p \leqslant p_{n'}} \frac{1}{p^{l_p}} \prod_{p > p_{n'}} \left(1 - \frac{n}{p^{l_p}}\right).$$

Now as $l_p \geqslant 2$, for $U \in \mathbb{N}^{>0}$ with $U > \max\{p'_n, n^2\}$ we have that

$$\prod_{p>U}\left(1-\frac{n}{p^{l_p}}\right) > \prod_{p>U}\left(1-\frac{1}{p^{\frac{3}{2}}}\right).$$

Hence, it follows from Euler's product formula that $\varepsilon > 0$. We now have

$$|\Psi^h_M(hs,ht)| \geqslant 2\varepsilon(ht-hs) - \prod_{p\leqslant M} p^{l_p}.$$

We note that ε is independent of the choice of M and h, and will serve as the promised ε in the statement of the lemma.

Next we obtain a upper bound on $|\Psi^h_M(s,t) \smallsetminus \Psi^h(s,t)|$ for $M > p_{n'}$ $h > 0$ and $\gcd(h, M!) = 1$. We arrange that $k > 0$ by replacing c by $-c$ and c' by $-c'$ if necessary. Note that an element $a \in \Psi^h_M(s,t) \smallsetminus \Psi^h(s,t)$ must be such that

$$hks + hc_i < ka + hc_i < hkt + hc_i \quad \text{for all } i \in \{1,\dots,n\}$$

and $ka+hc_i$ is a multiple of p^{l_p} for some $p > M$ and $i \in \{1,\dots,n\}$. For each p and $i \in \{1,\dots,n\}$, the number of multiples of p^{l_p} in $(hks+hc_i, hkt+hc_i)$ is

$$\text{either} \quad \lfloor hk(t-s)p^{-l_p}\rfloor \quad \text{or} \quad \lfloor hk(t-s)p^{-l_p}\rfloor + 1.$$

In the latter case, as $l_p \geqslant 2$ we moreover have

$$p^2 \leqslant |hks+hc_i| \quad \text{or} \quad p^2 \leqslant |hkt+hc_i|,$$

and so

$$p \leqslant \sqrt{|hks+hc_i|} + \sqrt{|hkt+hc_i|}.$$

As $l_p \geqslant 2$, we have $\lfloor hk(t-s)p^{-l_p}\rfloor \leqslant hk(t-s)p^{-2}$. Therefore we have that

$$|\Psi^h_M(s,t) \smallsetminus \Psi^h(s,t)| \leqslant h(t-s)\sum_{p>M}\frac{nk}{p^2} + \sum_{i=1}^{n}\sqrt{|hks+hc_i|} + \sqrt{|hkt+hc_i|}.$$

We now obtain N and C as in the statement of the lemma. Note that

$$\sum_{p>T} p^{-2} \leqslant \sum_{n>T} n^{-2} = O(T^{-1}).$$

Using this, we obtain $N \in \mathbb{N}^{>0}$ such that $N > p_{n'}$ and $\sum_{p>N} knp^{-2} < \varepsilon$ where ε is from the preceding paragraph. Set $C = -\prod_{p\leqslant N} p^{l_p}$. Combining the estimations from the preceding two paragraphs for $M = N$ it is easy to see that ε, N, C are as desired. □

Remark 3.1. The above weak lower bound is all we need for our purpose. We expect that a stronger estimate can be obtained using modifications of available techniques in the literature; see for example [**59**].

Corollary 3.2. *For all $c \in \mathbb{Z}$, there is $a \in \mathbb{Z}$ such that*

$$a + c \in \mathrm{SF}^{\mathbb{Z}} \quad \textit{and} \quad a + c + 1 \in \mathrm{SF}^{\mathbb{Z}}.$$

PROOF. We have that for all $c \in \mathbb{Z}$, $\psi(x, c) = (x + c \in \mathrm{SF}^{\mathbb{Z}}) \wedge (x + c + 1 \in \mathrm{SF}^{\mathbb{Z}})$ is a locally satisfiable $\mathbb{Z}$-system. Applying Lemma 3.9 for $h = 1$, $s = 0$, and t sufficiently large we see there is a solution $a \in \mathbb{Z}$ for $\psi(x, c)$. □

We next prove the main theorem of the section:

Theorem 3.1. *The* $\mathrm{Sf}^*_{\mathbb{Z}}$*-model* $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$*, the* $\mathrm{Sf}^*_{\mathbb{Q}}$*-model* $(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$*, and the* $\mathrm{OSf}^*_{\mathbb{Q}}$*-model* $(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ *are generic.*

PROOF. We get the first part of the theorem by applying Lemma 3.9 for $h = 1$, $s = 0$, and t sufficiently large. As the second part of the theorem follows easily from the third part, it remains to show that the $\mathrm{OSf}^*_{\mathbb{Q}}$-model $(\mathbb{Q}, <, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ is generic. Throughout this proof, suppose $\psi(x, z, z)$ is a special formula and $\psi(x, c, c')$ is a $\mathbb{Q}$-system which is nontrivial and locally satisfiable in $\mathbb{Q}$. Our job is to show that the $\mathbb{Q}$-system $\psi(x, c, c')$ has a solution in the $\mathbb{Q}$-interval $(b, b')^{\mathbb{Q}}$ for an arbitrary choice of $b, b' \in \mathbb{Q}$ such that $b < b'$.

We first reduce to the special case where $\psi(x, c, c')$ is also a $\mathbb{Z}$-system which is nontrivial and locally satisfiable in $\mathbb{Z}$. Let B be the boundary of $\psi(x, z, z')$ and for each p, let $\psi_p(x, z, z')$ be the associated p-condition of $\psi(x, z, z')$. Using the assumption that $\psi(x, c, c')$ is locally satisfiable $\mathbb{Q}$-system, for each $p < B$ we obtain $a_p \in \mathbb{Q}$ such that $\psi_p(a_p, c, c')$ holds. Let $h > 0$ be such that

$$hc \in \mathbb{Z}^n, hc' \in \mathbb{Z}^{n'} \text{ and } ha_p \in \mathbb{Z} \text{ for all } p < B.$$

Then by the choice of h , Lemma 3.6, and Lemma 3.7, the h-conjugate $\psi^h(x, hc, hc')$ of $\psi(x, c, c')$ is a $\mathbb{Z}$-system which is nontrivial and locally satisfiable in $\mathbb{Z}$. On the other hand, $\psi(x, c, c')$ has a solution in a interval $(b, b')^{\mathbb{Q}}$ if and only if

$$\psi^h(x, hc, hc') \text{ has a solution in } (hb, hb')^{\mathbb{Q}}.$$

Hence, by replacing $\psi(x, z, z')$ with $\psi^h(x, z, z')$, $\psi(x, c, c')$ with $\psi^h(x, hc, hc')$, and $(b, b')^{\mathbb{Q}}$ with $(hb, hb')^{\mathbb{Q}}$ if necessary we get the desired reduction.

We show $\psi(x, c, c')$ has a solution in the $\mathbb{Q}$-interval $(b, b')^{\mathbb{Q}}$ for the special case in the preceding paragraph. By an argument similar to the preceding paragraph, it suffices to show that for some $h \neq 0$, $\psi^h(x, hc, hc')$ has a solution in $(hb, hb')^{\mathbb{Q}}$. Applying Lemma 3.9 for $s = b$, $t = b'$, and h sufficiently large satisfying the condition of the lemma, we get the desired conclusion. □

3.2. Logical Tameness

We will next prove that $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ admit quantifier elimination. We first need a technical lemma saying that over $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$, quantifier free formulas are not much more complicated than special formulas.

Lemma 3.10. *Suppose $\varphi(x,y)$ is a quantifier-free L^*_{u}-formula. Then $\varphi(x,y)$ is equivalent over $\mathrm{Sf}^*_{\mathbb{Z}}$ to a disjunction of quantifier-free formulas of the form*

$$\rho(y) \wedge \varepsilon(x,y) \wedge \psi(x, t(y), t'(y))$$

where

(i) *$t(y)$ and $t'(y)$ are tuples of L^*_{u}-terms with length n and n' respectively;*
(ii) *$\rho(y)$ is a quantifier-free L^*_{u}-formula, $\varepsilon(x,y)$ an equational-condition, $\psi(x,z,z')$ a special formula.*

The corresponding statement with $\mathrm{Sf}^*_{\mathbb{Z}}$ replaced by $\mathrm{Sf}^*_{\mathbb{Q}}$ also holds.

Proof. Let $\varphi(x,y)$ be a quantifier-free L^*_{u}-formula. We will use the following disjunction observation several times in our proof: If $\varphi(x,y)$ is a finite disjunction of quantifier-free L^*_{u}-formulas and we have proven the desired statement for each of those, then the desired statement for $\varphi(x,y)$ follows. In particular, we can assume that $\varphi(x,y)$ is the conjunction

$$\rho(y) \wedge \varepsilon(x,y) \wedge \bigwedge_p \eta_p(x,y) \wedge \bigwedge_{i=1}^{n}(k_i x + t_i(y) \in P_{m_i}) \wedge \bigwedge_{i=1}^{n'}(k'_i x + t'_i(y) \notin P_{m'_i})$$

where $\rho(y)$ is a quantifier-free L^*_{u}-formula, $\varepsilon(x,y)$ is an equational condition, $k_1,\dots,k_n$ and $k'_1,\dots,k'_{n'}$ are in $\mathbb{Z} \setminus \{0\}$, $m_1,\dots,m_n$ and $m'_1,\dots,m'_{n'}$ are in $\mathbb{N}^{\geq 1}$, $t_1(y),\dots,t_n(y)$ and $t'_1(y),\dots,t'_n(y)$ are L^*_{u}-terms with variables in y, $\eta_p(x,y)$ is a p-condition for each p, and $\eta_p(x,y)$ is trivial for all but finitely many p.

We make further reductions to the form of $\varphi(x,y)$. Set $t(y) = (t_1(y),\dots,t_n(y))$ and $(t'_1(y),\dots,t'_{n'}(y))$. Using the disjunction observation and the fact that $(x+y_j \in P_1) \vee (x+y_j \notin P_1)$ is a tautology for every component y_j of y, we can assume that either $x+y_j \in P_1$ or $x+y_j \notin P_1$ are among the conjuncts of $\varphi(x,y)$, and so y_j is among the components of $t(y)$ or $t'(y)$. Then we obtain for each prime p a p-condition $\theta_p(x,z,z')$ such that $\theta_p(x,t(y),t(y'))$ is logically equivalent to $\eta_p(x,y)$. Let $\xi(x,z,z')$ be the formula

$$\bigwedge_p \theta_p(x,z,z') \wedge \bigwedge_{i=1}^{n}(k_i x + z_i \in P_{m_i}) \wedge \bigwedge_{i=1}^{n'}(k'_i x + z'_i \notin P_{m'_i}).$$

Clearly, $\varphi(x,y)$ is equivalent to the formula $\rho(y) \wedge \varepsilon(x,y) \wedge \xi(x,t(y),t'(y))$, so we can assume that $\varphi(x,y)$ is the latter.

We need a small observation. For a p-condition $\theta_p(z)$ and $h \neq 0$, we will show that there is another p-condition $\eta_p(z)$ such that over $\mathrm{Sf}^*_{\mathbb{Z}}$ and $\mathrm{Sf}^*_{\mathbb{Q}}$,

$$\eta_p(z_1, \dots, z_{i-1}, hz_i, z_{i+1}, \dots, z_n) \text{ is equivalent to } \theta_p(z).$$

For the special case where $\theta_p(z)$ is $t(z) \in U_{p,l}$, the conclusion follows from Lemma 3.1(iii), Lemma 3.4(iii) and the fact that there is an L^*_{u}-term $t'(z)$ such that $t'(z, \dots, z_{i-1}, hz_i, z_{i+1}, \dots, z_n) = ht(z)$. The statement of the paragraph follows easily from this special case.

With $\varphi(x, y)$ as in the end of the second paragraph, we further reduce the main statement to the special case where there is $k \neq 0$ such that $k_i = k'_{i'} = k$ for all $i \in \{1, \dots, n\}$ and $i' \in \{1, \dots, n'\}$. Choose $k \neq 0$ to be a common multiple of $k_1, \dots, k_n$ and $k'_1, \dots k'_{n'}$. Then by Lemma 3.1(vi) and Lemma 3.4(iii), we have for each $i \in \{1, \dots, n\}$ that

$$k_i x + z_i \in P_{m_i} \text{ is equivalent to } (kx + kk_i^{-1} z_i \in P_{kk_i^{-1} m_i}) \text{ over either } \mathrm{Sf}^*_{\mathbb{Z}} \text{ or } \mathrm{Sf}^*_{\mathbb{Q}}.$$

We have a similar observation for k and $k'_{i'}$ with $i' \in \{1, \dots, n'\}$. The desired reduction easily follows from these observations and the preceding paragraph.

Continuing with the reduction in the preceding paragraph, we next arrange that there is $m > 0$ such that $m_i = m'_{i'} = m$ for all $i \in \{1, \dots, n\}$ and $i' \in \{1, \dots, n'\}$. Let m be a common multiple of $m_1, \dots, m_n$ and $m'_1, \dots m'_{n'}$. By Lemma 3.1(v, vi) and Lemma 3.4(iii), we have for $i \in \{1, \dots, n\}$ that over either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$

$$kx + z_i \in P_{m_i} \text{ is equivalent to } kx + z_i \in P_m \wedge \bigwedge_{p \mid \frac{m}{m_i}} kx + z_i \notin U_{p, 2+v_p(m_i)}$$

and for $i' \in \{1, \dots, n'\}$ that over either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$

$$kx + z'_{i'} \notin P_{m'_{i'}} \text{ is equivalent to } kx + z'_{i'} \notin P_m \vee \bigvee_{p \mid \frac{m}{m'_{i'}}} kx + z'_{i'} \in U_{p, 2+v_p(m'_{i'})}.$$

It follows that $\varphi(x, y)$ is equivalent to a disjunction of formulas of the form we are aiming for. The desired conclusion of the lemma follows from the disjunction observation. $\square$

Corollary 3.3. *Suppose $\varphi(x, y)$ is a quantifier-free L^*_{ou} formula. Then $\varphi(x, y)$ is equivalent over $\mathrm{OSf}^*_{\mathbb{Q}}$ to a disjunction of quantifier-free formulas of the form*

$$\rho(y) \wedge \lambda(x, y) \wedge \psi(x, t(y), t'(y))$$

where

(i) *$t(y)$ and $t'(y)$ are tuples of L^*_{ou}-terms with length n and n' respectively;*

(ii) *$\rho(y)$ is a quantifier-free L^*_{ou}-formula, $\lambda(x, y)$ an order condition, $\psi(x, z, z')$ a special formula.*

In the next lemma, we show a "local quantifier elimination" result.

Lemma 3.11. *If $\varphi(x,z)$ is a p-condition, then over either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$, the formula $\exists x\varphi(x,z)$ is equivalent to a p-condition $\psi(z)$.*

Proof. If $\varphi(x,z)$ is a p-condition, then it is a boolean combination of atomic formulas of the form $kx+t(z)\in U_{p,l}$ where $t(z)$ is an L^*_{u}-term. Let l_p be the largest value of l occurring in an atomic formula in $\varphi(x,z)$ and $S=\{1\leqslant m<p^{l_p}\mid \exists x\varphi(x,m)\}$. Then by Lemma 3.1 (i), $\exists x\varphi(x,z)$ is equivalent to the p-condition $\bigvee_{m\in S}(z\equiv_{p^l} m)$ over $\mathrm{Sf}^*_{\mathbb{Q}}$.

Now, we proceed to prove the statement for models of $\mathrm{Sf}^*_{\mathbb{Q}}$. Throughout the rest of the proof, suppose $\varphi(x,z)$ is a p-condition, k, k', l, l' are in $\mathbb{Z}$, and $t(z),t'(z)$ are L^*_{u}-terms. First, we consider the case where $\varphi(x,z)$ is a p-condition of the form $kx+t(z)\in U_{p,l}$. The case $k=0$ is trivial. If $k\neq 0$, then $\exists x(kx+t(z)\in U_{p,l})$ is tautological over $\mathrm{Sf}^*_{\mathbb{Q}}$ following from (Q1) in the definition of $\mathrm{Sf}^*_{\mathbb{Q}}$ and Lemma 3.4(i).

We next consider the case where $\varphi(x,z)$ is a finite conjunction of p-conditions in $L^*_{\mathrm{u}}(x,z)$ such that one of the conjuncts is $kx+t(z)\in U_{p,l}$ with $k\neq 0$ and the other conjuncts are either of the form $k'x+t'(z)\in U_{p,l'}$ or of the form $k'x+t'(z)\notin U_{p,l'}$ where we do allow l' to vary. It follows from Lemma 3.4(i) that if $k=k'$, $l\geqslant l'$, then

$$k'x+t'(z)\in U_{p,l'} \text{ if and only if } t(z)-t'(z)\in U_{p,l'}.$$

So we have means to replace conjuncts of $\varphi(x,z)$ by terms independent of the variable x. However, the above will not work if $k\neq k'$ or $l<l'$. By Lemma 3.4(iii), across models of $\mathrm{Sf}^*_{\mathbb{Q}}$, we have that

$$kx+t(z)\in U_{p,l} \text{ if and only if } hkx+ht(z)\in U_{p,l+v_p(h)} \quad \text{for all } h\neq 0.$$

From this observation, it is easy to see that we can resolve the issue of having $k\neq k'$. By Lemma 3.4(i,ii), across models of $\mathrm{Sf}^*_{\mathbb{Q}}$, we have that

$$kx+t(z)\in U_{p,l} \text{ if and only if } \bigvee_{i=1}^{p^m} kz+t(z)+ip^l\in U_{p,l+m} \text{ for all } l\geqslant 0 \text{ and all } m.$$

Using the preceding two observations we resolve the issue of having $l<l'$. The statement of the lemma for this case then follows from the first paragraph.

We now prove the full lemma. It suffices to consider the case where $\varphi(x,z)$ is a conjunction of atomic formulas. In view of the preceding paragraph, we reduce further to the case where $\varphi(x,z)$ is of the form

$$\bigwedge_{i=1}^{m} kx+t_i(z)\notin U_{p,l_i}$$

We now show that $\exists\varphi(x,z)$ is a tautology over $\mathrm{Sf}^*_{\mathbb{Q}}$ and thus complete the proof. Suppose $(G,\mathcal{U}^G,\mathcal{P}^G)\models \mathrm{Sf}^*_{\mathbb{Q}}$ and $c\in G^n$. It suffices to find $a\in G$ such that the p-condition $ka+t_i(c)\notin U^G_{p,l_i}$ holds for all $i\in\{1,\dots,m\}$. Without loss of generality, we assume that $t_1(c),\dots,t_{m'}(c)$

are not in $U^G_{p,l}$ for all l and that $t_{m'+1}(c), \dots, t_m(c)$ are in U^G_{p,l_0} for some l_0 such that $l_0 < l_i$ for all $i \in \{1, \dots, m\}$. Using 3.4(ii), choose a such that $ka \in U^G_{p,l_0-1} \setminus U^G_{p,l_0}$. It follows from Lemma 3.4(i) that a is as desired. $\square$

Theorem 3.2. *The theories* $\mathrm{SF}^*_{\mathbb{Z}}$*,* $\mathrm{SF}^*_{\mathbb{Q}}$*, and* $\mathrm{OSF}^*_{\mathbb{Q}}$ *admit quantifier elimination.*

Proof. As the three situations are very similar, we will only present here the proof that $\mathrm{OSF}^*_{\mathbb{Q}}$ admits quantifier elimination. The proof for $\mathrm{SF}^*_{\mathbb{Z}}$ and $\mathrm{SF}^*_{\mathbb{Q}}$ are simpler as there is no ordering involved. Along the way we point out the necessary modifications needed to get the proof for $\mathrm{SF}^*_{\mathbb{Z}}$ and $\mathrm{SF}^*_{\mathbb{Q}}$. Fix $\mathrm{OSF}^*_{\mathbb{Q}}$-models $(G, <, \mathcal{U}^G, \mathcal{P}^G)$ and $(H, <, \mathcal{U}^H, \mathcal{P}^H)$ such that the latter is $|G|^+$-saturated. Suppose

$$f \text{ is a partial } L^*_{\mathrm{ou}}\text{-embedding from } (G, <, \mathcal{U}^G, \mathcal{P}^G) \text{ to } (H, <, \mathcal{U}^H, \mathcal{P}^H),$$

in other words, f is an L^*_{ou}-embedding of an L^*_{ou}-substructure of $(G, <, \mathcal{U}^G, \mathcal{P}^G)$ into $(H, <, \mathcal{U}^H, \mathcal{P}^H)$. By a standard test, it suffices to show that if $\mathrm{Domain}(f) \neq G$, then there is a partial L^*_{ou}-embedding from $(G, <, \mathcal{U}^G, \mathcal{P}^G)$ to $(H, <, \mathcal{U}^H, \mathcal{P}^H)$ which properly extends f. For the corresponding statements with $\mathrm{SF}^*_{\mathbb{Z}}$ or $\mathrm{SF}^*_{\mathbb{Q}}$, we need to consider instead $(G, \mathcal{U}^G, \mathcal{P}^G)$ and $(H, \mathcal{U}^H, \mathcal{P}^H)$ depending on the situation.

We remind the reader that our choice of language includes a symbol for additive inverse, and so $\mathrm{Domain}(f)$ is automatically a subgroup of G. Suppose $\mathrm{Domain}(f)$ is not a pure subgroup of G, that is, there is an element $\mathrm{Domain}(f)$ which is n-divisible in G but not n-divisible in $\mathrm{Domain}(f)$ for some $n > 0$. Then there is p and a in $G \setminus \mathrm{Domain}(f)$ such that $pa \in \mathrm{Domain}(f)$. Using divisibility of H, we get $b \in H$ such that $pb = f(pa)$. Let g be the extension of f given by

$$ka + a' \mapsto kb + f(a') \quad \text{ for } k \in \{1, \dots, p-1\} \text{ and } a' \in \mathrm{Domain}(f).$$

It is routine to check that g is an ordered group isomorphism from $\langle \mathrm{Domain}(f), a\rangle$ to $\langle \mathrm{Image}(f), b\rangle$. It is also easy to check using Lemma 3.4(iii) that $ka + a' \in U^G_{p,l}$ if and only if $kb + f(a') \in U^G_{p,l}$ and $ka + a' \in P^G_m$ if and only if $kb + f(a') \in U^G_m$ for all k, l, m, and $a' \in \mathrm{Domain}(f)$. Hence,

$$g \text{ is a partial } L^*_{\mathrm{ou}}\text{-embedding from } (G, <, \mathcal{U}^G, \mathcal{P}^G) \text{ to } (H, <, \mathcal{U}^H, \mathcal{P}^H).$$

Clearly, g properly extends f, so the desired conclusion follows. The proof for $\mathrm{SF}^*_{\mathbb{Q}}$ is the same but without the verification that the ordering is preserved. The situation for $\mathrm{SF}^*_{\mathbb{Z}}$ is slightly different as H is not divisible. However, pa is in $pG = U^G_{p,1}$, and so $f(pa)$ is in $U^H_{p,1} = pH$. The proof proceeds similarly using 3.1(4-6).

The remaining case is when $\mathrm{Domain}(f) \neq G$ is a pure subgroup of G. Let a be in $G \smallsetminus \mathrm{Domain}(f)$. We need to find b in $H \smallsetminus \mathrm{Image}(f)$ such that

$$\mathrm{qftp}_{L^*_{\mathrm{ou}}}(a \mid \mathrm{Domain}(f)) = \mathrm{qftp}_{L^*_{\mathrm{ou}}}(b \mid \mathrm{Image}(f)).$$

By the fact that $\mathrm{Domain}(f)$ is pure in G, and Corollary 3.3, $\mathrm{qftp}_{L^*_{\mathrm{ou}}}(a \mid \mathrm{Domain}(f))$ is isolated by formulas of the form

$$\rho(b) \wedge \lambda(x, b) \wedge \psi(x, t(b), t'(b))$$

where $\rho(y)$ is a quantifier-free L^*_{ou}-formula, $\lambda(x, y)$ is an order condition, $\psi(x, z, z')$ a special formula, $t(y)$ and $t'(y)$ are tuples of L^*_{ou}-terms of suitable length, b is a tuple of elements of $\mathrm{Domain}(f)$ of suitable length, and $\psi(x, t(b), t'(b))$ is a nontrivial $\mathrm{Domain}(f)$-system. As $\mathrm{Domain}(f)$ is a pure subgroup of G, we can moreover arrange that $\lambda(x, b)$ is simply the formula $b_1 < x < b_2$. Since f is an L^*_{ou}-embedding, $\rho(f(b))$ holds, $f(b_1) < f(b_2)$, and $\psi\big(x, t(f(b)), t'(f(b))\big)$ is a nontrivial $\mathrm{Image}(f)$-system. Using the fact that $(H, <, \mathcal{U}^H, \mathcal{P}^H)$ is $|G|^+$-saturated, the problem reduces to showing that

$$\psi\Big(x, f\big(t(b)\big), f\big(t'(b)\big)\Big) \text{ has a solution in the interval } [f(b_1), f(b_2)]^H.$$

As $\psi(x, t(b), t'(b))$ is satisfiable in G, it is locally satisfiable in G by Lemma 3.5. For each p, let $\psi_p(x, z, z')$ be the associated p-condition of $\psi(x, z, z')$. By Lemma 3.11, for all p, the formula $\exists x \psi_p(x, z, z')$ is equivalent over $\mathrm{Sf}^*_{\mathbb{Q}}$ to a quantifier free formula in $L^*_{\mathrm{u}}(z, z')$. Hence, $\exists x \psi_p\big(x, f(c), f(c')\big)$ holds in $(H, <, \mathcal{U}^H, \mathcal{P}^H)$ for all p. Thus,

$$\text{the } \mathrm{Image}(f)\text{-system } \psi\Big(x, f\big(t(b)\big), f\big(t'(b)\big)\Big) \text{ is locally satisfiable in } H.$$

The desired conclusion follows from the genericity of $(H, <, \mathcal{U}^H, \mathcal{P}^H)$. The proofs for $\mathrm{SF}^*_{\mathbb{Z}}$ and $\mathrm{SF}^*_{\mathbb{Q}}$ are similar. However, we have there the formula $\bigwedge_{i=1}^k x \neq b_i$ with $k \leqslant |b|$ instead of the formula $b_1 < x < b_2$, Lemma 3.10 instead of Corollary 3.3, and the corresponding notion of genericity instead of the current one. ☐

Corollary 3.4. *The theory* $\mathrm{SF}^*_{\mathbb{Z}}$ *is a recursive axiomatization of* $\mathrm{Th}(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$*, and is therefore decidable. Similar statements hold for* $\mathrm{SF}^*_{\mathbb{Q}}$ *in relation to* $\mathrm{Th}(\mathbb{Q}, \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$ *and* $\mathrm{OSF}^*_{\mathbb{Q}}$ *in relation to* $\mathrm{Th}(\mathbb{Q}, < \mathcal{U}^{\mathbb{Q}}, \mathcal{P}^{\mathbb{Q}})$.

Proof. By Lemma 3.1(ii), the relative divisible closure of 1 in an arbitrary model $(G, \mathcal{U}^G, \mathcal{P}^G)$ of $\mathrm{SF}^*_{\mathbb{Z}}$ is an isomorphic copy of $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$. Hence by Theorem 3.2, $\mathrm{SF}^*_{\mathbb{Z}}$ is complete, and on the other hand $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}}) \models \mathrm{SF}^*_{\mathbb{Z}}$ by Theorem 3.1. The first statement of the corollary follows. The justification of the second statement is obtained in a similar fashion. ☐

We will next deduce consequence for the structures $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$, $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$, and $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ in the original language.

Theorem 3.3. *The theory of* $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ *is model complete and decidable.*

Proof. For all p, $l \geqslant 0$, $m > 0$, and all $a \in \mathbb{Z}$, we have the following:

(1) $a \in U^{\mathbb{Z}}_{p,l}$ if and only there is $b \in \mathbb{Z}$ such that $p^l b = a$;
(2) $a \notin U^{\mathbb{Z}}_{p,l}$ if and only if for some $i \in \{1, \dots, p^l - 1\}$, there is $b \in \mathbb{Z}$ such that $p^l b = a + i$;
(3) $a \in P^{\mathbb{Z}}_m$ if and only if for some $d \mid m$, there is $b \in \mathbb{Z}$ such that $a = bd$ and $b \in \mathrm{SF}^{\mathbb{Z}}$;
(4) $a \notin P^{\mathbb{Z}}_m$ if and only if for all $d \mid m$, either for some $i \in \{1, \dots, d-1\}$, there is $b \in \mathbb{Z}$ such that $db = a + i$ or there is $b \in \mathbb{Z}$ such that $a = bd$ and $b \notin \mathrm{SF}^{\mathbb{Z}}$.

As $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}}) \vDash \mathrm{SF}^*_{\mathbb{Z}}$, it then follows from Theorem 3.2 and the above observation that every 0-definable set in $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ is existentially 0-definable. Hence, the theory of $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ is model complete. The decidability of $\mathrm{Th}(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ is immediate from the preceding corollary. $\square$

Lemma 3.12. *Suppose* $a \in \mathbb{Q}$ *has* $v_p(a) < 0$. *Then there is* $\varepsilon \in \mathbb{Q}$ *such that* $v_p(\varepsilon) \geqslant 0$ *and* $a + \varepsilon \in \mathrm{SF}^{\mathbb{Q}}$.

Proof. Suppose a is as stated. If $a \in \mathrm{SF}^{\mathbb{Q}}$ we can choose $\varepsilon = 0$, so suppose a is in $\mathbb{Q} \setminus \mathrm{SF}^{\mathbb{Q}}$. We can also arrange that $a > 0$. Then there are $m, n, k \in \mathbb{N}^{\geqslant 1}$ such that

$$a = \frac{m}{np^k}, \ (m, n) = 1, \ (m, p) = 1, \text{ and } (n, p) = 1.$$

It suffices to show there is $b \in \mathbb{Z}$ such that $m + p^k b$ is a square-free integer as then

$$a + \frac{b}{n} = \frac{m + p^k b}{np^k} \in \mathrm{SF}^{\mathbb{Q}}.$$

For all prime l, it is easy to check that there is $b_l \in \mathbb{Z}$ such that $p^k b_l + m \notin U^{\mathbb{Q}}_{l,2}$. The conclusion then follows from the genericity of $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$ as established in Theorem 3.1. $\square$

Corollary 3.5. *For all* p *and* l, $U^{\mathbb{Q}}_{p,l}$ *is universally* 0*-definable in* $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$.

Proof. We will instead show that $\mathbb{Q} \setminus U^{\mathbb{Q}}_{p,l} = \{a : v_p(a) < l\}$ is existentially 0-definable for all p and l. As $\mathbb{Q} \setminus U^{\mathbb{Q}}_{p,l+n} = p^n(\mathbb{Q} \setminus U^{\mathbb{Q}}_{p,l})$ for all p, l, and n, it suffices to show the statement for $l = 0$. Fix a prime p. By the preceding lemma we have that for all a, $v_p(a) < 0$ if and only if

$$\text{there is } \varepsilon \text{ such that } v_p(\varepsilon) \geqslant 0, a + \varepsilon \in \mathrm{SF}^{\mathbb{Q}} \text{ and } v_p(a + \varepsilon) < 0.$$

We recall that $\{\varepsilon : v_p(\varepsilon) \geqslant 0\}$ is existentially 0-definable by Lemma 3.3. Also, for all $a' \in \mathrm{SF}^{\mathbb{Q}}$, we have that $v_p(a') < 0$ is equivalent to $p^2 a' \in \mathrm{SF}^{\mathbb{Q}}$. The conclusion hence follows. $\square$

Theorem 3.4. *The theories of* $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ *and* $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ *are model complete and decidable.*

Proof. We show that the L_{u}-theory of $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ and L_{ou}-theory of $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ is model complete and decidable. The proof is almost exactly the same as that of part 1 of Theorem 1.2. It follows from Lemma 3.3 and Corollary 3.5 that for all p and l, the sets $U^{\mathbb{Q}}_{p,l}$ are existentially and universally 0-definable in $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$. For all m, $P_m^{\mathbb{Q}} = m\mathrm{SF}^{\mathbb{Q}}$ and $\mathbb{Q} \smallsetminus P_m^{\mathbb{Q}} = m(\mathbb{Q} \smallsetminus \mathrm{SF}^{\mathbb{Q}})$ are clearly existentially 0-definable. The conclusion follows. $\square$

Next, we will show that the L_{ou}-theory of $(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$ is bi-interpretable with arithmetic. The proof follow closely the arguments from [**6**]. In fact, we can slightly modify Corollary 3.6 to use essentially the same proof at the cost of replacing n^2 with $n^2 + n$.

Lemma 3.13. *Let $c_1, \ldots, c_n$ be an increasing sequence of natural numbers, assume that for all primes p, there is a solution to the system of congruence inequations*

$$x + c_i \notin U_{p,2}^{\mathbb{Z}} \text{ for all } i \in \{1, \ldots, n\}.$$

Then there is $a \in \mathbb{N}$ such that $a + c_1, \ldots, a + c_n$ are consecutive square-free integers.

Proof. Suppose $c_1, \ldots, c_n$ are as given. Let $c'_1, \ldots, c'_{n'}$ be the listing in increasing order of elements in the set of $c \in \mathbb{N}$ such that $c_1 \leqslant c \leqslant c_n$ and $c \neq c_i$ for $i \in \{1, \ldots, n\}$. The conclusion that there are infinitely many a such that

$$\bigwedge_{i=1}^{n} (a + c_i \in \mathrm{SF}^{\mathbb{Z}}) \wedge \bigwedge_{i=1}^{n'} (a + c'_i \notin \mathrm{SF}^{\mathbb{Z}})$$

follows from the assumptions about $c_1, \ldots, c_n$ and the genericity of $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$ as established in Theorem 3.1. $\square$

Corollary 3.6. *For all $n \in \mathbb{N}^{>0}$, there is $a \in \mathbb{N}$ such that $a+1, a+4, \ldots, a+n^2$ are consecutive square-free integers .*

Proof. For each p, we can obtain $a \in \{1, 2, \ldots, p^2 - 1\}$ such that

$$a \not\equiv_{p^2} -m^2 \text{ for all } m.$$

Hence, for any given $n > 0$ and p, the p-condition $\bigwedge_{i=1}^{n} (x + i^2 \notin U_{p,2}^{\mathbb{Z}})$ has a solution. The result now follows immediately from the preceding lemma. $\square$

Theorem 3.5. *The theory of $(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$ defines multiplication.*

Proof. It suffices to show that $(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$ interprets multiplication on $\mathbb{N}$. Let T be the set of $(a, b) \in \mathbb{N}^2$ such that for some $n \in \mathbb{N}^{\geqslant 1}$,

$$b = a + n^2 \text{ and } a + 1, a + 4, \ldots, a + n^2 \text{ are consecutive square-free integers.}$$

The set T is definable in $(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$ as $(a, b) \in T$ and $b \neq a + 1$ if and only if $a < b$, $a + 1$ and $a + 4$ are consecutive square-free integers, b is square-free, and whenever c, d, and e are consecutive square-free integers with $a < c < d < e \leqslant b$, we have that

$$(e - d) - (d - c) = 2.$$

Let S be the set $\{n^2 : n \in \mathbb{N}\}$. If $c = 0$ or there are a, b such that $(a,b) \in T$ and $b - a = c$, then $c = n^2$ for some n. Conversely, if $c = n^2$, then either $c = 0$ or by Corollary 3.6,

$$\text{there is } (a,b) \in T \text{ with } b - a = c.$$

Therefore, S is definable in $(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$. The map $n \mapsto n^2$ in $\mathbb{N}$ is definable in $(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$ as $b = a^2$ if and only if $b \in S$ and whenever $c \in S$ is such that $c > b$ and b, c are consecutive in S, we have that $c - b = 2a + 1$. Finally, $c = ba$ if and only if $2c = (b+a)^2 - b^2 - a^2$. Thus, multiplication on $\mathbb{N}$ is definable in $(\mathbb{Z}, <, \mathrm{SF}^{\mathbb{Z}})$. $\square$

3.3. Combinatorial Tameness

As the theories $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ are complete, it is convenient to work in the so-called monster models, that is, models which are very saturated and homogeneous. Until the end of the chapter, let $(\mathbb{G}, \mathcal{U}^{\mathbb{G}}, \mathcal{P}^{\mathbb{G}})$ be a monster model of either $\mathrm{SF}^*_{\mathbb{Z}}$ or $\mathrm{SF}^*_{\mathbb{Q}}$ depending on the situation. In the latter case, we suppose $(\mathbb{G}, <, \mathcal{U}^{\mathbb{G}}, \mathcal{P}^{\mathbb{G}})$ is a monster model of $\mathrm{OSF}^*_{\mathbb{Q}}$. We assume that κ, A and I have small cardinalities compared to $\mathbb{G}$.

Our general strategy to prove the tameness of $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ is to link them to the corresponding "local" facts. The next lemma says that $\mathrm{SF}^*_{\mathbb{Z}}$ is "locally" supersimple of U-rank 1.

Lemma 3.14. *Suppose* $(\mathbb{G}, \mathcal{U}^{\mathbb{G}}, \mathcal{P}^{\mathbb{G}}) \vDash \mathrm{SF}^*_{\mathbb{Z}}$, $\theta_p(x,y)$ *is a consistent p-condition, and b is in* $\mathbb{G}^{|y|}$. *Then* $\theta_p(x,b)$ *does not divide over any base set* $A \subseteq \mathbb{G}$.

Proof. Suppose $(\mathbb{G}, \mathcal{U}^{\mathbb{G}}, \mathcal{P}^{\mathbb{G}})$, $\theta_p(x,b)$ are as stated, and A is a small subset of $\mathbb{G}$. Suppose I is an infinite ordered set and $(\sigma_i)_{i\in I}$ a family of L^*_{u}-automorphisms of $(\mathbb{G}, \mathcal{U}^{\mathbb{G}}, \mathcal{P}^{\mathbb{G}})$ such that $(\sigma_i b)_{i\in I}$ is indiscernible over A. By the monstrosity of $\mathbb{G}$, the problem reduces to showing that the set $\{\theta_p(x, \sigma_i b) : i \in I\}$ is consistent. It is easy to see from Lemma 3.1(i, ii) that for some l, $\theta_p(x,b)$ defines a nonempty finite union of translations of $U^{\mathbb{G}}_{p,l}$, which is a set definable over the empty-set. Then $\theta_p(x, \sigma_i b)$ defines the same set for all $i \in I$, and so $\bigcap_{i\in I} \theta_p(x, \sigma_i b) \neq \emptyset$. The conclusion follows. $\square$

Theorem 3.6. *The theory of* $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ *is supersimple of U-rank* 1 *and k-independent for all* $k \in \mathbb{N}^{\geqslant 1}$.

Proof. We first show that $\mathrm{Th}(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ is supersimple of U-rank 1; see [**44**, p. 36] for a definition of U-rank or SU-rank. By the fact that $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ have the same definable sets as $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{G}})$ and Corollary 3.4, we can replace $\mathrm{Th}(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ with $\mathrm{SF}^*_{\mathbb{Z}}$. Suppose $(\mathbb{G}, \mathcal{U}^{\mathbb{G}}, \mathcal{P}^{\mathbb{G}}) \vDash \mathrm{SF}^*_{\mathbb{Z}}$. Our job is to show that every $L^*_{\mathrm{u}}(\mathbb{G})$-formula $\varphi(x,b)$ which forks over a small subset A of $\mathbb{G}$ must define a finite set in $\mathbb{G}$. We can easily reduce to the case that $\varphi(x,b)$ divides over A. Moreover, we can assume that $\varphi(x,b)$ is quantifier free by Theorem 3.2 which states that $(\mathbb{G},\mathcal{U}^{\mathbb{G}},\mathcal{P}^{\mathbb{G}})$ admits quantifier elimination. Using Lemma 3.10, we can also arrange that $\varphi(x,b)$ has the form

$$\rho(b)\wedge\varepsilon(x,b)\wedge\psi(x,t(b),t'(b))$$

where $\rho(y)$ is a quantifier-free formula, $\varepsilon(x,y)$ is an equational condition, $t(y)$ and $t'(y)$ are tuples of L^*_{u}-terms with length n and n' respectively, and $\psi(x,z,z')$ is a special formula.

Suppose to the contrary that $\varphi(x,b)$ divives over A but $\varphi(x,b)$ defines an infinite set in $\mathbb{G}$. From the first assumption, we get an infinite ordering I and a family $(\sigma_i)_{i\in I}$ of L^*_{u}-automorphisms of $(\mathbb{G},\mathcal{U}^{\mathbb{G}},\mathcal{P}^{\mathbb{G}})$ such that $(\sigma_i b)_{i\in I}$ is indiscernible over A and $\bigwedge_{i\in I}\varphi(x,\sigma_i b)$ is inconsistent. As $\varphi(x,b)$ defines an infinite set in $\mathbb{G}$, we get from the second assumption that $\rho(b)$ holds in $\mathbb{G}$, $\varepsilon(x,b)$ defines a cofinite set in $\mathbb{G}$, and $\psi(x,t(b),t'(b))$ defines an infinite hence non-empty set in $\mathbb{G}$. As $(\sigma_i b)_{i\in I}$ is indiscernible, we have that $\rho(\sigma_i b)$ holds in $\mathbb{G}$ and $\varepsilon(x,\sigma_i b)$ defines a cofinite set in $\mathbb{G}$ for all $i\in I$. Using the saturatedness of $\mathbb{G}$, we get a finite set $\Delta\subseteq I$ such that

$$\theta_\Delta(x) \coloneqq \bigwedge_{i\in\Delta}\psi\big(x,t(\sigma_i b),t'(\sigma_i b)\big)\text{ defines a finite set in }\mathbb{G}.$$

As $\theta_\Delta(x)$ is a conjunction of $\mathbb{G}$-systems given by the same special formula, it is easy to see that $\theta_\Delta(x)$ is also a $\mathbb{G}$-system.

We will show that $\theta_\Delta(x)$ defines an infinite set and thus obtain the desired contradiction. As $(\mathbb{G},\mathcal{U}^{\mathbb{G}},\mathcal{P}^{\mathbb{G}})$ is a model of $\mathrm{SF}^*_{\mathbb{Z}}$ and hence generic, it suffices to show that $\theta_\Delta(x)$ is non-trivial and locally satisfiable. As $\varphi(x,b)$ is consistent, $t(b)$ has no common components with $t'(b)$. The assumption that $(\sigma_i b)_{i\in I}$ is indiscernible gives us that $t(\sigma_i b)$ has no common components with $t'(\sigma_j b)$ for all i and j in I. It follows that $\theta_\Delta(x)$ is non-trivial. For each p, let $\psi_p(x,z,z')$ be the associated p-condition of $\psi(x,z,z')$. For all p, we have that $\psi_p(x,z,z')$ defines a nonempty set and consequently by Lemma 3.14,

$$\bigwedge_{i\in\Delta}\psi_p(x,t(\sigma_i b),t'(\sigma_i b))\text{ defines a nonempty set in }\mathbb{G}.$$

We easily check that the above means $\theta_\Delta(x)$ is p-satisfiable for all p. Thus $\theta_\Delta(x)$ is locally satisfiable which completes our proof that $\mathrm{Th}(\mathbb{Z},\mathrm{SF}^{\mathbb{Z}})$ has U-rank 1.

We will next prove that $\mathrm{Th}(\mathbb{Z},\mathrm{SF}^{\mathbb{Z}})$ is k-independent for all $k>0$; see [**17**] for a definition of k-independence. The proof is almost the exact replica of the proof in [**45**] except the necessary modifications taken in the current paragraph. Suppose $l>0$, S is an arbitrary subset of $\{0,\dots,l-1\}$. Our first step is to show that there are $a,d\in\mathbb{N}$ such that for $t\in\{0,\dots,l-1\}$,

$$a+td\text{ is square-free}\quad\text{if and only if}\quad t\text{ is in }S.$$

Let $n = |S|$ and $n' = l - n$, and let $c \in \mathbb{Z}^n$ be the increasing listing of elements in S and $c' \in \mathbb{Z}^{n'}$ the increasing listing of elements in $\{0, \dots, l-1\} \smallsetminus S$. Choose $d = (l!)^2$. We need to find a such that

$$\bigwedge_{i=1}^{n}(a + c_i d \in \mathrm{SF}^{\mathbb{Z}}) \wedge \bigwedge_{i=1}^{n'}(a + c'_i d \notin \mathrm{SF}^{\mathbb{Z}}).$$

For $p \leqslant l$, if $a_p \notin p^2\mathbb{Z} = U^{\mathbb{Z}}_{p,2}$, then $a_p + c_i d \notin p^2\mathbb{Z}$ for all $i \in \{1, \dots, n\}$. For $p > l$, it is easy to see that $0 + c_i d \notin p^2\mathbb{Z}$ for all $i \in \{1, \dots, n\}$. The desired conclusion follows from the genericity of $(\mathbb{Z}, \mathcal{U}^{\mathbb{Z}}, \mathcal{P}^{\mathbb{Z}})$.

Fix $k > 0$. We construct an explicit L_{u}-formula which witnesses the k-independence of $\mathrm{Th}(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$. Let $y = (y_0, \dots, y_{k-1})$ and let $\varphi(x, y)$ be a quantifier-free L^*_{u}-formula such that for all $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^k$,

$$\varphi(a, b) \text{ if and only if } a + b_0 + \cdots + b_{k-1} \in \mathrm{SF}^{\mathbb{Z}} \quad \text{where } b = (b_0, \dots, b_{k-1}).$$

We will show that for any given $n > 0$, there are families $(a_\Delta)_{\Delta \subseteq \{0,\dots,n-1\}^k}$ and $(b_{ij})_{0 \leqslant i < k, 0 \leqslant j < n}$ of integers such that

$$\varphi(a_\Delta, b_{0,j_0}, \dots, b_{k-1,j_{k-1}}) \text{ if and only if } (j_0, \dots, j_{k-1}) \in \Delta.$$

Let $f : \mathcal{P}(\{0, \dots, n-1\}^k) \to \{0, \dots, 2^{(n^k)} - 1\}$ be an arbitrary bijection. Let g be the bijection from $\{0, \dots, n-1\}^k$ to $\{0, \dots, n^k - 1\}$ such that if b and b' are in $\{0, \dots, n-1\}^k$ and $b <_{\mathrm{lex}} b'$, then $g(b) < g(b')$. More explicitly, we have

$$g(j_0, \dots, j_{k-1}) = j_0 n^{k-1} + j_1 n^{k-2} + \cdots + j_{k-1} \text{ for } (j_0, \dots, j_{k-1}) \in \{0, \dots, n-1\}^k.$$

It follows from the preceding paragraph that we can find an arithmetic progression $(c_i)_{i \in \{0,\dots,n^k 2^{(n^k)}-1\}}$ such that for all $\Delta \subseteq \{0, \dots, n-1\}^k$ and $(j_0, \dots, j_{k-1})$ in $\{0, \dots, n-1\}^k$, we have that

$$c_{f(\Delta)n^k + g(j_0,\dots,j_{k-1})} \in \mathrm{SF}^{\mathbb{Z}} \text{ if and only if } (j_0, \dots, j_{k-1}) \in \Delta.$$

Suppose $d = c_1 - c_0$. Set $b_{ij} = djn^{k-i-1}$ for $i \in \{0, \dots, k-1\}$ and $j \in \{0, \dots, n-1\}$, and set $a_\Delta = c_{f(\Delta)n^k}$ for $\Delta \subseteq \{0, \dots, n-1\}^k$. We have

$$c_{f(\Delta)n^k + g(j_0,\dots,j_{k-1})} = c_{f(\Delta)n^k} + dg(j_0, \dots, j_{k-1}) = a_\Delta + b_{0,j_0} + \cdots + b_{k-1,j_{k-1}}.$$

The conclusion thus follows. □

Lemma 3.15. *Every p-condition $\theta_p(x, y)$ is stable in $\mathrm{SF}^*_{\mathbb{Q}}$.*

Proof. Suppose $\theta_p(x, y)$ is as in the statement of the lemma. It is clear that if $\theta_p(x, y)$ does not contain the variable x, then it is stable. As stability is preserved under taking boolean combinations, we can reduce to the case where $\theta_p(x, y)$ is $kx + t(y) \in U_{p,l}$ with $k \neq 0$. We note that for any b and b' in $\mathbb{G}^{|y|}$, the sets defined by $\theta_p(x, b)$ and $\theta_p(x, b')$ are either the same or disjoint. It follows easily that $\theta_p(x, y)$ does not have the order property; in other words, $\theta_p(x,y)$ is stable. Alternatively, the desired conclusion also follows from the fact that $(\mathbb{Q},\mathcal{U}^{\mathbb{Q}})$ is an abelian structure and hence stable; see [**86**, p. 49] for the relevant definition and result. $\square$

Theorem 3.7. *The theory of* $(\mathbb{Q},\mathrm{SF}^{\mathbb{Q}})$ *is simple but not supersimple, and is k-independent for all $k \in \mathbb{N}^{\geqslant 1}$.*

Proof. We first show that $\mathrm{Th}(\mathbb{Q},\mathrm{SF}^{\mathbb{Q}})$ is simple. By the fact that $(\mathbb{Q},\mathrm{SF}^{\mathbb{Q}})$ has the same definable sets as $(\mathbb{Q},\mathcal{U}^{\mathbb{Q}},\mathcal{P}^{\mathbb{Q}})$ and Corollary 3.4, we can replace $\mathrm{Th}(\mathbb{Q},\mathrm{SF}^{\mathbb{Q}})$ with $\mathrm{SF}^*_{\mathbb{Q}}$. Towards a contradiction, suppose that the latter is not simple. We obtain as in a formula $\varphi(x,y)$ witnessing the tree property of $\mathrm{SF}^*_{\mathbb{Q}}$; see [**44**, pp. 24-25] for the definition and proof that this is one of the equivalent characterizations of simplicity. We can arrange that $\varphi(x,y)$ is quantifier-free by Theorem 3.2. Disjunction preserves simplicity of formulas [**9**, pp. 22-23], so using Lemma 3.10 we can arrange that $\varphi(x,y)$ is of the form

$$\rho(y) \wedge \varepsilon(x,y) \wedge \psi(x,t(y),t'(y))$$

where $\rho(y)$ is a quantifier-free L^*_{u}-formula, $\varepsilon(x,y)$ is an equational-condition, $t(y)$ and $t'(y)$ are tuples of L^*_{u}-terms with lengths n and n' respectively, and $\psi(x,z,z')$ is a special formula. Let $(\mathbb{G},\mathcal{U}^{\mathbb{G}},\mathcal{P}^{\mathbb{G}}) \models \mathrm{SF}^*_{\mathbb{Q}}$. Then there is $b \in \mathbb{G}^k$, an uncountable cardinal κ, and a tree $(\sigma_s)_{s\in\omega^{<\kappa}}$ of L^*_{u}-automorphisms of $(\mathbb{G},\mathcal{U}^{\mathbb{G}},\mathcal{P}^{\mathbb{G}})$ with the following properties:

(1) for all $s \in \omega^{<\kappa}$, $\{\varphi(x,\sigma_{s\frown(i)}b) : i \in \omega\}$ is inconsistent;
(2) for all $\hat{s} \in \omega^{\kappa}$, $\{\varphi(x,\sigma_{\hat{s}\restriction\alpha}b) : \alpha < \kappa\}$ is consistent;
(3) for every $\alpha < \kappa$ and $s \in \omega^{\alpha}$, the sequence of trees $\big((\sigma_{s\frown(i)\frown s'}b)_{s'\in\omega^{<\kappa}}\big)_{i\in\omega}$ is indiscernible.

More precisely, we can get b, κ, and $(\sigma_t)_{t\in\omega^{<\kappa}}$ satisfying (1) and (2) from the fact that $\varphi(x,y)$ witness the tree property of $\mathrm{SF}^*_{\mathbb{Q}}$, a standard Ramsey arguments, and the monstrosity of $(\mathbb{G},\mathcal{U}^{\mathbb{G}},\mathcal{P}^{\mathbb{G}})$. We can then arrange that (3) also holds using results in [**48**]; a direct argument is also straightforward.

We deduce the desired contradiction by showing that there is $s \in \omega^{<\kappa}$ such that $\{\varphi(x,\sigma_{s\frown(i)}b) : i \in \omega\}$ is consistent. From (1-3), we get for all $s \in \omega^{<\kappa}$ that $\rho(\sigma_s b)$ holds and $\varepsilon(x,\sigma_s b)$ defines a cofinite set. By montrosity of $\mathbb{G}$, it suffices to find $s \in \omega^{<\kappa}$ such that any finite conjunction of $\{\psi\big(x,t(\sigma_{s\frown(i)}b),t'(\sigma_{s\frown(i)}b)\big) : i \in \omega\}$ defines an infinite set in $\mathbb{G}$. For $s \in \omega^{<\kappa}$ and a finite $\Delta \subseteq \omega$, set

$$\theta_{s,\Delta}(x) \coloneqq \bigwedge_{i\in\Delta} \psi\big(x,t(\sigma_{s\frown(i)}b),t'(\sigma_{s\frown(i)}b)\big).$$

As κ is uncountable, it suffices to show for fixed Δ that for all but countably many $\alpha < \kappa$ and all $s \in \omega^{\alpha}$, the formula $\theta_{s,\Delta}(x)$ defines an infinite set in $\mathbb{G}$.

Note that $\theta_{s,\Delta}(x)$ is a conjunction of $\mathbb{G}$-systems given by the same special formula, so $\theta_{s,\Delta}(x)$ is also a $\mathbb{G}$-system. By the genericity of $\mathrm{SF}^*_{\mathbb{Q}}$ established in Theorem 3.1, we need to check that for all but countably many $\alpha < \kappa$ and all $s \in \omega^\alpha$, the $\mathbb{G}$-system $\theta_{s,\Delta}(x)$ is nontrivial and locally satisfiable. By (2), $\varphi(x,b)$ is consistent, and so is $\psi(x,t(b),t'(b))$. This implies in particular that $t(b)$ and $t'(b)$ have no common components. It then follows from (3) that for $s \in \omega^{<\kappa}$ and $i,j \in \omega$,

$$t(\sigma_{s^\frown(i)}b) \text{ and } t'(\sigma_{s^\frown(j)}b) \text{ have no common elements .}$$

Hence, $\theta_{s,\Delta}(x)$ is nontrivial for all $s \in \omega^{<\kappa}$. Let $\psi_p(x,z,z')$ be the associated p-condition of $\psi(x,z,z')$. We then get from (2) that $\{\psi_p(x,\sigma_{\hat{s}\restriction\alpha}b) : \alpha < \kappa\}$ is consistent for all $\hat{s} \in \omega^\kappa$. By Lemma 3.15, the formula $\psi_p(x,t(y),t'(y))$ is stable and hence does not witness the tree property. It follows that for all but finitely many $\alpha < \kappa$ and all $s \in \omega^\alpha$, the set

$$\{\psi_p(x,t(\sigma_{s^\frown(i)}b),t'(\sigma_{s^\frown(i)}b)) : i \in \omega\} \text{ is consistent.}$$

For such s, we have that $\theta_{s,\Delta}(x)$ is p-satisfiable. So for all but countably many $\alpha < \kappa$ and all $s \in \omega^\alpha$, $\theta_{s,\Delta}(x)$ is locally satisfiable which completes the proof that $\mathrm{Th}(\mathbb{Q},\mathrm{SF}^{\mathbb{Q}})$ is simple.

We next prove that $\mathrm{Th}(\mathbb{Q},\mathrm{SF}^{\mathbb{Q}})$ is not strong which implies that it is not supersimple; for the definition of strength and the relation to supersimplicity see [**1**]. Again, we can replace $\mathrm{Th}(\mathbb{Q},\mathrm{SF}^{\mathbb{Q}})$ by $\mathrm{SF}^*_{\mathbb{Q}}$ using Proposition 3.1 and Corollary 3.4. For each p, let $\varphi_p(x,y)$ with $|y| = 1$ be the formula $x - y \in U_{p,0}$. For all p and n, set $b_{p,i} = p^{-i}$. We will show that $\big(\varphi_p(x,y),(b_{p,i})_{i\in\mathbb{N}}\big)$ forms an inp-pattern of infinite depth in $(\mathbb{Q},\mathcal{U}^{\mathbb{Q}},\mathcal{P}^{\mathbb{Q}})$. For distinct i and j in $\mathbb{N}$, we have that $p^i - p^j \notin U^{\mathbb{Q}}_{p,0}$ which implies that $\varphi_p(x,b_{p,i}) \wedge \varphi_p(x,b_{p,i})$ is inconsistent. On the other hand, if S is a finite set of primes, then by the weak approximation theorem $\bigwedge_{i\in S}\varphi_p(x,b_{p,f(p)})$ is consistent for all $f : S \to \mathbb{N}$. The desired conclusion follows.

Finally, we note that $(\mathbb{Z},\mathcal{U}^{\mathbb{Z}},\mathcal{P}^{\mathbb{Z}})$ is a substructure of $(\mathbb{Q},\mathcal{U}^{\mathbb{Q}},\mathcal{P}^{\mathbb{Q}})$, the former theory admits quantifier elimination and has IP_k for all $k > 0$. Therefore, the latter also has IP_k for all $k > 0$. In fact, the construction in part 2 of the proof of Theorem 3.6 carries through. $\square$

Lemma 3.16. *Any order-condition has* NIP *in* $\mathrm{OSF}^*_{\mathbb{Q}}$.

Proof. The statement immediately follows from the fact that every order condition is a formula in the language of ordered groups and the fact that the reduct of any model of $\mathrm{OSF}^*_{\mathbb{Q}}$ to this language is an ordered abelian group, which has NIP; see for example [**73**, p. 151] $\square$

Theorem 3.8. *The theory* $(\mathbb{Q},<,\mathrm{SF}^{\mathbb{Q}})$ *has* NTP_2 *but is not strong, and is k-independent for all $k \in \mathbb{N}^{\geqslant 1}$.*

Proof. In the proof of part 2 of Theorem 3.7, we have shown that $\mathrm{Th}(\mathbb{Q},\mathrm{SF}^{\mathbb{Q}})$ is not strong and is k-independent for all $k > 0$, so the corresponding conclusions for $\mathrm{Th}(\mathbb{Q},<,\mathrm{SF}^{\mathbb{Q}})$ also follow. It remains to show that $\mathrm{Th}(\mathbb{Q},<,\mathrm{SF}^{\mathbb{Q}})$ has NTP_2. The proof is essentially the same as the proof that $\mathrm{Th}(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$ is simple, but with extra complications coming from the ordering. By Proposition 3.1 and Corollary 3.4, we can replace $\mathrm{Th}(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$ with $\mathrm{OSF}^*_{\mathbb{Q}}$. Towards a contradiction, we obtain as in [**16**, pp. 700-701] a formula $\varphi(x,y)$ witnessing the that $\mathrm{OSF}_{\mathbb{Q}}$ has TP_2. We can arrange that $\varphi(x,y)$ is quantifier-free by Theorem 3.2. Disjunctions of formulas with NTP_2 again have NTP_2[**16**, p. 701], so using Lemma 3.3 we can arrange that $\varphi(x,y)$ is of the form

$$\rho(y) \wedge \lambda(x,y) \wedge \psi(x, t(y), t'(y))$$

where $\rho(y)$ is a quantifier-free L^*_{ou}-formula, $\lambda(x,y)$ an order condition, $\psi(x,z,z')$ a special formula, and $t(y)$ and $t'(y)$ are tuples of L^*_{ou}-terms with length n and n' respectively. Then there is $b \in \mathbb{G}^k$ and an array $(\sigma_{ij})_{i\in\omega, j\in\omega}$ of L^*_{ou}-automorphisms of $(\mathbb{G}, <, \mathcal{U}^{\mathbb{G}}, \mathcal{P}^{\mathbb{G}})$ with the following properties:

(1) for all $i \in \omega$, $\{\varphi(x, \sigma_{ij}b) : j \in \omega\}$ is inconsistent;
(2) for all $f : \omega \to \omega$, $\{\varphi(x, \sigma_{if(i)}b) : i \in \omega\}$ is consistent;
(3) for all $i \in \omega$, $(\sigma_{ij}b)_{j\in\omega}$ is indiscernible over $\{\sigma_{i'j}b : i' \in \omega, i' \neq i, j \in \omega\}$;
(4) the sequence of "rows" $((\sigma_{ij}b)_{j\in\omega})_{i\in\omega}$ is indiscernible.

We could get b, ω, and $(\sigma_{ij})_{i\in\omega,j\in\omega}$ as above from the definition of NTP_2, Ramsey arguments, and the monstrosity of $(\mathbb{G}, \mathcal{U}^{\mathbb{G}}, \mathcal{P}^{\mathbb{G}})$; see also [**16**, p. 697] for the type of argument we need to get (3).

We deduce that the set $\{\varphi(x, \sigma_{ij}b) : j \in \omega\}$ is consistent for all $i \in \omega$, which is the desired contradiction. We get from (2) that $\rho(\sigma_{ij}b)$ holds for all $i \in \omega$ and $j \in \omega$. Hence, it suffices to show for all $i \in \omega$ that

$$\{\lambda(x, \sigma_{ij}b) \wedge \psi(x, t(\sigma_{ij}b), t'(\sigma_{ij}b)) : j \in \omega\} \text{ is consistent.}$$

The order condition $\lambda(x,y)$ has NIP by Lemma 3.16, and so it has NTP_2. Using conditions (2-4), we get that

$$\{\lambda(x, \sigma_{ij}b) : j \in \omega\} \text{ is consistent } \text{ for all } i \in \omega.$$

Hence, any finite conjunction from $\{\lambda(x, \sigma_{ij}b) : j \in \omega\}$ contains an interval for all $i \in \omega$. For $i \in \omega$ and a finite $\Delta \subseteq \omega$, set

$$\theta_{i,\Delta}(x) \coloneqq \bigwedge_{j\in\Delta} \psi\big(x, t(\sigma_{ij}b), t'(\sigma_{ij}b)\big).$$

It suffices to show that $\theta_{i,\Delta}(x)$ defines a non-empty set in every non-empty $\mathbb{G}$-interval.

We have that $\theta_{i,\Delta}(x)$ is a conjunction of $\mathbb{G}$-system given by the same special formula, and so is again a $\mathbb{G}$-system. By the genericity of $\mathrm{OSF}^*_{\mathbb{Q}}$, the problem reduces to showing $\theta_{i,\Delta}(x)$ is nontrivial and locally satisfiable. By (2), $\varphi(x,b)$ is consistent, and so is $\psi(x, t(b), t'(b))$. This implies in particular that $t(b)$ and $t'(b)$ have no common components. It then follows from (3) that for $i \in \omega$ and distinct $j, j' \in \omega$,

$$t(\sigma_{ij} b) \text{ and } t'(\sigma_{ij'} b) \text{ have no common elements.}$$

Hence, $\theta_{i,\Delta}(x)$ is nontrivial for all $i \in \omega$. Let $\psi_p(x, z, z')$ be the associated p-condition of $\psi(x, z, z')$. We then get from (2) that $\{\psi_p(x, \sigma_{if(i)} b) : i \in \omega\}$ is consistent for all $f : \omega \to \omega$. By Lemma 3.15, the formula $\psi_p(x, t(y), t'(y))$ is stable and hence has NTP$_2$. It follows that for all but finitely many $i \in \omega$ the set

$$\{\psi_p(x, t(\sigma_{ij} b), t'(\sigma_{ij} b)) : j \in \omega\} \text{ is consistent.}$$

Combining with (4), we get that $\theta_{i,\Delta}(x)$ is p-satisfiable for all p which completes the proof. □

Corollary 3.7. *The set* $\mathbb{Z}$ *is not definable in* $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$*.*

Proof. Towards a contradiction, suppose $\mathbb{Z}$ is definable in $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$. Then by Theorem 3.5, $(\mathbb{N}, +, \times, <, 0, 1)$ is interpretable in $(\mathbb{Q}, <, \mathrm{SF}^{\mathbb{Q}})$. It then follows from Theorem 3.8 that $(\mathbb{N}, +, \times, <, 0, 1)$ has NTP$_2$, but this is well-known to be false. □

CHAPTER 4

A family of dp-minimal expansions of the additive group $\mathbb{Z}$

We treat this chapter as the continuation of the corresponding summary in the introduction and keep the notational conventions, definitions, and statements of theorems given there. We also assume throughout this chapter that G is an additive abelian group, $(H,<)$ is a linearly ordered additive abelian group, j range over the integers, the operation $+$ on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathbb{Q}$, and $\mathbb{R}$ are the standard ones, and likewise for $\times$ and the ordering $<$ on all the above except $\mathbb{Z}^2$. If $<$ is a linear order on a set M and $a, b \in M$, set $[a,b)_M = \{t \in M : a \leqslant t < b\}$, likewise for other intervals.

A circular order $\lhd$ on the underlying set of an additive abelian group G is **additive** if $\lhd$ is preserved under the group operation. In this case, we call the combined structure $(G, \lhd)$ a **circularly ordered abelian group**.

Suppose u is an element in $H^{>0}$ such that $(nu)_{n>0}$ is cofinal in $(H,<)$, and $\pi : H \to G$ induces an isomorphism from $H/\langle u\rangle$ to G. Define the relation $\lhd$ on G by:

$$\lhd(\pi(a), \pi(b), \pi(c)) \text{ if } a<b<c \text{ or } b<c<a \text{ or } c<a<b \quad \text{for } a,b,c \in [0,u)_H.$$

We can easily check that $\lhd$ is an additive circular ordering on G. We call $(H,u,<)$ as above a **universal cover** of $(G,\lhd)$ and π a **covering map**.

The above three definitions were already given for multiplicative groups in Section 1.2 and Section 2.1.2. Even though there is no real difference between additive and multiplicative abelian groups, we decided to repeat the definitions as we find it mentally helpful to distinct the additive and multiplicative cases. Lemma 4.1 below is simply a restatement of Proposition 2.4, so we will not provide a proof.

Lemma 4.1. *Suppose $(G,\lhd)$ is a circularly ordered abelian group. Then $(G,\lhd)$ has a universal cover $(H,u,<)$ which is unique up to unique isomorphism. Moreover, $(G,\lhd)$ is isomorphic to $([0,u)_H; \tilde{+}, \tilde{\lhd})$ where $\tilde{+}$ and $\tilde{\lhd}$ are definable in $(H,u,<)$.*

Lemma 4.1 gives us a correspondence between additive circular orderings on $\mathbb{Z}$ and additive linear orderings on $\mathbb{Z}^2$:

Proposition 4.1. *Let $(\mathbb{Z},\lhd)$ be a circularly ordered group. Then there is a linear order $<$ on $\mathbb{Z}^2$ such that a universal cover of $(\mathbb{Z},\lhd)$ is $(\mathbb{Z}^2,u,<)$ with $u=(1,0)$.*

Proof. Suppose $(\mathbb{Z}, \triangleleft)$ is as above and $(H, u, <)$ is its universal cover. Then $\mathbb{Z}$ is $(H/\langle u\rangle$. Using also the fact that $\langle u\rangle$ is isomorphic to $\mathbb{Z}$, we arrange that H is $\mathbb{Z}^2$. Choose $v \in \mathbb{Z}^2$ such that v is mapped to 1 in $\mathbb{Z}$ under the quotient map. Then $\langle u, v\rangle = \mathbb{Z}^2$, and so by a change of basis we can arrange that $u = (1, 0)$. ☐

The dp-minimality of the circularly ordered groups $(\mathbb{Z}, \triangleleft)$ can be established rather quickly using a criterion in [**41**]:

Proof of Theorem 1.7. By the last statement of Lemma 4.1 and Proposition 4.1, it suffices to check that every linearly ordered group $(\mathbb{Z}^2, <)$ is dp-minimal. We have that

$$|\mathbb{Z}^2/n\mathbb{Z}^2| = n^2 < \infty.$$

The desired conclusion follows from the criterion in [**41**, Proposition 5.1]. ☐

So far it is still possible that every circularly ordered group $(\mathbb{Z}, <)$ is a reduct of a known dp-minimal expansion of $\mathbb{Z}$. Toward showing that this is not the case, we need a more explicit description of the additive circular orders on $\mathbb{Z}$.

Define the circular ordering $\triangleleft_+$ on $\mathbb{Z}$ by setting $\triangleleft_+(j, k, l)$ if and only if $j < k < l$ or $l < j < k$ or $k < l < j$. We define the opposite circular ordering $\triangleleft_-$ on $\mathbb{Z}$ by setting

$$\triangleleft_-(j, k, l) \text{ if and only if } \triangleleft_+(-j, -k, -l).$$

We observe that $\triangleleft_+$ and $\triangleleft_-$ are distinct, but $(\mathbb{Z}, \triangleleft_+)$ and $(\mathbb{Z}, \triangleleft_-)$ are isomorphic via the map $k \mapsto -k$ and both have $(\mathbb{Z}^2, <_{\text{lex}})$ as a universal cover where $<_{\text{lex}}$ is the usual lexicographic ordering on $\mathbb{Z}^2$. It is easy to see that both $(\mathbb{Z}, \triangleleft_+)$ and $(\mathbb{Z}, \triangleleft_-)$ are definably equivalent with $(\mathbb{Z}, <)$.

Let $(\mathbb{R}/\mathbb{Z}, \triangleleft)$ be the circularly ordered group with a universal cover $(\mathbb{R}, 1, <)$ and such that $\triangleleft(0 + \mathbb{Z}, 1/4 + \mathbb{Z}, 1/2 + \mathbb{Z})$ holds. We call $(\mathbb{R}/\mathbb{Z}, \triangleleft)$ the **positively oriented circle**. For $a, b \in \mathbb{R}$ such that $a - b \notin \mathbb{Z}$, we set $[a, b)_{\mathbb{R}/\mathbb{Z}}$ to be the set

$$\{t \in \mathbb{R}/\mathbb{Z} : t = a + \mathbb{Z} \text{ or } \triangleleft(a + \mathbb{Z}, t, b + \mathbb{Z})\}.$$

Let α be in $\mathbb{R} \setminus \mathbb{Q}$. Define the additive circular ordering $\triangleleft_\alpha$ on $\mathbb{Z}$ by setting

$$\triangleleft_\alpha(j, k, l) \text{ if and only if } \triangleleft(\alpha j + \mathbb{Z}, \alpha k + \mathbb{Z}, \alpha l + \mathbb{Z}).$$

In other words, $\triangleleft_\alpha$ is the pull-back of $\triangleleft$ by the character $\chi_\alpha : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}, l \mapsto \alpha l + \mathbb{Z}$. As before, we observe that $\triangleleft_\alpha$ and $\triangleleft_{-\alpha}$ are distinct. However, $(\mathbb{Z}+, \triangleleft_\alpha)$ and $(\mathbb{Z}, \triangleleft_{-\alpha})$ are isomorphic via the map $k \mapsto -k$ and both have $(\mathbb{Z}^2, <_\alpha)$ as a universal cover with $<_\alpha$ the pull-back of the ordering $<$ on $\mathbb{R}$ by the group embedding

$$\psi_\alpha : \mathbb{Z}^2 \to \mathbb{R}, \quad (k, l) \mapsto k + \alpha l.$$

We also note that $(\mathbb{Z}^2, <_\alpha)$ is not isomorphic to $(\mathbb{Z}^2, <_{\text{lex}})$ as the former is archimedean and the latter is not. It follows that $(\mathbb{Z}, \lhd_\alpha)$ is not isomorphic to $(\mathbb{Z}, \lhd_+)$ and $(\mathbb{Z}, \lhd_-)$.
The following result is essentialy the well-known classification of linearly ordered group expanding $\mathbb{Z}^2$ up to isomorphism:

Lemma 4.2. *Suppose $(\mathbb{Z}^2, <)$ is a linearly ordered group such that $(nu)_{n>0}$ is cofinal in $\mathbb{Z}^2$ with $u = (1,0)$. Then $(\mathbb{Z}^2, u, <)$ is isomorphic to either $(\mathbb{Z}^2, u, <_{\text{lex}})$ or $(\mathbb{Z}^2, u, <_\alpha)$ for a unique $\alpha \in [0, 1/2)_{\mathbb{R}\setminus\mathbb{Q}}$.*

Proof. Suppose $(\mathbb{Z}^2, <)$ and u are as stated above. Using the fact that $(nu)_{n>0}$ is cofinal in $\mathbb{Z}^2$, we obtain k such that $ku < (0,1) < (k+1)u$. Let v be $(0,1) - ku$ if $2ku < (0,2) < (2k+1)u$ and let u be $(k+1)u - (0,1)$ otherwise. Then

$$\langle u, v\rangle = \mathbb{Z}^2 \text{ and } 0 < 2v < u.$$

If $(nv)_{n>0}$ is not cofinal in $\mathbb{Z}^2$, then it is easy to see that the map

$$\mathbb{Z}^2 \to \mathbb{Z}^2, \quad ku + lv \mapsto (k, l)$$

is an ordered group isomorphism from $(\mathbb{Z}^2, u, <)$ to $(\mathbb{Z}^2, u, <_{\text{lex}})$. Now suppose $(nv)_{n>0}$ is cofinal in $\mathbb{Z}^2$. Then set

$$\alpha = \sup\left\{\frac{m}{n} : m, n > 0 \text{ and } mu < nv\right\}.$$

It is easy to check that $\alpha \in [0, 1/2)_{\mathbb{R}\setminus\mathbb{Q}}$ and that the map $\mathbb{Z}^2 \to \mathbb{Z}^2, ku + lv \mapsto (k, l)$ is an isomorphism from $(\mathbb{Z}^2, u, <)$ to $(\mathbb{Z}^2, u, <_\alpha)$.

Finally, suppose α and β are in $[0, 1/2)_{\mathbb{R}\setminus\mathbb{Q}}$ and f is an isomorphism from $(\mathbb{Z}^2, u, <_\alpha)$ to $(\mathbb{Z}^2, u, <_\beta)$ with $u = (1, 0)$. Let $v = (0, 1)$. Then

$$\langle u, f(v)\rangle = \mathbb{Z}^2 \text{ and } 0 < 2f(v) < u.$$

The former condition implies $f(v)$ is either $(k, 1)$ or $(k, -1)$ for some k. Combining with the latter condition, we get $f(v) = (0, 1)$, and so $f = \mathrm{id}_{\mathbb{Z}^2}$. It follows easily from the definition of $<_\alpha$ and $<_\beta$ that $\alpha = \beta$. $\square$

We deduce a classification of additive circular orders on $\mathbb{Z}$:

Proposition 4.2. *Every additive circular order on $\mathbb{Z}$ is either $\lhd_+$, $\lhd_-$, or $\lhd_\alpha$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, for $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$, $\lhd_\alpha = \lhd_\beta$ if and only if $\alpha - \beta \in \mathbb{Z}$.*

Proof. Suppose $\lhd$ is an additive circular order on $\mathbb{Z}$. It follows from Proposition 4.1 and Lemma 4.2 that $(\mathbb{Z}, \lhd)$ is isomorphic to either $(\mathbb{Z}, \lhd_+)$ or $(\mathbb{Z}, \lhd_\alpha)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Note that the only group automorphism of $\mathbb{Z}$ are $\mathrm{id}_{\mathbb{Z}}$ and $k \mapsto -k$. The latter maps $\lhd_+$ to $\lhd_-$ and $\lhd_\alpha$ to $\lhd_{-\alpha}$ for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The first statement of the proposition follows.

The backward direction of the second statement follows from the easy observations that $\lhd_\alpha = \lhd_{\alpha+1}$. For the forward direction of the second statement, suppose $\alpha, \beta \in \mathbb{R} \smallsetminus \mathbb{Q}$ and $\lhd_\alpha = \lhd_\beta$. In particular, this implies that

$$(\mathbb{Z}, \lhd_{-\alpha}) \cong (\mathbb{Z}, \lhd_\alpha) \cong (\mathbb{Z}, \lhd_\beta) \cong (\mathbb{Z}, \lhd_{-\beta}).$$

By the backward direction of the second statement, we can arrange that α and β are in $[-1/2, 1/2)_{\mathbb{R}\smallsetminus\mathbb{Q}}$. If both α and β are in $[0, 1/2)_{\mathbb{R}\smallsetminus\mathbb{Q}}$, then it follows from Lemma 4.2 that $\alpha = \beta$. If both α and β are in $[-1/2, 0)_{\mathbb{R}\smallsetminus\mathbb{Q}}$, a similar argument shows that $-\alpha = -\beta$, and so $\alpha = \beta$. Finally, suppose one out of α, β is in $[-1/2, 0)_{\mathbb{R}\smallsetminus\mathbb{Q}}$ and the other is in $[0, 1/2)_{\mathbb{R}\smallsetminus\mathbb{Q}}$. A similar argument as the previous cases give us that $\alpha = -\beta$. However, $\lhd_\alpha$ is always different from $\lhd_{-\alpha}$, so this last case never happens. $\square$

We also need a well-known result of Kronecker: If $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ is a $\mathbb{Q}$-linearly independent tuple of variables, then

$$\big(\alpha_1 m + \mathbb{Z} \dots, \alpha_n m + \mathbb{Z}\big)_{m>0} \text{ is dense in } (\mathbb{R}/\mathbb{Z})^n,$$

where the latter is equipped with the obvious topology. See also [**79**] for another instance where a phenomenon of this type is of central importance in dealing with circular orders.

Theorem 4.1. *Let α be in $\mathbb{R} \smallsetminus \mathbb{Q}$. Then $(\mathbb{Z}, \lhd_\alpha)$ is a reduct of neither $(\mathbb{Z}, <)$ nor $(\mathbb{Z}, \prec_p)$ for any prime p.*

Proof. Suppose the notations are as given. We will show that $X = \{k : \lhd(0, k, 1)\}$ is definable neither in $(\mathbb{Z}, <)$ nor $(\mathbb{Z}, \prec_p)$. By [**24**, Remark 3.2], any subset of $\mathbb{Z}$ definable in $(\mathbb{Z}, \prec_p)$ is definable in $\mathbb{Z}$. Hence, it suffices to show that X is not definable in $(\mathbb{Z}, <)$.

Toward a contradiction, suppose X is definable in $(\mathbb{Z}, <)$. By the one-dimensional case of Kronecker's approximation theorem, we get that both X and $\mathbb{Z} \smallsetminus X$ are infinite. It then follows easily from the quantifier elimination for $(\mathbb{Z}, <)$ that there is $k \neq 0$ and l such that

$$\{km + l : m > 0\} \subseteq \mathbb{Z} \smallsetminus X.$$

On the other hand, by Kronecker's approximation theorem again, we have that $X \cap \{km + l : m > 0\} \neq \varnothing$ for all $k \neq 0$ and all l, which is absurd. $\square$

4.1. Unary definable sets and definable equivalence

We now show that if $\alpha, \beta \in \mathbb{R} \smallsetminus \mathbb{Q}$ are $\mathbb{Q}$-linearly independent then $(\mathbb{Z}, \lhd_\alpha)$ does not define $\lhd_\beta$. This follows from a characterization of unary definable sets in a circularly ordered expansion of $\mathbb{Z}$ and Kronecker's approximation theorem.

Let $\lhd$ be a circular order on a set G. A subset J of G is **convex** (with respect to $\lhd$) if whenever $a, b \in J$ are distinct we either have $\{t : \lhd(a,t,b)\} \subseteq J$ or $\{t : \lhd(b,t,a)\} \subseteq J$. Intervals are convex, and it is easy to see that the union of a nested family of convex sets is convex.

Lemma 4.3. *Let $(G, \lhd)$ be densely circularly ordered abelian group with universal cover $(H, u, <)$ and covering map $\pi : H \to G$. If $J \subseteq H$ is convex (with respect to $<$) then $\pi(J)$ is convex (with respect to $\lhd$).*

Proof. Let $J \subseteq H$ be convex. Then J is the union of a nested family of closed intervals $\{I_a : a \in L\}$, i.e. we either have $I_a \subseteq I_b$ or $I_b \subseteq I_a$ for all $a, b \in L$. It follows that $\pi(J)$ is the union of the nested family $\{\pi(I_a) : a \in L\}$. It suffices to show that $\pi(J)$ is convex when J is a closed interval. Suppose $J = [g, h]$.

We first suppose $h - g \geqslant u$. Then $[0, u]_H \subseteq J - g$. The restriction of π to $[0, u]_H$ is a surjection so $\pi(J - g) = G$. As $\pi(J - g) = \pi(J) - \pi(g)$, we have $\pi(J) = G + \pi(g) = G$. So in particular $\pi(J)$ is convex. Now suppose $h - g < u$. Then $J - g \subseteq [0, u]_H$. It follows that

$$\pi(J - g) = \{t \in G : \lhd(0, t, \pi(g - h))\}$$

so $\pi(J - g)$ is convex. Then $\pi(J) = \pi(J - g) + \pi(g)$ is a translate of a convex set and is hence convex. $\square$

Suppose $(G, \ldots)$ expands either a linear order $<$ or a circular order $\lhd$; convexity in the definitions below is with respect to either $<$ or $\lhd$. A **tmc-set** is a translation of a multiple of a convex subset of G, that is, a subset of G the form $a + mJ$ with $a \in G$ and convex $J \subseteq G$. A **cnc-set** is a set of the form $J \cap (a + nG)$ with convex $J \subseteq G$ and $a \in G$.

We say that $(G, \ldots)$ is **tmc-minimal** if every definable unary set is a finite union of tmc-sets and that $(G, \ldots)$ is **cnc-minimal** if every definable unary set is a finite union of cnc-sets. These two notions coincide for linearly ordered groups.

Lemma 4.4. *Suppose that $(G, <)$ is a linearly ordered group. Then the collection of tmc-sets and the collection of cnc-sets coincide.*

Proof. Let $X \subseteq G$ be an cnc-set. Let $X = I \cap A$ for a convex $I \subseteq G$ and $A = a + nG$. Let $J = \{g \in G : ng \in I - a\}$. Monotonocity of $g \mapsto ng$ implies J is convex as $I - a$ is convex. The definition of J implies $g \in J$ if and only if $a + ng \in I$. As $a + ng \in A$ for all $g \in G$ we have $g \in J$ if and only if $a + ng \in I \cap A$. So $X = a + nJ$.

Conversely, suppose J is convex. A translate of an cnc-set is an cnc-set, so it suffices to show nJ is an cnc-set. Let I be the convex hull of nJ. Then $nJ \subseteq I \cap nG$. We show the other inclusion. Suppose $g \in G$ and $ng \in I$. Then $nh \leqslant ng \leqslant nh'$ for some $nh, nh' \in nJ$. Then $h \leqslant g \leqslant h'$, so $g \in J$ as $h, h' \in J$ and J is convex. Thus $ng \in nJ$. $\square$

In circularly ordered abelian groups there may be tmc-sets which are not cnc-sets. More precisely, it can be shown that there are tmc-sets which are not even finite unions of cnc-sets. An example is the set $\{2k : \alpha k \in [0, 1/2) + \mathbb{Z}\}$ in the structure $(\mathbb{Z}, \lhd_\alpha)$ with $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$. As this will not be used later, we leave the proof to the interested readers.

Lemma 4.5. *If $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ then $(\mathbb{Z}^2, <_\alpha)$ is cnc-minimal.*

Proof. The structure $(\mathbb{Z}^2, <_\alpha)$ admits quantifier elimination in the extended language where we add a predicate symbol defining $n\mathbb{Z}$ for each n. See [**88**], for example. It follows that any definable subset of $\mathbb{Z}^2$ is a finite union of finite intersections of sets of one of the following types:

(1) $\{t : k_1 t + a <_\alpha k_2 t + b\}$ for some k_1, k_2 and $a, b \in \mathbb{Z}^2$,
(2) $\{t : k_1 t + a \geqslant_\alpha k_2 t + b\}$ for some k_1, k_2 and $a, b \in \mathbb{Z}^2$,
(3) $\{t : kt + a \in n\mathbb{Z}^2\}$ for some k, n and $a \in \mathbb{Z}^2$,
(4) $\{t : kt + a \notin n\mathbb{Z}^2\}$ for some k, n and $a \in \mathbb{Z}^2$,
(5) $\{t : k_1 t + a_1 = k_2 t + a_2\}$ for some k_1, k_2 and $a_1, a_2 \in \mathbb{Z}^2$,
(6) $\{t : k_1 t + a_1 \neq k_2 t + a_2\}$ for some k_1, k_2 and $a_1, a_2 \in \mathbb{Z}^2$.

We show that any finite intersection of sets of type (1)-(6) is a finite union of cnc-sets. Every set of type (1) or (2) is either upwards or downwards closed. It follows that any intersection of such sets is convex.

Suppose $A = \{t : kt + a \in n\mathbb{Z}^2\}$. Suppose A is nonempty and $t' \in A$. Then $kt + a \in n\mathbb{Z}^2$ if and only if

$$(kt + a) - (kt' + a) = k(t - t') \in n\mathbb{Z}^2.$$

For any m we have $km \in n\mathbb{Z}$ if and only if m is in $N\mathbb{Z}$ where $N = n/\gcd(k, n)$. So $t \in A$ if and only if $t - t' \in N\mathbb{Z}^2$, equivalently if $t \in N\mathbb{Z}^2 + t'$. So A is a coset of a subgroup of the form $N\mathbb{Z}^2$. So any finite intersection of sets of type (3) and (4) is a boolean combination of cosets of subgroups of the form $n\mathbb{Z}^2$. As $|\mathbb{Z}^2/n\mathbb{Z}^2| < \infty$, a complement of a coset of a subgroup of the form $n\mathbb{Z}^2$ is a finite union of such cosets. It follows that any boolean combination of cosets of subgroups of the form $n\mathbb{Z}^2$ is a finite union of such cosets.

We have shown that a finite intersection of sets of type (1)-(4) is an intersection of a convex set by a finite union of cosets of subgroups of the form $n\mathbb{Z}^2$. It follows that any finite intersection of sets of type (1)-(4) is a finite union of cnc-sets.

Any set of type (5) or (6) is either empty, $\mathbb{Z}^2$, a singleton, or the complement of a singleton. It follows that any finite intersection of such sets is either finite or co-finite. Suppose that X is a finite union of cnc-sets. The intersection of a X and a finite set is finite, hence is a finite union of cnc-sets. It is easy to see that the intersection of X and a co-finite set is a finite union of cnc-sets. $\square$

Theorem 4.2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $(\mathbb{Z}, \triangleleft_\alpha)$ is tmc-minimal.*

Proof. Suppose $X \subseteq \mathbb{Z}$ is definable. Set

$$Y = \pi^{-1}(X) \cap [0, u)_{\mathbb{Z}^2}.$$

Then $X = \pi(Y)$ and Y is a finite union of cnc-sets $Y_1, \ldots, Y_k$ by 4.5. As

$$\pi(Y) = \pi(Y_1) \cup \ldots \cup \pi(Y_k)$$

we may assume Y is an cnc-set. Applying Lemma 4.4 we suppose that $Y = a + nJ$ for $a \in \mathbb{Z}^2$ and convex $J \subseteq \mathbb{Z}^2$. As π is a homomorphism we have

$$X = \pi(Y) = \pi(a) + n\pi(J).$$

It follows from Lemma 4.3 that $\pi(J)$ is convex. Thus X is a tmc-set. $\square$

We say that $X \subseteq \mathbb{Z}$ is $\triangleleft_\alpha$-dense if it is dense with respect to the obvious topology induced by $\triangleleft_\alpha$.

Lemma 4.6. *Suppose, α and β in $\mathbb{R} \setminus \mathbb{Q}$ are $\mathbb{Q}$-linearly independent and $J_\beta \subseteq \mathbb{Z}$ is $\triangleleft_\beta$-convex and infinite, fix $n \geqslant 1, k$. Then $k + nJ_\beta$ is $\triangleleft_\alpha$-dense.*

Proof. Suppose $X \subseteq \mathbb{Z}$ is $\triangleleft_\alpha$-dense. It follows by elementary topology that the image of X under the map $l \mapsto k + nl$ is $\triangleleft_\alpha$-dense in $k + n\mathbb{Z}$. As $k + n\mathbb{Z}$ is $\triangleleft_\alpha$-dense, it follows that $k + nX$ is $\triangleleft_\alpha$-dense. It therefore suffices to show that J_β is dense with respect to the topology induced by $\triangleleft_\alpha$. We show that J_β intersects an arbitrary infinite $\triangleleft_\alpha$-convex $J_\alpha \subseteq \mathbb{Z}$. Let J'_α and J'_β be $\triangleleft$-convex subsets of $\mathbb{R}/\mathbb{Z}$ such that $J_\alpha = \chi_\alpha^{-1}(J'_\alpha)$ and $J_\beta = \chi_\beta^{-1}(J'_\beta)$. Then J'_α, J'_β are infinite and so have nonempty interior. It follows from $\mathbb{Q}$-linear independence of α and β and Kronecker's theorem that

$$\{(\chi_\alpha(m), \chi_\beta(m)) : m \in \mathbb{Z}\} \text{ is dense in } (\mathbb{R}/\mathbb{Z})^2.$$

In particular, there is $m \in \mathbb{Z}$ such that $\big(\chi_\alpha(m), \chi_\beta(m)\big) \in J'_\alpha \times J'_\beta$. Then m is in $J_\alpha \cap J_\beta$, which implies that the latter is non-empty. $\square$

Corollary 4.1. *Suppose $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ are $\mathbb{Q}$-linearly independent. Then there is a $(\mathbb{Z}, \triangleleft_\beta)$-definable subset of $\mathbb{Z}$ which is not definable in $(\mathbb{Z}, \triangleleft_\alpha)$.*

Proof. Suppose that α and β are $\mathbb{Q}$-linearly independent elements of $\mathbb{R} \setminus \mathbb{Q}$. Let J_α be an infinite $\triangleleft_\alpha$-convex set definable in $(\mathbb{Z}, \triangleleft_\alpha)$ with infinite complement. Suppose $\mathbb{Z} \setminus J_\alpha$ is definable in $(\mathbb{Z}, \triangleleft_\beta)$. It follows from tmc-minmality of the latter that $\mathbb{Z} \setminus J_\alpha \supseteq k + nJ_\beta$ where

J_β is $\triangleleft_\beta$-covex and $n \geqslant 1$. Lemma 4.6 shows that $k + nJ_\beta$ is $\triangleleft_\alpha$-dense and thus intersects J_α, contradiction. $\square$

As consequence of Corollary 4.1 we obtain uncountably many definably distinct dp-minimal expansions of $\mathbb{Z}$.

Corollary 4.2. *There are continumn many pairwise definably distinct circularly ordered groups expanding $\mathbb{Z}$.*

We now show that if $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ are $\mathbb{Q}$-linearly dependent then $\triangleleft_\beta$ is $(\mathbb{Z}, \triangleleft_\alpha)$-definable. It follows that $(\mathbb{Z}, \triangleleft_\alpha)$ and $(\mathbb{Z}, \triangleleft_\beta)$ are definably equivalent if and only if α and β are $\mathbb{Q}$-linearly dependent, i,e, if $\beta = q\alpha + r$ for some $q, r \in \mathbb{Q}$. This requires several steps.

Lemma 4.7. *Suppose α is in $\mathbb{R} \setminus \mathbb{Q}$, n is in $\mathbb{N}^{\geqslant 1}$, and r is in $\{0, \dots, n-1\}$. Then the set*

$$\{l \ : \ \alpha l + \mathbb{Z} \in [r/n, (r+1)/n) + \mathbb{Z}\}$$

is definable in $(\mathbb{Z}, \triangleleft_\alpha)$.

Proof. Let the notation be as given and $(\mathbb{R}/\mathbb{Z}, \triangleleft)$ be the oriented circle. We have that $\alpha l + \mathbb{Z}$ is in $[r/n, (r+1)/n) + \mathbb{Z}$ if and only if $(\alpha i l + \mathbb{Z})_{i=0}^{n}$ "winds" r times around $\mathbb{R}/\mathbb{Z}$, that is,

$$\triangleleft\big(0 + \mathbb{Z}, \alpha(i+1)l + \mathbb{Z}, \alpha i l + \mathbb{Z}\big) \text{ holds for exactly } r \text{ values of } i \in \{1, \dots, n-1\}.$$

The desired conclusion follows. $\square$

Corollary 4.3. *If α and β are in $\mathbb{R} \setminus \mathbb{Q}$ and $\beta = \alpha + m/n$ with $n \geqslant 1$, then $\triangleleft_\beta$ is definable in $(\mathbb{Z}, \triangleleft_\alpha)$.*

Proof. Suppose α is in $\mathbb{R} \setminus \mathbb{Q}$. Note that $\triangleleft_{-\alpha}(j, k, l)$ if and only if $\triangleleft_\alpha(-j, -k, -l)$, so $\triangleleft_{-\alpha}$ is definable in $(\mathbb{Z}, \triangleleft_\alpha)$. As $\alpha - m/n = -(-\alpha + m/n)$ is suffices to treat the case when $m \geqslant 1$. It suffices to treat the case $\beta = \alpha + 1/n$ and then apply this case m times to get the general case.

Suppose α, β are in $\mathbb{R} \setminus \mathbb{Q}$ and $\beta = \alpha + 1/n$ with $n \geqslant 1$. As $\triangleleft_\alpha$ is additive it suffices to show that the set of pairs (k, l) such that $\triangleleft_\beta(0, k, l)$ is definable in $(\mathbb{Z}, \triangleleft_\alpha)$. Let $(\mathbb{R}/\mathbb{Z}, \triangleleft)$ be the positively oriented circle. By definition, $\triangleleft_\beta(0, k, l)$ is equivalent to $\triangleleft(0 + \mathbb{Z}, \beta k + \mathbb{Z}, \beta l + \mathbb{Z})$. The latter holds if and only if either there are $r, s \in \{0, \dots, n-1\}$ with $r < s$ such that

$$\beta k + \mathbb{Z} \in [r/n, (r+1)/n) + \mathbb{Z} \quad \text{and} \quad \beta l + \mathbb{Z} \in [s/n, (s+1)/n) + \mathbb{Z}$$

or there is $r \in \{0, \dots, n-1\}$ such that

$$\beta k + \mathbb{Z}, \beta l + \mathbb{Z} \in [r/n, (r+1)/n) + \mathbb{Z} \text{ and } \triangleleft\big(0, \beta n k + \mathbb{Z}, \beta n l + \mathbb{Z}\big).$$

For all $a \in \mathbb{R}$, we have that $a + k/n + \mathbb{Z} \in [r/n, (r+1)/n) + \mathbb{Z}$ holds if an only if a is in $[r'/n, (r'+1)/n) + \mathbb{Z}$ with $r' \in \{0, \dots, n-1\}$ and $r' + k \equiv r \pmod{n}$. Hence, it follows from $\beta = \alpha + 1/n$ that $\beta k + \mathbb{Z} \in [r/n, (r+1)/n) + \mathbb{Z}$ is equivalent to

$$\alpha k + \mathbb{Z} \in [r'/n, (r'+1)/n) + \mathbb{Z} \;\text{ with } r' \in \{0, \dots, n-1\} \text{ and } r' + k \equiv r \pmod{n}.$$

On the other hand, as $n\beta = n\alpha + 1$, so we get

$$\lhd(0+\mathbb{Z}, \beta nk + \mathbb{Z}, \beta nl + \mathbb{Z}) \;\text{ is equivalent to }\; \lhd(0+\mathbb{Z}, \alpha nk + \mathbb{Z}, \alpha nl + \mathbb{Z}).$$

By definition of $\lhd_\alpha$, the latter holds if and only if $\lhd_\alpha(0, nk, nl)$. Combining with Lemma 4.7 we get the desired conclusion. □

Lemma 4.8. *Suppose α is in $[0,1)_{\mathbb{R}\setminus\mathbb{Q}}$, m, n are in $\mathbb{N}^{\geqslant 1}$, and r is in $\{0, \dots, n-1\}$. Then the set*

$$\{l \;:\; \alpha l + \mathbb{Z} \in [0, r\alpha/n) + \mathbb{Z}\}$$

is definable in $(\mathbb{Z}, \lhd_\alpha)$.

Proof. Suppose α,n, and r are as given and $(\mathbb{R}/\mathbb{Z}, \lhd)$ is the positively oriented circle. We note that $\alpha l + \mathbb{Z}$ is in $[0, \alpha/n) + \mathbb{Z}$ if and only if $\lhd(0+\mathbb{Z}, \alpha nl + \mathbb{Z}, \alpha + \mathbb{Z})$ and $(\alpha il + \mathbb{Z})_{i=0}^{n}$ does not"winds" around $\mathbb{R}/\mathbb{Z}$, that is,

$$\lhd(0+\mathbb{Z}, \alpha il + \mathbb{Z}, \alpha(i+1)l + \mathbb{Z}) \text{ for all } i \in \{1, \dots, n-1\}.$$

Recall that by definition $\lhd(\alpha j + \mathbb{Z}, \alpha k + \mathbb{Z}, \alpha l + \mathbb{Z})$ if and only if $\lhd_\alpha(j, k, l)$. Hence,

$$\{l \;:\; \alpha l + \mathbb{Z} \in [0, \alpha/n) + \mathbb{Z}\} \text{ is definable in } (\mathbb{Z}, \lhd_\alpha).$$

The conclusion follow the easy observation that $\alpha l + \mathbb{Z}$ is in $[0, r\alpha/n) + \mathbb{Z}$ if and only if $\lhd_\alpha(0, l, rk)$ for some $k \in [0, \alpha/n) + \mathbb{Z}$. □

Corollary 4.4. *Suppose α is in $[0,1)_{\mathbb{R}\setminus\mathbb{Q}}$, n is in $\mathbb{N}^{\geqslant 1}$, and $\beta = m\alpha/n$. Then $\lhd_\beta$ is definable in $(\mathbb{Z}, \lhd_\alpha)$.*

Proof. As $\chi_{\alpha/n}(mk) = \chi_{m\alpha/n}(k)$ for all k we have $\lhd_{m\alpha/n}(i, j, l)$ if and only if $\lhd_{\alpha/n}(mi, mj, ml)$. It therefore suffices to treat the case $\beta = \alpha/n$. For any given k and $r \in \{0, 1, \dots, n\}$, let

$$X_{k,r} = \{l : \alpha l + \mathbb{Z} \in [0, k\alpha + r\alpha/n) + \mathbb{Z}\}.$$

We first prove that $X_{k,r}$ is definable in $(\mathbb{Z}, \lhd_\alpha)$ for all k and r as above. This is true for $r = 0$ as $l \in X_{k,0}$ if and only if either $l = 0$ or $\lhd(0+\mathbb{Z}, l\alpha + \mathbb{Z}, k\alpha + \mathbb{Z})$. The later is equivalent to $\lhd_\alpha(0, k, l)$ by definition. The case where $k = 0$ is just the preceding lemma. In general, we have that

$$X_{k,r} = \begin{cases} X_{k,0} \cup (k + X_{0,r}) & \text{if } X_{k,0} \cap (k + X_{0,r}) = \varnothing, \\ X_{k,0} \cap (k + X_{0,r}) & \text{otherwise.} \end{cases}$$

Let r, s be in $\{0, \ldots, n-1\}$. We have that $\triangleleft_\beta(0, kn+r, ln+s)$ is equivalent to $\triangleleft(0+\mathbb{Z}, \beta(kn+r)+\mathbb{Z}, \beta(ln+s)+\mathbb{Z})$ by definition. The latter holds if and only if $kn+r$, $ln+s$, and 0 are all distinct and $X_{k,r} \subseteq X_{l,r}$. The conclusion follows. $\square$

Corollary 4.3 and Corollary 4.4 show that $\triangleleft_\beta$ is definable in $(\mathbb{Z}, \triangleleft_\alpha)$ whenever $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ are $\mathbb{Q}$-linearly dependent. Combining with Corollary 4.1 we get:

Theorem 4.3. *Suppose α and β are in $\mathbb{R} \setminus \mathbb{Q}$. Then $(\mathbb{Z}, \triangleleft_\alpha)$ and $(\mathbb{Z}, \triangleleft_\beta)$ are definably equivalent if and only if α, β are $\mathbb{Q}$-linearly dependent.*

Finally, we give an example of a dp-minimal expansion of $\mathbb{Z}$ which defines uncountably many subsets of $\mathbb{Z}$. Let $\mathcal{M} = (M, \ldots)$ be a structure and $\mathcal{N} = (N, \ldots)$ be a highly saturated elementary expansion of $\mathcal{M}$. Then a subset of M^k is **externally definable** if it is of the form $A \cap M^k$ where $A \subseteq N^k$ is definable in $\mathcal{N}$. A standard saturation argument shows that the collection of externally definable sets does not depend on the choice of $\mathcal{N}$. The **Shelah expansion** of $\mathcal{M}$ is the expansion $\mathcal{M}^{\mathrm{Sh}}$ of $\mathcal{M}$ obtained by adding a predicate defining every externally definable subset of every M^k. It was shown in [**70**] that $\mathcal{M}^{\mathrm{Sh}}$ is NIP whenever $\mathcal{M}$ is, see also [**72**, Chapter 3]. It was observed in [**62**, 3.8] that the main theorem of [**70**] also shows that $\mathcal{M}^{\mathrm{Sh}}$ is dp-minimal whenever $\mathcal{M}$ is dp-minimal. In particular $(\mathbb{Z}, \triangleleft_\alpha)^{\mathrm{Sh}}$ is dp-minimal for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Proposition 4.3. *Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $(\mathbb{Z}, \triangleleft_\alpha)^{\mathrm{Sh}}$ defines uncountably many distinct subsets of $\mathbb{Z}$ and has uncountably many definably distinct reducts.*

Proof. If $\mathcal{M}, \mathcal{N}$ are as above, and $\mathcal{M}$ expands a linear or circular order then it is easy to see that any convex subset of M is of the form $I \cap M$ for an interval $I \subseteq N$. It follows that $(\mathbb{Z}, \triangleleft_\alpha)^{\mathrm{Sh}}$ defines every $\triangleleft_\alpha$-convex subset of $\mathbb{Z}$ and thus defines uncountably many subsets of $\mathbb{Z}$. Any reduct of $(\mathbb{Z}, \triangleleft_\alpha)^{\mathrm{Sh}}$ to a countable language defines only countably many subsets of $\mathbb{Z}$, it follows that $(\mathbb{Z}, \triangleleft_\alpha)^{\mathrm{Sh}}$ has uncountably many definably distinct reducts. $\square$

Part 2

Abstract partially random structures

CHAPTER 5

Preliminaries

Throughout, L is a language with S the set of sorts, and $\mathcal{M}$ is an L-structure. Concepts like variables, functions, formulas, etc. are by default with respect to L. We refer to $L(\varnothing)$-definable sets and $L(M)$-definable sets simply as L-definable sets and $\mathcal{M}$-definable set.

When the structure in question is the monster model for a complete theory, we boldface the relevant notations, i.e., writing $\boldsymbol{\mathcal{M}}$ instead of $\mathcal{M}$ and $\boldsymbol{M}$ instead of M. When discussing a monster model, we adopt the usual convention that all models of $\mathrm{Th}(\boldsymbol{\mathcal{M}})$ are small elementary substructures of $\boldsymbol{\mathcal{M}}$, and all sets of parameters are small subsets of $\boldsymbol{M}$.

We often work with multiple languages with the same set of sorts S. In these cases, we may define the union and intersection of the languages in the obvious manner and use tuples of variables without specifying the language. Whenever we consider multiple reducts of a structure, we decorate these reducts with the same decorations as their languages. For example, if $L_0 \subseteq L_1$ are languages, we denote an L_1-structure by $\mathcal{M}_1$, and we denote its reduct $\mathcal{M}_1|_{L_0}$ to L_0 by $\mathcal{M}_0$. In this situation, we write "in $\mathcal{M}_0$" to denote that we are evaluating some concept in the reduct.

In this chapter, we review background material and establish general results for later use which are not specific to the context of abstract partially random structures. The reader may skip to Chapter 6 and refer back to this chapter as needed.

5.1. Flat formulas

A formula is **atomic flat** if it is of the form $x = y$, $R(x_1, \dots, x_n)$, or $f(x_1, \dots, x_n) = y$, where R is an n-ary relation symbol and f is an n-ary function symbol. Here $x, y, x_1, \dots, x_n$ are single variables, which need not be distinct.

A **flat literal** is an atomic flat formula or the negation of an atomic flat formula. The **flat diagram** $\mathrm{fdiag}(\mathcal{A})$ of an L-structure $\mathcal{A}$ is the set of all flat literal $L(A)$-sentences true in $\mathcal{A}$.

A **flat formula** is a conjunction of finitely many flat literals. An **E$\flat$-formula** is a formula of the form $\exists y\, \varphi(x, y)$, where $\varphi(x, y)$ is flat and $\vDash \forall x\, \exists^{\leqslant 1} y\, \varphi(x, y)$.

Remark 5.1. The class of E$\flat$-formulas is closed (up to equivalence) under finite conjunction: the conjunction of the E$\flat$-formulas $\exists y_1\, \varphi_1(x, y_2)$ and $\exists y_2\, \varphi_2(x, y_2)$ is equivalent to the E$\flat$-formula

$$\exists y_1 y_2\, (\varphi_1(x, y_1) \wedge \varphi_2(x, y_2)).$$

The following lemma essentially appears as Theorem 2.6.1 in [**37**]. Note Hodges uses the term "unnested" instead of "flat".

Lemma 5.1. *Every literal (atomic or negated atomic formula) is logically equivalent to an E$\flat$-formula.*

Proof. We first show that for any term $t(x)$, with variables $x = (x_1, \dots, x_n)$, there is an associated E$\flat$-formula $\varphi_t(x, y)$ such that $\varphi_t(x, y)$ is logically equivalent to $t(x) = y$. We apply induction on terms. For the base case where $t(x)$ is the variable x_k, we let $\varphi_t(x, y)$ be $x_k = y$. Now suppose $t_1(x), \dots, t_m(x)$ are terms and f is an m-ary function symbol. Then $\varphi_{f(t_1,\dots,t_m)}$ is the E$\flat$-formula equivalent to

$$\exists z_1 \dots z_m \left[\bigwedge_{i=1}^{m} \varphi_{t_i}(x, z_i) \wedge (f(z_1, \dots, z_m) = y) \right].$$

We now show that every atomic or negated atomic formula is equivalent to an E$\flat$-formula. Suppose $t_1(x), \dots, t_m(x)$ are terms and R is either an m-ary relation symbol or $=$ (in the latter case, we have $m = 2$). Then the atomic formula $R(t_1(x), \dots, t_m(x))$ is equivalent to

$$\exists y_1 \dots \exists y_m \left[\bigwedge_{i=1}^{m} \varphi_{t_i}(x, y_i) \wedge R(y_1, \dots, y_m) \right].$$

Negated atomic formulas can be treated similarly. $\square$

Corollary 5.1. *Every quantifier-free formula is logically equivalent to a finite disjunction of E$\flat$-formulas.*

Proof. Suppose $\varphi(x)$ is quantifier-free. Then $\varphi(x)$ is equivalent to a formula in disjunctive normal form, i.e., a finite disjunction of finite conjunctions of literals. Applying Lemma 5.1 to each literal and using Remark 5.1, we find that $\varphi(x)$ is equivalent to a finite disjunction of E$\flat$-formulas. $\square$

5.2. $\mathcal{K}$-completeness

In this section, T is an L-theory and $\mathcal{K}$ is a class of pairs $(\mathcal{A}, \mathcal{M})$, where $\mathcal{M} \models T$ and $\mathcal{A}$ is a substructure of $\mathcal{M}$.

We say that T is **$\mathcal{K}$-complete** if for all $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$, every embedding from $\mathcal{A}$ to another T-model is partial elementary: if $f : \mathcal{A} \to \mathcal{N}$ is an embedding and $\mathcal{N} \models T$, then $\mathcal{M} \models \varphi(a)$ if and only if $\mathcal{N} \models \varphi(f(a))$ for any formula $\varphi(x)$ and any $a \in A^x$.

Remark 5.2. The terminology $\mathcal{K}$-complete comes from the following equivalent definition: T is $\mathcal{K}$-complete if and only if for all $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$,

$$T \cup \mathrm{fdiag}(\mathcal{A}) \vDash \mathrm{Th}_{L(A)}(\mathcal{M}),$$

i.e., $T \cup \mathrm{fdiag}(\mathcal{A})$ is a complete $L(A)$-theory. Indeed, if $\mathcal{N}$ is an $L(A)$-structure, then $\mathcal{N} \vDash \mathrm{fdiag}(\mathcal{A})$ if and only if the obvious map $\mathcal{A} \to \mathcal{N}$ is an embedding.

Suppose T is $\mathcal{K}$-complete. If $\mathcal{K}$ is the class of pairs $(\mathcal{M}, \mathcal{M})$ such that $\mathcal{M} \vDash T$, then T is **model-complete**. We say T is **substructure-complete** if $\mathcal{K}$ is the class of all pairs $(\mathcal{A}, \mathcal{M})$ such that $\mathcal{A}$ is a substructure of $\mathcal{M}$. If cl is a closure operator on T-models and $\mathcal{K}$ is the class of all pairs $(\mathcal{A}, \mathcal{M})$ such that $\mathcal{A}$ is a cl-closed substructure of $\mathcal{M}$, i.e., $\mathrm{cl}(A) = A$, then we say T is **cl-complete**.

The class of T-models has the **$\mathcal{K}$-amalgamation property** if whenever $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$, $\mathcal{N} \vDash T$, and $f: \mathcal{A} \to \mathcal{N}$ is an embedding, then there is an elementary extension $\mathcal{N} \preccurlyeq \mathcal{N}'$ and an elementary embedding $f': \mathcal{M} \to \mathcal{N}'$ such that $f'|_A = f$, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f'"'] & \mathcal{N}' \\ \mathcal{A} \arrow[u, "\subseteq"] \arrow[r, "f"] & \mathcal{N} \arrow[u, "\preccurlyeq"'] \end{tikzcd}$$

If, in the situation above, we can choose $\mathcal{N}'$ and f' with the further condition that

$$f'(M) \cap N = f'(A) = f(A),$$

then the class of T-models has the **disjoint $\mathcal{K}$-amalgamation property**.

Proposition 5.1. *The theory T is $\mathcal{K}$-complete if and only if the class of T-models has the $\mathcal{K}$-amalgamation property. Further, if T is $\mathcal{K}$-complete, then $\mathcal{A}$ is algebraically closed in $\mathcal{M}$ for all $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$ if and only if the class of T-models has the disjoint $\mathcal{K}$-amalgamation property.*

Proof. We prove the first equivalence. Suppose T is $\mathcal{K}$-complete. The $\mathcal{K}$-amalgamation property follows from [**37**, Theorem 6.4.1].

Conversely, suppose the class of T-models has the $\mathcal{K}$-amalgamation property. If $\mathcal{M}$ and $\mathcal{N}$ are T-models, $\mathcal{A} \subseteq \mathcal{M}$ is in $\mathcal{K}$, and $f: \mathcal{A} \to \mathcal{N}$ is an embedding, then there is an elementary extension $\mathcal{N} \preccurlyeq \mathcal{N}'$ and an elementary embedding $f': \mathcal{M} \to \mathcal{N}'$ such that $f'|_A = f$. For any L-formula $\varphi(x)$ and $a \in A^x$, $\mathcal{M} \vDash \varphi(a)$ if and only if $\mathcal{N}' \vDash \varphi(f'(a))$ if and only if $\mathcal{N} \vDash \varphi(f(a))$. So f is partial elementary. Thus T is $\mathcal{K}$-complete.

Now, assuming T is $\mathcal{K}$-complete, we prove the second equivalence. If every structure in $\mathcal{K}$ is algebraically closed, then the class of T-models has the disjoint $\mathcal{K}$-amalgamation property, by [**37**, Theorem 6.4.5].

Conversely, suppose the class of T-models has the disjoint $\mathcal{K}$-amalgamation property. Assume towards a contradiction that $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$ and A is not algebraically closed in $\mathcal{M}$. Then there is some $c \in M \setminus A$ such that $\mathrm{tp}(c/A)$ has exactly k realizations $c_1, \ldots, c_k$ in $M \setminus A$. Taking $\mathcal{N} = \mathcal{M}$ and $f = \mathrm{id}_A$ in the disjoint $\mathcal{K}$-amalgamation property, there is an elementary extension $\mathcal{M} \preccurlyeq \mathcal{M}'$ and an elementary embedding $f' : \mathcal{M} \to \mathcal{M}'$ which is the identity on A and satisfies $f'(M) \cap M = A$. Then $\mathrm{tp}(c/A)$ has $2k$ distinct realizations $c_1, \ldots, c_k, f'(c_1), \ldots, f'(c_k)$ in $\mathcal{M}'$, contradiction. $\square$

We recall some classical facts about model-completeness and model companions.

Fact 5.1 ([**37**], Theorem 6.5.9, Exercise 6.5.5)**.** *The following are equivalent:*

1. *T admits an $\forall\exists$-axiomatization.*
2. *The class of T-models is closed under unions of chains.*
3. *The class of T-models is closed under directed colimits (in the category of L-structures and embeddings).*

If one of the above equivalent conditions are satisfied, we say that T is **inductive**.

Fact 5.2 ([**37**], Theorem 8.3.3)**.** *Every model-complete theory is inductive.*

An L-theory T^* is a **model companion** of T if T^* is model-complete, every T-model embeds into a T^*-model, and every T^*-model embeds into a T-model.

Fact 5.3 ([**37**], Theorem 8.2.1, Theorem 8.3.6)**.** *Suppose T is inductive. Then:*

1. *Every T-model embeds into an existentially closed T-model.*
2. *T has a model companion if and only if the class of existentially closed T-models is elementary.*
3. *If T has a model companion T^*, then T^* is the theory of existentially closed T-models.*

Model-completeness has a syntactic equivalent: every L-formula is T-equivalent to an existential (hence also universal) formula [**37**, Theorem 8.3.1(e)].

Substructure-completeness also has a syntactic equivalent: quantifier elimination. This follows from [**37**, Theorem 8.4.1] and Proposition 5.1 above.

Many of the theories we consider are acl-complete. Unfortunately, there does not seem to be a natural syntactic equivalent to acl-completeness. We introduce a slightly stronger notion, bcl-completeness, which does have a syntactic equivalent.

An L-formula $\varphi(x, y)$ is **bounded in** y with bound k (with respect to T) if

$$T \models \forall x \, \exists^{\leqslant k} y \, \varphi(x, y).$$

A formula $\exists y \, \psi(x, y)$ is **boundedly existential (b.e.)** (with respect to T) if $\psi(x, y)$ is quantifier-free and bounded in y. We allow y to be the empty tuple of variables, so every quantifier-free formula is b.e. (with bound $k = 1$, by convention). The E$\flat$-formulas from Section 5.1 are also b.e. with bound $k = 1$ with respect to the empty theory.

Remark 5.3. The class of b.e. formulas is closed (up to T-equivalence) under conjunction: if $\exists y \, \psi_1(x, y_1)$ and $\exists y_2 \, \psi_2(x, y_2)$ are b.e. with bounds k_1 and k_2 on y_1 and y_2 respectively, then

$$(\exists y_1 \, \psi_1(x, y_1)) \wedge (\exists y_2 \, \psi_2(x, y_2))$$

is T-equivalent to

$$\exists y_1 y_2 \, (\psi_1(x, y_1) \wedge \psi_2(x, y_2)),$$

which is b.e. with bound $k_1 \cdot k_2$ on $y_1 y_2$.

Suppose $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. The **boundedly existential algebraic closure** of A in $\mathcal{M}$, denoted $\mathrm{bcl}(A)$, is the set of all b in M such that $\mathcal{M} \models \exists z \, \varphi(a, b, z)$ for some quantifier-free L-formula $\varphi(x, y, z)$ bounded in yz and some $a \in A^x$.

Remark 5.4. The formula $\varphi(x, y, z)$ is bounded in yz if and only if it is bounded in z and $\exists z \, \varphi(x, y, z)$ is bounded in y. As a consequence, $b \in \mathrm{bcl}(A)$ if and only if b satisfies a b.e. formula $\exists z \, \varphi(y, z)$ with parameters from A, which is bounded in y. Such a formula is algebraic, so $\mathrm{bcl}(A) \subseteq \mathrm{acl}(A)$.

Lemma 5.2. *If* $A \subseteq \mathcal{M}$ *then* $\langle A \rangle \subseteq \mathrm{bcl}(A)$*. Furthermore,* bcl *is a closure operator.*

Proof. Fix $A \subseteq \mathcal{M}$. Suppose $b \in \langle A \rangle$. Then $t(a) = b$ for a term $t(x)$ and a tuple a from A. Then the formula $t(x) = y$ is b.e. (taking z to be the empty tuple of variables) and bounded in y (with bound 1), so it witnesses $b \in \mathrm{bcl}(A)$ by Remark 5.4.

It follows that $A \subseteq \mathrm{bcl}(A)$, and it is clear that $A \subseteq B$ implies $\mathrm{bcl}(A) \subseteq \mathrm{bcl}(B)$. It remains to show bcl is idempotent.

Suppose $b \in \mathrm{bcl}(\mathrm{bcl}(A))$. Then $\mathcal{M} \models \exists z \, \varphi(a, b, z)$ for some quantifier-free formula $\varphi(x, y, z)$ which is bounded in yz and some tuple $a = (a_1, \ldots, a_n)$ from $\mathrm{bcl}(A)$. For each $1 \leqslant j \leqslant n$, since a_j is in $\mathrm{bcl}(A)$, $\mathcal{M} \models \exists z_j \, \psi_j(d_j, a_j, z_j)$ for some quantifier-free formula $\psi_j(w_j, x_j, z_j)$ which is bounded in $x_j z_j$, and some tuple d_j from A.

Then the quantifier-free formula

$$\left(\bigwedge_{j=1}^{n} \psi_j(w_j, x_j, z_j) \right) \wedge \varphi(x_1, \ldots, x_n, y, z)$$

is bounded in $x_1 \dots x_n y z_1 \dots z_n z$ (by the product of the bounds for φ and the ψ_j), and

$$\mathcal{M} \vDash \exists x_1 \dots x_n z_1 \dots z_n z \left(\bigwedge_{j=1}^{n} \psi_j(d_j, x_j, z_j) \right) \wedge \varphi(x_1, \dots, x_n, b, z),$$

so $b \in \mathrm{bcl}(A)$. $\square$

Remark 5.5. Every model is acl-closed, every acl-closed set is bcl-closed, and every bcl-closed set is a substructure, therefore:

$$\text{QE} \Leftrightarrow \text{substructure-complete} \Rightarrow \text{bcl-complete} \Rightarrow \text{acl-complete} \Rightarrow \text{model-complete.}$$

Theorem 5.1 clarifies the relationship between acl- and bcl-completeness and provides the promised syntactic equivalent to bcl-completeness.

Theorem 5.1. *The following are equivalent:*

(1) *Every L-formula is T-equivalent to a finite disjunction of b.e. formulas.*
(2) *T is* acl*-complete and* $\mathrm{acl} = \mathrm{bcl}$ *in T-models.*
(3) *T is* bcl*-complete.*

Proof. We assume (1) and prove (2). We first show acl and bcl agree. Suppose $A \subseteq \mathcal{M} \vDash T$ and $b \in \mathrm{acl}(A)$, witnessed by an algebraic formula $\varphi(a, y)$ with parameters a from A. Suppose there are exactly k tuples in M^y satisfying $\varphi(a, y)$. Let $\varphi'(x, y)$ be the formula

$$\varphi(x, y) \wedge \exists^{\leq k} y' \, \varphi(x, y'),$$

and note $\varphi'(x, y)$ is bounded in y. By assumption, $\varphi'(x, y)$ is equivalent to a finite disjunction of boundedly existential formulas, so there is some boundedly existential formula $\psi(x, y)$ such that $T \vDash \psi(x, y) \to \varphi'(x, y)$ and $\mathcal{M} \vDash \psi(a, b)$. Since $\varphi'(x, y)$ is bounded in y, so is $\psi(x, y)$, and hence $b \in \mathrm{bcl}(A)$ by Remark 5.4.

We continue to assume (1) and show T is acl-complete. Suppose $\mathcal{A}$ is an algebraically closed substructure of $\mathcal{M} \vDash T$ and $f \colon \mathcal{A} \to \mathcal{N} \vDash T$ is an embedding. We show that for any formula $\varphi(x)$, if $\mathcal{M} \vDash \varphi(a)$, where $a \in A^x$, then $\mathcal{N} \vDash \varphi(f(a))$. By our assumption, $\varphi(x)$ is equivalent to a finite disjunction of boundedly existential formulas, so there is some boundedly existential formula $\exists y \, \psi(x, y)$ such that

$$T \vDash (\exists y \, \psi(x, y)) \to \varphi(x) \quad \text{and} \quad \mathcal{M} \vDash \exists y \, \psi(a, y).$$

Let $b \in M^y$ be a witness for the existential quantifier. Then each component of the tuple b is in $\mathrm{acl}(a) \subseteq A$, since A is algebraically closed. And ψ is quantifier-free, so $\mathcal{N} \vDash \psi(f(a), f(b))$, and hence $\mathcal{N} \vDash \varphi(f(a))$.

It is clear that (2) implies (3).

We now assume (3) and prove (1). For any finite tuple of variables x, let Δ_x be the set of boundedly existential formulas with free variables from x.

Claim: For all models $\mathcal{M}$ and $\mathcal{N}$ of T and all tuples $a \in M^x$ and $a' \in N^x$, if $\mathrm{tp}_{\Delta_x}(a) \subseteq \mathrm{tp}_{\Delta_x}(a')$, then $\mathrm{tp}(a) = \mathrm{tp}(a')$.

Proof of claim: Suppose that $\mathcal{M}$ and $\mathcal{N}$ are models of T, $a \in M^x$, $a' \in N^x$, and $\mathrm{tp}_{\Delta_x}(a) \subseteq \mathrm{tp}_{\Delta_x}(a')$. Let y be a tuple of variables enumerating the elements of $\mathrm{bcl}(a)$ which are not in a. Let $p(x,y) = \mathrm{qftp}(\mathrm{bcl}(a))$, and let $q(x) = \mathrm{tp}(a')$. We claim that $T \cup p(x,y) \cup q(x)$ is consistent.

Let $b = (b_1, \dots, b_n)$ be a finite tuple from $\mathrm{bcl}(a)$ which is disjoint from a, and let $\psi(x,y')$ be a quantifier-free formula such that $\mathcal{M} \models \psi(a,b)$ (where $y' = (y_1, \dots, y_n)$ is the finite subtuple of y enumerating b).

For each $1 \leqslant j \leqslant n$, the fact that $b_j \in \mathrm{bcl}(a)$ is witnessed by $\mathcal{M} \models \exists z_j\, \varphi_j(a, b_j, z_j)$, where $\varphi_j(x, y_j, z_j)$ is quantifier-free and bounded in $y_j z_j$. Letting $z = (z_1, \dots, z_n)$, the conjunction $\bigwedge_{j=1}^n \varphi_j(x, y_j, z_j)$ is a quantifier-free formula $\varphi(x, y', z)$ which is bounded in $y'z$. It follows that $\varphi(x,y',z) \wedge \psi(x,y')$ is also bounded in $y'z$, and $\mathcal{M} \models \exists z\, (\varphi(a,b,z) \wedge \psi(a,b))$. Then

$$\exists y'z\, (\varphi(x,y',z) \wedge \psi(x,y')) \in \mathrm{tp}_{\Delta_x}(a) \subseteq \mathrm{tp}_{\Delta_x}(a'),$$

so $\mathcal{N} \models \exists y'z\, (\varphi(a',y',z) \wedge \psi(a',y'))$. Letting $b' \in N_{y'}$ be a witness for the first block of existential quantifiers, $\mathcal{N} \models \psi(a',b')$, so $T \cup \{\psi(x,y')\} \cup q(x)$ is consistent.

By compactness, $T \cup p(x,y) \cup q(x)$ is consistent, so there exists a model $\mathcal{N}' \models T$, a tuple $a'' \in (N')^x$ realizing $q(x)$, and an embedding $f\colon \mathrm{bcl}(a) \to \mathcal{N}'$ such that $f(a) = a''$. By bcl-completeness, we have $\mathrm{tp}(a) = \mathrm{tp}(a'') = \mathrm{tp}(a')$, as was to be shown.

Having established the claim, we conclude with a standard compactness argument. Let $\varphi(x)$ be an L-formula. Suppose $\mathcal{M} \models T$ and $\mathcal{M} \models \varphi(a)$. Let $p_a(x) = \mathrm{tp}_{\Delta_x}(a)$. By the claim, $T \cup p_a(x) \cup \{\neg\varphi(x)\}$ is inconsistent. Since $p_a(x)$ is closed under finite conjunctions (up to equivalence) by Remark 5.3, there is a formula $\psi_a(x) \in p_a(x)$ such that $T \models \psi_a(x) \to \varphi(x)$.

Now

$$T \cup \{\varphi(x)\} \cup \{\neg\psi_a(x) \mid \mathcal{M} \models T \text{ and } \mathcal{M} \models \varphi(a)\}$$

is inconsistent, so there are finitely many $a_1, \dots, a_n$ such that $T \models \varphi \to (\bigvee_{i=1}^n \psi_{a_i}(x))$. Since also $T \models (\bigvee_{i=1}^n \psi_{a_i}(x)) \to \varphi(x)$, we have shown that $\varphi(x)$ is T-equivalent to $\bigvee_{i=1}^n \psi_{a_i}(x)$. $\square$

It may be surprising that acl-completeness does not already imply every formula is equivalent to a finite disjunction of b.e. formulas, i.e., acl-completeness is not equivalent to bcl-completeness. We give a counterexample.

Example 5.1. Let L be the language with a single unary function symbol f. We denote by $E(x, y)$ the equivalence relation defined by $f(x) = f(y)$. We say an element of an L-structure is **special** if it is in the image of f. Let T be the theory asserting the following:

(1) Models of T are nonempty.
(2) There are no cycles, i.e., for all $n \geqslant 1$, $\forall x\, f^n(x) \neq x$.
(3) Each E-class is infinite and contains exactly one special element.

Every T-model can be decomposed into a disjoint union of **connected components**, each of which is a chain of E-classes, $(C_n)_{n\in\mathbb{Z}}$, such that each class C_n contains a unique special element a_n, and $f(b) = a_n$ for all $b \in C_{n-1}$.

Let A be a subset of a T-model. Then $\operatorname{acl}(A)$ consists of A, together with the $\mathbb{Z}$-indexed chain of special elements in each connected component which meets A. But $\operatorname{bcl}(A)$ is just the substructure generated by A: it only contains the special elements from E-classes further along in the chain than some element of A. Indeed, if a_n is the unique special element in class C_n, $a_n \notin A$, and no element of A is in any class C_m with $m < n$ in the same connected component, then a_n does not satisfy any bounded and b.e. formula with parameters from A.

It is not hard to show that T is acl-complete (and hence complete, since $\operatorname{acl}(\varnothing) = \varnothing$), but not bcl-complete. For an explicit example of a formula which is not equivalent to a disjunction of b.e. formulas, consider the formula

$$\exists y\, f(y) = x$$

defining the special elements.

5.3. Existential bi-interpretations

Here we set our notation for interpretations and related notions. We will then show that existential bi-interpretations preserve the property of being existentially closed, and hence restrict to bi-interpretations between model companions, when these exist.

Let T be an L-theory, and let T' be an L'-theory. An **interpretation** of T' in T, $F\colon T \rightsquigarrow T'$, consists of the following data:

(1) For every sort s' in L', an L-formula $\varphi_{s'}(x_{s'})$ and an L-formula $E_{s'}(x_{s'}, x^*_{s'})$.
(2) For every relation symbol R' in L' of type $(s'_1, \dots, s'_n)$ in L', an L-formula $\varphi_{R'}(x_{s'_1}, \dots, x_{s'_n})$.
(3) For every function symbol f' in L' of type $(s'_1, \dots, s'_n) \to s'$ in L', an L-formula $\varphi_{f'}(x_{s'_1}, \dots, x_{s'_n}, x_{s'})$.

We then require that for every model $\mathcal{M} \models T$, the formulas above define an L'-structure $\mathcal{M}'$ in the natural way, such that $\mathcal{M}' \models T'$. See [**37**, Section 5.3] for details. We sometimes denote $\mathcal{M}'$ by $F(\mathcal{M})$. For every sort s' in L', we write $\pi_{s'}$ for the surjective quotient map $\varphi_{s'}(\mathcal{M}) \to M'_{s'}$.

An interpretation $F{:}T \rightsquigarrow T'$ is an **existential interpretation** if for each sort s' in L', the L-formula $\varphi_{s'}(x_{s'})$ is T-equivalent to an existential formula, and all other formulas involved in the interpretation and their negations (i.e., the formulas $E_{s'}$, $\neg E_{s'}$, $\varphi_{R'}$, $\neg\varphi_{R'}$, $\varphi_{f'}$, and $\neg\varphi_{f'}$) are also T-equivalent to existential formulas.

Lemma 5.3. *Suppose $F{:}T \rightsquigarrow T'$ is a existential interpretation. Let $\varphi'(y)$ be a quantifier-free L'-formula, where $y = (y_1, \dots, y_n)$ and y_i is a variable of sort s'_i. Then there is an existential L-formula $\widehat{\varphi}(x_{s'_1}, \dots, x_{s'_n})$ such that for every $\mathcal{M} \models T$ and every tuple $a = (a_1, \dots, a_n)$ with $a_i \in \varphi_{s'_i}(\mathcal{M})$, $\mathcal{M} \models \widehat{\varphi}(a)$ if and only if $F(\mathcal{M}) \models \varphi'(\pi_{s_1}(a_1), \dots, \pi_{s_n}(a_n))$.*

Proof. By Corollary 5.1, $\varphi'(y)$ is equivalent to a finite disjunction of E$\flat$ formulas. The rest of the proof is as in [**37**, Theorem 5.3.2]. The fact that the formulas $E_{s'}$, $\neg E_{s'}$, $\varphi_{R'}$, $\neg\varphi_{R'}$, $\varphi_{f'}$, and $\neg\varphi_{f'}$ are existential implies that flat literal L'-formulas can be pulled back to existential L-formulas, and the fact that the formulas φ'_s are existential is used in the inductive step to handle existential quantifiers. $\square$

A **bi-interpretation** (F, G, η, η') between T and T' consists of an interpretation $F{:}T \rightsquigarrow T'$, an interpretation $G{:}T' \rightsquigarrow T$, together with L-formulas and L'-formulas defining for each $\mathcal{M} \models T$ and each $\mathcal{N}' \models T'$ isomorphisms

$$\eta_{\mathcal{M}} : \mathcal{M} \to G(F(\mathcal{M})) \quad \text{and} \quad \eta'_{\mathcal{N}'} : \mathcal{N}' \to F(G(\mathcal{N}')).$$

See [**37**, Section 5.4] for the precise definition. Such a bi-interpretation is **existential** if F and G are each existential interpretations, and moreover the aforementioned L-formulas and L'-formulas are existential. If there is an existential bi-interpretation between T and T', we say that T and T' are existentially bi-interpretable. The following is [**37**, Exercise 5.4.3]:

Lemma 5.4. *Suppose $F{:}T \rightsquigarrow T'$ is existential. Then F induces a functor from the category of models of T and embeddings to the category of models of T' and embeddings. Suppose moreover that (F, G, η, η') is an existential bi-interpretation from T to T'. Then the induced functors form an equivalence of categories; in particular, if $f : \mathcal{M} \to \mathcal{N}$ is an L-embedding, then the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{N} \arrow[r, "\eta_{\mathcal{N}}"] & G(F(\mathcal{N})) \\
\mathcal{M} \arrow[u, "f"] \arrow[r, "\eta_{\mathcal{M}}"] & G(F(\mathcal{M})) \arrow[u, "G(F(f))"']
\end{tikzcd}$$

We next prove the main result of this section:

Proposition 5.2. *Suppose T and T' are existentially bi-interpretable. Then $\mathcal{M}$ is an existentially closed model of T if and only if $F(\mathcal{M})$ is an existentially closed model of T'.*

Proof. Let (F, G, η, η') be an existential bi-interpretation of between T and T'. It suffices to show that if $F(\mathcal{M})$ is an existentially closed model of T', then $\mathcal{M}$ is an existentially closed model of T. Indeed, by symmetry it follows that if $G(\mathcal{N}')$ is an existentially closed model of T, then $\mathcal{N}'$ is an existentially closed model of T'. And then, since $\eta_{\mathcal{M}} : \mathcal{M} \to G(F(\mathcal{M}))$ is an isomorphism, if $\mathcal{M}$ is existentially closed, then $F(\mathcal{M})$ is existentially closed.

So assume that $F(\mathcal{M})$ is an existentially closed model of T'. Let $f : \mathcal{M} \to \mathcal{N}$ be an embedding of T-models, and let $\varphi(y)$ be a quantifier-free formula with parameters from $\mathcal{M}$ which is satisfied in $\mathcal{N}$. By commutativity of the diagram in Lemma 5.4, after moving the parameters of $\varphi(y)$ into $G(F(\mathcal{M}))$ by the isomorphism $\eta_{\mathcal{M}}$, we find that $\varphi(y)$ is satisfied in $G(F(\mathcal{N}))$, and it suffices to show that it is satisfied in $G(F(\mathcal{M}))$.

By Lemma 5.3, there is an existential L'-formula $\widehat{\varphi}'(x)$ with parameters from $F(\mathcal{M})$ such that $F(\mathcal{N}) \models \widehat{\varphi}'(a)$ if and only if $G(F(\mathcal{N})) \models \varphi(b)$, where b is the image of a under the appropriate π_s quotient maps. Writing $\widehat{\varphi}'(x)$ as $\exists z\, \psi'(x, z)$, we have $F(\mathcal{N}) \models \psi'(a, c)$ for some c, where a is any preimage of the tuple from $G(F(\mathcal{N}))$ satisfying $\varphi(y)$. But since $F(\mathcal{M})$ is existentially closed, there are some a^* and c^* in $F(\mathcal{M})$ such that $\mathcal{M} \models \psi'(a^*, c^*)$, so $\mathcal{M} \models \widehat{\varphi}'(a^*)$, and it follows that $\varphi(y)$ is satisfied in $G(F(\mathcal{M}))$, as desired. □

Corollary 5.2. *Suppose T and T' are inductive, and T has a model companion T^*. If (F, G, η, η') is an existential bi-interpretation between T and T', then T' has a model companion $(T')^*$, and (F, G, η, η') restricts to an existential bi-interpretation between T^* and $(T')^*$.*

Proof. By [**37**, Theorem 5.3.2], for every L-sentence $\varphi \in T^*$, there is an L'-sentence φ' such that for all $\mathcal{M}' \models T'$, $\mathcal{M}' \models \varphi'$ if and only if $G(\mathcal{M}) \models \varphi$. Let $(T')^* = T' \cup \{\varphi' \mid \varphi \in T^*\}$. Then $\mathcal{M}' \models (T')^*$ if and only if $\mathcal{M}' \models T'$ and $G(\mathcal{M}') \models T^*$. By Proposition 5.2, $\mathcal{M}' \models (T')^*$ if and only if $\mathcal{M}'$ is an existentially closed model of T'. So $(T')^*$ is the model companion of T'. And Proposition 5.2 further implies that $\mathcal{M} \models T^*$ if and only if $F(\mathcal{M}) \models (T')^*$. So (F, G, η, η') restricts to an existential bi-interpretation between the model companions. □

5.4. Stationary independence relations

In this section, T is a complete L-theory, L' is a first order language extending L, and T' is a complete L'-theory extending T. Let $\boldsymbol{\mathcal{M}}'$ be a monster model of T' and $\boldsymbol{\mathcal{M}}$ be the L-reduct of $\boldsymbol{\mathcal{M}}'$, so $\boldsymbol{\mathcal{M}}$ is a monster model of T.

Let $\mathop{\smile\!\!\!\!\!\!|}$ be a ternary relation on small subsets of $\boldsymbol{\mathcal{M}}$. We consider the following properties that $\mathop{\smile\!\!\!\!\!\!|}$ may satisfy. The first three are specific to T, while the fourth concerns the relationship between T and T'. We let A, B, and C range over arbitrary small subsets of $\boldsymbol{\mathcal{M}}$.

(1) **Invariance:** If σ is an automorphism of $\boldsymbol{\mathcal{M}}$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $\sigma(A) \mathop{\smile\!\!\!\!\!\!|}_{\sigma(C)} \sigma(B)$.

(2) **Algebraic independence:** If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, then

$$\mathrm{acl}_L(AC) \cap \mathrm{acl}_L(BC) = \mathrm{acl}_L(C).$$

(3) **Stationarity (over algebraically closed sets):** If $C = \mathrm{acl}_L(C)$, $\mathrm{tp}_L(A/C) = \mathrm{tp}_L(A^*/C)$, $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, and $A^* \mathop{\smile\!\!\!\!\!\!\mid}_C B$, then $\mathrm{tp}_L(A/BC) = \mathrm{tp}_L(A^*/BC)$.

(4) **Full existence (over algebraically closed sets) in T':** If $C = \mathrm{acl}_{L'}(C)$ then there exists A^* with $\mathrm{tp}_{L'}(A^*/C) = \mathrm{tp}_{L'}(A/C)$ and $A^* \mathop{\smile\!\!\!\!\!\!\mid}_C B$ in $\mathfrak{M}$.

For brevity, we omit the parenthetical "(over algebraically closed sets)" in properties (3) and (4).

We say $\mathop{\smile\!\!\!\!\!\!\mid}$ is a **stationary independence relation** in T if it satisfies invariance, algebraic independence, and stationarity. In particular, a stationary independence relation identifies, for every L-type $p(x) \in S_x(C)$ with $C = \mathrm{acl}_L(C)$ and every set B, a unique "independent" extension of $p(x)$ in $S_x(BC)$.

Our definition of a stationary independence relation differs from that introduced in [**78**]. Most natural stationary independence relations satisfy additional axioms (symmetry, monotonicity, etc.). We only require the axioms listed above.

Forking independence $\mathop{\smile\!\!\!\!\!\!\mid}^f$ in a stable theory with weak elimination of imaginaries is the most familiar stationary independence relation, and this is the relation we will use in most examples. However, as the next example shows, there are also non-trivial examples in unstable theories.

Example 5.2. Suppose L contains a single binary relation E, and T is the theory of the random graph (the Fraïssé limit of the class of finite graphs). Define:

$$A \mathop{\smile\!\!\!\!\!\!\mid}^E_C B \iff A \cap B \subseteq C \text{ and } aEb \text{ for all } a \in A \setminus C \text{ and } b \in B \setminus C$$

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{\not E}_C B \iff A \cap B \subseteq C \text{ and } \neg aEb \text{ for all } a \in A \setminus C \text{ and } b \in B \setminus C.$$

Both $\mathop{\smile\!\!\!\!\!\!\mid}^E$ and $\mathop{\smile\!\!\!\!\!\!\mid}^{\not E}$ are stationary independence relations in T.

Now let $L' = \{E, P\}$, where P is a unary predicate, and let T' be the theory of the Fraïssé limit of the class of finite graphs with a predicate P naming a clique. T' extends T and has quantifier elimination, and $\mathrm{acl}_{L'}(A) = A$ for all sets A.

Then $\mathop{\smile\!\!\!\!\!\!\mid}^E$ has full existence in T'. Indeed, for any A, B, and C, let $p(x) = \mathrm{tp}_{L'}(A/C)$, where $x = (x_a)_{a \in A}$ is a tuple of variables enumerating A. The type $p(x) \cup \{x_a E b \mid a \in A \setminus C \text{ and } b \in B \setminus C\}$ is consistent, and for any realization A^* of this type, we have $A^* \mathop{\smile\!\!\!\!\!\!\mid}^E_C B$ in $\mathfrak{M}$.

On the other hand, let a and b be any two elements of $\mathfrak{M}'$ satisfying P. Then for any realization a^* of $\mathrm{tp}_{L'}(a/\varnothing)$, we have $P(a^*)$, so $a^* E b$, and $a^* \not\mathop{\smile\!\!\!\!\!\!|}^{\not E}_{\varnothing} b$ in $\mathfrak{M}$. So $\mathop{\smile\!\!\!\!\!\!|}^{\not E}$ does not have full existence in T'.

The remainder of this section is devoted to the proof that when T is stable with weak elimination of imaginaries, the stationary independence relation $\mathop{\smile\!\!\!\!\!\!|}^{f}$ in T always has full existence in T'. We first recall some definitions. T has **stable forking** if whenever a complete type $p(x)$ over B forks over $A \subseteq B$, then there is a stable formula $\delta(x, y)$ such that $\delta(x, b) \in p(x)$ and $\delta(x, b)$ forks over A. Every theory with stable forking is simple; the converse is the Stable Forking Conjecture, which remains open (see [**49**]).

We recall a few variations on the notion of elimination of imaginaries (see [**10**]).

(1) T has **elimination of imaginaries** if every $a \in \mathfrak{M}^{\mathrm{eq}}$ is interdefinable with some $b \in \mathfrak{M}$, i.e., $a \in \mathrm{dcl}^{\mathrm{eq}}(b)$ and $b \in \mathrm{dcl}^{\mathrm{eq}}(a)$.
(2) T has **weak elimination of imaginaries** if for every $a \in \mathfrak{M}^{\mathrm{eq}}$ there is some $b \in \mathfrak{M}$ such that $a \in \mathrm{dcl}^{\mathrm{eq}}(b)$ and $b \in \mathrm{acl}^{\mathrm{eq}}(a)$.
(3) T has **geometric elimination of imaginaries** if every $a \in \mathfrak{M}^{\mathrm{eq}}$ is interalgebraic with some $b \in \mathfrak{M}$, i.e., $a \in \mathrm{acl}^{\mathrm{eq}}(b)$ and $b \in \mathrm{acl}^{\mathrm{eq}}(a)$.

Let $\delta(x, y)$ be a formula. An **instance** of δ is a formula $\delta(x, b)$ with $b \in \boldsymbol{M}^y$, and a **δ-formula** is a Boolean combination of instances of δ. A **global δ-type** is a maximal consistent set of δ-formulas with parameters from $\boldsymbol{M}$. We denote by $S_\delta(\boldsymbol{M})$ the Stone space of global δ-types.

The following lemma is a well-known fact about the existence of weak canonical bases for δ-types when $\delta(x, y)$ is stable.

Lemma 5.5. *Suppose T has geometric elimination of imaginaries, and $\delta(x, y)$ is a stable formula. For any $q(x) \in S_\delta(\boldsymbol{M})$, there exists a tuple d such that:*

(1) *$q(x)$ has finite orbit under automorphisms of $\mathfrak{M}$ fixing d.*
(2) *d has finite orbit under automorphisms of $\mathfrak{M}$ fixing $q(x)$.*
(3) *$q(x)$ does not divide over d.*

If T has weak elimination of imaginaries, we can arrange that d is fixed by automorphisms of $\mathfrak{M}$ fixing $q(x)$. And if T has elimination of imaginaries, we can further arrange that $q(x)$ is fixed by automorphisms of $\mathfrak{M}$ fixing d.

Proof. Let $e \in \mathfrak{M}^{\mathrm{eq}}$ be the canonical base for $q(x)$. Then $q(x)$ is fixed by all automorphisms fixing e, e is fixed by all automorphisms fixing $q(x)$, and $q(x)$ does not divide over e. By geometric elimination of imaginaries, e is interalgebraic with a real tuple d, and (1), (2), and

(3) follow immediately. The cases when T has elimination of imaginaries or weak elimination of imaginaries are similar. $\square$

The following lemma is essentially the same idea as [**64**, Lemma 3], which itself makes use of key ideas from [**40**, Lemmas 5.5 and 5.8].

Lemma 5.6. *Suppose T has stable forking and geometric elimination of imaginaries. Then $\mathop{\smile\!\!\!\!\!|}^f$ in T has full existence in T'.*

Proof. Suppose towards a contradiction that there exist sets A, B, and C in $\mathfrak{M}'$ such that $C = \mathrm{acl}_{L'}(C)$, and for any A^* with $\mathrm{tp}_{L'}(A^*/C) = \mathrm{tp}_{L'}(A/C)$, $A^* \not\mathop{\smile\!\!\!\!\!|}^f_C B$ in $\mathfrak{M}$. We may assume $C \subseteq B$. Let $p(x) = \mathrm{tp}_{L'}(A/C)$. Since T has stable forking, the fact that $\mathrm{tp}_L(A^*/B)$ forks over C is always witnessed by a stable L-formula. So the partial type

$$p(x) \cup \{\neg\delta(x,b) \mid \delta(x,y) \in L \text{ is stable, and } \delta(x,b) \text{ forks over } C \text{ in } \mathfrak{M}\}$$

is not satisfiable in $\mathfrak{M}'$. By saturation and compactness, we may assume that A is finite and x is a finite tuple of variables. And as stable formulas and forking formulas are closed under disjunctions, there is an $L'(C)$-formula $\varphi(x) \in p(x)$, a stable L-formula $\delta(x,y)$, and $b \in \boldsymbol{M}^y$ such that $\delta(x,b)$ forks over C, and

$$\mathfrak{M}' \models \forall x\,(\varphi(x) \to \delta(x,b)).$$

Since forking and dividing agree in simple theories [**9**, Prop. 5.17], $\delta(x,b)$ divides over C.

Let $[\varphi]$ be the set of all δ-types in $S_\delta(\boldsymbol{M})$ which are consistent with $\varphi(x)$. This is a closed set in $S_\delta(\boldsymbol{M})$: it consists of all global δ-types $r(x)$ such that $\chi(x) \in r(x)$ whenever $\chi(x)$ is a δ-formula and $\varphi(\mathfrak{M}') \subseteq \chi(\mathfrak{M}')$. In particular, if $r(x) \in [\varphi]$, then $\delta(x,b) \in r(x)$. Since δ is stable, $[\varphi]$ contains finitely many points of maximal Cantor-Bendixson rank. Let $q(x)$ be such a point.

Let d be the weak canonical base for $q(x)$ obtained in Lemma 5.5. Since $[\varphi]$ is fixed setwise by any L'-automorphism fixing C, $q(x)$ has finitely many conjugates under such automorphisms. It follows that d too has finitely many conjugates, so $d \in C$, as C is algebraically closed in $\mathcal{M}'$. But then $q(x)$ does not divide over C, contradicting the fact that $\delta(x,b) \in q(x)$. $\square$

Remark 5.6. The following counterexample shows the assumptions of geometric elimination of imaginaries in T and $C = \mathrm{acl}(C)$ in $\mathfrak{M}'$ (not just in $\mathfrak{M}$) in Lemma 5.6 are necessary. Let T be the theory of an equivalence relation with infinitely many infinite classes. Let T' be the expansion of this theory by a single unary predicate P naming one of the classes. Let a and b be two elements of the class named by P in $\mathfrak{M}'$, and let $C = \varnothing$ (which is algebraically closed in $\mathfrak{M}$ and $\mathfrak{M}'$). For any a^* such that $\mathrm{tp}_{L'}(a^*/\varnothing) = \mathrm{tp}_{L'}(a/\varnothing)$, we have a^*Eb, and xEb forks over $\varnothing$ in $\mathfrak{M}$. To fix this, we move to $\mathfrak{M}^{\mathrm{eq}}$, so we have another sort containing names for all the E-classes. Note that $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$ in $\mathfrak{M}$ still doesn't contain any of these names. But $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$ in $\mathfrak{M}'$ contains the name for the class named by P, since it is fixed by L'-automorphisms. And we recover the lemma, since xEb does not fork over the name for the E-class of b.

Remark 5.7. It is also possible for Lemma 5.6 to fail when there are unstable forking formulas. Let T be be the theory of $(\mathbb{Q}, <)$ and T' be the expansion of T by a unary predicate P defining an open interval (p, p'), where $p < p'$ are irrational reals. Let $b_1 < a < b_2$ be elements of $\mathfrak{M}'$ such that $a \in P$ and $b_1, b_2 \notin P$. Let $C = \varnothing$ (which is algebraically closed in $\mathfrak{M}'$). Then for any realization a^* of $\mathrm{tp}_{L'}(a/\varnothing)$, we have $a^* \not\mathop{\smile\!\!\!\!\!\mid}^f_{\varnothing} b_1 b_2$ in $\mathfrak{M}$, witnessed by the formula $b_1 < x < b_2$.

Remark 5.8. It is not possible to strengthen the conclusion of Lemma 5.6 to the following: For all small sets A, B, and C, such that $C = \mathrm{acl}_{L'}(C)$, and for any A'' such that $\mathrm{tp}_L(A''/C) = \mathrm{tp}_L(A/C)$ and $A'' \mathop{\smile\!\!\!\!\!\mid}^f_C B$ in $\mathfrak{M}$, there exists A' with $\mathrm{tp}_{L'}(A'/C) = \mathrm{tp}_{L'}(A/C)$ and $\mathrm{tp}_L(A'/BC) = \mathrm{tp}_L(A''/BC)$.

That is, while it is possible to find a realization A' of $\mathrm{tp}_{L'}(A/C)$ such that $\mathrm{tp}_L(A'/BC)$ is a nonforking extension of $\mathrm{tp}_L(A/C)$, it is not possible in general to obtain an arbitrary nonforking extension of $\mathrm{tp}_L(A/C)$ in this way.

For a counterexample, consider the theories T and T' from Example 5.2 above. T has stable forking and geometric elimination of imaginaries. Let a and b be elements of the clique defined by P in $\mathfrak{M}'$, and let $C = \varnothing$ (which is algebraically closed in $\mathfrak{M}'$). Let a'' be any element such that $\mathfrak{M}' \vDash \neg a''Eb$, and note that $a'' \mathop{\smile\!\!\!\!\!\mid}^f_{\varnothing} b$ and $\mathrm{tp}_L(a''/\varnothing) = \mathrm{tp}_L(a/\varnothing)$ (there is only one 1-type over the empty set with respect to T). But for any a' with $\mathrm{tp}_{L'}(a'/\varnothing) = \mathrm{tp}_{L'}(a/\varnothing)$, $\mathfrak{M}' \vDash P(a')$, so $a'Eb$, and $\mathrm{tp}_L(a'/b) \neq \mathrm{tp}_L(a''/b)$.

We have seen that the hypotheses of stable forking (and hence simplicity) and geometric elimination of imaginaries in T are sufficient to ensure that $\mathop{\smile\!\!\!\!\!\mid}^f$ has full existentence in T', with no further assumptions on T'. But we would also like $\mathop{\smile\!\!\!\!\!\mid}^f$ to be a stationary independence relation in T.

In a simple theory T, $\mathop{\smile\!\!\!\!\!\mid}^f$ satisfies stationarity over $\mathrm{acl}^{\mathrm{eq}}$-closed sets if and only if T is stable [**9**, Ch. 11]. And a stable theory has weak elimination of imaginaries if and only if it has geometric elimination of imaginaries and $\mathop{\smile\!\!\!\!\!\mid}^f$ satisfies stationarity over acl-closed sets [**10**, Prop. 3.2 and 3.4]. So stability with weak elimination of imaginaries is the natural hypothesis on T in the following proposition.

Proposition 5.3. *If T is stable with weak elimination of imaginaries, then $\mathop{\smile\!\!\!\!\!\mid}^f$ is a stationary independence relation in T which has full existence in T'.*

5.4.1. NSOP$_1$**.** Dzamonja and Shelah [**29**] introduced NSOP$_1$. Let $\leqslant$ be the lexicographic order on $2^{<\omega}$ and let $\nu\hat{\eta}$ be the usual concatenation of $\nu, \eta \in 2^{<\omega}$. Let T be a theory. A formula $\varphi(x; y)$ has **SOP**$_1$ (relative to T) if there are tuples $(a_\eta)_{\eta\in 2^{<\omega}}$ in a model of T such that:

(1) $\nu\hat{}0 \leqslant \eta$ implies $\{\varphi(x; a_\eta), \varphi(x; a_{\nu\hat{}1})\}$ is inconsistent for all $\nu, \eta \in 2^{<\omega}$,
(2) $\{\varphi(x; a_{\sigma|_n}) \mid n \in \omega\}$ is consistent for all $\sigma \in 2^\omega$.

We say that T is NSOP$_1$ if no formula has SOP$_1$ relative to T.

We recall the definition of Kim independence, due to Ramsey, and review some foundational results, most of which are due to Kaplan and Ramsey. Suppose T is complete, let $\mathfrak{M}$ be a monster model of T, and let $\mathcal{M} \preccurlyeq \mathfrak{M}$ be a small submodel.

A global type $q(y) \in S_y(\mathfrak{M})$ is M**-invariant** if for any formula $\psi(y, z)$ and any elements $c \equiv_M c'$ of $\boldsymbol{M}_z$, we have $\psi(y, c) \in q$ if and only if $\psi(y, c') \in q$. A sequence $(b_i)_{k\in\omega}$ is a **Morley sequence** for q over M if b_k realizes the restriction of $q(y)$ to $Mb_0 \ldots b_{k-1}$ for all i. Suppose $q(y)$ is a global M-invariant type extending $\mathrm{tp}(b/M)$ and $(b_i)_{i\in\omega}$ is a Morley sequence for q over M. The formula $\varphi(x, b)$ q-divides over M if $\{\varphi(x, b_i)\}_{i\in\omega}$ is inconsistent. The formula $\varphi(x, b)$ **Kim divides** over M if it q-divides over M for some global M-invariant type q extending $\mathrm{tp}(b/M)$. A formula **Kim forks** over M if it implies a disjunction of formulas which Kim divide over M. We write $A \mathop{\smile\!\!\!\!\!|}^K_M B$ (read "A is Kim independent from B over M") to mean that no formula in $\mathrm{tp}(A/MB)$ Kim forks over M.

These definitions are made over a submodel M of $\mathcal{M}$, rather than over an arbitrary small set A of parameters, as a type over A need not extend to a global A-invariant type.

Theorem 5.2 ([**44**] Theorem 3.15)**.** *Suppose* T *is* NSOP$_1$. *If* $\varphi(x, b)$ *q-divides for* some *global* M*-invariant type* q *extending* $\mathrm{tp}(b/M)$*, then* $\varphi(x, b)$ *q-divides for* every *global* M*-invariant type* q *extending* $\mathrm{tp}(b/M)$.

Theorem 5.2 is a version of Kim's lemma for Kim independence in NSOP$_1$ theories. Kim's lemma was originally proven for forking in simple theories.

Theorem 5.3 ([**44**] Proposition 3.19)**.** *Suppose* T *is* NSOP$_1$. *If* $\varphi_i(x, b_i)$ *Kim divides over* M *for all* $1 \leqslant i \leqslant n$*, then* $\bigvee_{i=1}^n \varphi_i(x, b_i)$ *Kim divides over* M.

Theorem 5.3 shows Kim forking and Kim dividing agree.

Theorem 5.4 ([**44**] Corollary 5.17)**.** *If* T *is* NSOP$_1$*, then for all* A, B *and submodels* M*,*

$$A \mathop{\smile\!\!\!\!\!|}^K_M B \quad \textit{if and only if} \quad \mathrm{acl}(MA) \mathop{\smile\!\!\!\!\!|}^K_M \mathrm{acl}(MB).$$

Theorem 5.5 below characterizes simple theories among NSOP_1 theories. Kim independence satisfies **base monotonicity over models** if $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M Nb$ implies $a \mathop{\smile\!\!\!\!\!\!\mid}^K_N b$ for all $M \preccurlyeq N$.

Theorem 5.5. *Suppose T is* NSOP_1. *Then the following are equivalent:*

(1) *T is simple,*
(2) *$\mathop{\smile\!\!\!\!\!\!\mid}^f_M = \mathop{\smile\!\!\!\!\!\!\mid}^K_M$ for all submodels M.*
(3) *T is NTP_2,*
(4) *Kim independence satisfies base monotonicity over models.*

The equivalence of $(1), (2)$, and (4) above follows from Proposition 8.4 and Proposition 8.8 of [**44**]. The equivalence of (1) and (3) is Corollary 8.5 of [**44**], but it also follows immediately from the well-known facts that a non-simple theory has TP_1 or TP_2 [**69**], any NSOP_1 theory is NSOP_2 [**29**], and NTP_1 is equivalent to NSOP_2 [**47**].

Theorem 5.6 ([**44**] Theorem 9.1)**.** *Suppose $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies the following for all A, A', B, B' and all submodels M, M':*

(1) ***Invariance**: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $MAB \equiv M'A'B'$, then $A' \mathop{\smile\!\!\!\!\!\!\mid}_{M'} B'$.*
(2) ***Existence**: $A \mathop{\smile\!\!\!\!\!\!\mid}_M M$*
(3) ***Monotonicity**: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!\mid}_M B'$.*
(4) ***Symmetry**: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$, then $B \mathop{\smile\!\!\!\!\!\!\mid}_M A$.*
(5) ***The independence theorem**: If $A \equiv_M A'$, $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$, $A' \mathop{\smile\!\!\!\!\!\!\mid}_M C$, and $B \mathop{\smile\!\!\!\!\!\!\mid}_M C$, then there exists A'' such that $A'' \equiv_{MB} A$, $A'' \equiv_{MC} A'$, and $A'' \mathop{\smile\!\!\!\!\!\!\mid}_M BC$.*
(6) ***Strong finite character**: If $A \not\mathop{\smile\!\!\!\!\!\!\mid}_M B$, then there is a formula $\varphi(x, b, m) \in \mathrm{tp}(A/MB)$ such that for any c such that $\mathfrak{M} \models \varphi(c, b, m)$, we have $c \not\mathop{\smile\!\!\!\!\!\!\mid}_M b$.*

Then T is NSOP_1. *If $\mathop{\smile\!\!\!\!\!\!\mid}$ additionally satisfies*

(7) ***Witnessing**: If $A \not\mathop{\smile\!\!\!\!\!\!\mid}_M B$, then there is a formula $\varphi(x, b, m) \in \mathrm{tp}(A/MB)$ which Kim divides over M.*

Then $\mathop{\smile\!\!\!\!\!\!\mid}_M = \mathop{\smile\!\!\!\!\!\!\mid}^K_M$ for all M.

Theorem 5.6 gives a positive axiomatic characterization of NSOP_1. An earlier version of this criterion appeared in [**18**].

Remark 5.9. If T is NSOP_1, then Kim independence satisfies all of the properties in Theorem 5.6. The only nontrivial properties are symmetry ([**44**] Theorem 5.16) and the independence theorem ([**44**] Theorem 6.5).

Now suppose, that $L \subseteq L'$, T is a complete L-theory, T' is a complete L'-theory extending T, and $\mathfrak{M}$ and $\mathfrak{M}'$ are monster models of T and T', respectively. We consider the relationship between independence in $\mathfrak{M}$ and $\mathfrak{M}'$.

It is not clear from the definition that Kim dividing is preserved under reducts, since the property of being an M-invariant type is not preserved under reducts in general. However, Theorem 5.2 shows that Kim dividing is always witnessed by q-dividing for a global type q which is finitely satisfiable in M, and this property is preserved under reducts. This gives us the following lemma.

Lemma 5.7. *If T' is* NSOP_1 *then:*

1. *T is* NSOP_1.
2. *Let $\mathbb{M} \preccurlyeq \mathbb{M}'$, and let $\varphi(x,b)$ be an L-formula. Then $\varphi(x,b)$ Kim divides over M in $\mathbb{M}$ if and only if it Kim divides over M in $\mathbb{M}'$.*
3. *Kim independence is preserved by reducts: if $A \mathop{\smile\!\!\!\!\!\!\perp}^K_M B$ in $\mathbb{M}'$, then also $A \mathop{\smile\!\!\!\!\!\!\perp}^K_M B$ in $\mathbb{M}$.*

Proof. For (1), the fact that NSOP_1 is preserved by reducts is clear from the definition: any formula with SOP_1 relative to T also has SOP_1 relative to T'.

For (2), fix a global L'-type q' extending $\mathrm{tp}_{L'}(b/M)$, which is finitely satisfiable in M (hence M-invariant). Let $(b_i)_{i\in\omega}$ be a Morley sequence for q' over M. Let q be the restriction of q' to L. Then q is also finitely satisfiable in M (hence M-invariant) and extends $\mathrm{tp}_L(b/M)$, and $(b_i)_{i\in\omega}$ is a Morley sequence for q over M. By Theorem 5.2 and (1), $\varphi(x,b)$ Kim divides over M in $\mathbb{M}$ if and only if $\{\varphi(x,b_i)\}_{i\in\omega}$ is inconsistent if and only if $\varphi(x,b)$ Kim divides over M in $\mathbb{M}'$.

For (3), suppose $A \mathop{\smile\!\!\!\!\!\!\perp}^K_M B$ in $\mathbb{M}'$. Then no formula in $\mathrm{tp}_{L'}(A/MB)$ Kim divides over M in $\mathbb{M}'$, so in particular, by (2), no formula in $\mathrm{tp}_L(A/MB)$ Kim divides over M in $\mathbb{M}$. So $A \mathop{\smile\!\!\!\!\!\!\perp}^K_M B$ in $\mathbb{M}$. $\square$

Define the relation $\mathop{\smile\!\!\!\!\!\!\perp}^r$, **independence in the reduct**, in $\mathbb{M}'$:

$$a \mathop{\smile\!\!\!\!\!\!\perp}^r_C b \Leftrightarrow \mathrm{acl}'(Ca) \mathop{\smile\!\!\!\!\!\!\perp}^f_{\mathrm{acl}'(C)} \mathrm{acl}'(Cb) \text{ in } \mathbb{M}$$

where acl' is the algebraic closure operator in $\mathbb{M}'$.

Note that if L is the language of equality and T is the theory of an infinite set then $\mathop{\smile\!\!\!\!\!\!\perp}^r = \mathop{\smile\!\!\!\!\!\!\perp}^a$ in $\mathbb{M}'$, where $\mathop{\smile\!\!\!\!\!\!\perp}^a$ is **algebraic independence**:

$$a \mathop{\smile\!\!\!\!\!\!\perp}^a_C b \Leftrightarrow \mathrm{acl}(Ca) \cap \mathrm{acl}(Cb) = \mathrm{acl}(C).$$

Strengthened versions of extension and the independence theorem, adding additional instances of algebraic independence to the conclusion, were established for Kim independence in NSOP_1 theories in [**52**]. Theorem 5.7 is a modified version of these results, with $\mathop{\smile\!\!\!\!\!\!\perp}^a$ replaced by $\mathop{\smile\!\!\!\!\!\!\perp}^r$, and additional hypotheses on T coming from Proposition 5.3. The proof will be given in [**53**].

Theorem 5.7. *Suppose T' is* NSOP_1 *and T is simple with stable forking and geometric elimination of imaginaries. Then we have the following:*

(1) *Reasonable extension: For all $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ and for all c, there exists a' such that $a' \equiv_{Mb} a$, $a' \mathop{\smile\!\!\!\!\!\mid}^K_M bc$, and $a' \mathop{\smile\!\!\!\!\!\mid}^r_{Mb} c$;*

(2) *Reasonable independence: If $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\mid}^K_M c$, $b \mathop{\smile\!\!\!\!\!\mid}^K_M c$, and $a \equiv_M a'$, then there exists a'' such that $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$, and $a'' \mathop{\smile\!\!\!\!\!\mid}^K_M bc$, and further $a \mathop{\smile\!\!\!\!\!\mid}^r_{Mc} b$, $a \mathop{\smile\!\!\!\!\!\mid}^r_{Mb} c$, and $b \mathop{\smile\!\!\!\!\!\mid}^r_{Ma} c$.*

CHAPTER 6

Interpolative structures and interpolative fusions

In addition to the notation convention of Chapter 5, we also assume the following throughout this chapter. let $L_\cap$ be a language and let $(L_i)_{i\in I}$ be a nonempty family of languages, all with the same set S of sorts, such that $L_i \cap L_j = L_\cap$ for all distinct $i, j \in I$. Let T_i be a (possibly incomplete) L_i-theory for each $i \in I$, and assume that each T_i has the same set $T_\cap$ of $L_\cap$-consequences. This assumption is quite mild: given an arbitrary family of L_i-theories $(T_i)_{i\in I}$, we can extend each T_i with the set of all $L_\cap$-consequences of $\bigcup_{i\in I} T_i$. Set

$$L_\cup = \bigcup_{i\in I} L_i \quad \text{and} \quad T_\cup = \bigcup_{i\in I} T_i,$$

and assume that $T_\cup$ is consistent. Alternatively, we could assume that $T_\cap$ is consistent, as these two assumptions are equivalent by Corollary 6.1 below and the assumption that the theories T_i have the same set of $T_\cap$-consequences.

Let $\mathcal{M}_\cup$ be an $L_\cup$-structure. Suppose $J \subseteq I$ is finite and $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for all $i \in J$. Then $(X_i)_{i\in J}$ is **separated** if there is a family $(X^i)_{i\in J}$ of $\mathcal{M}_\cap$-definable subsets of M^x such that

$$X_i \subseteq X^i \text{ for all } i \in J, \text{and } \bigcap_{i\in J} X^i = \varnothing.$$

We say $\mathcal{M}_\cup$ is **interpolative** if for all families $(X_i)_{i\in J}$ such that $J \subseteq I$ is finite and $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for all $i \in J$, $(X_i)_{i\in J}$ is separated if and only if $\bigcap_{i\in J} X_i \neq \varnothing$. Note that this generalizes the setting in the introduction.

When the class of interpolative $T_\cup$-models is elementary, we denote the theory of this class by $T^*_\cup$ and call it the **interpolative fusion** of $(T_i)_{i\in I}$ over $T_\cap$. In this case, we say that "$T^*_\cup$ exists".

Remark 6.1. The notion of interpolative structure is rather robust. If we change languages in a way that does not change the class of definable sets (with parameters), then the class of interpolative $L_\cup$-structures is not affected. In particular:

(1) An interpolative structure $\mathcal{M}_\cup$ remains so after adding new constant symbols naming elements of M to all the languages $L_\Box$ for $\Box \in I \cup \{\cup, \cap\}$.

(2) Suppose $L^\diamond_\Box$ is a definitional expansion of $L_\Box$ for $\Box \in I \cup \{\cap\}$, $L^\diamond_i \cap L^\diamond_j = L^\diamond_\cap$ for distinct i and j in I, and $L^\diamond_\cup = \bigcup_{i\in I} L^\diamond_i$ is the resulting definitional expansion of $L_\cup$. Then any

$L_\cup$-structure $\mathcal{M}_\cup$ has a canonical expansion $\mathcal{M}_\cup^\diamond$ to an $L_\cup^\diamond$-structure. And $\mathcal{M}_\cup$ is an interpolative $L_\cup$-structure if and only if $\mathcal{M}_\cup^\diamond$ is an interpolative $L_\cup^\diamond$-structure.

(3) An interpolative $\mathcal{M}_\cup$-structure remains so after replacing each function symbol f in each of the languages $L_\square$ for $\square \in I \cup \{\cup, \cap\}$ by a relation symbol R_f, interpreted as the graph of the interpretation of f in $\mathcal{M}_\cup$.

(4) Suppose $\mathcal{M}_\cup$ is an $L_\cup$-structure. Moving to $\mathcal{M}_\cap^{\mathrm{eq}}$ involves the introduction of new sorts and function symbols for quotients by $L_\cap$-definable equivalence relations on M. For all $\square \in I \cup \{\cup, \cap\}$, let $L_\square^{\cap-\mathrm{eq}}$ be the language obtained by adding these new sorts and function symbols to $L_\square$ (note that we do not add quotients by L_i-definable equivalence relations), and let $\mathcal{M}_\square^{\cap-\mathrm{eq}}$ be the natural expansion of $\mathcal{M}_\square$ to $L_\square^{\cap-\mathrm{eq}}$. Then $\mathcal{M}_\cup$ is interpolative if and only if $\mathcal{M}_\cup^{\cap-\mathrm{eq}}$ is interpolative. This follows from the fact that if $X_\square$ is an $\mathcal{M}_\square^{\cap-\mathrm{eq}}$-definable set in one of the new sorts, corresponding to the quotient of M^x by an $L_\cap$-definable equivalence relation, then the preimage of $X_\square$ under the quotient is $\mathcal{M}_\square$-definable.

The name "interpolative fusion" is inspired by a connection to the classical Craig interpolation theorem, which we recall now (see, for example, [**37**, Theorem 6.6.3]). It is well-known that in the context of first-order logic, the Craig interpolation theorem is equivalent to Robinson's joint consistency theorem.

Theorem 6.1. *Suppose L_1 and L_2 are first order languages with intersection $L_\cap$ and φ_i is an L_i-sentence for $i \in \{1,2\}$. If $\models (\varphi_1 \to \varphi_2)$ then there is an $L_\cap$-sentence ψ such that $\models (\varphi_1 \to \psi)$ and $\models (\psi \to \varphi_2)$. Equivalently: $\{\varphi_1, \varphi_2\}$ is inconsistent if and only if there is an $L_\cap$-sentence ψ such that $\models (\varphi_1 \to \psi)$ and $\models (\varphi_2 \to \neg\psi)$.*

We make extensive use of the following easy generalization of Theorem 6.1.

Corollary 6.1. *For each $i \in I$, let $\Sigma_i(x)$ be a set of L_i-formulas. If $\bigcup_{i \in I} \Sigma_i(x)$ is inconsistent, then there is a finite subset $J \subseteq I$ and an $L_\cap$-formula $\varphi^i(x)$ for each $i \in J$ such that:*

$$\Sigma_i(x) \models \varphi^i(x) \text{ for all } i \in J, \text{ and } \{\varphi^i(x) \mid i \in J\} \text{ is inconsistent.}$$

Proof. Using the standard trick of introducing a new constant for each free variable, we reduce to the case when x is the empty tuple of variables. We may also assume that the sets Σ_i are closed under conjunction. By compactness, if $\bigcup_{i \in I} \Sigma_i$ is inconsistent, then there is a nonempty finite subset $J \subseteq I$ and a formula $\varphi_i \in \Sigma_i$ for all $i \in J$ such that $\{\varphi_i \mid i \in J\}$ is inconsistent.

We argue by induction on the size of J. For the sake of notational simplicity, we suppose $J = \{1, \dots, n\}$. If $n = 1$, then we choose φ^1 to be the contradictory $L_\cap$-formula $\bot$. Suppose $n \geqslant 2$. Then $(\varphi_1 \wedge \dots \wedge \varphi_{n-1})$ is an $(L_1 \cup \dots \cup L_{n-1})$-sentence and the set

$$\{(\varphi_1 \wedge \dots \wedge \varphi_{n-1}), \varphi_n\} \text{ is inconsistent.}$$

Applying Theorem 6.1, we get a sentence ψ in $L_n \cap (L_1 \cup \ldots \cup L_n) = L_\cap$ such that

$$\vDash (\varphi_1 \wedge \ldots \wedge \varphi_{n-1}) \to \psi \quad \text{and} \quad \vDash \varphi_n \to \neg\psi.$$

Then $\{\varphi_i \wedge \neg\psi \mid i \leqslant n-1\}$ is inconsistent and $\varphi_i \wedge \neg\psi$ is an L_i-sentence for $1 \leqslant i \leqslant n-1$. Applying induction, we choose for each $1 \leqslant i \leqslant n-1$ an L-sentence θ^i such that

$$\vDash (\varphi_i \wedge \neg\psi) \to \theta^i \text{ for all } 1 \leqslant i \leqslant n-1, \text{and } \vDash \neg(\theta^1 \wedge \ldots \wedge \theta^{n-1}).$$

Finally, set φ^i to be $(\psi \vee \theta^i)$ for $1 \leqslant i \leqslant n-1$ and φ^n to be $\neg\psi$. It is easy to check that all the desired conditions are satisfied. □

The following consistency condition for types follows immediately from Corollary 6.1. This is the generalization to our context of Robinson's joint consistency theorem.

Corollary 6.2. *Let $p(x)$ be a complete $L_\cap$-type, and for all $i \in I$, let $p_i(x)$ be a complete L_i-type such that $p(x) \subseteq p_i(x)$. Then $\bigcup_{i\in I} p_i(x)$ is consistent.*

The following lemma says that any family of definable sets which is not separated has "potentially" nonempty intersection.

Lemma 6.1. *Let $\mathcal{M}_\cup$ be an $L_\cup$-structure, and suppose $J \subseteq I$ is finite and $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for all $i \in J$. The family $(X_i)_{i\in J}$ is separated if and only if for every $L_\cup$-structure $\mathcal{N}_\cup$ such that $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$, $\bigcap_{i\in J} X_i(\mathcal{N}_\cup) = \varnothing$.*

Proof. Suppose $(X_i)_{i\in J}$ is separated. Then there are $\mathcal{M}_\cap$-definable $X^1, \ldots, X^n$ such that $X_i \subseteq X^i$ for all $i \in J$ and $\bigcap_{i\in J} X^n = \varnothing$. Suppose $\mathcal{N}_\cup$ is a $T_\cup$-model satisfying $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$. Then $X_i(\mathcal{N}_\cup) \subseteq X^i(\mathcal{N}_\cup)$ for all $i \in J$ and $\bigcap_{i\in J} X^i(\mathcal{N}_\cup) = \varnothing$, so also $\bigcap_{i\in J} X_i(\mathcal{N}_\cup) = \varnothing$.

Conversely, suppose that for every $L_\cup$-structure $\mathcal{N}_\cup$ such that $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$, $\bigcap_{i\in J} X_i(\mathcal{N}_\cup) = \varnothing$. For each $i \in J$, let $\varphi_i(x,b)$ be an $L_i(M)$-formula defining X_i. Then the partial type

$$\bigcup_{i\in I} \mathrm{Ediag}(\mathcal{M}_i) \cup \bigcup_{i\in J} \varphi_i(x,b) \quad \text{is inconsistent.}$$

By compactness, there is a finite subset $J' \subseteq I$ with $J \subseteq J'$, a finite tuple $c \in M^y$ and a formula $\psi_i(b,c) \in \mathrm{Ediag}(\mathcal{M}_i)$ for each $i \in J'$ such that

$$\{\psi_i(b,c) \mid i \in J'\} \cup \{\varphi_i(x,b) \mid i \in J\} \quad \text{is inconsistent.}$$

Let φ_i be the true formula $\top$ when $i \in J' \smallsetminus J$, and define $\varphi_i'(x,y,z) = \varphi_i(x,y) \wedge \psi_i(y,z)$ for all $i \in J'$. Note that since $\mathcal{M}_i \vDash \psi_i(b,c)$,

$$\varphi_i(\mathcal{M}_\cup, b) = \varphi_i'(\mathcal{M}_\cup, b, c).$$

Applying Lemma 6.1, we obtain an an inconsistent family $\{\theta_i(x,y,z) \mid i \in J'\}$ of $L_\cap$-formulas such that $\models \varphi'_i(x,y,z) \to \theta_i(x,y,z)$ for each $i \in J'$. It follows that

$$\varphi_i(\mathcal{M}_\cup, b, c) \subseteq \theta_i(\mathcal{M}_\cup, b, c) \text{ for all } i \in J', \text{ and } \bigcap_{i \in J'} \theta_i(\mathcal{M}_\cup, b, c) = \varnothing.$$

But since $\varphi_i(\mathcal{M}_\cup, b, c) = M^x$ when $i \in J' \setminus J$, and also $\theta_i(\mathcal{M}_\cup, b, c) = M^x$ when $i \in J' \setminus J$. So already $\bigcap_{i \in J} \theta_i(\mathcal{M}_\cup, b, c) = \varnothing$, and the family $(\theta_i(\mathcal{M}_\cup, b, c))_{i \in J}$ separates $(X_i)_{i \in J}$. □

We now show that interpolative models of $T_\cup$ can be thought of as "relatively existentially closed" models of $T_\cup$, and the interpolative fusion $T_\cup^*$ can be thought of as the "relative model companion" of $T_\cup$.

Theorem 6.2. *Suppose $\mathcal{M}_\cup$ is a model of $T_\cup$.*

(1) *$\mathcal{M}_\cup$ is interpolative if and only if for all $\mathcal{N}_\cup$ such that $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$,*

$$\mathcal{N}_\cup \models \exists x\, \varphi_\cup(x) \quad \textit{implies} \quad \mathcal{M}_\cup \models \exists x\, \varphi_\cup(x)$$

whenever $\varphi_\cup(x)$ is a Boolean combination of L_i-formulas with parameters from M.

(2) *There exists an interpolative $L_\cup$-structure $\mathcal{N}_\cup$ such that $\mathcal{M}_\cup \subseteq \mathcal{N}_\cup$, and $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$.*

(3) *If $T_\cup^*$ exists, $\mathcal{M}_\cup \models T_\cup^*$, $\mathcal{N}_\cup \models T_\cup^*$, and $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$, then $\mathcal{M} \preccurlyeq \mathcal{N}$.*

Proof. Part (1) is a restatement of Lemma 6.1. Part (2) can be proven by an elementary chain argument, similar to the proof of Fact 5.3(1), by iteratively applying Lemma 6.1 to add solutions to families of definable sets which are not separated.

We now prove part (3), assuming $T_\cup^*$ exists. By Morleyizing each T_i, and replacing each function symbol with its graph, we can arrange for each $i \in I$ that T_i admits quantifier-elimination and L_i only contains relation symbols, without changing the class of interpolative structures or the relation of elementary substructure (see [**37**, Theorem 2.6.5] and Remark 6.1, and see Section 9.4 below for a more careful treatment of Morleyization). Then, since each T_i is model-complete, whenever $\mathcal{M}_\cup \subseteq \mathcal{N}_\cup$ are both models of $T_\cup$, we have

$$\mathcal{M}_i \preccurlyeq \mathcal{N}_i \quad \text{for all } i \in I.$$

And since there are no function symbols in $L_\cup$, every quantifier-free $L_\cup$-formula is logically equivalent to a Boolean combination of L_i-formulas. So it follows from (1) that $\mathcal{M}_\cup$ is interpolative if and only if it is existentially closed in the class of $T_\cup$-models. By Facts 5.1 and 5.2, each T_i has an axiomatization by $\forall\exists$-sentences, so $T_\cup$ does too. Hence $T_\cup$ is inductive and Fact 5.3 applies: $T_\cup^*$ is the model companion of $T_\cup$. The desired conclusion then follows from model-completeness of $T_\cup^*$. □

The proof of Proposition 6.2 shows that if T_i admits quantifier eliminations and L_i only contains relation symbols for each $i \in I$, then the interpolative models of $T_\cup$ are just its existentially closed models of $T_\cup$ and the interpolative fusion of $T_\cup$ is just its model companion. This is also true in a slightly more general situation.

Remark 6.2. Any flat literal $L_\cup$-formula (see Section 5.1) is an L_i-formula for some $i \in I$. This trivial observation has two important consequences:

(1) If $\varphi(x)$ is a flat $L_\cup$-formula, then there is some finite $J \subseteq I$ and a flat L_i-formula $\varphi_i(x)$ for all $i \in J$ such that $\varphi(x)$ is logically equivalent to $\bigwedge_{i\in J} \varphi_i(x)$.

(2) If $\mathcal{A}_\cup$ is an $L_\cup$-structure, then $\mathrm{fdiag}(\mathcal{A}_\cup) = \bigcup_{i\in I} \mathrm{fdiag}_{L_i}(\mathcal{A}_i)$.

Theorem 6.3. *Suppose each T_i is model-complete. Then $\mathcal{M}_\cup \models T_\cup$ is interpolative if and only if it is existentially closed in the class of $T_\cup$-models. Hence, $T^*_\cup$ is precisely the model companion of $T_\cup$, if either of these exists.*

Proof. We prove the first statement. Let $\mathcal{M} \models T_\cup$ be existentially closed. Suppose $J \subseteq I$ is finite and $\varphi_i(x)$ is an $L_i(M)$-formula for each $i \in J$ such that $(\varphi_i(\mathcal{M}))_{i\in J}$ is not separated. We may assume each $\varphi_i(x)$ is existential as T_i is model-complete. Lemma 6.1 gives a $T_\cup$-model $\mathcal{N}$ extending $\mathcal{M}$ such that $\mathcal{N} \models \exists x \bigwedge_{i\in J} \varphi_i(x)$. As $\mathcal{M}$ is existentially closed and each φ_i is existential, we have $\mathcal{M} \models \exists x \bigwedge_{i\in J} \varphi_i(x)$. Thus $\mathcal{M}$ is interpolative.

Now suppose $\mathcal{M} \models T_\cup$ is interpolative. Suppose $\psi(x)$ is a quantifier-free $L_\cup(M)$-formula and $\mathcal{N}$ is a $T_\cup$-model extending $\mathcal{M}$ such that $\mathcal{N} \models \exists x\, \psi(x)$. Applying Corollary 5.1, $\psi(x)$ is logically equivalent to a finite disjunction of E$\flat$-formulas $\bigvee_{k=1}^{n} \exists y_k\, \psi_k(x, y_k)$. Then for some k, $\mathcal{N} \models \exists x\, \exists y_k\, \psi_k(x, y_k)$. By Remark 6.2, the flat $L_\cup(M)$-formula $\psi_k(x, y_k)$ is equivalent to a conjunction $\bigwedge_{i\in J} \varphi_i(x, y_k)$ where $J \subseteq I$ is finite and $\varphi_i(x, y_k)$ is a flat $L_i(M)$-formula for each $i \in J$. So $\mathcal{N} \models \exists x\, \exists y_k \bigwedge_{i\in J} \varphi_i(x, y_k)$. As each T_i is model-complete, we have $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$. By Lemma 6.1, the sets defined by $\varphi_i(x, y_k)$ are not separated, and since $\mathcal{M}$ is interpolative, $\mathcal{M} \models \exists x\, \exists y_k \bigwedge_{i\in J} \varphi_i(x, y_k)$. So $\mathcal{M} \models \exists x\, \exists y_k \psi_k(x, y_k)$, and $\mathcal{M} \models \exists x\, \psi(x)$.

By Facts 5.1 and 5.2, each T_i has an axiomatization by $\forall\exists$-sentences, so $T_\cup$ does too. Hence $T_\cup$ is inductive and Fact 5.3 applies. The second statement then follows from the first statement. $\square$

CHAPTER 7

Examples of interpolative fusions

In this chapter, we continue to adopt the notational conventions of Chapter 6. We show that several theories previously studied in the literature are interpolative fusions or bi-interpretable with interpolative fusions. This can be explained by two phenomena:

(1) Model theorists often study model companions of theories of interest.
(2) Many natural theories are either equal to or bi-intepretable with a union of two or more simpler theories.

If a theory T is a union of model-complete theories, then Theorem 6.3 identifies the model companion of T with the interpolative fusion of these theories. It turns out that the context of interpolative fusions includes a wider breadth of examples than one might initially expect, since Corollary 5.2 implies that if T is merely existentially bi-interpretable with a union of model-complete theories, then the model companion of T is existentially bi-interpretable with the interpolative fusion of these theories.

The general theory of interpolative fusions developed in Chapters 8 and 9 will allow us to recover many known results about these examples.

7.1. Disjoint unions of theories

In this section we assume $L_\cap = \varnothing$, so the languages L_i are pairwise disjoint. Note that equality is a primitive logical symbol, so $T_\cap$ is the theory of a (usually infinite) set with equality. The following result is proven in Winkler's thesis [**89**].

Theorem 7.1. *Suppose each T_i is model-complete and eliminates $\exists^\infty$. Then $T_\cup$ has a model companion.*

By Theorem 6.3, the model companion in this case is precisely $T^*_\cup$. So Theorem 7.1 provides us with the simplest class of interpolative fusions. Since we can Morleyize each theory T_i without changing the class of interpolative models (see Remark 6.1), we can do without the assumption of model-completeness.

Corollary 7.1. *Suppose T_i eliminates $\exists^\infty$. Then $T^*_\cup$ exists.*

A special case is the expansion by a generic unary predicate defined in [**11**] and [**28**]. This deserves special mention as it often serves as a good toy example.

Suppose L is a one-sorted language, $\mathcal{M}$ is an infinite L-structure, and P is a unary predicate on $\mathcal{M}$ which is not in L. An $\mathcal{M}$-definable set $X \subseteq M^n$ is said to be **large** if there is a tuple $(a_1, \dots, a_n) \in X(\boldsymbol{\mathcal{M}})$ such that

$$a_i \notin M \text{ for all } i \quad \text{and} \quad a_i \neq a_j \text{ for all } i \neq j.$$

The predicate P is **generic** if and only if the following holds: For every large $\mathcal{M}$-definable $X \subseteq M^n$ and every $S \subseteq \{1, \dots, n\}$, there exists $(a_1, \dots, a_n) \in X$ such that for all $1 \leqslant k \leqslant n$,

$$a_k \in P \text{ if and only if } k \in S.$$

Equivalently, every large $\mathcal{M}$-definable subset of M^n intersects every subset of the form $S_1 \times \dots \times S_n$ where $S_i \in \{P, M \setminus P\}$ for $1 \leqslant i \leqslant n$.

Let $L_{\mathrm{u}} = L \cup \{P\}$. Let T be an L-theory with no finite models, and let T_{u} be T viewed as an L_{u}-theory, so that the models of T_{u} are the L_{u}-structures $(\mathcal{M}, P)$, where $\mathcal{M} \models T$ and P is an arbitrary predicate on $\mathcal{M}$. The following is shown in [**11**].

Theorem 7.2. *Suppose T is model-complete and eliminates $\exists^\infty$. Then T_{u} has a model companion T_{u}^*. Moreover, the models of T_{u}^* are precisely the L_{u}-structures $(\mathcal{M}, P)$, where $\mathcal{M} \models T$ and P is a generic predicate on $\mathcal{M}$.*

We can realize T_{u}^* as an interpolative fusion of two theories in disjoint languages as follows. Let $I = \{1, 2\}$, $L_1 = L$, and $L_2 = \{P\}$. Then we have $L_\cap = \varnothing$ and $L_\cup = L_{\mathrm{u}}$. Let $T_1 = T$, and let T_2 be the L_2-theory such that $(M; P) \models T_2$ if and only if $P \subseteq M$ is both infinite and coinfinite. It is easy to see that T_1 and T_2 have a common set of $L_\cap$-consequences $T_\cap$, which is simply the theory of infinite sets. The theory $T_\cup$ properly extends T_{u}, and every model of T_{u} can be embedded into a model of $T_\cup$, so T_{u}^* is also the model companion of $T_\cup$. The theory T_1 is model-complete by assumption, and it is also easy to check that T_2 is model-complete. So $T_\cup^* = T_{\mathrm{u}}^*$ by Theorem 6.3.

By Morleyization, we get the following restatement of Theorem 7.2 in our context, without assuming model-completeness.

Corollary 7.2. *Suppose T_1 is an L_1-theory which eliminates $\exists^\infty$, and T_2 is the theory of an infinite and coinfinite predicate in the language $L_2 = \{P\}$. Then $T_\cup^*$ exists. Moreover, the models of $T_\cup^*$ are precisely the $L_\cup$-structures $(\mathcal{M}, P)$, where $\mathcal{M} \models T$ and P is a generic predicate on $\mathcal{M}$.*

7.2. Fields with multiple independent valuations

The theory of algebraically closed fields with multiple independent valuations studied in [**82, 43**] is an interpolative fusion of copies of the theory ACVF of algebraically closed valued fields. This is one instance of a large class of examples coming from expansions of algebraically closed fields by extra structure (e.g. valuations, derivations, automorphisms, etc.) in multiple independent ways.

A valuation v on a field K is **trivial** if the v-topology on K is discrete, equivalently if every element of K lies in the valuation ring of v. In this section, all valuations are non-trivial. Two valuations are **independent** if they induce distinct topologies.

Suppose K is a field and $(v_i)_{i\in I}$ is a family of valuations on K. For $i \in I$, let R_i be the valuation ring $\{a \in K : v_i(a) \geqslant 0\}$ of v_i. Note that v_i can be recovered from its valuation ring R_i. We view K as a structure in a language consisting of the language of rings together with a unary predicate naming R_i for each $i \in I$. We set this to be our $L_\cup$. Then $L_\cap$ is the language of rings, and $L_i = L_\cap \cup \{R_i\}$ for each $i \in I$. Note that the only difference between L_i and L_j when $i \neq j$ is the name of the relation symbol.

Let each T_i be the L_i-theory of algebraically closed valued fields, and let $T_\cap$ be the common set of $L_\cap$-consequences of T_i for $i \in I$. By well-known results about algebraically closed valued fields (often treated in slightly different languages), each T_i is model-complete; see for example [**36**, Theorem 2.1.1]. Hence, $T^*_\cup$ is the model companion of $T_\cup$ if either of these exist by Theorem 6.3.

The following Theorem 7.3 can be found in [**43**]. The first statement is a special case of [**81**, 3.1.6], so this was known earlier.

Theorem 7.3. *$T_\cup$ has a model companion $T^*_\cup$. Moreover, $(K, (R_i)_{i\in I})$ is a model of $T^*_\cup$ if and only if $(K; (R_i)_{i\in I}) \models T_\cup$ and the valuations $(v_i)_{i\in I}$ are pairwise independent.*

Let T^-_i be the L_i-theory of valued fields for $i \in I$, and let $T^-_\cup = \bigcup_{i\in I} T^-_i$. As every valuation on a field can be extended to a valuation on its algebraic closure, every model of $T^-_\cup$ can be embedded into a model of $T_\cup$. So $T^*_\cup$ is also the model companion of $T^-_\cup$.

7.3. The group of integers with p-adic valuations

In a similar spirit, we can consider the additive group of integers $(\mathbb{Z}; 0, +, -)$ equipped with multiple p-adic valuations, as studied in [**24**].

Let I be the set of primes. We let v_p be the p-adic valuation on $\mathbb{Z}$ for $p \in I$, and declare $k \preccurlyeq_p l$ if $v_p(k) \leqslant v_p(l)$. Note that v_p can be recovered from $\preccurlyeq_p$. We view $\mathbb{Z}$ as a structure in a language extending the language of additive groups by a binary relation $\preccurlyeq_p$ for each prime p. This is our $L_\cup$. Then we set $L_\cap$ to be the language of additive groups and $L_p = L_\cap \cup \{\preccurlyeq_p\}$ for each $p \in I$.

For p in I, set $T_p = \mathrm{Th}(\mathbb{Z}, 0, +, -, \preccurlyeq_p)$. Then the theories T_p with $p \in I$ have a common set of $L_\cap$-consequences, which is simply $\mathrm{Th}(\mathbb{Z}; 0, +, -)$. The theory T_p is model-complete. This was proven independently by Guingnot [**33**] and Mariaule [**57**], but it can also be deduced from a more general result in [**24**]. By Theorem 6.3, $T^*_\cup$ is the model companion of $T_\cup$, if either of these exists. The following was shown in [**24**].

Theorem 7.4. *$T_\cup$ is model-complete, and so $T_\cup = T^*_\cup$.*

This naturally raises the following question.

Question. When is a union of model-complete theories model-complete? In other words, under what conditions on the theories T_i is every model of $T_\cup$ interpolative?

See Example 9.2 below for another example of this phenomenon.

7.4. Fields with multiplicative circular orders

The next example was considered by the second author in [**79**], and the original motivation for developing a general theory of interpolative fusions came from the idea of unifying this example with the examples in Section 7.2. This example illustrates that interpolative fusions can arise naturally in contexts often described as "pseudo-random" in mathematics. Here, the pseudo-randomness comes from number-theoretic results on character sums over finite fields.

A **circular order** on an abelian group G is a ternary relation $\lhd$ on G which is invariant under the group operation and satisfies the following for all $a, b, c \in G$:

(1) If $\lhd(a, b, c)$, then $\lhd(b, c, a)$.
(2) If $\lhd(a, b, c)$, then not $\lhd(c, b, a)$.
(3) If $\lhd(a, b, c)$ and $\lhd(a, c, d)$, then $\lhd(a, b, d)$.
(4) If a, b, c are distinct, then either $\lhd(a, b, c)$ or $\lhd(c, b, a)$.

An example to keep in mind is the multiplicative group $\mathbb{T}$ of complex numbers with norm 1, thought of as the unit circle in the complex plane, together with the circular order $\lhd^+$ of positive orientation.

A **multiplicative circular order** on a field F is a circular order on the multiplicative group $F^\times$, viewed as a ternary relation on F. Let ACFO^- be the theory whose models are $(F, \lhd)$, where F is an algebraically closed field (viewed as a structure in the language of rings), and $\lhd$ is multiplicative circular order on F. The following result is essentially shown in [**79**].

Theorem 7.5. ACFO$^-$ *has a model companion* ACFO. *Moreover, if* $\overline{\mathbb{F}}_p$ *is the field-theoretic algebraic closure of the prime field of characteristic* $p > 0$, *and* $\lhd$ *is any multiplicative circular order on* $\overline{\mathbb{F}}_p$, *then* $(\overline{\mathbb{F}}_p, \lhd)$ *is a model of* ACFO.

It can be shown that for any multiplicative circular order $\lhd$ on $\overline{\mathbb{F}}_p$, there is an injective group homomorphism $\chi : \overline{\mathbb{F}}_p^\times \to \mathbb{T}$ such that $\lhd$ is the preimage of $\lhd^+$, i.e.,

$$\lhd(a,b,c) \text{ if and only if } \lhd^+(\chi(a),\chi(b),\chi(c)) \quad \text{for } a,b,c \in \overline{\mathbb{F}}_p^\times.$$

The proof of the second statement of Theorem 7.5 in [**79**] proceeds by exploiting this connection and results on character sums over finite fields mentioned earlier.

We next explain how to realize ACFO as an interpolative fusion. Let $L_\cup = \{+,-,\times,0,1,\lhd\}$ be the language of ACFO. Let $L_1 = \{+,-,\times,0,1\}$ be the language of rings, and let $L_2 = \{\times,0,1,\lhd\}$. Then $L_\cap = L_1 \cap L_2 = \{\times,0,1\}$.

Let T_1 be ACF, and let T_2 be the L_2-consequences of ACFO. Then ACFO$^- \subseteq T_\cup$ and $T_\cup \subseteq$ ACFO, and so ACFO is the model companion of $T_\cup$. Each completion of the theory T_2 is model complete, and T_1 and T_2 have the same set of $L_\cap$-consequences; see [**79**] for the details. Thus $T^*_\cup$ = ACFO.

In fact, the proof of the existence of the model companion ACFO in [**79**] proceeded by developing a notion of interpolative model of ACFO$^-$ (called "generic" in [**79**]) and concluding that the interpolative fusion (the theory ACFO of the "generic" models) is the model companion of ACFO$^-$. So the story here is told backward.

We end with a few remarks.

Remark 7.1. The reader might wonder why we do not consider fields with additive cyclic orders. An infinite field of characteristic $p > 0$ does not admit an additive circular order, because every element is p-torsion. In characteristic 0, the theory of algebraically closed fields with an additive circular order is consistent, but we believe that it does not have a model companion.

We also expect that some aspects of the results above still hold if we replace the role of the theory of algebraically closed fields with the theory of pseudo-finite fields. Note that in this case $T^*_\cup$ is not model-complete in its natural language, as the theory of pseudo-finite fields is not model-complete in the language of rings. Hence, this would be a natural example of an interpolative fusion which is not a model companion.

7.5. Skolemizations

In this section, we treat another construction from Winkler's thesis [**89**]. Let L be a one-sorted language, and let T be an L-theory with only infinite models. Suppose $\varphi(x,y)$ is an L-formula, where y is a single variable and x is a tuple of variables of length $n > 0$, such that

$$T \models \forall x \exists^{\geqslant k} y\, \varphi(x,y) \text{ for all } k.$$

Let $L_+ = L \cup \{f\}$, where f is a new n-ary function symbol, and let

$$T_+ = T \cup \{\forall x\, \varphi(x, f(x))\}.$$

Then T_+ is the "φ-Skolemization" of T. Theorem 7.6 was shown in [**89**].

Theorem 7.6. *If T is model-complete and eliminates $\exists^\infty$, then T_+ has a model companion T_+^*, the generic φ-Skolemization of T.*

We will show that T_+ is existentially bi-interpretable with a union of two theories, one of which is is existentially bi-interpretable with T, and the other of which is interpretable in the theory of an infinite set. This will imply, by Corollary 5.2, that T_+^* is existentially bi-interpretable with the interpolative fusion of these theories.

Suppose $(\mathcal{M}, f) \models T_+$. Let $E \subseteq M^{n+1}$ be $\varphi(\mathcal{M})$, let $p_x : E \to M^n$ and $p_y : E \to M$ be the projection on the first n coordinates and the last coordinate, respectively, and let $g : M^n \to E$ be the function $a \mapsto (a, f(a))$. Note that p_x is an infinite-to-one surjection onto M^n, and g is a section of p_x. We consider $(\mathcal{M}, E; p_x, p_y, g)$ as a structure in a two-sorted language consisting of a copy of L for $\mathcal{M}$, together with function symbols p_x, p_y, and g. Let this be $L_\cup$, let L_1 be the sublanguage of $L_\cup$ without g, and let L_2 be the sublanguage of $L_\cup$ containing only p_x and g (without p_y and the copy of L). Then $L_\cap$ contains only p_x.

Let T_1 be the L_1-theory whose models are $(\mathcal{M}, E; p_x, p_y)$ such that $\mathcal{M} \models T$, and $e \mapsto (p_x(e), p_y(e))$ is a bijection from E to $\varphi(\mathcal{M})$. It is easy to see that T_1 is existentially bi-interpretable with T. Let T_2 be the L_2-theory whose models are $(M, E; p_x, g)$ such that M and E are infinite sets, p_x is an infinite-to-one surjection $E \to M^n$, and g is a section of p_x. This theory T_2 is interpretable in the theory of an infinite set M: let $E = M^{n+1}$, let p_x be the projection on the first n coordinates, and define $g(a_0, \ldots, a_{n-1}) = (a_0, \ldots, a_{n-1}, a_0)$. Then $T_\cap$ is the $L_\cap$-theory whose models are $(M, E; p_x)$ where M and E are infinite and p_x is an infinite-to-one surjection $E \to M^n$.

Now $T_\cup$ is the theory whose models are $(\mathcal{M}, E; p_x, p_y, g)$, where $\mathcal{M} \models T$, $e \mapsto (p_x(e), p_y(e))$ is a bijection from E to $\varphi(\mathcal{M})$, and $g\colon M^n \to E$ is a section of p_x. We have seen above how to obtain such a structure from a model of T_+. And conversely, given a model of $T_\cup$, we can recover the Skolem function f as $p_y \circ g$. So the following theorem follows easily, by Corollary 5.2.

Theorem 7.7. *T_+ is existentially bi-interpretable with $T_\cup$. Hence, T_+ has a model companion T_+^* if and only if $T_\cup$ has a model companion $T_\cup^*$. Moreover, T_+^* and $T_\cup^*$ are existentially bi-interpretable whenever they both exist.*

In [**89**], Winkler handles the case of simultaneously adding Skolem functions for an arbitrary family of formulas, and he does not impose the restriction that every set defined by an instance of the formula φ is infinite. It is possible to adjust our construction to handle this more general context, but the technical difficulties would obscure the main point of the example.

Remark 7.2. In the notation above, if φ is $\top$, then T_+ is the theory of models of T expanded by an arbitrary new n-ary function f, and T_+^* is the "generic expansion" of T by f. In the special case that T is the theory of an infinite set, T_+ is the theory T_n of a "random n-ary function". It follows from the discussion above that T_n is existentially bi-interpretable with a union of two theories, each of which is interpretable in the theory of an infinite set. In [**52**], Ramsey and the first author showed that T_n is NSOP_1 (but not simple when $n \geqslant 2$), and more generally that if T is NSOP_1, then any generic Skolemization of T or generic expansion of T by new function symbols is NSOP_1. We will show how to recover these results from general results about interpolative fusions in the next paper [**53**].

7.6. Graphs

We now illustrate how to obtain "random n-ary relations" in the context of interpolative fusions (compare with the "random n-ary functions" in Remark 7.2 above). In particular, we show that the theory of the random graph is existentially bi-interpretable with an interpolative fusion of two model-complete theories, each of which is interpretable in the theory of an infinite set.

Let L be the language of graphs and T be the the theory of (undirected, loopless) infinite graphs with infinitely many edges. Suppose $(V, E) \models T$. Let S_V be the quotient $\{(v_1, v_2) \in V^2 : v_1 \neq v_2\}/\sim$ where the equivalence relation $\sim$ is defined by

$$(v_1, v_2) \sim (v_1', v_2') \quad \text{if and only if} \quad \{v_1, v_2\} = \{v_1', v_2'\}.$$

Let $\pi_V : \{(v_1, v_2) \in V^2 : v_1 \neq v_2\} \to S_V$ be the quotient map seen as a relation on $V^2 \times S_V$, and let E_V be the image of E under π_V, seen as a relation on S_V. One can observe that the two-sorted structure $(V, S_V; \pi_V, E_V)$ is essentially equivalent to (V, E). Indeed, for distinct v_1 and v_2 in V, (v_1, v_2) is in E if and only if $\pi_V(v_1, v_2)$ is in E_V. On the other hand, $(V, S_V; \pi_V, E_V)$

can be seen as built up from the two components $(V, S_V; \pi_V)$ and $(V, S_V; E_V)$ agreeing on the common part (V, S_V).

The observations in the preceding paragraph translate into model-theoretic language. Let $(V, S_V; \pi_V, E_V)$ be as above. Choose the obvious languages L_1 and L_2 for $(V, S_V; \pi_V)$ and $(V, S_V; E_V)$. Then with $I = \{1, 2\}$, $(V, S_V; \pi_V, E_V)$ and (V, S_V) are an $L_\cup$-structure and an $L_\cap$-structure, respectively. Let T_1 be the L_1-theory such that $(V, S; \pi) \models T_1$ if V, S are infinite sets and $\pi : \{(v_1, v_2) \in V^2 : v_1 \neq v_2\} \to S$ has

$$\pi(a) = \pi(b) \quad \text{if and only if} \quad a \sim b.$$

Let T_2 be the L_2-theory such that $(V, S; E) \models T_2$ when V, S are infinite sets and E is an infinite subset of S. The theories T_1 and T_2 are easily seen to be model-complete and interpretable in the theory of an infinite set. The constructions of $(V, S_V; \pi_V, E_V)$ from (V, E) and vice versa are very simple, so we can easily verify that they form an existential bi-interpretation between T and $T_\cup$.

It is well known that the theory of graphs has a model companion, the theory of the random graph. We obtain:

Theorem 7.8. *The theory T is existentially bi-interpretable with the theory $T_\cup$. Hence, $T^*_\cup$ has a model companion which is existentially bi-interpretable with the theory of the random graph.*

This example can be easily modified to show that the theory of the random n-hypergraph, random directed graph, and random bipartite graph are all bi-interpretable with an interpolative fusion of two theories, each of which is interpretable in the theory of an infinite set.

7.7. Structures and fields with automorphisms

In this section T is a one-sorted model-complete consistent L-theory. Let L_{Aut} be the extension of L by a new unary function symbol and T_{Aut} be the theory such that $(\mathcal{M}, \sigma) \models T_{\text{Aut}}$ if and only if $\mathcal{M} \models T$ and σ is an automorphism of $\mathcal{M}$. We will show that T_{Aut} is existentially bi-interpretable with the union of two theories each of which is existentially bi-interpretable with T. This brings generic automorphisms as defined in [**11**] into our framework.

Let $(\mathcal{M}, \sigma) \models T$, and set $I = \{1, 2\}$. We can view $(\mathcal{M}, \mathcal{M}; \text{id}_\mathcal{M}, \sigma)$ as a structure in a two-sorted language consisting of two disjoint copies of L for the two copies of $\mathcal{M}$ and two function symbols for $\text{id}_\mathcal{M}$ and σ respectively. Set this to be our $L_\cup$. Let L_1, L_2, and $L_\cap$ to be the sublanguages of $L_\cup$ for the reducts $(\mathcal{M}, \mathcal{M}; \text{id}_\mathcal{M})$, $(\mathcal{M}, \mathcal{M}; \sigma)$, and $(\mathcal{M}, \mathcal{M})$ respectively.

We note that L_1 differs from L_2 only in the name of the function symbol. Let T_1 be the L_1-theory whose models are $(\mathcal{M}, \mathcal{N}; f)$ with $\mathcal{M} \models T$, $\mathcal{N} \models T$, and $f : \mathcal{M} \to \mathcal{N}$ is an L-isomorphism. Obtain T_2 from T_1 by replacing the function symbol from L_1 with the corresponding function symbol from L_2.

Proposition 7.1. *The theories T_1 and T_2 are each existentially bi-interpretable with T.*

PROOF. As T_2 is a copy of T_1, it suffices to prove the statement for T_1. If $(\mathcal{M}, \mathcal{N}; f) \models T_1$, then $\mathcal{M} \models T$. If $\mathcal{M} \models T$, then $(\mathcal{M}, \mathcal{M}; \mathrm{id}_{\mathcal{M}}) \models T_1$. The two constructions above can be easily turned into existential mutual interpretation between T_1 and T.

Applying the first construction above followed by the second construction above to $(\mathcal{M}, \mathcal{N}; f) \models T_1$ gives us $(\mathcal{M}, \mathcal{M}; \mathrm{id}_{\mathcal{M}})$. It is easy to see that $(\mathrm{id}_{\mathcal{M}}, f^{-1})$ is an isomorphism from $(\mathcal{M}, \mathcal{N}; f)$ to $(\mathcal{M}, \mathcal{M}; \mathrm{id}_{\mathcal{M}})$ in this case. Applying the second construction above followed by the first construction above to $\mathcal{M} \models T$ gives us back $\mathcal{M}$, so $\mathrm{id}_{\mathcal{M}}$ is already the desired isomorphism. It is easy to see that these isomorphism can be defined by existential formulas in the respective languages. Moreover, the choice of these formulas can be made independent of the choice of $(\mathcal{M}, \mathcal{N}; f) \models T_1$ and $\mathcal{M} \models T$. $\square$

The existential bi-interpretation between T_1 and T above restricts to a mutual interpretation between $T_\cap$ and T. But $T_\cap$ and T are not bi-interpretable.

It is easy to see that T_1 and T_2 are inductive. So by Corollary 5.2, T_1 and T_2 are both model-complete. Hence, $T^*_\cup$ is the model companion of $T_\cup$ if either of these exists by Theorem 6.3. We prove the main result of this section:

Theorem 7.9. *The theory T_{Aut} is existentially bi-interpretable with $T_\cup$. Hence, T_{Aut} has a model companion T^*_{Aut} if and only if the interpolative fusion $T^*_\cup$ exists. Moreover, T^*_{Aut} and $T^*_\cup$ are existentially bi-interpretable whenever they exist.*

PROOF. Applying Corollary 5.2 and the easy fact that T_{Aut} is inductive, we get the second and third claims from the first statement. So it remains to prove the first statement. If $(\mathcal{M}; \sigma) \models T_{\mathrm{Aut}}$, then $(\mathcal{M}, \mathcal{M}; \mathrm{id}_{\mathcal{M}}, \sigma) \models T_\cup$. Suppose $(\mathcal{M}, \mathcal{N}; f, g)$ is a model of $T_\cup$. Then $(\mathcal{M}, f^{-1} \circ g) \models T_{\mathrm{Aut}}$. It is easy to see that the two constructions above can be turned into existential mutual interpretation between T_{Aut} and $T_\cup$.

Applying the first construction above followed by the second construction above to $(\mathcal{M}, \sigma) \models T_{\mathrm{Aut}}$ gives us back $(\mathcal{M}, \sigma)$, so $\mathrm{id}_{\mathcal{M}}$ is already the desired isomorphism. Applying the second construction above followed by the first construction above to $(\mathcal{M}, \mathcal{N}; f, g) \models T_\cup$ gives us back $(\mathcal{M}, \mathcal{M}; \mathrm{id}_{\mathcal{M}}, f^{-1} \circ g)$. Then $(\mathrm{id}_{\mathcal{M}}, f^{-1})$ is an isomorphism from $(\mathcal{M}, \mathcal{N}; f, g)$ to $(\mathcal{M}, \mathcal{M}; \mathrm{id}_{\mathcal{M}}, f^{-1} \circ g)$ in this case. It is easy to see that these isomorphisms can be defined by existential formulas in the respective languages. Moreover, the choice of these formulas can be made independent of the choice of $(\mathcal{M}, \mathcal{N}; f) \models T_1$ and $(\mathcal{M}; \sigma) \models T_{\mathrm{Aut}}$. $\square$

The existence of a model companion of T_{Aut} is tied to classification-theoretic issues. If T has the strict order property then T_{Aut} does not have a model companion [**46**]. It is conjectured that if T is unstable then T_{Aut} does not have a model companion. Baldwin and Shelah [**5**] gave necessary and sufficient conditions for T_{Aut} to admit a model companion when T is stable.

In the special case where T is ACF, it is well-known that T_{Aut} has a model companion, called ACFA. This important theory is treated in [**13, 56**] and many other places. Hence, we get the following as a corollary of Theorem 7.9:

Corollary 7.3. *If T = ACF, then the interpolative fusion $T^*_\cup$ exists and is existentially bi-interpretable with* ACFA.

Following our motivational theme that many mathematical structures that exhibit randomness in some sense can be treated in the context of interpolative fusions, it is natural to ask whether this is true of pseudofinite fields, i.e, infinite field which is a model of the theory of finite fields. We do not see a way to make the theory of pseudofinite fields bi-interpretable with an interpolative fusion. Proposition 7.2 below implies that the theory of pseudo-finite fields is interpretable in ACFA, and hence is interpretable in an interpolative fusion. This is a folklore result which follows from unpublished work of Hrushovski [**38**]. We include here a direct short proof for the sake of completeness.

The fixed field of a difference field (L, σ) is the subfield $\{x \in L : \sigma(x) = x\}$.

Proposition 7.2. *A field is pseudofinite if and only if it is elementarily equivalent to the fixed field of a model of* ACFA.

Proof. See [**56**, Theorem 6] for a proof of the fact that the fixed field of a model of ACFA is pseudofinite. Suppose k is a pseudofinite field. Let K be an algebraic closure of k, and let σ be some automorphism of K with fixed field k. As ACFA is the model companion of T_{Aut} when T is the theory of fields, there is an ACFA-model (K', σ') such that (K, σ) is a sub-difference field of (K', σ'). Let F be the fixed field of (K', σ'). Then k is a subfield of F, and the (field-theoretic) algebraic closure of k inside of F is equal to k. It follows that the algebraic closure of the prime subfield of F agrees with that of k. A well known theorem of Ax (see for example [**12**, Theorem 1]) implies that K and F are elementarily equivalent. $\square$

7.8. Differential fields and $\mathcal{D}$-fields

We treat the $\mathcal{D}$-fields formalism developed in [**60**], a framework which generalizes both differential fields and difference fields. As special cases, we show that the theories DCF_0 (the model companion of the theory of differential fields of characteristic 0) and ACFA_0 (the model companion of the theory of difference fields of characteristic 0) are each bi-interpretable with an interpolative fusion of two theories, each of which is bi-interpretable with ACF_0. In the case of ACFA_0, this provides an alternative presentation as an interpolative fusion to the one described in Section 7.7.

In this subsection, all rings are commutative with unit. If K is a field, a K-algebra is a pair (A, ρ), where A is a ring and $\rho\colon K \to A$ is a ring homomorphism. Note that ρ is necessarily injective unless A is the zero ring. The homomorphism ρ makes A a vector space over K with left multiplication by elements in K given by

$$a \cdot r := \rho(a)r \quad \text{for } a \in K \text{ and } r \in A.$$

We denote this K-vector space as $V(A,\rho)$. A K-algebra (A,ρ) is finite if $V(A,\rho)$ has finite dimension. In particular, (K, id_K) is a finite K-algebra. A K-algebra homomorphism $(A,\rho) \to (A',\rho')$ is a ring homomorphism $f\colon A \to A'$ such that $f \circ \rho = \rho'$.

We fix a field F, a non-zero finite F-algebra $(\mathcal{D}_F, \rho_F)$, an F-algebra homomorphism $\pi_F\colon \mathcal{D}_F \to F$ (in other words, π_F is a ring homomorphism with $\pi_F \circ \rho_F = \mathrm{id}_F$), and a basis $e = (e_0, \dots, e_m)$ of $V(\mathcal{D}_F, \rho_F)$ such that $\pi_F(e_0) = 1_F$ and $\pi_F(e_i) = 0_F$ for all $i \in \{1, \dots, m\}$.

Now suppose K is a field extending F. We will define objects parallel to those in the preceding paragraph by extension of scalars. Identifying K with the F-algebra (K, ι) where $\iota\colon F \to K$ is the inclusion map, we define the K-algebra $(\mathcal{D}_K, \rho_K)$ and a K-algebra homomorphism $\pi_K\colon \mathcal{D}_K \to K$ by setting:

$$\mathcal{D}_K = K \otimes_F \mathcal{D}_F, \quad \rho_K = \mathrm{id}_K \otimes_F \rho_F, \text{ and } \quad \pi_K = \mathrm{id}_K \otimes_F \pi_F.$$

Identifying $\mathcal{D}_F$ with its image in $\mathcal{D}_K$ under the injective map $a \mapsto 1_K \otimes a$, it is easy to see that $(\mathcal{D}_K, \rho_K)$ is a non-zero finite K-algebra and e is a basis for $V(\mathcal{D}_K, \rho_K)$ satisfying

$$\pi_K(e_0) = 1_K \text{ and } \pi_K(e_i) = 0_K \quad \text{for all } i \in \{1, \dots, m\}.$$

It follows that any $a \in \mathcal{D}_K$ can be written as $\rho_K(a_0)e_0 + \dots + \rho_K(a_m)e_m$ for unique elements $a_0, \dots, a_m \in K$, and $\pi_K(a) = \pi_K(\rho_K(a_0)) = a_0$.

Suppose $\partial_i : K \to K$ are functions for $i \in \{1, \dots, m\}$, and the map

$$\delta_K : K \to \mathcal{D}_K, \quad a \mapsto \rho_K(a)e_0 + \rho_K(\partial_1(a))e_1 + \ldots + \rho_K(\partial_m(a))e_m$$

is an F-algebra homomorphism $(K,\iota) \to (\mathcal{D}_K, \rho_K \circ \iota)$. In this case, we call $(K, \partial_1, \dots, \partial_m)$ a **$\mathcal{D}$-field**. Note in particular that δ_K is a section of π_K, since for all $a \in K$, $\pi_K(\delta_K(a)) = \pi_K(\rho_K(a)) = a$.

Example 7.1. We show how the framework above generalizes differential fields and difference fields.

(1) Let $F = \mathbb{Q}$, $\mathcal{D}_{\mathbb{Q}} = \mathbb{Q}[\varepsilon]/(\varepsilon^2)$, $\rho(a) = a + 0\varepsilon$, $\pi(a+b\varepsilon) = a$, and $e = (1, \varepsilon)$. For any field K of characteristic 0, $\mathcal{D}_K \cong K[\varepsilon]/(\varepsilon^2)$. If $\delta \colon K \to K$ is a function, then the map $\delta_K \colon K \to \mathcal{D}_K$ given by $a \mapsto a + \partial(a)\varepsilon$ is a $\mathbb{Q}$-algebra homomorphism if and only if ∂ is a derivation on K. So a $\mathcal{D}$-field in this case is the same thing as a differential field of characteristic 0.

(2) Let $F = \mathbb{Q}$, $\mathcal{D}_{\mathbb{Q}} = \mathbb{Q} \times \mathbb{Q}$, $\rho(a) = (a,a)$, $\pi(a,b) = a$, and $e = ((1,0),(0,1))$. For any field K of characteristic 0, $\mathcal{D}_K \cong K \times K$. If $\sigma \colon K \to K$ is a function, then the map $\delta_K \colon K \to \mathcal{D}_K$ given by $a \mapsto (a, \sigma(a))$ is a $\mathbb{Q}$-algebra homomorphism if and only if σ is a field endomorphism. So a $\mathcal{D}$-field in this case is the same thing as a difference field of characteristic 0.

The key to viewing a $\mathcal{D}$-field as built up from two simpler structures is to see the two F-algebra homomorphisms ρ_K and δ_K in a more symmetric way. As we have seen, both ρ_K and δ_K are sections of π_K. Remark 7.3 below tells us that even more is true:

Remark 7.3. Since e is a basis of $V(\mathcal{D}_F, \rho_F)$, there are uniquely determined elements c_{ijk} in F for $0 \leqslant i,j,k \leqslant m$ such that

$$e_i e_j = \sum_{k=0}^{m} \rho_F(c_{ijk}) e_k \quad \text{for all } 0 \leqslant i,j \leqslant m.$$

And since $\pi_F \circ \rho_F = \mathrm{id}_F$, there are uniquely determined elements d_k in F for $1 \leqslant k \leqslant m$ such that

$$1_{\mathcal{D}_F} = \rho_F(1_F) = e_0 + \sum_{k=1}^{m} \rho_F(d_k) e_k.$$

Note that the constants c_{ijk} and d_k determine the multiplicative structure of $\mathcal{D}_F$.

Let $(K, \partial_1, \dots, \partial_m)$ be a $\mathcal{D}$-field. Then as ρ_K and δ_K are both F-algebra homomorphisms, and the constants c_{ijk} and d_k are in F,

$$e_i e_j = \sum_{k=0}^{n} \rho_K(c_{ijk}) e_k = \sum_{k=0}^{n} \partial_K(c_{ijk}) e_k \quad \text{for all } 0 \leqslant i,j \leqslant m.$$

Likewise, $1_{\mathcal{D}_K} = \rho_K(1_K) = \delta_K(1_K) = e_0 + \sum_{k=1}^{m} \rho_K(d_k) e_k = e_0 + \sum_{k=1}^{m} \delta_K(d_k) e_k$.

The F-algebra homomorphisms ρ_K and δ_K in a $\mathcal{D}$-field are still different in one important respect: e is a basis of $V(\mathcal{D}_K, \rho_K)$ but might not be a basis for $V(\mathcal{D}_K, \delta_K)$. Proposition 7.3 tells us that e is a basis for $V(\mathcal{D}_K, \delta_K)$ if and only if the $\mathcal{D}$-field is *inversive*, a condition introduced in [**60**]. When $\mathcal{D}_F$ is a local F-algebra (e.g., in the case of the dual numbers $F[\varepsilon]/(\varepsilon^2)$), the inversive assumption is the trivial requirement that $\pi_K \circ \delta_K = \mathrm{id}_K$ is a field automorphism of K, so e is here automatically a basis for $V(\mathcal{D}_K, \delta_K)$. The reader who is primarily interested in this case may prefer to skip directly to Proposition 7.3.

In the general case, every finite F-algebra is isomorphic to a product of finite local F-algebras. So there exist $n \geqslant 0$ and finite local F-algebras $(\mathcal{D}_F^j, \rho_F^j)$ for $0 \leqslant j \leqslant n$ such that $\mathcal{D}_F$ is isomorphic as a ring to $\prod_{j=0}^n \mathcal{D}_F^j$, and for $0 \leqslant i \leqslant n$, we have

$$\rho_F^j = \theta_F^j \circ \rho_F \quad \text{where } \theta_F^j \colon \mathcal{D}_F \to \mathcal{D}_F^j \text{ is the projection map.}$$

For each $j \in \{0, \dots, n\}$, $\mathcal{D}_F^j$ has a unique maximal ideal $\mathfrak{m}_F^j$ and a residue map $\pi_F^j \colon \mathcal{D}_F^j \to \mathcal{D}_F^j/\mathfrak{m}_F^j$. Then as j ranges over $\{0, \dots, n\}$, $(\theta_F^j)^{-1}(\mathfrak{m}_F^j)$ ranges over the $n+1$ maximal ideals of $\mathcal{D}_F$. Let $\mathfrak{m}_F = \ker(\pi_F)$. Then $\mathfrak{m}_F$ is a maximal ideal of $\mathcal{D}_F$, and so $\mathfrak{m}_F = (\theta_F^j)^{-1}(\mathfrak{m}_F^j)$ for some $j \in \{0, \dots, n\}$. Without loss of generality, we assume $j = 0$, i.e., $\mathfrak{m}_F = (\theta_F^0)^{-1}(\mathfrak{m}_F^0)$. It follows that $\mathcal{D}_F/\mathfrak{m}_F \cong \mathcal{D}_F^0/\mathfrak{m}_F^0$, and since π_F is surjective onto F, the composition $\pi_F^0 \circ \rho_F^0 : F \to \mathcal{D}_F^0/\mathfrak{m}_F^0$ is an isomorphism. We make the further assumption that for all $j \in \{1, \dots, n\}$, the composition $\pi_F^j \circ \rho_F^j : F \to \mathcal{D}_F^j/\mathfrak{m}_F^j$ is an isomorphism. This assumption, together with the fact that we work with a base field F instead of an arbitrary ring, corresponds to Assumptions 4.1 in [**60**]. The assumption holds trivially when $n = 0$, or equivalently, when $\mathcal{D}_F$ is a local F-algebra. Note that $\mathcal{D}_F^i/\mathfrak{m}_F^j$ is necessarily a finite field extension of F, so the assumption also holds trivially if F is algebraically closed.

The entire discussion above is preserved under tensor product with K. Explicitly:

(1) With $\mathcal{D}_K^j = K \otimes_F \mathcal{D}_F^j$, $\theta_K^j = \mathrm{id}_K \otimes_F \theta_F^j$, and $\rho_K^j = \mathrm{id}_K \otimes_F \rho_F^j$ for $j \in \{1, \dots, n\}$, each $(\mathcal{D}_K^j, \rho_K^j)$ is a finite local K-algebra, and $\mathcal{D}_K \cong \prod_{j=0}^n \mathcal{D}_K^j$ as K-algebras, with the θ_K^j as projection maps. In particular, $\rho_K^j = \theta_K^j \circ \rho_K$ for $j \in \{1, \dots, n\}$. We identify $\mathcal{D}_F^j$ with its image in $\mathcal{D}_K^j$ under the injective map $a \mapsto 1_K \otimes a$.
(2) The unique maximal ideal of $\mathcal{D}_K^j$ is $\mathfrak{m}_K^j = K \otimes_F \mathfrak{m}_F^j$, and $\pi_K^j = \mathrm{id}_K \otimes_F \pi_F^j$ is the residue map $\mathcal{D}_K^j \to \mathcal{D}_K^j/\mathfrak{m}_K^j$ for all $0 \leqslant i \leqslant n$.
(3) $\mathfrak{m}_K = K \otimes_F \mathfrak{m}_F = \ker(\pi_K)$ is equal to $(\theta_K^0)^{-1}(\mathfrak{m}_K^0)$, and $\pi_K^j \circ \rho_K^j : K \to \mathcal{D}_K^i/\mathfrak{m}_K^j$ is an isomorphism for $j \in \{0, \dots, n\}$.

For $j \in \{0, \dots, n\}$, let $\delta_K^j = \theta_K^j \circ \delta_K \colon K \to \mathcal{D}_K^j$. Then δ_K^j is an F-algebra homomorphism, but not necessarily a K-algebra homomorphism. Since $(\pi_K^j \circ \rho_K^j) \colon K \to \mathcal{D}_K^j$ is an isomorphism, we obtain an F-algebra endomorphism

$$\sigma_j = (\pi_K^j \circ \rho_K^j)^{-1} \circ (\pi_K^j \circ \delta_K^j) \colon K \to K.$$

When $j = 0$, $\sigma_0 = \mathrm{id}_K$, since δ_K is a section of π_K and $\mathcal{D}_K/\mathfrak{m}_K \cong \mathcal{D}_K^0/\mathfrak{m}_K^0$. We call $(\sigma_1, \dots, \sigma_n)$ the **associated endomorphisms** of the $\mathcal{D}$-field $(K, \partial_1, \dots, \partial_m)$. We say that the $\mathcal{D}$-field is **inversive** if each of its associated endomorphisms is an automorphism, equivalently if $\pi_K^j \circ \delta_K^j$ is surjective for all $j \in \{1, \dots, n\}$.

Remark 7.4. Continuing Example 7.1, we will consider what these notions mean in the cases of differential fields and difference fields.

(1) Every $\mathcal{D}$-field is trivially inversive when $n = 0$, or equivalently, when $\mathcal{D}_F$ is a local F-algebra. So in particular, differential fields of characteristic 0 are always inversive.
(2) The finite $\mathbb{Q}$-algebra $\mathbb{Q} \times \mathbb{Q}$ is a product of two finite local $\mathbb{Q}$-algebras, namely $\mathbb{Q}$ and $\mathbb{Q}$, $\pi_{\mathbb{Q}}^0$ and $\pi_{\mathbb{Q}}^1$ are the projections onto the first and second factors, and $\pi_{\mathbb{Q}}^j \circ \rho_{\mathbb{Q}}^j = \mathrm{id}_{\mathbb{Q}}$ for $j \in \{0, 1\}$. So a difference field (K, σ) has one associated endomorphism, namely $\pi_K^1 \circ \delta_K = \sigma$, and (K, σ) is inversive if and only if σ is an automorphism.

The next result provides the promised alternative characterization of inversive $\mathcal{D}$-fields. The reader who is only interested in the case where $\mathcal{D}_F$ is a local F-algebra might read the proof below in the following way: In that special case, $n = 0$, $\pi_K^j = \pi_K$, and $\delta_K^j = \delta_K$ for all $j \in \{0, \dots, n\}$, so the only use of the inversive hypothesis in the forward direction of the proof is not necessary.

Proposition 7.3. *Suppose $(K, \partial_1, \dots, \partial_m)$ is a $\mathcal{D}$-field and $\delta_K : K \to \mathcal{D}_K$ is the associated F-algebra homomorphism. Then the following are equivalent:*

(1) *$(K, \partial_1, \dots, \partial_m)$ is inversive*
(2) *e is a basis of the K-vector space $V(\mathcal{D}_K, \delta_K)$.*

Hence, if $\mathcal{D}_F$ is a local F-algebra, then e is a basis of the K-vector space $V(\mathcal{D}_K, \delta_K)$.

Proof. For the forward direction, suppose $(K, \partial_1, \dots, \partial_m)$ is inversive. We reduce the problem to finding for each $j \in \{0, \dots, n\}$ a basis $e^j = (e_0^j, \dots, e_{m_j}^j)$ of $V(\mathcal{D}_F^j, \rho_F^j)$ such that e^j is also a basis of $V(\mathcal{D}_K^j, \delta_K^j)$. Then for all $j \in \{0, \dots, n\}$ and $i \in \{0, \dots, m_j\}$, let $\tilde{e}_i^j$ be the element in $\mathcal{D}_F$ satisfying

$$\theta_F^j(\tilde{e}_i^j) = e_i^j \text{ and } \theta_F^{j'}(\tilde{e}_i^j) = 0 \text{ for } j' \in \{0, \dots, n\} \setminus \{j\}.$$

Since $V(\mathcal{D}_F, \rho_F) \cong \bigoplus_{j=0}^n V(\mathcal{D}_F^j, \rho_F^j)$ and $V(\mathcal{D}_K, \delta_K) \cong \bigoplus_{j=0}^n V(\mathcal{D}_K^j, \delta_K^j)$, we have that $\tilde{e} = ((\tilde{e}_i^j)_{i=0}^{m_j})_{j=0}^n$ is a basis for both vector spaces. It follows that $\tilde{e}$ and e have the same cardinality, so it suffices to show that e spans $V(\mathcal{D}_K, \delta_K)$. But this is clear, since each component of $\tilde{e}$ can be written as an F-linear combination of e, and δ_K is an F-algebra homomorphism.

Next we explain how to obtain the basis e^j for a fixed $j \in \{1, \dots, n\}$. Note that for all $l \leqslant 0$, $(\mathfrak{m}_F^j)^l$ is a subspace of $V(\mathcal{D}_F^j, \rho_F^j)$, and $(\mathfrak{m}_F^j)^l = 0$ when l is large enough. Let be e^j

be any basis of $V(\mathcal{D}_F^j, \rho_F^j)$ such that for each $l \geqslant 0$, a basis of $(\mathfrak{m}_F^j)^l$ can be chosen from the components of e^j. This can be done by taking a basis of $(\mathfrak{m}_F^j)^l$ for the largest l such that $(\mathfrak{m}_F^j)^l \neq 0$, extending it to a basis of $(\mathfrak{m}_F^j)^{l-1}$ for the same l, and continuing in the same fashion until we reach $(\mathfrak{m}_F^j)^0 = \mathcal{D}_F^j$.

Extending scalars to K, we have that e^j is a basis of $V(\mathcal{D}_K^j, \rho_K^j)$ such that for each $l \geqslant 0$, a basis of $(\mathfrak{m}_K^j)^l$ can be chosen from the components of e^j. It remains to show that e^j is also a basis of $V(\mathcal{D}_K^j, \delta_K^j)$. Fix some l such that $(\mathfrak{m}_K^j)^l \neq 0$. Permuting the components of e^j if necessary, we suppose $e_0^j, \dots, e_k^j$ are the only components of e^j which are in $(\mathfrak{m}_K^j)^l \setminus (\mathfrak{m}_K^j)^{l+1}$. Then if $r \in (\mathfrak{m}_K^j)^l$, there are unique $b_0, \dots, b_k \in K$ such that

$$r - \rho_K^j(b_0)e_0^j - \dots - \rho_K^j(b_k)e_k^j \text{ is in } (\mathfrak{m}_K^j)^{l+1}.$$

We reduce the problem to showing for arbitrary $r \in (m_K^j)^l$ that there are unique $a_0, \dots, a_k \in K$ such that

$$r - \delta_K^j(a_0)e_0^j - \dots - \delta_K^j(a_k)e_k^j \text{ is in } (\mathfrak{m}_K^j)^{l+1}.$$

If this is true, then an easy induction argument shows that for any $r \in \mathcal{D}_K^j$, there are unique $c_0, \dots, c_{m_j} \in K$ such that

$$r = \sum_{i=0}^{m_j} \delta_K^j(c_i)e_i^j.$$

So fix $r \in (m_K^j)^l$. Let $b_0, \dots, b_k$ be the unique elements of K such that $r - \rho_K^j(b_0)e_0^j - \dots - \rho_K^j(b_k)e_k^j$ is in $(\mathfrak{m}_K^j)^{l+1}$. As $(K, \partial_1, \dots, \partial_m)$ is inversive, we have that $(\pi_K^j \circ \delta_K^j)$ is a field isomorphism. Hence, there are unique $a_0, \dots, a_k \in K$ such that

$$(\pi_K^j \circ \delta_K^j)(a_i) = (\pi_K^j \circ \rho_K^j)(b_i) \text{ for } i \in \{0, \dots, k\}.$$

It follows that $\rho_K^j(b_i) - \delta_K^j(a_i) \in \mathfrak{m}_K^j$ for $i \in \{0, \dots, k\}$, so

$$(\rho_K^j(b_0) - \delta_K^j(a_0))e_0^j + \dots + (\rho_K^j(b_k) - \delta_K^j(a_k))e_k^j \text{ is in } (\mathfrak{m}_K^j)^{l+1},$$

and hence

$$r - \delta_K^j(a_0)e_0^j - \dots - \delta_K^j(a_k)e_k^j \text{ is in } (\mathfrak{m}_K^j)^{l+1}.$$

For uniqueness, suppose $a_0', \dots, a_k' \in K$ also satisfy the conclusion. As $(\pi_K^j \circ \rho_K^j)$ is a field isomorphism, running the construction above backwards gives us $b_0', \dots, b_k' \in K$ such that $(\pi_K^j \circ \delta_K^j)(a_i') = (\pi_K^j \circ \rho_K^j)(b_i')$ for $i \in \{0, \dots, k\}$, and $r - \rho_K^j(b_0')e_0^j - \dots - \rho_K^j(b_k')e_k^j$ is in $(\mathfrak{m}_K^j)^{l+1}$. Hence, $b_i' = b_i$ for $i \in \{0, \dots, k\}$. It follows that $a_i' = a_i$ for $i \in \{0, \dots, k\}$, which gives us the desired uniqueness.

For the backward direction, suppose e is a basis of $V(\mathcal{D}_K, \delta_K)$. Consider

$$f : \mathcal{D}_K \to \mathcal{D}_K, \quad \sum_{i=0}^{m} \delta_K(a_i)e_i \mapsto \sum_{i=0}^{m} \rho_K(a_i)e_i$$

where a_i is in K for $i \in \{0, \dots m\}$. It is easy to check that f is a K-algebra isomorphism from $(\mathcal{D}_K, \delta_K)$ to $(\mathcal{D}_K, \rho_K)$. For any $j \in \{0, \dots, n\}$, f induces an isomorphism $(\mathcal{D}^j_K, \delta^j_K) \cong (\mathcal{D}^{j'}_K, \rho^{j'}_K)$ for some $j' \in \{0, \dots, n\}$ (in fact, it is not hard to show that we must have $j = j'$, but we do not need to use this). Since the composition $\pi^{j'}_K \circ \rho^{j'}_K : K \to \mathcal{D}^{j'}_K / \mathfrak{m}^{j'}_K$ is a field isomorphism, it follows that $\pi^j_K \circ \delta^j_K : K \to \mathcal{D}^j_K / \mathfrak{m}^j_K$ is also a field isomorphism. So $(K, \partial_1, \dots, \partial_m)$ is inversive. $\square$

The proposition above suggests how to find an existential bi-interpretation between the theory of inversive $\mathcal{D}$-fields and a union of two theories, each of which is existentially bi-interpretable with the theory of fields extending F. We now spell out the details.

In the rest of the section, we will never seriously encounter the situation where two different F-algebras have the same underlying ring. Therefore, we will suppress the F-embedding ρ, refer to an F-algebra (A, ρ) as the F-algebra A, and refer to $\rho(c)$ for $c \in F$ as c_A. Note that the data specified by the ring A and $(c_A)_{c \in F}$ is completely equivalent to the data of (A, ρ). We can then view an F-algebra A as a structure in a language extending the language of rings by constants for the elements c_A as c ranges over F. We will refer to this as the language of F-algebras.

Let $(K, \partial_1, \dots, \partial_m)$ be a $\mathcal{D}$-field and δ_K its associated F-algebra homomorphism. Then we can view $(K, \mathcal{D}_K; \pi_K, e, \rho_K, \delta_K)$ naturally as a structure in a two-sorted language which consists of two copies of the language of F-algebras for K and $\mathcal{D}_K$, constant symbols for the components of e, and function symbols for ρ_K and δ_K. We set this language to be $L_\cup$. Let L_1, L_2, and $L_\cap$ be the sublanguages corresponding to the reducts $(K, \mathcal{D}_K; \pi_K, e, \rho_K)$, $(K, \mathcal{D}_K; \pi_K, e, \delta_K)$, and $(K, \mathcal{D}_K; \pi_K, e)$, respectively. We note that L_1 and L_2 only differ in the names of the function symbols ρ_K and δ_K.

Let T_1^- be the L_1-theory whose models $(K, A; \pi, u, \rho)$ satisfy the following conditions:

(1) K and A are F-algebras, and K is a field.
(2) $\rho : K \to A$ is an embedding of F-algebras.
(3) $u = (u_0, \dots, u_m)$ is a basis of $V(A, \rho)$ such that

$$u_i u_j = \sum_{i,j,k} (c_{ijk})_A u_k$$

and

$$1_A = u_0 + \sum_{i=1}^{m} (d_i)_A u_i$$

(4) $\pi : A \to K$ is an F-algebra homomorphism with $\pi(u_0) = 1_K$ and $\pi(u_i) = 0_K$ for $i \in \{1, \dots, m\}$

Let T_2^- be the copy of T_1^- obtained by replacing L_1-symbols with L_2-symbols. Set $T_\cup^- = T_1^- \cup T_2^-$, and let $T_\cap^-$ be the set of $L_\cap$-consequence of T_1^- (equivalently, of T_2^-). Suppose $(K, \partial_1, \dots, \partial_m)$ be a $\mathcal{D}$-field and δ_K its associated F-algebra homomorphism. It is easy to see that

$$(K, \mathcal{D}_K; \pi_K, e, \rho_K) \models T_1^- \quad \text{and} \quad (K, \mathcal{D}_K; \pi_K, e) \models T_\cap^-.$$

If $(K, \partial_1, \dots, \partial_m)$ is inversive, it follows from Proposition 7.3 that

$$(K, \mathcal{D}_K; \pi_K, e, \delta_K) \models T_2^- \quad \text{and} \quad (K, \mathcal{D}_K; \pi_K, e, \rho_K, \delta_K) \models T_\cup^-.$$

The following lemma can be easily verified.

Lemma 7.1. *Suppose $(K, A; \pi, u, \rho) \models T_1^-$. Then*

$$f : A \to \mathcal{D}_K, \quad \sum_{i=0}^{m} \rho(a_i) u_i \mapsto \sum_{i=0}^{m} \rho_K(a_i) e_i$$

is K-algebra isomorphism which moreover induces an L_1-isomorphism from $(K, A; \pi, u, \rho)$ to $(K, \mathcal{D}_K; \pi_K, e, \rho_K)$ which we also denote by f.

We will view a field extending F as a structure in the language of F-algebras.

Proposition 7.4. *Both T_1^- and T_2^- are existentially bi-interpretable with the theory of fields extending F.*

Proof. It suffices to prove the statement for T_1^-. If $(K, A; \pi, u, \rho)$ is a model of T_1^-, then K is a field extending F. Conversely, if K is a field extending F, then since $V(\mathcal{D}_K, \rho_K)$ is isomorphic to K^m as a K-vector space, we can define a K-algebra (K^m, ρ) such that $(K, K^m; \pi, e, \rho) \models T_1^-$, where e is the standard basis of K^m and π is the projection on the first coordinate. It is easy to see that these constructions correspond to existential interpretations. If we start with K and apply the second construction followed by the first, we get K back, and id_K is the required isomorphism. And if we start with $(K, A; \pi, u, \rho) \models T_1^-$ and apply the first construction followed by the second, we obtain the required isomorphism from Lemma 7.1, since both structures are isomorphic to $(K, \mathcal{D}_K; \pi_k, e, \rho_K)$. It is also easy to check that this can be defined by an existential formula chosen independently of $(K, A; \pi, u, \rho)$. Thus T_1 and the theory of fields are existentially bi-interpretable. $\square$

The existential bi-interpretation in Proposition 7.4 restricts to a mutual interpretation between $T_\cap^-$ and the theory of fields extending F. But this is not a bi-interpretation, due to the fact that ρ_K is not definable in the structure $(K, \mathcal{D}_K; \pi_K, e)$. In fact, if $(K, A; \pi, e) \models T_\cap^-$, it does not necessarily follow that A is isomorphic to $\mathcal{D}_K$ as an K-algebra (or even as a ring).

The model companion of the theory of fields extending F is the theory of algebraically closed fields extending F. Applying Corollary 5.2, we get the following.

Corollary 7.4. *The theories T_1^- and T_2^- have model companions T_1 and T_2. The theories T_1 and T_2 are each existentially bi-interpretable with the theory of algebraically closed fields extending F.*

For the rest of the section, we let T_1 and T_2 be as described in Corollary 7.4. Let $T_\cup^- = T_1^- \cup T_2^-$ and $T_\cup = T_1 \cup T_2$.

We view a $\mathcal{D}$-field $(K, \partial_1, \dots, \partial_n)$ as a structure in a language extending the language of F-algebras by adding function symbols for $\partial_1, \dots, \partial_n$. In [**60**], it is verified that the class of $\mathcal{D}$-fields and the class of inversive $\mathcal{D}$-fields are elementary. It follows that we can also axiomatize the theory of algebraically closed $\mathcal{D}$-fields ($\mathcal{D}$-fields whose underlying fields are algebraically closed).

Theorem 7.10. *The theory of inversive $\mathcal{D}$-fields is existentially bi-interpretable with $T_\cup^-$, and the theory of algebraically closed inversive $\mathcal{D}$-fields is existentially bi-interpretable with $T_\cup$. Hence, the theory of algebraically closed inversive $\mathcal{D}$-fields has a model companion if and only if $T_\cup^*$ exists. Moreover, this model companion is bi-interpretable with $T_\cup^*$ whenever they both exist.*

Proof. We will only prove the first claim, as the other claims are immediate consequences. If $(K, A; \pi, u, \rho, \delta) \models T_\cup^-$, then since u is a basis for $V(A, \rho)$ and δ and ρ are both sections of π, we have that for any $a \in K$, there exist unique $d_1, \dots, d_m \in K$ such that

$$\delta(a) = \rho(a)u_0 + \rho(d_1)u_1 + \dots + \rho(d_m)u_m.$$

We define $\partial_i(a) = d_i$ for all $i \in \{1, \dots, m\}$. Then it follows from Proposition 7.3 that $(K, \partial_1, \dots, \partial_m)$ is an inversive $\mathcal{D}$-field. Conversely, suppose $(K, \partial_1, \dots, \partial_m)$ is an inversive $\mathcal{D}$-field and δ_K is the associated F-algebra homomorphism. Then by Proposition 7.3, and encoding an isomorphic copy of $\mathcal{D}_K$ with domain K^m as in the proof of Proposition 7.4, we have $(K, \mathcal{D}_K; \pi_K, e, \rho_K, \delta_K) \models T_\cup^-$. It is easy to see that the two constructions above describe existential interpretations between $T_\cup^-$ and the theory of inversive $\mathcal{D}$-fields.

If $(K, A; \pi, u, \rho, \delta) \models T_\cup^-$, then applying the first construction followed by the second construction in the preceding paragraph gives us $(K, \mathcal{D}_K; \pi_K, e, \rho_K, \delta_K)$, and a calculation shows that (id_K, f) is an isomorphism from $(K, A; \pi, u, \rho, \delta)$ to $(K, \mathcal{D}_K; \pi_K, e, \rho_K, \delta_K)$ where f is the function in Lemma 7.1. If $(K, \partial_1, \dots, \partial_m)$ is an inversive $\mathcal{D}$-field, then applying the second construction followed by the first construction in the preceding paragraph gives us back $(K, \partial_1, \dots, \partial_m)$, and id_K is already the desired isomorphism. It is also easy to check that there are existential formulas chosen independently from $(K, A; \pi, u, \rho, \delta)$ and $(K, \partial_1, \dots, \partial_m)$ that define these isomorphisms. Thus the two theories are existentially bi-interpretable. □

In [**60**], it is shown that when $\mathrm{char}(F) = 0$, every $\mathcal{D}$-field can be embedded into an algebraically closed inversive $\mathcal{D}$-field, and the theory of $\mathcal{D}$-fields has a model companion. Hence, we get the following corollary.

Corollary 7.5. *If* $\mathrm{char}(F) = 0$*, then the model companion of the theory of* $\mathcal{D}$*-fields is bi-interpretable with* $T^*_\cup$*.*

In the special cases from Example 7.1, $\mathcal{D}$-fields are simply differential fields of characteristic 0 or difference fields of characteristic 0, and the theory of algebraically closed fields extending $\mathbb{Q}$ is simply ACF_0, so we obtain the following consequence.

Corollary 7.6. *The theories* DCF_0 *and* ACFA_0 *are each bi-interpretable with an interpolative fusion of two theories* T_1 *and* T_2*, each of which is bi-interpretable with* ACF_0*.*

CHAPTER 8

Existence results

Throughout this chapter, we assume in addition to the notational conventions of Chapters 5 and 6 that L' is a first-order language also with S the set of sorts and $L \subseteq L'$, $\mathcal{M}$ and $\mathcal{M}'$ are an L-structure and an L'-structure both with underlying collection of sorts M, T' is an L'-theory, and T is the set of L-consequences of T'.

The goal of the chapter is to provide sufficient conditions for the existence of the interpolative fusion with an eye toward natural examples. For this purpose, it is useful to find simpler characterizations of interpolative $T_\cup$-models in various settings. In Section 8.1, we accomplish this in the setting when $T_\cap$ admits an ordinal-valued dimension function, highlighting the notions of approximability and definability of pseudo-denseness in the expansions T_i. In Section 8.2, we show how to relativize these conditions to collections of definable sets we call pseudo-cells. In the remaining sections, we investigate these notions under additional hypotheses on $T_\cap$ such as $\aleph_0$-stability and o-minimality.

Our general theory allows us to recover many results on the existence of model companions in the literature. In fact, the existing proofs of these results can be thought of as specializations of the general arguments developed here. This is an imprecise claim which cannot be rigorously justified, but we will demonstrate what we mean by revisiting the earlier examples from Chapter 7.

8.1. The pseudo-topological axioms

In an interpolative structure $\mathcal{M}_\cup$, any finite family of L_i-definable sets which is not separated has nonempty intersection. Heuristically, if each X_i is "large" in a some fixed $X_\cap$, then $(X_i)_{i\in I}$ cannot be separated. When $T_\cap$ has a reasonable notion of dimension we can make this idea precise. The setting has a certain topological flavor, hence the name.

Throughout this section, we assume the existence of a function dim, which assigns an ordinal or the formal symbol $-\infty$ to each $\mathcal{M}$-definable set so that for all $\mathcal{M}$-definable $X, X' \subseteq M^x$:

(1) $\dim(X \cup X') = \max\{\dim X, \dim X'\}$,
(2) $\dim X = -\infty$ if and only if $X = \varnothing$,
(3) $\dim X = 0$ if and only if X is nonempty and finite,

We call such a function dim an **ordinal rank** on $\mathcal{M}$. A function on the collection of definable sets in T-models that restricts to an ordinal rank on each T-model, and such that $\dim X(\mathcal{M}) = \dim X(\mathcal{N})$ for all $\mathcal{M} \models T$, $\mathcal{M}$-definable sets X, and elementary extensions $\mathcal{N}$ of $\mathcal{M}$, is called an **ordinal rank** on T. In all cases of interest T defines dim in families. Note that when T is complete, an ordinal rank on T is essentially the same as an ordinal rank on the monster model of T.

We can equip any theory with a trivial ordinal rank by declaring $\dim(X) = 1$ whenever X is infinite. Tame theories are generally equipped with a natural (often canonical) ordinal rank. Examples are $\aleph_0$-stable theories with Morley rank, superstable theories with U-rank, and supersimple theories with SU-rank.

Let X be a definable subset of M^x and A be an arbitrary subset of M^x. Then A is **pseudo-dense** in X if A intersects every nonempty definable $X' \subseteq X$ such that $\dim X' = \dim X$. We call X a **pseudo-closure** of A if $A \subseteq X$ and A is pseudo-dense in X. The following lemma collects a few easy facts about pseudo-denseness, the proofs of which we leave to the readers.

Lemma 8.1. *Let X and X' be $\mathcal{M}$-definable subsets of M^x, and let A be an arbitrary subset of M^x. Then:*

(1) *When X is finite, A is pseudo-dense in X if and only if $X \subseteq A$.*
(2) *If A is pseudo-dense in X, $X' \subseteq X$, and $\dim X' = \dim X$, then A is pseudo-dense in X'.*
(3) *If $X^1, \dots, X^n \subseteq X$ are $\mathcal{M}$-definable, with $\dim X^i = \dim X$ for all i, and*

$$\dim X \triangle (X^1 \cup \dots \cup X^n) < \dim X,$$

then A is pseudo-dense in X if and only if A is pseudo-dense in each X^i.

If in addition X is a pseudo-closure of A, then:

(4) *$A \subseteq X'$ implies $\dim X \leqslant \dim X'$.*
(5) *If X' is another pseudo-closure of A, then $\dim(X \triangle X') < \dim X = \dim X'$.*
(6) *If $A \subseteq X' \subseteq X$ then X' is a pseudo-closure of A.*

Suppose $\mathcal{M}'$ is an expansion of $\mathcal{M}$. Then $\mathcal{M}'$ is **approximable** over $\mathcal{M}$ (with respect to dim) if every $\mathcal{M}'$-definable set admits an $\mathcal{M}$-definable pseudo-closure.

The definition above admits an obvious generalization to theories. If T is equipped with an ordinal rank, we say that T' is **approximable over** T if $\mathcal{M}'$ is approximable over $\mathcal{M} = \mathcal{M}' \restriction L$ for all $\mathcal{M}' \models T'$.

Proposition 8.1. *Suppose $J \subseteq I$ is finite and $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for all $i \in J$. If there is an $\mathcal{M}_\cap$-definable set X in which each X_i is pseudo-dense, then $(X_i)_{i \in J}$ is not separated. The converse implication holds provided $\mathcal{M}_i$ is approximable over $\mathcal{M}_\cap$ for all $i \in J$.*

Proof. For the first statement, suppose X is a nonempty $\mathcal{M}_\cap$-definable subset of M^x in which each X_i is pseudo-dense, and $(X^i)_{i\in J}$ is a family of $\mathcal{M}_\cap$-definable sets satisfying $X_i \subseteq X^i$ for each $i \in J$. As X_i is pseudo-dense in X and disjoint from $X \setminus X^i$, we have $\dim X \setminus X^i < \dim X$ for all $i \in J$. Hence,

$$\dim \bigcup_{i\in J}(X \setminus X^i) < \dim X.$$

Thus $\dim \bigcap_{i\in J} X^i \geqslant \dim X$, so $\bigcap_{i\in J} X^i$ is nonempty.

Now suppose $\mathcal{M}_i$ is approximable over $\mathcal{M}_\cap$ for each $i \in J$. Simplifying notation, we let $J = \{1,\dots,n\}$. Suppose X_i is an $\mathcal{M}_i$-definable set for each $1 \leqslant i \leqslant n$, and suppose there is no $\mathcal{M}_\cap$-definable set Z in which all of the X_i are pseudo-dense. We show $(X_i)_{i=1}^n$ is separated by applying simultaneous transfinite induction to $d_1,\dots,d_n$ where d_i is the dimension of any pseudo-closure of X_i.

Let X^i be a pseudo-closure of X_i for each i and let

$$Z = X^1 \cap \ldots \cap X^n.$$

If $\dim X^j = -\infty$ for some $j \in J$, then X^j and Z are both empty, so $(X^i)_{i=1}^n$ separates $(X_i)_{i=1}^n$. If $\dim X^i = \dim Z$ for each i, then Lemma 8.1(2) shows each X_i is pseudo-dense in Z, contradiction. After re-arranging the X_i if necessary we suppose $\dim Z < \dim X^1$. Let $Y_1 = X_1 \cap Z$. As $(X_i)_{i=1}^n$ cannot be simultaneously pseudo-dense in an $\mathcal{M}_\cap$-definable set, it follows that $Y_1, X_2, \dots, X_n$ cannot be simultaneously pseudo-dense in an $\mathcal{M}_\cap$-definable set. As the dimension of any pseudo-closure of Y_1 is strictly less then the dimension of X^1, an application of the inductive hypothesis provides $\mathcal{M}_\cap$-definable sets $Y^1,\dots,Y^n$ separating $Y_1, X_2, \dots, X_n$. It is easy to see

$$Y^1 \cup (X^1 \setminus Z), Y^2 \cap X^2, \dots, Y^n \cap X^n$$

separates $X_1,\dots,X_n$, which completes the proof. $\square$

We say $\mathcal{M}_\cup$ is **approximately interpolative** if whenever $J \subseteq I$ is finite, $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for $i \in J$, and $(X_i)_{i\in J}$ are simultaneously pseudo-dense in some nonempty $\mathcal{M}_\cap$-definable set, then $\bigcap_{i\in J} X_i \neq \emptyset$. As we will see in the later parts of Chapter 8, this definition is very close in spirit to the definitions of generic predicates in [**11**], generic automorphisms in [**13**], algebraically closed fields with independent valuations in [**42**], and algebraically closed fields with generic multiplicative circular order in [**79**].

The following corollary is an immediate consequence of Proposition 8.1.

Corollary 8.1. *If $\mathcal{M}_\cup$ is interpolative, then it is approximately interpolative. The converse also holds if $\mathcal{M}_i$ is approximable over $\mathcal{M}_\cap$ for each $i \in I$.*

We say that T' **defines pseudo-denseness** over T if for every L-formula $\varphi(x, y)$ and every L'-formula $\varphi'(x, z)$, there is an L'-formula $\delta'(y, z)$ such that if $\mathcal{M}' \models T'$, $b \in M^y$, and $c \in M^z$, then

$$\varphi'(\mathcal{M}', c) \text{ is pseudo-dense in } \varphi(\mathcal{M}', b) \text{ if and only if } \mathcal{M}' \models \delta(b, c).$$

Theorem 8.1. *Suppose* dim *is an ordinal rank on* $T_\cap$. *Then:*

(1) *If T_i defines pseudo-denseness over $T_\cap$ for all $i \in I$, then the class of approximately interpolative $T_\cup$-models is elementary.*
(2) *If, in addition, T_i is approximable over $T_\cap$ for all $i \in I$, then $T^*_\cup$ exists.*

Proof. We first prove (1). Let $\varphi_\cap(x, y)$ be an $L_\cap$-formula, let $J \subseteq I$ be finite, and let $\varphi_i(x, z_i)$ be an L_i-formula for each $i \in J$. Let $\delta_i(y, z_i)$ be an L_i-formula defining pseudo-denseness for $\varphi_\cap(x, y)$ and $\varphi_i(x, z_i)$. For simplicity, we assume $J = \{1, \ldots, n\}$. Then we have the following axiom:

$$\forall y, z_1, \ldots, z_n \left(\left(\bigwedge_{i=1}^{n} \delta_i(y, z_i) \right) \to \exists x \bigwedge_{i=1}^{n} \varphi_i(x, z_i) \right).$$

Then $T_\cup$, together with one such axiom for each choice of $\varphi_\cap(x, y)$, J, and $\varphi_i(x, z_i)$ for $i \in J$ as above, axiomatizes the class of approximately interpolative $T_\cup$-models. Statement (2) follows from statement (1) and Corollary 8.1. □

We refer to the axiomatization given in the proof of Theorem 8.1 as the **pseudo-topological axioms**.

Remark 8.1. We shall see that many examples where $T^*_\cup$ exists can be viewed as special cases of Theorem 8.1. On the other hand, there are also interesting examples which are not covered by Theorem 8.1. In Proposition 9.7, we consider another sufficient condition for the existence of $T^*_\cup$, which does not assume any notion of dimension on $T_\cap$. Therefore, this lies completely outside the framework of Chapter 8. Below, we will revisit the example from Section 7.3. Here, there is a good notion of dimension on $T_\cap$, and $T^*_\cup$ exists, but none of the T_i are approximable over $T_\cap$, so this is not a special case of Theorem 8.1.

Consider the setting of Section 7.3. Let dim be the canonical rank on the additive group of integers, which coincides with U-rank, acl-dimension, etc; see for example [**20**].

Proposition 8.2. *Suppose* $(Z; 0, +, -, \preccurlyeq_p)$ *is an* $\aleph_1$*-saturated elementary extension of* $(\mathbb{Z}; 0, +, -, \preccurlyeq_p)$. *Then* $(Z; 0, +, - \preccurlyeq_p)$ *is not approximable over* $(Z; 0, +, -)$.

Proof. Let N be an element of Z such that $k \preccurlyeq_p N$ for all $k \in \mathbb{Z}$. We show that

$$E := \{z \in Z \mid N \preccurlyeq_p z\}$$

does not have a pseudo-closure in Z. We make use of the fact that a $(Z;0,+,-)$-definable subset of Z is one-dimensional if and only if it is infinite. Quantifier elimination for $(Z;+,0,1)$ implies that every $(Z;0,+,-)$-definable subset of Z is a finite union of sets of the form $(kZ+l)\setminus F$ for $k,l\in\mathbb{Z}$ and finite F. This is also a special case of Conant's quasi-coset decomposition.

Thus, if E has a pseudo-closure then E is pseudo-dense in $kZ+l$ for some $k,l\in\mathbb{Z}$. We fix $k\in\mathbb{Z}$ and $l\in\{0,\dots,k-1\}$, and we show E is not pseudo-dense in $kZ+l$. As E is a subgroup of Z and $kZ+l$ is a coset of a subgroup, E and $kZ+l$ are disjoint when $l\neq 0$, so it suffices to treat the case when $l=0$. Then $E\subseteq kZ$. Let $k'=pk$ so $v_p(k')=v_p(k)+1$. Then $E\subseteq k'Z\subseteq kZ$. As $v_p(k'm)\geqslant v_p(k)+1$ and $v_p(k'm+k)=v_p(k)$ for all $m\in\mathbb{Z}$, $k'Z+k$ is disjoint from $k'Z$. Thus, $k'Z+k$ is a one-dimensional definable subset of kZ which is disjoint from E. Hence E is not pseudo-dense in kZ. $\square$

Remark 8.2. One can in fact show that $(\mathbb{Z};0,+,-,\preccurlyeq_p)$ is not approximable over $(\mathbb{Z};0,+,-)$ by applying the "quasi-coset" decomposition of $(\mathbb{Z};0,+,-)$-definable sets given in [**20**, Theorem 4.10] to show that

$$\{(k,l)\in\mathbb{Z}^2 : k\preccurlyeq_p l\}$$

does not have a pseudo-closure in $\mathbb{Z}^2$. This presents some technical difficulties so we do not include it here. As every $(\mathbb{Z};0,+,-,\preccurlyeq_p)$-definable subset of $\mathbb{Z}$ is $(\mathbb{Z};0,+,-)$-definable [**24**], we must pass to an elementary extension to obtain a unary set without a pseudo-closure.

The following two issues deserve further investigation. First, when the class of approximately interpolative $T_\cup$-models is elementary, we could call the theory of this class the approximate interpolative fusion. Can we say anything interesting about the model theory of the approximate interpolative fusion in cases when not all the T_i are approximable over $T_\cap$? Second, Theorem 8.1 tells us that defining pseudo-denseness is the key sufficient property for existence of the approximate interpolative fusion. We believe the converse may also be true when $T_\cap$ defines dimension, but we currently do not have a proof.

8.2. Relativization to pseudo-cells

Sometimes it is enough to check the sufficient conditions of the previous section for a sufficiently rich collection $\mathcal{C}$ of $\mathcal{M}_\cap$-definable sets. We call such a $\mathcal{C}$ a pseudo-cell collection.

Suppose $\mathcal{M}$ is an L-structure equipped with an ordinal rank dim and $\mathcal{C}$ is a collection of $\mathcal{M}$-definable sets. We say that an $\mathcal{M}$-definable set X admits a **$\mathcal{C}$-decomposition** if there is a finite family $(X_j)_{j\in J}$ from $\mathcal{C}$ such that

$$\dim\Big(X \vartriangle \bigcup_{j\in J} X^j\Big) < \dim X.$$

An $\mathcal{M}$-definable set X admits a **$\mathcal{C}$-patching** if there is a finite family $(X^j, Y^j, f^j)_{j \in J}$ such that for all $j, j' \in J$:

(1) Y^j is in $\mathcal{C}$.
(2) $f^j : X^j \to Y^j$ is an $\mathcal{M}$-definable bijection.
(3) And finally,

$$\dim\Big(X \triangle \bigcup_{j \in J} X^j\Big) < \dim X.$$

We say $\mathcal{C}$ is a **pseudo-cell collection** for $\mathcal{M}$ if either every $\mathcal{M}$-definable set admits a $\mathcal{C}$-decomposition or dim is preserved under $\mathcal{M}$-definable bijections and every $\mathcal{M}$-definable set admits a $\mathcal{C}$-patching. Examples include the collection of irreducible varieties in an algebraically closed field and the collection of cells in an o-minimal structure.

The definition above naturally extends to theories. Let T be an L-theory equipped with an ordinal rank dim and $\mathcal{C}$ a collection of definable sets in T-models. We say that $\mathcal{C}$ is a **pseudo-cell collection** for T if for all $\mathcal{M} \models T$, $\mathcal{C} \cap \mathrm{Def}(\mathcal{M})$ is a pseudo-cell collection for $\mathcal{M}$.

Suppose dim is an ordinal rank on $\mathcal{M}_\cap$ and $\mathcal{C}$ is a collection of $\mathcal{M}_\cap$-definable sets. We say $\mathcal{M}_\cup$ is **$\mathcal{C}$-approximately interpolative** if for all finite $J \subseteq I$, $X_\cap \in \mathcal{C}$, and $(X_i)_{i \in J}$, where X_i is $\mathcal{M}_i$-definable and pseudo-dense in $X_\cap$, we have $\bigcap_{i \in J} X_i \neq \emptyset$. Clearly, if $\mathcal{M}_\cup$ is approximately interpolative then it is $\mathcal{C}$-approximately interpolative. The following proposition gives situations where the converse is true. We omit the straightforward proof.

Proposition 8.3. *Suppose $\mathcal{C}$ is a collection of pseudo-cells in $\mathcal{M}_\cap$. Then we have the following:*

(1) *$\mathcal{M}_\cup$ is approximately interpolative if and only if it is $\mathcal{C}$-approximately interpolative.*
(2) *If moreover $\mathcal{M}_i$ is approximable over $\mathcal{M}_\cap$ for all $i \in I$, then $\mathcal{M}_\cup$ is interpolative if and only if it is $\mathcal{C}$-approximately interpolative.*

Let $\mathcal{C}$ be a collection of definable sets in T-models. We say that T **defines $\mathcal{C}$-membership** if for every L-formula $\varphi(x, y)$ there is an L-formula $\gamma(y)$ such that for all $\mathcal{M} \models T$ and $b \in M^y$,

$$\varphi(\mathcal{M}, b) \text{ is in } \mathcal{C} \text{ if and only if } \mathcal{M} \models \gamma(b).$$

We say that T' **defines pseudo-denseness over** $\mathcal{C}$ if for every L'-formula $\varphi(x, y)$ and every L-formula $\varphi(x, z)$, there is an L'-formula $\delta'(y, z)$ such that if $\mathcal{M}' \models T'$ and $c \in M^y$ with $\varphi(\mathcal{M}', c) \in \mathcal{C}$, then

$$\varphi'(\mathcal{M}', b) \text{ is pseudo-dense in } \varphi(\mathcal{M}', c) \text{ if and only if } \mathcal{M}' \models \delta'(b, c).$$

Theorem 8.2. *Suppose* dim *is an ordinal rank on $T_\cap$, $\mathcal{C}$ is a collection of definable sets of $T_\cap$-models such that $T_\cap$ defines $\mathcal{C}$-membership, and T_i defines pseudo-denseness over $\mathcal{C}$ for $i \in I$. Then we have the following:*

1. *The class of $\mathcal{C}$-approximately interpolative $T_\cup$-models is elementary.*
2. *If $\mathcal{C}$ is a pseudo-cell collection for $T_\cap$, then the class of approximately interpolative $T_\cup$-models is elementary.*
3. *If, in addition, T_i is approximable over $T_\cap$ for each $i \in I$, then the interpolative fusion exists.*

Proof. We first prove statement (1). Let $\varphi_\cap(x, y)$ be an $L_\cap$-formula, let $J \subseteq I$ be finite, and let $\varphi_i(x, z_i)$ be an L_i-formula for each $i \in J$. Let $\gamma_\cap(y)$ be an $L_\cap$-formula defining $\mathcal{C}$-membership for $\varphi_\cap(x, y)$ and $\delta_i(y, z_i)$ an L_i-formula defining pseudo-denseness over $\mathcal{C}$ for $\varphi_\cap(x, y)$ and $\varphi_i(x, z_i)$ for each $i \in J$. For simplicity, we assume $J = \{1, \dots, n\}$. Then we have the following axiom:

$$\forall y, z_1, \dots, z_n \left(\left(\gamma_\cap(y) \wedge \bigwedge_{i=1}^{n} \delta_i(y, z_i) \right) \to \exists x \bigwedge_{i=1}^{n} \varphi_i(x, z_i) \right).$$

Then $T_\cup$, together with one axiom of the above form for each choice of $\varphi_\cap(x, y)$, J, and $\varphi_i(x, z_i)$ for $i \in J$ as above, axiomatizes the class of $\mathcal{C}$-approximately interpolative $T_\cup$-models. Assertions (2) and (3) follow immediately from Proposition 8.3. $\square$

The axiomatization given in the proof of Theorem 8.2 is slightly different than that of Theorem 8.1. They are nevertheless very similar in spirit, so we also refer to the former as the **pseudo-topological axioms**.

Clearly, if T' defines pseudo-denseness over T, then T' defines pseudo-denseness over any collection $\mathcal{C}$ of definable sets of T-models. The converse is true when the dimension is definable.

We say T **defines dimension** if for every ordinal α, and every L-formula $\varphi(x, y)$, there is an L-formula $\delta_\alpha(x, y)$ such that for all $\mathcal{M} \models T$ and $b \in M^y$

$$\dim \varphi(\mathcal{M}, b) = \alpha \quad \text{if and only if} \quad \mathcal{M} \models \delta_\alpha(b).$$

We leave the straightforward proof of the following proposition to the reader.

Proposition 8.4. *Suppose $\mathcal{C}$ is a collection of pseudo-cells, T defines $\mathcal{C}$-membership and dimension, and T' defines pseudo-denseness over $\mathcal{C}$. Then T' defines pseudo-denseness over T.*

8.3. Tame topological base

If $\mathcal{M}$ is o-minimal, then $\mathcal{M}'$ is approximable over $\mathcal{M}$ if and only if the closure of every $\mathcal{M}'$-definable set is $\mathcal{M}$-definable. This equivalence only depends on two well-known facts from o-minimality. One of these is known as the frontier inequality, and we refer to the other as the residue inequality. We explore these issues in an abstract setting below.

A **definable topology** $\mathcal{T}$ on $\mathcal{M}$ consists of a topology $\mathcal{T}_x$ on each M^x, for which there is an L-formula $\varphi(x,y)$ such that $\{\varphi(\mathcal{M},a) : a \in M^y\}$ is an open basis for $\mathcal{T}_x$. Note that we also obtain a definable topology on every structure elementarily equivalent to $\mathcal{M}$.

For the rest of Section 8.3, we suppose $\mathcal{T}$ is a definable topology on $\mathcal{M}$ and dim is an ordinal rank on $T = \mathrm{Th}(\mathcal{M})$, such that T defines dimension.

Let A be a subset of M^x. We denote by $\mathrm{cl}(A)$ the closure of A with respect to $\mathcal{T}_x$. The **frontier** of A, $\mathrm{fr}(A)$, is defined as $\mathrm{cl}(A) \setminus A$. Since $\mathcal{T}$ is a definable topology, the interior, closure, and frontier of a definable subset of M^x are definable. We say that A has **nonempty interior** in $X \subseteq M^x$ if there is an open $U \subseteq M^x$ such that $U \cap X \subseteq A$.

In general there need be no connection between pseudo-denseness and $\mathcal{T}$-denseness. We give conditions under which the two naturally relate. We say $\mathcal{M}$ satisfies the **frontier inequality** if

$$\dim \mathrm{fr}(X) < \dim X \quad \text{for all definable } X.$$

This is a strong assumption which in particular implies, by a straight-forward induction on dimension, that every definable set is a Boolean combination of open definable sets.

Lemma 8.2. *Suppose $\mathcal{M}$ satisfies the frontier inequality and $X' \subseteq X$ are $\mathcal{M}$-definable sets. If $\dim X' = \dim X$, then X' has nonempty interior in X.*

Proof. If X' has empty interior in X, then $X \setminus X'$ is dense in X, and so $X' \subseteq X \subseteq \mathrm{cl}(X \setminus X')$. In particular, $X' \subseteq \mathrm{fr}(X \setminus X')$. The frontier inequality implies $\dim X' < \dim X \setminus X' \leqslant \dim X$. $\square$

Lemma 8.3. *The following are equivalent:*

1. *$\mathcal{M}$ satisfies the frontier inequality.*
2. *If $A \subseteq M^x$ is dense in a definable $X \subseteq M^x$ then A is pseudo-dense in X.*

Proof. Suppose that $\mathcal{M}$ satisfies the frontier inequality and that $A \subseteq M^x$ is dense in a definable set $X \subseteq M^x$. Suppose $X' \subseteq X$ is definable and $\dim X' = \dim X$. Lemma 8.2 implies that X' has nonempty interior in X. Thus A intersects X'. It follows that A is pseudo-dense in X.

Conversely, assume (2), and let $X \subseteq M^x$ be definable. Since X is dense in $\mathrm{cl}(X)$, X is also pseudo-dense in $\mathrm{cl}(X)$. But since X does not intersect $\mathrm{fr}(X)$, we have $\dim \mathrm{fr}(X) < \dim \mathrm{cl}(X)$. It follows that $\dim X = \dim \mathrm{cl}(X)$, so the frontier inequality holds. □

The converse to (2) above almost always fails for general definable sets X. For example, if $A \subseteq M^x$ is an infinite definable set and $p \in M^x$ does not lie in $\mathrm{cl}(A)$, then A is pseudo-dense in $X = A \cup \{p\}$ but not dense in X. However, the converse to (2) does hold for certain definable sets, which we call dimensionally pure.

Let $X \subseteq M^x$ be definable. Given $p \in X$, we define

$$\dim_p X = \min\{\dim(U \cap X) : U \text{ is a definable neighborhood of } p\}.$$

We say that X is **dimensionally pure** if $\dim_p X = \dim X$ for all $p \in X$. Equivalently, X is dimensionally pure if and only if $\dim U = \dim X$ for all $U \subseteq X$ such that U is definable, nonempty, and open in X.

Lemma 8.4. *Suppose $X \subseteq M^x$ is definable. Then the following are equivalent:*

(1) *X is dimensionally pure.*
(2) *If a subset A of M^x is pseudo-dense in X, then A is dense in X.*

PROOF. Suppose X is not dimensionally pure. Let U be a definable nonempty open subset of X such that $\dim U < \dim X$. Then $X \setminus U$ is pseudo-dense in X and not dense in X.

Suppose X is dimensionally pure and A is pseudo-dense in X. Suppose U is a nonempty open subset of X. Then there is a definable nonempty open subset U' of U. Then $\dim U' = \dim X$, so A intersects U'. Hence A is dense in X. □

The following proposition gives another characterization of dimensionally pure sets. We will not use this characterization, so we leave its proof to the reader.

Proposition 8.5. *Suppose $X \subseteq M^x$ is definable. If X is dimensionally pure, then there are no definable sets X^1 and X^2 such that $X = X^1 \cup X^2$, X^1 and X^2 are closed in X, neither X^1 nor X^2 contains the other, and $\dim X^1 \neq \dim X^2$. If $\mathcal{M}$ satisfies the frontier inequality, then the converse holds.*

For a definable $X \subseteq M^x$, we define the **essence** of X, $\mathrm{es}(X)$, and the **residue** of X, $\mathrm{rs}(X)$:

$$\mathrm{es}(X) = \{p \in X : \dim_p X = \dim X\}$$
$$\mathrm{rs}(X) = \{p \in X : \dim_p X < \dim X\}$$

As $\mathcal{T}_x$ admits a definable basis, and T defines dimension, it follows that $\mathrm{es}(X)$ and $\mathrm{rs}(X)$ are definable.

We say that $\mathcal{M}$ satisfies the **residue inequality** if

$$\dim \operatorname{rs}(X) < \dim X \quad \text{for all definable } X.$$

Note that the residue inequality implies that all definable discrete sets are finite.

Lemma 8.5. *If $\mathcal{M}$ satisfies the residue inequality, then for all definable $X \subseteq M^x$, $\operatorname{es}(X)$ is dimensionally pure.*

Proof. As $X = \operatorname{rs}(X) \cup \operatorname{es}(X)$ and $\dim \operatorname{rs}(X) < \dim X$, we have $\dim \operatorname{es}(X) = \dim(X)$. Now suppose $p \in \operatorname{es}(X)$ and U is a definable neighborhood of p. Then we have $\dim_p X = \dim X$ and $\dim(U \cap X) = \dim X$. But

$$(U \cap X) = (U \cap \operatorname{rs}(X)) \cup (U \cap \operatorname{es}(X)),$$

and $\dim(U \cap \operatorname{rs}(X)) \leqslant \dim \operatorname{rs}(X) < \dim X$, so $\dim(U \cap \operatorname{es}(X)) = \dim X = \dim \operatorname{es}(X)$. Hence $\dim_p \operatorname{es}(X) = \dim \operatorname{es}(X)$, as was to be shown. $\square$

We will not use the following proposition, but we include it here, since it provides additional motivation for the residue inequality.

Proposition 8.6. *$\mathcal{M}$ satisfies the residue inequality if and only if every definable set is a finite disjoint union of dimensionally pure definable sets.*

Proof. Suppose first that $\mathcal{M}$ satisfies the residue inequality. Let $X \subseteq M^x$ be definable. We argue by induction on $\dim X$. If $\dim X = -\infty$, then $X = \varnothing$ and the conclusion holds vacuously. Otherwise, X is the disjoint union of $\operatorname{es}(X)$ and $\operatorname{rs}(X)$. By Lemma 8.5, $\operatorname{es}(X)$ is dimensionally pure, and by the residue inequality $\dim \operatorname{rs}(X) < \dim X$, so by induction $\operatorname{rs}(X)$ is a finite disjoint union of dimensionally pure definable sets.

Conversely, for any definable set X, suppose that X is a disjoint union of dimensionally pure definable sets $Y_1, \ldots, Y_m$. We will show that $\dim \operatorname{rs}(X) < \dim X$. We may assume without loss of generality that $1 \leqslant j \leqslant m$ is such that

$$\dim Y_k = \dim X \text{ when } k \leqslant j \quad \text{and} \quad \dim Y_k < \dim X \text{ when } k > j.$$

Let $p \in \operatorname{rs}(X)$, and suppose for contradiction that $p \in Y_k$ for some $k \leqslant j$. Then since Y_k is dimensionally pure, $\dim_p Y_k = \dim Y_k = \dim X$, so for any definable neighborhood U of p,

$$\dim X = \dim(U \cap Y_k) \leqslant \dim(U \cap X) \leqslant \dim X.$$

So $\dim_p X = \dim X$, contradicting the fact that $p \in \operatorname{rs}(X)$. Thus $\operatorname{rs}(X) \subseteq \bigcup_{k>j} Y_k$, and $\dim \operatorname{rs}(X) \leqslant \dim \bigcup_{k>j} Y_k < \dim X$. $\square$

We say $\mathcal{T}$ is $\dim$-**compatible** if $\mathcal{M}$ satisfies both the frontier inequality and the residue inequality. **For the remainder of Section 8.3,** $\dim$ **is an ordinal rank on** $T = \operatorname{Th}(\mathcal{M})$

such that T defines dimension, and $\mathcal{T}$ is a dim**-compatible definable topology on $\mathcal{M}$.** Definability of the dimension and the topology ensure that dim-compatibility is an elementary property, i.e., the topology on any model of T is dim-compatible.

Proposition 8.7. *Suppose $X \subseteq M^x$ is definable and $A \subseteq M^x$. Then A is pseudo-dense in X if and only if A is dense in* $\mathrm{es}(X)$*.*

Proof. Since $\dim \mathrm{rs}(X) < \dim X$ and $\dim \mathrm{es}(X) = \dim X$, A is pseudo-dense in X if and only if A is pseudo-dense in $\mathrm{es}(X)$. The equivalence then follows from Lemma 8.3, Lemma 8.4, and Lemma 8.5. $\square$

Proposition 8.8. *Any expansion T' of T defines pseudo-denseness over T.*

Proof. Suppose $\mathcal{M}$ is a T-model and $\mathcal{M}'$ is a T'-model expanding $\mathcal{M}$. Suppose $(X_b)_{b \in M^y}$ and $(X'_c)_{c \in M^z}$ are families of subsets of M^x, which are $\mathcal{M}$-definable and $\mathcal{M}'$-definable, respectively. By Proposition 8.7, X'_c is pseudo-dense in X_b if and only if X'_c is dense in $\mathrm{es}(X_b)$.

Using definability of the topology and dimension, essences of definable sets are uniformly definable, i.e., there is an $\mathcal{M}$-definable family $(Y_b)_{b \in M^y}$ such that $Y_b = \mathrm{es}(X_b)$ for all $b \in M^y$. Thus X'_c is pseudo-dense in X_b if and only if X'_c is dense in Y_b. And using definability of the topology, the set of all (b, c) such that X'_c is dense in Y^b is definable. $\square$

Proposition 8.9. *Suppose $\mathcal{M}'$ expands $\mathcal{M}$. Then $\mathcal{M}'$ is approximable over $\mathcal{M}$ if and only if the closure of any $\mathcal{M}'$-definable set is $\mathcal{M}$-definable.*

Proof. Suppose that the closure of any $\mathcal{M}'$-definable set is $\mathcal{M}$-definable. Then for any $\mathcal{M}'$-definable $X \subseteq M^x$, $\mathrm{cl}(X)$ is a pseudo-closure of X by Lemma 8.3.

Conversely, suppose $\mathcal{M}'$ is approximable over $\mathcal{M}$ and $X' \subseteq M^x$ is $\mathcal{M}'$-definable. Let X be a pseudo-closure of X'. We apply induction to the dimension of X. If $\dim X = -\infty$, then X' is empty and trivially $\mathcal{M}$-definable. Now suppose $\dim X \geqslant 0$. We have

$$\mathrm{cl}(X') = \mathrm{cl}(X' \cap \mathrm{es}(X)) \cup \mathrm{cl}(X' \cap \mathrm{rs}(X)).$$

Since X' is pseudo-dense in X, X' is dense in $\mathrm{es}(X)$ by Proposition 8.7. It follows that $\mathrm{cl}(X' \cap \mathrm{es}(X)) = \mathrm{cl}(\mathrm{es}(X))$, which is $\mathcal{M}$-definable. As $(X' \cap \mathrm{rs}(X)) \subseteq \mathrm{rs}(X)$, any pseudo-closure of $(X' \cap \mathrm{rs}(X))$ has dimension at most $\dim \mathrm{rs}(X) < \dim X$. So $\mathrm{cl}(X' \cap \mathrm{rs}(X))$ is $\mathcal{M}$-definable by induction. Thus $\mathrm{cl}(X')$ is a union of two $\mathcal{M}$-definable sets and is therefore $\mathcal{M}$-definable. $\square$

We conclude this section by giving examples of structures with compatible definable topologies. In each case $\mathcal{T}$ and dim are canonical, so we do not describe them in detail. And in each case the existence of dimensionally pure decompositions (and hence the residue inequality, by Proposition 8.6) follows from the appropriate cell decomposition or "weak cell decomposition" result. In different settings, cells (or "weak cells") have different definitions, but they are easily seen to be dimensionally pure in each case.

The most familiar case is when $\mathcal{M}$ is an o-minimal expansion of a dense linear order, see [**84**]. Similarly, it follows from [**76**, Proposition 4.1,4.3] that if $\mathcal{M}$ is a dp-minimal expansion of a divisible ordered abelian group then the usual order topology is compatible. This covers the case when $\mathcal{M}$ is an expansion of an ordered abelian group with weakly o-minimal theory. It is shown in Johnson's thesis [**43**] that a dp-minimal, non strongly minimal, expansion of a field admits a definable field topology and it is shown in [**76**] that this topology is compatible. It follows in particular that a C-minimal expansion of an algebraically closed field, or a P-minimal expansion of a p-adically closed field admits a compatible definable topology. It was previously shown in [**22**] that P-minimal expansions of p-adically closed fields satisfy the frontier inequality and admit dimensionally pure decompositions.

We say that T is an **open core** of T' if the closure of every T'-definable set in every T'-model $\mathcal{M}'$ is $\mathcal{M} = \mathcal{M}'|L$ definable. Proposition 8.8 and Proposition 8.9 together yield the following theorem.

Theorem 8.3. *If $T_\cap$ admits an ordinal rank* dim *and a* dim*-compatible definable topology, and $T_\cap$ is an open core of T_i for each $i \in I$, then $T^*_\cup$ exists. In particular, if $T_\cap$ is an o-minimal expansion of a dense linear order or a p-minimal expansion of a p-adically closed field, and $T_\cap$ is an open core of T_i for each $i \in I$, then $T^*_\cup$ exists.*

We give a concrete example of Theorem 8.3. Suppose $T_\cap$ is a complete and model complete o-minimal theory that extends the theory of ordered abelian groups. For each $i \in I$, let T_i be the theory of a T-model $\mathcal{N}$ equipped with a unary predicate R_i defining a dense elementary substructure of $\mathcal{N}$. Then T_i is model complete by [**83**, Thm 1] and $T_\cap$ is an open core of T_i [**27**, Section 5]. Applying Theorem 8.3, we see that the theory $T_\cup$ of a T-model $\mathcal{N}$ equipped with a family $(R_i)_{i\in I}$ of unary predicates defining dense elementary substructures of $\mathcal{N}$ has a model companion.

8.4. $\aleph_0$-stable base

We assume throughout this section that T is $\aleph_0$-stable and dim is Morley rank on T. We write mult for Morley degree on T.

Suppose X^1 and X^2 are $\mathcal{M}$-definable subsets of M^x. Then X^1 is **almost a subset** of X^2, if

$$\dim(X^1 \setminus X^2) < \dim(X^1),$$

and X^1 is **almost equal** to X^2, if X^1 is almost a subset of X^2 and vice versa. An $\mathcal{M}$-definable subset X of M^x is **almost irreducible** if whenever $X = X^1 \cup X^2$ for $\mathcal{M}$-definable X^1 and X^2, we have X is almost equal to X^1 or to X^2. Any $\mathcal{M}$-definable set of Morley degree one is almost irreducible, and the converse holds when $\mathrm{Th}(\mathcal{M})$ defines Morley rank or when $\mathcal{M}$ is $\aleph_0$-saturated.

The following easy proposition is the main advantage of assuming that $T_\cap$ is $\aleph_0$-stable in our setting.

Lemma 8.6. *Suppose A is a subset of M^x. Then an $\mathcal{M}$-definable set $X \subseteq M^x$ is a pseudo-closure of A if and only if $A \subseteq X$ and*

$$(\dim X, \mathrm{mult} X) \leqslant_{\mathrm{Lex}} (\dim X', \mathrm{mult} X')$$

for all $\mathcal{M}$-definable $X' \subseteq M^x$ with $A \subseteq X'$.

Proof. By standard properties of Morley rank and degree in $\aleph_0$-stable theories, for any $\mathcal{M}$-definable X and X', if $(\dim X', \mathrm{mult} X') <_{\mathrm{Lex}} (\dim X, \mathrm{mult} X)$, then $\dim(X \setminus X') = \dim X$. If $X' \subseteq X$, then the converse is true.

Let X be a pseudo-closure of A, so $A \subseteq X$, and suppose for contradiction that there is some $\mathcal{M}$-definable $X' \subseteq M^x$ with $A \subseteq X'$ and $(\dim X', \mathrm{mult} X') <_{\mathrm{Lex}} (\dim X, \mathrm{mult} X)$. Then $\dim(X \setminus X') = \dim X$, but $A \cap (X \setminus X') = \varnothing$, contradicting the fact that A is pseudo-dense in X.

Conversely, suppose $A \subseteq X$ and $(\dim X, \mathrm{mult} X)$ is minimal in the lexicographic order among $\mathcal{M}$-definable sets containing A. Then for any $\mathcal{M}$-definable $X' \subseteq X$ with $\dim X' = \dim X$, $(\dim(X \setminus X'), \mathrm{mult}(X \setminus X')) <_{\mathrm{Lex}} (\dim X, \mathrm{mult} X)$. It follows that $A \notin (X \setminus X')$, so $A \cap X' \neq \varnothing$. Hence X is a pseudo-closure of A. □

The preceding lemma has the following important immediate consequence for the approximability condition in this setting.

Proposition 8.10. *Every $A \subseteq M^x$ has a pseudo-closure. Hence every expansion of $\mathcal{M}$ is approximable over $\mathcal{M}$ and every expansion of T is approximable over T.*

Proof. This is an immediate consequence of Lemma 8.6, using the fact that the lexicographic order on pairs $(\dim X, \mathrm{mult} X)$ is a well-order. □

Corollary 8.2. *If $\mathrm{Th}(\mathcal{M}_\cap)$ is $\aleph_0$-stable and $\dim$ is Morley rank, then $\mathcal{M}_\cup$ is interpolative if and only it is approximately interpolative.*

As a demonstration of the material developed so far, we will revisit the example of difference fields as presented in Section 7.7. Suppose K is a model of ACF. We say that $V \subseteq K^x$ is a

irreducible if it is K-definable and irreducible with respect to the Zariski topology on K, or equivalently, V is a quasi-affine variety.

Suppose K and K' are algebraically closed fields and $f : K \to K'$ is a field isomorphism. Then $(K, K'; f)$ is a model of the theory T_1 (or equivalently of T_2) in Corollary 7.3. As in the proof of Proposition 7.1, $(K, K'; f)$ is isomorphic to $(K, K; \mathrm{id}_K)$ via the map (id_K, f^{-1}). If $Z \subseteq K^m \times (K')^n$, set

$$(\mathrm{id}_K, f^{-1})(Z) = \{(a, f^{-1}(b)) \mid (a, b) \in Z\} \subseteq K^{m+n}.$$

Then $Z \subseteq K^m \times (K')^n$ is $(K, K'; f)$-definable if and only if $(\mathrm{id}_K, f^{-1})(Z)$ is K-definable. Hence, we can liberally import concepts and results from definable sets in ACF to definable sets in $(K, K'; f)$. In particular, we say Z is **irreducible** if $(\mathrm{id}_K, f^{-1})(Z)$ is irreducible. Likewise, we say Z is **Zariski-closed** in Z' if $(\mathrm{id}_K, f^{-1})(Z)$ is Zariski-closed in $(\mathrm{id}_K, f^{-1})(Z')$.

The remark below follows easily from quantifier elimination in ACF.

Remark 8.3. Suppose K and K' are algebraically closed fields.

(1) Every (K, K')-definable subset of $K^m \times (K')^n$ is a finite union of sets of the form $V \times V'$ where $V \subseteq K^m$ is an irreducible K-definable set and $V' \subseteq (K')^m$ is an irreducible K'-definable set.

(2) If $f : K \to K'$ is a field isomorphism, then every $(K, K'; f)$-definable set is a finite union of irreducible $(K, K'; f)$-definable sets.

(3) If $V \subseteq K^m$ is an irreducible K-definable set, and $V \subseteq (K')^n$ is an irreducible K'-definable set, then $V \times V'$ is an irreducible $(K, K'; f)$-definable set.

Recall that $T_\cap$ is the theory of pairs (K, K'), where K and K' are algebraically closed fields. It is easy to see that this theory is $\aleph_0$-stable. We write dim for Morley rank on (K, K') and mult for Morley degree on (K, K'). The following facts are easy to verify.

Remark 8.4. If $V \subseteq K^m$ is (K, K')-definable, then $\dim(V)$ and $\mathrm{mult}(V)$ are equal to the dimension and multiplicity of V considered as a K-definable set relative to ACF, and similarly for $V' \subseteq (K')^n$. If $V \subseteq K^m$ is K-definable and $V \subseteq (K')^{m'}$ is K'-definable, then $\dim(V \times V') = \dim(V) + \dim(V')$. If V and V' are irreducible, then $V \times V'$ is irreducible and $\mathrm{mult}(V \times V') = 1$. The collection of all such irreducible sets $V \times V'$ is a pseudo-cell collection for $T_\cap$.

These observations lead immediately to a characterization of pseudo-denseness.

Lemma 8.7. *Suppose $V \subseteq K^x$ and $V' \subseteq (K')^y$ are irreducible, π and π' are the coordinate projections from $V \times V'$ to V and V', and $Z \subseteq V \times V'$ is an irreducible definable set in*

$(K, K'; f)$. *Then Z is pseudo-dense in $V \times V'$ if and only if $\pi(Z)$ is Zarski-dense in V and $\pi'(Z)$ is Zarski-dense in V'.*

Proof. Suppose $\pi(Z)$ is Zariski-dense in V and $\pi'(Z)$ is Zariski-dense in V'. Using Remark 8.3, Z has a pseudo closure of the form

$$(W_1 \times W_1') \cup \ldots \cup (W_n \times W_n')$$

where W_i is an irreducible K-definable subset of V, and W_i' is an irreducible K'-definable subset of V' for $i \in \{1, \ldots, n\}$. Using Lemma 8.6 and replacing the relevant sets with their Zariski-closures in V and V' if necessary, we can arrange that W_i is closed in V, W_i' is closed in V', and hence $W_i \times W_i'$ is closed in $V \times V'$ for $i \in \{1, \ldots, n\}$. As Z is irreducible, we must have

$$Z \subseteq W_i \times W_i' \quad \text{for a single } i \in \{1, \ldots, n\}.$$

Thus Z has a pseudo-closure of the form $W \times W'$ with $W \subseteq V$ and $W' \subseteq V'$ irreducible. As $\pi(Z) \subseteq W$, and $\pi'(Z) \subseteq W'$, W is Zariski-dense in V and W' is Zariski-dense in V'. Applying Remark 8.4, we get $\dim(W \times W') = \dim(V \times V')$. Since $\operatorname{mult}(V \times V') = 1$, it follows from Lemma 8.6 that $V \times V'$ is a pseudo-closure of Z. In particular, Z is pseudo-dense in $V \times V'$.

Now suppose Z is pseudo-dense in $V \times V'$. Let $W \subseteq V$ be a pseudo-closure of $\pi(Z)$, and let $W' \subseteq V'$ be a pseudo-closure of $\pi'(Z)$. As $V \times V'$ is a pseudo-closure of Z, we must have $\dim(V \times V') \leqslant \dim(W \times W')$. This forces $\dim W = \dim V$ and $\dim W' = \dim V'$. Since V and V' are irreducible, $\operatorname{mult}(V) = \operatorname{mult}(V') = 1$. Applying Lemma 8.6, $\pi(Z)$ is pseudo-dense in V, and $\pi'(Z)$ is pseudo-dense in V'. Since V and V' are irreducible, $\pi(Z)$ is Zariski-dense in V and $\pi'(Z)$ is Zariski-dense in V', as desired. □

Suppose K is an algebraically closed field and σ is an automorphism of K. We say that σ is a **generic automorphism** if for all irreducible K-definable sets $V \subseteq K^m$ and $Z \subseteq V \times \sigma(V)$ such that the projections of Z onto V and $\sigma(V)$ are Zariski-dense in V and $\sigma(V)$ respectively, we can find $a \in V$ such that

$$(a, \sigma(a)) \in Z.$$

It is well known that $(K; \sigma) \models \mathrm{ACFA}$ if and only if $K \models \mathrm{ACF}$ and σ is a generic automorphism. This description of models of ACFA is often referred to as Hrushovksi's *geometric characterization/axioms* [**56**]. The following proposition clarifies the relationship between this and our pseudo-topological characterization/axioms.

Proposition 8.11. *Suppose K is an algebraically closed field and σ is an automorphism of K. Then the following statements are equivalent:*

(1) *σ is a generic automorphism.*

(2) *With $I = \{1, 2\}$, $(K, K; \mathrm{id}_K)$ viewed as an L_1-structure, and $(K, K; \sigma)$ viewed as an L_2 structure for L_1 and L_2 as in Corollary 7.3, the $L_\cup$-structure $(K, K; \mathrm{id}_K, \sigma)$ is approximately interpolative.*

PROOF. We first show the backward direction. Suppose (2) holds, and let V and Z be as in the definition of a generic automorphism. Set $V' = \sigma(V)$. Applying Lemma 8.7, we find that Z as a $(K, K; \mathrm{id}_K)$-definable set is pseudo-dense in $V \times V'$. Also by Lemma 8.7, $\{(a, \sigma(a)) : a \in K\}$ as a $(K, K; \sigma)$-definable set is pseudo-dense in $V \times V'$. Therefore $Z \cap \{(a, \sigma(a)) : a \in K\} \neq \varnothing$ by (2), which is the desired conclusion.

Conversely, suppose (1) holds. By Remark 8.4, (K, K)-definable sets of the form $V \times V'$ with V and V' irreducible form a pseudo-cell collection. Hence, it suffices to fix V and V', and show that $X \cap Y \neq \varnothing$ whenever X is a $(K, K; \mathrm{id}_K)$-definable set, Y is a $(K, K; \sigma)$-definable set, and X and Y are each pseudo-dense in $V \times V'$. Using Remark 8.3, we can reduce to the case when X and Y are irreducible. Let $Y^* = (\mathrm{id}_K, \sigma^{-1})(Y)$. Our job is to show that there is $(a, b) \in V \times V'$ with $(a, b) \in X$ and $(a, \sigma^{-1}(b)) \in Y^*$. By Lemma 8.7, we have that the projections of Y^* and X onto V, of Y^* onto $\sigma^{-1}(V')$, and of X onto V' have Zariski-dense image. Using generic flatness (see [**32**] or [**61**]), we can arrange that these maps are flat by shrinking X, Y^*, V, and V' if necessary. Let $Y^* \times_V X$ be the fiber product of Y^* and X over V. Then

$$Y^* \times_V X \subseteq \left(V \times \sigma^{-1}(V')\right) \times_V (V \times V') = V \times \sigma^{-1}(V') \times V'.$$

As flatness is preserved under base-change and composition, the obvious maps from $Y^* \times_V X$ to $\sigma^{-1}(V')$ and to V' are flat. Let $\tilde{Z}$ be an irreducible component of $Y^* \times_V X$ and Z the image of $\tilde{Z}$ in $\sigma^{-1}(V) \times V'$. As flat maps are open [**35**, Exercise III.9.1], the image of the projections of $\tilde{Z}$ and hence of Z onto $\sigma^{-1}(V')$ and onto V' contain Zariski-open subsets of $\sigma^{-1}(V')$ and V' respectively. By (1), Z contains a point of the form $(\sigma^{-1}(b), b)$. Hence, there is a point of the form $(a, \sigma^{-1}(b), b)$ in $Y^* \times_V X$. This implies that $(a, \sigma^{-1}(b))$ is in Y^* and (a, b) is in X, which is our desired conclusion. $\square$

Proposition 8.12 below, combined with Theorem 7.9, allows us to recover the fact that the theory of difference fields has a model companion.

Proposition 8.12. *Let T_1 and T_2 be as in Corollary 7.3. Then $T_\cup^*$ exists.*

PROOF. By Proposition 7.1, T_1 and T_2 are bi-interpretable with ACF, so in particular they are $\aleph_0$-stable. It follows that $T_\cap$ is also $\aleph_0$-stable, so T_1 and T_2 are automatically approximable over $T_\cap$. T_1 and T_2 also define pseudo-denseness over $T_\cap$, using Lemma 8.7 and the fact that ACF defines dimension and irreducibility. Thus $T_\cup^*$ exists by Theorem 8.1. $\square$

Remark 8.5. Note that the proof of Proposition 8.12 does not use Proposition 8.11. It follows from Corollary 7.3 that $(K;\sigma) \models \mathrm{ACFA}$ if and only if $(K,K;\mathrm{id}_K,\sigma) \models T^*_\cup$ where $T^*_\cup$ is as in Proposition 8.12. Combining with the fact the $(K;\sigma) \models \mathrm{ACFA}$ if and only if σ is a generic automorphism, and the fact that the models of $T^*_\cup$ are the approximately interpolative models of $T_\cup$, we get an alternative proof of Proposition 8.11.

The proof we gave for Proposition 8.11 is purely at the level of structures and not theories. This is technically harder but done to make a point: In addition to recovering the fact that various theories in the literature have a model companion, the material we develop in this section is the common abstraction of the proofs in the literature that these theories have model companions.

Having finished our discussion of ACFA, we now return to the general case. In Proposition 8.2, we gave a concrete example of an expansion of $T = \mathrm{Th}(\mathbb{Z};0,+,-)$ which is not approximable over T. It is well known that T is superstable but not $\aleph_0$-stable, so this demonstrates that superstability is not sufficient for Proposition 8.10. For the reader who is still looking for a free ride outside of the $\aleph_0$-stable context, Proposition 8.13 will dash this hope.

If $\dim_1, \dim_2$ are ordinal ranks on an $L^\diamond$-theory $T^\diamond$ then we say $\dim_1$ is **smaller than** $\dim_2$ if $\dim_1 X \leqslant \dim_2 X$ for all definable sets X.

Lemma 8.8. *The theory $T^\diamond$ is $\aleph_0$-stable if and only if it admits an ordinal rank* $\dim$ *such that for every $T^\diamond$-model $\mathcal{M}^\diamond$, $\mathcal{M}^\diamond$-definable set X, and family $(X_n)_{n\in\mathbb{N}}$ of pairwise disjoint $\mathcal{M}^\diamond$-definable subsets of X, we have* $\dim X_n < \dim X$ *for some n. If $T^\diamond$ is $\aleph_0$-stable, then Morley rank is the smallest ordinal rank with this property.*

Proof. It is well-known that Morley rank RM is an ordinal rank satisfying the hypotheses when $T^\diamond$ is $\aleph_0$-stable. Suppose dim is an ordinal rank satisfying the hypotheses. We will show that $\mathrm{RM}(X) \leqslant \dim X$ for all $\mathcal{M}^\diamond$-definable sets X in $T^\diamond$-models $\mathcal{M}^\diamond$. This implies that RM is ordinal valued and hence that $T^\diamond$ is $\aleph_0$-stable.

As RM and dim are preserved in elementary extensions, it suffices to fix an $\aleph_0$-saturated $T^\diamond$-model $\mathcal{M}^\diamond$ and show $\mathrm{RM}(X) \leqslant \dim(X)$ for all $\mathcal{M}^\diamond$-definable sets X. We show by induction on ordinals α that if $\alpha \leqslant \mathrm{RM}(X)$, then $\alpha \leqslant \dim(X)$. If $0 \leqslant \mathrm{RM}(X)$, then X is nonempty, so $0 \leqslant \dim(X)$. If α is a limit ordinal and $\alpha \leqslant \mathrm{RM}(X)$, then $\beta \leqslant \mathrm{RM}(X)$ for all $\beta < \alpha$, so by induction $\beta \leqslant \dim(X)$ for all $\beta < \alpha$, and hence $\alpha \leqslant \dim(X)$. If $\alpha = \beta + 1$ is a successor ordinal and $\alpha \leqslant \mathrm{RM}(X)$, then since $\mathcal{M}^\diamond$ is $\aleph_0$-saturated, there are pairwise disjoint $\mathcal{M}$-definable subsets $(X_n)_{n\in\mathbb{N}}$ of X such that $\beta \leqslant \mathrm{RM}(X_n)$ for all n. By induction, $\beta \leqslant \dim(X_n)$ for all n, and by our assumption on dim there is some n such that $\dim(X_n) < \dim(X)$. So $\alpha \leqslant \dim(X)$. □

Proposition 8.13. *Suppose $L^\diamond$ is countable and $\dim^\diamond$ is an ordinal rank on a complete $L^\diamond$-theory $T^\diamond$. If $T^\diamond$ is not $\aleph_0$-stable, then there is an expansion of $T^\diamond$ which is not approximable over $T^\diamond$.*

Proof. Suppose $T^\diamond$ is not $\aleph_0$-stable. Applying Lemma 8.8, we obtain a $T^\diamond$-model $\mathcal{M}^\diamond$, an $\mathcal{M}^\diamond$-definable set X with $\dim^\diamond X = \alpha$, and a sequence $(X_n)_{n\in\mathbb{N}}$ of pairwise disjoint $\mathcal{M}^\diamond$-definable subsets of X such that $\dim^\diamond X_n = \alpha$ for all n. Since X and each X_n are definable with parameters from a countable elementary submodel, we may assume $\mathcal{M}^\diamond$ is countable.

Given $S \subseteq \mathbb{N}$, let $A_S = \bigcup_{n\in S} X_n$. We show that A_S does not have a pseudo-closure for uncountably many $S \subseteq \mathbb{N}$. Suppose $S \subseteq \mathbb{N}$ is nonempty and X' is a pseudo-closure of A_S. As $A_S \subseteq X$, we have $\dim^\diamond X' \leqslant \alpha$. As S is nonempty, we have $X_n \subseteq X'$ for some n, so $\dim^\diamond X' \geqslant \alpha$. Thus any pseudo-closure X' of A_S has $\dim^\diamond X' = \alpha$.

Now suppose $S, S' \subseteq \mathbb{N}$ are nonempty and $S \nsubseteq S'$. We show any pseudo-closure of A_S is not a pseudo-closure of $A_{S'}$. Fix $n \in S \setminus S'$ and suppose X' is a pseudo-closure of A_S. Then $\dim^\diamond X' = \alpha$, X_n is an $\mathcal{M}^\diamond$-definable subset of X' with $\dim^\diamond X_n = \alpha$, but X_n is disjoint from $A_{S'}$. Thus X' is not a pseudo-closure of $A_{S'}$.

Let $\mathfrak{J}$ be an uncountable collection of nonempty subsets of $\mathbb{N}$ such that $S \nsubseteq S'$ for all distinct $S, S' \in \mathfrak{J}$. If $S, S' \in \mathfrak{J}$ are distinct, then A_S and $A_{S'}$ cannot have a common pseudo-closure. As $\mathcal{M}^\diamond$ and L are countable, there are only countably many $\mathcal{M}^\diamond$-definable sets, so there are uncountably many $S \in \mathfrak{J}$ such that A_S does not have a pseudo-closure. The expansion of $\mathcal{M}^\diamond$ by a predicate defining any such A_S is not approximable over $\mathcal{M}^\diamond$. It follows that the theory of this expansion is not approximable over $T^\diamond$. ☐

We next give a useful characterization of definability of pseudo-denseness over an $\aleph_0$-stable theory. Lemma 8.6 motivates the following definition. Suppose M' is a model of T', $\mathcal{M} = \mathcal{M}' {\upharpoonright} L$, and $X' \subseteq M^x$ is $\mathcal{M}'$-definable. Define

$$\dim' X' = \dim X \quad \text{and} \quad \mathrm{mult}' X' = \mathrm{mult} X$$

where X is a pseudo-closure of X'. The following corollary is an immediate consequence of Lemma 8.6.

Lemma 8.9. *For $A \subseteq M^x$ and $\mathcal{M}$-definable $X \subseteq M^x$, we have the following:*

(1) *A is pseudo-dense in X if and only if we have both $\dim'(X \cap A) = \dim(X)$ and $\mathrm{mult}'(X \cap A) = \mathrm{mult}(X)$.*

(2) *If X is almost irreducible, then A is pseudo-dense in X if and only if $\dim'(X \cap A)$ is the same as $\dim(X)$.*

In general $\dim'$ might not be an ordinal rank on T' as $\dim'(X')$ might be different from $\dim'(X'(\mathcal{N}'))$ where $\mathcal{N}'$ is an elementary extension of $\mathcal{M}'$. When T defines Morley rank, we can easily check that $\dim'$ is an ordinal rank on T', which we will refer to as the **induced rank** on T'.

We say T **defines multiplicity** (or has the **DMP**) if for all L-formulas $\varphi(x, y)$, ordinals α, and n, there is an L-formula $\mu_{\alpha,n}(y)$ such that for all $\mathcal{M} \models T$ and $b \in M^y$ we have that

$$\mathcal{M} \models \mu_{\alpha,n}(b) \text{ if and only if } \dim \varphi(\mathcal{M}, b) = \alpha \text{ and mult } \varphi(\mathcal{M}, b) = n.$$

In particular, if T defines multiplicity, then T defines Morley rank, and the induced rank on T' is well-defined.

Proposition 8.14. *Suppose T defines multiplicity. Then T' defines pseudo-denseness over T if and only if T' defines induced rank.*

Proof. Suppose T' defines pseudo-denseness and $\varphi'(x, y)$ is an L'-formula. Let $(X'_{b'})_{b' \in Y'}$ be a family of subsets of M^x defined by $\varphi'(x, y)$. Using the assumption that T' defines pseudo-denseness and a standard compactness argument, we obtain a family $(X_c)_{c \in Z}$ defined by a formula whose choice might depend on $\varphi'(x, y)$ but not on $\mathcal{M}'$, such that for every $b' \in Y$, X'_b has a pseudo-closure which in a member of the family $(X_c)_{c \in Z}$. It follows from Proposition 8.10 that $\dim'(X'_{b'}) = \alpha$ for $b' \in Y$ if and only there is $c \in Z$ such that X'_b is pseudo-dense in X_c and $\dim(X_c) = \alpha$. As T defines multiplicity and T' defines pseudo-denseness, it follows that T' defines induced rank.

Now suppose T' defines induced rank. Let $\mathcal{C}$ be the collection of almost irreducible subsets of T-models. Then $\mathcal{C}$ is a collection of pseudo-cells for T. As T defines multiplicity, T defines $\mathcal{C}$-membership. So by Proposition 8.4, it suffices to show T' defines pseudo-denseness over $\mathcal{C}$. Let $(X'_{b'})_{b' \in Y'}$ and $(X_c)_{c \in Z}$ be a families defined by an L'-formula $\varphi'(x, y)$ and an L-formula $\varphi(x, z)$. It follows from Lemma 8.6 that when X_c is in $\mathcal{C}$, $X'_{b'}$ is pseudo-dense in X_c if and only if $\dim'(X \cap X') = \dim(X)$. The desired conclusion follows. $\square$

Remark 8.6. If T defines Morley rank, then mult' is preserved under elementary extensions, so we may speak of induced multiplicity on T'. There is also an analogue of Proposition 8.14 which involves both $\dim'$ and mult': Suppose T defines Morley rank. Then T' defines pseudo-denseness if and only if T' defines induced rank and induced multiplicity. We do not include it here as we do not have an application in mind.

Theorem 8.4. *Suppose $T_\cap$ is $\aleph_0$-stable and defines multiplicity. If each T_i defines induced rank, then $T^*_\cup$ exists.*

Proof. This is an immediate consequence of Theorem 8.1, Proposition 8.10, and Proposition 8.14. $\square$

Proposition 8.14 and Theorem 8.4 are mainly of interest because there are several situations where the induced rank on T' is a natural notion of dimension, and its definability follows from our general knowledge about T'. Proposition 8.15 below presents a general class of such situations.

The **algebraic dimension** $\mathrm{adim}(X)$ of an $\mathcal{N}$-definable set X is the maximal k for which there is $a = (a_1, \ldots, a_n) \in X(\boldsymbol{\mathcal{N}})$ such that (after permuting coordinates) $a_1, \ldots, a_k$ are acl-independent over N. It is well-known that algebraic dimension is an ordinal rank on $\mathrm{Th}(\mathcal{N})$, which coincides with Morley rank for strongly minimal theories. The following fact is also well known (see [**11**, Lemma 2.2]).

Fact 8.1. *A theory defines algebraic dimension if and only if it eliminates $\exists^\infty$.*

Proposition 8.15. *Suppose T is strongly minimal and* acl' *agrees with* acl *in all T'-models. Then T' defines induced rank if and only if T' eliminates $\exists^\infty$.*

PROOF. Suppose $\mathcal{M}' \models T'$, and $\mathcal{M} = \mathcal{M}' \restriction L$. Since T is strongly minimal, $\dim = \mathrm{adim}$. We write $\dim'$ for the induced rank on T' and adim' for the algebraic dimension in $\mathcal{M}'$. Using Fact 8.1, it suffices to show that $\dim' = \mathrm{adim}'$.

If X' is an arbitrary $\mathcal{M}'$-definable subset of M^x,

$$\dim'(X') = \min\{\mathrm{adim}(X) \mid X \subseteq M^x \text{ is } \mathcal{M}\text{-definable, and } X' \subseteq X\}.$$

As $\mathrm{acl}' = \mathrm{acl}$, whenever $a \in X'(\boldsymbol{\mathcal{M}}')$ has k components which are acl'-independent over M, these components are also acl-independent over M, and we have $a \in X(\boldsymbol{\mathcal{M}}')$ for any $\mathcal{M}$-definable X such that $X' \subseteq X$. Hence, $\mathrm{adim}'(X') \leqslant \dim'(X')$.

Conversely, let $X \subseteq M^x$ be a pseudo-closure of X', and $n = \mathrm{adim}(X)$. Then X' is not contained in any $\mathcal{M}$-definable set of smaller dimension. Since the set of $\mathcal{M}$-definable sets of dimension less than n is closed under finite unions, by compactness there is some $a' \in X'(\boldsymbol{\mathcal{M}}')$ which is not contained in any $\mathcal{M}$-definable set of dimension less than n. If a' does not have n components which are acl'-independent over M, then since $\mathrm{acl}' = \mathrm{acl}$, this dependence is witnessed by $a' \in Y$, where Y is $\mathcal{M}$-definable and $\mathrm{adim}(Y) < n$. This contradicts the choice of a'. $\square$

As a demonstration of Proposition 8.14 and Proposition 8.15, we will revisit the theory of algebraically closed fields with multiple valuations described in Section 7.2 and show that this has a model companion. We need the following fact about algebraically closed valued fields, which can be found in [**82**].

Fact 8.2. *Suppose K is an algebraically closed field and $R \subseteq K$ is a nontrivial valuation ring. Then the model-theoretic algebraic closure in $(K; R)$ agrees with the field-theoretic algebraic closure in K (which agrees with the model-theoretic algebraic closure in K).*

Applying Proposition 8.14, Proposition 8.15, and Fact 8.2 we recover the promised fact, which is also the first part of Theorem 7.3.

Proposition 8.16. *Suppose $L_\cap$ is the language of rings, and for each $i \in I$, L_i extends $L_\cap$ by a unary relation symbol, T_i is the theory whose models are $(K; R_i)$ with $K \vDash \mathrm{ACF}$ and R_i a nontrivial valuation ring on K. Then $T^*_\cup$ exists.*

In [**81**] and [**43**], the strategy to show that the theory of algebraically closed fields with multiple valuations has a model companion involves:

(1) Identifying a class $\mathcal{C}$ of "generic" algebraically closed fields with multiple valuations.
(2) Showing that $\mathcal{C}$ consists precisely of the existentially closed models of the theory of algebraically closed fields with multiple valuations.
(3) Showing that $\mathcal{C}$ is first-order axiomatizable.

For an algebraically closed field K and a family $(R_i)_{i\in I}$ of nontrivial valuation rings on K, we say $(K; (R_i)_{i\in I})$ is **generic** if whenever $V \subseteq K^m$ is Zariski-closed and irreducible, $J \subseteq I$ is finite, $U_i \subseteq K^m$ is v_i-open in in V for $i \in J$, we have $\bigcap_{i\in J} U_i \neq \varnothing$.

We will show that this notion of genericity agrees with our notion of approximately interpolative structure. This is a special case of the notions of genericity in [**81**] and [**43**]: they work in a more general setting and use different terminology. We need the following lemma about algebraically closed valued fields.

Lemma 8.10. *Suppose K is an algebraically closed field, $R \subseteq K$ is a nontrivial valuation ring, $V \subseteq K^m$ is irreducible, and $X \subseteq V$ is $(K; R)$-definable. Let v be the valuation associated to R and $\dim$ be the acl-dimension on $(K; R)$. Then the following are equivalent:*

(1) $\dim X = \dim V$.
(2) *X is Zariski-dense (equivalently pseudo-dense) in V.*
(3) *X has nonempty interior in the v-topology on V.*

Proof. Fact 8.2 together with Lemma 8.6 shows that (1) and (2) are equivalent. The proof of [**82**, Proposition 2.18] shows that (2) implies (3). As every Zariski-closed set is v-closed, it follows that any subset of V which is not Zariski-dense in V has empty interior in the v-topology on V. □

Proposition 8.17. *Suppose $(K; (R_i)_{i\in I})$ has $K \vDash \mathrm{ACF}$ and R_i a nontrivial valuation ring on K for $i \in I$. Then following are equivalent:*

(1) *$(K; (R_i)_{i\in I})$ is generic.*
(2) *With $L_\cap$ the language of rings and L_i extending $L_\cap$ by a unary relation symbol for each $i \in I$, $(K; (R_i)_{i\in I})$ as an $L_\cup$-structure is approximately interpolative.*

Proof. The backward direction follows immediately from Lemma 8.10. It is easy to see that the collection of irreducible varieties forms a pseudo-cell collection for ACF. Applying Lemma 8.10 again, we get the forward direction. ☐

Remark 8.7. Proposition 8.17 can alternatively be obtained as a consequence of Theorem 7.3, Corollary 8.1, the fact that $T_\cap$ is $\aleph_0$-stable, and the result from [**81**] and [**43**] that the generic models are the existentially closed models of $T_\cup$. The current proof of Proposition 8.17 again illustrates the point made in Remark 8.5 that the material we develop in this section is the common abstraction of the proofs in the literature that various theories have model companions

Note that a separate argument is needed to show that $(K;(R_i)_{i\in I})$ as in Proposition 8.17 is generic if and only if $(v_i)_{i\in I}$ is an independent family of valuations. We do not include a proof of this result here, as we found no other way except to essentially repeat the argument in [**43**].

In the same spirit but more closely related to the notion of induced dimension, we show how the definition of generic predicates is related to approximately interpolative structures. The proof that $T^*_\cup$ exists for this example must wait until Section 8.5.

Proposition 8.18. *Suppose $\mathcal{M}$ is an infinite one-sorted L-structure and P is a unary predicate on $\mathcal{M}$ which is not in L. Set $I = \{1,2\}$, and let $L_1 = L$ and $L_2 = \{P\}$. Then the following are equivalent:*

(1) *P is a generic predicate.*
(2) *$\mathcal{M}_\cup$ is approximately interpolative.*

Proof. Note that $T_\cap$ is the strongly minimal theory of an infinite set with no structure, so dim = adim. Let $\mathcal{C}$ be the collection of M^n as n ranges over $\mathbb{N}$. From the fact that $T_\cap$ admits quantifier elimination, it is easy to deduce that $\mathcal{C}$ is a pseudo-cell collection. Therefore, by Proposition 8.3, it suffices to show that P is a generic predicate if and only if $\mathcal{M}_\cup$ is $\mathcal{C}$-approximately interpolative, i.e., $X_1 \cap X_2 \neq \varnothing$ whenever the $\mathcal{M}_1$-definable set $X_1 \subseteq M^n$ and the $\mathcal{M}_2$-definable set $X_2 \subseteq M^n$ are pseudo-dense in M^n.

We first show that an $\mathcal{M}_1$-definable set $X_1 \subseteq M^n$ is large if and only if X_1 is pseudo-dense in M^n. Let adim$'$ be the induced dimension on $\mathcal{M}_1$. As algebraic closure in $T_\cap$ is trivial, it follows directly from the definitions that an $\mathcal{M}_1$-definable subset X_1 of M^n is large if and only if $\operatorname{adim}'(X_1) = n$. On the other hand, as $\operatorname{adim}'(X_1) < n$ if and only if X_1 is contained in an M-definable set of Morley rank $< n$, and M^n has Morley degree 1 (as an M-definable set), it follows by Lemma 8.6 that X_1 is pseudo-dense in M^n if and only if $\operatorname{adim}'(X_1) = n$.

On the other hand, it follows from quantifier elimination that an $\mathcal{M}_2$-definable set $X_2 \subseteq M^n$ is pseudo-dense in M^n if and only if it differs by an $\mathcal{M}_\cap$-definable set of smaller dimension from a set of the form $\prod_{i=1}^n S_i$, where $S_i \in \{P, M \smallsetminus P\}$ for all $1 \leqslant i \leqslant n$. So $\mathcal{M}_\cup$ is $\mathcal{C}$-approximately interpolative if and only if every large $\mathcal{M}_1$-definable set meets every set of this form, as desired. $\square$

Remark 8.8. The notion of genericity introduced in [**79**] is also very close in spirit to the notion of approximately interpolative structure. It is also possible to prove that these notions are equivalent in the same fashion as Proposition 8.11, Proposition 8.17, and Proposition 8.18, but that is outside the scope of this paper.

8.5. Toward $\aleph_0$-categorical base

Throughout this section, we assume L has finitely many sorts and T is $\aleph_0$-stable, $\aleph_0$-categorical, weakly eliminates imaginaries, and has no finite models. We write dim for Morley rank on T and mult for Morley degree on T. We make extensive use of Proposition 8.10, which ensures that every subset of a model of T. Despite this, we consider this section more of a first step toward developing the theory of interpolative fusions over an $\aleph_0$-categorical base, rather than a continuation of the preceding section. A full-fledged theory should also cover Proposition 9.7.

The $\aleph_0$-stable assumption also gives us the following "inductive" procedure to check whether a subset is pseudo-dense in an almost irreducible set.

Lemma 8.11. *Suppose $X \subseteq M^x$ is almost irreducible, $\mathcal{D}$ is a collection of almost irreducible subsets of M^x such that any almost irreducible subset of M^x is almost equal to an element in $\mathcal{D}$, and A is a subset of M^x. For $\alpha < \dim X$, let $\mathcal{D}_\alpha(A, X)$ be the collection of almost irreducible $X_\alpha \in \mathcal{D}$ such that*

$$\dim X_\alpha = \alpha,\ A \textit{ is pseudo-dense in } X_\alpha, \textit{ and } X_\alpha \textit{ is almost a subset of } X.$$

If $\mathcal{D}_\beta(A, X) = \varnothing$ for all $\alpha < \beta < \dim X$, then we have the following:

1. *If $\mathcal{D}_\alpha(A, X)$ is infinite up to almost equality, then A is pseudo-dense in X.*
2. *If $\mathcal{D}_\alpha(A, X)$ is finite up to almost equality, $X_\alpha^1, \ldots, X_\alpha^n$ are the representatives of the almost equality classes, and*
$$A' := A \smallsetminus \bigcup_{i=1}^n X_\alpha^i,$$
then $\mathcal{D}_\beta(A', X) = \varnothing$ for all $\alpha \leqslant \beta < \dim X$, and A is pseudo-dense in X if and only if A' is.

Proof. As $\mathcal{M}$ is $\aleph_0$-stable, $A \cap X$ has a pseudo-closure Y which is a subset of X by Proposition 8.10. Suppose $\mathcal{D}_\beta(A, X) = \varnothing$ for all $\alpha < \beta < \dim X$. Then either $\dim Y \leqslant \alpha$ or $\dim Y = \dim X$. If $\mathcal{D}_\alpha(A, X)$ is infinite up to almost equality, then $\dim Y > \alpha$, and so

$\dim Y = \dim X$. The latter implies A is pseudo-dense in X by Lemma 8.6. Thus we get statement (1).

Now suppose $X^1_\alpha, \dots, X^n_\alpha$ and A' are as stated in (2). Since A' is a subset of A, $\mathcal{D}_\beta(A', X)$ is a subset of $\mathcal{D}_\beta(A, X)$ for all β. So in particular, $\mathcal{D}_\beta(A', X) = \varnothing$ for all $\alpha < \beta < \dim X$. Suppose X_α is an element of $\mathcal{D}_\alpha(A', X)$. Then A is also pseudo-dense in X_α and so X_α is almost equal to X^i_α with $i \in \{1, \dots, n\}$. As $X^i_\alpha \cap A' = \varnothing$, X^i_α and X_α are both almost irreducible, and $\dim X^i_\alpha = \dim X_\alpha$, it follows from Lemma 8.1 that A' is not pseudo-dense in X_α which is absurd. Thus,

$$\mathcal{D}_\alpha(A', X) = \varnothing \quad \text{for all } \alpha \leqslant \beta < \dim X.$$

If A' is pseudo-dense in X then clearly A is. Suppose A' is not pseudo-dense in X. Then $A' \cap X$ has a pseudo-closure Y' with $\dim Y' < \dim X$. It follows that A has a pseudo-closure Y which is a subset of $Y' \cup X^1_\alpha \cup \dots \cup X^n_\alpha$. It is easy to see that $\dim Y < \dim X$, and so A is not pseudo-dense in X. We have thus obtained all the desired conclusions in (2). $\square$

The lemma above is hardly useful if the purpose is defining pseudo-denseness for a general $\aleph_0$-stable theory. The issue is that many of the objects involved in the previous lemma are not definable. Remarkably, many of them are definable when we additionally assume T is $\aleph_0$-categorical. We recall a number of facts about $\aleph_0$-stable and $\aleph_0$-categorical theories.

Fact 8.3. *The first two statements below only require $\aleph_0$-categoricity:*

1. *T is complete.*
2. *For all finite x, there are finitely many formula $\varphi(x)$ up to T equivalence.*
3. *T defines multiplicity.*
4. *([**14**], Theorem 5.1) $\mathcal{M}$ has finite Morley rank, that is, for all finite x, $\dim M^x < \omega$.*
5. *([**14**], Theorem 6.3) if x is a single variable, and $p \in S^x(\mathcal{M})$, then p is definable over $M^x \times M^x$.*

We now prove a key lemma that does not hold outside of the $\aleph_0$-categorical setting.

Lemma 8.12. *For each finite x there is an L-formula $\psi(x, z)$ such that whenever $\mathcal{M} \models T$ and $\mathcal{D} = (X_c)_{c \in Z}$ is the family of subsets of M^x defined by $\psi(x, z)$, we have that every member of $\mathcal{D}$ is almost irreducible and every almost irreducible subset of M^x is almost equal to a member of $\mathcal{D}$.*

Proof. Fix $\mathcal{M} \models T$ of the given T, and a finite tuple x of variables. We reduce the problem to finding a formula $\psi(x, z)$ independent of the choice of $\mathcal{M}$ such that with $\mathcal{D} = (X_c)_{c \in Z}$ the family of subsets of M^x defined by $\psi(x, z)$, every almost irreducible X is almost equal to X_c for some $c \in M^z$. The analogous statement also hold in other models of T as T is complete.

As T defines multiplicity, we can modify $\psi(x,z)$ to exclude the X_c which are not almost irreducible.

We reduce the problem further to showing that every almost irreducible $X \subseteq M^x$ is almost equal to a subset of M^x which is $\mathcal{M}$-definable over some element of M^w with $|w| = 2|x|$. Suppose we have done so. By Fact 8.3(2), there are finitely many formulas $\psi_1(x,w),\dots,\psi_l(x,w)$ such that every L-formula in variables (x,w) is T-equivalent to one of these. By routine manipulation, we can get a finite tuple z of variables and a formula $\psi(x,z)$ such that for all $i \in \{1,\dots,l\}$ and $d \in M^w$, there is $c \in M^z$ with $\psi_i(\mathcal{M},d) = \psi(\mathcal{M},c)$. Hence, we obtained the desired reduction.

Let $p \in S^x(M)$ be the generic type of X and p^{eq} the unique element of $S^x(M^{\mathrm{eq}})$ extending p. By merging the sorts, we can arrange that $|x| = 1$. By Fact 8.3(5), there is $c \in M^2$ such that p is definable over c. Hence p^{eq} is definable over c and therefore stationary over $\mathrm{acl}^{\mathrm{eq}}(c)$. It follows that

$$q = p^{\mathrm{eq}} \restriction S^x(\mathrm{acl}^{\mathrm{eq}}(c)) \text{ has } \mathrm{mult}(q) = 1.$$

Let $X' \subseteq M^x$ be defined by a minimal formula of q. Then X' is $\mathcal{M}^{\mathrm{eq}}$-definable over $\mathrm{acl}^{\mathrm{eq}}(c)$ and X' is almost equal to X. Let $X'_1,\dots,X'_l$ be all the finitely many conjugates of X' by $\mathrm{Aut}(\mathcal{M}/c)$. Then $\bigcap_{i=1}^{l} X'_i$ is $\mathcal{M}$-definable over c and is almost equal to X which is the desired conclusion. □

A **function up-to-permutation** from $Z \subseteq M^z$ to M^w is a relation $f \subseteq Z \times M^w$ satisfying the following two conditions:

(1) For all $c \in Z$, there is $d \in M^w$ such that $(c,d) \in f$.
(2) If (c,d) and (c,d') are both in f, then d is a permutation of d'.

A function up-to-permutation f determines an ordinary function $\tilde{f} : Z \to M^w/\sim$, where $\sim$ is the equivalence relation defined by permutations. We are interested in f instead of $\tilde{f}$, as it is possible that f is $\mathcal{M}$-definable while $\tilde{f}$ is only $\mathcal{M}^{\mathrm{eq}}$-definable. For $C \subseteq Z$, we will write $f(Z)$ for the set

$$\{d \in M^w \mid \text{ there is } c \in C \text{ such that } (c,d) \in f\}.$$

It is easy to observe that $|\tilde{f}(Z)| \leqslant |f(Z)| \leqslant |w|!|\tilde{f}(Z)|$ with $\tilde{f}$ as above. In particular, $f(Z)$ is finite if and only if $\tilde{f}(Z)$ is.

The following fact only uses the assumption that T is complete and weakly eliminates imaginaries.

Fact 8.4. *For all $\mathcal{M} \models T$, 0-definable $Z \subseteq M^z$, and 0-definable equivalence relation $R \subseteq Z^2$, there is w and a 0-definable function up-to-permutation from Z to M^w such that cRc' in Z if and only if $f(c) = f(c')$. Moreover, the choice of formula defining f can be made depending only on the choices of L-formulas defining Z and R but not on the choice of $\mathcal{M}$.*

Proposition 8.19. *The theory T' defines pseudo-denseness over T if and only if T' eliminates $\exists^\infty$.*

Proof. For the forward direction, suppose T and T' are fixed, T' defines pseudo-denseness, $\varphi'(x,y)$ is an L'-formula, $\mathcal{M}' \vDash T'$, $\mathcal{M} = \mathcal{M}' \upharpoonright L$, $(X'_b)_{b \in Y'}$ is the family of subsets of M^x defined by $\varphi'(x,y)$. Our job is to show that the set of $b \in Y'$ with infinite X'_b can be defined by a formula whose choice might depend on $\varphi(x,y)$ but does not depend on $\mathcal{M}'$. Let $\mathcal{D} = (X_c)_{c \in Z}$ be the family of subsets of M^x defined by an L-formula $\psi(x,z)$ as described in Lemma 8.12. Note that X'_b is infinite if and only if there is $c \in Z$ such that

$$X'_b \text{ is pseudo-dense in } X_c \text{ and } \dim(X_c) > 0.$$

By assumption, the set of pairs (b,c) with X'_b pseudo-dense in X_c can be defined by a formula whose choice does not depend on $\mathcal{M}'$. By Fact 8.3, T defines multiplicity. In particular, the set of $c \in Z$ with $\dim X_c > 0$ can be defined by an L-formula whose choice does not depend on $\mathcal{M}'$. The desired conclusion follows.

For the backward implication, suppose T and T' are fixed, T' eliminates $\exists^\infty$, $\varphi'(x,y)$ and $\psi(x,z)$ are an L'-formula and an L-formula, $\mathcal{M}' \vDash T'$, $\mathcal{M} = \mathcal{M}' \upharpoonright L$, and $(X'_b)_{b \in Y'}$ and $(X_c)_{c \in Z}$ are the families of subsets of M^x defined by $\varphi'(x,y)$ and $\psi(x,z)$. Set

$$\mathfrak{Pd} = \{(b,c) \in M^{(y,z)} \mid X'_b \text{ is pseudo-dense in } X_c\}.$$

We need to show that $\mathfrak{Pd}$ can be defined by an L'-formula whose choice might depend on $\varphi'(x,y)$ and $\psi(x,z)$ but not on $\mathcal{M}'$.

We first reduce to the special case where $\psi(x,z)$ is a formula as described in Lemma 8.12. Let $\delta(x,w)$ be a formula as described in Lemma 8.12 and $(X_d)_{d \in W}$ the family of subsets of M^x defined by $\delta(x,w)$, and suppose we have proven the corresponding statement for $\delta(x,w)$. We note that X'_b is pseudo-dense in X_c for $b \in Y'$ and $c \in Z$ if and only if for all $d \in W$ with X_d almost a subset of X_c and $\dim X_d = \dim X_c$, we have X'_b is pseudo-dense in X_d. The desired reduction follows from the special case and Fact 8.3, which states that T defines multiplicity.

We next make a further reduction. Note that by the reduction in the preceding paragraph, $\mathcal{D} = (X_c)_{c \in Z}$ is a family as described in Lemma 8.11, so we will set ourselves up to apply this lemma. For $\alpha < \dim M^x$, $b \in Y$, and $c \in Z$, we define $\mathfrak{D}_{\alpha,b,c}$ to be the set of $d \in Z$ such that $\dim X_d = \alpha$, X'_b is pseudo-dense in X_d, and X_d is almost a subset of X_c. In other words, if $\mathcal{D}_\alpha(X'_b, X_c)$ is defined as in Lemma 8.11, then

$$d \text{ is in } \mathfrak{D}_{\alpha,b,c} \quad \text{if and only if} \quad X_d \text{ is in } \mathcal{D}_\alpha(X'_b, X_c).$$

Set $\mathfrak{Pd}^0$ to be the set of $(b,c) \in \mathfrak{Pd}$ with $\dim X_c = 0$. For $\alpha < \gamma \leqslant \dim M^x$, set $\mathfrak{Pd}^\gamma = \{(b,c) \in \mathfrak{Pd} \mid \dim X_c = \gamma\}$ and set

$$\mathfrak{Pd}^\gamma_\alpha = \{(b,c) \in \mathfrak{Pd} \mid \dim X_c = \gamma \text{ and } \mathfrak{D}_{\beta,b,c} = \varnothing \text{ for all } \alpha < \beta < \gamma\}.$$

We reduce the problem further to showing $\mathfrak{Pd}^\gamma_\alpha$ can be defined by an L'-formula whose choice is independent of $\mathcal{M}'$ for all $\alpha < \gamma \leqslant \dim M^x$. Note that $(b,c) \in M^{(y,z)}$ is in $\mathfrak{Pd}^0$ if and only if $X_c \subseteq X'_b$ and $\dim(X_c) = 0$, so $\mathfrak{Pd}^0$ can be defined by a formula whose choice is independent of $\mathcal{M}'$. Moreover, $\mathfrak{Pd} = \bigcup_{\beta < \dim M^x} \mathfrak{Pd}^\beta$ and $\mathfrak{Pd}^\beta = \mathfrak{Pd}^\beta_{\beta-1}$, so by Fact 8.3(4) we obtained the desired reduction.

We will show the statement in the previous paragraph by lexicographic induction on (γ, α). We first settle some simple cases. For $\gamma = 1$ and $\alpha = 0$, the condition $\mathfrak{D}_{\beta,b,c} = \varnothing$ for all $\alpha < \beta < \gamma$ is vacuous, and the desired conclusion follows from the fact that T defines multiplicity and T' eliminates $\exists^\infty$. Suppose we have proven the statement for all smaller values of γ. It follows from $\mathfrak{Pd}^\beta = \mathfrak{Pd}^\beta_{\beta-1}$ that for all $\beta < \gamma$, $\mathfrak{Pd}^\beta$ can be defined by an L'-formula whose choice is independent of $\mathcal{M}'$. Let

$$Z_\gamma = \{c \in Z \mid \dim(X_c) = \gamma\}.$$

Note for $\beta < \gamma$ and $(b,c) \in Y \times Z_\gamma$ that $d \in M^z$ is in $\mathfrak{D}_{\beta,b,c}$ if and only if $\dim X_d = \beta$ and $(b,d) \in \mathfrak{Pd}^\gamma_\beta$. Using the fact that T defines multiplicity, we get for each $\beta < \gamma$ that the family $(\mathfrak{D}_{\beta,b,c})_{(b,c) \in Y \times Z_\gamma}$ can be defined by a formula independent of the choice of $\mathcal{M}'$. We get from Lemma 8.11 that $(b,c) \in M^{(y,z)}$ is in $\mathfrak{Pd}^\gamma_0$ if and only if

$$\dim X_c = \gamma, \quad \mathfrak{D}_{\beta,b,c} = \varnothing \text{ for all } 0 < \beta < \gamma, \quad \text{ and } X'_b \text{ is infinite.}$$

Hence, $\mathfrak{Pd}^\gamma_0$ can be defined by an L'-formula independent of the choice of $\mathcal{M}'$ by the assumption that T' eliminates $\exists^\infty$ and Fact 8.3(3).

Suppose $0 < \alpha < \gamma \leqslant \dim M^x$ and we have shown the statement for all lexicographic lesser values of (γ, α) not just for the formula $\varphi(x,y)$ but also for any similar chosen $\varphi^*(x, y^*)$. From the argument in the preceding paragraph, $\mathfrak{Pd}^0, \dots, \mathfrak{Pd}^{\gamma-1}$ and $(\mathfrak{D}_{\beta,b,d})_{(b,c) \in Y \times Z_\gamma}$ for each $\beta < \gamma$ can be defined by formulas independent of the choice of $\mathcal{M}'$. By the assumption that T weakly eliminates $\exists^\infty$ and Fact 8.4, there is w and a L-definable function up-to-permutation f from Z to M^w defined by a formula whose choice does not depend on $\mathcal{M}'$ such that for all d_1 and d_2 in Z,

$$f(d_1) = f(d_2) \text{ if and only if } X_{d_1} \text{ is almost equal to } X_{d_2}.$$

In particular, the family $(f(\mathfrak{D}_{\alpha,b,c}))_{(b,c)\in Y\times Z_\gamma}$ can be defined by a formula whose choice does not depend on $\mathcal{M}'$. As T' eliminates $\exists^\infty$, there is n such that

$$|f(\mathfrak{D}_{\alpha,b,c})| > n|w|! \text{ implies } f(\mathfrak{D}_{\alpha,b,c}) \text{ is infinite.}$$

Now let Y^* be the set of $b^* = (b, c, d_1, \dots, d_n)$ in $Y \times Z \times \dots \times Z$ where the product by Z is taken $n+1$-times such that the following properties hold:

(1) $c \in Z_\gamma$ and $\mathfrak{D}_{\beta,b,c} = \varnothing$ for all $0 < \beta < \gamma$.
(2) $f(\mathfrak{D}_{\alpha,b,c})$ is finite.
(3) $\dim X_{d_i} = \alpha$ and X'_b is pseudo-dense in Z_{d_i} for $i \in \{1, \dots, n\}$.
(4) If $\dim X_d = \alpha$ and X'_b is pseudo-dense in Z_d for some $d \in Z$, then X_d is almost equal to X_{d_i} for some $i \in \{1, \dots, n\}$.

For each $b^* \in Y^*$, set

$$X'_{b^*} = X'_b \setminus \bigcup_{i=1}^{n} X_{d_i}.$$

Then by the induction hypothesis and Fact 8.3(3) the family $(X'_{b^*})_{b^*\in Y^*}$ can be defined by a formula $\varphi^*(x, y^*)$ whose choice does not depend on $\mathcal{M}'$. We obtain $\mathfrak{Pd}^{*\gamma}_{\alpha-1}$ from $\varphi^*(x, y^*)$ in the same fashion as we get $\mathfrak{Pd}^{\gamma}_{\alpha-1}$ from $\varphi(x, y)$. The induction hypothesis implies that $\mathfrak{Pd}^{*\gamma}_{\alpha-1}$ can be defined by formulas whose choice does not depend on $\mathcal{M}'$. It follows from Lemma 8.11 that $(b, c) \in \mathfrak{Pd}^{\gamma}_{\alpha}$ if and only if $\dim Z_c = \gamma$ and $\mathfrak{D}_{\beta,b,c} = \varnothing$ for all $\alpha < \beta < \gamma$ and either of the following hold:

(1) $f(\mathfrak{D}_{\alpha,b,c})$ is infinite.
(2) There are $d_1, \dots, d_n$ in Z such that $b^* = (b, c, d_1, \dots, d_n)$ is in Y^* and

$$X'_{b^*} \text{ is in } \mathfrak{Pd}^{*\gamma}_{\alpha-1}.$$

Thus $\mathfrak{Pd}^{\gamma}_{\alpha}$ can be defined by a formula whose choice does not depend on $\mathcal{M}'$ which completes the proof. □

Combining Theorem 8.1, Proposition 8.10, and Proposition 8.19, we have proven the following theorem.

Theorem 8.5. *Suppose L has finitely many sorts, $T_\cap$ is an $\aleph_0$-stable and $\aleph_0$-categorial theory with no finite models which weakly eliminates imaginaries, and T_i eliminates $\exists^\infty$ for all $i \in I$. Then $T^*_\cup$ exists.*

The conditions of Theorem 8.5 are satisfied for instance when $L_\cap = \varnothing$ and $T_\cap$ is the theory of infinite sets. Hence, we recover Winkler's "prehistoric results" on interpolative fusions, Theorem 7.1 and Corollary 7.1. The following proposition combined with Theorem 7.7 allows us to recover Theorem 7.6, the other result from Winkler.

Proposition 8.20. *Suppose $I = \{1, 2\}$, T_1 and T_2 are as in Section 7.5, and T_1 eliminates $\exists^\infty$. Then $T^*_\cup$ exists.*

Proof. We will verify the conditions of Theorem 8.5 to show that $T^*_\cup$ exists. We have assumed that T_1 eliminates $\exists^\infty$. We saw in Section 7.5 that T_2 is interpretable in the theory of an infinite set, and hence so is its reduct $T_\cap$. So both T_2 and $T_\cap$ are $\aleph_0$-categorical and $\aleph_0$-stable. It follows that T_2 eliminates $\exists^\infty$. It is also easy to see that $T_\cap$ admits weak elimination of imaginaries. $\square$

A similar argument allows us to deduce from Theorem 7.8 the fact that the theory of graphs has a model companion. We also get from Theorem 7.9 that if T satisfies the conditions of this section, then T_{Aut} has a model companion. We will leave the details to the reader.

The theory T_q of vector spaces over the finite field $\mathbb{F}_q$ with q elements is $\aleph_0$-stable, $\aleph_0$-categorical, and weakly eliminates imaginaries. Thus any theory T' extending T_q defines pseudo-denseness if and only if it eliminates $\exists^\infty$. This does not generalize to vector spaces over characteristic zero fields, which are $\aleph_0$-stable and weakly eliminate imaginaries, but are not $\aleph_0$-categorical. For example, let T be the theory of torsion-free divisible abelian groups (vector spaces over $\mathbb{Q}$). Let T' be ACF_0, and note that T' is an expansion of T. Then T' does not define pseudo-denseness over T. Suppose $\mathcal{M}'$ is an $\aleph_1$-saturated model of T'. Let

$$L = \{(a, b, c) \in \boldsymbol{M}^3 : ab = c\}$$

and consider the definable family $\{L_a : a \in M\}$ where $L_a = \{(b, c) \in \boldsymbol{M}^2 : ab = c\}$. We leave the easy verification of the following to the reader:

Lemma 8.13. *Fix $a \in \boldsymbol{M}$. Then L_a is pseudo-dense in $\boldsymbol{M}^2$ if and only if $a \notin \mathbb{Q}$.*

As $\mathbb{Q}$ is countable and infinite it cannot be a definable set in an $\aleph_1$-saturated structure. Thus $\mathcal{M}'$ does not define pseudo-denseness over $(\boldsymbol{M}; +)$.

There is a natural rank rk on any $\aleph_0$-categorical theory, described in [**75**, Section 2.3] and [**15**, Section 2.2.1]. This rank is known to agree with thorn rank on $\aleph_0$-categorical structures, so it is an ordinal rank on rosy $\aleph_0$-categorical theories. A special case of Theorem 8.3 is that any expansion of the theory DLO of dense linear orders defines pseudo-denseness over DLO with respect to rk (which agrees with the usual o-minimal dimension over DLO). This fact, together with Proposition 8.19, and recent groundbreaking work on NIP $\aleph_0$-categorical structures [**75, 74**] motivates the following question.

Question. Suppose T is NIP, $\aleph_0$-categorical, and rosy. If T' eliminates $\exists^\infty$ than must T define psuedo-denseness over T (with respect to rk)?

Unfortunately rk does not necessarily agree with Morley rank on $\aleph_0$-stable, $\aleph_0$-categorical theories. One might hope that an approach to Question 8.5 would synthesize the ideas of Section 8.5 and Section 8.3.

CHAPTER 9

Preservation results

Throughout this chapter, we use the notational conventions of Chapters 5 and 6. We also fix I, languages $L_\square$ and theories $T_\square$ for $\square \in I \cup \{\cup, \cap\}$, and assume $T^*_\cup$ exists.

We seek to understand when properties of the theories T_i are preserved in passing to the interpolative fusion $T^*_\cup$. We have already seen a close connection between interpolative fusions and model completeness, which we reformulate as a preservation result in the brief Section 9.1 below. In order to understand definable sets and types, we often want something stronger than model-completeness, so Section 9.2 and 9.3 are devoted to $\mathcal{K}$-completeness of $T^*_\cup$ for various classes $\mathcal{K}$ (see Section 5.2).

Remark 9.1. Much of this chapter is devoted to $\mathcal{K}$-completeness of $T^*_\cup$ for various classes $\mathcal{K}$ (see Section 5.2). By Remarks 5.2 and 6.2, if $T^*_\cup$ is $\mathcal{K}$-complete, then for any pair $(\mathcal{A}_\cup, \mathcal{M}_\cup) \in \mathcal{K}$,

$$T^*_\cup \cup \bigcup_{i \in I} \mathrm{fdiag}_{L_i}(\mathcal{A}_i) \models \mathrm{Th}_{L_\cup(A)}(\mathcal{M}_\cup).$$

This allows us to understand certain $L_\cup$-types in terms of quantifier-free L_i-types.

Many of the results in this chapter contain the hypothesis "suppose $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all $i \in I$". When $T_\cap$ or T_i is incomplete, we mean that this property holds in all consistent completions of these theories. By Proposition 5.3, this hypothesis is always satisfied by $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}^f$ when $T_\cap$ is stable with weak elimination of imaginaries. For example, this applies when $T_\cap$ is the theory of an infinite set or the theory of algebraically closed fields. In the general case, elimination of imaginaries for $T_\cap$ is easily arranged (see Remark 6.1).

9.1. Preservation of model-completeness

We interpret Theorem 6.3 as a first preservation result.

Theorem 9.1. *Suppose each T_i is model-complete. Then $T^*_\cup$ is model-complete, and every $L_\cup$-formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas of the form*

$$\exists y \bigwedge_{i \in J} \varphi_i(x, y),$$

where $J \subseteq I$ is finite and each $\varphi_i(x, y)$ is a flat L_i-formula.

Proof. The first assertion follows immediately from Theorem 6.3. Since $T^*_\cup$ is model-complete, $\psi(x)$ is $T^*_\cup$-equivalent to an existential $L_\cup$-formula $\exists z\,\varphi(x,z)$. By Corollary 5.1, $\varphi(x,z)$ is equivalent to a finite disjunction of E$\flat$-formulas. Distributing the quantifier $\exists z$ over the disjunction and applying Remark 6.2 yields the desired result. $\square$

9.2. Preservation of acl- and bcl-completeness

Given $\square \in I \cup \{\cup,\cap\}$, let $\mathrm{acl}_\square(A)$ be the $\mathcal{M}_\square$-algebraic closure of a subset A of a $T^*_\cup$-model $\mathcal{M}_\cup$. The **combined closure**, $\mathrm{ccl}(A)$, of a subset A of $\mathcal{M}_\cup$ is the smallest set containing A which is acl_i-closed for each $i \in I$. More concretely, $b \in \mathrm{ccl}(A)$ if and only if

$$b \in \mathrm{acl}_{i_n}(\dots(\mathrm{acl}_{i_1}(A))\dots) \text{ for some } i_1,\dots,i_n \in I.$$

Theorem 9.2. *Suppose $T_\cap$ admits a stationary independence relation $\perp$ which satisfies full existence in T_i for all i. If each T_i is* acl-*complete then $T^*_\cup$ is* acl-*complete and* $\mathrm{acl}_\cup = \mathrm{ccl}$.

Proof. Theorem 9.1 shows $T^*_\cup$ is model-complete. In order to apply Proposition 5.1, we will show that the class of $T^*_\cup$-models has the disjoint ccl-amalgamation property.

So suppose $\mathcal{A}_\cup$ is a ccl-closed substructure of a $T^*_\cup$-model $\mathcal{M}_\cup$ and $f\colon\mathcal{A}_\cup \to \mathcal{N}_\cup \models T^*_\cup$ is an embedding. Let $\boldsymbol{\mathcal{M}}_\cup$ be a monster model of $\widehat{T}_\cup = \mathrm{Th}_{L_\cup}(\mathcal{N}_\cup)$, so $\mathcal{N}_\cup$ is an elementary substructure of $\boldsymbol{\mathcal{M}}_\cup$. Let $A' = f(A) \subseteq N$. Let $p_\square(x) = \mathrm{tp}_{L_\square}(M/A)$ for each $\square \in I \cup \{\cap\}$, where x is a tuple of variables enumerating M. By acl-completeness of T_i, $f\colon\mathcal{A}_i \to \mathcal{N}_i$ is partial elementary for all $i \in I$, so $f\colon\mathcal{A}_\cap \to \mathcal{N}_\cap$ is also partial elementary, and we can replace the parameters from A in $p_\square(x)$ by their images under f, obtaining a consistent type $p'_\square(x)$ over A' for all $\square \in I \cup \{\cap\}$.

Fix $i \in I$. Since A is algebraically closed in $\mathcal{M}_i$, A' is algebraically closed in $\boldsymbol{\mathcal{M}}_i$. By full existence for $\perp$ in T_i, there is a realization M'_i of $p'_i(x)$ in $\boldsymbol{\mathcal{M}}_i$ such that $M'_i \perp_{A'} N$ in $\boldsymbol{\mathcal{M}}_i$. Let $q_i(x) = \mathrm{tp}_{L_i}(M'_i/N)$.

For all $i,j \in I$, $\mathrm{tp}_{L_\cap}(M'_i/A') = \mathrm{tp}_{L_\cap}(M'_j/A') = p'_\cap(x)$, so by stationarity for $\perp$, $\mathrm{tp}_{L_\cap}(M'_i/N) = \mathrm{tp}_{L_\cap}(M'_j/N)$. Let $q_\cap(x)$ be this common type, so $q_\cap(x) \subseteq q_i(x)$ for all i. We claim that $\bigcup_{i\in I} q_i(x)$ is realized an an elementary extension of $\mathcal{N}_\cup$.

By Lemma 6.2, the partial $L_\cup(N)$-type

$$\bigcup_{i\in I}(\mathrm{Ediag}(\mathcal{N}_i) \cup q_i(x))$$

is consistent, since each $L_i(N)$-type $(\mathrm{Ediag}(\mathcal{N}_i) \cup q_i(x))$ contains the complete $L_\cap(N)$-type $(\mathrm{Ediag}(\mathcal{N}_\cap) \cup q_\cap(x))$. Suppose it is realized by M'' in $\mathcal{N}'_\cup$. Then M'' is the domain of a substructure $\mathcal{M}''_\cup$ isomorphic to $\mathcal{M}_\cup$ via the enumeration of both structures by the variables x. Let $f'\colon\mathcal{M}_\cup \to \mathcal{M}''_\cup$ be this isomorphism. Also $\mathcal{N}_i \preccurlyeq \mathcal{N}'_i$ for all $i \in I$, and in particular $\mathcal{N}'_\cup \models T_\cup$. Since $T_\cup$ is inductive, there is an extension $\mathcal{N}^*_\cup$ of $\mathcal{N}'_\cup$ such that $\mathcal{N}^*_\cup$ is existentially closed, i.e., $\mathcal{N}^*_\cup \models T^*_\cup$. Since each T_i is model-complete, we have $\mathcal{N}'_i \preccurlyeq \mathcal{N}^*_i$ for all $i \in I$, so M'' satisfies $\bigcup_{i\in I} q_i(x)$ in $\mathcal{N}^*_\cup$. And since $T^*_\cup$ is model-complete, $\mathcal{N}_\cup \preccurlyeq \mathcal{N}^*_\cup$.

We view $\mathcal{N}^*_\cup$ as an elementary substructure of $\boldsymbol{\mathcal{M}}_\cup$, and we view f' as an embedding $\mathcal{M}_\cup \to \mathcal{N}^*_\cup$. If $a \in A$, then a is enumerated by a variable x_a from x, and the formula $x_a = a$ is in $p_\cap(x)$. So $f'(a)$ satisfies the formula $x_a = f(a)$. This establishes the amalgamation property.

For the disjoint amalgamation property, note that we have $M'' \mathop{\perp}_{A'} N$ in $\boldsymbol{\mathcal{M}}_\cap$, so by algebraic independence for $\perp$, $M'' \cap N = A'$, and hence $f'(M) \cap N = f(A)$.

By Proposition 5.1, $T^*_\cup$ is ccl-complete and every ccl-closed substructure is $\mathrm{acl}_\cup$-closed. It follows that for any set $B \subseteq \mathcal{M} \models T$, $\mathrm{acl}_\cup(B) \subseteq \mathrm{ccl}(B)$.

For the converse, it suffices to show $\mathrm{acl}_\cup(B)$ is acl_i-closed for all $i \in I$. Indeed,

$$\mathrm{acl}_i(\mathrm{acl}_\cup(B)) \subseteq \mathrm{acl}_\cup(\mathrm{acl}_\cup(B)) = \mathrm{acl}_\cup(B).$$

So $\mathrm{acl}_\cup = \mathrm{ccl}$, and $T^*_\cup$ is acl-complete. □

Corollary 9.1. *Assume $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i. Suppose each T_i is* bcl*-complete. Then $T^*_\cup$ is* bcl *complete and every $L_\cup$-formula is $T^*_\cup$-equivalent to a finite disjunction of b.e. formulas of the form*

$$\exists y \bigwedge_{i\in J} \varphi_i(x,y),$$

where $J \subseteq I$ is finite and $\varphi_i(x,y)$ is a flat L_i formula for all $i \in J$.

Proof. Theorem 5.1 implies T_i is acl-complete and $\mathrm{bcl}_i = \mathrm{acl}_i$ for all $i \in I$. We have $\mathrm{bcl}_\cup(A) \subseteq \mathrm{acl}_\cup(A)$ for any subset A of a $T_\cup$-model. But also, for all $i \in I$,

$$\begin{aligned}\mathrm{acl}_i(\mathrm{bcl}_\cup(A)) &= \mathrm{bcl}_i(\mathrm{bcl}_\cup(A))\\ &\subseteq \mathrm{bcl}_\cup(\mathrm{bcl}_\cup(A))\\ &= \mathrm{bcl}_\cup(A).\end{aligned}$$

So $\mathrm{bcl}_\cup(A)$ is acl_i-closed for all $i \in I$, hence $\mathrm{ccl}(A) \subseteq \mathrm{bcl}_\cup(A)$.

Theorem 9.2 implies $T^*_\cup$ is acl-complete and $\mathrm{ccl}(A) = \mathrm{bcl}_\cup(A) = \mathrm{acl}_\cup(A)$. Applying Theorem 5.1 again, $T^*_\cup$ is bcl-complete.

It remains to characterize $L_\cup$-formulas up to equivalence. Theorem 5.1 shows every $L_\cup$-formula is $T^*_\cup$-equivalent to a finite disjunction of b.e. formulas. Let $\exists y\, \psi(x,y)$ be a b.e. formula appearing in the disjunction. By Corollary 5.1, the quantifier-free formula $\psi(x,y)$ is equivalent to a finite disjunction of E$\flat$-formulas $\bigvee_{j=1}^m \exists z_j\, \theta_j(x,y,z_j)$. Distributing the quantifier $\exists y$ over the disjunction, we find that $\exists y \exists z_j\, \theta_j(x,y,z_j)$ is a b.e. formula. Applying Remark 6.2 to the flat formula $\theta_j(x,y,z_j)$ yields the result. □

We conclude with two counterexamples showing that the hypotheses on $T_\cap$ are necessary for acl-completeness of interpolative fusions. In the first example $T_\cap$ is unstable with elimination of imaginaries, and in the second example $T_\cap$ is stable but fails weak elimination of imaginaries. In neither example does $T_\cap$ admit a stationary independence relation which satisfies full existence in T_i for all i.

Example 9.1. Let $L_\cap = \{\leqslant\}$ and L_i be the expansion of $L_\cap$ by a unary predicate P_i for $i \in \{1,2\}$. Let $T_\cap = \mathrm{DLO}$, and T_i be the theory of a dense linear order equipped with a downwards closed supremum-less set defined by P_i for $i \in \{1,2\}$. Then $T^*_\cup$ exists and has exactly two completions: an $L_\cup$-structure $\mathcal{M}_\cup$ is a $T^*_\cup$-model if and only if we either have $P_1(\mathcal{M}_\cup) \subsetneq P_2(\mathcal{M}_\cup)$ or $P_2(\mathcal{M}_\cup) \subsetneq P_1(\mathcal{M}_\cup)$. In either kind of model $\varnothing$ is easily seen to be algebraically closed. The completions of $T^*_\cup$ are not determined by $\mathrm{fdiag}_{L_1}(\varnothing) \cup \mathrm{fdiag}_{L_2}(\varnothing)$, so $T^*_\cup$ is not acl-complete.

Example 9.2. Let $L_\cap = \{E\}$ where E is a binary relation symbol. Let $L_i = \{E, P_i\}$ where P_i is unary for $i \in \{1,2\}$. Let $T_\cap$ be the theory of an equivalence relation with infinitely many infinite classes. Let T_i be the theory of a $T_\cap$-model with a distinguished equivalence class named by P_i. Then every model of $T_\cup$ is interpolative, so $T^*_\cup = T_\cup$. A $T^*_\cup$-model $\mathcal{M}_\cup$ may have $P_1(\mathcal{M}_\cup) = P_2(\mathcal{M}_\cup)$ or $P_1(\mathcal{M}_\cup) \neq P_2(\mathcal{M}_\cup)$, so $T^*_\cup$ has exactly two completions. Again, $\mathrm{acl}_\cup(\varnothing) = \varnothing$ and the completions are not determined by $\mathrm{fdiag}_{L_1}(\varnothing) \cup \mathrm{fdiag}_{L_2}(\varnothing)$, so $T^*_\cup$ is not acl-complete.

9.3. Preservation of quantifier elimination

When is quantifier elimination is preserved in interpolative fusions? In contrast to preservation of model-completeness, acl-completeness, and bcl-completeness, we cannot obtain quantifier elimination in $T^*_\cup$ without tight control on algebraic closure in the T_i. In this section we will assume each T_i admits quantifier elimination, hence T_i is bcl-complete and $\mathrm{bcl}_i = \mathrm{acl}_i$ for all $i \in I$ by Theorem 5.1.

Theorem 9.3 below is motivated by some comments in the introduction of [**60**] on the failure of quantifier elimination in ACFA.

Theorem 9.3. *Assume $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i. Suppose every T_i has quantifier elimination, and*

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) \quad \textit{and} \quad \mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$$

for all $L_\cup$-substructures A of $T^*_\cup$-models and all $i \in I$. Then $T^*_\cup$ has quantifier elimination.

Proof. Theorem 9.2 shows $T^*_\cup$ is ccl-complete. We will show $T^*_\cup$ is substructure complete. Suppose $\mathcal{A}_\cup$ is an $L_\cup$-substructure of a $T^*_\cup$-model $\mathcal{M}_\cup$, $\mathcal{N}_\cup$ is another model of $T^*_\cup$, $f\colon \mathcal{A}_\cup \to \mathcal{N}_\cup$

is an embedding. Any $\mathrm{acl}_\cap$-closed subset of M is acl_i-closed for all $i \in I$. Hence,

$$\mathrm{acl}_\cap(A) = \mathrm{acl}_i(A) = \mathrm{ccl}(A) \quad \text{for all } i \in I.$$

As each T_i is substructure complete, f is partial elementary $\mathcal{M}_i \to \mathcal{N}_i$, so f extends to a partial elementary map $g_i\colon \mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) \to \mathcal{N}_i$.

Fix $j \in I$. For all $i \in I$, $g_i^{-1} \circ g_j$ is an $L_\cap$-automorphism of $\mathrm{acl}_\cap(A)$ fixing A pointwise, so in fact it is an L_i-automorphism of $\mathrm{acl}_\cap(A)$ by the assumption on the automorphism groups. It follows that $g_j = g_i \circ (g_i^{-1} \circ g_j)$ is an L_i-embedding $\mathrm{acl}_\cap(A) \to \mathcal{N}_i$. Since i was arbitrary, g_j is an $L_\cup$-embedding. But $\mathrm{acl}_\cap(A) = \mathrm{ccl}(A)$, so by ccl-completeness, g_j is partial elementary $\mathcal{M}_\cup \to \mathcal{N}_\cup$, and hence so is $g_j|_A = f$. □

We prefer hypothesis which can be checked language-by-language, i.e., which refer only to properties of T_i, $T_\cap$, and the relationship between T_i and $T_\cap$ rather than how T_i and T_j relate when $i \neq j$, or how T_i relates to $T_\cup$. The hypothesis of Theorem 9.3 is not strictly language-by-language, because it refers to an arbitrary $L_\cup$-substructure A. However, there are several natural strengthenings of this hypothesis which are language-by-language. One is to simply assume the hypothesis of Theorem 9.3 for all sets A. Simpler language-by-language criteria are given in the following corollaries.

Corollary 9.2. *Assume $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i. Suppose each T_i admits quantifier elimination. If either of the following conditions hold for all sets A, then $T_\cup^*$ has quantifier elimination:*

(1) $\mathrm{acl}_i(A) = \langle A\rangle_{L_i}$ *for all* $i \in I$.
(2) $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ *for all* $i \in I$.

Proof. We apply Theorem 9.3, so assume $A = \langle A\rangle_{L_\cup}$.

(1) We have $A \subseteq \mathrm{acl}_\cap(A) \subseteq \mathrm{acl}_i(A) = \langle A\rangle_{L_i} = A$.
(2) We have $\mathrm{dcl}_\cap(A) \subseteq \mathrm{acl}_\cap(A) \subseteq \mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$.

In both cases, the group $\mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A)$ is already trivial, so its subgroup $\mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$ is too. □

Corollary 9.3. *Assume $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i. Suppose each T_i admits quantifier elimination and a universal axiomization. Then $T_\cup^*$ has quantifier elimination.*

Proof. Every L_i-substructure of a model of T_i is an elementary substructure, and hence acl_i-closed, so we can apply Corollary 9.2(1). □

9.4. Consequences for general interpolative fusions

Many of the results above can be translated to the general case (when the T_i are not model-complete) by Morleyization. This allows us to understand $L_\cup$-formulas and complete $L_\cup$-types relative to L_i-formulas and complete L_i-types.

To set notation: For each i, Morleyization gives a definitional expansion $L_i^\diamond$ of L_i and an extension $T_i^\diamond$ of T_i by axioms defining the new symbols in $L_i^\diamond$. We assume that the new symbols in $L_i^\diamond$ and $L_j^\diamond$ are distinct for $i \neq j$, so that $L_i^\diamond \cap L_j^\diamond = L_\cap$. It follows that each $T_i^\diamond$ has the same set of $L_\cap$ consequences, namely $T_\cap$. We let $L_\cup^\diamond = \bigcup_{i\in I} L_i^\diamond$ and $T_\cup^\diamond = \bigcup_{i\in I} T_i^\diamond$. Then every $T_\cup$-model $\mathcal{M}_\cup$ has a canonical expansion to a $T_\cup^\diamond$-model $\mathcal{M}_\cup^\diamond$, and by Remark 6.1, $\mathcal{M}_\cup$ is interpolative if and only if $\mathcal{M}_\cup^\diamond$ is interpolative.

Proposition 9.1. (1) *Every formula $\psi(x)$ is $T_\cup^*$-equivalent to a finite disjunction of formulas of the form*

$$\exists y \bigwedge_{i\in J} \varphi_i(x,y)$$

where $J \subseteq I$ is finite and $\varphi_i(x,y)$ is an L_i-formula for all $i \in J$.

(2) *If $\mathcal{M}_\cup$ is a $T_\cup^*$-model, then*

$$T_\cup^* \cup \bigcup_{i\in I} \mathrm{Ediag}_{L_i}(\mathcal{M}) \models \mathrm{Ediag}_{L_\cup}(\mathcal{M}).$$

Proof. For (1), each Morleyized theory $T_i^\diamond$ has quantifier elimination, hence is model-complete, so we can apply Theorem 9.1 to the theory $(T_\cup^\diamond)^*$ of interpolative $T_\cup^\diamond$ models. This says $(T_\cup^\diamond)^*$ is model-complete, and $\psi(x)$ is $(T_\cup^\diamond)^*$-equivalent to a finite disjunction of formulas of the form $\exists y \bigwedge_{i\in J} \varphi_i(x,y)$, where each $\varphi_i(x,y)$ is a flat $L_i^\diamond$-formula. But since $L_i^\diamond$ is a definitional expansion of L_i, each formula $\varphi_i(x,y)$ can be translated back to an L_i-formula.

For (2), since $(T_\cup^\diamond)^*$ is model-complete, we have

$$(T_\cup^*)^\diamond \cup \mathrm{fdiag}_{L_\cup^\diamond}(\mathcal{M}) \models \mathrm{Ediag}_{L_\cup^\diamond}(\mathcal{M}).$$

But $\mathrm{fdiag}_{L_\cup^\diamond}(\mathcal{M}) = \bigcup_{i\in I} \mathrm{fdiag}_{L_i^\diamond}(\mathcal{M})$, and $\mathrm{fdiag}_{L_i^\diamond}(\mathcal{M})$ is completely determined by $\mathrm{Ediag}_{L_i}(\mathcal{M})$, so the result follows. □

We note that Proposition 9.1(2) is simply a restatement of Proposition 6.2(3), which we think of as "relative model-completeness".

We will now establish a sequence of variants on Proposition 9.1, with stronger hypotheses and stronger conclusions.

Proposition 9.2. *Assume $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i. Then:*

(1) *Every formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas of the form*

$$\exists y \bigwedge_{i \in J} \varphi_i(x, y)$$

where $J \subseteq I$ is finite, $\varphi_i(x,y)$ is an L_i-formula for all $i \in J$, and $\bigwedge_{i\in J} \varphi_i(x,y)$ is bounded in y.

(2) *If A is an algebraically closed subset of a $T^*_\cup$-model M, then*

$$T^*_\cup \cup \bigcup_{i \in I} \mathrm{Th}_{L_i(A)}(\mathcal{M}) \vDash \mathrm{Th}_{L_\cup(A)}(\mathcal{M}).$$

Proof. Just as in the proof of Proposition 9.1, but this time using the fact that $(T^\diamond_\cup)^*$ is bcl-complete and applying Corollary 9.1. □

It follows from Proposition 9.2 that if $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i, then the completions of $T^*_\cup$ are determined by the L_i-types of $\mathrm{acl}_\cup(\varnothing)$ for all i.

Proposition 9.3. *Assume $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i. Suppose further that*

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) \quad \textit{and} \quad \mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$$

for all $L_\cup$-substructures A of $T^*_\cup$-models and all $i \in I$. Then:

(1) *Every formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas*

$$\exists y \bigwedge_{i \in J} \varphi_i(x, y)$$

where $J \subseteq I$ is finite, $\varphi_i(x,y)$ is an L_i-formula for all $i \in J$, and $\bigwedge_{i\in J} \varphi_i(x,y)$ is bounded in y with bound 1.

(2) *If A is an $L_\cup$-substructure of a $T^*_\cup$-model $\mathcal{M}$ then*

$$T^*_\cup \cup \bigcup_{i \in I} \mathrm{Th}_{L_i(A)}(\mathcal{M}) \vDash \mathrm{Th}_{L_\cup(A)}(\mathcal{M}).$$

Proof. Observing that Morleyization does not affect our hypotheses about acl_i and $\mathrm{acl}_\cap$, we find that $(T^\diamond_\cup)^*$ has quantifier elimination, by Theorem 9.3. This gives us (2) as in the proof of Proposition 9.1.

For (1), note that $\psi(x)$ is $(T^\diamond_\cup)^*$-equivalent to a quantifier-free formula. The result then follows from Corollary 5.1 and Remark 6.2. □

Remark 9.2. As in Corollary 9.2(1), we can replace the hypotheses of Proposition 9.3 with: $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i, and for all sets A and all $i \in I$, $\mathrm{acl}_i(A) = \langle A \rangle_{L_i}$. The assumption $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ gives us something stronger, see Remark 9.3 below.

With a slightly strong assumption, we can get true relative quantifier elimination down to L_i-formulas in $T_\cup^*$.

Proposition 9.4. *Assume $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i. Suppose further that*

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) \quad \textit{and} \quad \mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$$

for all sets A and all $i \in I$. Then:

(1) *Every formula is $T_\cup^*$-equivalent to a Boolean combination of L_i-formulas.*
(2) *For any subset A of a $T_\cup^*$-models $\mathcal{M}$,*

$$T_\cup^* \cup \bigcup_{i \in I} \mathrm{Th}_{L_i(A)}(\mathcal{M}) \models \mathrm{Th}_{L_\cup(A)}(\mathcal{M}).$$

Proof. We first move to a relational language by replacing all function symbols by their graphs. Then we proceed just as in the proof of Proposition 9.3, noting that when $L_\cup^\diamond$ is relational, a quantifier-free $L_\cup^\diamond$-formula is already a Boolean combination of $L_i^\diamond$-formulas. ☐

Remark 9.3. Once again, as in Corollary 9.2(2), we can replace the hypotheses of Proposition 9.4 with: $T_\cap$ admits a stationary independence relation which satisfies full existence in T_i for all i, and $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ for all sets A and all $i \in I$. The assumption $\mathrm{acl}_i(A) = \langle A \rangle_{L_i}$ does not suffice for this, because this condition is lost when moving to a relational language.

9.5. Preservation of stability and NIP

In this section we give applications of some of the technical work above.

Proposition 9.5. *Assume the hypotheses of Proposition 9.4. If each T_i is stable (NIP), then $T_\cup^*$ is stable (NIP).*

Proof. This follows immediately from Proposition 9.4(1) as stable (NIP) formulas are closed under Boolean combinations. ☐

We can also use Proposition 9.4(2) to count types.

Proposition 9.6. *Assume the hypotheses of Proposition 9.4, and suppose that I is finite. If each T_i is stable in κ, then $T_\cup^*$ is stable in κ. As a consequence, if each T_i is $\aleph_0$-stable then $T_\cup^*$ is $\aleph_0$-stable, and if each T_i is superstable, then $T_\cup^*$ is superstable.*

Proof. We consider $S_x(A)$, where x is a finite tuple of variables, $A \subseteq \mathcal{M} \models T_\cup^*$, and $|A| \leqslant \kappa$. By Proposition 9.4(2), a type in $S_x(A)$ is completely determined by its restrictions to L_i for all $i \in I$. Since the number of L_i-types over A in the variables x is at most κ, we have $|S_x(A)| \leqslant \prod_{i \in I} \kappa = \kappa$, since I is finite. ☐

We do not expect to obtain preservation of stability or NIP without strong restrictions on acl, as in the hypotheses of Proposition 9.4. The proofs of Propositions 9.5 and 9.6 do not apply to other classification-theoretic properties such as simplicity, NSOP_1, and NTP_2, as these properties are not characterized by counting types, and formulas with these properties are not closed under Boolean combinations in general. However, we can obtain preservation results for some of these properties under more general hypotheses. These results will be contained in future papers, beginning with [**53**].

Corollary 7.1, Proposition 9.4, the fact that a theory with trivial algebraic closure eliminates $\exists^\infty$, and Proposition 9.5, together imply Corollary 9.4.

Corollary 9.4. *Suppose* acl_i *is trivial for all* $i \in I$*. Then* $T^*_\cup$ *exists. If* $\mathcal{M}_\cup \models T^*_\cup$ *then every* $\mathcal{M}_\cup$*-definable set is a Boolean combination of* $\mathcal{M}_i$*-definable sets for various* $i \in I$*. If each* T_i *is additionally stable (NIP) then* $T^*_\cup$ *is stable (NIP).*

The special case of Corollary 9.4 when T_2 is the theory of dense linear orders is proven in [**71**, Corollary 1.2].

9.6. Preservation of $\aleph_0$-categoricity

In this section, we do not assume that the interpolative fusion $T^*_\cup$ exists. Applying the preservation results above, we show that $T^*_\cup$ exists and is $\aleph_0$-categorical provided that certain hypotheses, including $\aleph_0$-categoricity, on the T_i hold. This section is closely related to work of Pillay and Tsuboi [**64**].

Proposition 9.7. *Assume* $T_\cap$ *admits a stationary independence relation which satisfies full existence in* T_i *for all* i*. Assume also that all languages have only finitely many sorts. Suppose that each* T_i *is* $\aleph_0$*-categorical and that there is some* $i^* \in I$ *such that* $\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A)$ *for all* $i \neq i^*$*. Then the interpolative fusion exists.*

Proof. A $T_\cup$-model $\mathcal{M}_\cup$ has the **joint consistency property** if for every finite $B \subseteq M$ such that $B = \mathrm{acl}_{i^*}(B)$ and every family $(p_i(x))_{i\in J}$ such that J is a finite subset of I, $p_i(x)$ is a complete L_i-type over B for all $i \in J$, and the p_i have a common restriction $p_\cap(x)$ to $L_\cap$, then $\bigcup_{i\in I} p_i(x)$ is realized in $\mathcal{M}_\cup$.

Note that the joint consistency property is elementary. Indeed, by $\aleph_0$-categoricity, there is an L_{i^*}-formula $\psi(y)$ expressing the property that the set B enumerated by a tuple b is acl_{i^*}-closed. Since B is finite, every complete type $p_i(x)$ over B is isolated by a single formula. And the property that the L_i-formula $\varphi_i(x, b)$ isolates a complete L_i-type over B whose restriction to $L_\cap$ is isolated by the $L_\cap$-formula $\varphi_\cap(x, b)$ is definable by a formula

$\theta_{\varphi_i,\varphi_\cap}(b)$. So the class of $T_\cup$-models with the joint consistency property is axiomatized by $T_\cup$ together with sentences of the form

$$\forall y \left[\left(\psi(y) \wedge \bigwedge_{i\in J} \theta_{\varphi_i,\varphi_\cap}(y)\right) \to \exists x \bigwedge_{i\in J} \varphi_i(x,y)\right].$$

It remains to show that a structure $\mathcal{M}_\cup$ is interpolative if and only if it has the joint consistency property. So suppose $\mathcal{M}_\cup$ is interpolative, let B and $(p_i(x))_{i\in J}$ be as in the definition of the joint consistency property, and suppose for contradiction that $\bigcup_{i\in J} p_i(x)$ is not realized in $\mathcal{M}_\cup$. Note that since B is acl_{i^*}-closed, it is also acl_i-closed for all $i \neq i^*$, since $\mathrm{acl}_i(B) = \mathrm{acl}_\cap(B) \subseteq \mathrm{acl}_{i^*}(B) = B$.

Each $p_i(x)$ is isolated by a single L_i-formula $\varphi_i(x,b)$, and

$$\mathcal{M}_\cup \models \neg\exists x \bigwedge_{i\in J} \varphi_i(x,b).$$

It follows that the φ_i are separated by a family of $L_\cap$-formulas $(\psi^i(x,c_i))_{i\in J}$. Let $C = B \cup \{c_i \mid i \in J\}$. By full existence for $\mathop{\smile\!\!\!\!\!\!\mid}$ in T_i, since B is acl_i-closed, $p_i(x)$ has an extension to a type $q_i(x)$ over C such that for any realization a_i of $q_i(x)$, $a \mathop{\smile\!\!\!\!\!\!\mid}_B C$. By stationarity, the types $q_i(x)$ have a common restriction $q_\cap$ to $L_\cap$. Now for all $i \in J$, since $\varphi_i(x,b) \in p_i(x)$, $\psi^i(x,c_i) \in q_i(x)$, and hence $\psi^i(x,c_i) \in q_\cap(x)$. This is a contradiction, since $\{\psi^i(x,c_i) \mid i \in J\}$ is inconsistent.

Conversely, suppose $\mathcal{M}_\cup$ has the joint consistency property. Let $(\varphi_i(x,a_i))_{i\in J}$ be a family of formulas which are not separated. Let $B = \mathrm{acl}_{i^*}((a_i)_{i\in J})$. Since T_{i^*} is $\aleph_0$-categorical, and there are only finitely many sorts, B is finite. For each $i \in J$, there is an $L_\cap$-formula $\psi^i(x,b)$ such that $\mathcal{M}_\cup \models \psi^i(a,b)$ if and only if $\mathrm{tp}_{L_\cap}(a/B)$ is consistent with $\varphi_i(x,a_i)$ (we may take $\psi^i(x,b)$ to be the disjunction of formulas isolating each of the finitely many such types). Since the formulas $\psi^i(x,b)$ do not separate the formulas $\varphi_i(x,a_i)$, there must be some element $a \in M^x$ satisfying $\bigwedge_{i\in J} \psi^i(x,b)$. Then $p_\cap(x) = \mathrm{tp}_{L_\cap}(a/B)$ is consistent with each $\varphi_i(x,a_i)$, so $p_\cap(x) \cup \{\varphi_i(x,a_i)\}$ can be extended to a complete L_i-type $p_i(x)$ over B. By the joint consistency property, there is some element in M^x realizing $\bigcup_{i\in J} p_i(x)$, and in particular satisfying $\bigwedge_{i\in J} \varphi_i(x,a_i)$. □

A type-counting argument as in Proposition 9.6 now gives preservation of $\aleph_0$-categoricity.

Theorem 9.4. *Assume the hypotheses of Proposition 9.7, and let $T_\cup^*$ be the interpolative fusion. Assume additionally that I is finite. Then every completion of $T_\cup^*$ is $\aleph_0$-categorical.*

Proof. Let $\widehat{T}$ be a completion of $T_\cup^*$. It suffices to show that for any finite tuple of variables x, there are only finitely many $L_\cup$-types over the empty set in the variables x relative to $\widehat{T}$. Since $\mathrm{acl}_\cup = \mathrm{acl}_{i^*}$ is uniformly locally finite, there is an upper bound m on the size of $\mathrm{acl}_\cup(a)$ for any tuple $a \in M^x$ when $M \models \widehat{T}$.

By Proposition 9.2, $\mathrm{tp}_{L_\cup}(\mathrm{acl}_\cup(a))$ is determined by $\bigcup_{i\in I}\mathrm{tp}_{L_i}(\mathrm{acl}_\cup(a))$. So the number of possible $L_\cup$-types of a is bounded above by the product over all i of the number of L_i-types of m-tuples relative to T_i. This is finite, since I is finite and each T_i is $\aleph_0$-categorical. $\square$

The presentation of the $\aleph_0$-categorical theory of the random graph as an interpolative fusion in Section 7.6 illustrates Theorem 9.4. Indeed, letting T_1 and T_2 be as in Section 7.6, $T_\cap$ is the theory of two infinite sets with no extra structure, which is stable with weak elimination of imaginaries, and algebraic closure in T_2 is trivial and thus agrees with algebraic closure in $T_\cap$.

We recover the following result of Pillay and Tsuboi.

Corollary 9.5 ([**64**, Corollary 5])**.** *Assume $T_\cap$ is stable with weak elimination of imaginaries. Let $I = \{1,2\}$, suppose T_1 and T_2 are $\aleph_0$-categorical single-sorted theories, and suppose $\mathrm{acl}_1(A) = \mathrm{acl}_\cap(A)$ for all $A \subseteq \boldsymbol{M}_1$. Then $T_\cup$ admits an $\aleph_0$-categorical completion.*

9.7. Preservation of NSOP$_1$

We fix a completion $\widehat{T}$ of $T^*_\cup$ and a monster model $\mathfrak{M}_\cup \models \widehat{T}$. We also assume that $T_\cap$ is stable with weak elimination of imaginaries, and we will additionally need to assume that $T_\cap$ has 3-uniqueness.

Let T be a stable theory. Suppose a_1, a_2, and a_3 are tuples enumerating algebraically closed sets, which are pairwise forking independent over a common algebraically closed subset A. For $1 \leqslant i < j \leqslant 3$, let a_{ij} be a tuple enumerating $\mathrm{acl}(a_i, a_j)$. Then T has **3-uniqueness** if $\mathrm{tp}(a_{12}a_{13}a_{23})$ is uniquely determined by $\mathrm{tp}(a_{12}) \cup \mathrm{tp}(a_{13}) \cup \mathrm{tp}(a_{23})$.

Hrushvoski [**39**] showed that a stable theory has 3-uniqueness if and only if it eliminates *generalized imaginaries*. Generalized imaginaries correspond to definable groupoids. Ordinary amalgamation over algebraically closed sets in the sense of Proposition 5.3 requires weak elimination of imaginaries in $T_\cap$. It is therefore natural that independent 3-amalgamation in $\widehat{T}$ (the independence theorem, the main component in showing $\widehat{T}$ is NSOP$_1$) requires elimination of generalized imaginaries in $T_\cap$.

If $\widehat{T}$ is NSOP$_1$, what relationship does $\mathop{\smile\!\!\!\!\!|}^K$ in $\mathfrak{M}_\cup$ have to $\mathop{\smile\!\!\!\!\!|}^K$ in $\mathfrak{M}_i$ for $i \in I$? Note that $A \mathop{\smile\!\!\!\!\!|}^K_M B$ implies $\mathrm{acl}_\cup(MA) \mathop{\smile\!\!\!\!\!|}^K_M \mathrm{acl}_\cup(MB)$ in $\mathfrak{M}$ by Theorem 5.4. Then by Lemma 5.7, we have $\mathrm{acl}_\cup(MA) \mathop{\smile\!\!\!\!\!|}^K_M \mathrm{acl}_\cup(MB)$ in $\mathfrak{M}_i$ for all i. It is reasonable to hope that Kim forking between $\mathrm{acl}_\cup(MA)$ and $\mathrm{acl}_\cup(MB)$ in some $\mathfrak{M}_i$ is the only source of Kim forking between A and B in $\mathfrak{M}_\cup$.

For all A, B and M, we declare:

$$A \underset{M}{\perp} B \Leftrightarrow \mathrm{acl}_{\cup}(MA) \underset{M}{\perp}^{K} \mathrm{acl}_{\cup}(MB) \text{ in } \mathbf{M}_i \text{ for all } i \in I.$$

We use the axiomatic characterization of Theorem 5.6 to show $\widehat{T}$ is NSOP$_1$ and $\perp = \perp^K$.

Theorem 9.5. *Suppose that each T_i is NSOP$_1$ and $T_\cap$ has 3-uniqueness. Then $\widehat{T}$ is NSOP$_1$ and $\perp_M = \perp^K_M$ for all $\mathcal{M} \prec \mathbf{M}_\cup$.*

Proof. We show $\perp$ satisfies the properties listed in Theorem 5.6.

Invariance, existence, monotonicity, symmetry: Clear from the definition, using the corresponding properties of Kim independence in each $\mathbf{M}_i$.

The independence theorem: We are given A, A', B, C and $\mathcal{M}$ such that $\mathrm{tp}_{L_\cup}(A/M) = \mathrm{tp}_{L_\cup}(A'/M)$, $A \perp_M B$, $A' \perp_M C$, and $B \perp_M C$. By adding elements to A, A', B, and C, we may assume $A = \mathrm{acl}_\cup(MA)$, $A' = \mathrm{acl}_\cup(MA")$, $B = \mathrm{acl}_\cup(MB)$, and $C = \mathrm{acl}_\cup(MC)$. Then by definition of $\perp$, we have, for all $i \in I$, $\mathrm{tp}_{L_i}(A/M) = \mathrm{tp}_{L_i}(A'/M)$, $A \perp^K_M B$, $A' \perp^K_M C$, and $B \perp^K_M C$ in $\mathbf{M}_i$.

Let $B' = \mathrm{cl}(AB)$, $C' = \mathrm{cl}(A'C)$, and $D = \mathrm{cl}(BC)$. Let $f\colon A \to A'$ be the bijection established by the equality of types $\mathrm{tp}_{L_\cup}(A/M) = \mathrm{tp}_{L_\cup}(A'/M)$.

For all $\square \in I \cup \{\cap\}$, let $\Sigma_\square$ be the partial $L_\square$-type

$$\begin{aligned} &T_\square \cup \Delta^{L_\square}_{B'} \cup \Delta^{L_\square}_{C'} \cup \Delta^{L_\square}_{D} \cup \{x_a = x_{f(a)} \mid a \in A\} \\ &\quad \cup \{\neg\delta(\overline{x}_a, \overline{x}_d) \mid \overline{a} \in A, \overline{d} \in D, \delta(\overline{x}_a, \overline{d}) \text{ Kim divides over } M \text{ in } \mathbf{M}_\square\} \end{aligned}$$

In the last part of the definition of $\Sigma_\square$ above, $\overline{x}_a$ and $\overline{x}_d$ are the variables representing $\overline{a}$ and $\overline{d}$ in the diagrams.

I claim it suffices to show $\bigcup_{i\in I} \Sigma_i$ is consistent. Indeed, any model N of $\bigcup_{i\in I} \Sigma_i$ is a model of $T_\cup$, and hence embeds in a model M' of $T^*_\cup$. Since the induced embedding $M \to M'$ is elementary, $M' \models \widehat{T}$, and we can embed M' in $\mathbf{M}$ in a way which maps the elements named by $(x_d)_{d\in D}$ to D (indeed, these elements satisfy $\Delta^{L_\cup}_D$, and hence the $L_\cup$-type of D in M', since D is closed). Then taking A'' to be the image of the elements named by $(x_a)_{a\in A}$, we have that $\mathrm{tp}_{L_\cup}(A''B/M) = \mathrm{tp}_{L_\cup}(AB/M)$, since $\mathrm{cl}(A''B)$ satisfies $\Delta^{L_\cup}_{B'}$, and $\mathrm{tp}_{L_\cup}(A''C/M) = \mathrm{tp}_{L_\cup}(A'C/M)$, since $\mathrm{cl}(A''C)$ satisfies $\Delta^{L_\cup}_{C'}$. And finally $A'' \perp^K_M D$ in $\mathbf{M}_i$ for all $i \in I$, thanks to quantifier-elimination and the definition of Σ_i.

By Robinson joint consistency, we just need to show that $\Sigma_\cap$, which is equal to $\Sigma_i \cap \Sigma_j$ for all $i \neq j$, has a completion $\Sigma^*_\cap$ which is consistent with each Σ_i. Let $\Sigma^*_\cap$ be the partial

$L_\cap$-type:

$$\begin{aligned} T_\cap \cup \Delta^{L_\cap}_{B'} \cup \Delta^{L_\cap}_{C'} \cup \Delta^{L_\cap}_{D} \cup \{x_a = x_{f(a)} \mid a \in A\} \\ \cup \{\neg\delta(\overline{x}_a, \overline{x}_d) \mid \overline{a} \in A, \overline{d} \in D, \delta(\overline{x}_a, \overline{d}) \text{ Kim divides over } M \text{ in } \mathfrak{M}_\cap\} \\ \cup \{\neg\delta(\overline{x}_b, \overline{x}_c) \mid \overline{b} \in B', \overline{c} \in C', \delta(\overline{x}_b, \overline{c}) \text{ forks over } A' \text{ in } \mathfrak{M}_\cap\} \\ \cup \{\neg\delta(\overline{x}_b, \overline{x}_d) \mid \overline{b} \in B', \overline{d} \in D, \delta(\overline{x}_b, \overline{d}) \text{ forks over } B \text{ in } \mathfrak{M}_\cap\} \\ \cup \{\neg\delta(\overline{x}_c, \overline{x}_d) \mid \overline{c} \in C', \overline{d} \in D, \delta(\overline{x}_c, \overline{d}) \text{ forks over } C \text{ in } \mathfrak{M}_\cap\} \end{aligned}$$

First, I claim that $\Sigma^*_\cap$ is consistent with Σ_i for all i. We begin by applying Theorem 5.7(2) in $\mathfrak{M}_i$. This gives us A'' such that $A'' \equiv_{MB} A$, $A'' \equiv_{MC} A'$, and $A'' \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} BC$, and further $B \mathop{\smile\!\!\!\!\!\!\mid}^{r}_{MA''} C$, $A'' \mathop{\smile\!\!\!\!\!\!\mid}^{r}_{MB} C$, and $A'' \mathop{\smile\!\!\!\!\!\!\mid}^{r}_{MC} B$.

Naming $B_i = \mathrm{acl}_i(A''B)$, $C_i = \mathrm{acl}_i(A''C)$, and $D_i = \mathrm{acl}_i(BC)$, the instances of $\mathop{\smile\!\!\!\!\!\!\mid}^{r}$ mean that $B_i \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{A''} C_i$, $B_i \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} D_i$, and $C_i \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{C} D_i$ in $\mathfrak{M}_\cap$. We also have that $A'' \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} D_i$, and B_i, C_i, and D_i satisfy the subsets of $\Delta^{L_i}_{B'}$, $\Delta^{L_i}_{C'}$, and $\Delta^{L_i}_{D}$ on the variables which enumerate these sets.

By Theorem 5.7(1), after moving by an automorphism over MD_i, we can find $D'' \equiv_{MD_i} D$ (so in particular D'' satisfies $\Delta^{L_i}_{D}$) such that $A'' \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} D''$ and $A'' \mathop{\smile\!\!\!\!\!\!\mid}^{r}_{MD_i} D''$. Since B_i and C_i are both subsets of $\mathrm{acl}_i(MA''D_i)$, we have $B_i \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{MD_i} D''$ and $B_i \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{MD_i} D''$ in $\mathfrak{M}_\cap$. By transitivity for $\mathop{\smile\!\!\!\!\!\!\mid}^{f}$, $B_i \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} D''$ and $C_i \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{C} D''$.

Since B_i is acl_i-closed, we can find a realization B'' of $\Delta^{L_i}_{B'}$ over B_i such that $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B_i} C_iD''$. In particular, by transitivity of $\mathop{\smile\!\!\!\!\!\!\mid}^{f}$, $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{A''} C_i$ and $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} D''$.

Similarly, since C_i is acl_i, closed, we can find a realization C'' of $\Delta^{L_i}_{C'}$ over C_i such that $C'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{C_i} B''D''$. In particular, by transitivity of $\mathop{\smile\!\!\!\!\!\!\mid}^{f}$, $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{A''} C''$ and $C'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{C} D''$.

All in all, $B''C''D''$ satisfies $\Sigma_i \cup \Sigma^*_\cap$.

Having shown consistency, it remains to show that $\Sigma^*_\cap$ is complete. To do this, we will apply 3-uniqueness twice. Let $B'_\cap = \mathrm{acl}_\cap(AB)$, $C'_\cap = \mathrm{acl}_\cap(A'C)$, and $D_\cap = \mathrm{acl}_\cap(BC)$. By 3-uniqueness, the restriction of $\Sigma_\cap$ to the variables labeling elements of $B'_\cap \cup C'_\cap \cup D_\cap$ is complete. Now we have to handle the rest of B', C', and D. So suppose we have any realization of $\Sigma^*_\cap$. We may assume the variables labeling D are interpreted by D, so as above we name by A'' the interpretation of the variables labeling A, and set $B'' = \mathrm{cl}(A''B)$ and $C'' = \mathrm{cl}(A''C)$, and similarly for $B''_\cap$ and $C''_\cap$.

Let $E = \mathrm{acl}_\cap(B''_\cap C''_\cap D_\cap) = \mathrm{acl}_\cap(A''BC)$. We will use E as the base algebraically closed set for another application of 3-uniqueness. By the extra non-forking conditions added to $\Sigma^*_\cap$, we have $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{A''} C''$, $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} D$, and $C'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{C} D$. Using base monotonicity on the left and right, we have $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{A''BC} C''$, $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{A''BC} D$, and $C'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{A''BC} D$, so $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{E} C''$, $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{E} D$, and $C'' \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{E} D$. By stationarity, this information determines $\mathrm{tp}_{L_\cap}(B''C'')$, $\mathrm{tp}_{L_\cap}(B''D)$, and $\mathrm{tp}_{L_\cap}(C''D)$ uniquely, and by 3-uniqueness, these types determine $\mathrm{tp}_{L_\cap}(B''C''D)$ uniquely.

Strong finite character: Suppose $A \not\mathop{\smile\!\!\!\!\!|}_M B$. Then for some $i \in I$, we have $\mathrm{acl}(MA) \not\mathop{\smile\!\!\!\!\!|}^K_M \mathrm{acl}(MB)$ in $\mathbf{M}_i$. So there is some $a' \in \mathrm{acl}(MA)$ and $b' \in \mathrm{acl}(MB)$ such that $a' \not\mathop{\smile\!\!\!\!\!|}^K_M b'$ in $\mathbf{M}_i$. Let $\varphi(x', b', m)$ be an L_i-formula in $\mathrm{tp}_{L_i}(a'/Mb')$ which Kim divides over M in $\mathbf{M}_i$, let $\psi(x', a, m)$ be an $L_\cup$-formula isolating $\mathrm{tp}_{L_\cup}(a'/MA)$, and let $\theta(y', b, m)$ be an $L_\cup$-formula isolating $\mathrm{tp}_{L_\cup}(b'/MB)$. Note that by replacing ψ with $\psi(x', a, m) \wedge (\exists^{\leqslant k} x' \, \psi(x', a, m))$ for some k, we may assume $\psi(x', c, m)$ has only finitely many realizations for any c.

I claim the following formula $\chi(x, b, m)$ witnesses strong finite character:

$$\exists x' \, \exists y' \, [\varphi(x', y', m) \wedge \psi(x', x, m) \wedge \theta(y', b, m)].$$

Certainly we have $\chi(x, b, m) \in \mathrm{tp}_{L_\cup}(A/MB)$. Suppose we are given c such that $\mathbf{M} \vDash \chi(c, b, m)$. Then picking witnesses c' and b'' for the existential quantifiers, we have that $c' \in \mathrm{acl}_\cup(Mc)$ (since $\mathbf{M} \vDash \psi(c', c, m)$) and $b'' \in \mathrm{acl}_\cup(Mb)$ (since $\mathbf{M} \vDash \theta(b'', b, m)$). Further, $b'' \equiv_{MB} b'$, so $\varphi(x', b'', m)$ Kim divides over M in $\mathbf{M}_i$. Since $\mathbf{M} \vDash \varphi(c', b'', m)$, we have $c' \not\mathop{\smile\!\!\!\!\!|}^K_M b''$ in $\mathbf{M}_i$, so $c \not\mathop{\smile\!\!\!\!\!|}_M b$.

At this point, we can conclude $\widehat{T}$ is NSOP$_1$. To get the characterization of $\mathop{\smile\!\!\!\!\!|}^K$, we need to check one more property.

Witnessing: Suppose again $A \not\mathop{\smile\!\!\!\!\!|}_M B$. We use the same notation as in the proof of strong finite character, and we seek to show that $\chi(x, b, m)$ Kim divides over M in $\mathbf{M}$.

If not, then using Theorem 5.3, we can find a complete $L_\cup$-type $p(x)$ over Mb which contains $\chi(x, b, m)$ but does not Kim divide. Let e realize this type. Then we have $e \mathop{\smile\!\!\!\!\!|}^K_M b$ in $\mathbf{M}$, so by Theorem 5.4, $\mathrm{acl}_\cup(Me) \mathop{\smile\!\!\!\!\!|}^K_M \mathrm{acl}_\cup(Mb)$ in $\mathbf{M}$. But since $\mathbf{M} \vDash \chi(e, b, m)$, there is some $e' \in \mathrm{acl}_\cup(Me)$ and some $b'' \in \mathrm{acl}_\cup(Mb)$ such that $\mathbf{M} \vDash \varphi(e', b'', m)$. This is a contradiction, since by Lemma 5.7 and the fact that $\mathrm{tp}_{L_\cup}(b''/M) = \mathrm{tp}_{L_\cup}(b'/M)$, $\varphi(x', b'', m)$ Kim divides over M in $\mathbf{M}$. ☐

9.8. Preservation of simplicity

Further, we get preservation of simplicity whenever algebraicity in $\mathbf{M}_\cup$ agrees with algebraicity in $\mathbf{M}_i$ for all i. In other words, a failure of simplicity in the interpolative fusion of simple theories T_i always comes from nontrivial interactions between algebraic closures.

Proposition 9.8. *Suppose each T_i is simple, $T_\cap$ has 3-uniqueness, and for all $i \in I$,*

$$\mathrm{acl}_i(N\mathrm{acl}_\cup(Ma)) = \mathrm{acl}_\cup(Na)$$

whenever $a \mathop{\smile\!\!\!\!\!|}^K_M N$ and $M \prec N \prec \mathbf{M}_\cup$. Then $\widehat{T}$ is simple.

Proof. By Theorem 9.5, $\widehat{T}$ is NSOP$_1$, and we obtain a characterization of Kim independence from the proof. By Theorem 5.5, it suffices to show that $\mathop{\smile\!\!\!\!\!|}^K$ satisfies base monotonicity over models in $\mathbf{M}_\cup$.

So fix $M \prec N \prec \mathbf{M}_\cup$ and $a \mathop{\smile\!\!\!\!\!\mid}^K_M Nb$. Then, for all $i \in I$, since T_i is simple, we have

$$\begin{aligned}
\mathrm{acl}_\cup(Ma) \mathop{\downarrow}^K_M \mathrm{acl}_\cup(Nb) &\Rightarrow \mathrm{acl}_\cup(Ma) \mathop{\downarrow}^f_M \mathrm{acl}_\cup(Nb) \\
&\Rightarrow \mathrm{acl}_\cup(Ma) \mathop{\downarrow}^f_N \mathrm{acl}_\cup(Nb) \\
&\Rightarrow \mathrm{acl}_i(N\mathrm{acl}_\cup(Ma)) \mathop{\downarrow}^f_N \mathrm{acl}_\cup(Nb) \\
&\Rightarrow \mathrm{acl}_\cup(Na) \mathop{\downarrow}^K_N \mathrm{acl}_\cup(Nb).
\end{aligned}$$

So $a \mathop{\downarrow}^K_N b$ in $\mathbf{M}_\cup$, as desired. □

Remark 9.4. In the statement of Corollary 9.8, we have put the weakest possible hypothesis on the interaction between cl and acl_i. In a typical application, we will actually have $\mathrm{cl} = \mathrm{acl}_i$ for all $i \in I$.

But it is worth noting that the weaker condition $\mathrm{cl}(Mab) = \mathrm{acl}_i(\mathrm{cl}(Ma)\mathrm{cl}(Mb))$ for all $i \in I$ suffices. This condition essentially says that cl has no binary algebraic dependencies that are not already present in every acl_i, and it will allow us to recover the theorem that when T is stable with weak elimination of imaginaries, the theory T_A of T with a generic automorphism is simple.

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