EXTREMAL PROBLEMS FOR CYCLES IN GRAPHS AND HYPERGRAPHS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2019

Urbana, Illinois

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Abstract

In this thesis, we study several generalizations of Turán type problems in graphs and hypergraphs. In particular, we focus on graphs and hypergraphs without long cycles or long paths, extending famous results of Erdős and Gallai. Results include bounds on the size of such objects as well as stability theorems about the structure of extremal and "almost" extremal objects.

To my parents.

Acknowledgments

I would first like to thank my advisor, Sasha Kostochka. I am very grateful for his kindness, patience, knowledge, and professionalism.

There are also a number of professors throughout my graduate and undergraduate careers that have helped and inspired me: Zoltán Füredi, Jozsi Balogh, Anush Tserunyan, Doug West, Cun-Quan Zhang, David Kinderlehrer, Philipp Hieronymi, Wes Pegden, Misha Lavrov, and Steve Bradlow. Thank you for your encouragement and your conversations. I have learned so much from all of you.

I would like to thank the many friends that I have made through mathematics, among them: Ravi Donepudi, Derek Jung, Adam Wagner, Anton Bernshteyn, Josh Wen, George Shakan, Yong Escobar, Lina Li, Xujun Liu, Zhiyu Wang, Mike Tait, Nigel Pynn-Coates, Xiao Li, Ningchuan Zhang, Minh Tran, and William Balderrama. Finally, thank you to Gryphon Drake, my best friend.

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LIST OF ABBREVIATIONS AND NOTATIONS

$\emptyset$	the empty set
$[n]$	for $n \in \mathbb{N}$, $[n] := \{1, \ldots, n\}$
$\alpha(G)$	the independence number of a graph G
$\chi(G)$	the chromatic number of a graph G
$E(G)$	the edge set of a (hyper)graph G
$e(G)$	$e(G) := \|E(G)\|$
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Z}$	the set of integers
$V(G)$	the vertex set of a (hyper)graph G
$v(G)$	$v(G) := \|V(G)\|$
$c(G)$	the circumference of G, i.e., the length of a longest cycle in G
K_n	the complete graph on n vertices
C_n	the cycle on n vertices
P_n	the path on n vertices
$N(v)$	the neighborhood of a vertex, i.e., $\{u \in V(G) : uv \in E(G)\}$
$d(v)$	the degree of a vertex, i.e., $\|N(v)\|$

Chapter 1

Introduction

Extremal combinatorics studies the maximum size of a finite object that satisfies a given set of constraints. In other words, what is a bound on the size of an object such that anything larger must contain some "bad" property. Some typical questions in extremal combinatorics are the following:

1. What is the maximum size of a family of subsets in $\{1, \ldots, n\}$ such that that no two subsets are disjoint?
2. How large is a largest subset of $\{1, \ldots, n\}$ that does not contain an increasing sequence of k integers such that consecutive elements have the same difference?
3. How many people must one invite to a party such that there exists either k people who either all know each other or all don't know each other?

The first question is the subject of the Erdős-Ko-Rado Theorem [EKR61] for intersecting families, an important result in extremal set theory. The second question is answered by Szemerédi's Theorem [Sze75] for arithmetic progressions, for which Szemerédi proved his famous Regularity Lemma [Sze78]. Notably, this question also attracted the attention of mathematicians working in harmonic analysis, number theory, ergodic theory, and probability, although Szemerédi's proof was purely combinatorial in nature. The last question is the central question of Ramsey theory [Ram29], a well studied topic in combinatorics which has its roots in model theory.

In this thesis, we study extremal problems in graphs and hypergraphs. A cornerstone of extremal graph theory is the Turán problem. Let F be a fixed graph. We say a graph G is F-free if it does not contain a subgraph isomorphic to G. Then we denote

$$\mathrm{ex}(n, F) = \max\{e(G) : G \text{ is an } n\text{-vertex, } F\text{-free graph}\}.$$

The first result of this nature was Mantel's theorem [Man07] which states that every graph with more than $n^2/4$ edges must contain a triangle (a copy of a K_3). Furthermore, the complete bipartite graph with partite sets of equal size has $n^2/4$ and no triangles. Mantel's theorem was later generalized by Turán.

Theorem 1 (Turán [Tur41]). *For $n > r \geq 3$, the n-vertex K_r-free graph with the maximum number of edges is complete $(r-1)$-partite. In particular, we have $ex(n, K_r) = (1 - \frac{1}{r-1})\frac{n^2}{2}$.*

We call $\mathrm{ex}(n, F)$ the *Turán number of F*. Thus any n-vertex graph with more than $ex(n, F)$ edges must contain a copy of F. We say that an n-vertex F-free graph G is *extremal* if it is F-free and with exactly $ex(n, F)$ edges.

In the case of Turán's theorem, the extremal K_r-free graph is the complete, balanced $(r-1)$-partite graph.

We will also consider the *stability* of the extremal examples. Roughly speaking, we study the structure of n-vertex F-free graphs with "almost" $ex(n, F)$ edges. Stability occurs if all such graphs have structure "close to" that of one of the extremal examples. For instance, we have the following stability version of Turán's theorem.

Theorem 2 (Erdős–Simonovits [ES66]). *For every $\epsilon > 0$, there exists a $\delta > 0$ and n_0 such that for every $n \geq n_0$, if G is an n-vertex, K_r-free graph with at least $(1 - \frac{1}{r-1} - \delta)\frac{n^2}{2}$ edges, then G has an induced subgraph on at least $(1 - \epsilon)n$ vertices that is $(r-1)$-partite.*

In Chapter 2, we study *nonhamiltonian graphs*, i.e., n-vertex graphs containing no cycles of length n. Conditions for forcing hamiltonicity is a classical topic of study. In particular, hamiltonicity is a *monotone* property—that is, any graph that is obtained from a hamiltonian graph by adding more edges is also hamiltonian. In this manner, most nonhamiltonian graphs have "few" edges. Our results extend a theorem of Erdős [Erd62b], giving a characterization of the nonhamiltonian graphs with "many" edges. Furthermore, for some fixed subgraph T, we determine the nonhamiltonian graphs maximizing the number of copies of T. This notion of forbidding some subgraph F and counting the maximum number of copies of another graph T has recently been popularized by Alon and Shikhelman [AS16] and a number of other papers.

In Chapters 3 and 4, we consider graphs without long cycles, i.e., graphs with bounded circumference, beginning with the classical Erdős–Gallai theorem. Again we have that containing long cycles is a monotone property. We first prove a stability theorem for graphs with bounded circumference, extending a result of [FKV16]. Our results give a complete characterization of all such graphs. Next, we count the maximum number of cliques in graphs without long cycles. In particular, we show that the graphs with bounded circumference that maximize the number of edges also maximize the number of copies of K_r's for any $r \geq 3$. These results imply analogous results for graphs without long paths.

After proving the cliques-counting result for graphs with bounded circumference, it was thought that the result may have application in counting the maximum number of hyper-edges in hypergraphs without long cycles. We explore this in Chapters 5-8, using a general notion of cycles in hypergraphs due to Berge. The notion of a Berge cycle in a hypergraph can also be generalized to other so-called Berge subgraphs in hypergraphs. This was first introduced recently by Gerbner and Palmer [GP17]. For some graph F, we study the extremal hypergraphs which forbid copies of Berge F's. In particular, we explore reductions from extremal hypergraph problems to extremal graph problems.

In the final chapters, Chapters 6-8, we return to the study of hypergraphs with bounded circumference, finding sharp bounds for the maximum number edges in r-uniform hypergraphs without Berge cycles of length k or longer. In particular, these bounds depend on the relationship between r and k. Chapter 6 is devoted to the case where r is small compared to k, and Chapter 7 when r is large. Finally, in Chapter 8, we study 2-connected hypergraphs with bounded circumference, obtaining an upper bound for the number of edges that is significantly smaller than the case without conditions on connectivity.

1.1 Turán numbers for bipartite graphs

The Turán numbers for graphs with chromatic number at least 3 are known, up to a small error term.

Theorem 3 (Erdős–Stone–Simonovits [ES46, ES66])**.** *Let F be any graph with chromatic number $\chi(F) \geq 3$. Then $e(G) \leq (1 - \frac{1}{r-1})\binom{n}{2} + o(n^2)$.*

The remaining interesting case, graphs with $\chi(F) = 2$ includes several classes of graphs such as paths, (even) cycles, complete bipartite graphs, trees, etc. In particular, determining the Turán number of even cycles is a well studied open problem.

Problem 4. *Determine* $\mathrm{ex}(n, C_{2k})$.

An upper bound of $O(n^{1+1/k})$ was first proved by Erdős in an unpublished manuscript. Lower bounds of matching order were proved for $k = 2, 3, 5$. However it is still open to prove matching lower bounds for all other k.
In [EG59], Erdős and Gallai determined $ex(n, P_k)$, where P_k is the path on k vertices.

Theorem 5 (Erdős and Gallai [EG59])**.** *For $k \geq 3$, $ex(n, P_k) \leq \frac{1}{2}(k-2)n$.*

This result can be proved as a corollary of a stronger theorem.

Theorem 6 (Erdős and Gallai [EG59])**.** *Let G be an n-vertex graph with more than $\frac{1}{2}(k-1)(n-1)$ edges, $k \geq 3$. Then G contains a cycle of length at least k.*

To obtain the result for paths, suppose G is an n-vertex graph with no copy of P_k. Add a new vertex v adjacent to all vertices in G, and let this new graph be G'. Then G' is an $n+1$-vertex graph with no cycle of length $k+1$ or longer, and so $e(G)+n = e(G') \leq \frac{1}{2}kn$ edges.

Both results are sharp with the following extremal examples: for Theorem 6, when $k-2$ divides $n-1$, take any connected n-vertex graph whose blocks (maximal connected subgraphs with no cut vertices) are cliques of order $k-1$. For Corollary 5, when $k-1$ divides $n-1$, take the n-vertex graph whose connected components are cliques of order $k-1$.

There have been several alternate proofs and sharpenings of the Erdős-Gallai theorem including results by Woodall [Woo76], Lewin [Lew75], Faudree and Schelp[FS75b, FS75a], and Kopylov [Kop77] – see [FS13] for further details.

The strongest version was that of Kopylov who improved the Erdős–Gallai bound for 2-connected graphs. To state the theorem, we first introduce the family of extremal graphs. Fix $k \geq 4$, $n \geq k$, $\frac{k}{2} > a \geq 1$. Define the n-vertex graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three sets A, B, C such that $|A| = a$, $|B| = n-k+a$ and $|C| = k-2a$ and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$.

Note that when $a \geq 2$, $H_{n,k,a}$ is 2-connected, has no cycle of length k or longer, and $e(H_{n,k,a}) = \binom{k-a}{2} + (n-k+a)a$.

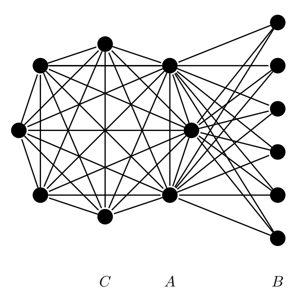

Figure 1.1: $H_{14,11,3}$

Theorem 7 (Kopylov [Kop77]). *Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. If G is a 2-connected n-vertex graph with*

$$e(G) \geq \max\{e(H_{n,k,2}, e(H_{n,k,t})\},$$

then either G has a cycle of length at least k, or $G = H_{n,k,2}$, or $G = H_{n,k,t}$.

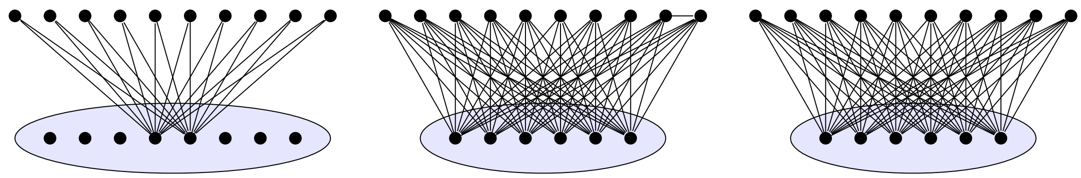

Figure 1.2: $H_{n,k,2}, H_{n,k,t}(k = 2t+1), H_{n,k,t}(k = 2t+2)$; ovals denote complete subgraphs of sizes $k-2$, t, and t, respectively.

1.2 Main theorems

1.2.1 Results for nonhamiltonian graphs

In 1961, Ore [Ore61] determined the Turán number of the hamiltonian cycle: $ex(n, C_n) = \binom{n-1}{2} + 1$. The extremal example is a clique of order $n-1$ and one vertex of degree 1. In this example, we see that there cannot exist a hamiltonian cycle because each vertex in a cycle must have degree at least 2.

In [Erd62b], Erdős showed that if G is an n-vertex, nonhamiltonian graph with minimum degree at least d, then

$$e(G) \leq \max\{\binom{n-d}{2} + d^2, \binom{\lceil (n+1)/2 \rceil}{2} + \lfloor (n-1)/2 \rfloor^2\}.$$

Furthermore the graphs $H_{n,n,d}$ and $H_{n,n,\lfloor (n-1)/2 \rfloor}$ are nonhamiltonian with $\binom{n-d}{2} + d^2$ and $\binom{\lceil (n+1)/2 \rceil}{2} + \lfloor (n-1)/2 \rfloor^2$ edges respectively.

We show a stability version of Erdős' theorem. Let $K'_{n,d}$ be the graph composed of a K_{n-d} and a K_{d+1} sharing exactly one vertex.

Theorem 8 (Füredi, Kostochka, Luo [FKL17])**.** *Let $n \geq 3$ and $d \leq \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ such that*

$$e(G) > \max\{\binom{n-d-1}{2} + (d+1)^2, \binom{\lceil (n+1)/2 \rceil}{2} + \lfloor (n-1)/2 \rfloor^2\}.$$

Then G is a subgraph of either $H_{n,n,d}$ or $K'_{n,d}$.

Next, we consider a recently popular generalization of the Turán number (see [AS16], for instance). For graphs T, F, and G, let $N(G, T)$ denote the number of copies of T in G, and let $ex(n, T, F)$ denote the maximum number of copies of a graph T in an n-vertex, F-free graph.

We show that among all sufficiently large nonhamiltonian graphs with minimum degree at least d, $H_{n,n,d}$ not only maximizes the number of edges but also the number of any fixed small subgraph.

Theorem 9 (Füredi, Kostochka, Luo [FKL18b]). *For every graph T with $t := |V(T)| \geq 3$, any $d \in \mathbb{N}$, and any $n \geq n_0(d,t) := 4dt + 3d^2 + 5t$, if G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$, then $N(G,T) \leq N(H_{n,n,d},T)$.*

Finally, we show one more "step" of stability, akin to Theorem 8. In this version, we find stability not only in the number of edges but for the number of cliques of any size. Roughly speaking, we give a characterization of all nonhamiltonian graphs with minimum degree at least d and *almost maximal* number of copies of cliques.

Theorem 10 (Füredi, Kostochka, Luo [FKL18b]). *Let $n \geq 3$ and $1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ such that there exists $k \geq 2$ for which*

$$N(G,K_k) > \max\{\binom{n-d-2}{k} + (d+2)\binom{d+2}{k-1}, \binom{\lceil (n+1)/2 \rceil}{k} + \lfloor (n-1)/2 \rfloor \binom{\lfloor (n-1)/2 \rfloor}{k-1}\}.$$

Then G is a subgraph of one of 7 extremal graphs.

See Chapter 2 for the proofs of this Theorems 8, 9, and 10.

1.2.2 Results for graphs with bounded circumference

In Chapters 3 and 4, we prove two extensions of the Erdős–Gallai theorem. In particular, we prove a stability version and a clique-counting version of Kopylov's theorem.

Previously, Füredi, Kostochka, and Verstraëte [FKV16] proved a stability version of Kopylov's theorem for large graphs. They gave a characterization for all 2-connected graphs without cycles of length k or longer and *almost* the maximum number of edges. Namely they showed that every such graph must be contained as a subgraph in $H_{n,k,\lfloor (k-1)/2 \rfloor}$ or must contain a large star forest. Their result applied to graphs with at least $n \geq 3k/2$ vertices.

In [FKLV18], we modified their original approach to extend the result for all n, with the extra condition that dense graphs with bounded circumference may also be subgraphs of $H_{n,k,2}$.

Theorem 11 (Füredi, Kostochka, Luo, Verstraëte [FKLV18]). *Let $t \geq 4$ and $k \in \{2t+1, 2t+2\}$, so that $k \geq 9$. If G is a 2-connected graph on $n \geq k$ vertices and $c(G) < k$, then either $e(G) \leq \max\{e(H_{n,k,t-1}), e(H_{n,k,3})\}$ or*

(*a*) $k = 2t+1$ *and* $G \subseteq H_{n,k,t}$ *or*

(*b*) $k = 2t+2$ *and* $G - A$ *is a star forest for some* $A \subseteq V(G)$ *of size at most* t.

(*c*) $G \subseteq H_{n,k,2}$.

See Chapter 3 for the proof of Theorem 11.
Next, we study a special case of the $ex(n, T, F)$ where $T = K_s$. As a generalization of the Erdős–Gallai theorem, we show that the same graphs without cycles of length k or longer that maximize the number of edges also maximize the number of cliques of any size.
Our main theorem which implies this is a generalization of Kopylov's theorem for 2-connected graphs.

Theorem 12 (Luo [Luo18]). *Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. If G is a 2-connected n-vertex graph with circumference less than k, then*

$$N(G, K_s) \leq \max\{N(H_{n,k,2}, K_s), N(H_{n,k,t}, K_s)\}.$$

This result implies the analogous result for non-2-connected graphs as well as graphs without long paths.
See Chapter 4 for the proof of Theorem 12.

1.2.3 Results for hypergraphs with bounded circumference

In the final half of this thesis, we study hypergraphs without long Berge cycles. A *Berge cycle* of length ℓ in a hypergraph is a set of ℓ vertices $\{v_1, \ldots, v_\ell\}$ and ℓ edges $\{e_1, \ldots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo ℓ. Note that some other notions of cycles in hypergraphs include so-called loose or tight cycles, both of which are special cases of Berge cycles.
Similarly, a *Berge path* of length ℓ is a set of $\ell + 1$ vertices $\{v_1, \ldots, v_{\ell+1}\}$ and ℓ edges $\{e_1, \ldots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for $i \leq \ell$.
In [GKL16], Győri, Katona, and Lemons proved an analogue of the Erdős–Gallai theorem for paths in hypergraphs. They gave a bound for the maximum number of edges in an r-uniform graph without a Berge path of length k. For r small compared to k, they showed that the extremal hypergraphs were the natural hypergraph version of the extremal examples of graphs without paths of length k (i.e., components isomorphic to $K_k^{(r)}$). The large r case differed slightly—the extremal examples were hypergraphs where each component contained $r + 1$ vertices and $k - 1$ edges.
We prove an analogous result for hypergraphs without Berge cycles of length k or longer. Again, our results are broken up into the small r case and the large r case.

Theorem 13 (Füredi, Kostochka, Luo [FKL18a]). *Let $r \geq 3$ and $k \geq r+3$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$.*

Furthermore, the bound $\frac{n-1}{k-2}\binom{k-1}{r}$ is achieved whenever $\mathcal{H}$ is a hypergraph with each block isomorphic to $K_{k-1}^{(r)}$.

See Chapter 6 for a proof of Theorem 13.

Theorem 14 (Kostochka, Luo [KL18]). *Let $k \geq 3$ and $r \geq k+1$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq \frac{(k-1)(n-1)}{r}$.*

Furthermore, the bound $\frac{(k-1)(n-1)}{r}$ is achieved whenever $\mathcal{H}$ is a hypergraph with each block containing $r+1$ vertices and $k-1$ edges.
See Chapter 7 for a proof of Theorem 14.
Finally, we consider the case of 2-connected hypergraphs. In the graph case, we get a stronger upper bound for the number of edges in 2-connected graphs with bounded circumference than in non-2-connected graphs. It turns out that the same phenomenon holds for hypergraphs, with a significantly better bound.
For our notation, v is a *cut vertex* of a connected hypergraph if the hypergraph obtained by removing v and shrinking all edges that contained v is disconnected. We say that a e is a *cut edge* of a connected hypergraph if the hypergraph obtained by removing e is disconnected. We say that a connected hypergraph is *2-connected* if it contains neither a cut vertex nor a cut edge.
Our result holds when the bound on the circumference is large compared to the uniformity of the hypergraph, and when the number of vertices is sufficiently large.

Theorem 15 (Füredi, Kostochka, Luo [FKL19]). *Let $k \geq 4r \geq 12$. There exists some $n_{k,r}$ such that if $n \geq n_{k,r}$ and $\mathcal{H}$ is an n-vertex 2-connected r-graph with no Berge cycle of length k or longer, then*

$$e(\mathcal{H}) \leq \binom{\lceil (k+1)/2 \rceil}{r} + (n - \lceil (k+1)/2 \rceil)\binom{\lfloor (k-1)/2 \rfloor}{r-1}.$$

See Chapter 8 for a proof of Theorem 15.

Chapter 2

Graphs without Hamiltonian cycles

As a special case of the bounded circumference problem, we first consider n-vertex graphs without cycles of length n (i.e., nonhamiltonian graphs). In particular, we prove Theorems 8, 9, and 10. The results of this chapter are joint work with Zoltán Füredi and Alexandr Kostochka [FKL17, FKL18b].

2.1 Introduction

Ore [Ore61] proved the following Turán-type result:

Theorem 16 (Ore [Ore61]). *If G is a nonhamiltonian graph on n vertices, then $e(G) \leq \binom{n-1}{2} + 1$.*

This bound is achieved only for the n-vertex graph obtained from the complete graph K_{n-1} by adding a vertex of degree 1. Erdős [Erd62b] refined the bound in terms of the minimum degree of the graph:

Theorem 17 (Erdős [Erd62b]). *Let n, d be integers with $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, and set $h(n,d) := \binom{n-d}{2} + d^2$. If G is a nonhamiltonian graph on n vertices with minimum degree $\delta(G) \geq d$, then*

$$e(G) \leq \max\left\{h(n,d), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\} =: e(n,d).$$

This bound is sharp for all $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

To show the sharpness of the bound, for $n, d \in \mathbb{N}$ with $d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, consider the graph $H_{n,d}$ obtained from a copy of K_{n-d}, say with vertex set A, by adding d vertices of degree d each of which is adjacent to the same d vertices in A. An example of $H_{11,3}$ is below.
By construction, $H_{n,d}$ has minimum degree d, is nonhamiltonian, and $e(H_{n,d}) = \binom{n-d}{2} + d^2 = h(n,d)$. Elementary calculation shows that $h(n,d) > h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)$ in the range $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ if and only if $d < (n+1)/6$ and n is odd or $d < (n+4)/6$ and n is even. Hence there

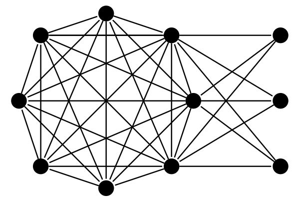

Figure 2.1: $H_{11,3}$

exists a $d_0 := d_0(n)$ such that

$$e(n,1) > e(n,2) > \cdots > e(n,d_0) = e(n,d_0+1) = \cdots = e(n, \left\lfloor \frac{n-1}{2} \right\rfloor),$$

where $d_0(n) := \lceil \frac{n+1}{6} \rceil$ if n is odd, and $d_0(n) := \lceil \frac{n+4}{6} \rceil$ if n is even. Let $H'_{n,d}$ denote the graph that is an edge-disjoint union of two complete graphs K_{n-d} and K_{d+1} sharing one vertex.

2.2 A stability theorem for dense nonhamiltonian graphs

We first present the a refinement of Theorem 17. The following is a refinement of the statement of Theorem 8.

Theorem 18. *Let $n \geq 3$ and $d \leq \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ such that*

$$e(G) > e(n,d+1) = \max\left\{h(n,d+1), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\}. \tag{2.1}$$

(So we have $d < d_0(n)$.) Then G is a subgraph of either $H_{n,d}$ or $H'_{n,d}$.

This is a stability result in the sense that for $d < n/6$, each 2-connected, nonhamilitonian n-vertex graph with minimum degree at least d and "close" to $h(n,d)$ edges is a subgraph of the extremal graph $H_{n,d}$. Note that $h(n,d) - h(n,d+1) = n - 3d - 2$ is at least $n/2$ for $d < d_0 - 1$. Note also that $e(H'_{n,d}) > e(n,d+1)$ only when $d = O(\sqrt{n})$.
We will use the following well-known theorems of Pósa.

Theorem 19 (Pósa [P62]). *Let $n \geq 3$. If G is a nonhamiltonian n-vertex graph, then there exists $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ such that G has a set of k vertices with degree at most k.*

Theorem 20 (Pósa [P63]). *Let $n \geq 3$, $1 \leq \ell < n$ and let G be an n-vertex graph such that $d(u) + d(v) \geq n + \ell$ for every non-edge uv in G. Then for every linear forest F with ℓ edges contained in G, the graph G has a hamiltonian cycle containing all edges of F.*

2.3 Proof of Theorem 18

Call a graph G *saturated* if G is nonhamiltonian but for each $uv \notin E(G)$, $G+uv$ has a hamiltonian cycle. Ore's proof [Ore61] of Dirac's Theorem [Dir52] yields that

$$\textit{for every } n\textit{-vertex saturated graph } G \textit{ and for each } uv \notin E(G),\ d(u)+d(v) \leq n-1. \tag{2.2}$$

First we show two facts on saturated graphs with many edges.

Lemma 21. *Let G be a saturated n-vertex graph with $e(G) > h(n, \lfloor \frac{n-1}{2} \rfloor)$. Then for some $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, $V(G)$ contains a subset D of k vertices of degree at most k such that $G-D$ is a complete graph.*

Proof. Since G is nonhamiltonian, by Theorem 19, there exists some $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ such that G has k vertices with degree at most k. Pick the maximum such k, and let D be the set of the vertices with degree at most k. Since $e(G) > h(n, \lfloor \frac{n-1}{2} \rfloor)$, $k < \lfloor \frac{n-1}{2} \rfloor$. So, by the maximality of k, $|D| = k$.
Suppose there exist $x, y \in V(G) - D$ such that $xy \notin E(G)$. Among all such pairs, choose x and y with the maximum $d(x)$. Since $y \notin D$, $d(y) > k$. Let $D' := V(G) - N(x) - \{x\}$ and $k' := |D'| = n-1-d(x)$. By (2.2),

$$d(z) \leq n-1-d(x) = k' \ \textit{ for all } \ z \in D'. \tag{2.3}$$

So D' is a set of k' vertices of degree at most k'. Since $y \in D'$, $k' \geq d(y) > k$. Thus by the maximality of k, we get $k' = n-1-d(x) > \lfloor \frac{n-1}{2} \rfloor$. Equivalently, $d(x) < \lceil \frac{n-1}{2} \rceil$. For all $z \in D' + \{x\}$, either $z \in D$ where $d(z) \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, or $z \in V(G) - D$, and so $d(z) \leq d(x) \leq \lfloor \frac{n-1}{2} \rfloor$. It follows that $e(G) \leq h(n, \lfloor \frac{n-1}{2} \rfloor)$, a contradiction. □

Lemma 22. *Under the conditions of Lemma 21, if $k = \delta(G)$, then $G = H_{n,\delta(G)}$ or $G = H'_{n,\delta(G)}$.*

Proof. Set $d := \delta(G)$, and let D be a set of d vertices with degree at most d. Let $u \in D$. Since $\delta(G) \geq |D| = d$, u has a neighbor $w \in V(G) - D$. Consider any $v \in D - \{u\}$. By Lemma 21, w is adjacent to all of $V(G) - D - \{w\}$. It also is adjacent to u, therefore its degree is at least $n-d$. We obtain

$$d(w) + d(v) \geq (n-d) + d = n.$$

Then by (2.2), w is adjacent to v, and hence w is adjacent to all vertices of D.

Let W be the set of vertices in $V(G) - D$ having a neighbor in D. We have obtained that $W \neq \emptyset$ and

$$N(u) \cap (V(G) - D) = W \text{ for all } u \in D. \tag{2.4}$$

Let $G' = G[D \cup W]$. If $|W| = 1$, then $G = H'_{n,d}$. If $|V(G')| = 2d$, then by (2.4), each vertex $u \in D$ has the same d neighbors in $V(G) - D$. Because $d(u) = d$, D is an independent set. Thus $G = H_{n,d}$. Otherwise, $d + 2 \le |V(G')| \le 2d - 1$, $|D| \ge 2$.
Fix a pair of vertices $w_1, w_2 \in W$. For any $x, y \in V(G')$,

$$d(x) + d(y) \ge d + d \ge |V(G')| + 1.$$

Therefore by Theorem 20, G' has a hamiltonian cycle C that uses the edge w_1w_2. Since $G'' := G - (V(G') - \{w_1, w_2\})$ is a complete graph, it contains a hamiltonian w_1, w_2-path P. Then $P \cup (C - w_1w_2)$ is a hamiltonian cycle of G, a contradiction. $\square$

Proof of Theorem 18. Suppose that an n-vertex, nonhamiltonian graph G satisfies the constraints of Theorem 18 for some $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$. We may assume G is saturated, since if a graph containing G is a subgraph of $H_{n,d}$ or $H'_{n,d}$, then G is as well.
By Lemma 21, G has a set D of $k \le \lfloor \frac{n-1}{2} \rfloor$ vertices with degree at most k such that $G - D$ is a complete graph. Therefore $e(G) \le \binom{n-k}{2} + k^2 = h(n,k)$. If $k > d$, then $e(G) \le \max\{h(n, d+1), h(n, \lfloor \frac{n-1}{2} \rfloor)\} = e(n, d+1)$, a contradiction. Thus $k \le d$. Furthermore, $k \ge \delta(G) \ge d$, and hence $k = d$. Also, since $e(G) > h(n, \lfloor \frac{n-1}{2} \rfloor))$, we have $d + 1 \le d_0(n) \le (n+8)/6$. Applying Lemma 22 completes the proof. $\square$

2.4 Counting subgraphs in nonhamiltonian graphs

One of the main results of this section shows that when n is large enough with respect to d and t, among then n-vertex nonhamiltonian graphs with minimum degree at least d, $H_{n,d}$ not only has the most edges but also contains the most copies of *any* t-vertex graph. This is an instance of a generalization of the Turán problem called *subgraph density problem*: for $n \in \mathbb{N}$ and graphs F and H, let $ex(n, F, H)$ denote the maximum possible number of (unlabeled) copies of F in an n-vertex H-free graph. When $F = K_2$, we have the usual extremal number $ex(n, F, H) = ex(n, H)$.
Some notable results on the function $ex(n, F, H)$ for various combinations of F and H were obtained in [Erd62a, BG08, AS16, Grz12, HHK$^+$13, FO17]. In particular, Erdős [Erd62a] determined $ex(n, K_s, K_t)$, Bollobás and Győri [BG08] found the order of magnitude of $ex(n, C_3, C_5)$, Alon and Shikhelman [AS16] presented a series of bounds on $ex(n, F, H)$ for

different classes of F and H.
In this chapter, we study the maximum number of copies of F in nonhamiltonian n-vertex graphs, i.e. $ex(n, F, C_n)$. For two graphs G and F, let $N(G, F)$ denote the number of *labeled* copies of F that are subgraphs of G, i.e., the number of injections $\phi : V(F) \to V(G)$ such that for each $xy \in E(F)$, $\phi(x)\phi(y) \in E(G)$. Since for every F and H, $|Aut(F)|\, ex(n, F, H)$ is the maximum of $N(G, F)$ over the n-vertex graphs G not containing H, some of our results are in the language of labeled copies of F in G. For $k \in \mathbb{N}$, let $N_k(G)$ denote the number of unlabeled copies of K_k's in G. Since $|Aut(K_k)| = k!$, we have $N_k(G) = N(G, K_k)/k!$.

2.5 Results for counting subgraphs

As an extension of Theorem 17, we show that for each fixed graph F and any d, if n is large enough with respect to $|V(F)|$ and d, then among all n-vertex nonhamiltonian graphs with minimum degree at least d, $H_{n,d}$ contains the maximum number of copies of F.
The following is a refinement of the statement of Theorem 9.

Theorem 23. *For every graph F with $t := |V(F)| \geq 3$, any $d \in \mathbb{N}$, and any $n \geq n_0(d, t) := 4dt + 3d^2 + 5t$, if G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$, then $N(G, F) \leq N(H_{n,d}, F)$.*

On the other hand, if F is a star $K_{1,t-1}$ and $n \leq dt - d$, then $H_{n,d}$ does not maximize $N(G, F)$. At the end of Section 2.7 we show that in this case, $N(H_{n,\lfloor (n-1)/2 \rfloor}, F) > N(H_{n,d}, F)$. So, the bound on $n_0(d, t)$ in Theorem 23 has the right order of magnitude when $d = O(t)$.
An immediate corollary of Theorem 23 is the following generalization of Theorem 16

Corollary 24. *For every graph F with $t := |V(F)| \geq 3$ and any $n \geq n_0(t) := 9t + 3$, if G is an n-vertex nonhamiltonian graph, then $N(G, F) \leq N(H_{n,1}, F)$.*

We consider the case that F is a clique in more detail. For $n, k \in \mathbb{N}$, define on the interval $[1, \lfloor (n-1)/2 \rfloor]$ the function

$$h_k(n, x) := \binom{n-x}{k} + x\binom{x}{k-1}. \tag{2.5}$$

We use the convention that for $a \in \mathbb{R}$, $b \in \mathbb{N}$, $\binom{a}{b}$ is the polynomial $\frac{1}{b!} a \times (a-1) \times \ldots \times (a-b+1)$ if $a \geq b - 1$ and 0 otherwise.
By considering the second derivative, one can check that for any fixed k and n, $h_k(n, x)$ as a function of x is convex on $[1, \lfloor (n-1)/2 \rfloor]$, hence it attains its maximum at one of

the endpoints, $x = 1$ or $x = \lfloor (n-1)/2 \rfloor$. When $k = 2$, $h_2(n, x) = h(n, x)$. We prove the following generalization of Theorem 17.

Theorem 25. *Let n, d, k be integers with $1 \le d \le \left\lfloor \frac{n-1}{2} \right\rfloor$ and $k \ge 2$. If G is a nonhamiltonian graph on n vertices with minimum degree $\delta(G) \ge d$, then the number $N_k(G)$ of k-cliques in G satisfies*

$$N_k(G) \le \max\left\{ h_k(n, d), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\}.$$

Again, graphs $H_{n,d}$ and $H_{n,\lfloor (n-1)/2 \rfloor}$ are sharpness examples for the theorem.
Finally, we present a stability version of Theorem 25. To state the result, we first define the family of extremal graphs.
Fix $d \le \lfloor (n-1)/2 \rfloor$. In addition to graphs $H_{n,d}$ and $K'_{n,d}$ defined above, define $H'_{n,d}$: $V(H'_{n,d}) = A \cup B$, where A induces a complete graph on $n-d-1$ vertices, B is a set of $d+1$ vertices that induce exactly one edge, and there exists a set of vertices $\{a_1, \ldots, a_d\} \subseteq A$ such that for all $b \in B$, $N(b) - B = \{a_1, \ldots, a_d\}$. Note that contracting the edge in $H'_{n,d}[B]$ yields $H_{n-1,d}$. These graphs are illustrated below.

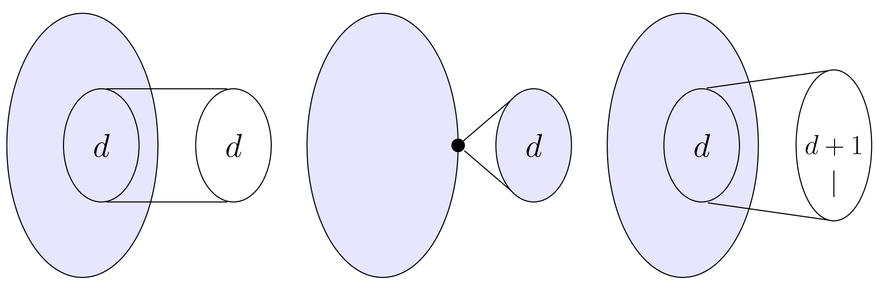

Figure 2.2: Graphs $H_{n,d}$ (left), $K'_{n,d}$ (center), and $H'_{n,d}$ (right), where shaded background indicates a complete graph.

We also have two more extremal graphs for the cases $d = 2$ or $d = 3$. Define the nonhamiltonian n-vertex graph $G'_{n,2}$ with minimum degree 2 as follows: $V(G'_{n,2}) = A \cup B$ where A induces a clique or order $n-3$, $B = \{b_1, b_2, b_3\}$ is an independent set of order 3, and there exists $\{a_1, a_2, a_3, x\} \subseteq A$ such that $N(b_i) = \{a_i, x\}$ for $i \in \{1, 2, 3\}$ (see the graph on the left in Fig. 3).
The nonhamiltonian n-vertex graph $F_{n,3}$ with minimum degree 3 has vertex set $A \cup B$, where A induces a clique of order $n-4$, B induces a perfect matching on 4 vertices, and each of the vertices in B is adjacent to the same two vertices in A (see the graph on the right in Fig. 3).
The following is a refinement of the statement of Theorem 10.

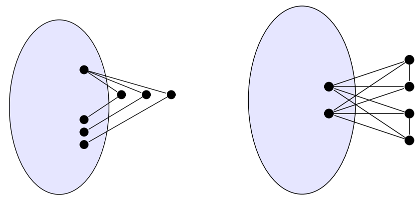

Figure 2.3: Graphs $G'_{n,2}$ (left) and $F_{n,3}$ (right).

Theorem 26. *Let $n \geq 3$ and $1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ such that there exists $k \geq 2$ for which*

$$N_k(G) > \max\left\{ h_k(n, d+2), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\}. \tag{2.6}$$

Let $\mathcal{H}_{n,d} := \{H_{n,d}, H_{n,d+1}, K'_{n,d}, K'_{n,d+1}, H'_{n,d}\}$.
(i) If $d = 2$, then G is a subgraph of $G'_{n,2}$ or of a graph in $\mathcal{H}_{n,2}$;
(ii) if $d = 3$, then G is a subgraph of $F_{n,3}$ or of a graph in $\mathcal{H}_{n,3}$;
(iii) if $d = 1$ or $4 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$, then G is a subgraph of a graph in $\mathcal{H}_{n,d}$.

The result is sharp because $H_{n,d+2}$ has $h_k(n, d+2)$ copies of K_k, minimum degree $d+2 > d$, is nonhamiltonian and is not contained in any graph in $\mathcal{H}_{n,d} \cup \{G'_{n,2}, F_{n,3}\}$.
The outline for the rest of the chapter is as follows: in Section 6 we present some structural results for graphs that are edge-maximal nonhamiltonian to be used in the proofs of the main theorems, in Section 7 we prove Theorem 23, in Section 8 we prove Theorem 25 and give a cliques version of Theorem 18, and in Section 9 we prove Theorem 26.

2.6 Structural results for saturated nonhamiltonian graphs

We will use two structural results for saturated graphs.

Lemma 27. *Let G be a saturated n-vertex graph with $N_k(G) > h_k(n, \lfloor \frac{n-1}{2} \rfloor)$ for some $k \geq 2$. Then for some $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$, $V(G)$ contains a subset D of r vertices of degree at most r such that $G - D$ is a complete graph.*

Proof. Since G is nonhamiltonian, by Theorem 19, there exists some $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$ such that G has r vertices with degree at most r. Pick the maximum such r, and let D be the set of the vertices with degree at most r. Since $N_k(G) > h_k(n, \lfloor \frac{n-1}{2} \rfloor)$, $r < \lfloor \frac{n-1}{2} \rfloor$. So, by the maximality of r, $|D| = r$.
Suppose there exist $x, y \in V(G) - D$ such that $xy \notin E(G)$. Among all such pairs, choose x and y with the maximum $d(x)$ and subject to this, the maximum $d(y)$. Let $D' :=$

$V(G) - N(x) - \{x\}$. Consider any vertex $z \in D'$. If $z \in D$, then $d(z) \leq r < d(y)$. If $z \notin D$, then $d(z) \leq d(y)$ by the choice of y. So D' is a set of $n-1-d(x)$ vertices of degree at most $d(y)$. By (2.2), $|D'| \geq d(y)$. By the maximality of r, we have $d(y) > \lfloor (n-1)/2 \rfloor$. Since $d(x) \geq d(y)$, we get $d(x) + d(y) \geq 2d(y) \geq n$, contradicting (2.2). $\square$

2.7 Maximizing the number of copies of a given graph and a proof of Theorem 23

In order to prove Theorem 23, we first show that for any fixed graph F and any d, if n is large then of the two extremal graphs in Lemma 22, $H_{n,d}$ contains at least as many copies of F as $K'_{n,d}$.

Lemma 28. *For any $d, t, n \in \mathbb{N}$ with $n \geq 2dt + d + t$ and any graph F with $t = |V(F)|$ we have $N(K'_{n,d}, F) \leq N(H_{n,d}, F)$.*

Proof. Fix F and $t = |V(F)|$. Let $K'_{n,d} = A \cup B$ where A and B are cliques of order $n-d$ and $d+1$ respectively and $A \cap B = \{v^*\}$, the cut vertex of $K'_{n,d}$. Also, let D denote the independent set of order d in $H_{n,d}$. We may assume $d \geq 2$, because $H_{n,1} = K'_{n,1}$. If x is an isolated vertex of F then for any n-vertex graph G we have $N(G,F) = (n-t+1)N(G, F-x)$. So it is enough to prove the case $\delta(F) \geq 1$, and we may also assume $t \geq 3$.
Because both $K'_{n,d}[A]$ and $H_{n,d} - D$ are cliques of order $n-d$, the number of embeddings of F into $K'_{n,d}[A]$ is the same as the number of embeddings of F into $H_{n,d} - D$. So it remains to compare only the number of embeddings in $\Phi := \{\varphi : V(F) \to V(K'_{n,d})$ such that $\varphi(F)$ intersects $B - v^*\}$ to the number of embeddings in $\Psi := \{\psi : V(F) \to V(H_{n,d})$ such that $\psi(F)$ intersects $D\}$.
Let $C \cup \overline{C}$ be a partition of the vertex set $V(F)$, $s := |C|$. Define the following classes of Φ and Ψ

— $\Phi(C) := \{\varphi : V(F) \to V(K'_{n,d})$ such that $\varphi(C)$ intersects $B - v^*$, $\varphi(C) \subseteq B$, and $\varphi(\overline{C}) \subseteq V - B\}$,

— $\Psi(C) := \{\psi : V(F) \to V(H_{n,d})$ such that $\psi(C)$ intersects D, $\psi(C) \subseteq (D \cup N(D))$, and $\psi(\overline{C}) \subseteq V - (D \cup N(D))\}$.

By these definitions, if $C \neq C'$ then $\Phi(C) \cap \Phi(C') = \emptyset$, and $\Psi(C) \cap \Psi(C') = \emptyset$. Also $\bigcup_{\emptyset \neq C \subseteq V(F)} \Phi(C) = \Phi$. We claim that for every $C \neq \emptyset$,

$$|\Phi(C)| \leq |\Psi(C)|. \tag{2.7}$$

Summing up the number of embeddings over all choices for C will prove the lemma. If $\Phi(C) = \emptyset$, then (2.7) obviously holds. So from now on, we consider the cases when $\Phi(C)$ is not empty, implying $1 \le s \le d+1$.

Case 1: There is an F-edge joining $\overline{C}$ and C. So there is a vertex $v \in C$ with $N_F(v) \cap \overline{C} \ne \emptyset$. Then for every mapping $\varphi \in \Phi(C)$, the vertex v must be mapped to v^* in $K'_{n,d}$, $\varphi(v) = v^*$. So this vertex v is uniquely determined by C. Also, $\varphi(C) \cap (B - v^*) \ne \emptyset$ implies $s \ge 2$. The rest of C can be mapped arbitrarily to $B - v^*$ and $\overline{C}$ can be mapped arbitrarily to $A - v^*$. We obtained that $|\Phi(C)| = (d)_{s-1}(n-d-1)_{t-s}$.

To obtain a lower bound for $|\Psi(C)|$, we construct mappings $\psi \in \Psi(C)$ as follows. Let $\psi(v) = x \in N(D)$ (there are d possibilities), then map some vertex of $C - v$ to a vertex $y \in D$ (there are $(s-1)d$ possibilities). Since $N + y$ forms a clique of order $d+1$ we may embed the rest of C into $N - v$ in $(d-1)_{s-2}$ ways and finish embedding of F into $H_{n,d}$ by arbitrarily placing the vertices of $\overline{C}$ to $V - (D \cup N(D))$. We obtained that $|\Psi(C)| \ge d^2(s-1)(d-1)_{s-2}(n-2d)_{t-s} = d(s-1)(d)_{s-1}(n-2d)_{t-s}$.

Since $s \ge 2$ we have that

$$
\begin{aligned}
\frac{|\Psi(C)|}{|\Phi(C)|} \ge \frac{d(s-1)(d)_{s-1}(n-2d)_{t-s}}{(d)_{s-1}(n-d-1)_{t-s}} &\ge d(2-1)\left(\frac{n-2d+1-t+s}{n-d-t+s}\right)^{t-s} \\
&= d\left(1 - \frac{d-1}{n-d-t+s}\right)^{t-s} \\
&\ge d\left(1 - \frac{(d-1)(t-s)}{n-d-t+s}\right) \\
&\ge d\left(1 - \frac{(d-1)t}{n-d-t}\right) \\
&> 1 \text{ when } n > dt + d + t.
\end{aligned}
$$

Case 2: C and $\overline{C}$ are not connected in F. We may assume $s \ge 2$ since C is a union of components with $\delta(F) \ge 1$. In $K'_{n,d}$ there are at exactly $(d+1)_s(n-d-1)_{t-s}$ ways to embed F into B so that only C is mapped into B and $\overline{C}$ goes to $A - v^*$, i.e., $|\Phi(C)| = (d+1)_s(n-d-1)_{t-s}$.

To obtain a lower bound for $|\Psi(C)|$, we construct mappings $\psi \in \Psi(C)$ as follows. Select any vertex $v \in C$ and map it to some vertex in D (there are sd possibilities), then map $C - v$ into $N(D)$ (there are $(d)_{s-1}$ possibilities) and finish embedding of F into $H_{n,d}$ by arbitrarily placing the vertices of $\overline{C}$ to $V - (D \cup N(D))$. We obtained that $|\Psi(C)| \ge ds(d)_{s-1}(n-2d)_{t-s}$.

We have

$$
\begin{aligned}
\frac{|\Psi(C)|}{|\Phi(C)|} \ge \frac{ds(d)_{s-1}(n-2d)_{t-s}}{(d+1)_s(n-d-1)_{t-s}} &\ge \frac{ds}{d+1}\left(1 - \frac{(d-1)t}{n-d-t}\right) \\
&\ge \frac{2d}{d+1}\left(1 - \frac{(d-1)t}{n-d-t}\right) \text{ because } s \ge 2 \\
&> 1 \text{ when } n > 2dt + d + t.
\end{aligned}
$$

$\square$

We are now ready to prove Theorem 23.

Theorem 23. *For every graph F with $t := |V(F)| \geq 3$, any $d \in \mathbb{N}$, and any $n \geq n_0(d,t) := 4dt + 3d^2 + 5t$, if G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$, then $N(G,F) \leq N(H_{n,d},F)$.*

Proof. Let $d \geq 1$. Fix a graph F with $|V(F)| \geq 3$ (if $|V(F)| = 2$, then either $F = K_2$ or $F = \overline{K}_2$). The case where G has isolated vertices can be handled by induction on the number of isolated vertices, hence we may assume each vertex has degree at least 1. Set

$$n_0 = 4dt + 3d^2 + 5t. \tag{2.8}$$

Fix a nonhamiltonian graph G with $|V(G)| = n \geq n_0$ and $\delta(G) \geq d$ such that $N(G,F) > N(H_{n,d},F) \geq (n-d)_t$. We may assume that G is saturated, as the number of copies of F can only increase when we add edges to G.

Because $n \geq 4dt + t$ by (2.8),

$$\begin{aligned} \frac{(n-d)_t}{(n)_t} &\geq \left(\frac{n-d-t}{n-t}\right)^t = \left(1 - \frac{d}{n-t}\right)^t \\ &\geq 1 - \frac{dt}{n-t} \geq 1 - \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

So, $(n-d)_t \geq \frac{3}{4}(n)_t$.

By mapping edge xy of F to an edge of G in two labeled ways, we get that $N(G,F)$ satisfies

$$2e(G)(n-2)_{t-2} \geq N(G,F) \geq (n-d)_t \geq \frac{3}{4}(n)_t,$$

This yields the loose upper bound

$$e(G) \geq \frac{3}{4}\binom{n}{2} > h_2(n, \lfloor (n-1)/2 \rfloor). \tag{2.9}$$

By Pósa's theorem (Theorem 19), there exists some $d \leq r \leq \lfloor (n-1)/2 \rfloor$ such that G contains a set R of r vertices with degree at most r. Furthermore by (2.9), $r < d_0$. So by integrality, $r \leq d_0 - 1 \leq (n+3)/6$. If $r = d$, then by Lemma 22, either $G = H_{n,d}$ or $G = K'_{n,d}$. By Lemma 28 and (2.8), $G = H_{n,d}$, a contradiction. So we have $r \geq d+1$.

Let $\mathcal{I}$ denote the family of all nonempty independent sets in F. For $I \in \mathcal{I}$, let $i = i(I) := |I|$ and $j = j(I) := |N_F(I)|$. Since F has no isolated vertices, $j(I) \geq 1$ and so $i \leq t-1$ for each $I \in \mathcal{I}$. Let $\Phi(I)$ denote the set of embeddings $\varphi : V(F) \to V(G)$ such that $\phi(I) \subseteq R$ and I is a maximum independent subset of $\phi^{-1}(R \cap \varphi(F))$. Note that $\varphi(I)$ is not necessarily independent in G. We show that

$$|\Phi(I)| \le (r)_i r(n-r)_{t-i-1}. \tag{2.10}$$

Indeed, there are $(r)_i$ ways to choose $\phi(I) \subseteq R$. After that, since each vertex in R has at most r neighbors in G, there are at most r^j ways to embed $N_F(I)$ into G. By the maximality of I, all vertices of $F - I - N_F(I)$ should be mapped to $V(G) - R$. There are at most $(n-r)_{t-i-j}$ to do it. Hence $|\Phi(I)| \le (r)_i r^j (n-r)_{t-i-j}$. Since $2r+t \le 2(d_0-1)+t < n$, this implies (2.10).

Since each $\varphi : V(F) \to V(G)$ with $\varphi(V(F)) \cap R \ne \emptyset$ belongs to $\Phi(I)$ for some nonempty $I \in \mathcal{I}$, (2.10) implies

$$N(G,F) \le (n-r)_t + \sum_{\emptyset \ne I \in \mathcal{I}} |\Phi(I)| \le (n-r)_t + \sum_{i=1}^{t-1} \binom{t}{i} (r)_i r (n-r)_{t-i-1}. \tag{2.11}$$

Hence

$$\begin{aligned}
\frac{N(G,F)}{N(H_{n,d},F)} &\le \frac{(n-r)_t + \sum_{i=1}^{t-1}\binom{t}{i}(r)_i r(n-r)_{t-i-1}}{(n-d)_t} \\
&\le \frac{(n-r)_t}{(n-d)_t} + \frac{1}{(n-d)_t} \times \frac{r}{n-r-t+2} \sum_{i=1}^{t-1} \binom{t}{i} (r)_i (n-r)_{t-i} \\
&= \frac{(n-r)_t}{(n-d)_t} + \frac{(n)_t - (n-r)_t - (r)_t}{(n-d)_t} \times \frac{r}{n-r-t+2} \\
&\le \frac{(n-r)_t}{(n-d)_t} \times \frac{n-t+2-2r}{n-t+2-r} + \frac{(n)_t}{(n-d)_t} \times \frac{r}{n-t+2-r} := f(r).
\end{aligned}$$

Given fixed n, d, t, we claim that the real function $f(r)$ is convex for $0 < r < (n-t+2)/2$. Indeed, the first term $g(r) := \frac{(n-r)_t}{(n-d)_t} \times \frac{n-t+2-2r}{n-t+2-r}$ is a product of t linear terms in each of which r has a negative coefficient (note that the $n-t+2-r$ term cancels out with a factor of $n-r-t+2$ in $(n-r)_t$). Applying product rule, the first derivative g' is a sum of t products, each with $t-1$ linear terms. For $r < (n-t+2)/2$, each of these products is negative, thus $g'(r) < 0$. Finally, applying product rule again, g'' is the sum of $t(t-1)$ products. For $r < (n-t+2)/2$ each of the products is positive, thus $g''(r) > 0$.

Similarly, the second factor of the second term (as a real function of r, of the form $r/(c-r)$) is convex for $r < n-t+2$.

We conclude that in the interval $[d+1, (n+3)/6]$ the function $f(r)$ takes its maximum either at one of the endpoints $r = d+1$ or $r = (n+3)/6$. We claim that $f(r) < 1$ at both end points.

In case of $r = d+1$ the first factor of the first term equals $(n-d-t)/(n-d)$. To get an upper bound for the first factor of the second term one can use the inequality $\prod(1+x_i) < 1 + 2\sum x_i$ which holds for any number of non-negative x_i's if $0 < \sum x_i \leq 1$. Because $dt/(n-d-t+1) \leq 1$ by (2.8), we obtain that

$$\begin{aligned}
f(d+1) &< \frac{n-d-t}{n-d} \times \frac{n-t-2d}{n-t-d+1} + \left(1 + \frac{2dt}{n-d-t+1}\right) \times \frac{d+1}{n-t-d+1} \\
&= \left(1 - \frac{t}{n-d}\right) \times \left(1 - \frac{d+1}{n-t-d+1}\right) + \left(\frac{d+1}{n-t-d+1}\right) + \left(\frac{2dt(d+1)}{(n-t-d+1)^2}\right) \\
&= 1 - \frac{t}{n-d} + \frac{t}{n-d} \times \frac{d+1}{n-t-d+1} + \frac{t}{n-d} \times \frac{2d(d+1)}{n-t-d+1} \times \frac{n-d}{n-t-d+1} \\
&= 1 - \frac{t}{n-d} \times \left(1 - \frac{d+1}{n-t-d+1} - \frac{2d(d+1)}{n-t-d+1} \times \left(1 + \frac{t-1}{n-t-d+1}\right)\right) \\
&< 1 - \frac{t}{n-d} \times (1 - \frac{1}{4t} - \frac{2}{3}(1 + \frac{1}{4d})) \\
&\leq 1 - \frac{t}{n-d} \times (1 - 1/12 - 2/3 \times 5/4) \\
&< 1.
\end{aligned}$$

Here we used that $n \geq 3d^2 + 2d + t$ and $n \geq 4dt + 5t + d$ by (2.8), $t \geq 3$, and $d \geq 1$. To bound $f(r)$ for other values of r, let us use $1 + x \leq e^x$ (true for all x). We get

$$f(r) < \exp\left\{-\frac{(r-d)t}{n-d-t+1}\right\} + \frac{r}{n-r-t+2} \times \exp\left\{\frac{dt}{n-d-t+1}\right\}.$$

When $r = (n+3)/6$, $t \geq 3$, and $n \geq 24d$ by (2.8), the first term is at most $e^{-18/46} = 0.676...$. Moreover, for $n \geq 9t$ (2.8) (therefore $n \geq 27$) we get that $\frac{r}{n-r-t+2}$ is maximized when t is maximized, i.e., when $t = n/9$. The whole term is at most $(3n+9)/(13n+27) \times e^{1/4} \leq 5/21 \times e^{1/4} = 0.305...$, so in this range, $f((n+3)/6) < 1$.
By the convexity of $f(r)$, we have $N(G,F) < N(H_{n,d},F)$. □

When F is a star, then it is easy to determine $\max N(G,F)$ for all n.

Claim 29. *Suppose $F = K_{1,t-1}$ with $t := |V(F)| \geq 3$, and $t \leq n$ and d are integers with $1 \leq d \leq \lfloor (n-1)/2 \rfloor$. If G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$, then*

$$N(G,F) \leq \max\left\{N(H_{n,d},F), N(H_{n,\lfloor (n-1)/2 \rfloor},F)\right\}, \tag{2.12}$$

and equality holds if and only if $G \in \left\{H_{n,d}, H_{n,\lfloor (n-1)/2 \rfloor}\right\}$.

Proof. The number of copies of stars in a graph G depends only on the degree sequence of the graph: if a vertex v of a graph G has degree $d(v)$, then there are $(d(v))_{t-1}$ labeled copies of F in G where v is the center vertex. We have

$$N(G,F) = \sum_{v\in V(G)} \binom{d(v)}{t-1}. \tag{2.13}$$

Since G is nonhamiltonian, Pósa's theorem yields an $r \le \lfloor (n-1)/2 \rfloor$, and an r-set $R \subset V(G)$ such that $d_G(v) \le r$ for all $v \in R$. Take the minimum such r, then there exists a vertex $v \in R$ with $\deg(v) = r$. We may also suppose that G is edge-maximal nonhamiltonian, so Ore's condition (2.2) holds. It implies that $\deg(w) \le n-r-1$ for all $w \notin N(v)$. Altogether we obtain that G has r vertices of degree at most r, at least $n-2r$ vertices (those in $V(G) - R - N(v)$) of degree at most $(n-r-1)$. This implies that the right hand side of (2.13) is at most

$$r \times (r)_{t-1} + (n-2r) \times (n-r-1)_{t-1} + r \times (n-1)_{t-1} = N(H_{n,r}, F).$$

(Here equality holds only if $G = H_{n,r}$). Note that $r \in [d, \lfloor \frac{1}{2}(n-1) \rfloor]$. Since for given n and t the function $N(H_{n,r}, F)$ is strictly convex in r, it takes its maximum at one of the endpoints of the interval. $\square$

Remark 30. *As it was mentioned in Section 2.5, $O(dt)$ is the right order for $n_0(d,t)$ when $d = O(t)$.*

To see this, fix $d \in \mathbb{N}$ and let F be the star on $t \ge 3$ vertices. If $d < \lfloor (n-1)/2 \rfloor$, $t \le n$ and $n \le dt - d$, then $H_{n,\lfloor (n-1)/2 \rfloor}$ contains more copies of F than $H_{n,d}$ does, the maximum in (2.12) is reached for $r = \lfloor (n-1)/2 \rfloor$. We present the calculation below only for $2d+7 \le n \le dt-d$, the case $2d+3 \le n \le 2d+6$ can be checked by hand by plugging n into the first line of the formula below. We can proceed as follows.

$$
\begin{aligned}
N(H_{n,\lfloor (n-1)/2\rfloor}, F) - N(H_{n,d}, F) &= \Big(\lfloor (n-1)/2\rfloor (n-1)_{t-1} + \lceil (n+1)/2\rceil (\lfloor (n-1)/2\rfloor)_{t-1}\Big) \\
&\quad -\Big(d(n-1)_{t-1} + (n-2d)(n-d-1)_{t-1} + d(d)_{t-1}\Big) \\
&= \Big(\lfloor (n-1)/2\rfloor - d\Big)(n-1)_{t-1} - (n-2d)(n-d-1)_{t-1} \\
&\quad +\lceil (n+1)/2\rceil (\lfloor (n-1)/2\rfloor)_{t-1} - d(d)_{t-1} \\
&> \Big(\lfloor (n-1)/2\rfloor - d\Big)(n-1)_{t-1} \\
&\quad -\Big((n-2d)(1-d/n)^{t-1}\Big)(n-1)_{t-1} \\
&> (n-1)_{t-1}\Big(\lfloor (n-1)/2\rfloor - d - (n-2d)e^{-(dt-d)/n}\Big) \\
&\geq (n-1)_{t-1}\left(\lfloor (n-1)/2\rfloor - d - (n-2d)/e\right) \\
&\geq 0.
\end{aligned}
$$

2.8 Theorem 25 and a stability version of it

In general, it is difficult to calculate the exact value of $N(H_{n,d}, F)$ for a fixed graph F. However, when $F = K_k$, we have $N(H_{n,d}, K_k) = h_k(n,d)k!$. Recall Theorem 25:
Let n, d, k be integers with $1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$ and $k \geq 2$. If G is a nonhamiltonian graph on n vertices with minimum degree $\delta(G) \geq d$, then

$$N_k(G) \leq \max\left\{h_k(n,d), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\}.$$

Proof of Theorem 25. By Theorem 19, because G is nonhamiltonian, there exists an $r \geq d$ such that G has r vertices of degree at most r. Denote this set of vertices by D. Then $N_k(G - D) \leq \binom{n-r}{k}$, and every vertex in D is contained in at most $\binom{r}{k-1}$ copies of K_k. Hence $N_k(G) \leq h_k(n,r)$. The theorem follows from the convexity of $h_k(n,x)$. $\square$
Our older stability theorem (Theorem 18) also translates into the the language of cliques, giving a stability theorem for Theorem 25:

Theorem 31. *Let $n \geq 3$, and $d \leq \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ and there exists a $k \geq 2$ such that*

$$N_k(G) > \max\left\{h_k(n,d+1), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\}. \tag{2.14}$$

Then G is a subgraph of either $H_{n,d}$ or $K'_{n,d}$.

Proof. Take an edge-maximum counterexample G (so we may assume G is saturated). By Lemma 27, G has a set D of $r \le \lfloor (n-1)/2 \rfloor$ vertices such that $G - D$ is a complete graph. If $r \ge d+1$, then $N_k(G) \le \max\left\{h_k(n, d+1), h_k(n, \lfloor \frac{n-1}{2} \rfloor)\right\}$. Thus $r = d$, and we may apply Lemma 22. □

2.9 Discussion and proof of Theorem 26

What happens when we consider n-vertex nonhamiltonian graphs with minimum degree at least d and less than $e(n, d+1)$ but more than $e(n, d+2)$ edges?
Note that for $d < d_0(n) - 2$,

$$e(n, d) - e(n, d+2) = 2n - 6d - 7,$$

which is greater than n. Theorem 26 answers the question above in a more general form—in terms of k-cliques instead of edges. In other words, we classify all n-vertex nonhamiltonian graphs with more than $\max\left\{h_k(n, d+2), h_k(n, \lfloor \frac{n-1}{2} \rfloor)\right\}$ copies of K_k.
As in Lemma 31, such G can be a subgraph of $H_{n,d}$ or $K'_{n,d}$. Also, G can be a subgraph of $H_{n,d+1}$ or $K'_{n,d+1}$. Recall the graphs $H_{n,d}, K'_{n,d}, H'_{n,d}, G'_{n,2}$, and $F_{n,3}$ defined earlier:

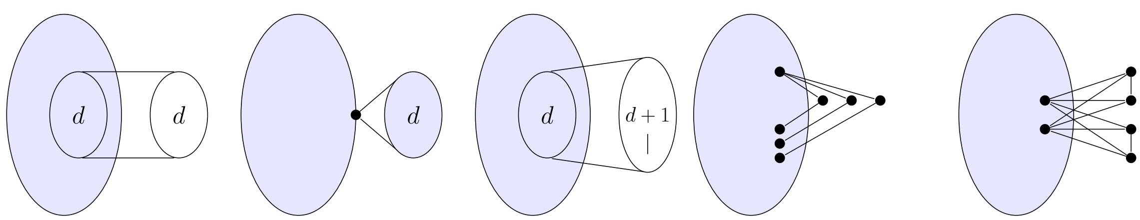

Figure 2.4: Graphs $H_{n,d}, K'_{n,d}, H'_{n,d}, G'_{n,2}$, and $F_{n,3}$.

Theorem 26. *Let $n \ge 3$ and $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \ge d$ such that exists a $k \ge 2$ for which*

$$N_k(G) > \max\left\{h_k(n, d+2), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\}.$$

Let $\mathcal{H}_{n,d} := \{H_{n,d}, H_{n,d+1}, K'_{n,d}, K'_{n,d+1}, H'_{n,d}\}$.
(i) If $d = 2$, then G is a subgraph of $G'_{n,2}$ or of a graph in $\mathcal{H}_{n,2}$;
(ii) if $d = 3$, then G is a subgraph of $F_{n,3}$ or of a graph in $\mathcal{H}_{n,3}$;
(iii) if $d = 1$ or $4 \le d \le \lfloor \frac{n-1}{2} \rfloor$, then G is a subgraph of a graph in $\mathcal{H}_{n,d}$.

Proof. Suppose G is a counterexample to Theorem 26 with the most edges. Then G is saturated. In particular, degree condition (2.2) holds for G. So by Lemma 27, there exists

an $d \le r \le \lfloor (n-1)/2 \rfloor$ such that $V(G)$ contains a subset D of r vertices of degree at most r and $G - D$ is a complete graph.
If $r \ge d+2$, then because $h_k(n,x)$ is convex, $N_k(G) \le h_k(n,r) \le \max\left\{h_k(n,d+2), h_k(n,\lfloor\frac{n-1}{2}\rfloor)\right\}$. Therefore either $r = d$ or $r = d+1$. In the case that $r = d$ (and so $r = \delta(G)$), Lemma 22 implies that $G \subseteq H_{n,d}$. So we may assume that $r = d+1$.
If $\delta(G) \ge d+1$, then we simply apply Theorem 18 with $d+1$ in place of d and get $G \subseteq H_{n,d+1}$ or $G \subseteq K'_{n,d+1}$. So, from now on we may assume

$$\delta(G) = d. \tag{2.15}$$

Now (2.15) implies that our theorem holds for $d = 1$, since each graph with minimum degree exactly 1 is a subgraph of $H_{n,1}$. So, below $2 \le d \le \lfloor\frac{n-1}{2}\rfloor$.
Let $N := N(D) - D \subseteq V(G) - D$. The next claim will be used many times throughout the proof.

Lemma 32. *(a) If there exists a vertex $v \in D$ such that $d(v) = d+1$, then $N(v) - D = N$. (b) If there exists a vertex $u \in N$ such that u has at least 2 neighbors in D, then u is adjacent to all vertices in D.*

Proof. If $v \in D$, $d(v) = d+1$ and some $u \in N$ is not adjacent to v, then $d(v) + d(u) \ge d + 1 + (n-d-2) + 1 = n$. A contradiction to (2.2) proves (a).
Similarly, if $u \in N$ has at least 2 neighbors in D but is not adjacent to some $v \in D$, then $d(v) + d(u) \ge d + (n-d-2) + 2 = n$, again contradicting (2.2). □

Define $S := \{u \in V(G) - D : u \in N(v) \text{ for all } v \in D\}$, $s := |S|$, and $S' := V(G) - D - S$. By Lemma 32 (b), each vertex in S' has at most one neighbor in D. So, for each $v \in D$, call the neighbors of v in S' *the private neighbors of* v.

We claim that

$$D \textit{ is not independent.} \tag{2.16}$$

Indeed, assume D is independent. If there exists a vertex $v \in D$ with $d(v) = d+1$, then by Lemma 32 (a), $N(v) - D = N$. So, because D is independent, $G \subseteq H_{n,d+1}$. Assume now that every vertex $v \in D$ has degree d, and let $D = \{v_1, \dots, v_{d+1}\}$.
If $s \ge d$, then because each $v_i \in D$ has degree d, $s = d$ and $N = S$. Then $G \subseteq H_{n,d+1}$. If $s \le d-2$, then each vertex $v_i \in D$ has at least two private neighbors in S'; call these private neighbors x_{v_i} and y_{v_i}. The path $x_{v_1}v_1y_{v_1}x_{v_2}v_2y_{v_2}\dots x_{v_{d+1}}v_{d+1}y_{v_{d+1}}$ contains all vertices in D and can be extended to a hamiltonian cycle of G, a contradiction.
Finally, suppose $s = d-1$. Then every vertex $v_i \in D$ has exactly one private neighbor. Therefore $G = G'_{n,d}$ where $G'_{n,d}$ is composed of a clique A of order $n-d-1$ and an

independent set $D = \{v_1, \ldots, v_{d+1}\}$, and there exists a set $S \subset A$ of size $d-1$ and distinct vertices $z_1, \ldots, z_{d+1}$ such that for $1 \leq i \leq d+1$, $N(v_i) = S \cup z_i$. Graph $G'_{n,d}$ is illustrated in Fig. 5.

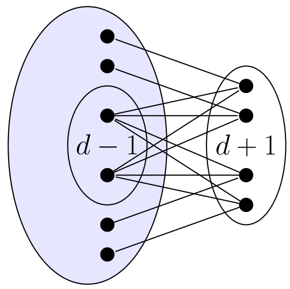

Figure 2.5: $G'_{n,d}$.

For $d = 2$, we conclude that $G \subseteq G'_{n,2}$, as claimed, and for $d \geq 3$, we get a contradiction since $G'_{n,d}$ is hamiltonian. This proves (2.16).

Call a vertex $v \in D$ *open* if it has at least two private neighbors, *half-open* if it has exactly one private neighbor, and *closed* if it has no private neighbors.
We say that *paths* $P_1, \ldots, P_q$ *partition* D, if these paths are vertex-disjoint and $V(P_1) \cup \ldots \cup V(P_q) = D$. The idea of the proof is as follows: because $G - D$ is a complete graph, each path with endpoints in $G - D$ that covers all vertices of D can be extended to a hamiltonian cycle of G. So such a path does not exist, which implies that too few paths cannot partition D:

Lemma 33. *If $s \geq 2$ then the minimum number of paths in $G[D]$ partitioning D is at least s.*

Proof. Suppose D can be partitioned into $\ell \leq s-1$ paths $P_1, \ldots, P_\ell$ in $G[D]$. Let $S = \{z_1, \ldots, z_s\}$. Then $P = z_1 P_1 z_2 \ldots z_\ell P_\ell z_{\ell+1}$ is a path with endpoints in $V(G) - D$ that covers D. Because $V(G) - D$ forms a clique, we can find a $z_1, z_{\ell+1}$ - path P' in $G - D$ that covers $V(G) - D - \{z_2, \ldots, z_\ell\}$. Then $P \cup P'$ is a hamiltonian cycle of G, a contradiction. $\square$

Sometimes, to get a contradiction with Lemma 33 we will use our information on vertex degrees in $G[D]$:

Lemma 34. *Let H be a graph on r vertices such that for every nonedge xy of H, $d(x) + d(y) \geq r - t$ for some t. Then $V(H)$ can be partitioned into a set of at most t paths. In other words, there exist t disjoint paths $P_1, \ldots, P_t$ with $V(H) = \bigcup_{i=1}^{t} V(P_i)$.*

Proof. Construct the graph H' by adding a clique T of size t to H so that every vertex of T is adjacent to each vertex in $V(H)$. For each nonedge $x, y \in H'$,

$$d_{H'}(x) + d_{H'}(y) \geq (r - t) + t + t = r + t = |V(H')|.$$

By Ore's theorem, H' has a hamiltonian cycle C'. Then $C' - T$ is a set of at most t paths in H that cover all vertices of H. $\square$

The next simple fact will be quite useful.

Lemma 35. *If $G[D]$ contains an open vertex, then all other vertices are closed.*

Proof. Suppose $G[D]$ has an open vertex v and another open or half-open vertex u. Let v', v'' be some private neighbors of v in S' and u' be a neighbor of u in S'. By the maximality of G, graph $G + vu'$ has a hamiltonian cycle. In other words, G has a hamiltonian path $v_1v_2 \dots v_n$, where $v_1 = v$ and $v_n = u'$. Let $V' = \{v_i : vv_{i+1} \in E(G)\}$. Since G has no hamiltonian cycle, $V' \cap N(u') = \emptyset$.
Since $d(v) + d(u') = n - 1$, we have $V(G) = V' \cup N(u') + u'$. Suppose that $v' = v_i$ and $v'' = v_j$. Then $v_{i-1}, v_{j-1} \in V'$, and $v_{i-1}, v_{j-1} \notin N(u')$. But among the neighbors of v_i and v_j, only v is not adjacent to u', a contradiction. $\square$

Now we show that S is non-empty and not too large.

Lemma 36. $s \geq 1$.

Proof. Suppose $S = \emptyset$. If D has an open vertex v, then by Lemma 35, all other vertices are closed. In this case, v is the only vertex of D with neighbors outside of D, and hence $G \subseteq K'_{n,d}$, in which v is the cut vertex. Also if D has at most one half-open vertex v, then similarly $G \subseteq K'_{n,d}$.
So suppose that D contains no open vertices but has two half-open vertices u and v with private neighbors z_u and z_v respectively. Then $\delta(G[D]) \geq d - 1$. By Pósa's Theorem, if $d \geq 4$, then $G[D]$ has a hamiltonian v, u-path. This path together with any hamiltonian z_u, z_v-path in the complete graph $G - D$ and the edges uz_u and vz_v forms a hamiltonian cycle in G, a contradiction.
If $d = 3$, then by Dirac's Theorem, $G[D]$ has a hamiltonian cycle, i.e. a 4-cycle, say C. If we can choose our half-open v and u consecutive on C, then $C - uv$ is a hamiltonian v, u-path in $G[D]$, and we finish as in the previous paragraph. Otherwise, we may assume that $C = vxuy$, where x and y are closed. In this case, $d_{G[D]}(x) = d_{G[D]}(y) = 3$, thus $xy \in E(G)$. So we again have a hamiltonian v, u-path, namely $vxyu$, in $G[D]$. Finally, if $d = 2$, then $|D| = 3$, and $G[D]$ is either a 3-vertex path whose endpoints are half-open or a 3-cycle. In both cases, $G[D]$ again has a hamiltonian path whose ends are half-open. $\square$

Lemma 37. $s \leq d-3$.

Proof. Since by (2.15), $\delta(G) = d$, we have $s \leq d$. Suppose $s \in \{d-2, d-1, d\}$.
Case 1: All vertices in D have degree d.
Case 1.1: $s = d$. Then $G \subseteq H_{n,d+1}$.
Case 1.2: $s = d-1$. In this case, each vertex in graph $G[D]$ has degree 0 or 1. By (2.16), $G[D]$ induces a non-empty matching, possibly with some isolated vertices. Let m denote the number of edges in $G[D]$.
If $m \geq 3$, then the number of components in $G[D]$ is less than s, contradicting Lemma 33. Suppose now $m = 2$, and the edges in the matching are x_1y_1 and x_2y_2. Then $d \geq 3$. If $d = 3$, then $D = \{x_1, x_2, y_1, y_2\}$ and $G = F_{n,3}$ (see Fig 3 (right)). If $d \geq 4$, then $G[D]$ has an isolated vertex, say x_3. This x_3 has a private neighbor $w \in S'$. Then $|S + w| = d$ which is more than the number of components of $G[D]$ and we can construct a path from w to S visiting all components of $G[D]$.
Finally, suppose $G[D]$ has exactly one edge, say x_1y_1. Recall that $d \geq 2$. Graph $G[D]$ has $d-1$ isolated vertices, say $x_2, \dots, x_d$. Each of x_i for $2 \leq i \leq d$ has a private neighbor u_i in S'. Let $S = \{z_1, \dots, z_{d-1}\}$. If $d = 2$, then $S = \{z_1\}$, $N(D) = \{z_1, u_2\}$ and hence $G \subset H'_{n,2}$. So in this case the theorem holds for G. If $d \geq 3$, then G contains a path $u_dx_dz_{d-1}x_{d-1}z_{d-2}x_{d-2}\dots z_2x_1y_1z_1x_2u_2$ from u_d to u_2 that covers D.
Case 1.3: $s = d-2$. Since $s \geq 1$, $d \geq 3$. Every vertex in $G[D]$ has degree at most 2, i.e., $G[D]$ is a union of paths, isolated vertices, and cycles. Each isolated vertex has at least 2 private neighbors in S'. Each endpoint of a path in $G[D]$ has one private neighbor in S'. Thus we can find disjoint paths from S' to S' that cover all isolated vertices and paths in $G[D]$ and all are disjoint from S. Hence if the number c of cycles in $G[D]$ is less than $d-2$, then we have a set of disjoint paths from $V(G) - D$ to $V(G) - D$ that cover D (and this set can be extended to a hamiltonian cycle in G). Since each cycle has at least 3 vertices and $|D| = d+1$, if $c \geq d-2$, then $(d+1)/3 \geq d-2$, which is possible only when $d < 4$, i.e. $d = 3$. Moreover, then $G[D] = C_3 \cup K_1$ and $S = N$ is a single vertex. But then $G \subseteq K'_{n,3}$.
Case 2: There exists a vertex $v^* \in D$ with $d(v^*) = d+1$. By Lemma 32 (a), $N = N(v^*) - D$, and so G has at most one open or half-open vertex. Furthermore,

$$\text{if } G \text{ has an open or half-open vertex, then it is } v^*, \text{ and by Lemma 32, there are no other vertices of degree } d+1. \tag{2.17}$$

Case 2.1: $s = d$. If v^* is not closed, then it has a private neighbor $x \in S'$, and the neighborhood of each other vertex of D is exactly S. Furthermore, since $d(v^*) = d+1$, v^* has no neighbors outside of $D + \{x\}$. This implies that D is independent, contradicting (2.16). If v^* is closed (i.e., $N = S$), then $G[D]$ has maximum degree 1. Therefore $G[D]$ is a matching with at least one edge (coming from v^*) plus some isolated vertices. If this matching has

at least 2 edges, then the number of components in $G[D]$ is less than s, contradicting Lemma 33. If $G[D]$ has exactly one edge, then $G \subseteq H'_{n,d}$.
Case 2.2: $s = d-1$. If v^* is open, then $d_{G[D]}(v^*) = 0$ and by (2.17), each other vertex in D has exactly one neighbor in D. In particular, d is even. Therefore $G[D - v^*]$ has $d/2$ components. When $d \geq 3$ and d is even, $d/2 \leq s-1$ and we can find a path from S to S that covers $D - v^*$, and extend this path using two neighbors of v^* in S' to a path from $V(G) - D$ to $V(G) - D$ covering D. Suppose $d = 2$, $D = \{v^*, x, y\}$ and $S = \{z\}$. Then z is a cut vertex separating $\{x, y\}$ from the rest of G, and hence $G \subseteq K'_{n,2}$.
If v^* is half-open, then by (2.17), each other vertex in D is closed and hence has exactly one neighbor in D. Let $x \in S'$ be the private neighbor of v^*. Then $G[D]$ is 1-regular and therefore has exactly $(d+1)/2$ components, in particular, d is odd. If $d \geq 2$ and is odd, then $(d+1)/2 \leq d-1 = s$, and so we can find a path from x to S that covers D.
Finally, if v^* is closed, then by (2.17), every vertex of $G[D]$ is closed and has degree 1 or 2, and v^* has degree 2 in $G[D]$. Then $G[D]$ has at most $\lfloor d/2 \rfloor$ components, which is less than s when $d \geq 3$. If $d = 2$, then $s = 1$ and the unique vertex z in S is a cut vertex separating D from the rest of G. This means $G \subseteq K'_{n,3}$.
Case 2.3: $s = d-2$. Since $s \geq 1$, $d \geq 3$. If v^* is open, then $d_{G[D]}(v^*) = 1$ and by (2.17), each other vertex in D is closed and has exactly two neighbors in D. But this is not possible, since the degree sum of the vertices in $G[D]$ must be even. If v^* is half-open with a neighbor $x \in S'$, then $G[D]$ is 2-regular. Thus $G[D]$ is a union of cycles and has at most $\lfloor (d+1)/3 \rfloor$ components. When $d \geq 4$, this is less than s, contradicting Lemma 33. If $d = 3$, then $s = 1$ and the unique vertex z in S is a cut vertex separating D from the rest of G. This means $G \subseteq K'_{n,4}$.
If v^* is closed, then $d_{G[D]}(v^*) = 3$ and $\delta(G[D]) \geq 2$. So, for any vertices x, y in $G[D]$,

$$d_{G[D]}(x) + d_{G[D]}(y) \geq 4 \geq (d+1) - (d-2-1) = |V(G[D])| - (s-1).$$

By Lemma 34, if $s \geq 2$, then we can partition $G[D]$ into $s-1$ paths $P_1, ..., P_{s-1}$. This would contradict Lemma 33. So suppose $s = 1$ and $d = 3$. Then as in the previous paragraph, $G \subseteq K'_{n,4}$. □

Next we will show that we cannot have $2 \leq s \leq d-3$.

Lemma 38. $s = 1$.

Proof. Suppose $s = d-k$ where $3 \leq k \leq d-2$.
Case 1: $G[D]$ has an open vertex v. By Lemma 35, every other vertex in D is closed. Let

$G' = G[D] - v$. Then $\delta(G') \geq k-1$ and $|V(G')| = d$. In particular, for any $x, y \in D - v$,

$$d_{G'}(x) + d_{G'}(y) \geq 2k - 2 \geq k + 1 = d - (d - k - 1) = |V(G')| - (s - 1).$$

By Lemma 34, we can find a path from S to S in G containing all of $V(G')$. Because v is open, this path can be extended to a path from $V(G) - D$ to $V(G) - D$ including v, and then extended to a hamiltonian cycle of G.

Case 2: D has no open vertices and $4 \leq k \leq d-2$. Then $\delta(G[D]) \geq k-1$ and again for any $x, y \in D$, $d_{G[D]}(x) + d_{G[D]}(y) \geq 2k-2$. For $k \geq 4$, $2k-2 \geq k+2 = (d+1) - (d-k-1) = |D| - (s-1)$. Since $k \leq d-2$, by Lemma 34, $G[D]$ can be partitioned into $s-1$ paths, contradicting Lemma 33.

Case 3: D has no open vertices and $s = d - 3 \geq 2$. If there is at most one half-open vertex, then for any nonadjacent vertices $x, y \in D$, $d_{G[D]}(x) + d_{G[D]}(y) \geq 2 + 3 = 5 \geq (d+1) - (d-3-1)$, and we are done as in Case 2.

So we may assume G has at least 2 half-open vertices. Let D' be the set of half-open vertices in D. If $D' \neq D$, let $v^* \in D - D'$. Define a subset D^- as follows: If $|D'| \geq 3$, then let $D^- = D'$, otherwise, let $D^- = D' + v^*$. Let G' be the graph obtained from $G[D]$ by adding a new vertex w adjacent to all vertices in D^-. Then $|V(G')| = d+2$ and $\delta(G') \geq 3$. In particular, for any $x, y \in V(G'), d_{G'}(x) + d_{G'}(y) \geq 6 \geq (d+2) - (d-3-1) = |V(G')| - (s-1)$. By Lemma 34, $V(G')$ can be partitioned into $s-1$ disjoint paths $P_1, \ldots, P_{s-1}$. We may assume that $w \in P_1$. If w is an endpoint of P_1, then D can also be partitioned into $s-1$ disjoint paths $P_1 - w, P_2, \ldots, P_{s-1}$ in $G[D]$, a contradiction to Lemma 33.

Otherwise, let $P_1 = x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k$ where $x_i = w$. Since every vertex in $(D^-) - v^*$ is half-open and $N_{G'}(w) = D^-$, we may assume that x_{i-1} is half-open and thus has a neighbor $y \in S'$. Let $S = \{z_1, \ldots, z_{d-3}\}$. Then

$$y x_{i-1} x_{i-2} \ldots x_1 z_1 x_{i+1} \ldots x_k z_2 P_2 z_3 \ldots z_{d-4} P_{d-4} z_{d-3}$$

is a path in G with endpoints in $V(G) - D$ that covers D. $\square$

Now we may finish the proof of Theorem 26. By Lemmas 36–38, $s = 1$, say, $S = \{z_1\}$. Furthermore, by Lemma 37,

$$d \geq 3 + s = 4. \tag{2.18}$$

Case 1: D has an open vertex v. Then by Lemma 35, every other vertex of D is closed. Since $s = 1$, each $u \in D - v$ has degree $d - 1$ in $G[D]$. If v has no neighbors in D, then $G[D] - v$ is a clique of order d, and $G \subseteq K'_{n,d}$. Otherwise, since $d \geq 4$, by Dirac's Theorem, $G[D] - v$ has a hamiltonian cycle, say C. Using C and an edge from v to C, we obtain a hamiltonian path P in $G[D]$ starting with v. Let $v' \in S'$ be a neighbor of v. Then $v'Pz_1$ is a path from S' to S that covers D, a contradiction.

Case 2: D has a half-open vertex but no open vertices. It is enough to prove that

$$G[D] \text{ has a hamiltonian path } P \text{ starting with a half-open vertex } v, \tag{2.19}$$

since such a P can be extended to a hamiltonian cycle in G through z_1 and the private neighbor of v. If $d \geq 5$, then for any $x, y \in D$,

$$d_{G[D]}(x) + d_{G[D]}(y) \geq d - 2 + d - 2 = 2d - 4 \geq d + 1 = |V(G[D])|.$$

Hence by Ore's Theorem, $G[D]$ has a hamiltonian cycle, and hence (2.19) holds.

If $d < 5$ then by (2.18), $d = 4$. So $G[D]$ has 5 vertices and minimum degree at least 2. By Lemma 34, we can find a hamiltonian path P of $G[D]$, say $v_1v_2v_3v_4v_5$. If at least one of v_1, v_5 is half-open or $v_1v_5 \in E(G)$, then (2.19) holds. Otherwise, each of v_1, v_5 has 3 neighbors in D, which means $N(v_1) \cap D = N(v_5) \cap D = \{v_2, v_3, v_4\}$. But then $G[D]$ has hamiltonian cycle $v_1v_2v_5v_4v_3v_1$, and again (2.19) holds.

Case 3: All vertices in D are closed. Then $G \subseteq K'_{n,d+1}$, a contradiction. This proves the theorem. $\square$

Chapter 3

A stability theorem for graphs with bounded circumference

3.1 Introduction

In this section, we prove Theorem 11, a stability result for graphs with no cycles of length k or longer. This theorem is a strengthening of the Erdős–Gallai Theorem (Theorem 6) and Kopylov's theorem (Theorem 7). This is joint work with Zoltán Füredi, Alexandr Kostochka, and Jacques Verstraëte [FKLV18].

3.2 Stability results

Recall the definition of graphs $H_{n,k,a}$: let $n \geq k$ and $1 \leq a < \frac{1}{2}k$. The vertex set of $H_{n,k,a}$ is the union of three disjoint sets A, B, and C such that $|A| = a$, $|B| = n - k + a$ and $|C| = k - 2a$, and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$ (Fig. 1 shows $H_{14,11,3}$). Let

$$h(n, k, a) := e(H_{n,k,a}) = \binom{k-a}{2} + a(n - k + a).$$

Kopylov [Kop77] showed that the extremal 2-connected n-vertex graphs with no cycles of

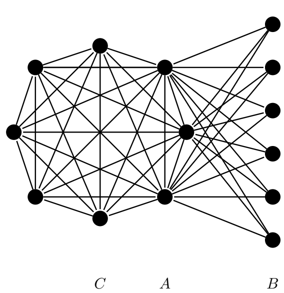

Figure 3.1: $H_{14,11,3}$.

length at least k are $G = H_{n,k,2}$ and $G = H_{n,k,t}$: the first has more edges for small n, and the second has more edges for large n.

Füredi, Kostochka, and Verstraëte proved in [FKV16] a stability version of Theorems 6 and 7 for n-vertex 2-connected graphs with $n \geq 3k/2$, but the problem remained open for $n < 3k/2$ when $k \geq 9$. The main result of [FKV16] was the following:

Theorem 39 (Füredi, Kostochka, Verstraëte [FKV16])**.** *Let $t \geq 2$ and $n \geq 3t$ and $k \in \{2t+1, 2t+2\}$. Let G be a 2-connected n-vertex graph $c(G) < k$. Then $e(G) \leq h(n,k,t-1)$ unless*

- (a) $k = 2t+1$, $k \neq 7$, *and* $G \subseteq H_{n,k,t}$ *or*
- (b) $k = 2t+2$ *or* $k = 7$, *and* $G - A$ *is a star forest for some* $A \subseteq V(G)$ *of size at most* t.

The paper [FKV16] also describes the 2-connected n-vertex graphs G with $e(G) > h(n,k,t-1)$ and $c(G) < k \leq 8$ for all $n \geq k$. In particular, for $k < 8$, each such graph satisfies either (a) or (b) of Theorem 39.

Together with the cases for $k \leq 8$, this result gives a full description of the 2-connected n-vertex graphs G with $c(G) < k$ and 'many' edges for all k and n.

The following is a refinement of the statement of Theorem 11.

Theorem 40. *Let $t \geq 4$ and $k \in \{2t+1, 2t+2\}$, so that $k \geq 9$. If G is a 2-connected graph on $n \geq k$ vertices and $c(G) < k$, then either $e(G) \leq \max\{h(n,k,t-1), h(n,k,3)\}$ or*

- (a) $k = 2t+1$ *and* $G \subseteq H_{n,k,t}$ *or*
- (b) $k = 2t+2$ *and* $G - A$ *is a star forest for some* $A \subseteq V(G)$ *of size at most* t.
- (c) $G \subseteq H_{n,k,2}$.

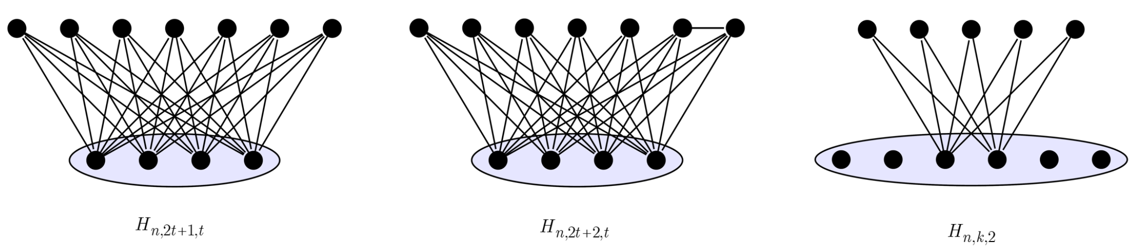

Figure 3.2: Ovals denote complete subgraphs of order t, t, and $k-2$.

Note that

$$h(n,k,t) - h(n,k,t-1) = \begin{cases} n-t-3 & \text{if } k = 2t+1, \\ n-t-5 & \text{if } k = 2t+2, \end{cases}$$

and

$$h(n,k,2) - h(n,k,3) = k - n - 3.$$

We consider the case $e(G) > h(n, k, t-1)$ whenever n is large compared to k (and t), and $e(G) > h(n, k, 3)$ whenever n is small. We state these exact bounds in Section 3.
Also, note that the case $n < k$ is trivial and the case $k \leq 8$ was fully resolved in [FKV16].
We will reuse many slightly modified lemmas from [FKV16] in the proof of the main result. As such, when introducing such lemmas, instead of repeating the proofs word-for-word, we provide brief proof sketches and a reference to the corresponding full proof in [FKV16] for the interested reader.

3.2.1 A more detailed form of the stability result

In order to prove Theorem 40, we need a more detailed description of the graphs satisfying (b) in the theorem that do not contain 'long' cycles. For this, we introduce four families of graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$ that (apart from $\mathcal{G}_1$) are identical to the families introduced in [FKV16]. In the definitions below we use $t = \lfloor (k-1)/2 \rfloor$.
Let $\mathcal{G}_1(n, k) = \{H_{n,k,t}, H_{n,k,2}\}$. Each $G \in \mathcal{G}_2(n, k)$ is defined by a partition $V(G) = A \cup B \cup C$ and two vertices $a_1 \in A$, $b_1 \in B$ such that

— $|A| = t$,
— $G[A] = K_t$,
— $G[B]$ is the empty graph,
— $G(A, B)$ is a complete bipartite graph, and
— $N(c) = \{a_1, b_1\}$ for every $c \in C$.

Every graph $G \in \mathcal{G}_3(n, k)$ is defined by a partition $V(G) = A \cup B \cup J$ such that $|A| = t$, $G[A] = K_t$, $G(A, B)$ is a complete bipartite graph, and

— $G[J]$ has more than one component,
— all components of $G[J]$ are stars with at least two vertices each,
— there is a 2-element subset A' of A such that $N(J) \cap (A \cup B) = A'$,
— for every component S of $G[J]$ with at least 3 vertices, all leaves of S have degree 2 in G and are adjacent to the same vertex $a(S)$ in A'.

The class $\mathcal{G}_4(n, k)$ is empty unless $k = 10$. Each graph $H \in \mathcal{G}_4(n, 10)$ has a 3-vertex set A such that $H[A] = K_3$ and $H - A$ is a star forest such that if a component S of $H - A$ has more than two vertices then all its leaves have degree 2 in H and are adjacent to the same vertex $a(S)$ in A.
These classes are illustrated in Figure 3.
Now we define $\mathcal{G}(n, k)$ as follows:

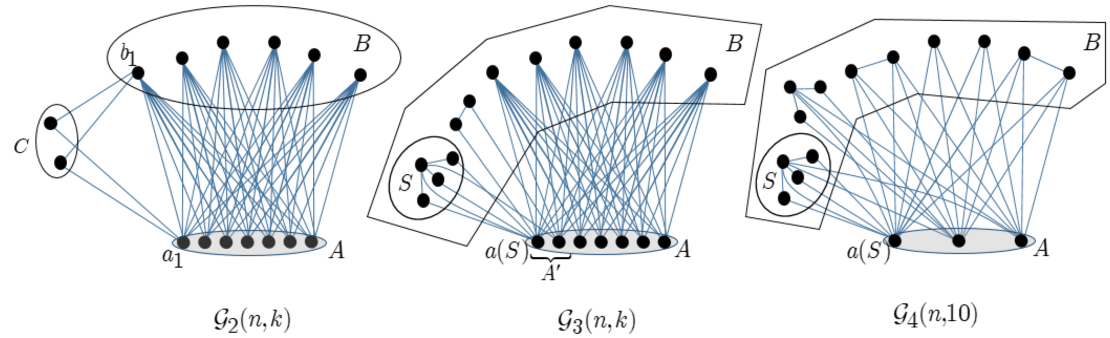

Figure 3.3: Examples of graphs in classes $\mathcal{G}_2(n,k)$, $\mathcal{G}_3(n,k)$, and $\mathcal{G}_4(n,10)$, respectively.

(1) if k is odd, then $\mathcal{G}(n,k) = \mathcal{G}_1(n,k) = \{H_{n,k,t}, H_{n,k,2}\}$;
(2) if k is even and $k \neq 10$, then $\mathcal{G}(n,k) = \mathcal{G}_1(n,k) \cup \mathcal{G}_2(n,k) \cup \mathcal{G}_3(n,k)$;
(3) if $k = 10$, then $\mathcal{G}(n,k) = \mathcal{G}_1(n,10) \cup \mathcal{G}_2(n,10) \cup \mathcal{G}_3(n,10) \cup \mathcal{G}_4(n,10)$.

In these terms, we get the following refinement of Theorem 40:

Theorem 41. (Main Stability Theorem) *Let $k \geq 9$, $n \geq k$ and $t = \left\lfloor \frac{1}{2}(k-1) \right\rfloor$. Let G be an n-vertex 2-connected graph with no cycle of length at least k. Then either $e(G) \leq \max\{h(n,k,t-1), h(n,k,3)\}$ or G is a subgraph of a graph in $\mathcal{G}(n,k)$.*

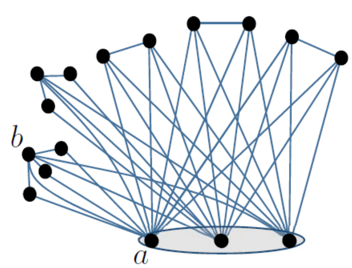

Figure 3.4: The set $\{a,b\}$ forms a separating set of the graph.

Since every graph in $\mathcal{G}_2(n,k) \cup \mathcal{G}_3(n,k)$ and many graphs in $\mathcal{G}_4(n,k)$ have a separating set of size 2 (see Figure 4), the theorem implies the following simpler statement for 3-connected graphs:

Corollary 42. *Let $k \in \{2t+1, 2t+2\}$ where $k \geq 9$. If G is a 3-connected graph on $n \geq k$ vertices and $c(G) < k$, then either $e(G) \leq \max\{h(n,k,t-1), h(n,k,3)\}$ or*

(1) *$G \subseteq H_{n,k,t}$, or*

(2) *$k = 10$ and G is a subgraph of some graph $H \in \mathcal{G}_4(n, 10)$ such that each component of $H - A$ has at most 2 vertices.*

3.3 The setup and ideas

3.3.1 Small dense subgraphs

First we define some more graph classes (also defined identically to [FKV16]). For a graph F and a nonnegative integer s, we denote by $\mathcal{K}^{-s}(F)$ the family of graphs obtained from F by deleting at most s edges.

Let $F_0 = F_0(t)$ denote the complete bipartite graph $K_{t,t+1}$ with partite sets A and B where $|A| = t$ and $|B| = t + 1$. Let $\mathcal{F}_0 = \mathcal{K}^{-t+3}(F_0)$, i.e., the family of subgraphs of $K_{t,t+1}$ with at least $t(t + 1) - t + 3$ edges.

Let $F_1 = F_1(t)$ denote the complete bipartite graph $K_{t,t+2}$ with partite sets A and B where $|A| = t$ and $|B| = t + 2$. Let $\mathcal{F}_1 = \mathcal{K}^{-t+4}(F_1)$, i.e., the family of subgraphs of $K_{t,t+2}$ with at least $t(t + 2) - t + 4$ edges.

Let $\mathcal{F}_2$ denote the family of graphs obtained from a graph in $\mathcal{K}^{-t+4}(F_1)$ by subdividing an edge a_1b_1 with a new vertex c_1, where $a_1 \in A$ and $b_1 \in B$. Note that any member $H \in \mathcal{F}_2$ has at least $|A||B| - (t - 3)$ edges between A and B and the pair a_1b_1 is not an edge.

Let $F_3 = F_3(t, t')$ denote the complete bipartite graph $K_{t,t'}$ with partite sets A and B where $|A| = t$ and $|B| = t'$. Take a graph from $\mathcal{K}^{-t+4}(F_3)$, select two non-empty subsets $A_1,\ A_2 \subseteq A$ with $|A_1 \cup A_2| \geq 3$ such that $A_1 \cap A_2 = \emptyset$ if $\min\{|A_1|, |A_2|\} = 1$, add two vertices c_1 and c_2, join them to each other and add the edges from c_i to the elements of A_i, $(i = 1, 2)$. The class of obtained graphs is denoted by $\mathcal{F}(A, B, A_1, A_2)$. The family $\mathcal{F}_3$ consists of these graphs when $|A| = |B| = t$, $|A_1| = |A_2| = 2$ and $A_1 \cap A_2 = \emptyset$. In particular, $\mathcal{F}_3(4)$ consists of exactly one graph, call it $F_3(4)$.

Graph F_4 has vertex set $A \cup B$, where $A = \{a_1, a_2, a_3\}$ and $B := \{b_1, b_2, \ldots, b_6\}$ are disjoint. Its edges are the edges of the complete bipartite graph $K(A, B)$ and three extra edges b_1b_2, b_3b_4, and b_5b_6 (see Figure 4 below). Define F_4' as the (only) member of $\mathcal{F}(A, B, A_1, A_2)$ such that $|A| = |B| = t = 4$, $A_1 = A_2$, and $|A_i| = 3$. Let $\mathcal{F}_4 := \{F_4, F_4'\}$, which is defined only for $t = 4$.

Define $\mathcal{F}(k) := \begin{cases} \mathcal{F}_0, & \text{if } k \text{ is odd,} \\ \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4, & \text{if } k \text{ is even.} \end{cases}$

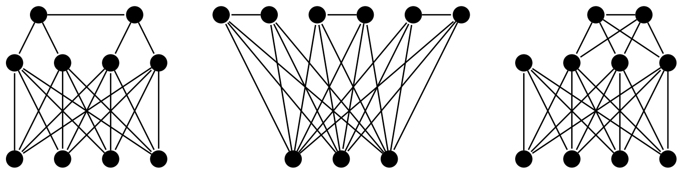

Figure 3.5: Graphs $F_3(4), F_4$, and F_4'.

3.3.2 Proof idea

In order to employ a stronger induction assumption, we will prove the following slightly stronger version of Theorem 41 claiming that the graphs in question contain dense graphs from $\mathcal{F}(k)$:

Theorem 41′ *Let $t \geq 4$, $k \in \{2t+1, 2t+2\}$, and $n \geq k$. Let G be an n-vertex 2-connected graph with no cycle of length at least k. Then either $e(G) \leq \max\{h(n,k,t-1), h(n,k,3)\}$ or*

(a) $G \subseteq H_{n,k,2}$, or

(b) G is contained in a graph in $\mathcal{G}(n,k) - \{H_{n,k,2}\}$, and G contains a subgraph $H \in \mathcal{F}(k)$, where $\mathcal{G}(n,k)$ is as defined in Section 2.3.

The method of the proof is a variation of that of [FKV16] for larger n as well as Kopylov's disintegration method for n close to k. We take an n-vertex graph G satisfying the hypothesis of Theorem 41′, and iteratively contract edges in a certain way so that each intermediate graph still satisfies the hypothesis. We consider the final graph of this process G_m on m vertices and show that G_m satisfies Theorem 41′. We will use two instrumental lemmas from [FKV16].

Lemma 43 (Main lemma on contraction, Lemma 4.9 in [FKV16])**.** *Let $k \geq 9$ and suppose F and F' are 2-connected graphs such that $F = F'/xy$ and $c(F') < k$. If F contains a subgraph $H \in \mathcal{F}(k)$, then F' also contains a subgraph $H' \in \mathcal{F}(k)$.*

This lemma shows that if G_m contains a subgraph $H \in \mathcal{F}(k)$, then the original graph G also contains a subgraph in $\mathcal{F}(k)$. The second result concludes that the original graph $G = G_n$ must satisfy (b) of Theorem 41′. For the full proof of the lemma, we refer the reader to [FKV16]. Below we include a brief sketch of the proof.

Lemma 44 ([FKV16](Subsection 4.5))**.** *Let $k \geq 9$, and let G be a 2-connected graph with $c(G) < k$ and $e(G) > h(n,k,t-1)$. If G contains a subgraph $H \in \mathcal{F}(k)$, then G is a subgraph of a graph in $\mathcal{G}(n,k) - \{H_{n,k,2}\}$.*

Sketch of proof. Consider a component of S of $G - H$. Because G is 2-connected, S has at least two neighbors, say x and y in H. Let ℓ be the length of a longest (x,y)-path P

such that all internal vertices in P are in S. When k is odd, since H is “close” to $K_{t,t+1}$, it contains a long path P' from x to y. Thus if ℓ is too large, $P' \cup P$ yields a cycle of length k or longer, a contradiction. Then one can show that $\ell = 2$ (edges). That is, each path from H to H that goes through S has only one internal vertex. Thus $|V(S)| = 1$ and moreover, x and y both lie in the partite set of of H of size t. This shows that $G \subseteq H_{n,k,t}$. The case for k even is handled similarly (but with more subcases; in particular we have $\ell \leq 3$). We obtain that either $G \subseteq H_{n,k,t}$ or the components of $G - H$ are star forests that connect to H in the ways described in the classes $\mathcal{G}_i(n,k)$, $i \in \{2,3,4\}$, otherwise G would contain a cycle of length k or longer. □

We will split the proof into the cases of small n and large n. The following observations can be obtained by simple calculations (for $t \geq 4$):

k	$h(n,k,3) \geq h(n,k,t-1)$	$h(n,k,2) \geq h(n,k,t-1)$
$2t+1$	If and only if $n \leq k + (t-5)/2$	If and only if $n \leq k + t/2 - 1$
$2t+2$	If and only if $n \leq k + (t-3)/2$	If and only if $n \leq k + t/2$

In the case of large n we will contract an edge such that the new graph still has more than $h(n-1,k,t-1)$ edges. In order to apply induction, we also need the number of edges to be greater than $h(n-1,k,3)$. To guarantee this, we pick the cutoffs for the two cases $n \leq k + (t-1)/2$ and $n > k + (t-1)/2$ (therefore $n - 1 > k + (t-3)/2$).

3.4 Tools

3.4.1 Classical theorems

Theorem 45 (Erdős [Erd62b]). *Let $d \geq 1$ and $n > 2d$ be integers, and*

$$\ell_{n,d} = \max\left\{\binom{n-d}{2} + d^2, \binom{\lceil \frac{n+1}{2} \rceil}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2\right\}.$$

Then every n-vertex graph G with $\delta(G) \geq d$ and $e(G) > \ell_{n,d}$ is hamiltonian. □

Theorem 46 (Chvátal [Chv72]). *Let $n \geq 3$ and G be an n-vertex graph with vertex degrees $d_1 \leq d_2 \leq \ldots \leq d_n$. If G is not hamiltonian, then there is some $i < n/2$ such that $d_i \leq i$ and $d_{n-i} < n - i$.* □

Theorem 47 (Kopylov [Kop77]). *If G is 2-connected and P is an x,y-path of ℓ vertices, then $c(G) \geq \min\{\ell, d(x,P) + d(y,P)\}$.* □

3.4.2 Claims on contractions

A helpful tool will be the following lemma from [FKV16] on contraction.

Lemma 48 (Lemma 3.2 in [FKV16]). *Let $n \geq 4$ and let G be an n-vertex 2-connected graph. For every $v \in V(G)$, there exists $w \in N(v)$ such that G/vw is 2-connected.* $\square$

For an edge xy in a graph H, let $T_H(xy)$ denote the number of triangles containing xy. Let $T(H) = \min\{T_H(xy) : xy \in E(H)\}$. When we contract an edge uv in a graph H, the degree of every $x \in V(H) - u - v$ either does not change or decreases by 1. Also if $u * v$ is the vertex created upon contraction, then the degree of $u * v$ in H/uv is at least $\max\{d_H(u), d_H(v)\} - 1$. Thus

$$d_{H/uv}(w) \geq d_H(w) - 1 \text{ for any } w \in V(H) \text{ and } uv \in E(H). \text{ Also } d_{H/uv}(u * v) \geq d_H(u) - 1. \tag{3.1}$$

Similarly,

$$T(H/uv) \geq T(H) - 1 \text{ for every graph } H \text{ and } uv \in E(H). \tag{3.2}$$

We will use the following analog of Lemma 3.3 in [FKV16].[1]

Lemma 49. *Let h be a positive integer. Suppose a 2-connected graph G is obtained from a 2-connected graph G' by contracting edge xy into $x * y$ chosen using the following rules:*
(i) one of x, y, say x is a vertex of the minimum degree in G';
(ii) $T_{G'}(xy)$ is the minimum among the edges xu incident with x such that G'/xu is 2-connected. If G has at least h vertices of degree at most h, then either $G' = K_{h+2}$ or
(a) G' also has a vertex of degree at most h, and
(b) G' has at least $h+1$ vertices of degree at most $h+1$.

Proof. Note that in (ii), such edges exist by Lemma 48. Since G is 2-connected, $h \geq 2$. Below for a positive integer s and a graph H, by $V_{\leq s}(H)$ we denote the set of vertices of degree at most s in H. Then by (3.1), each $v \in V_{\leq h}(G) - x * y$ is also in $V_{\leq h+1}(G')$. Moreover, then by (i),

$$x \in V_{\leq h+1}(G'). \tag{3.3}$$

Thus if $x * y \notin V_{\leq h}(G)$, then (b) follows. But if $x * y \in V_{\leq h}(G)$, then by (3.1), also $y \in V_{\leq h+1}(G')$. So, again (b) holds.
If $V_{\leq h-1}(G) \neq \emptyset$, then (a) holds by (3.1). So, if (a) does not hold, then

$$\text{each } v \in V_{\leq h}(G) - x * y \text{ has degree } h+1 \text{ in } G' \text{ and is adjacent to both } x \text{ and } y \text{ in } G'. \tag{3.4}$$

[1] The difference between our analog and the original Lemma 3.3 in [FKV16] is small: the rules we are following are slightly different, and we prove the additional property (b).

Case 1: $|V_{\leq h}(G) - x*y| \geq h$. Then by (3.3), $d_{G'}(x) = h+1$. This in turn yields $N_{G'}(x) = V_{\leq h}(G) + y$. Since G' is 2-connected, each $v \in V_{\leq h}(G) - x*y$ is not a cut vertex. Furthermore, $\{x, v\}$ is not a cut set. If it was, because y is a common neighbor of all neighbors of x, all neighbors of x must be in the same component as y in $G' - x - v$. It follows that

$$\text{for every } v \in V_{\leq h}(G) - x*y,\ G'/vx \text{ is 2-connected.} \tag{3.5}$$

If $uv \notin E(G)$ for some $u, v \in V_{\leq h}(G)$, then by (3.5) and (i), we would contract the edge xu rather than xy. Thus $G'[V_{\leq h}(G) \cup \{x, y\}] = K_{h+2}$ and so either $G' = K_{h+2}$ or y is a cut vertex in G', as claimed.

Case 2: $|V_{\leq h}(G) - x*y| = h-1$. Then $x*y \in V_{\leq h}(G)$. This means $d_{G'}(x) = d_{G'}(y) = h+1$ and $N_{G'}[x] = N_{G'}[y]$. So by (3.4), there is $z \in V(G)$ such that $N_{G'}[x] = N_{G'}[y] = V_{\leq h}(G) \cup \{x, y, z\}$. Again (3.5) holds (for the same reason that $N_{G'}[x] \subseteq N_{G'}[y]$). Thus similarly $vu \in E(G')$ for every $v \in V_{\leq h}(G) - x*y$ and every $u \in V_{\leq h}(G) + z$. Hence $G'[V_{\leq h}(G) \cup \{x, y, z\}] = K_{h+2}$ and either $G' = K_{h+2}$ or z is a cut vertex in G', as claimed. $\square$

3.4.3 A property of graphs in $\mathcal{F}(k)$

A useful feature of graphs in $\mathcal{F}(k)$ is the following.

Lemma 50. *Let $k \geq 9$ and $n \geq k$. Let F be an n-vertex graph contained in $H_{n,k,t}$ with $e(F) > h(n, k, t-1)$. Then F contains a graph in $\mathcal{F}(k)$.*

Proof. Assume the sets A, B, C to be as in the definition of $H_{n,k,t}$. We will use induction on n.

Case 1: $k = 2t+1$. If $n = k$, then $F \in \mathcal{K}^{-t+3}(H_{k,k,t})$ because $h(k,k,t) - h(k,k,t-1) - 1 = t-3$. Thus, since $H_{k,k,t} \supseteq F_0(t)$, F contains a subgraph in $\mathcal{F}_0$. Suppose now the lemma holds for all $k \leq n' < n$. If $\delta(F) \geq t$, then each $v \in V(F) - A$ is adjacent to every $u \in A$. Hence F contains $K_{t,n-t}$. If $\delta(F) < t$, then since A is dominating and $n > 2t$, there is $v \in V(F) - A$ with $d_F(v) \leq t-1$. Then $F - v \subseteq H_{n-1,k,t}$, and we are done by induction.

Case 2: $k = 2t+2$. Let $C = \{c_1, c_2\}$. If $n = k$ then as in Case 1,

$$e(H_{k,k,t}) - e(F) \leq h(k,k,t) - h(k,k,t-1) - 1 = t-4,$$

i.e., $F \in \mathcal{K}^{-t+4}(H_{k,k,t})$. Since $F_1(t) \subseteq H_{k,k,t}$, F contains a subgraph in $\mathcal{F}_1$. Suppose now the lemma holds for all $k \leq n' < n$. If $\delta(F) < t$, then there is $v \in V(F) - A$ with $d_F(v) \leq t-1$. Then $F - v \subseteq H_{n-1,k,t}$, and we are done by induction.

Finally, suppose $\delta(F) \geq t$. So, each $v \in B$ is adjacent to every $u \in A$ and each of c_1, c_2 has at least $t-1$ neighbors in A. Since $|B \cup \{c_1\}| \geq n-t-1 \geq t+2$, F contains a member of $\mathcal{K}^{-1}(F_1(t))$. Thus F contains a member of $\mathcal{F}_1$ unless $t = 4$, $n = 2t + 3$ and c_1 has a nonneighbor $x \in A$. But then $c_1c_2 \in E(F)$, and so F contains either $F_3(4)$ or F_4'. □

3.5 Proof of Theorem 41′

Let $n \geq k \geq 9$ and suppose Theorem 41′ holds for all graphs with n' vertices where $k \leq n' < n$. Suppose further that

$$G \textit{ is an } n\textit{-vertex } 2\textit{-connected graph with } c(G) < k \textit{ and } e(G) > \max\{h(n,k,t-1), h(n,k,3)\}. \tag{3.6}$$

3.5.1 Contraction procedures

If $n > k$, we iteratively construct a sequence of graphs $G_n, G_{n-1}, ...G_m$ where $G_n = G$ and $|V(G_j)| = j$ for all $m \leq j \leq n$. In [FKV16], the following **Basic Procedure** (BP) was used:
At the beginning of each round, for some $j : k \leq j \leq n$, we have a j-vertex 2-connected graph G_j with $e(G_j) > h(j,k,t-1)$.

(R1) If $j = k$, then we stop.

(R2) If there is an edge uv with $T_{G_j}(uv) \leq t-2$ such that G_j/uv is 2-connected, choose one such edge so that
(i) $T_{G_j}(uv)$ is minimum, and subject to this
(ii) uv is incident to a vertex of minimum possible degree.
Then obtain G_{j-1} by contracting uv.

(R3) If (R2) does not hold, $j \geq k + t - 1$ and there is $xy \in E(G_j)$ such that $G_j - x - y$ has at least 3 components and one of the components, say H_1 is a K_{t-1}, then let $G_{j-t+1} = G_j - V(H_1)$.

(R4) If neither (R2) nor (R3) occurs, then we stop.

Remark 5.1. By definition, (R3) applies only when $j \geq k+t-1$. As observed in [FKV16], if $j \leq 3t-2$, then a j-vertex graph G_j with a 2-vertex set $\{x,y\}$ separating the graph into at least 3 components cannot have $T_{G_j}(uv) \geq t-1$ for every edge uv. It also was calculated there that if $3t-1 \leq j \leq 3t$, then any j-vertex graph G' with such 2-vertex set $\{x,y\}$ and $T_{G'}(uv) \geq t-1$ for every edge uv has at most $h(j,k,t-1)$ edges and so cannot be G_j.
In this version, we use a quite similar **Modified Basic Procedure** (MBP): start with a 2-connected, n-vertex graph $G = G_n$ with $e(G) > h(n,k,t-1)$ and $c(G) < k$. Then

(MR0) if $\delta(G_j) \geq t$, then apply the rules (R1)–(R4) of (BP) given above;

(MR1) if $\delta(G_j) \leq t-1$ and $j = k$, then stop;

(MR2) otherwise, pick a vertex v of smallest degree, contract an edge vu with the minimum $T_{G_j}(vu)$ among the edges vu such that G_j/vu is 2-connected, and set $G_{j-1} = G_j/uv$.

3.5.2 Proof of Theorem 41′ for the case $n \leq k + (t-1)/2$

Let G satisfy (3.6). Apply to G the Modified Basic Procedure (MBP) starting from $G_n = G$. Denote by G_m the terminating graph of MBP. By Remark 1, (R3) was never applied, since $k + (t-1)/2 < k + t - 1$. Therefore

$$\textit{for each } m \leq j < n, \textit{ graph } G_j \textit{ is obtained from } G_{j+1} \textit{ by contracting an edge.} \tag{3.7}$$

Then G_j is 2-connected and $c(G_j) \leq c(G) < k$ for each $m \leq j \leq n$. By construction, after each contraction, we lose at most $t-1$ edges. It follows that $e(G_m) > h(m, k, t-1)$.
Suppose first that $m > k$. Then the same argument as in [FKV16] gives us the following structural result:

Lemma 51 (Proposition 4.2 in [FKV16])**.** *Let $m > k \geq 9$ and $n \geq k$.*

- *If $k \neq 10$, then $G_m \subseteq H_{m,k,t}$.*
- *If $k = 10$, then $G_m \subseteq H_{m,k,t}$ or $G_m \supseteq F_4$.*

Again we sketch the proof briefly and refer the reader to [FKV16] for the full proof.
Sketch of proof. If $\delta(G_m) \leq t-1$, then either Rule (R2) or Rule (MR2) applies to G_m, so Procedure MBP does not stop, contradicting the definition of m. Thus $\delta(G_m) \geq t$. Since G_m is 2-connected, $c(G_m) \geq 2\delta(G_m) \geq 2t$. So if k is even, $c(G_m) \in \{2t, 2t+1\}$, and if k is odd, $c(G_m) = 2t$. For simplicity in this sketch, we only consider the odd case.
Let $C = v_1, \ldots, v_{2t}$ be a longest cycle in G_m. Because we could not apply rule (R2), for each edge v_iv_{i+1} in C, either v_iv_{i+1} is contained in at least $t-1$ triangles, or the set $\{v_i, v_{i+1}\}$ is separating in G_m. In the latter case, we show that C can be extended to a longer cycle. Thus the former holds. If $v_iv_{i+1}z$ is a triangle, then $z \in V(C)$, otherwise we get a longer cycle by including z. Thus we have shown that the induced subgraph $G[V(C)]$ has many edges, and furthermore it can be shown that $G[V(C)]$ is 3-connected. We then apply a structural theorem for 3-connected graphs due to Enomoto [Eno84] (see, e.g. Theorem 2.7 in [FKV16]) that yields three possible cases for the structure of $G[V(C)]$. In the first case, $\overline{K_t} + \overline{K_t} \subseteq G_m[V(C)] \subseteq K_t + \overline{K_t}$. In this case, by considering the connected components of $G_m - V(C)$ and the ways they connect to C, similarly to the proof of Lemma 44, we

obtain $G_m \subseteq H_{m,k,t}$. In the other two cases, we either obtain $c(G) \geq k$ or $q < 2t$, a contradiction. $\square$

Since $F_4 \in \mathcal{F}(k)$, if $k = 10$ and $G_m \supseteq F_4$, then G_m contains a subgraph in $\mathcal{F}(k)$. Otherwise, by Lemmas 50 and 51, again G_m has a subgraph in $\mathcal{F}(k)$. Then by (3.7) and Lemma 43, for every $m \leq j \leq n$, graph G_j contains a subgraph $H_j \in \mathcal{F}(k)$. In particular, $G = G_n$ contains such a subgraph. Thus by Lemma 44, G satisfies Theorem 41$'$.
So, below we assume

$$m = k. \tag{3.8}$$

Since $c(G_k) < k$, G_k does not have a hamiltonian cycle. Let $d_1 \leq d_2 \leq ... \leq d_k$ be the vertex degrees of G_k. By Theorem 46, there exists some $2 \leq i \leq t$ such that $d_i \leq i$ and $d_{k-i} < k - i$. Let $r = r(G_k)$ be the smallest such i.
Let R be a set of r vertices of degree at most r in G_k. Then

$$e(G_k) \leq r^2 + e(G_k - R) \leq r^2 + \binom{k-r}{2}.$$

For $k = 2t+1$, $r^2 + \binom{k-r}{2} > h(n, k, t-1)$ only when $r = t$ or $r < (t+4)/3$, and for $k = 2t+2$, when $r = t$ or $r < (t+6)/3$. If $r = r(G_k) = t$, then repeating the argument in [FKV16] yields:

Lemma 52 (Lemma 4.4 in [FKV16])**.** *If $r(G_k) = t$ then $G_k \subseteq H_{k,k,t}$.*

Sketch of proof. Since $c(G_k) < k$, G_k is nonhamiltonian. Let G' be the hamiltonian closure of G_k. Then $r(G')$ exists, and furthermore, $r(G') \geq r(G_k)$. Thus $r(G') = t$. Our goal is to show that $G' \subseteq H_{k,k,t}$. Let $V(G') = \{v_1, \ldots, v_k\}$ and $d'_i = d_{G'}(v_i)$ for $i = 1, \ldots, k$. Rename the vertices of G' so that $d'_1 \leq \ldots \leq d'_k$. By the definition of $r(G') = t$, $d'_1 \leq \ldots d'_t \leq t$. Let $A = \{v_k, v_{k-1}, \ldots, v_{k-t+1}\}$. If any vertex in A has too small degree, then we show $e(G_k) \leq h(k, k, t-1)$, a contradiction. Since G' is hamiltonian-closed, for each nonedge $xy \notin E(G')$,

$$d(x) + d(y) \leq |V(G')| - 1 = k - 1. \tag{3.9}$$

Using this, we show that $G'[A] = K_t$. Next, we consider the edges between $G' - A$ and A. If there are many non-edges, then applying (3.9) for each non-edge yields that $e(G') \leq h(k, k, t-1)$, so we finally show that every vertex in A but at most one is adjacent to every other vertex in G'. We focus here on the case that every vertex in A is adjacent to every other vertex. Then the neighborhood of every vertex of degree at most t is exactly A. If k is odd, we show that also $d'_{t+1} = t$ and so $G' = H_{k,k,t}$, since the vertices of $G' - A$ must form an independent set. The even case is proved similarly, but with more subcases. $\square$

By Lemmas 50, 43, and 44, $G \subseteq H_{n,k,t}$ and contains some subgraph in $\mathcal{F}(k)$. This finishes the case $r = t$.
So we may assume that

$$\text{if } k = 2t + 1 \text{ then } r < (t + 4)/3, \text{ and if } k = 2t + 2 \text{ then } r < (t + 6)/3. \tag{3.10}$$

Our next goal is to show that G contains a large "core", i.e., a subgraph with large minimum degree. For this, we recall the notion of *disintegration* used by Kopylov [Kop77].
Definition: *For a natural number α and a graph G, the* α-disintegration *of a graph G is the process of iteratively removing from G the vertices with degree at most α until the resulting graph has minimum degree at least $\alpha + 1$. This resulting subgraph $H = H(G, \alpha)$ will be called the* α-core *of G.*
It is well known that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion.

Claim 53. *The t-core $H(G, t)$ of G is nonempty.*

Proof of Claim 53: We may assume that for all $m \le j < n$, graph G_j was obtained from G_{j+1} by contracting edge $x_j y_j$, where $d_{G_{j+1}}(x_j) \le d_{G_{j+1}}(y_j)$. By Rule (MR2), $d_{G_{j+1}}(x_j) = \delta(G_{j+1})$, provided that $\delta(G_{j+1}) \le t - 1$.
By definition, $|V_{\le r}(G_k)| \ge r$. So by Lemma 49 (applied several times), for each $k+1 \le j \le k + t - r$, because each G_j is not a complete graph (otherwise it would have a hamiltonian cycle),

$$\delta(G_j) \le j - k + r - 1 \ \text{ and } |V_{\le j-k+r}(G_j)| \ge j - k + r. \tag{3.11}$$

To show that

$$\delta(G_j) \le t - 1 \text{ for all } k \le j \le n, \tag{3.12}$$

by (3.11) and (3.10), it is enough to observe that

$$\delta(G_j) \le j - k + r - 1 \le (n - k) + r - 1 \le \frac{t-1}{2} + \frac{t+6}{3} - 1 = \frac{5t+3}{6} < t.$$

We will apply a version of t-disintegration in which we first manually remove a sequence of vertices and count the number of edges they cover. By (3.12) and (MR2), $d_{G_n}(x_{n-1}) = \delta(G_n) \le n - k + r - 1$. Let $v_n := x_{n-1}$. Then $G - v_n$ is a subgraph of G_{n-1}. If $x_{n-2} \ne x_{n-1} * y_{n-1}$ in G_{n-1}, then let $v_{n-1} := x_{n-2}$, otherwise let $v_{n-1} := y_{n-1}$. In both cases, $d_{G-v_n}(v_{n-1}) \le n - k + r - 2$. We continue in this way until $j = k$: each time we delete from the graph $G - v_n - \ldots - v_{j+1}$ the unique survived vertex v_j that was in the preimage of x_{j-1} when we obtained G_{j-1} from G_j. Graph $G - v_n - ... - v_{k+1}$ has $r \ge 2$ vertices of degree at most r. We additionally delete 2 such vertices v_k and v_{k-1}. Altogether, we have lost at most $(r + n - k - 1) + (r + n - k - 2) + ... + r + 2r$ edges in the deletions.

Finally, apply t-disintegration to the remaining graph on $k-2 \in \{2t-1, 2t\}$ vertices. Suppose that the resulting graph is empty.

Case 1: $n = k$. Then
$$e(G) \le r + r + t(2t-1-t) + \binom{t}{2},$$
where $r+r$ edges are from v_k and v_{k-1}, and after deleting v_k and v_{k-1}, every vertex deleted removes at most t edges, until we reach the final t vertices which altogether span at most $\binom{t}{2}$ edges.

For $k = 2t+1$,
$$h(k,k,t-1) - e(G) \ge \binom{2t+1-(t-1)}{2} + (t-1)^2 - \left[r + r + t(2t-1-t) + \binom{t}{2}\right] = t+2-2r,$$
which is nonnegative for $r < (t+3)/3$. Therefore $e(G) \le h(k,k,t-1)$, a contradiction.

Similarly, if $k = 2t+2$,
$$e(G) \le r + r + t(2t-t) + \binom{t}{2},$$
and
$$h(k,k,t-1) - e(G) \ge \binom{2t+2-(t-1)}{2} + (t-1)^2 - [r + r + t(2t-t) + \binom{t}{2}] = t+4-2r,$$
which is nonnegative when $r < (t+6)/3$.

Case 2: $k < n \le k + (t-1)/2$. Then for $k = 2t+1$,
$$\begin{aligned}
e(G) &\le \big[(r+n-k-1) + (r+n-k-2) + \ldots + r\big] + 2r + t(2t-1-t) + \binom{t}{2} \\
&\le \big[(t-1) + (t-1) + \ldots + (t-1)\big] + h(k,k,t-1) \\
&= (t-1)(n-k) + h(k,k,t-1) \\
&= h(n,k,t-1),
\end{aligned}$$
where the last inequality holds because $r + n - k - 1 \le t - 1$.

Similarly, for $k = 2t+2$,
$$\begin{aligned}
e(G) &\le \big[(r+n-k-1) + (r+n-k-2) + \ldots + r\big] + 2r + t(2t-t) + \binom{t}{2} \\
&\le (n-k)(t-1) + h(k,k,t-1) \\
&= h(n,k,t-1).
\end{aligned}$$

This contradiction completes the proof of Claim 53. $\square$

For the rest of the proof of Theorem 41′ for $n \le k + (t-1)/2$, we will follow the method of Kopylov in [Kop77] to show that $G \subseteq H_{n,k,2}$. Let G^* be the k-closure of G. That is, add edges to G until adding any additional edge creates a cycle of length at least k. In particular, for any non-edge xy of G^*, there is an (x, y)-path in G^* with at least $k-1$ edges. Because G has a nonempty t-core, and G^* contains G as a subgraph, G^* also has a nonempty t-core (which contains the t-core of G). Let $H = H(G^*, t)$ denote the t-core of G^*. We will show that

$$H \text{ is a complete graph.} \tag{3.13}$$

Indeed, suppose (3.13) does not hold. Choose a longest path P of G^* whose terminal vertices $x \in V(H)$ and $y \in V(H)$ are nonadjacent. By the maximality of P, every neighbor of x in H is in P. The same holds for y. Hence $d_P(x)+d_P(y) = d_H(x)+d_H(y) \geq 2(t+1) > k$, and also P has $k-1$ edges. By Theorem 47, $c(G^*) \geq k$, a contradiction. This proves (3.13).
Let $\ell = |V(H)|$. Because every vertex in H has degree at least $t+1$, $\ell \geq t+2$. Furthermore, if $\ell \geq k-1$, then G^* has a clique K of size at least $k-1$. Because G^* is 2-connected, we can extend a $(k-1)$-cycle of K to include at least one vertex in $G^* - H'$, giving us a cycle of length at least k. It follows that

$$t + 2 \leq \ell \leq k - 2, \tag{3.14}$$

and therefore $k - \ell < t$. Apply $(k - \ell)$-disintegration to G^*, and denote by H' the resulting graph. By construction, $H \subseteq H'$.
Case 1: There exists $v \in V(H') - V(H)$. Since $v \notin V(H)$, there exists a nonedge between a vertex in H and a vertex in $H' - H$. Pick a longest path P with terminal vertices $x \in V(H')$ and $y \in V(H)$. Then $d_P(x) + d_P(y) \geq (k - \ell + 1) + (\ell - 1) = k$, and therefore $c(G^*) \geq k$.
Case 2: $H = H'$. Then

$$e(G^*) \leq \binom{\ell}{2} + (n-\ell)(k-\ell) = h(n, k, k-\ell).$$

If $3 \leq (k - \ell) \leq t - 1$, then $e(G) \leq \max\{h(n, k, 3), h(n, k, t-1)\}$, so by (3.14), $k - \ell = 2$, and H is the complete graph with $k-2$ vertices. Let $D = V(G^*) - V(H)$. If there is an edge xy in $G^*[D]$, then because G^* is 2-connected, there exist two vertex-disjoint paths, P_1 and P_2, from $\{x, y\}$ to H such that P_1 and P_2 only intersect $\{x, y\} \cup H$ at the beginning and end of the paths. Let a and b be the terminal vertices of P_1 and P_2 respectively that lie in H. Let P be any (a, b)-hamiltonian path of H. Then $P_1 \cup P \cup P_2 + xy$ is a cycle of length at least k in G^*, a contradiction.
Therefore D is an independent set, and since G^* is 2-connected, each vertex of D has degree 2. Suppose there exists $u, v \in D$ where $N(u) \neq N(v)$. Let $N(u) = \{a, b\}, N(v) = \{c, d\}$ where it is possible that $b = c$. Then we can find a cycle C of H that covers $V(H)$ which

contains edges ab and cd. Then $C - ab - cd + ua + ub + vc + vd$ is a cycle of length k in G^*. Thus for every $v \in D$, $N(v) = \{a, b\}$ for some $a, b \in H$. I.e., $G^* = H_{n,k,2}$, and thus $G \subseteq H_{n,k,2}$. This completes the proof of Theorem 41′ for the case $n \le k + (t-1)/2$. $\square$

3.5.3 Proof of Theorem 41′ for all n

We use induction on n with the base case $n \le k + (t-1)/2$. Suppose $n \ge k + t/2$ and for all $k \le n' < n$, Theorem 41′ holds. Let G be a 2-connected graph G with n vertices such that

$$e(G) > \max\{h(n,k,t-1), h(n,k,3)\} \text{ and } c(G) < k. \tag{3.15}$$

Apply one step of Procedure BP. If (R4) was applied (so neither (R2) nor (R3) applies to G), then $G_m = G$ (with G_m defined as in the previous case). By Lemmas 51, 50, and 44, the theorem holds.

Therefore we may assume that either (R2) or (R3) was applied. Let G^- be the resulting graph. Then $c(G^-) < k$, and G^- is 2-connected.

Claim 54.

$$e(G^-) > \max\{h(|V(G^-)|, k, t-1), h(|V(G^-)|, k, 3)\}. \tag{3.16}$$

Proof. If (R2) was applied, i.e., $G^- = G/uv$ for some edge uv, then

$$e(G^-) \ge e(G) - (t-1) > h(n-1, k, t-1) \ge h(n-1, k, 3),$$

so (3.16) holds. Therefore we may assume that (R3) was applied to obtain G^-. Then $n \ge k + t - 1$ and $e(G) - e(G^-) = \binom{t+1}{2} - 1$. So by (3.15),

$$e(G^-) > h(n,k,t-1) - \binom{t+1}{2} + 1. \tag{3.17}$$

The right hand side of (3.17) equals $h(n-(t-1), k, t-1) + t^2/2 - 5t/2 + 2$ which is at least $h(n-(t-1), k, t-1)$ for $t \ge 4$, proving the first part of (3.16).

We now show that also $e(G^-) > h(n-(t-1), k, 3)$. Indeed, for $k = 2t+1$,

$$e(G^-) - h(n-(t-1), k, 3) > \binom{t+2}{2} + (t-1)(n-t-2) - \binom{t+1}{2} + 1$$
$$- \left[\binom{2t-2}{2} + 3(n-(t-1)-(2t-2))\right] \ge 0 \text{ when } n \ge 3t.$$

Similarly, for $k = 2t + 2$,

$$e(G^-) - h(n-(t-1), k, 3) > \binom{t+3}{2} + (t-1)(n-t-3) - \binom{t+1}{2} + 1$$

$$- \left[\binom{2t-1}{2} + 3(n-(t-1)-(2t-1)) \right] > 0 \text{ when } n \geq 3t+1.$$

Thus if $n \geq 3t+1$, then (3.16) is proved. But if $n \in \{3t-1, 3t\}$ then by Remark 5.1, no graph to which (R3) applied may have more than $h(n, k, t-1)$ edges. $\square$

By (3.16), we may apply induction to G^-. So G^- satisfies either (a) $G^- \subseteq H_{|V(G^-)|,k,2}$, or (b) G^- is contained in a graph in $\mathcal{G}(n,k) - H_{|V(G^-)|,k,2}$ and contains a subgraph $H \in \mathcal{F}(k)$. Suppose first that G^- satisfies (b). If (R3) was applied to obtain G^- from G, then because G^- contains a subgraph $H \in \mathcal{F}(k)$ and $G^- \subseteq G$, G also contains H. If (R2) was applied, then by Lemma 43, G contains a subgraph $H' \in \mathcal{F}(k)$. In either case, Lemma 44 implies that G is a subgraph of a graph in $\mathcal{G}(n,k) - H_{n,k,2}$.
So we may assume that (a) holds, that is, G^- is a subgraph of $H_{|V(G^-)|,k,2}$. Because $\delta(G^-) \leq 2$, $\delta(G) \leq 3$, and so G has edges in at most $2 \leq t-2$ triangles. Therefore (R2) was applied to obtain G^-, where $G/uv = G^-$. Let D be an independent set of vertices of G^- of size $(n-1)-(k-2)$ with $N(D) = \{a, b\}$ for some $a, b \in V(G^-)$. Since $T_{G^-}(xa), T_{G^-}(xb) \leq 1$ for every $x \in D$, we have that $T_G(uv) \leq 2$ with equality only if $T(G) = 2$ where $T(G) = \min_{xy \in E(G)} T_G(xy)$.
We want to show that $T_G(uv) \leq 1$. If not, suppose first that $u * v \in D \subseteq V(G^-)$. Then there exists $x \in D - u * v$, and x and $u * v$ are not adjacent in G^-. Therefore x was not in a triangle with u and v in G, and hence $T_G(xa) = T_{G^-}(xa) \leq 1$, so the edge xa should have been contracted instead. Otherwise if $u * v \notin D$, at least one of $\{a, b\}$, say a, is not $u * v$. If $T(G) = 2$, then for every $x \in D \subseteq V(G)$, $T_G(xa) = 2$, therefore each such edge xa was in a triangle with uv in G. Then $T_G(uv) \geq |D| = (n-1)-(k-2) \geq k + t/2 - 1 - k + 2 \geq 3$, a contradiction.
Thus $T_G(uv) \leq 1$ and $e(G) \leq 2 + e(G^-) \leq 2 + h(n-1, k, 2) = h(n, k, 2)$. But for $n \geq k + t/2$, we have $h(n, k, t-1) \geq h(n, k, 2)$, a contradiction. This completes the proof of Theorem 41′ and therefore the proof of the main result. $\square$

Chapter 4

Counting cliques in graphs with bounded circumference

4.1 Introduction

In this chapter, we prove Theorem 12, a generalization of Kopylov's theorem (Theorem 7) which counts the maximum number of cliques in a 2-connected graph without cycles of length k or longer. This work can be found in [Luo18].

4.2 Clique counting results

Definition. Let $f_s(n,k,a) := \binom{k-a}{s} + (n-k+a)\binom{a}{s-1}$, where $f_2(n,k,a) = e(H_{n,k,a})$.
By considering the second derivative, one can check that $f_s(n,k,a)$ is convex in a in the domain $[1, \lfloor (k-1)/2 \rfloor]$, thus it attains its maximum at one of the endpoints $a = 1$ or $a = \lfloor (k-1)/2 \rfloor$.
We again consider a generalized Turán-type problem. Recall that the function $ex(n,T,H)$ denotes the maximum number of (unlabeled) copies of T in an H-free graph on n vertices. When $T = K_2$, we have the usual extremal number $ex(n,T,H) = ex(n,H)$.
Definition. For $s \geq 2$, let $N_s(G)$ denote the number of unlabeled copies of K_s in G, e.g., $N_2(G) = e(G)$.
The following is a refinement of the statement of Theorem 12.

Theorem 55. *Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. If G is a 2-connected n-vertex graph with circumference less than k, then*

$$N_s(G) \leq \max\{f_s(n,k,2), f_s(n,k,t)\}.$$

Again, this theorem is sharp with the same extremal examples $H_{n,k,2}$ and $H_{n,k,t}$.
This theorem implies the cliques version of Theorem 6:

Corollary 56. *Let $n \geq k \geq 4$. If G is an n-vertex graph with circumference less than k,* *then*

$$N_s(G) \le \frac{n-1}{k-2}\binom{k-1}{s}.$$

Unlike the edges case, Theorem 55 unfortunately does not easily imply $ex(n, K_s, P_k)$. However, a Kopylov-style argument very similar to the proof of Theorem 55 gives the result for paths.

Theorem 57. *Let $n \ge k \ge 4$ and let G be an n-vertex connected graph with no path on k vertices. Let $t = \lfloor (k-2)/2 \rfloor$. Then $N_s(G) \le \max\{f_s(n, k-1, 1), f_s(n, k-1, t)\}$.*

We have sharpness examples $H_{n,k-1,1}$ and $H_{n,k-1,t}$. Finally, using induction on the number of components gives the following result:

Corollary 58. $ex(n, K_s, P_k) = \frac{n}{k-1}\binom{k-1}{s}$.

And the same extremal examples as for Corollary 5 apply.
The proofs for Corollary 56, Theorem 57, and Theorem 58 are given in Section 4 of this chapter. We first prove Theorem 55.

4.3 Proof of Theorem 55

Let G be an edge-maximal counterexample. Then G is *k-closed*, i.e., adding any additional edge to G creates a cycle of length at least k. In particular, for any nonadjacent vertices x and y of G, there exists a path of at least $k-1$ edges between x and y. We will use the following lemma:

Lemma 59 (Kopylov [Kop77])**.** *Let G be a 2-connected n-vertex graph with a path P of m edges with endpoints x and y. For $v \in V(G)$, let $d_P(v) = |N(v) \cap V(P)|$. Then G contains a cycle of length at least* $\min\{m+1, d_P(x) + d_P(y)\}$.

Our first goal is to show that G contains a large "core", i.e., a subgraph with large minimum degree. For this, we use the notion of *disintegration*.
Recall the definition of α-disintegration.
Definition: For a natural number α and a graph G, the *α-disintegration* of a graph G is the process of iteratively removing from G the vertices with degree at most α until the resulting graph has minimum degree at least $\alpha+1$ or is empty. This resulting subgraph $H = H(G, \alpha)$ will be called the $(\alpha+1)$-*core* of G. It is well known that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion (for instance, see [PSW96]).
Let $H(G, t)$ denote the $(t+1)$-core of G, i.e., the resulting graph of applying t-disintegration to G. We claim that

$$H(G,t) \textit{ is nonempty.}$$

Suppose $H(G,t)$ is empty. In the disintegration process, every time a vertex of degree at most t is removed, we delete at most $\binom{t}{s-1}$ copies of K_s. For the last $\ell \leq t$ vertices, we remove at most $\binom{\ell-1}{s-1}$ copies of K_s with each deletion. Thus

$$\begin{aligned} N_s(G) &\leq (n-t)\binom{t}{s-1} + \binom{t-1}{s-1} + \binom{t-2}{s-1} + \ldots + \binom{0}{s-1} \\ &= (n-t)\binom{t}{s-1} + \binom{t}{s} \\ &= (n-(t+1))\binom{t}{s-1} + \binom{t+1}{s} \\ &\leq f_s(n,k,t), \end{aligned}$$

a contradiction.
Therefore $H(G,t)$ is nonempty. Next we show that

$H(G,t)$ is a complete graph.

If there exists a nonedge of $H(G,t)$, then in G, there is a path of length at least $k-1$ edges with these vertices as its endpoints. Among all nonadjacent pairs of vertices in $H(G,t)$, choose x, y such that there is a longest path P in G with endpoints x and y. By maximality of P, all neighbors of x in $H(G,t)$ lie in P: if x has a neighbor $x' \in H(G,t) - P$, then either $x'y \in E(G)$ and $x'P$ is a cycle of length at least k, or $x'y \notin E(G)$ and so $x'P$ is a longer path. Similar for y. Hence, by Lemma 59, G has a cycle of length at least $\min\{k, d_P(x) + d_P(y)\} = \min\{k, 2(t+1)\} = k$, a contradiction.
Now let $r = |V(H(G,t)|$. Each vertex in $H(G,t)$ has degree at least $t+1$, so $r \geq t+2$. Also, if $r \geq k-1$, as G is 2-connected and $H(G,t)$ is a clique, we can extend a path on r vertices of $H(G,t)$ to a cycle of length at least $r+1 \geq k$, a contradiction. Therefore $t+2 \leq r \leq k-2$. In particular, $2 \leq k-r \leq t$. Apply $(k-r)$-disintegration to G, and let $H(G,k-r)$ be the resulting graph. Then $H(G,t) \subseteq H(G,k-r)$.
If $H(G,t) = H(G,k-r)$, then

$$N_s(G) \leq \binom{r}{s} + (n-r)\binom{k-r}{s-1} = f_s(n,k,k-r) \leq \max\{f_s(n,k,2), f_s(n,k,t)\}$$

by the convexity of f_s. Therefore, $H(G,t)$ is a proper subgraph of $H(G,k-r)$, and there must be a nonedge between a vertex in $H(G,t)$ and a vertex in $H(G,k-r)$. Among all such pairs, choose $x \in H(G,t)$ and $y \in H(G,k-r)$ to have a longest path P between them. As before, P contains at least $k-1$ edges, and each neighbor of x in $H(G,t)$ and each neighbor of y in $H(G,k-r)$ lie in P. Then G contains a cycle of length at least

$\min\{k,(r-1)+(k-r+1)\}=k$, a contradiction. $\square$

4.4 Proofs for general graphs and graphs without long paths

Proof of Corollary 56. Define $g_s(n,k)=\frac{n-1}{k-2}\binom{k-1}{s}$ and $t=\lfloor\frac{k-1}{2}\rfloor$. One can check that when $n\geq k$,

$$g_s(n,k)\geq\max\{f_s(n,k,t),f_s(n,k,2)\}.$$

Fix a graph G on n vertices with circumference less than k. If G is disconnected, simply apply induction to each component of G to obtain the desired result. Therefore we may assume G is connected. We induct on the number of blocks of G. First suppose $k\geq 5$. If G is a block, i.e., 2-connected, then either $n\leq k-1$, and so $N_s(G)\leq\binom{|V(G)|}{s}\leq g_s(n,k)$, or $n\geq k$, and so by Theorem 55, $N_s(G)\leq\max\{f_s(n,k,t),f_s(n,k,2)\}\leq g_s(n,k)$. Otherwise, consider the *block-cut tree* of G—the tree whose vertices correspond to blocks of G such that two vertices in the tree are adjacent if and only if the corresponding blocks in G share a vertex. Let B_1 be a block in G corresponding to a leaf-vertex in the block-cut tree such that B_1 and its complement are connected by the cut vertex v. Set $B_2=G-B_1+\{v\}$. Apply the induction hypothesis to B_1 and B_2 to obtain

$$\begin{aligned}N_s(G) &= N_s(B_1)+N_s(B_2)\leq g_s(|B_1|,k)+g_s(n-|B_1|+1,k)\\ &= \frac{|B_1|-1}{k-2}\binom{k-1}{s}+\frac{(n-|B_1|+1)-1}{k-2}\binom{k-1}{s}\\ &= g_s(n,k).\end{aligned}$$

If $k=4$, then either G is a forest or G has circumference 3. In the second case, each block of G is either a triangle or an edge. Thus $N_s(G)\leq g_s(n,k)$ in both cases. $\square$

The proof of Theorem 57 follows the same steps as the proof of Theorem 55. As some details here will be omitted to prevent repetition, it is advised that the reader first reads the proof of Theorem 55.

Proof of Theorem 57. Suppose for contradiction that $N_s(G)>\max\{f_s(n,k-1,1),f_s(n,k-1,t)\}$ where $t=\lfloor(k-2)/2\rfloor$. Let G_0 be the graph obtained by adding a dominating vertex v_0 adjacent to all of $V(G)$. Then G_0 is 2-connected, has $n+1$ vertices, and contains no cycle of length $k+1$ or greater. Let G' be the $k+1$-closure of G_0 (i.e., add edges to G_0 until any additional edge creates a cycle of length at least $k+1$). Denote by $N'_s(G')$ the number of K_s's in G' that do not contain v_0. Thus $N'_s(G')\geq N'_s(G_0)=N_s(G)$. Apply $(t+1)$-disintegration to G', where if necessary, we delete v_0 last. Let $H(G',t+1)$ be the resulting graph of the disintegration. If $H(G',t+1)$ is empty, then at the time of deletion each vertex has at most t neighbors that are not v_0. Hence

$$N'_s(G') \leq (n-(t+1))\binom{t}{s-1} + \binom{t+1}{s} \leq f_s(n,k-1,t),$$

a contradiction.

The same argument as in the proof of Theorem 55 also shows that $H(G',t+1)$ is a complete graph, otherwise there would be a cycle of length at least $2(t+2) \geq (k-1)+2$ in G'. Note that v_0 must be contained in $H(G',t+1)$ as it is adjacent to all vertices in G'. Set $|V(H(G',t+1))| = r$ where $t+3 \leq r \leq k-1$ (and so $k-r \geq 1$). In particular, $(k+1)-r \leq t+1$. Apply $(k+1-r)$-disintegration to G'. If $H(G',t+1) \neq H(G',k+1-r)$, then again we can find a cycle of length at least $(r-1)+k+2-r = k+1$. Otherwise, suppose $H(G',t+1) = H(G',k+1-r)$. In $H(G',t+1)$, the number of s-cliques that do not include v_0 is $\binom{r-1}{s}$, and in $V(G)-V(H(G',k+1-r))$, every vertex had at most $k-r$ neighbors that were not v_0 at the time of its deletion. We have

$$\begin{aligned} N'_s(G') &\leq \binom{r-1}{s} + (n+1-r)\binom{k-r}{s-1} \\ &= f_s(n,k-1,k-r) \leq \max\{f_s(n,k-1,1), f_s(n,k-1,t)\}, \end{aligned}$$

a contradiction. □

Proof of Corollary 58. Define $h_s(n,k) = \frac{n}{k-1}\binom{k-1}{s}$, and note that when $n \geq k$,

$$h_s(n,k) \geq \max\{f_s(n,k-1,t), f_s(n,k-1,1)\}.$$

We induct on the number of components in G. First suppose $k \geq 4$. If G is connected, then either $n \leq k-1$, in which case $N_s(G) \leq \binom{|V(G)|}{s} \leq h_s(n,k)$, or $n \geq k$ and $N_s(G) \leq \max\{f_s(n,k-1,1), f_s(n,k-1,t)\} \leq h_s(n,k)$. Otherwise if G is not connected, let C_1 be a component of G. Then $N_s(G) = N_s(C_1) + N_s(G-C_1) \leq h_s(|C_1|,k) + h_s(n-|C_1|,k) = h_s(n,k)$.

If $k=3$ (the cases $k \leq 2$ are not interesting), then the longest path in G has two vertices. It follows that G is the union of a matching and isolated vertices. Therefore $N_s(G) \leq h_s(n,k)$. □

Chapter 5

Berge hypergraphs

5.1 Introduction

In this chapter, we consider a generalization of the Turán problem for hypergraphs.
An early notion of cycles and paths in hypergraphs is due to Berge.

Definition 60. *A* **Berge cycle** *of length ℓ in a hypergraph is a set of ℓ distinct vertices $\{v_1, \ldots, v_\ell\}$ and ℓ distinct edges $\{e_1, \ldots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo ℓ.*
A **Berge path** *of length ℓ in a hypergraph in a hypergraph is a set of $\ell + 1$ vertices $\{v_1, \ldots, v_{\ell+1}\}$ and ℓ hyperedges $\{e_1, \ldots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq \ell$.*

We wish to generalize this notion to other graphs.
Fix a graph F. We say an r-uniform hypergraph $\mathcal{H}$ is a *Berge F* if there exists a bijective mapping $f : E(F) \to E(\mathcal{H})$ such that for every $xy \in E(F)$, $xy \subseteq f(xy)$.
We consider the following function:

$$\mathrm{ex}_r(n, F) := \max\{e(\mathcal{H}) : \mathcal{H} \subseteq \binom{[n]}{r}, \mathcal{H} \text{ is Berge } F\text{-free}\}.$$

Note that when forbidding Berge copies of graphs, we are actually forbidding a *family of subhypergraphs*, rather than a single subhypergraph.
The so-called Berge Turán number has recently become a popular area of research. In the following sections, we demonstrate the relationship between extremal hypergraph problems and generalizations of Turán problems for graphs. These lemmas will be helpful tools for proving extremal results in hypergraphs.

5.2 Reduction to graphs

Let F be a fixed graph. Recall the following parameter for graphs:

$$\mathrm{ex}(n, K_r, F) := \max\{N(G, K_r) : |V(G)| = n, G \text{ is } F\text{-free}\},$$

where $N(G, K_r)$ is the number of copies of K_r in G.
Gerbner and Palmer [GP17] proved upper and lower bounds for $\mathrm{ex}_r(n, F)$ in terms of Turán numbers for graphs.

Theorem 61 (Gerbner and Palmer [GP17])**.** *Fix $r \geq 2$ and let F be any graph. Then*

$$ex(n, F) \leq \mathrm{ex}_r(n, F) \leq \mathrm{ex}(n, K_r, F) + \mathrm{ex}(n, F).$$

In [FKL18a], Füredi, Kostochka, and I proved a strengthening of the result.

Definition 62. *For a hypergraph $\mathcal{H}$, a* **system of distinct representative pairs (SDRP) of** $\mathcal{H}$ *is a set of distinct pairs $A = \{\{x_1, y_1\}, \ldots, \{x_s, y_s\}\}$ and a set of distinct hyperedges $\mathcal{A} = \{f_1, \ldots f_s\}$ of $\mathcal{H}$ such that for all $1 \leq i \leq s$*
— *$\{x_i, y_i\} \subseteq f_i$, and*
— *$\{x_i, y_i\}$ is not contained in any $f \in \mathcal{H} - \{f_1, \ldots, f_s\}$.*

Let $\mathcal{H}$ be a hypergraph and p be an integer. The *p-shadow*, $\partial_p\mathcal{H}$, is the collection of the p-sets that lie in some edge of $\mathcal{H}$. In particular, we will often consider the 2-shadow $\partial_2\mathcal{H}$ of a r-uniform hypergraph $\mathcal{H}$ in which each edge of $\mathcal{H}$ yields a clique on r vertices.

Lemma 63. *Let $\mathcal{H}$ be a hypergraph, let $(A, \mathcal{A})$ be an* SDRP *of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$ and let $B = \partial_2\mathcal{B}$ be the 2-shadow of $\mathcal{B}$. For a subset $S \subseteq B$, let $\mathcal{B}_S$ denote the set of hyperedges that contain at least one edge of S. Then for all nonempty $S \subseteq B$, $|S| < |\mathcal{B}_S|$.*

Proof. Suppose there exists a nonempty set $S \subseteq B$ such that $|S| \geq |\mathcal{B}_S|$. Choose a smallest such S.
We claim that $|S| = |\mathcal{B}_S|$. Indeed, if $|S| > |\mathcal{B}_S|$ then $|S| \geq 2$ because $\mathcal{B}_S \neq \emptyset$ by definition. Take any edge $e \in S$. The set $S \setminus e$ is nonempty and $|S \setminus e| = |S| - 1 \geq |\mathcal{B}_S| \geq |\mathcal{B}_{S\setminus e}|$, a contradiction to the minimality of S.
Consider the case $|S| = |\mathcal{B}_S|$. By the minimality of S, each subset $S' \subset S$ satisfies $|S'| < |\mathcal{B}_{S'}|$. Therefore by Hall's theorem, one can find a bijective mapping of S to $\mathcal{B}_S$, where say the edge $e_i \in S$ gets mapped to hyperedge f_i in $\mathcal{B}_S$ for $1 \leq j \leq |S|$. Then $(A \cup \{e_i, \ldots, e_{|S|}\}, \mathcal{A} \cup \{f_1, \ldots, f_{|S|}\})$ is a larger SDRP of $\mathcal{H}$, a contradiction. $\square$

Lemma 64. *Let $\mathcal{H}$ be a hypergraph and let $(A, \mathcal{A})$ be an* SDRP *of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$, $B = \partial_2\mathcal{B}$, and let G be the graph on $V(\mathcal{H})$ with edge set $A \cup B$. If G contains a copy of a graph F, then $\mathcal{H}$ contains a Berge F on the same base vertex set.*

Proof. Let $\{v_1, \ldots, v_p\}$ and $\{e_1, \ldots, e_q\}$ be a set of vertices and a set of edges forming a copy of F in G such that the edges $e_1, \ldots, e_b$ belong to B. By Lemma 63, each subset S of $\{e_1, \ldots, e_b\}$ satisfies $|S| < |\mathcal{B}_S|$. So we may apply Hall's Theorem to match each of these e_i's to a hyperedge $f_i \in \mathcal{B}$. The edges $e_i \in A$ can be matched to distinct edges of $\mathcal{A}$ given by the SDRP. Since $\mathcal{A} \cap \mathcal{B} = \emptyset$ this yields a Berge F in $\mathcal{H}$ on the same base vertex set. □

We note that this Lemma 64 was proved independently by Gerbner, Methuku, and Palmer [GMP18]. We have $|\mathcal{H}| = |A| + |\mathcal{B}|$. Note that the number of r-edges in $\mathcal{B}$ is at most the number of copies of K_r in its 2-shadow. Therefore Lemma 64 gives a new proof for the result of Gerbner and Palmer [GP17].

5.3 Forbidding Berge paths and long Berge cycles

Recently, several interesting results were obtained for Berge paths and cycles. Notably, the results depend on the relationship between k and r.

Theorem 65 (Győri, Katona, and Lemons [GKL16])**.** *Let $\mathcal{H}$ be an n-vertex r-graph with no Berge path of length k. If $r \geq k \geq 3$, then $e(\mathcal{H}) \leq \frac{(k-1)n}{r+1}$. If $k > r+1 > 3$, then $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}$.*

Later, the remaining case $k = r+1$ was resolved by Davoodi, Győri, Methuku, and Tompkins [DGMT18].
Furthermore, the bounds in Theorem 65 and in [DGMT18] are sharp for each k and r for infinitely many n.
Győri, Methuku, Salia, Tompkins, and Vizer [GMS$^+$18] proved an asymptotic version of the Erdős–Gallai theorem for Berge paths in *connected* hypergraphs whenever r is fixed and n and k tend to infinity.

Theorem 66 (Győri, Methuku, Salia, Tompkins, and Vizer [GMS$^+$18])**.** *Let r be given. Let $\mathcal{H}_{n,k}$ be a largest r-uniform connected n-vertex hypergraph with no Berge path of length k. Then*
$$\lim_{k\to\infty}\left(\lim_{n\to\infty}\frac{e(\mathcal{H}_{n,k})}{k^{r-1}n}\right) = \frac{1}{2^{r-1}(r-1)!}.$$

In the following chapters, we will present analogous results for hypergraphs without long Berge cycles. The exact result for $k \geq r+3$ was obtained in [FKL18a]:

Theorem 67 (Füredi, Kostochka and Luo [FKL18a])**.** *Let $k \geq r+3 \geq 6$, and let $\mathcal{H}$ be an n-vertex r-graph with no Berge cycles of length k or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$.*

The case of $k \leq r-1$ was resolved by Kostochka and Luo [KL18].

Theorem 68 (Kostochka and Luo [KL18]). *Let $k \geq 4, r \geq k+1$, and let $\mathcal{H}$ be an n-vertex r-graph with no Berge cycles of length k or longer. Then $e(\mathcal{H}) \leq \frac{(k-1)(n-1)}{r}$.*

Later, Ergemlidze, Győri, Methuku, Salia, Thompkins, and Zamora [EGM+18] extended the results to $k \in \{r+1, r+2\}$, and Győri, Lemons, Salia, and Zamora [GLSZ18] extended the results to $k = r$.

Theorem 69 (Ergemlidze et al. [EGM+18]). *If $k \geq 4$ and $\mathcal{H}$ is an n-vertex r-graph with no Berge cycles of length k or longer, then $k = r+1$ and $e(\mathcal{H}) \leq n-1$, or $k = r+2$ and $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$.*

Theorem 70 (Győri et al. [GLSZ18]). *If $r \geq 3$ and $\mathcal{H}$ is an n-vertex r-graph with no Berge cycles of length r or longer, then $e(\mathcal{H}) \leq \max\{\lfloor \frac{n-1}{r} \rfloor (r-1), n-r+1\}$.*

Furthermore, stronger bounds for 2-connected hypergraphs were proved in [KL18] and [FKL19].

Chapter 6

Hypergraphs with bounded circumference and small uniformity

6.1 Introduction

In this chapter, we prove Theorem 13 which provides an upper bound for the maximum number of hyperedges in hypergraphs without long cycles, where the uniformity of the hypergraphs is small. This can be viewed as a hypergraph version of the Erdős–Gallai Theorem (Theorem 6). We will make use of Systems of Distinct Representatives (SDRPs) which were introduced in the previous section. This is joint work with Zoltán Füredi and Alexandr Kostochka.

6.2 Results for avoiding long Berge cycles

Our main result is an analogue of the Erdős–Gallai theorem on cycles for r-graphs. The following is a refinement of the statement of Theorem 13.

Theorem 71. *Let $r \geq 3$ and $k \geq r+3$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$. Moreover, equality is achieved if and only if $\partial_2\mathcal{H}$ is connected and for every block D of $\partial_2\mathcal{H}$, $D = K_{k-1}$ and $\mathcal{H}[D] = K_{k-1}^{(r)}$.*

Note that a Berge cycle can only be contained in the vertices of a single block of the 2-shadow. Hence the aforementioned sharpness examples cannot contain Berge cycles of length k or longer.

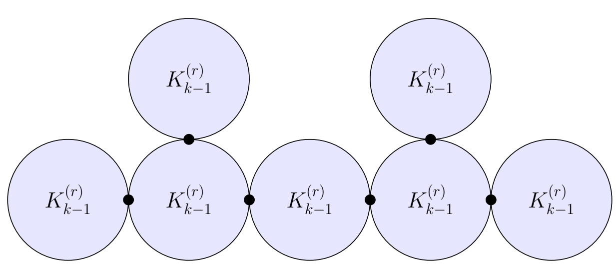

Figure 6.1: An extremal example for Theorem 71.

For convenience, below we will use notation

$$C_r(k) := \frac{1}{k-2}\binom{k-1}{r}. \tag{6.1}$$

(So $C_2(k)(n-1) = (k-1)(n-1)/2$.) Theorem 71 yields the following implication for paths.

Corollary 72. *Let $r \geq 3$ and $n \geq k+1 \geq r+4$. If $\mathcal{H}$ is a connected n-vertex r-graph with no Berge path of length k, then $e(\mathcal{H}) \leq C_r(k)(n-1)$.*

This gives a $\frac{k-2}{k-r}$ times stronger bound than Theorem 34 for connected r-graphs for all $r \geq 3$ and $n \geq k+1 \geq r+4$ and not only for sufficiently large k and n. In particular, Corollary 72 implies the following slight sharpening of Theorem 34 for $k \geq r+3$ in which we also describe the extremal hypergraphs.

Corollary 73. *Let $r \geq 3$ and $n \geq k \geq r+3$. If $\mathcal{H}$ is an n-vertex r-graph with no Berge path of length k, then $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}$ with equality only if every component of $\mathcal{H}$ is the complete r-graph $K_k^{(r)}$.*

6.3 Kopylov's Theorem and two inequalities

Recall again the definition of α-distintegration.

Definition: For a natural number α and a graph G, the *α-disintegration* of a graph G is the process of iteratively removing from G the vertices with degree at most α until the resulting graph has minimum degree at least $\alpha+1$ or is empty. This resulting subgraph $H(G,\alpha)$ will be called the *$(\alpha+1)$-core* of G. It is well known (and easy) that $H(G,\alpha)$ is unique and does not depend on the order of vertex deletion. If $H(G,\alpha)$ is the empty graph, then we say H is *α-disintegrable.*

We use a consequence of Kopylov [Kop77] about the structure of graphs without long cycles. We state it in the form that we need.

Theorem 74 (Kopylov [Kop77])**.** *Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. Suppose that G is a 2-connected n-vertex graph with no cycle of length at least k. Suppose that it is saturated, i.e., for every nonedge xy the graph $G \cup \{xy\}$ has a cycle of length at least k. Then either*

(74.1) *the t-core $H(G,t)$ is empty, i.e., G is t-disintegrable; or*

(74.2) *$|H(G,t)| = s$ for some $t+2 \leq s \leq k-2$, it is a complete graph on s vertices, and $H(G,t) = H(G,k-s)$, i.e., the rest of the vertices can be removed by a $(k-s)$-disintegration.*

Note that in the second case $2 \leq k-s \leq t$.

Lemma 75. *Let k, r, t, s, a nonnegative integers, and suppose $k \geq r+3 \geq 6$, $t = \lfloor (k-1)/2 \rfloor$, and $0 \leq a \leq s \leq t$. Then*

$$a + \binom{s-a}{r-1} \leq \frac{1}{k-2}\binom{k-1}{r} := C_r(k).$$

This is the part of the proof where we use $k \geq r+3$ because this inequality does not hold for $k = r+2$ (then the right hand side is $(r+1)/r$ while the left hand side could be as large as $\lfloor (r+1)/2 \rfloor$).

Proof. Keeping k, r, t, s fixed the left hand side is a convex function of a (defined on the integers $0 \leq a \leq s$). It takes its maximum either at $a = s$ or $a = 0$. So the left hand side is at most $\max\{s, \binom{s}{r-1}\}$. This is at most $\max\{t, \binom{t}{r-1}\}$. We have eliminated the variables a and s.
We claim that $t \leq \frac{1}{k-2}\binom{k-1}{r}$. Indeed, keeping k, t fixed, the right hand side is minimized when $r = k-3$, and then it equals to $(k-1)/2$. This is at least $\lfloor (k-1)/2 \rfloor = t$.
Finally, we claim that $\binom{t}{r-1} \leq \frac{1}{k-2}\binom{k-1}{r}$. If $t < r-1$, then there is nothing to prove. For $t \geq r-1$ rearranging the inequality we get

$$r \leq \frac{k-1}{t} \times \frac{k-3}{t-1} \times \cdots \times \frac{k-r}{t-r+2}.$$

Each fraction on the right hand side is at least 2. Since $r < 2^{r-1}$, we are done. $\square$

Lemma 76. *Let $w, r \geq 2$, $k \geq r+3$ and let $\mathcal{H}$ be a w-vertex r-graph. Let $\overline{\partial_2 \mathcal{H}}$ denote the family of pairs of $V(\mathcal{H})$ not contained in any member of $\mathcal{H}$ (i.e., the complement of the 2-shadow). Then*

$$|\mathcal{H}| + |\overline{\partial_2 \mathcal{H}}| \leq a_r(w) := \begin{cases} \binom{w}{2} & \text{for } 2 \leq w \leq r+2, \\ \binom{w}{r} & \text{for } r+2 \leq w. \end{cases}$$

Moreover, for $2 \leq w \leq k-1$, $|\mathcal{H}| + |\overline{\partial_2 \mathcal{H}}| = a_r(w)$ if and only if $w = k-1$ and either $w > r+2$ and $\mathcal{H}$ is complete, or $w = r+2$ and one of $\mathcal{H}$ or $\overline{\partial_2 \mathcal{H}}$ is complete.
Also, if $2 \leq w \leq k-1$, we have $a_r(w) \leq (w-1)\binom{k-1}{r}/(k-2) = C_r(k)(w-1)$.

Proof. The case of $w \geq r+2$ is a corollary of the classical Kruskal-Katona theorem, but one can give a direct proof by a double counting. If $\overline{\partial_2 \mathcal{H}}$ is empty, then $|\mathcal{H}| = \binom{w}{r}$ if and only if $\mathcal{H} = \binom{V(\mathcal{H})}{r}$. Otherwise, let $\overline{\mathcal{H}}$ denote the r-subsets of $V(\mathcal{H})$ that are not members of $\mathcal{H}$, $\overline{\mathcal{H}} = \binom{V(\mathcal{H})}{r} \setminus \mathcal{H}$. Each pair of $\overline{\partial_2 \mathcal{H}}$ is contained in $\binom{w-2}{r-2}$ members of $\overline{\mathcal{H}}$ and each $e \in \overline{\mathcal{H}}$

contains at most $\binom{r}{2}$ edges of $\overline{\partial_2 \mathcal{H}}$. We obtain

$$|\overline{\partial_2 \mathcal{H}}| \binom{w-2}{r-2} \le |\overline{\mathcal{H}}| \binom{r}{2}.$$

Since $\binom{w-2}{r-2} \ge \binom{r}{r-2} = \binom{r}{2}$, $|\overline{\partial_2 \mathcal{H}}| \le |\overline{\mathcal{H}}|$ with equality only when $w = r+2$. Furthermore, if $\overline{\partial_2 \mathcal{H}}$ and $\mathcal{H}$ are both nonempty, then for any $xy \in \overline{\partial_2 \mathcal{H}}$ and $uv \in \partial_2 \mathcal{H}$ (with possibly $x = u$), any r-tuple e containing $\{x, y\} \cup \{u, v\}$ is in $\overline{\mathcal{H}}$ but contributes strictly less than $\binom{r}{2}$ edges to $\overline{\partial_2 \mathcal{H}}$, implying $|\overline{\partial_2 \mathcal{H}}| < |\overline{\mathcal{H}}|$. This completes the proof of the case.
The case $w \le r+1$ is easy, and the calculation showing $a_r(w) \le C_r(k)(w-1)$ with equality only if $w = k-1$ is standard. □

6.4 Proof of Theorem 71, the main upper bound

Proof. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices with no Berge cycle of length k or longer ($k \ge r+3 \ge 6$). Let $(A, \mathcal{A})$ be an SDRP of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$, $B = \partial_2 \mathcal{B}$. By Lemma 64 the graph G with edge set $A \cup B$ does not contain a cycle of length k or longer.
Let $V_1, V_2, \ldots, V_p$ be the vertex sets of the standard (and unique) decomposition of G into 2-connected blocks of sizes $n_1, n_2, \ldots, n_p$. Then the graph $A \cup B$ restricted to V_i, denoted by G_i, is either a 2-connected graph or a single edge (in the latter case $n_i = 2$), each edge from $A \cup B$ is contained in a single G_i, and $\sum_{i=1}^{p}(n_i - 1) \le (n-1)$.
This decomposition yields a decomposition of $A = A_1 \cup A_2 \cup \cdots \cup A_p$ and $B = B_1 \cup B_2 \cup \cdots \cup B_p$, $A_i \cup B_i = E(G_i)$. If an edge $e \in B_i$ is contained in $f \in \mathcal{B}$, then $f \subseteq V_i$ (because f induces a 2-connected graph K_r in B), so the block-decomposition of G naturally extends to $\mathcal{B}$, $\mathcal{B}_i := \{f \in \mathcal{B} : f \subseteq V_i\}$ and we have $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$, and $B_i = \partial_2 \mathcal{B}_i$.
We claim that for each i,

$$|A_i| + |\mathcal{B}_i| \le C_r(k)(n_i - 1), \tag{6.2}$$

and hence

$$|\mathcal{H}| = |A| + |\mathcal{B}| = \sum_{i=1}^{p} |A_i| + |\mathcal{B}_i| \le \sum_{i=1}^{p} C_r(k)(n_i - 1) \le C_r(k)(n-1),$$

completing the proof.
To prove (6.2) observe that the case $n_i \le k-1$ immediately follows from Lemma 76. From now on, suppose that $n_i \ge k$.
Consider the graph G_i and, if necessary, add edges to it to make it a saturated graph with no cycle of length k or longer. Let the resulting graph be G'. Kopylov's Theorem (Theorem 74)

can be applied to G'. If G' is t-disintegrable, then make $(n_i - k + 2)$ disintegration steps and let W be the remaining vertices of V_i ($|W| = k-2$). For the edges of A_i and $\mathcal{B}_i$ contained in W we use Lemma 76 to see that

$$|A_i[W]| + |\mathcal{B}_i[W]| < C_r(k)(|W| - 1).$$

In the t-disintegration steps, we iteratively remove vertices with degree at most t until we arrive to W. When we remove a vertex v with degree $s \leq t$ from G', a of its incident edges are from A, and the remaining $s - a$ incident edges eliminate at most $\binom{s-a}{r-1}$ hyperedges from $\mathcal{B}_i$ containing v. Therefore v contributes at most $a + \binom{s-a}{r-1} \leq C_r(k)$ (by Lemma 75) to $|\mathcal{B}_i| + |A_i|$.

It follows that

$$|A_i| + |\mathcal{B}_i| < \left(\sum_{v \in G' - W} C_r(k) \right) + C_r(k)(|W| - 1) = C_r(k)(n_i - 1).$$

This completes this case.

Next consider the case (74.2), $W := V(H(G,t))$, $|W| = s \leq k-2$. We proceed as in the previous case, making $(n_i - s)$ disintegration steps. Apply Lemma 76 for $|A_i[W]| + |\mathcal{B}_i[W]|$ and Lemma 75 for the $(k-s)$-disintegration steps (where $k - s \leq t$) to get the desired upper bound (with strict inequality). This completes the proof of (6.2).

The extremal systems. Suppose that $e(\mathcal{H}) = |A| + |\mathcal{B}| = C_r(k)(n-1)$. Then $\sum_{i=1}^{p}(n_i - 1) = n-1$ (so $A \cup B$ is connected) and $|A_i| + |\mathcal{B}_i| = C_r(k)(n_i - 1)$ for each $1 \leq i \leq p$. From the previous proof and Lemma 76, we see that this holds if and only if for each i, $n_i = k-1$, and either $\mathcal{B}_i$ or A_i is complete. In particular, this implies that each block of $A \cup B$ is a K_{k-1}. We will show that each G_i corresponds to a block in $\mathcal{H}$ that is $K_{k-1}^{(r)}$ with vertex set V_i.

In the case that $\mathcal{B}_i$ is complete for all $1 \leq i \leq p$, we are done. Otherwise, if some A_i is complete ($n_i = k-1 = r+2$ by Lemma 76) then there are $\binom{k-1}{2} = \binom{k-1}{k-3} = \binom{k-1}{r}$ hyperedges in $\mathcal{A}$ intersecting V_i in at least two vertices. If all such hyperedges are contained in V_i, again we get $\mathcal{H}[V_i] = K_{k-1}^{(r)}$. So suppose there exists a $f \in \mathcal{A}$ which is paired with an edge $xy \in A_i$ in the SDRP, but for some $z \notin V_i$, $\{x, y, z\} \subseteq f$. Then z belongs to another block G_j of $A \cup B$. In $A \cup B$, there exists a path from x to z covering $V_i \cup V_j$ which avoids the edge xy. Thus by Lemma 64, there is a Berge path from x to z with at least $2(k-1)-1$ base vertices which avoids the hyperedge f (since edge xy was avoided). Adding f to this path yields a Berge cycle of length $2(k-1) - 1 > k$, a contradiction. $\square$

6.5 Corollaries for paths

In order to be self-contained, we present a short proof of a lemma by Győri, Katona, and Lemons [GKL16].

Lemma 77 (Győri, Katona, and Lemons [GKL16])**.** *Let $\mathcal{H}$ be a connected hypergraph with no Berge path of length k. If there is a Berge cycle of length k on the vertices $v_1, \dots, v_k$ then these vertices constitute a component of $\mathcal{H}$.*

Proof. Let $V = \{v_1, \dots, v_k\}$, $E = \{e_1, \dots, e_k\}$ form the Berge cycle in $\mathcal{H}$. If some edge, say e_1 contains a vertex v_0 outside of V, then we have a path with vertex set $\{v_0, v_1, \dots, v_\ell\}$ and edge set E. Therefore each e_i is contained in V. Suppose $V \neq V(\mathcal{H})$. Since $\mathcal{H}$ is connected, there exists an edge $e_0 \in \mathcal{H}$ and a vertex $v_{k+1} \notin V$ such that for some $v_i \in V$, say $i = k$, $\{v_k, v_{k+1}\} \subseteq e_0$. Then $\{v_1, \dots, v_k, v_{k+1}\}, \{e_1, \dots, e_{k-1}, e_0\}$ is a Berge path of length k. □

Proof of Corollary 72. Suppose $n \geq k+1$ and $\mathcal{H}$ is a connected n-vertex r-graph with $e(\mathcal{H}) > C_r(k)(n-1)$. Then by Theorem 71, $\mathcal{H}$ has a Berge cycle of length $\ell \geq k$. If $\ell \geq k+1$, then removing any edge from the cycle yields a Berge path of length at least k. If $\ell = k$, then by Lemma 77, $\mathcal{H}$ again has a Berge path of length k. □

Now Theorem 71 together with Corollary 72 directly imply Corollary 73.

Proof of Corollary 73: Suppose $k \geq r+3 \geq 6$ and $\mathcal{H}$ is an r-graph. Let $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_s$ be the connected components of $\mathcal{H}$ and $|V(\mathcal{H}_i)| = n_i$ for $i = 1, \dots, s$.
If $n_i \leq k-1$, then $|\mathcal{H}_i| \leq \binom{n_i}{r} < \frac{n_i}{k}\binom{k}{r}$. If $n_i \geq k+1$, then by Corollary 72, $|\mathcal{H}_i| \leq C_r(k)(n_i - 1) < \frac{n_i}{k}\binom{k}{r}$. Finally, if $n_i = k$, then $|\mathcal{H}_i| \leq \binom{k}{r} = \frac{n_i}{k}\binom{k}{r}$, with equality only if $\mathcal{H}_i = K_k^{(r)}$. This proves the corollary. □

Chapter 7

Hypergraphs with bounded circumference and large uniformity

7.1 Introduction

In this chapter we prove Theorem 14, an analogue of the Erdős–Gallai theorem for long Berge cycles in hypergraphs with large uniformity. In particular, we get bounds for general hypergraphs as well as 2-connected hypergraphs with some lower rank r. The methods we use differ from those of the previous chapter. In much of this work, we focus instead on proving results in the incidence bipartite graph of a hypergraph. This is joint work with Alexandr Kostochka [KL18].

7.2 Notation and results

7.2.1 Hypergraph notation

The *lower rank* of a multi-hypergraph $\mathcal{H}$ is the size of a smallest edge of $\mathcal{H}$.
In view of the structure of our proof, it is more convenient to consider hypergraphs with lower rank at least r instead of r-uniform hypergraphs. It also yields formally stronger statements of the results.
The *incidence graph* $G(\mathcal{H})$ of a multi-hypergraph $\mathcal{H} = (V, E)$ is the bipartite graph with parts V and E where $v \in V$ is adjacent to $e \in E$ iff in $\mathcal{H}$ vertex v belongs to edge e.
There are several versions of connectivity of hypergraphs. We will call a multi-hypergraph $\mathcal{H}$ *2-connected* if the incidence graph $G(\mathcal{H})$ is 2-connected.
A *hyperblock* in a multi-hypergraph $\mathcal{H}$ is a maximal 2-connected sub-multi-hypergraph of $\mathcal{H}$.

Definition 78. *For integers r, k with $r \geq k+1$, we call a multi-hypergraph with lower rank at least r an* $(r+1, k-1)$-block *if it contains exactly $r+1$ vertices and $k-1$ hyperedges.*

Definition 79. *A multi-hypergraph $\mathcal{H}$ with lower rank at least r is an* $(r+1, k-1)$-block-tree *if*

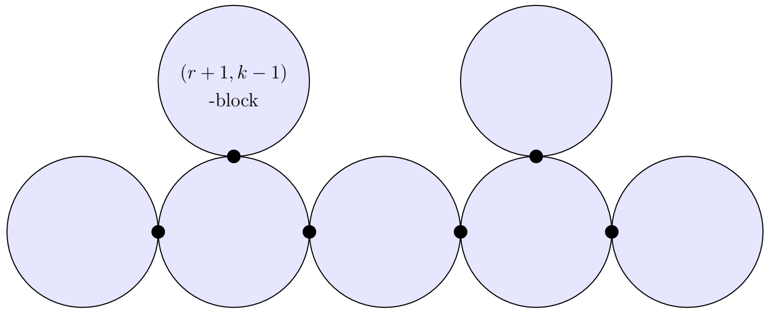

Figure 7.1: An $(r+1, k-1)$-block tree. Each hyperblock contains $r+1$ vertices and $k-1$ hyperedges.

(i) *every hyperblock of $\mathcal{H}$ is an $(r+1, k-1)$-block,*

(ii) *all cut-vertices of the incidence graph $G(\mathcal{H})$ of $\mathcal{H}$ are in V.*

An $(r+1, k-1)$-block cannot contain a Berge-cycle of length k or longer because it contains fewer than k edges. Therefore an $(r+1, k-1)$-block-tree also cannot contain such a cycle.

7.2.2 Results for hypergraphs

The following is a refinement of the statement of Theorem 14.

Theorem 80. *Let $k \geq 4, r \geq k+1$ and let $\mathcal{H}$ be an n-vertex multi-hypergraph such that $\mathcal{H}$ has lower rank at least r, and each edge of $\mathcal{H}$ has multiplicity at most $k-2$. If $\mathcal{H}$ has no Berge-cycles of length k or longer, then $e(\mathcal{H}) \leq \frac{(k-1)(n-1)}{r}$, and equality holds if and only if $\mathcal{H}$ is an $(r+1, k-1)$-block-tree.*

As a corollary of Theorem 80 we obtain a slight generalization of Theorem 65 [GKL16] (their result is for uniform hypergraphs without repeated edges):

Corollary 81. *Let $r \geq k+1 \geq 3$, and let $\mathcal{H}$ be an n-vertex multi-hypergraph such that $\mathcal{H}$ has lower rank at least r, and each edge of $\mathcal{H}$ has multiplicity at most $k-2$. If $\mathcal{H}$ has no Berge-paths of length k, then $e(\mathcal{H}) \leq \frac{(k-1)n}{r+1}$.*

Theorem 80 also implies the following analogue of the Erdős–Gallai theorem for cycles in r-uniform hypergraphs (without repeated edges).

Theorem 82 (Erdős–Gallai for hypergraphs)**.** *Let $k \geq 4, r \geq k+1$, and let $\mathcal{H}$ be an n-vertex r-graph with no Berge-cycles of length k or longer. Then $e(\mathcal{H}) \leq \frac{(k-1)(n-1)}{r}$. Furthermore, equality holds if and only if $\mathcal{H}$ is an $(r+1, k-1)$-block-tree.*

There is a phase transition when $r = k$. Let $\mathcal{H}$ be an r-uniform hypergraph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_{n-r+1}\}$, where $e_i = \{v_i\} \cup \{v_n, v_{n-1}, \ldots, v_{n-r+2}\}$. Then

the longest Berge-cycle in $\mathcal{H}$ has length $r-1$, and $m = n-(r-1)$. Thus when r is fixed and n is large, $e(\mathcal{H}) = n-r+1 > \frac{(r-1)(n-1)}{r}$. But when n is small, $\frac{(r-1)(n-1)}{r}$ is larger.
In [GLSZ18], it was proved that this construction and the aforementioned $(r+1, k-1)$-block-trees are optimal.
The key to our proof is a stronger version of Theorem 80 for multi-hypergraphs that are 2-connected.

Theorem 83. *Let $k \geq 4, r \geq k+1$ and let $\mathcal{H}$ be an n-vertex multi-hypergraph such that $\mathcal{H}$ is 2-connected, has lower rank at least r, and each edge of $\mathcal{H}$ has multiplicity at most $k-2$. If $\mathcal{H}$ contains no Berge-cycles of length k or longer, then*

$$e(\mathcal{H}) \leq \max\{k-1, \frac{k}{2r-k+2}(n-1)\}.$$

A proof similar to that of Corollary 81 (see the last section) yields the following result for paths in connected hypergraphs.

Corollary 84. *Let $r \geq k \geq 3$, and let $\mathcal{H}$ be a connected n-vertex r-graph with no Berge-path of length k. Then*

$$e(\mathcal{H}) \leq \max\{k-1, \frac{k}{2r-k+4}n\}.$$

Remark 85. *We do not know if the bound for $e(\mathcal{H})$ in Theorem 83 is sharp. But the following multi-hypergraph construction shows that when k is much smaller than r, our bound is asymptotically (when r tends to infinity) optimal: Let $k \geq 3$ be odd, $t \in \mathbb{N}$ and $V(\mathcal{H}_t) = \{a,b\} \cup V_1 \cup \ldots \cup V_t$ where $|V_i| = r-2$ for each $1 \leq i \leq t$, and the V_i's are pairwise disjoint. The edge set of $\mathcal{H}_t$ consists of $\frac{k-1}{2}$ copies of $V_i \cup \{a,b\}$ for each $1 \leq i \leq t$. Then each Berge-cycle in $\mathcal{H}_t$ intersects at most two V_i's and hence contains at most $k-1$ hyperedges. We also have*

$$e(\mathcal{H}_t) = \frac{k-1}{2}t = \frac{k-1}{2r-4}(n-2).$$

7.2.3 Results for bipartite graphs

By definition, a multi-hypergraph $\mathcal{H}$ has a cycle of length k if and only if the incidence graph $G(\mathcal{H})$ has a cycle of length $2k$. Also if $\mathcal{H}$ has lower rank r, then the degree of each vertex in one of the parts of $G(\mathcal{H})$ is at least r. In view of this we have studied bipartite graphs $G = (X, Y; E)$ with circumference at most $2k-2$ in which the degrees of all vertices in X are at least r. One of the main results (implying Theorem 80) is:

Theorem 86. *Let $k \geq 4, r \geq k+1$ and let $G = (X, Y; E)$ be a bipartite graph with $|X| = m$ and $|Y| = n$ such that $d(x) \geq r$ for every $x \in X$. Also suppose G has no blocks isomorphic* *to $K_{k-1,r}$. If $c(G) < 2k$, then $m \leq \frac{k-1}{r}(n-1)$. Moreover, if $m = \frac{k-1}{r}(n-1)$, then every block of G is a subgraph of $K_{k-1,r+1}$ and every cut vertex is in Y.*

The heart of the proof is the following stronger bound for 2-connected graphs.

Theorem 87. *Let $k \geq 4, r \geq k+1$ and let $G = (X, Y; E)$ be a bipartite 2-connected graph with $|X| = m$ and $|Y| = n$ such that $m \geq k$ and $d(x) \geq r$ for every $x \in X$. If $c(G) < 2k$, then $m \leq \frac{k}{2r-k+2}(n-1)$.*

In order to use induction on the number of blocks, we will prove a more general statement: We will allow some vertices in X to have degrees less than r and assign them a *deficiency*. Let $G = (X, Y; E)$ be a bipartite graph, and r be a positive integer. For a vertex $x \in X$, the *deficiency* of x is $D_G(x) := \max\{0, r - d_G(x)\}$. For a subset $X^* \subseteq X$, define the *deficiency of X^** as $D(G, X^*) := \sum_{x \in X^*} D_G(x)$.
In these terms our more general theorem is as follows.

Theorem 88 (Main Theorem)**.** *Let $k \geq 4$, $r \geq k+1$ and m, m^*, n be positive integers with $n \geq k$, $m \geq m^* \geq k-1$ and $m \geq k$. Let $G = (X, Y; E)$ be a bipartite 2-connected graph with parts X and Y, where $|X| = m$, $|Y| = n$, and let $X^* \subseteq X$ with $|X^*| = m^*$. If $c(G) < 2k$, then*

$$m^* \leq \frac{k}{2r-k+2}(n-1+D(G, X^*)). \tag{7.1}$$

7.3 Proof outline

As we discussed in the previous section, our main theorem is on bipartite graphs with circumference at most $2k-2$, based on a stronger result for 2-connected graphs.
In Section 4, we present a general theorem on the structure of 2-connected bipartite graphs with no long cycles and the most edges. In particular, we show that for $3 \leq d \leq (k-1)/2$, each such graph that is neither d-degenerate nor "too dense" contains a substructure that we call a "saturated crossing formation". In Section 5, we state the Main Theorem for 2-connected bipartite graphs that will be used to prove the inductive statement for general bipartite graphs. In Sections 5, 6, and 7 we show that if a graph is too sparse then it satisfies the Main Theorem, but if it is too dense, then it contains a long cycle. So our graphs must contain a path in saturated crossing formation, but we also prove that any graph that contains such a path satisfies our Main Theorem, a contradiction. In Section 8, we prove a bound on the size of X for general bipartite graphs, and in Section 9, we apply this bound to finally prove Theorem 80 for hypergraphs.

7.4 Structure of bipartite graphs without long cycles

Definition 89. *Let $P = v_1, \dots, v_p$ be a path with endpoints $x = v_1$ and $y = v_p$. For vertices v_i, v_j of P, let $P[v_i, v_j] = v_i, v_{i+1}, \dots, v_j$ if $i < j$, and $v_i, v_{i-1}, \dots, v_j$ if $i > j$.*

Definition 90. *Vertices v_i, v_j in P are called* **crossing neighbors** *if $i < j$, $v_i \in N(y), v_j \in N(x)$, and for each $i < \ell < j$, $v_\ell \notin N(x) \cup N(y)$. The edges xv_j, yv_i are called* **crossing edges**.

Definition 91. *For a set $S \subseteq V(P)$, define $S_P^+ := \{v_{i+1} : v_i \in S\}$ and $S_P^- := \{v_{i-1} : v_i \in S\}$. When there is no ambiguity, we will simply write $S^+ = S_P^+$ and $S^- = S_P^-$.*

Lemma 92. *Let G be a 2-connected bipartite graph, and let P be an (x, y)-path. Then either*

(a) P contains no crossing neighbors and G has a cycle of length at least $2(d_P(x)+d_P(y)-1)$, or

(b) x and y are in different partite sets of G and there exists a cycle of length at least

$$\min\{|V(P)|, 2(d_P(x) + d_P(y) - 1)\}$$

in G, or

(c) x and y are in the same partite set and there exists a cycle of length at least

$$\min\{|V(P)| - 1, 2(d_P(x) + d_P(y) - 2)\}$$

in G. Furthermore in all cases, we obtain a cycle that covers $N_P(x) \cup N_P(y)$.

Proof. Suppose first that P contains no crossing neighbors. Our proof is based off Bondy's theorem for general 2-connected graphs.
Let $P = v_1, \dots, v_p$ where $v_1 = x, v_p = y$. Let $t_0 = \max\{s : v_s \in N(x)\}$ and $u = \min\{s : v_s \in N(y)\}$, thus $t_0 \le u$. Iteratively construct paths $P_1, P_2, \dots$ as follows: given t_{r-1}, find s_r, t_r such that $s_r < t_{r-1} < t_r$ where t_r is as large as possible, and P_r is a path from v_{s_r} to v_{t_r} that is internally disjoint from P. It is always possible to find such a P_r because G is 2-connected. We stop at step ℓ at the first instance where $\ell > u$. Observe that for $r_1 < r_2$, paths P_{r_1} and P_{r_2} must be disjoint: if they share a vertex, then we would have chosen P_{r_1} to end at vertex v_{r_2}, contradicting the maximality of r_1. Also, $s_{r+1} \ge t_{r-1}$, otherwise we would choose P_{r+1} instead of P_r.
Now let $a = \min\{r : v_r \in N(x), r > s_1\}$, $b = \max\{r : v_r \in N(y), r < t_\ell\}$.

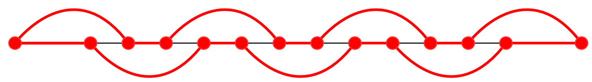

Figure 7.2: A cycle in a path without crossing neighbors.

If ℓ is odd, then we take the cycle

$$C_1 := P[x, vs_1] \cup P_1 \cup P[v_{t_1}, v_{s_3}] \cup P_3 \cup \ldots \cup P[v_{t_{\ell-2}}, v_{s_\ell}] \cup P_\ell \cup P[vt_\ell, y] \cup$$

$$\cup yv_u \cup P[v_u, v_{t_{\ell-1}}] \cup P_{\ell-1} \cup \ldots \cup P[v_{s_4}, v_{t_2}] \cup P_2 \cup P[v_{s_2}, v_{t_0}] \cup v_{t_0}x.$$

And if ℓ is even, we take the cycle

$$C_2 := P[x, v_{s_1}] \cup P_1 \cup P[v_{t_1}, v_{s_3}] \cup P_3 \cup \ldots \cup P[v_{t_{\ell-3}}, v_{s_{\ell-1}}] \cup P_{\ell-1} \cup P[v_{t_{\ell-1}}, v_u] \cup$$

$$\cup v_u y \cup P[y, v_{t_\ell}] \cup P_\ell \cup \ldots \cup P[v_{s_4}, v_{t_2}] \cup P_2 \cup P[v_{s_2}, v_{t_0}] \cup v_{t_0}x.$$

Both cycles cover $N_P(x) \cup N_P(y)$. If x and y are the same parity, since P contains no crossing neighbors, $|N_P(x) \cap N_P(y)| \leq 1$. Therefore P contains at least $d_P(x) + d_P(y) - 1$ even vertices, which implies $|V(C)| \geq 2(d_P(x)+d_P(y)-1)$ because G is bipartite. Otherwise, if x and y are different parities, then the neighbors of x and the successors of neighbors of y are disjoint, of the same parity, and are contained in C. Thus $|V(C)| \geq 2(d_P(x) + d_P(y))$, as desired.

Now suppose G has crossing neighbors v_i and v_j with $j < i$ and $xv_j, yv_i \in E(G)$. Let $C = P[x, v_i] \cup v_i y \cup P[y, v_j] \cup v_j x$. If $j = i + 1$, then C contains all vertices of P, as desired. If $j = i + 2$, then x and y must be the same parity, and C omits only vertex v_{i+1}. I.e., $|V(C)| = |V(P)| - 1$.

Consider first the case where x and y are different parities and each pair of crossing neighbors has at least 2 vertices between them. For every neighbor v_s of x in $P - \{v_j\}$ (note s is even), the odd vertex v_{s-1} is in C and is not a neighbor of y, otherwise v_{s-1} and v_s would form a pair of crossing neighbors. Also, each neighbor of y in P is in C. Thus C has at least $d_P(x) - 1 + d_P(y)$ odd vertices. That is, $|C| \geq 2(d_P(x) + d_P(y) - 1)$.

Now suppose x and y are the same parity and that crossing neighbors have at least 3 vertices between them. Let C be as before.

For any vertices $v_s \in N_P(x) - \{v_j\}$ and $v_t \in N_P(y) - \{v_i\}$, v_{s-1} and v_{t+1} are distinct and of the same parity (in this case, odd). Thus C contains at least $d_P(x) - 1 + d_P(y) - 1$ odd vertices. It follows that $|C| \geq 2(d_P(x) + d_P(y) - 2)$, as desired. $\square$

Definition 93. *Let G be a 2-connected bipartite graph, and let P be a path $v_1, \ldots, v_p$. We say that P is in* **crossing formation** *if there is a sequence of vertices $v_{i_0}, v_{i_1}, \ldots, v_{i_q}$ such that $v_i, v_{i'}$ are crossing neighbors if and only if $\{v_i, v_{i'}\} = \{v_{i_\ell}, v_{i_{\ell+1}}\}$ for some $0 \leq \ell \leq q-1$.*

Definition 94. *Let G be a 2-connected bipartite graph, and let P be an x, y-path $v_1, \ldots, v_p$. Say that P is in* **saturated crossing formation** *if*

1. *P is in crossing formation,*
2. *$G[\{v_1, v_2, \ldots, v_{i_0}\} \cup \{v_{i_0}, \ldots, v_{i_q}\}]$ and $G[\{v_{i_q}, \ldots, v_p\} \cup \{v_{i_0}, \ldots, v_{i_q}\}]$ are both complete bipartite,*
3. *if P has more than one pair of crossing neighbors, then each pair has exactly 3 vertices between them,*
4. *v_2 has a neighbor v_1' outside of P such that $N(v_1') = N(v_1)$, and v_{p-1} has a neighbor v_p' outside of P such that $N_P(v_p') = N_P(v_p)$,*
5. *for every even $h \leq i_0 - 2$ and every $u \in N_G(v_h)$, $N_G(u) \subseteq N_P(v_1)$, similarly, for every even $h \geq i_q + 2$ and every $w \in N_G(v_h)$, $N_G(w) \subseteq N_P(v_p)$; in particular, for every odd $g \leq i_0 - 1$, $N_G(v_g) \subseteq V(P)$ and for every odd $h \geq i_q + 1$, $N_G(v_h) \subseteq V(P)$.*

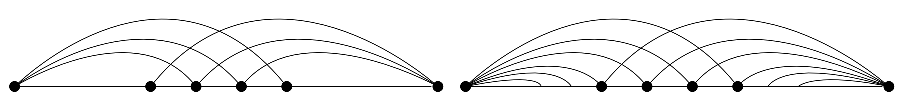

Figure 7.3: A crossing formation, a saturated crossing formation

Definition 95. *For a bipartite graph $G = A \cup B$, $\alpha \in \mathbb{N}$, and subsets $X^* \subseteq A$, $Y^* \subseteq B$, the* **$\alpha(X^*, Y^*)$-disintegration** *of G is the process of first deleting the vertices of $(A - X^*) \cup (B - Y^*)$ from G, then iteratively removing the remaining vertices of degree at most α until the resulting graph is either empty or has minimum degree at least $\alpha + 1$. Let $G_\alpha(X^*, Y^*)$ denote the result of applying $\alpha(X^*, Y^*)$-disintegration to G.*
In the case where $X^* = A$ and $Y^* = B$, in literature, $G_\alpha(X^*, Y^*)$ is commonly referred to as the $(\alpha + 1)$-core of G, that is, the unique maximum subgraph of G with minimum degree at least $\alpha + 1$.

Note that if $\alpha \geq (k-1)/2$, then $k - 1 - \alpha \leq \alpha$, so $G_\alpha(X^*, Y^*) \subseteq G_{k-1-\alpha}(X^*, Y^*)$.

Definition 96. *A bigraph $G = (X, Y; E)$ is $2k$-***saturated** *if $c(G) < 2k$, but for each $x \in X$ and $y \in Y$ with $xy \notin E(G)$, the graph $G + xy$ has a cycle of length at least $2k$.*

For example, if $s \leq k-1$ then for any t, the complete bipartite graph $K_{s,t}$ is $2k$-saturated, because it does not have $x \in X$ and $y \in Y$ with $xy \notin E(G)$

Theorem 97. *Fix $k \geq 3$ odd and let G be a 2-connected $2k$-saturated bipartite graph. For some $(k-1)/2 \leq \alpha \leq k-2$, and (X^*, Y^*), suppose there are $x \in V(G_\alpha(X^*, Y^*))$ and $y \in V(G_{k-1-\alpha}(X^*, Y^*))$ that are nonadjacent to each other. Let $P = v_1, \dots, v_p$ be a path with the following properties: (1) P is a longest path in G such that $v_1 \in G_\alpha(X^*, Y^*)$ and $v_p \in G_{k-1-\alpha}(X^*, Y^*)$, and (2) subject to the first condition, $\sum_{i=1}^{p} d_P(v_i)$ is maximized. Then P is in a saturated crossing formation.*

Proof. For simplicity, denote $G_\alpha = G_\alpha(X^*, Y^*)$ and $G_{k-1-\alpha} = G_{k-1-\alpha}(X^*, Y^*)$.
Because G is saturated and $xy \notin E(G)$, G contains an (x, y)-path with at least $2k$ vertices. Thus $p \geq 2k$. By the maximality of P, all neighbors of v_1 in G_α and all neighbors of v_p in $G_{k-1-\alpha}$ are in P. Thus

$$d_P(v_1) \geq \alpha + 1 \text{ and } d_P(v_p) \geq k - 1 - \alpha + 1. \tag{7.2}$$

By Lemma 92, G contains a cycle of length at least $2(d_P(x) + d_P(y) - 2) \geq 2((\alpha+1) + (k-1-\alpha+1) - 2) = 2(k-1)$. But $c(G) \leq 2k-2$, so P satisfies neither (a) nor (b) in Lemma 92. In particular, p is odd (so $p \geq 2k+1$), and G has crossing neighbors. Let v_i, v_j be a pair of crossing neighbors such that $v_i - v_j$ is minimized. Examining the proof of Lemma 92, each pair of crossing neighbors in P has at least 3 vertices between them. Furthermore, we obtain a cycle $C = P[v_1, v_i] \cup v_i v_p \cup P[v_p, v_j] \cup v_j v_1$ such that

I. $V(C) = V(P) - \{v_{i+1}, \dots, v_{j-1}\}$,

II. $|V(C)| = 2(d_P(x) + d_P(y) - 2)$, and

III. each odd vertex in C belongs to $N_P(v_1)^- \cup N_P(v_p)^+$ with $N_P(v_1)^- \cap N_P(v_p)^+ = \emptyset$.

In particular, since C misses only vertices between one pair of crossing neighbors, if C contains more than one pair of crossing neighbors, then each pair only contains 3 vertices between them, otherwise condition III. is violated. Thus Part 3 in the definition of saturated crossing formation holds.
First we show that

$$v_2 \text{ has a neighbor } v_1' \in G_\alpha \text{ outside of } P. \tag{7.3}$$

By Lemma 92, v_1 has exactly $\alpha + 1$ neighbors in P. So by the maximality of P, each of these neighbors must be in G_α. In particular, $v_2 \in V(G_\alpha)$, and so it has at least $\alpha + 1$ neighbors in G_α as well. Suppose that all of its neighbors in G_α are in P.
If v_2 has a neighbor $v_t \in N(v_p)^+$, then $P[v_2, v_{t-1}] \cup v_{t-1}v_p \cup P[v_p, v_t] \cup v_t v_2$ is a cycle of length at $|V(P)| - 1 \geq 2k$, a contradiction. So $N(v_2) \cap N(v_p)^+ = \emptyset$. Hence by fact

III, $N_{G_\alpha}(v_2) \subset N_P(v_1)^- \cup V(P-C)$. Since $v_{j-1} \in N_{G_\alpha}(v_1)^-$ but $v_{j-1} \notin V(C)$, we have $|N_P(v_1)^- \setminus V(P-C)| \leq \alpha$. Thus v_2 has a neighbor $v_h \in V(P-C)$. Furthermore, because $v_h \notin N(v_p)^+$, $i+3 \leq h \leq j-1$.

Let v_ℓ be the first neighbor of v_p that appears in P (so $v_{\ell-2} \in N(v_1)$ by III). Then the cycle

$$C' := v_2v_h \cup P[v_h, v_\ell] \cup v_\ell v_p \cup P[v_p, v_j] \cup v_jv_1 \cup v_1v_{\ell-2} \cup P[v_{\ell-2}, v_2]$$

has at least $|V(C)|+2$ vertices, a contradiction. This proves (7.3).

Consider the path $P' = v_1'v_2 \cup P[v_2, v_p]$. By definition, $|V(P')| = |V(P)|$. By the maximality of P, each neighbor of v_1' in G_α must also lie in P', and v_1' must have $\alpha+1$ such neighbors, otherwise we would apply Lemma 92 to get a longer cycle in G.

Suppose v_1' is adjacent to v_h for some $i+1 \leq h \leq j-1$. Then the cycle $P'[v_1', v_i] \cup v_iv_p \cup P'[v_p, v_h] \cup v_hv_1'$ is longer than C, a contradiction. Thus $N(v_1') \cap P \subseteq V(C)$. Then the analog of fact III for P' yields

$$N_P(v_1') = N_P(v_1). \tag{7.4}$$

By a symmetric argument, we get the analog of (7.3) and (7.4):

$$v_{p-1} \text{ has a neighbor } v_p' \text{ outside of } P \text{ such that } N_P(v_p') = N_P(v_p). \tag{7.5}$$

This shows that Part 4 of the definition of saturated crossing formation holds.

Again, let v_ℓ be the first neighbor of v_p in P. We claim that

$$\text{for each even } h \geq \ell, \text{ either } v_1v_h \notin E(G) \text{ or } v_1v_{h+2} \notin E(G). \tag{7.6}$$

Indeed, suppose $h \geq \ell$, $v_1v_h \in E(G)$ and $v_1v_{h+2} \in E(G)$. Then by (7.4), $v_1'v_h \in E(G)$ and by the definition of ℓ, $v_1v_{\ell-2} \in E(G)$. Then the cycle

$$C'' := v_1'v_h \cup P[v_h, v_\ell] \cup v_\ell v_p \cup P[v_p, v_{h+2}] \cup v_{h+2}v_1 \cup v_1v_{\ell-2} \cup P[v_{\ell-2}, v_2] \cup v_2v_1'$$

avoids only the vertices $v_{\ell-1}$ and v_{h+1} in P and includes $v_1' \notin P$. Thus $|V(C'')| \geq 2k$, a contradiction.

Similarly , if $v_{\ell'}$ is the last neighbor of v_1 in P, then

$$\text{for each even } h \leq \ell', \text{ either } v_pv_h \notin E(G) \text{ or } v_pv_{h-2} \notin E(G). \tag{7.7}$$

Together, (7.6) and (7.7) imply Part 1 of the definition of the saturated crossing formation holds, i.e. there is a sequence of vertices $v_{i_0}, v_{i_1}, , \ldots, v_{i_q}$ with $i_0 = i$ and $i_1 = j$ such that $v_r, v_{r'}$ are crossing neighbors if and only if $\{v_r, v_{r'}\} = \{v_{i_\ell}, v_{i_{\ell+1}}\}$ for some $0 \leq \ell \leq q-1$.

To see this, suppose there exists two pairs of crossing neighbors, $\{v_{a_1}, v_{b_1}\}$ and $\{v_{a_2}, v_{b_2}\}$ such that there are no other pairs of crossing neighbors between them, and $b_1 < a_2$. Then $a_2 \geq b_1 + 2$. Then by (7.6), $v_{b_1+2}v_1 \notin E(G)$. If $v_{b_1}v_p \notin E(G)$, then the vertex v_{b_1+1} violates fact III. Otherwise, if $v_{b_1}v_p \in E(G)$, then because there are no crossing pairs between $\{v_{a_1}, v_{b_2}\}$ and $\{v_{a_2}, v_{b_2}\}$, for each $b_1 < c \leq a_2$, $v_c v_1 \notin E(G)$. By condition III, this means each even vertex v_c between v_{b_1} and v_{a_2} belong to $N(v_p)$, contradicting (7.7).
Therefore we have proved that P is in crossing formation (Part 1 of the definition of saturated crossing formation). Let $v_{i_0}, \ldots, v_{i_q}$ be the set of crossing neighbors. By fact III,

$$\text{for each even } s \leq i_0,\ v_1v_s \in E(G), \text{ and for each even } t \geq i_q,\ v_pv_t \in E(G). \tag{7.8}$$

Next we will prove Part 5 in 3 steps. Our first step is to prove:

$$\text{For each odd } 1 \leq h < i_0,\ N(v_h) \subseteq N_P(v_1). \text{ In particular, } N(v_1) = N_P(v_1). \tag{7.9}$$

Indeed, suppose for some odd $1 \leq h < i_0$, vertex v_h has a neighbor $w \notin \{v_2, v_4, \ldots, v_{i_0-2}\} \cup \{v_{i_0}, \ldots, v_{i_q}\}$. Since G is 2-connected, $G - v_h$ contains a path $Q = w_1, \ldots, w_s$ from $w = w_1$ to $P - v_h + v_1'$ (possibly, $s = 1$ if $w \in P$) that is internally disjoint from P. If $w_s = v_1'$, then the path

$$P_1 = Q^{-1} \cup wv_h \cup P[v_1, v_h] \cup v_1v_{h+1} \cup P[v_{h+1}, v_p]$$

starts from $v_1' \in G_\alpha$ and is longer than P, a contradiction. Suppose now that $w_s = v_g$. Then $v_1' \notin V(Q)$. If $g > i_q$, then the cycle

$$C_1 = v_hw \cup Q \cup P[v_g, v_p] \cup v_pv_{2\lfloor (g-1)/2 \rfloor} \cup P[v_{2\lfloor (g-1)/2 \rfloor}, v_{h+1}] \cup v_{h+1}v_1 \cup P[v_1, v_h]$$

has at least $2k$ vertices, a contradiction. If $i_j < g \leq i_{j+1}$ for some $1 \leq j < q$, then the cycle

$$C_2 = v_hw \cup Q \cup P[v_g, v_p] \cup v_pv_{i_j} \cup P[v_{h+1}, v_{i_j}] \cup v_{h+1}v_1 \cup P[v_1, v_h]$$

is longer than C, unless $g = i_{j+1}$ and $s = 1$. But $g = i_{j+1}$ and $s = 1$ means $w = v_{i_{j+1}}$, contradicting the fact that $w \notin N_P(v_1)$. So suppose $1 \leq g \leq i_0$. If $|g - h| = 1$ then $s \geq 2$: if $s = 1$ then $Q = w = i_g \in N_P(v_1)$, a contradiction. But if $s \geq 2$, then replacing edge v_hv_g in P with Q, we obtain a longer (v_1, v_p)-path. So let $|g - h| \geq 2$. Since $v_1' \notin V(Q)$, if $g > h$, then the path

$$P_1 = P[v_1, v_{h-1}] \cup v_{h-1}, v_1' \cup v_1'v_{2\lfloor (g-1)/2 \rfloor} \cup P[v_{2\lfloor (g-1)/2 \rfloor}, v_h] \cup v_hw \cup Q \cup P[v_g, v_p]$$

has the same ends as P, but is longer than P, contradicting the choice of P. Similarly, if

$g < h$, then the path

$$P_2 = P[v_1, v_{2\lfloor(g-1)/2\rfloor}] \cup v_{2\lfloor(g-1)/2\rfloor}, v_1' \cup v_1'v_{h-1} \cup P[v_g, v_{h-1}] \cup Q \cup wv_h \cup P[v_h, v_p]$$

has the same ends as P, but is longer than P. This proves (7.9).
By the symmetry between v_1 and v_1', the same proof implies

$$N(v_1') = N_P(v_1). \tag{7.10}$$

Now let $h < i_0$ be even.

For any $g \in \{i_0, \ldots, p\} \setminus \{i_0, \ldots, i_q\}$, there is no v_h, v_g-path internally disjoint from P. (7.11)

Indeed, suppose such a path $Q = w_1, w_2, \ldots, w_s$ exists with $w_1 = v_h$ and $w_s = v_g$. If $i_j < g < i_{j+1}$ for some $1 \le j < q$, then the cycle

$$C_3 = Q \cup P[v_g, v_p] \cup v_p v_{i_j} \cup P[v_{h+2}, v_{i_j}] \cup v_{h+2}v_1 \cup P[v_1, v_h]$$

is longer than C unless $s = 2$ and $g = i_{j+1} - 1$. In this case, by (7.4), the cycle

$$v_h v_g \cup P[v_g, v_{i_0}] \cup v_{i_0} v_p \cup P[v_p, v_{g+1}] \cup v_{g+1} v_1 \cup P[v_1, v_{h-2}] \cup v_{h-2} v_1' \cup v_1' v_{i_0-2} \cup P[v_h, v_{i_0-2}]$$

has at least $2k$ vertices, a contradiction. So, suppose $g > i_q$. Then the cycle

$$C_4 = Q \cup P[v_g, v_p] \cup v_p v_{2\lfloor(g-1)/2\rfloor} \cup P[v_{2\lfloor(g-1)/2\rfloor}, v_{h+2}] \cup v_{h+2} v_1 \cup P[v_1, v_h]$$

has more than $2k - 2$ vertices, a contradiction. This proves (7.11).
To finish the proof of Part 5 by contradiction, suppose that for some even $h \le i_0 - 2$, vertex v_h has a neighbor u that has a neighbor $w \notin N_P(v_1)$. By (7.9), (7.10) and (7.11), $u \notin V(P) + v_1'$. Since u is in the same partite set of G as v_1, (7.11) implies that $w \notin V(P)$. Since G is 2-connected, $G - u$ has a path Q connecting w with $V(P) + v_1'$ internally disjoint from $P + v_1'$. Let $Q = w_1, \ldots, w_s$, where $w_1 = w$ and either $w_s = v_1'$ or $w_s = v_\ell \in V(P)$. By (7.9) and (7.10), $w_s \notin \{v_1, v_3, \ldots, v_{i_0-1}, v_1'\}$. So, in view of (7.11), $w_s = v_\ell \in V(P)$, where $\ell \in \{2, 4, \ldots, i_0 - 2\} \cup \{i_0, \ldots, i_q\}$. If $\ell \in \cup\{i_1, \ldots, i_q\}$, say $\ell = i_j$ then the cycle

$$C_5 = v_h u \cup uw \cup Q \cup P[v_{i_j}, v_p] \cup v_p v_{i_{j-1}} \cup P[v_{i_{j-1}}, v_{h+2}] \cup v_{h+2} v_1 \cup P[v_1, v_h]$$

is longer than C. The last possibility is that $1 \le g \le i_0$. Since G is 2-connected, we may assume that $g \ne h$ (indeed, if $g = h$, then $G - v_h$ has a path from $V(Q) - v_g + u$ to $V(P) + v_1'$ which together with a part of Q can play the role of Q). For definiteness, suppose $g > h$ (the case of $g < h$ works the same way with the roles of v_h and v_g switched). If $h < g \leq h+2$, then the path $P[v_1, v_h] \cup v_h u \cup uw \cup Q \cup P[v_g, v_p]$ has the same ends as P, but is longer. Let $g \geq h+3$. Then the path

$$v_1 v_{2\lfloor (g-1)/2 \rfloor} \cup P[v_{2\lfloor (g-1)/2 \rfloor}, v_{2\lceil (h+1)/2 \rceil}] \cup v_{2\lceil (h+1)/2 \rceil} v_1' \cup v_1' v_2 \cup P[v_2, v_h] \cup v_h u \cup uw \cup Q \cup P[v_g, v_p]$$

has the same ends as P, but is longer, a contradiction. Similarly, we obtain the symmetric part of Part 5.

Finally, we will show Part 2 of the definition of saturated crossing formation. Suppose there exists some odd $h \leq i_0 - 1$ such that for some $s \in \{i_0, \ldots, i_q\} \cup \{2, 4, \ldots, i_0 - 2\}$, $v_h v_s \notin E(G)$. By Part 5, $N(v_h) \subset N(v_1) = N(v_1')$. Also, $v_{h-1} v_1', v_{h+1} v_1' \in E(G)$ (by (7.8)), so we can replace v_h in P with v_1' to obtain a new path such that $\sum_{i=1}^{p} d_P(v_i) < \sum_{i=1}^{p} d_P(v_i) - d_P(v_h) + d_P(v_1')$, a contradiction. Together with the symmetric argument for the other side of P, we have shown that Part 2 of the definition of saturated crossing formation holds. $\square$

Let P be a path satisfying the conditions of Theorem 97. For simplicity, we denote

$$P = L \cup H_1 \cup \ldots \cup H_q \cup R$$

where $L = P[v_1, v_{i_0}]$, and $R = P[v_{i_q}, v_p]$, $V(H_1) \cap V(L) = \{v_{i_0}\}, V(H_t) \cap V(H_{t+1}) = \{v_{i_t}\}$ for all $1 \leq t \leq q-1$, and $V(H_q) \cap R = \{v_{i_q}\}$. Let $H := H_1 \cup \ldots \cup H_q$.

Lemma 98. *Let P satisfy the conditions of Theorem 97. Let $I := \{v_{i_0}, \ldots, v_{i_q}\}$, $L^o := L - I, R^o := R - I, H^o := H - I$. Then I separates L^o, R^o, and H^o. That is, L^o, R^o, and H^o are each in different connected components in $G - I$.*

Proof. Let $Q = z_1, z_2, \ldots, z_s$ be a shortest path that between vertices from two different sets in $\{L^o, R^o, H^o\}$. By minimality, Q only intersects P at z_1 and z_s. Also, $|V(Q)| \geq 3$ by Part 5 of the definition of saturated crossing formation.

Without loss of generality, $z_1 \in L^o$ (so $z_s \in H^o \cup R^o$). (Note that the case where Q goes from R^o to H^o is symmetric to the case from L^o to H^o.) By Part 5, since odd vertices in P only have neighbors in P, z_1 and z_s must be even. Also by Part 5, $N(z_2) \subseteq N(v_1) \subseteq L \cup X$. In particular, z_3 is in P, so we must have $z_3 = z_s$, but $z_s \in L \cup I$, where $(L \cup I) \cap (R^o \cup H^o) = \emptyset$, a contradiction. $\square$

Claim 99. *Under the conditions of Theorem 97, for any $0 \leq s < t \leq q$, let Q be a (v_{i_s}, v_{i_t})-path that is internally disjoint from P. Then*

1. *if P has exactly one pair of crossing neighbors (so $s = 0, t = 1$), then $|V(Q)| < k+1$.*

2. if P has multiple pairs of crossing neighbors, then $|V(Q)| < 6$.

Proof. First suppose P has only one pair of crossing neighbors. Then v_1 has $\alpha + 1 - 1$ neighbors in L. That is, $|V(L)| \geq 2\alpha \geq k-1$. If $|V(Q)| \geq k+1$, then the cycle $P[v_1, v_{i_0}] \cup Q \cup v_{i_1}v_1$ has length at least $k - 1 + k + 1 - 1 = 2k - 1$, a contradiction.
Otherwise, if P has more than one pair of crossing neighbors, then each pair has 3 vertices strictly between them in P. Suppose $|V(Q)| \geq 6$ (so there are at least 4 internal vertices). If $t = s+1$, then replacing $P[v_{i_s}, v_{i_{s+1}}]$ with Q gives a longer path with the same endpoints as P. So we may assume $s \geq t+2$. Then the cycle

$$P[v_1, v_{i_s}] \cup Q \cup P[v_{i_t}, v_p] \cup v_p v_{i_{t-1}} \cup P[v_{i_{t-1}}, v_{i_{s+1}}] \cup v_{i_{s+1}} v_1$$

has length at least $|V(P)| + 2$. □

Observe that because P is in crossing formation, $|V(L)|, |V(R)| \geq 4$ and $c(G) = 2(k-1) \geq |V(L)| + |V(R)| = 8$, thus $k \geq 5$.

7.5 The Main Lemma

Recall that the *deficiency* of a vertex $x \in X$ in a bipartite graph $G = (X, Y; E)$ is $D_G(x) := \max\{0, r - d_G(x)\}$. For a subset $X^* \subseteq X$, the *deficiency* of X^* as $D(G, X^*) := \sum_{x \in X^*} D_G(x)$.
Our goal is to eventually to prove the Main Theorem, Theorem 88.
The first big step is to prove the Main Lemma below that states roughly that graphs that contain a path in saturated crossing formation satisfy Theorem 88.

Lemma 100 (Main Lemma)**.** *Let $k \geq 5$ be odd, and let $G = (X, Y; E)$ and $X^* \subseteq X$ be a minimum (with respect to $|X|$) counterexample to Theorem 88. Fix any $X^* \subseteq X$ and set $Y = Y^*$. If $|Y| \geq k$ and P is a path as in the hypothesis of Theorem 97, then P is not in saturated crossing formation.*

7.5.1 Lemmas for induction

We first prove a series of lemmas. Often, we will use the following inductive argument:

Lemma 101. *Let $k \geq 4$. Let $G = (X, Y; E)$ and $X^* \subseteq X$ be a minimum (with respect to $|X|$) counterexample to Theorem 88. Suppose $|X| \geq k+1$, $|X^*| \geq k$, $|Y| \geq k$ and there exists a vertex $x \in X^*$ with $d(x) \leq k-2$. Then $G - x$ is not 2-connected.*

Proof. Suppose $G - x$ is 2-connected. As $d_G(x) \leq k-2$, we have $D_G(x) \geq r-k+2$. Since $|X - x| \geq k+1-1 = k$ and $|X^* - x| \geq k-1$, by the choice of G as a minimum counterexample, $G - x$ and $X^* - x$ satisfy

$$|X^*|-1 = |X^*-x| \leq \frac{k}{2r-k+2}(n-1+D(G', X^*-x)) \leq \frac{k}{2r-k+2}(n-1+D(G,X^*))-\frac{k(r-k+2)}{2r-k+2}.$$

Elementary calculation shows that $\frac{k}{2r-k+2}(r-k+2) \geq 1$ whenever $r \geq k-1$. Thus

$$|X^*| \leq \frac{k}{2r-k+2}(n-1+D(G,X^*)),$$

a contradiction.

□

Lemma 102. *Let $k \geq 4$. Let $G = (X, Y; E)$ and $X^* \subseteq X$ be a minimum (with respect to $|X|$) counterexample to Theorem 88. Suppose $|X| \geq k+1$, $|X^*| \geq k$, $|Y| \geq k$, and P is a path in G with an endpoint x. Suppose also that x has no neighbors outside of P, $d(x) \leq k-2$, and $x \in X^*$. Then there does not exist a vertex $x' \in V(G) - V(P)$ such that $N(x) \subseteq N(x')$.*

Proof. Suppose such a vertex x' exists. If $G - x$ is not 2-connected, then it contains a cut vertex v such that $(G-x)-v$ contains at least two components, C_1 and C_2, and v is the only vertex in $G - x$ with neighbors in both C_1 and C_2. Then in G, x and v form a cut set, and x and v are the only vertices in G with neighbors in both C_1 and C_2. As $N(x) \subseteq N(x')$, $v = x'$. Let y_1, and y_2 be neighbors of x such that $y_1 \in C_1$, and $y_2 \in C_2$. Because $N(x) \subseteq V(P)$, $y_1, y_2 \in V(P) - x$, but the path $P[y_1, y_2]$ is a (y_1, y_2)-path in G that avoids both x and x', a contradiction.

□

7.5.2 Paths in saturated crossing formation

Lemma 103. *Let $k \geq 5$ be odd, and let $G = (X, Y; E)$ and $X^* \subseteq X$ be a minimum (with respect to $|X|$) counterexample to Theorem 88. Fix any X^* and set $Y = Y^*$. If $|Y| \geq k$ and $P = v_1, \ldots, v_p$ is a path as in the hypothesis of Theorem 97, then the endpoints v_1 and v_p of P belong to the partite set Y of G.*

Proof. Suppose $v_1, v_p \in X$. By Lemma 92, one of the endpoints of P, say v_p, must satisfy $d_P(v_p) \leq \frac{k+1}{2}$. Since $v_2 \in G_\alpha(X^*) \subseteq X^*$, $v_{p-1} \in G_{k-1-\alpha}(X^*) \subseteq X^*$ and v_2 and v_{p-1} have no common neighbors by Lemma 98, we have $|X^*| \geq d_{G_\alpha}(v_2) + d_{G_{k-1-\alpha}}(v_{p-1}) \geq \alpha + 1 + k - \alpha = k+1$. Also, by Part 4 of the definition of saturated crossing formation, there exists a vertex $v_p' \in V(G) - V(P)$ with $N_P(v_p') = N_P(v_p)$. By Part 5 of the definition of saturated crossing formation, $N(v_p) = N_P(v_p)$, so $d(v_P) \leq k-2$. But the existence of v_p' contradicts Lemma 102. $\square$

Suppose $P = v_1, \ldots, v_p$ is in saturated crossing position. Denote $P = L \cup H_1 \cup ... \cup H_\ell \cup R$ as before.

Lemma 104. *Under the conditions of Theorem 97, let F be a component of $G - \{v_{i_0}, \ldots, v_{i_q}\}$ distinct from the components containing L and R. Then the $\frac{k-1}{2}(X^* \cap F, Y \cap F)$-disintegration of F is empty.*

Proof. Set $\alpha' = (k-1)/2$ and denote $F_{\alpha'} = G_{\alpha'}(X^* \cap F, Y \cap F) = G_{k-1-\alpha'}(X^* \cap F, Y \cap F)$. Because G is 2-connected, there are at least 2 neighbors of F in P, and so these neighbors must be contained in $\{v_{i_0}, \ldots, v_{i_q}\}$ by Lemma 98.
If $F_{\alpha'}$ is complete bipartite, then each part has size at least $\alpha' + 1 = (k+1)/2$, so we may find a path of length at least $k+1$ from some v_{i_s} to some v_{i_t} whose internal vertices are all from F, violating Lemma 99.
If $F_{\alpha'}$ is not complete bipartite, then fix a longest path $P_F = u_1, \ldots, u_{p'}$ with nonadjacent endpoints in $F_{\alpha'}$ such that $\sum_{i=1}^{p'} d_{P_F}(u_i)$ is maximized. Then by Theorem 97, P_F must be in saturated crossing formation. Again, u_1 has exactly $\alpha' + 1$ neighbors in $F_{\alpha'}$ in P_F. Furthermore, by Lemma 103, $u_1, u_{p'} \in Y$.
Denote $P_F = L' \cup H_1' \cup \ldots \cup H_{q'}' \cup R'$ where $H_1' \cap L' = \{u_{j_0}\}$, for each $0 \leq s \leq q'-1$, $H_s' \cap H_{s+1}' = u_{j_s}$, and $H_{q'}' \cap R' = \{u_{j_{q'}}\}$. There exists a cycle $C' = P_F[u_1, u_{j_0}] \cup u_{j_0} u_{p'} \cup P_F[u_{p'}, u_{j_1}] \cup u_{j_1} u_1$ which has length exactly $2(k-1)$.
Case 1: at most 1 vertex from $\{v_{i_0}, \ldots, v_{i_q}\}$ is contained in C'. Then because $\{v_{i_0}, \ldots, v_{i_q}\}$ separates F from $G - F$, C' never leaves $F \cup \{v_{i_0}, \ldots, v_{i_q}\}$. Choose two shortest disjoint paths P_s, P_t from $\{v_{i_0}, \ldots, v_{i_q}\}$ to $V(C')$ (possibly P_s or P_t may be a single vertex). Such paths exist because G is 2-connected. Furthermore, by choice of P_s and P_t, the paths each contain exactly one vertex from $\{v_{i_0}, \ldots, v_{i_q}\}$ and one vertex from C', and hence the paths cannot leave $F \cup \{v_{i_0}, \ldots, v_{i_q}\}$. Say P_s has endpoints v_{i_s} and $u_{s'} \in V(C')$ and P_t has endpoints v_{i_t} and $u_{t'} \in V(C')$.
Because $|V(C')| = 2(k-1)$, one of the $(u_{s'}, u_{t'})$-paths along C' must have at least $k-1$ edges, i.e., k vertices. Then because at least one of P_s or P_t has at least 2 vertices by the case, we have that $P_s \cup P_F[u_{s'}, u_{t'}] \cup P_t$ is a path of length at least $k+1$ from v_{i_s} to v_{i_t}, contradicting Lemma 99.
Case 2: at least 2 vertices from $\{v_{i_0}, \ldots, v_{i_q}\}$ are contained in C'. Let $N_F = \{v_{i_0}, \ldots, v_{i_q}\} \cap V(P_F)$. If any vertex $v_{i_s} \in N_F$ appears in L' or R', then because v_{i_s} is even, we have that $v_{i_s} \in N(u_1) \cup N(u_{p'})$. Therefore either $d_{P_F}(u_1) \geq (k+1)/2 + 1$ or $d_{P_F}(u_{p'}) \geq (k+1)/2 + 1$, which would give us a longer cycle by Lemma 92, a contradiction. Therefore we may assume

that each vertex in N_F appears in P_F strictly between some crossing neighbors. We will show that P_F has only one pair of crossing neighbors, in which case $N_F \cap V(C') = \emptyset$, leading to a contradiction.

Suppose not, then each pair of crossing neighbors have exactly 3 vertices strictly between them in P_F. Because each $v_{i_s} \in N_F$ is even, v_{i_s} must appear as the middle vertex between a pair of crossing neighbors, say $u_{j_{s'-1}}$ and $u_{j_{s'}}$, and there cannot be any other vertices from N_F in between these crossing neighbors. Furthermore, the predecessor and the successor of v_{i_s} in P_F belong to K since they are odd neighbors of $u_{j_{s'_1}}$ or $u_{j_{s'}}$ which are both in F. Thus P_F never leaves $K \cup \{v_{i_0}, \ldots, v_{i_q}\}$.

Suppose there exists $v_{i_s}, v_{i_t} \in N_F$ such that v_{i_s} and v_{i_t} appear between consecutive pairs of crossing neighbors in P_F. Say $v_{i_s} \in H'_{s'}$ and $v_{i_t} \in H'_{s'+1}$. Then the cycle $C'' = P_F[u_1, v_{i_s}] \cup [v_1, v_2, v_3] \cup P_F[v_{i_t}, u_{P'}] \cup u_{p'} u_{j_{s'}} \cup u_{j_{s'}} u_1$ omits only the successor of v_{i_s} and the predecessor of v_{i_t} in P_F and includes three additional vertices, v_1, v_2, v_3 from P. Therefore $|V(C'')| = |V(P_K)| - 2 + 3 > 2k$, a contradiction. Otherwise, each $v_{i_s}, v_{i_t} \in N_F$ appear in P_F between nonconsecutive pairs of crossing neighbors of P_F. Pick v_{i_s}, v_{i_t} such that no other vertex in N_F lies between them in P_F. Then $P_F[v_{i_s}, v_{i_t}]$ is a path with at least 9 vertices that is internally disjoint from P, contradicting Lemma 99. It follows that $F_{\alpha'}$ is empty. □

We are now ready to prove the Main Lemma.

Lemma 100. *Let $k \geq 5$ be odd, and let $G = (X, Y; E)$ and $X^* \subseteq X$ be a minimum (with respect to $|X|$) counterexample to Theorem 88. Fix any $X^* \subseteq X$ and set $Y = Y^*$. If $|Y| \geq k$ and P is a path as in the hypothesis of Theorem 97, then P is not in saturated crossing formation.*

Proof. Let $P = v_1, \ldots, v_p$ be the path in saturated crossing formation. Let C_L and C_R denote the connected components of $G - \{v_{i_0}, \ldots, v_{i_q}\}$ that contain $L - \{v_{i_0}\}$ and $R - \{v_{i_q}\}$ respectively. By Lemma 98, C_L and C_R are distinct. Let $D = C_L \cup C_R \cup \{v_{i_0}, \ldots, v_{i_q}\}$ and set $X' = X^* \cap D$, $n' = |Y \cap D|$.

By Lemma 103, $v_1, v_p \in Y$, and hence each odd vertex in P also belongs to Y. We first claim that there cannot be any X vertices in C_L outside of P: suppose such vertices exist, and pick a shortest path Q from $X - L$ to P with endpoints $x \in X - L$, $v \in L$. If v is odd, then $Q = vx$, but x is a neighbor of v outside of P, violating Part 5 of the definition of saturated crossing formation. If v is even, then Q contains at least 3 vertices, and the predecessor of v in Q is in X and has a neighbor outside of P, again violating Part 5. Therefore $X \cap C_L \subseteq L$. Similarly, $X \cap C_R \subseteq R$. It follows from Part 2 of the definition of saturated crossing formations that $X \cap D \subseteq N(v_1) \cup N(v_p)$.

Observe that we must have $X' = N(v_1) \cup N(v_p)$: by definition of saturated crossing formation, all neighbors of v_1 and all neighbors of v_p belong in $G_\alpha \cup G_{k-1-\alpha} \subseteq X^* \cap D = X'$.

Furthermore, $N(v_1) \cup N(v_p)$ contains all of the even vertices in $L \cup R \cup \{v_{i_0}, \ldots, v_{i_q}\}$ and therefore all of the X vertices in D. This proves that $X' = N(v_1) \cup N(v_p)$.
As v_1 and v_p share at least two neighbors (the crossing neighbors), we have

$$|X'| = |N(v_1) \cup N(v_p)| \leq (\alpha + 1) + (k - \alpha) - 2 = k - 1.$$

Furthermore, because v_2 and v_{p-1} share no neighbors,

$$n' + D(G, X') \geq d_G(v_2) + D_G(v_2) + d_G(v_{p-1}) + D_G(v_{p-1}) \geq 2r.$$

Putting these together, we have

$$\frac{|X'|}{n' - 1 + D(G, X')} \leq \frac{k-1}{2r-1},$$

and therefore

$$|X'| \leq \frac{k-1}{2r-1}(n' - 1 + D(G, X')) < \frac{k}{2r-k+2}(n' - 1 + D(G, X')). \tag{7.12}$$

Finally, for any component F of $G - \{v_{i_0}, \ldots v_{i_q}\}$ distinct from C_L and C_R, we have that the $\frac{k-1}{2}(X^* \cap F, Y \cap F)$-disintegration of F is empty by Lemma 104. Set $n_F = |Y \cap F|$. Each time we delete a vertex in the disintegration process, we delete at most $(k-1)/2$ edges until we reach the last $k-1$ vertices where there are at most $((k-1)/2)^2$ edges. Thus

$$e(F) \leq \frac{k-1}{2}(n_F + m_F - (k-1)) + \left(\frac{k-1}{2}\right)^2 = \frac{k-1}{2}(n_F + m_F - \frac{k-1}{2}).$$

As $e(F) \geq r|X^* \cap F| - D(G, (X^* \cap F))$, we have

$$|X^* \cap F| \leq \frac{\frac{k-1}{2}\left(n_F - \frac{k-1}{2}\right) + D(G, X^* \cap F)}{r - \frac{k-1}{2}} < \frac{k}{2r-k+2}(n_F + D(G, X^* \cap F)). \tag{7.13}$$

Combining (7.12) and (7.13),

$$\begin{aligned}
|X^*| &= |X'| + \sum_{F \neq C_L, C_R} |X^* \cap F| \\
&< \frac{k}{2r-k+2}(n' - 1 + D(G, X')) + \sum_{F \neq C_L, C_R} \frac{k}{2r-k+2}\left(n_F + D(G, X^* \cap F)\right) \\
&\leq \frac{k}{2r-k+2}(n - 1 + D(G, X^*)),
\end{aligned}$$

a contradiction. $\square$

7.6 Large complete bipartite subgraphs in extremal graphs

We will need three more lemmas to be used later in the Proof of Theorem 88.

Definition 105. *For a set U of vertices in a graph G, we say a U,U-*path *is a path whose ends are in U and all internal vertices are not in U.*

We will use several times the following simple property of 2-connected graphs.

Property 1. *Let G be a 2-connected graph, $U \subset V(G)$ with $|U| \geq 2$, and xy be an edge in $E(G)$ such that $\{x, y\} \not\subseteq U$. Then there is a U,U-path P containing xy.*

Lemma 106. *Let $m, n \geq k \geq 4$ be positive integers. Let $G = (X, Y; E)$ be a bipartite 2-connected graph with $|X| = m$, $|Y| = n$ and $c(G) < 2k$. Suppose G contains a copy K of $K_{k-1,k-2}$ with parts $A \subset X$ and $B \subset Y$ such that $|A| = k - 1$. Then*

$$|N(Y - B)| = 2 \quad \text{or} \quad |N(Y - B) \cap A| \leq 1. \tag{7.14}$$

Proof. Suppose that $G = (X, Y; E)$ is a bipartite 2-connected graph with $|X| = m \geq k$, $|Y| = n \geq k$ and $c(G) < 2k$ containing a copy $K = (A, B; E_1)$ of $K_{k-1,k-2}$ with $|A| = k-1$, $|B| = k-2$, $A \subset X$ and $B \subset Y$. Suppose further that (7.14) does not hold, i.e., that

$$|N(Y - B)| \geq 3 \quad \text{and} \quad |N(Y - B) \cap A| \geq 2.$$

First, we remark that

$$\text{each } A \cup B, A \cup B\text{-path in } G \text{ contains at most one vertex in } Y - B. \tag{7.15}$$

Indeed, if an $A \cup B, A \cup B$-path P contains two vertices in $Y - B$, then $G[A \cup B \cup V(P)]$ has a cycle C that contains $B \cup V(P)$. This C has at least k vertices in Y, and hence $|C| \geq 2k$, contradicting $c(G) < 2k$. This proves (7.15).

Case 1: There is $y_1 \in Y - B$ with $|N(y_1) \cap A| \geq 2$. Suppose $N(y_1) \cap A = \{a_1, \ldots, a_q\}$.
Case 1.1: $V(G) - A - B - y_1$ has an edge xy_2. By Property 1, there is an $(A \cup B, A \cup B)$-path P containing xy_1. Let $P = w_1 w_2 \ldots w_h$, $x = w_j$ and $y_1 = w_{j+1}$ for some $2 \leq j \leq h - 2$. By (7.15), $y_1 \notin P$ and $j = 2$. In particular, $w_1 \in B$. Let $P' = a_1 y_1 a_2$. Then $G[A \cup B \cup V(P) \cup \{y_1\}]$ has a cycle C containing $B \cup P \cup P'$ and hence at least k vertices in Y. So $|C| \geq 2k$, contradicting $c(G) < 2k$.

Case 1.2: $V(G) - A - B - y_1$ is an independent set. Then any $y_2 \in Y - A - y_1$ has at least two neighbors in A. So by the Case 1.1 for y_2 in place of y_1, $V(G) - A - B - y_2$ is an independent set. Then $V(G) - A - B$ is an independent set. In particular, any $x \in X - A$ has two neighbors in B. So, the graph $G[A \cup B \cup \{y_1, y_2, x\}]$ has no cycle containing $B \cup \{y_1, y_2\}$ only if $q = 2$ and $N(y_2) \cap A = \{a_1, a_2\}$. Trying each $y \in Y - B - y_1$ as y_2, we conclude that $\bigcup_{y \in Y} N(y) = \{a_1, a_2\}$, so (7.14) holds.

Case 2: $|N(y) \cap A| \leq 1$ for every $y \in Y - B$. Because $|N(Y - B) \cap A| \geq 2$, there are distinct $a_1, a_2 \in A$ adjacent to $Y - B$. Let $a_1y_1, a_2y_2 \in E(G)$ where $y_1, y_2 \in Y - B$. By the case, $y_2 \neq y_1$. By Property 1, for $j = 1, 2$ there is an $(A \cup B, A \cup B)$-path $P_j = w_{1,j}, w_{2,j}, \ldots, w_{h_j,j}$ containing a_jy_j. Since $a_j \in A$, we may assume $w_{1,j} = a_j$ and $w_{2,j} = y_j$. By the case $w_{3,j} \notin A$, and so $h_j \geq 4$. Furthermore, by (7.15), $w_{4,j} \in B$, and so $h_1 = h_2 = 4$. If $w_{3,2} = w_{3,1}$, then the path $a_1, y_1, w_{3,1}, y_2, a_2$ contradicts (7.15). So, paths P_1 and P_2 are internally disjoint and have at most one common end. Thus $G[A \cup B \cup V(P_1) \cup V(P_2)]$ has a cycle C containing $B \cup V(P_1) \cup V(P_2)$, which implies $|C| \geq 2k$, contradicting $c(G) < 2k$. □

Lemma 107. *Let H be a bipartite graph with parts A and B, where $|B| = g \geq 2$. Suppose H has no isolated vertices and for each $b \in B$, $d(b) \geq g$. Then either (i) $H = K_{g,g}$ or (ii) there exist disjoint paths $Q_1, \ldots, Q_\ell$ such that for each $1 \leq i \leq \ell$, Q_i has both ends in A, and $B \subset V(Q_1 \cup \ldots \cup Q_\ell)$.*

Proof. We proceed by induction. If $g = 2$ and $H \neq K_{2,2}$, then H contains either a P_5 or two disjoint copies of P_3, both of which satisfy (ii). Now let $g > 2$. Fix $ab \in E(G)$ such that $a \in A, b \in B$. Set $B' = B - \{b\}$ and $A' = N(B') - \{a\}$. Then $H' = H[B' \cup A']$ satisfies the conditions of the lemma for $g - 1$.
Suppose first that $H' = K_{g-1,g-1}$. Then because each vertex $b' \in B'$ has exactly $g - 1$ neighbors in H' and at least g neighbors in H, $b'a \in E(H)$ for each $b' \in B'$. If b has no neighbors outside $A' \cup \{a\}$, then $G = K_{a,a}$. Otherwise, if b has a neighbor $a' \in A - A' - \{a\}$, we may take any path P with $2g$ vertices starting with a and covering $A' \cup B'$ and append the edge ba' to P.
If $H' \neq K_{g-1,g-1}$, then let $Q'_1, \ldots, Q'_q$ be the set of paths satisfying (ii) for H'. If b has a neighbor $a' \in A - \{a\} - \bigcup_{i=1}^{q} V(Q'_i)$, then we take the set of paths $Q'_1, \ldots, Q'_q, aba'$. Otherwise, all neighbors of b are in $N(B') + a$. In particular, b has at least $g - 1$ neighbors distinct from a. But each Q'_i has fewer internal vertices in A than in B. Thus paths $Q'_1, \ldots, Q'_q$ together have at most $g - 2$ internal vertices in A. Thus b has a neighbor a' that is an end of a path, say of Q'_q. Then we append the path a', b, a to Q'_q. □

Lemma 108. *Let $G = (X, Y; E)$ and $X^* \subseteq X$ be a counterexample to Theorem 88 with minimum $|X|$. Then G cannot contain a complete bipartite subgraph $G' = K_{s,t}$ with parts*

$A \subseteq X^*$ *and* $B \subseteq Y$ *such that*

$$|A| = t \geq k \text{ and } |B| = s \text{ with } k/2 \leq s \leq k-2. \tag{7.16}$$

Proof. Suppose that such a $K_{s,t}$ exists. We may assume that s and t are largest possible, i.e. each $x \in X - A$ has a nonneighbor in B and each $y \in Y - B$ has a nonneighbor in A. Consider a mixed $(k-s, k-s-1)(X^*, Y)$-disintegration of G: we first delete all vertices from $X - X^*$ and then consecutively delete remaining vertices in X if their degrees in the current graph are at most $k-s$ and vertices in $Y - B$ if their degrees in the current graph are at most $k-s-1$[1]. Let G_0 be the resulting graph. If $G_0 = G' = K_{s,t}$, then by (7.16),

$$\begin{aligned} rm^* - D(G, X^*) \leq e(G) &\leq st + (m^* - t)(k-s) + (n-s)(k-s-1) \\ &= (2s-k)t + m^*(k-s) + (n-s)(k-s-1) \qquad (7.17) \\ &\leq ((2s-k) + (k-s))m^* + (n-s)(k-s-1) \\ &= sm^* + (n-s)(k-s-1), \end{aligned}$$

and hence

$$\begin{aligned} m^* &\leq \frac{(k-1-s)(n-s) + D(G,X^*)}{r-s} < \frac{(k-1-s)(n-1+D(G,X^*))}{r-k+s} \\ &\leq \frac{(k-1-(k/2))(n-1+D(G,X^*)}{r-(k/2)} = \frac{(k-2)(n-1+D(G,X^*)}{2r-k}, \end{aligned}$$

but $\frac{k-2}{2r-k} < \frac{k}{2r-k+2}$, a contradiction. Thus, suppose $G_0 \neq G''$, and the partite sets of G_0 are $A \cup A'$ and $B \cup B'$.

Since G is $2k$-saturated and G_0 is not complete bipartite, there exist paths with at least $2k$ vertices both ends of which are in $V(G_0)$ and at least one end in $A \cup B$. Among such paths choose a path $P = v_1, \ldots, v_p$ with $v_1 \in A \cup B$ so that

(P1) p is maximum possible,

(P2) modulo (P1), $d_G(v_p)$ is maximum, and

(P3) modulo (P1) and (P2), P has as many vertices from A as possible.

Our first observation is

$$v_1 \in A. \tag{7.18}$$

Indeed, if $v_1 \in B$, then by (P1), each $a \in A$ is in P and $d_P(v_p) \geq k-s$. Thus by $t \geq k$ and

[1]Note that we do not delete vertices in A even when $s = k/2$ and they have degree $k/2 = k-s$ in the current graph.

$s \leq k-2$,

$$d_P(v_1) + d_P(v_p) \geq t + (k-s) \geq k+2.$$

So by Lemma 1.2, $c(G) \geq 2k$, a contradiction.
By (7.18) and (P1),

$$B \subseteq V(P). \tag{7.19}$$

Case 1: $d_P(v_p) \geq k-s+1$. Then Lemma 2.1 implies

$$d_P(v_1) = s, d_P(v_p) = k-s+1,\ p \text{ is odd, and } P \text{ has crossing neighbors } v_{i_1} \text{ and } v_{i_2}. \tag{7.20}$$

Let $C = P[v_1, v_{i_1}] \cup v_{i_1}v_p \cup P[v_{i_2}, v_p] \cup v_{i_2}v_1$. By the choice of G, $|C| \leq 2k-2$. Since $N_P(v_1)^- \cap N_P(v_p)^+ = \emptyset$, $|N_P(v_1)^-| + |N_P(v_p)^+| \geq k+1$, and C does not contain only two vertices from $N_P(v_1)^- \cup N_P(v_p)^+$,

$$|C| = 2k-2 \text{ and each } v_i \in C \cap (A \cup A') \text{ is in } N_P(v_1)^- \cup N_P(v_p)^+. \tag{7.21}$$

By (7.19) and (7.20), $N_P(v_1) = B$. In particular, $v_2 \in B$. If $A \subset V(P)$, then by (7.16) and $s \leq k-2$, for the path $P' = P - v_1$ we have $d_{P'}(v_2) + d_{P'}(v_p) \geq |A - v_1| + (k-s+1) \geq (t-1) + 3 \geq k+2$. In this case, by Lemma 1.2, $c(G) \geq 2k$, a contradiction. Thus, there is a vertex $a \in A - V(P)$. So, if for some $3 \leq i \leq p-2$, vertices v_{i-1} and v_{i+1} are in B, then the path P'' obtained from P by replacing v_i with a has the same length and ends as P. Hence (P3) implies that

$$\text{if for some } 3 \leq i \leq p-2, \text{ vertices } v_{i-1} \text{ and } v_{i+1} \text{ are in } B, \text{ then } v_i \in A. \tag{7.22}$$

Case 1.1: v_1 has no neighbors outside of P. Thus $D_G(v_1) = r - s \geq r - (k-2)$. As v_1 is contained in the $K_{s,t}$ and has no neighbors outside of B, it is easy to see that $G - v_1$ is 2-connected. Since $|V(P)| \geq 2k$ (and therefore $|V(P)| \geq 2k+1$ since $|V(P)|$ is odd), $|X - v_1| \geq k$. Furthermore, since $A \subseteq X^*$, $|X^*| \geq k$ and so $|X^* - v_1| \geq k-1$. Applying Lemma 101 yields a contradiction.
Case 1.2: v_1 has a neighbor $z \in N(v_1) - V(P)$. Let $Q = w_1, \ldots, w_j$ be a path from $z = w_1$ to $P - v_1$ in $G - v_1$. Suppose $w_j = v_h$. Let $Q' = v_1 z \cup Q$. Since $z \notin V(P)$, $j \geq 2$. We claim that

$$\text{for } h-4 \leq g \leq h-1,\ v_g \notin N(v_p). \tag{7.23}$$

Indeed, otherwise the cycle $P[v_1, v_g] \cup v_g v_p \cup P[v_h, v_p] \cup Q'$ would have at least

$$2k+1-(h-g-1)+(j-1) \geq 2k+1-3+1 = 2k-1$$

vertices. This contradicts the choice of G.
Similarly, we show

$$v_h \notin P[v_{i_1+1}, v_{i_2}]. \tag{7.24}$$

Indeed, if $i_1 + 1 \leq h \leq i_2$, then the cycle $P[v_1, v_{i_1}] \cup v_{i_1}v_p \cup P[v_h, v_p] \cup Q'$ would have at least $|C| + 1$ vertices, which means at least $2k$ vertices.
Also

$$\{v_{h-2}, v_{h-1}\} \cap B = \emptyset. \text{ Since } v_{i_2} \in N_P(v_1) = B, \text{ this yields } h \notin \{i_2 + 1, i_2 + 2\}. \tag{7.25}$$

Indeed if $h - 2 \leq g \leq h - 1$ and $v_g \in B$, then the path $P[v_g, v_1] \cup Q' \cup P[v_h, v_p]$ starts from $v_g \in B$ and is longer than P (because if $g = h - 2$ then by parity, $j \geq 3$).
Let $\alpha \in \{0, 1\}$ be such that $h - 2 - \alpha$ is odd. Then by (7.25), $v_{h-2-\alpha} \notin B^- = N_P(v_1)^-$, by (7.23), $v_{h-2-\alpha} \notin N_P(v_p)^+$, and by (7.24) and (7.25), $v_{h-2-\alpha} \notin P[v_{i_1+1}, v_{i_2-1}]$. But this contradicts (7.21).

Case 2: $d_P(v_p) = k - s$ and $p \geq 2k + 2$. Then $v_p \in B'$ or k is even. Lemma 92 together with (7.18) implies

$$d_P(v_1) = s \text{ and } P \text{ has crossing neighbors } v_{i_1} \text{ and } v_{i_2}. \tag{7.26}$$

As in Case 1, let $C = P[v_1, v_{i_1}] \cup v_{i_1}v_p \cup P[v_p, v_{i_2}] \cup v_{i_2}v_1$. By the choice of G, $|C| \leq 2k - 2$. By (7.19) and the definition of C, $B \subset V(C)$ and only one vertex in $N_P(v_p)^+$ is not in C. So since $B \cap N_P(v_p)^+ = \emptyset$ and $|B| + |N_P(v_p)^+| = s + (k - s) = k$,

$$|C| = 2k - 2 \text{ and each } v_i \in C \cap Y \text{ is in } B \cup N_P(v_p)^+. \tag{7.27}$$

By (7.19) and (7.27), $N_P(v_1) = B$. In particular, $v_2 \in B$. Repeating the proof of (7.22), we derive that it holds also in our case.
Again, as in Case 1.1, if v_1 has no neighbors outside of P, we obtain $m^* \leq \frac{k}{2r-k+2}(n - 1 + D(G, X^*))$. So we may assume there exists $z \in N(v_1) - V(P)$ and a path $Q' = v_1, w_1, \ldots, w_j$ from v_1 through $w_1 = z$ to $P - v_1$ internally disjoint from P. Suppose $w_j = v_h$. Then repeating the proofs word by word, we derive that (7.23), (7.24) and (7.25) hold in our case, as well.
Let $\beta \in \{0, 1\}$ be such that $h-1-\beta$ is even. Then by (7.25), $v_{h-1-\beta} \notin B$, by (7.23), $v_{h-1-\beta} \notin N_P(v_p)^+$, and by (7.24) and (7.25), $v_{h-1-\beta} \notin P[v_{i_1+1}, v_{i_2-1}]$. But this contradicts (7.27).

Case 3: $d_P(v_p) = k - s$ and $p = 2k$. We claim that

$$A' = \emptyset \text{ and } d_{G_0}(b') = k - s \text{ for each } b' \in B'. \tag{7.28}$$

Indeed, if there is $a' \in A'$, then a' has a nonneighbor $b \in B$ and since G is saturated, it contains an (a', b)-path P' with at least $2k$ vertices. Choose P' to satisfy (P1), (P2), and (P3). By the case, P' has exactly $2k$ vertices, and $A \subseteq V(P')$ otherwise we could extend P'. But then $|V(P')| \geq 2|A| + 1 > 2k$, a contradiction. Similarly, if there is $b' \in B'$ with $d_{G_0}(b') \geq k - s + 1$, then b' has a nonneighbor $a \in A$, but any (a, b')-path P' with at least $2k$ vertices contradicts the choice of P in our case. This proves (7.28).
If $|B'| \leq s$, then by (7.28), instead of (7.6) we have

$$rm^* - D(G, X^*) \leq st + (m^* - t)(k - s) + (n - s)(k - s - 1) + s$$

$$= (2s - k)t + m^*(k - s) + n(k - s - 1) - s(k - s - 2),$$

which is maximized when $t = m^*$. Because $k - s - 2 \geq 0$, this yields

$$rm^* \leq ((2s - k) + (k - s))m^* + n(k - s - 1) + D(G, x^*) = sm^* + n(k - s - 1) + D(G, X^*),$$

and hence as before,

$$m^* \leq \frac{(k - s - 1)n + D(G, X^*)}{r - s} \leq \frac{k}{2r - k + 2}(n - 1 + D(G, X^*)),$$

a contradiction. So suppose $|B'| \geq s + 1$.
Recall that $d_A(v) = k - s$ for each $v \in B'$. By Lemma 107, either (i) there exists a set $A_1 \subset A$ with $|A_1| = k - s$ such that $N_A(v) = A_1$ for each $v \in B'$, or (ii) there exists $B'' \subset B'$ with $|B''| = k - s$ such that there exists a set of disjoint paths from A to A that covers B''. If (ii) holds, then because $G[A \cup B]$ is complete bipartite, we can extend the set of paths to a cycle containing $B \cup B''$. This cycle must have at least $2k$ vertices, a contradiction.
Therefore we may assume that for each $(k-s)$-subset B'' of B', we have $B'' \cup (A \cap N(B'')) = K_{k-s,k-s}$. Fix any $(k - s)$-subset $B'' \subset B'$. Let $G_2 = G[A \cup B \cup B'']$. Since G_2 is the union of $K_{s,t}$ and $K_{k-s,k-s}$ with the intersection A_1, it has the following property: for each $a^* \in A - A_1$, $a_1 \in A - a^*$ and $b \in B \cup B''$,

$$G_2 \text{ has an } (a^*, a_1)\text{-path with } 2k - 1 \text{ vertices and an } (a^*, b)\text{-path with } 2k - 2 \text{ vertices.} \tag{7.29}$$

Let $a^* \in A - A_1$. Since $a^* \notin A_1$, $N_{G_2}(a^*) = B$. If also $N_G(a^*) = B$, i.e., a^* has no neighbors outside of P, then again as in Case 1.1, we obtain $|X^*| \leq \frac{k}{2r-k+2}(n - 1 + D(G, X^*))$. So we may assume a^* has a neighbor, say $y \in Y$, outside of $B \cup B'$. Let $Q = a^*, y, w_1, \ldots, w_\ell$ be a path internally disjoint from $A \cup B \cup B'$ such that $w_\ell \in A \cup B \cup B'$. Such a path exists because G is 2-connected. Note that if $w_\ell \in B \cup B''$, then by parity, $\ell \geq 2$. Then Q together with a path in G_2 satisfying (7.29) forms a cycle of length at least $2k$. □

7.7 Proof of Theorem 88 for 2-connected graphs

Recall the statement of the Main Theorem for bipartite graphs.

Theorem 88. *Let $k \geq 4$, $r \geq k+1$ and m, m^*, n be positive integers with $n \geq k$, $m \geq m^* \geq k-1$ and $m \geq k$. Let $G = (X, Y; E)$ be a bipartite 2-connected graph with parts X and Y, where $|X| = m$, $|Y| = n$, and let $X^* \subseteq X$ with $|X^*| = m^*$. If $c(G) < 2k$, then*

$$m^* \leq \frac{k}{2r-k+2}(n-1+D(G,X^*)). \tag{7.30}$$

Proof. Let $G = (X, Y; E)$ and $X^* \subseteq X$ be an edge-maximal counterexample with minimum $|X|$. Note that adding edges to G can only decrease the deficiency while m^*, m, and n stay the same. So we may assume that G is $2k$-saturated, i.e., adding any additional edge connecting X with Y creates a cycle of length at least $2k$. Therefore,

$$\text{for any nonadjacent } x \in X \text{ and } y \in Y, \text{ there is an } (x,y)\text{-path on at least } 2k \text{ vertices.} \tag{7.31}$$

If $n \leq k-1$ then each vertex $x \in X$ has $D_G(x) \geq r-n \geq r-k+1$. Then for $r \geq k+1$, we get

$$\begin{aligned}\frac{k}{2r-k+2}(n-1+D(G,X^*)) &\geq \frac{k}{2r-k+2}(n-1+m^*(r-k+1)) \\ &\geq \frac{k}{2r-k+2}m^*(r-k+1) \geq m^*,\end{aligned}$$

where the last inequality holds whenever $k(r-k+1) \geq 2r-k+2$, i.e., whenever $r \geq k+\frac{2}{k-2}$. Thus we may assume from now on that $n \geq k$.

Our first claim is:

$$e(G) > \left\lfloor \frac{k-1}{2} \right\rfloor m^* + \left\lceil \frac{k-1}{2} \right\rceil (n-1). \tag{7.32}$$

Indeed, $e(G) \geq rm^* - D(G,X^*)$. So, if (7.32) fails and k is odd, then $rm^* - D(G,X^*) \leq \frac{k-1}{2}(m^* + (n-1))$. Solving for m^*, we get

$$m^* \leq \frac{(k-1)(n-1+D(G,X^*))}{2r-k+1}.$$

Since $r \geq k$, this yields (7.30), a contradiction to the choice of G. So suppose k is even. Then $rm^* + D(G,X^*) \leq \frac{k}{2}(m^* + (n-1)) - m^*$. Solving for m^* and using $k \geq 4$ and $r \geq k$, we get

$$m^* \leq \frac{k(n-1+D(G,X^*))}{2r-k+2},$$

and the theorem holds. This proves (7.32).

Apply a mixed $(\left\lfloor \frac{k-1}{2} \right\rfloor, \left\lceil \frac{k-1}{2} \right\rceil)(X^*, Y)$-disintegration to G, that is, first delete all vertices in

$X - X^*$ and then consecutively delete vertices of degree at most $\lfloor \frac{k-1}{2} \rfloor$ in X and vertices of degree at most $\lceil \frac{k-1}{2} \rceil$ in Y. Let G' be the resulting graph with parts $A \subseteq X^*$ and $B \subseteq Y$. Suppose first that G' is empty. Then at each step of the disintegration process, we lose at most $\lfloor \frac{k-1}{2} \rfloor$ edges if a vertex in X^* is deleted and at most $\lceil \frac{k-1}{2} \rceil$ edges if a vertex in Y is deleted. Furthermore, when we arrive to the last $\lfloor \frac{k-1}{2} \rfloor + \lceil \frac{k-1}{2} \rceil = k-1$ vertices in the disintegration process, there exists at most $\lfloor \frac{k-1}{2} \rfloor \cdot \lceil \frac{k-1}{2} \rceil$ edges. Thus

$$\begin{aligned} e(G) &\leq \left\lfloor \frac{k-1}{2} \right\rfloor m^* + \left\lceil \frac{k-1}{2} \right\rceil n - \left\lfloor \frac{k-1}{2} \right\rfloor (k-1) + \left\lfloor \frac{k-1}{2} \right\rfloor \cdot \left\lceil \frac{k-1}{2} \right\rceil \\ &= \left\lfloor \frac{k-1}{2} \right\rfloor m^* + \left\lceil \frac{k-1}{2} \right\rceil n - \left\lfloor \frac{k-1}{2} \right\rfloor^2 \leq \left\lfloor \frac{k-1}{2} \right\rfloor m^* + \left\lceil \frac{k-1}{2} \right\rceil (n-1), \end{aligned}$$

contradicting (7.32). Therefore G' is not empty,

$$d_{G'}(a) \geq 1 + \lfloor \tfrac{k-1}{2} \rfloor \textit{ for each } a \in A, \textit{ and } d_{G'}(b) \geq 1 + \lceil \tfrac{k-1}{2} \rceil \textit{ for each } b \in B. \tag{7.33}$$

Case 1: G' is a complete bipartite graph . Let $s = \min\{|A|, |B|\}$ and $t = \max\{|A|, |B|\}$. Since $c(G) \leq 2k-2$, by (7.33),

$$\tfrac{k}{2} \leq s \leq k-1, \text{ and if } s = |A|, \text{ then } s \geq \tfrac{k+1}{2}. \tag{7.34}$$

Moreover, suppose $s = k-1$. Then G contains a $K_{k-1,k-1}$ with parts A and B' where $B' \subseteq B$. Let $u \in G - (A \cup B')$. Such a vertex exists because $m, n \geq k$. Because G is 2-connected, there exists two internally disjoint paths P_1 and P_2 from u to $A \cup B'$ such that P_1 has endpoints u and u_1 and P_2 has endpoints u and u_2, and these paths only interesect $A \cup B'$ at u_1 and u_2 respectively. If $|V(P_1 \cup P_2)| \geq 4$, that is, $P_1 \cup P_2$ contains a vertex in $G - (A \cup B')$ other than u, then we may find a path P_3 in $A \cup B'$ of length $2k-2$ if u_1 and u_2 are in different partite sets, or of length $2k-3$ if they are in the same partite set. Then $P_1 \cup P_2 \cup P_3$ yields a cycle of length at least $2k$. Therefore $P_1 \cup P_2 = u_1, u, u_2$. Next let w be a vertex in $G - (A \cup B')$ in the opposite partite set than that of u (again such a vertex exists because $n, m \geq k$). Similarly, all internally disjoint paths Q_1, Q_2 connecting w to $A \cup B'$ must be of the form $Q_1 \cup Q_2 = w_1 w w_2$ for some $w_1, w_2 \in A \cup B'$. Thus we may find disjoint paths R_1 and R_2 partitioning $V(A \cup B')$ such that R_1 has endpoints u_1 and w_1 and R_2 has endpoints u_2 and w_2. Then $P_1 \cup R_2 \cup Q_2 \cup R_1$ yields a cycle of length $2k$, a contradiction. Therefore $s \leq k-2$.

Case 1.1: $s = |B|$.

For k odd, by (7.32) and the definition of G', $st > \frac{k-1}{2}(t+s-1)$. Solving for t and using (7.34), we have

$$t > \frac{(k-1)(s-1)}{2s-k+1} = \frac{k-1}{\frac{2(s-1)}{s-1} + \frac{2}{s-1} - \frac{k-1}{s-1}} = \frac{k-1}{2 - \frac{k-3}{s-1}} \geq \frac{k-1}{2 - \frac{k-3}{(k-2)-1}} = (k-1).$$

For k even, we instead get $st > \frac{k-2}{2}t + \frac{k}{2}(s-1)$ and so

$$t > \frac{k(s-1)}{2s-k+2} = \frac{k}{2 - \frac{k-4}{s-1}},$$

so $t \geq k$ except in the case where $s = k-2$ and $t = k-1$. So suppose $|B| = k-2$ and $|A| = k-1$.

By Lemma 106, either $|N(Y-B)| = 2$ or $|N(Y-B) \cap A| \leq 1$. Suppose the first case holds. Let $N(Y-B) = \{x_1, x_2\}$ so that each vertex in $X^* - \{x_1, x_2\}$ has neighbors only in B. Without loss of generality, first assume that $x_1 \in X^*$. Then

$$D(G, X^*) \geq D_G(x_1) + (|X^*| - 2)(r-k+2).$$

Also $n - 1 + D_G(x_1) \geq r-1$. Thus using the fact that $|X^*| \geq k-1$, we have

$$\frac{k(n-1+D(G,X^*))}{2r-k+2} - |X^*| \geq \frac{k(r-1+(|X^*|-2)(r-k+2))}{2r-k+2} - |X^*|$$

$$= \frac{k((|X^*|-1)(r-k+2)+k-3)}{2r-k+2} - |X^*| \geq \frac{k((k-2)(r-k+2)+k-3)}{2r-k+2} - (k-1) \geq 0.$$

where the last inequality holds whenever $r \geq k + \frac{2}{k(k-4)+2} - 2$. Therefore

$$|X^*| \leq \frac{k}{2r-k+2}(n-1+D(G,X^*)),$$

a contradiction. The case where X^* contains neither x_1 nor x_2 is similar (and easier) as we would have $D(G, X^*) = |X^*|(r-k+2)$.

So we may assume that $|N(Y-B) \cap A| \leq 1$ but $|N(Y-B)| \neq 2$. If $|X^*| = k-1$, i.e., $X^* = A$, then all vertices in X^* but at most one have neighbors only in B. Then just as in the previous case, we have

$$|X^*| \leq \frac{k}{2r-k+2}(n-1+D(G,X^*)).$$

Also, if $|X| = k$, then there is a single vertex $x' \in X - A$. Because $|N(Y-B) \cap A| \leq 1$ and G is 2-connected, all vertices in $Y - B$ must also be adjacent to x'. But then $|N(Y-B)| = 2$, a contradiction.

So we may assume that $|X^*| \geq k$ and $|X| \geq k+1$. Fix $x \in A - N(Y-B)$. Then x is contained in a $K_{k-1,k-2}$ subgraph of G and has no neighbors outside of this subgraph. It is easy to see then that $G-x$ is 2-connected. Furthermore, $|X^*-x| \geq k-1$, $|X-x| \geq k$ and $D(G-x, X^*-x) = D(G,X^*) - (r-k+2)$, contradicting Lemma 101. This completes the proof that $t = |A| \geq k$. Applying Lemma 108 completes the case.

Case 1.2: $s = |A|$. By (7.32), $s \geq \frac{k+1}{2}$. Apply the $(k-s)(X^*,Y)$-disintegration to G, and let G'' be the resulting graph. If $G'' \neq G'$, then there is some $u \in V(G'') - V(G')$ not adjacent to some $v \in V(G')$ in the other partite set. By (7.31), G contains a (u,v)-path P' on at least $2k$ vertices. Choose a path P'' of maximum length in G whose both endpoints are in G'' and at least one of them in G'. Let $P'' = v_1, \ldots, v_p$ where $v_1 \in V(G')$. In view of P', $p \geq 2k$. By the maximality of P'', all neighbors of v_p in G'' and all neighbors of v_1 in G' lie in P''. So, $d_{P''}(v_1) + d_{P''}(v_p) \geq (k-s+1)+s = k+1$. By Lemma 92, v_1 and v_p are in the same partite set (i.e. p is odd) and have crossing neighbors in P''. So, P'' satisfies the conditions of Theorem 97 for $\alpha = k-s$, therefore P'' is in crossing formation. But this contradicts the Main Lemma. Thus $G' = G''$, i.e., everything except for G' is removed in the weaker $(k-s)(X^*,Y)$-disintegration.

Next, we apply a mixed disintegration process to $G - (A \cup B)$ where vertices in $Y - B$ are removed (iteratively) if at the time of deletion they have at most $k-s-1$ neighbors within $X^* - A$, and vertices in $X * -A$ are removed if they have at most $k-s$ neighbors total. Let G''' be the resulting graph. We claim that also

$$G''' = G'. \tag{7.35}$$

Suppose not. Then there exists a non-edge between $A \cup B$ and $G''' - (A \cup B)$. Among such nonadjacent vertices, choose a pair $v_1 \in A \cup B$, $v_p \in G''' - (A \cup B)$ such that a path $P = v_1, \ldots, v_p$ between them is longest possible. Thus all neighbors of v_1 in $A \cup B$ are in P, and all neighbors of v_p in $G''' - (A \cup B)$ are in P.

First observe that if $v_1 \in A$, then by the maximality of P, all vertices in B (which are neighbors of v_1) appear in P. Thus by Lemma 92, we may find a cycle that contains all of B. Such a cycle contains at least $2|B| \geq 2k$ vertices, a contradiction. So we assume $v_1 \in B$. If $v_p \in X - A$, then because $d_P(v_p) \geq k-s+1$, Lemma 92 implies that G contains a cycle of length at least $2(s+(k-s+1)-1) = 2k$. Therefore $v_p \in Y - B$, and v_p has at least $k-s$ neighbors from $X - A$ in P. Since v_1 has s neighbors in A, we can find a cycle that covers $N_P(v_1) \cup N_P(v_p)$. Note that $N_P(v_1)$ and $N_P(v_p)$ are the same parity. Thus such a cycle has at least $2(k-s+s)$ vertices, a contradiction. Thus proves (7.35).

Case 1.2.1: $t \geq r$. For simplicity, let $D' = D(G, X^* - A)$ denote the deficiency of vertices in $X^* - A$.

For any v be any vertex in $X^* - A$, because v was deleted in the first disintegration, v has at most $k - s$ neighbors in B. Thus v has at most $(n - t) + (k - s)$ neighbors. Since $d(v) + D(v) \geq r$, $(n - t) + (k - s) + D(v) \geq r$ which implies that

$$n - t + D' \geq r - k + s \geq r - \frac{k-1}{2} \tag{7.36}$$

By (7.35), we obtain that $r(m^* - s) + D' \leq (k - s)(m^* - s) + (k - s - 1)(n - t)$. Solving for m^*, we get

$$m^* \leq \frac{k-s-1}{r-k+s}(n-t) + D' + s \leq \frac{k-s-1}{r-k+s}(n-t+D') + s.$$

We first show that for fixed r, k, n, t, D', the function $f(s) := \frac{k-s-1}{r-k+s}(n - t + D') + s$ is decreasing in s. Indeed, taking the first derivative, we have

$$f'(s) = \frac{-(r-k+s)-(k-s-1)}{(r-k+s)^2}(n-t+D') + 1 = \frac{-(r-1)(n-t+D')}{(r-k+s)^2} + 1.$$

Since $r-1 > r-k+s$ and $n-t+D' > r-k+s$, $\frac{-(r-1)(n-t)}{(r-k+s)^2} < -1$, therefore it is maximized at $s = \frac{k+1}{2}$.

$$\begin{aligned}
m^* &\leq \frac{(k-\frac{k+1}{2}-1)(n-t)-D'}{r-k+\frac{k+1}{2}} + \frac{k+1}{2} < \frac{(k-3)(n-t+D')}{2r-k+1} + \frac{k+1}{2} \\
&= \frac{(k-1)(n-t+D')}{2r-k+1} - \frac{2(n-t+D')}{2r-k+1} + \frac{k+1}{2} \\
&\leq \frac{(k-1)(n-1+D')}{2r-k+1} - \frac{(k-1)(t-1)}{2r-k+1} - \frac{2(r-\frac{k-1}{2})}{2r-k+1} + \frac{k+1}{2} \\
&\leq \frac{(k-1)(n-1+D')}{2r-k+1} - \frac{(k-1)(r-1)}{2r-k+1} + (\frac{k+1}{2} - 1) \\
&\leq \frac{(k-1)(n-1+D')}{2r-k+1} - \frac{(k-1)(r-1)}{2r-k+1} + \frac{(k-1)(r-\frac{k-1}{2})}{2r-k+1} < \frac{(k-1)(n-1+D')}{2r-k+1},
\end{aligned}$$

which is less than $\frac{k}{2r-k+1}(n - 1 + D(G, X^*))$.
Case 1.2.2: $t \leq r$. For simplicity, let $D = D(G, X^*)$. We have that $rm^* - D \leq e(G) \leq st + (k - s)(n - t) + (k - s)(m^* - s)$. Solving for m^*, we have

$$m^* \leq \frac{s(t-k+s)+(k-s)(n-t)+D}{r-k+s} \leq \frac{s(t-k+s)+(k-s)(n-t)+D}{r-k+s}.$$

Again, it can be shown that this function is decreasing with respect to s, and so it is maximized when $s = \frac{k+1}{2}$. Furthermore, the function is maximized whenever t is as large as possible, i.e., when $t = r$. Therefore

$$
\begin{aligned}
m^* &\le \frac{(k-1)(n-r+D)}{2r-k+1} + \frac{k+1}{2} \\
&= \left[\frac{k(n-r+D)}{2r-k+2} - \frac{(2r-2k+2)(n-r+D)}{(2r-k+2)(2r-k+1)}\right] + \frac{k+1}{2} \\
&= \frac{k(n-1+D)}{2r-k+2} - \frac{k(r-1)}{2r-k+2} - \frac{(2r-2k+2)(n-r+D)}{(2r-k+2)(2r-k+1)} + \frac{k+1}{2} \\
&\le \frac{k(n-1+D)}{2r-k+2} - \frac{k(r-1)}{2r-k+2} - \frac{(2r-k+2-k)(r-\frac{k-1}{2})}{(2r-k+2)(2r-k+1)} + \frac{k+1}{2} \\
&= \frac{k(n-1+D)}{2r-k+2} - \frac{k(r-1)}{2r-k+2} - \frac{1}{2}\left(1 - \frac{k}{2r-k+2}\right) + \frac{k+1}{2} \\
&= \frac{k(n-1+D)}{2r-k+2} - \frac{k(r-1-\frac{1}{2})}{2r-k+2} + \frac{k}{2} \\
&= \frac{k(n-1+D)}{2r-k+2} - \frac{k(r-1-\frac{1}{2})}{2r-k+2} + \frac{k(r-\frac{k-2}{2})}{2r-k+2} \le \frac{k(n-1+D)}{2r-k+2},
\end{aligned}
$$

a contradiction.[2]

Case 2: G' is not a complete bipartite graph. Let $P = u_1, \dots, u_q$ be a longest path in G whose both ends are in $V(G')$, and subject to this, $\sum_{i=1}^{q} d_P(u_i)$ is maximized. By (7.31) and the case, $q \ge 2k$. By the maximality of P, all neighbors of u_1 and of u_q in G' lie in P.

Case 2.1: k is odd. Then P satisfies the conditions of Theorem 97 for $\alpha = \frac{k-1}{2}$. But then P is in saturated crossing formation, contradicting the Main Lemma.

Case 2.2: k is even. By (7.33), $d_{G'}(a) \ge \frac{k}{2}$ for each $a \in A$ and $d_{G'}(b) \ge \frac{k+2}{2}$ for each $b \in B$. Since $c(G) \le 2k-2$, by Lemma 92,

$$
q \textit{ is odd, } \{u_1, u_q\} \subseteq A,\ d_{G'}(u_1) = d_P(u_1) \in \{\tfrac{k}{2}, \tfrac{k}{2}+1\}, d_P(u_q) = d_{G'}(u_q) \in \{\tfrac{k}{2}, \tfrac{k}{2}+1\}. \tag{7.37}
$$

First we show the following claim.

Claim 109. *Path P has a pair of crossing neighbors.*

Proof. Suppose not. Suppose the largest index of a neighbor of u_1 in P is j_1 and the smallest index of a neighbor of u_q in P is j_2. Since P has no crossing neighbors, $j_1 \le j_2$. If $j_1 < j_2$ or $d_{G'}(u_1) + d_{G'}(u_q) \ge k+1$, then by the "furthermore" part of Lemma 92, G has a cycle with at least k vertices in B, a contradiction to $c(G) \le 2k-2$. Thus $d_{G'}(u_1) = d_{G'}(u_q) = k/2$ and $j_1 = j_2$.

[2]Note that the last inequality holds whenever $k \ge 5$. If $k = 4$, then instead of $s = (k+1)/2$ we substitute $s = k/2$ and obtain the same inequality in the end.

By the definition of P, $q \geq 2k+1$. By symmetry, we may assume $j_1 \geq k+1$. Since k and j_1 are even, this yields

$$j_1 \geq k+2 \quad \textit{and hence } u_1 \textit{ has a nonneighbor } u_j \textit{ for some even } j < j_1. \tag{7.38}$$

Since G is 2-connected, $G - u_{j_1}$ has a path P_1 that is internally disjoint from P connecting $P[u_1, u_{j_1-1}]$ with $P[u_{j_1+1}, u_q]$. Among such paths, choose a path $P_1 = w_1, \ldots, w_\ell$ with $w_1 = u_{j_3}$ and $w_\ell = u_{j_4}$ so that $j_3 < j_1 < j_4$ and j_3 is as small as possible. Let j_5 be the smallest index such that $j_5 > j_3$ and $u_{j_5}u_1 \in E(G)$ and let j_6 be the largest index such that $j_6 < j_4$ and $u_{j_6}u_q \in E(G)$ Since $j_3 < j_1 = j_2 < j_4$, indices j_5 and j_6 are well defined and $j_5 \leq j_1 \leq j_6$. If $j_3 = 1$, then by the definition of j_1, $\ell \geq 3$ and hence $w_2 \in Y - V(P)$. Thus the cycle $P_1 \cup P[u_{j_4}, u_q] \cup u_q u_{j_6} \cup P[u_1, u_{j_6}]$ has at least k vertices in Y: $N_P(u_1) \cup N_P(u_q) \cup \{w_2\}$. This contradicts to $c(G) \leq 2k-2$. Therefore

$$j_3 \geq 2. \tag{7.39}$$

Cycle $C_1 = P[u_1, u_{j_3}] \cup P_1 \cup P[u_{j_4}, u_q] \cup u_q u_{j_6} \cup P[u_{j_5}, u_{j_6}] \cup u_{j_5} u_1$ contains $N_P(u_1) \cup N_P(u_q) \cup V(P_1)$. Since $c(G) \leq 2k-2$, $V(C_1) \cap Y = N_P(u_1) \cup N_P(u_q)$. In particular, $|V(C_1)| = 2|k/2 + k/2 - 1| = 2k-2$, and

$$\textit{for every even } 2 \leq j \leq j_3 \textit{ and } j_5 \leq j \leq j_1,\ u_1 u_j \in E(G), \tag{7.40}$$

and similarly for u_q,

$$\textit{for every even } j_1 \leq j \leq j_6 \textit{ and } j_4 \leq j \leq q-1,\ u_q u_j \in E(G). \tag{7.41}$$

From (7.38) and (7.40) we conclude

$$u_1 u_{j_5-2} \notin E(G). \tag{7.42}$$

Now we will show that

$$u_1 \textit{ has a neighbor outside of } P. \tag{7.43}$$

Suppose not. Since $N_P(u_1) \subseteq G'$, we have $u_2 \in G'$. In particular, u_2 has at least $\frac{k}{2}+1$ neighbors in G'.

If u_2 has a neighbor u_r in P that is a successor of some neighbor, say u_s of u_q, that is, $r = s+1$, then the cycle $u_2 u_{s+1} \cup P[u_{s+1}, u_q] \cup u_q u_s \cup P[u_s, u_1]$ has length $|V(P)| - 1 \geq 2k$, a contradiction.

Suppose that u_2 is adjacent to a vertex in u_r with $j_6 < r < j_4$. Then the cycle $C' = u_2 u_r \cup P[u_r, u_q] \cup u_q u_{j_6} \cup P[u_{j_6}, u_2]$ contains $N_P(u_1) \cup N_P(u_q) \cup \{u_{j_5-2}\}$. By (7.38) this means C' has at least k vertices in Y. Thus $c(G) \geq 2k$, a contradiction.

Therefore, by (7.41), if u_2 has neighbors in P, they appear in $P[u_1, \ldots, u_{j_1-1}]$. If u_2 has a neighbor u_r with $j_3 < r < j_5$, then the cycle $P[u_2, u_{j_3}] \cup P_1 \cup P[u_{j_4} u_q] \cup u_q u_{j_6} \cup P[u_{j_6}, u_r] \cup u_r u_2$ is longer than C_1 except when $u_r = u_{j_5-1}$. This implies that

$$\text{each neighbor of } u_2 \text{ in } P \text{ is a predecessor of a neighbor of } u_1. \tag{7.44}$$

Similarly,

$$\text{each neighbor of } u_{q-1} \text{ in } P \text{ is a successor of a neighbor uf } u_q. \tag{7.45}$$

Since $d_{G'}(u_2) \geq k/2 + 1$, and $d_G(u_1) = k/2$, u_2 has at least 1 neighbor outside of P in G'. Call this neighbor u_1'. By definition of G', $d_G(u_1') \geq k/2$. Let $P' = u_1' \cup P[u_2, u_q]$, where $|V(P')| = |V(P)|$. If u_1' has a neighbor $v \in V(G') - V(P')$, then the path $v \cup P'$ is a longer path with endpoints in G', contradicting the choice of P. Therefore, by Lemma 92, $d_{G'}(u_1') = d_{P'}(u_1') = d_P(u_1) = k/2$, and we may assume P' has no crossing neighbors. If u_1' has any neighbors in G outside of P, then we instead consider the path P' and vertex u_1' and arrive at (7.43).

Define the indices j_1', j_3', and j_5' as before in view of P' and u_1'. By symmetry, we have $j_1' = j_1$. Because $P' - u_1' = P - u_1$ and by (7.39), $j_3' = j_3$. Finally, if $j_5' < j_5$, then $C_2 = P[u_1', u_{j_3'}] \cup P_1 \cup P[u_{j_4}, u_q] \cup u_q u_{j_6} \cup P[u_{j_5'}, u_{j_6}] \cup u_{j_5'} u_1'$ is a longer cycle than C_1. If $j_5' > j_5$, then u_1' must have less than $k/2$ neighbors in P', a contradiction. Therefore $j_5' = j_5$ and again by symmetry, (7.40) and (7.41) hold for P' and u_1'.

Thus $N_G(u_1) \subseteq N_G(u_1')$. Note that $|X^* - u_1| \geq |N(u_2) \cup N(u_{q-1})| - 1 \geq k/2+1+k/2+1-1 = k+1$, where the last inequality holds because if u_2 and u_{q-1} shared a neighbor v (note that it cannot be in P by (7.44) and (7.45)) then $u_2 v u_{q-1} \cup P[u_2, u_{q-1}]$ is a cycle with $|V(P)| - 1 \geq 2k$ vertices, a contradiction. Applying Lemma 102 gives a contradiction, hence (7.43) holds.

Let $z_1 \in N_G(u_1) - V(P)$. As G is 2-connected, $G - u_1$ contains a path $P_2 = z_1, \ldots, z_m$ from z_1 to $V(P \cup P_1) - u_1$. By (7.39), $z_m \in V(P)$, say $z_m = u_{j_7}$. Again by (7.39), $j_7 \leq j_1$.

If $j_3 < j_7 \leq j_5$, then the cycle $C_2 = P[u_1, u_{j_3}] \cup P_1 \cup P[u_{j_4}, u_q] \cup u_q u_{j_6} \cup P[u_{j_7}, u_{j_6}] \cup P_2 \cup z_1 u_1$ contains at least k vertices from Y, namely, $N_P(u_1) \cup N_P(u_q) \cup \{z_1\}$, a contradiction. Suppose now that $j_7 \in \{2i-1, 2i\}$ for some $i \in \{2, \ldots, j_3/2\} \cup \{1 + j_5/2, \ldots, j_1/2\}$. By (7.40), $u_1 u_{2i-2} \in E(G)$ and so by (7.37), $u_{2i-2} \in V(G')$. But the path $P_3 = P[u_{2i-2}, u_1] \cup u_1 z_1 \cup P_2 \cup P[u_{j_7}, u_q]$ is longer than P, contradicting the choice of P. Finally, if $j_7 = 2$, then we instead take the path $u_1 z_1 \cup P_2 \cup P[u_2, u_q]$. This proves the claim. $\square$

Let u_{i_1} and u_{i_2} be the first occurring pair of crossing neighbors on P.

Claim 110. *If $|X^*| \geq k+1$, then every cycle in G containing $N(u_1)$ also contains u_1, and every cycle containing $N(u_q)$ also contains u_q.*

Proof. We prove the claim for u_1. The result for u_q follows by symmetry. Suppose there exists a cycle C that contains $N(u_1)$ but not u_1. If $G - u_1$ has a cut vertex v, then because G is 2-connected, $\{v, u_1\}$ is a cut set of G. Therefore there exist vertices $u_i, u_j \in N_G(u_1)$ that are in distinct components of $(G - u_1) - v$. Let P' be a segment of C from u_i to u_j not containing v. Then P' is a path from u_i to u_j in $(G - u_1) - v$, a contradiction. Therefore $G - u_1$ is 2-connected, contradicting Lemma 101. $\square$

Claim 111. *Each of u_1 and u_q has at least one neighbor outside of P.*

Proof. Similarly to the proof of Case 1 of Lemma 108, let $C = P[u_1, u_{i_1}] \cup u_{i_1} u_q \cup P[u_{i_2}, u_q] \cup u_{i_2} u_1$. By the choice of G, $|C| \leq 2k-2$. Since $N_P(u_1)^- \cap N_P(u_q)^+ = \emptyset$, $|N_P(u_1)^-| + |N_P(u_q)^+| \geq k$, and C does not contain only two vertices in $N_P(u_1)^- \cup N_P(u_q)^+$,

$$2k-2 \geq |C| \geq 2k-4 \text{ and } |(N_P(u_1)^- \cup N_P(u_q)^+) \cap C \cap A| \geq d_P(u_1) + d_P(u_q) - 2 \; .$$

This means

$$\text{either } |C| = 2k-4 \text{ and each } u_i \in C \cap A \text{ is in } N_P(u_1)^- \cup N_P(u_q)^+, \text{ or } |C| = 2k-2 \text{ and there is at most one } i_0 \text{ such that } u_{i_0} \in (C \cap A) - (N_P(u_1)^- \cup N_P(u_q)^+). \tag{7.46}$$

We will show that u_1 has a neighbor in G' outside of P. The result for u_q follows by symmetry.

Case 2.2.1: $|C| = 2k-4$. By Lemma 92, $d_P(u_1) = d_P(u_q) = k/2$. As before, the vertex u_2 is in G' and hence has at least $k/2+1$ neighbors in G'. Suppose first that all neighbors of u_2 in G' are in P. As in the previous case, u_2 cannot have a neighbor that is a successor of a neighbor of u_q. If u_2 has a neighbor u_r with $r \in \{i_1+1, \ldots, i_2-5\}$ then $u_2 u_r \cup P[u_r, u_q] \cup u_q u_{i_1} \cup P[u_{i_1}, u_2]$ is a cycle with length at least $|C| - 1 + 5 \geq 2k$, a contradiction.

Thus every neighbor of u_2 in G' is in $N(u_1)^- + u_{i_2-3}$. Also by symmetry $N_P(u_{q-1}) \subseteq N(u_q)^+ + u_{i_1+3}$, so $N_P(u_2)$ and $N_P(u_{q-1})$ intersect in at most one vertex. As before, u_2 and u_{q-1} cannot share a neighbor outside of P. Therefore

$$|X^*| \geq |N_{G'}(u_2) \cup N_{G'}(u_{q-1})| \geq (k/2+1) + (k/2+1) - 1 \geq k+1. \tag{7.47}$$

If $u_{i_2-1} \in N(u_2)$, then the cycle $P[u_2, u_{i_1}] \cup u_{i_1} u_q \cup P[u_q, u_{i_2-1}] \cup u_{i_2-1} u_2$ contradicts Claim 110. So $d_P(u_2) \leq |N(u_1)^- + u_{i_2-3} - u_{i_2-1}| = k/2$, hence u_2 has a neighbor u_1' in G' outside of P. If u_1' is adjacent to some vertex u_r with $r \in \{i_1+2, \ldots, i_2\}$, then the cycle $P'[u_1', u_{i_1}] \cup u_{i_1} u_q \cup P'[u_q, u_r] \cup u_r u_1'$ contradicts Claim 110. So by (7.46), $N_P(u_1') \subseteq$

$N(u_1) - u_{i_2}$, and hence $d_P(u_1') \leq k/2 - 1 < d(u_1')$, so u_1' has a neighbor outside of P. Then again we consider P' and u_1', and complete the case.

Case 2.2.2: $|C| = 2k-2$. Assume $N_{G'}(u_1)$(and $N_{G'}(u_q)) \subseteq V(P)$. Suppose first that $d_P(u_1) = k/2$. As before, we will show that u_2 must have a neighbor u_1' in G' outside of P. So suppose first that u_2 has no such neighbors.

If it exists, let u_{i_0} be the unique vertex in $C \cap A$ which is not contained in $N(u_1)^- \cup N(u_q)^+$. Note that in this case, $C \cap A$ contains all vertices in $N(u_1)^- \cup N(u_q)^+ - u_{i_1+1} - u_{i_2-1} + u_{i_0}$. In particular, $|N(u_1)^- \cup N(u_q)^+ - u_{i_1+1} - u_{i_2-1} + u_{i_0}| \leq k-1$ if and only if $d(u_1) = d(u_q) = k/2$. If u_2 is adjacent to a vertex u_r with $r \in \{i_1+1, \ldots, i_2-3\}$, then the cycle $P[u_2, u_{i_1}] \cup u_{i_1}u_q \cup P[u_q, u_r] \cup u_r u_2$ has at least $|C| + 2 \geq 2k$ vertices. Therefore $N_{G'}(u_2) \subseteq N(u_1)^- + u_{i_0}$. We obtain a similar result for u_{q-1} by symmetry, and again we get $|X^*| \geq k+1$.

We also have that $u_2 u_{i_2-1} \notin E(G)$, otherwise the cycle $P[u_2, u_{i_1}] \cup u_{i_1}u_q \cup P[u_1, u_{i_2-1}] \cup u_{i_2-1}u_2$ contradicts Claim 110. Therefore $d_P(u_2) \leq k/2$. This implies u_2 has a neighbor u_1' in G' outside of P. If u_1' has a neighbor outside of P, then we instead consider the path $P' = u_1' \cup P[u_2, u_q]$ and are done. As in the previous case, u_1' does not have neighbors in $\{u_{i_1+2}, \ldots, u_{i_2}\}$. Hence $N(u_1')$ is contained in C but u_1' is not, contradicting Claim 110 (applied to P' and u_1').

This completes the proof for u_1. By symmetry, we have that u_p also contains a neighbor outside of P. $\square$

For $j = 1$ and $j = q$, let $z_j \in N(v_j) - V(P)$. Since G is 2-connected, this implies that there is a path $Q_j = w_{1,j}, \ldots, w_{\ell_j,j}$ from u_j through $z_j = w_{2,j}$ to $P - u_j$ internally disjoint from P. Let $w_{\ell,j} = u_{h_j}$. If Q_1 and Q_q share a vertex outside of P, then G has a cycle containing P, a contradiction to $c(G) \leq 2k-2$. So, the only vertex common for Q_1 and Q_q could be u_{h_1} if it coincides with u_{h_q}. Also, $h_1 \geq 3$, since if $h_1 = 2$, then the path $Q_1 \cup P[u_2, u_q]$ is longer than P. Similarly, $h_q \leq q-2$.

We claim that

$$\text{for } h_1 - 4 \leq g \leq h_1 - 1,\ u_g \notin N(u_q) \text{ and for } h_q + 1 \leq g \leq h_q + 4,\ u_g \notin N(u_1)\ . \tag{7.48}$$

Indeed, by symmetry suppose $u_g u_q \in E(G)$ for some $h_1 - 4 \leq g \leq h_1 - 1$. Then the cycle $P[u_1, u_g] \cup u_g u_q \cup P[u_q, u_{h_1}] \cup Q_1$ would have at least

$$2k + 1 - (h_1 - g - 1) + (j_1 - 1) \geq 2k + 1 - 3 + 1 = 2k - 1$$

vertices. This contradicts $c(G) \leq 2k-2$.

Also

$$\{u_{h_1-2}, u_{h_1-1}, u_{h_q+1}, u_{h_q+2}\} \cap B = \emptyset. \tag{7.49}$$

Indeed if $h_1 - 2 \le g \le h_1 - 1$ and $u_g \in B$, then the path $P[u_g, u_1] \cup Q_1 \cup P[u_{h_1}, u_q]$ starts from $u_g \in B$ and is longer than P (because if $g = h_1 - 2$ then by parity, $\alpha_1 \ge 4$). The proof for $h_q + 1 \le g \le h_q + 2$ is symmetric.
Similarly to (7.24), we show

$$\begin{aligned}&\textit{(i) if } |C| = 2k-4, \textit{ then } u_{h_1} \notin P[u_{i_1+1}, u_{i_2-2}] \textit{ and } u_{h_q} \notin P[u_{i_1+2}, u_{i_2-1}];\\ &\textit{(ii) if } |C| = 2k-2, \textit{ then } u_{h_1} \notin P[u_{i_1+1}, u_{i_2}] \textit{ and } u_{h_q} \notin P[u_{i_1}, u_{i_2-1}]\ .\end{aligned} \tag{7.50}$$

Indeed, if for example, $i_1 + 1 \le h_1 \le i_2 - 2$, then the cycle $P[u_1, u_{i_1}] \cup u_{i_1}u_q \cup P[u_q, u_{h_1}] \cup Q_1$ would have at least $|C| + 3$ vertices, which means at least $2k$ vertices. All other possibilities are very similar.
Let λ be the odd integer in the set $\{h_1 - 3, h_1 - 2\}$. Similarly, let μ be the odd integer in the set $\{h_q + 2, h_q + 3\}$. By (7.48), $u_\lambda \notin P^+(N_P(u_q))$. By (7.46), we have the following cases.
First suppose $|C| = 2k - 4$ and each $u_i \in C \cap A$ is in $N_P(u_1)^- \cup N_P(u_q)^+$. Since $N_P(u_1)^- \cap N_P(u_q)^+ = \emptyset$ and each of u_1 and u_q has $k/2$ neighbors in B, this means

$$\textit{all neighbors of } u_1 \textit{ and } u_q \textit{ in } P \textit{ are in } B. \tag{7.51}$$

By (7.49) and (7.51), $u_\lambda \notin N_P(u_1)^-$. So by the case and the fact that λ is odd, $i_1 + 1 \le \lambda \le i_2 - 1$. This means $i_1 + 3 \le h_1 \le i_2 + 2$. Since $u_{i_2} \in B$, by (7.50)(i) and (7.49), $i_2 - 1 \le h_1 \le i_2$. Similarly, $i_1 \le h_q \le i_1 + 1$. Then the cycle

$$P[u_1, u_{h_q}] \cup Q_q \cup P[u_q, u_{h_1}] \cup Q_1$$

has length at least $|C| + 4$, contradicting $c(G) \le 2k - 2$.
Next, suppose $|C| = 2k-2$ and there is exactly one i_0 such that $u_{i_0} \in (C \cap A) - (N_P(u_1)^- \cup N_P(u_q)^+)$. As the case $|C| = 2k - 4$ this yields (7.51). By (7.49) and (7.51), $u_\lambda \notin N_P(u_1)^-$. So by the case and the fact that λ is odd, either $\lambda = i_0$, or $i_1 + 1 \le \lambda \le i_2 - 1$. If the latter holds, then $i_1 + 3 \le h_1 \le i_2 + 2$, which is impossible by (7.50)(ii) and (7.49). Thus $\lambda = i_0$. Similarly, we conclude $\mu = i_0$. In particular, $h_q < h_1$. Since $\lambda = \mu$ is odd, the cycle

$$P[u_1, u_{h_q}] \cup Q_q \cup P[u_q, u_{h_1}] \cup Q_1$$

has length at least $|V(P)| - 1 \ge 2k$, contradicting $c(G) \le 2k - 2$.
Finally, suppose $|C| = 2k-2$ and each $u_i \in C \cap A$ is in $N_P(u_1)^- \cup N_P(u_q)^+$. By Lemma 92, $d_P(u_1) + d_P(u_q) \le k+1$. So by the symmetry between u_1 and u_q, we may assume $d_P(u_1) = k/2$ and hence $N_P(u_1) = N_{G'}(u_1)$. Since i_0 does not exist, repeating the argument of Case 2.2.2, we get a contradiction even earlier. □

7.8 Proof of Theorem 86 for general graphs

For disjoint vertex sets X and Y, an (X,Y)-*frame* is a pair (G,X^*) where G is a bigraph with parts X and Y, and $X^* \subseteq X$.
A block G' in an (X,Y)-frame (G,X^*) with parts X' and Y' is *special* if all of the following holds:
(i) $G' = K_{k-1,r}$ with $|X'| = k-1$;
(ii) $X' \subseteq X^*$;
(iii) $N_G(x) = Y'$ for each $x \in X'$.
Let $Q(G,X^*)$ denote the number of special blocks in an (X,Y)-frame (G,X^*). Recall the definition of deficiency:

$$D(G,X^*) = \sum_{x\in X^*} D_G(x) = \sum_{x\in X^*} \max\{0, r - d_G(x)\}.$$

The following theorem implies Theorem 86.

Theorem 112. *Let $k \geq 4$, $r \geq k+1$ and m, m^*, n be positive integers with $m^* \leq m$. Let (G,X^*) be an (X,Y)-frame, where $|X| = m$, $|Y| = n$, and $|X^*| = m^*$, and G is $2k$-saturated. If $c(G) < 2k$, then*

$$m^* \leq \frac{k-1}{r}(n - 1 + D(G,X^*) + Q(G,X^*)). \tag{7.52}$$

Furthermore, equality holds if and only if G and X^* satisfy the following:
(i) G is connected;
(ii) all blocks of G are copies of either $K_{k-1,r}$ or $K_{k-1,r+1}$ with the partite set of size $k-1$ in X and all cut vertices of G in Y;
(iii) $X^* = X$;
(iv) $D(G,X^*) = 0$.

It is straightforward to check that the graphs described in (i)-(iv) are indeed sharpness examples to Theorem 112: suppose G has s blocks of the form $K_{k-1,r}$ and t of the form $K_{k-1,r+1}$. Then $m = m^* = (s+t)(k-1)$, $n = s(r-1) + tr + 1$, $D(G,X) = 0$, and $Q(G,X) = s$, since each $K_{k-1,r}$ block is special. Therefore

$$\frac{k-1}{r}(n-1+D(G,X)+Q(G,X)) = \frac{k-1}{r}(s(r-1)+tr+1-1+0+s) = \frac{k-1}{r}(r(s+t)) = m^*.$$

Proof of Theorem 112. Let (G,X^*) be a counterexample to the theorem with the fewest vertices in G. For short, let $D = D(G, X^*)$ and $Q = Q(G, X^*)$. By the definition of $D(x)$,

$$d(x) + D(x) \geq r \text{ for every } x \in X. \tag{7.53}$$

Case 1: G is 2-connected. If $m^* \geq k-1$ and $m \geq k$, then (7.52) follows Theorem 88. In fact, we get strict inequality as $\frac{k}{2r-k+2} < \frac{k-1}{r}$ whenever $k-1 < r$. Suppose $1 \leq m^* \leq k-2$ and $x \in X^*$. Then by (7.53),

$$n - 1 + D \geq d(x) + D(x) \geq r - 1,$$

so $\frac{k-1}{r}(n-1+D) \geq \frac{k-1}{r}(r-1) > k-2 \geq m$.
The last possibility is that $m^* = m = k-1$. If $n + D \geq r+1$, then (7.52) holds, so suppose $n + D = r$. Since $k - 1 \geq 2$, this together with (7.53), implies that $n = r$ and $D = 0$. Thus $G = K_{k-1,r}$ and $X^* = X$ which yields that G is a special block. Thus $Q = 1$ and so $n - 1 + D + Q = r$. This finishes Case 1.
Note that equality is obtained only in this subcase where $G = K_{k-1,r}, X = X^*$, and $D = 0$. Therefore G and X^* satisfy (i)-(iv).

Since Case 1 does not hold, G has a pendant block, say with vertex set B. Let b be the cut vertex in B, $X^*_B = X^* \cap B - b$, $m^*_B = |X^*_B|$, and $n_B = |B \cap Y|$. Furthermore, let $G_1 = G - (B - b)$ and $n_1 = |Y \cap V(G_1)|$.
Note that G_1 is 2k-saturated: as a cycle cannot span multiple blocks in a graph, if there exists an edge $xy \notin E(G_1)$ such that $G_1 + xy$ contains no cycle of length $2k$ or longer, then $G + xy$ also contains no cycle of length $2k$ or longer, contradicting that G is $2k$-saturated.
Case 2: $b \in Y$. Let $X^*_1 = X^* - X^*_B$. By the minimality of G,

$$|X^*_1| \leq \frac{k-1}{r}(n_1 - 1 + D(G_1, X^*_1) + Q(G_1, X^*_1)), \text{ and} \tag{7.54}$$

$$m^*_B \leq \frac{k-1}{r}(n_B - 1 + D(G[B], X^*_B) + Q(G[B], X^*_B)), \tag{7.55}$$

using $m^* = |X^*_1| + m^*_B$, we obtain

$$m^* \leq \frac{k-1}{r}\Big(n_1 + n_B - 2 + D(G_1, X^*_1) + D(G[B], X^*_B) + Q(G_1, X^*_1) + Q(G[B], X^*_B)\Big). \tag{7.56}$$

Since $n_1 + n_B - 2 = n - 1$, $D = D(G_1, X^*_1) + D(G[B], X^*_B)$ and $Q = Q(G_1, X^*_1) + Q(G[B], X^*_B)$, (7.56) implies (7.52).
Furthermore, if equality holds in (7.52), then we have equalities in both (7.54) and (7.55). Again by the minimality of G, frames B with X^*_B and G_1 with X^*_1 both satisfy (i)-(iv). In particular, we have $X^*_B = X \cap B$ and $X^*_1 = X - B$. Since $X^* = X^*_B \cup X^*_1 = X$, it follows

that G also satisfies (i)-(iv).
Case 3: $b \in X$ and $X_B^* = \emptyset$. By the minimality of G,

$$|X^*| \leq \frac{k-1}{r}(n_1 - 1 + D(G_1, X^*) + Q(G_1, X^*)). \tag{7.57}$$

Since $d_G(b) - d_{G_1}(b) \leq n_B$, $D(G_1, X^*) \leq D(G, X^*) + n_B$. If $Q(G_1, X^*) = Q(G, X^*)$, then (7.57) implies (7.52). Furthermore, suppose that equality holds in (7.52). Then equality also holds in (7.57), and $D(G, X^*) + n_B = D(G_1, X^*)$. By the minimality of G, G_1 and X^* satisfy (i)-(iv). In particular by (iv), $D(G_1, X^*) = 0$, contradicting that $D(G, X^*) + n_B = D(G_1, X^*)$.
So, suppose $Q(G_1, X^*) > Q(G, X^*)$. By Part (iii) of the definition of a special block, if this happens, then $Q(G_1, X^*) = Q(G, X^*)+1$ and the unique block B_1 that is special in (G_1, X^*) but not special in (G, X^*) contains b. This means $d_{G_1}(b) = r$ and hence $D(G_1, X^*) = D$. But $n_B \geq 1$, and so again (7.57) implies (7.52). Furthermore, if $n_B \geq 2$, then we obtain strict inequality in (7.52). If $n_B = 1$, say $Y \cap B = \{y\}$, then since B is 2-connected, B consists of a single edge yb attached to the special block B_1, where B_1 is a copy of $K_{k-1,r}$ with $|X \cap B_1| = k-1$. Note that if a block in G has a partite set of size $k-1$, then the longest cycle in G has length at most $2(k-1)$. Thus for any $x \in X \cap (B_1 - b)$, $c(G+xy) \leq 2(k-1)$, contradicting that G is $2k$-saturated.
Case 4: $b \in X$ and $1 \leq m_B^* \leq k-2$. Let $X_1^* = X^* - X_B^* - b$. By the minimality of G,

$$|X_1^*| < \frac{k-1}{r}(n_1 - 1 + D(G_1, X_1^*) + Q(G_1, X_1^*)). \tag{7.58}$$

Where note that we have strict inequality because X_1^* does not satisfy (iii) for G_1. Since $b \notin X_1^*$, $Q(G_1, X_1^*) = Q$. Let $x \in X_B^*$. By (7.53), $D(x) + n_B \geq r$. Thus by the case,

$$m^* \leq |X_B^* \cup \{b\}| + |X_1^*| < k - 1 + \frac{k-1}{r}(n_1 - 1 + D(G_1, X_1^*) + Q) \leq$$

$$k-1+\frac{k-1}{r}(n-n_B-1+(D-D(x))+Q) \leq k-1+\frac{k-1}{r}(n-1+D-r+Q) = \frac{k-1}{r}(n-1+D+Q),$$

as claimed.
Case 5: $b \in X$ and $m_B^* \geq k-1$. Let $X_1^* = X^* - X_B^* - b$. By the minimality of G, again (7.58) holds. Also, as in Case 4, $Q(G_1, X_1^*) = Q$ and $m^* \leq |X_1^*| + 1 + m_B^*$. Since $n = n_1 + n_B$, and $D(G_1, X_1^*) + D(G[B], X_B^*) \leq D$, in order for (G, X^*) to be a counterexample to the theorem, all this together yields

$$m_B^* + 1 > \frac{k-1}{r}(n_B + D(G[B], X_B^*)). \tag{7.59}$$

On the other hand, by Theorem 112,

$$m_B^* \le \frac{k}{2r-k+2}(n_B - 1 + D(G[B], X_B^*)).$$

Plugging this into (7.59), we get

$$\frac{k}{2r-k+2}(n_B - 1 + D(G[B], X_B^*)) + 1 > \frac{k-1}{r}(n_B + D(G[B], X_B^*)). \tag{7.60}$$

Since the coefficient at $n_B + D(G[B], X_B^*)$ in the left side of (7.60) is less than the one in the right side, and since by (7.53), $n_B + D(G[B], X_B^*) \ge r$, (7.60) implies

$$\frac{k}{2r-k+2}(r-1) + 1 > \frac{k-1}{r}r = k-1. \tag{7.61}$$

But (7.61) is equivalent to $k(r-1) > (k-2)(2r-k+2)$, which is not true when $r \ge k \ge 4$. □

As a corollary, we obtain the same result for graphs that are not $2k$-saturated.

Corollary 113. *Let $k \ge 4$, $r \ge k+1$ and m, m^*, n be positive integers with $m^* \le m$. Let (G, X^*) be an (X,Y)-frame, where $|X| = m$, $|Y| = n$, and $|X^*| = m^*$. If $c(G) < 2k$, then*

$$m^* \le \frac{k-1}{r}(n - 1 + D(G, X^*) + Q(G, X^*)). \tag{7.62}$$

Proof. Add edges to G until the resulting graph is $2k$-saturated. Call this graph G'. Note that when adding edges, the deficiency $D(G, X^*)$ of any $X^* \subseteq X$ cannot grow. That is, $D(G, X^*) \le D(G', X^*)$.

Applying Theorem 112 to G', we obtain

$$|X^*| \le \frac{k-1}{r}(n - 1 + D(G', X^*) + Q(G', X^*)).$$

If $Q(G', X^*) \le Q(G, X^*)$, then we're done. Otherwise suppose that $Q(G', X^*) = Q(G, X^*) + t$. This implies that there were t special blocks created when adding edges within the blocks of G. Let B be such a block that was not special in G but became special in G'. Then in G, $B \subsetneq K_{k-1,r}$. Thus some vertex $v \in B \cap X$ has $d_G(v) < r$ but $d_{G'}(v) = r$. Hence $D_G(v) > D_{G'}(v)$. It follows $D(G, X^*) \ge D(G', X^*) + t$.

Thus

$$|X^*| \le \frac{k-1}{r}(n - 1 + D(G', X^*) + Q(G, X^*) + t) \le \frac{k-1}{r}(n - 1 + D(G, X^*) + Q(G, X^*),$$

as desired. □

7.9 Proofs for hypergraphs: Theorem 80 and Corollary 81

Proof of Theorem 80. Let $\mathcal{H}$ be an n-vertex multi-hypergraph with lower rank r and edge multiplicty at most $k-2$. Let $G = G(\mathcal{H})$ be the incidence graph of $\mathcal{H}$ with parts $X = V(\mathcal{H})$ and $Y = E(\mathcal{H})$. By construction, since $\mathcal{H}$ has lower rank at least r, each $x \in X$ has $d_G(x) \geq r$. Therefore $D(G, X) = 0$. Also, G cannot contain a special block (i.e., $Q(G, X) = 0$) as such a block in G would correspond to a set of $k-1$ edges in $\mathcal{H}$ that are composed of the same r vertices. But we assumed that $\mathcal{H}$ has no edges with multiplicity greater than $k-2$.

Applying Theorem 112 to G with $X^* = X$, we obtain

$$e(\mathcal{H}) = |X| \leq \frac{k-1}{r}(n-1+D(G,X)+Q(G,X)) = \frac{k-1}{r}(n-1).$$

Finally, suppose equality holds. Add edges to G until it is $2k$-saturated. Let G' be the resulting graph. Again we have $Q(G', X) = Q(G, X) = 0$ and $D(G', X) = D(G, X) = 0$, therefore $|X| = \frac{k-1}{r}(n-1+D(G',X)+Q(G',X))$. Hence G' satisfies (i)-(iv) in the second part of the statement of Theorem 112. In particular, all blocks of G' are copies of $K_{k-1,r+1}$ with cut vertices in Y. Then in G within each block, every vertex $x \in X$ is adjacent to a subset of the $r+1$-partite set of size r or $r+1$. That is, each $K_{k-1,r+1}$ block in G' corresponds to an $(r+1, k-1)$-block in $\mathcal{H}$. This completes the proof of Theorem 80. □

Proof of Corollary 81. Recall that a Berge-path of length k has $k+1$ base vertices and k hyperedges. Suppose $\mathcal{H}$ satisfies the conditions of the corollary. We construct the multi-hypergraph $\mathcal{H}'$ by adding a new vertex x to $\mathcal{H}$ and extending each hyperedge of $\mathcal{H}$ to include x. Then $\mathcal{H}'$ has $n+1$ vertices, lower rank at least $r+1$, no edge with multiplicity at least $k-1$, and $e(\mathcal{H}') = e(\mathcal{H})$.

We claim that $\mathcal{H}'$ has no Berge-cycle of length k or longer. Suppose there exists such a cycle with edges $e_1, \ldots, e_\ell$ and base vertices $v_1, \ldots v_\ell$ and $\ell \geq k$. If $x \in \{v_1, \ldots, v_k\}$, say $x = v_1$, then since each edge in $\mathcal{H}'$ contains at least $r+1$ vertices, there exist distinct vertices $v_1' \in e_1 - \{v_1, \ldots, v_k\}$ and $v_{k+1}' \in e_k - \{v_1, \ldots, v_k\}$. For each $1 \leq i \leq \ell$, let $e_i' = e_i - \{x\}$. Then $e_1', \ldots, e_k'$ and $\{v_1', v_2, \ldots, v_k, v_{k+1}'\}$ form a Berge-path of length k. The case where $x \notin \{v_1, \ldots, v_k\}$ is similar (and simpler). Therefore, applying Theorem 82 to $\mathcal{H}'$, we obtain

$$e(\mathcal{H}) = e(\mathcal{H}') \leq \frac{k-1}{r+1}((n+1)-1),$$

as desired. □

Chapter 8

2-connected hypergraphs with bounded circumference

8.1 Introduction

In this chapter, we prove Theorem 15 which provides a better upper bound the number of edges in 2-connected hypergraphs without long cycles. Here we consider hypergraphs with upper rank r without cycles of length $k \geq r$ or longer. Our results are sharp for general hypergraphs for all $n \geq k \geq r$, and for r-uniform hypergraphs when $k \geq 4r$ and n is sufficiently large. This is joint work with Zoltán Füredi and Alexandr Kostochka [FKL19].

8.1.1 Basic definitions

The *upper rank* of a hypergraph $\mathcal{H}$ is the size of a largest edge. For brevity, instead of saying "a hypergraph of upper rank r" we will say "an r^--*graph*". When every edge has size r, i.e., $\mathcal{H}$ is r-uniform, we call $\mathcal{H}$ an "r-graph".
A hypergraph $\mathcal{H}$ is *Sperner* if no edge of $\mathcal{H}$ is contained in another edge. In particular, a Sperner hypergraph has no multiple edges, and all simple uniform hypergraphs are Sperner.

8.2 Results for hypergraphs and bipartite graphs

8.2.1 2-connected hypergraphs without long Berge cycles

Our goal is to prove a version of Kopylov's theorem for hypergraphs, i.e., to find the maximum number of edges in a 2-connected hypergraph with no Berge cycle of length k or greater.
Define
$$f(n,k,r,a) := \binom{k-a}{\min\{r, \lfloor \frac{k-a}{2} \rfloor\}} + (n-k+a)\binom{a}{\min\{r-1, \lfloor a/2 \rfloor\}}.$$

Also define

$$f^*(n,k,r,a) := \binom{k-a}{r} + (n-k+a)\binom{a}{r-1}.$$

Note that $f(n,k,r,a) = f^*(n,k,r,a)$ whenever $r \le \lfloor (k-a)/2 \rfloor$ and $r-1 \le \lfloor a/2 \rfloor$. Our main result is:

Theorem 114. *Let $n \ge k \ge r \ge 3$. If $\mathcal{H}$ is an n-vertex Sperner 2-connected r^--hypergraph with no Berge cycle of length k or longer, then $e(\mathcal{H}) \le \max\{f(n,k,r,\lfloor (k-1)/2 \rfloor), f(n,k,r,2)\}$.*

This bound is sharp. To see this, we construct a series of hypergraphs (not necessarily uniform).

Construction 115. *For $n \ge k \ge r$, $1 \le a \le \lfloor (k-1)/2 \rfloor$, let $\mathcal{H}_{n,k,r,a}$ be the hypergraph with vertex set $A \cup B \cup C$ such that $|A| = k-2a$, $|B| = a, |C| = n-(k-a)$. The edge set of $\mathcal{H}_{n,k,r,a}$ is the family*

$$\{e \subseteq A \cup B : |e| = \min\{r, \lfloor (k-a)/2 \rfloor\}\} \cup \{c \cup e' : c \in C, e' \subseteq B, |e'| = \min\{r-1, \lfloor a/2 \rfloor\}\}.$$

For $a \ge 2$, $\mathcal{H}_{n,k,r,a}$ is 2-connected and contains no Berge cycle of length k or longer. We have that $|E(\mathcal{H}_{n,k,r,a})| = f(n,k,r,a)$, which is maximized when $a = \lfloor (k-1)/2 \rfloor$ or $a = 2$ by the convexity of f (as a function of a). Furthermore, when $r \le \lfloor (k-a)/2 \rfloor$ and $r-1 \le \lfloor a/2 \rfloor$, $\mathcal{H}_{n,k,r,a}$ is r-uniform with $f^*(n,k,r,a)$ edges.
For integers $k \ge r$, let $n_{k,r}$ be the smallest positive integer n such that $f(n,k,r,\lfloor (k-1)/2 \rfloor) \ge f(n,k,r,2)$. Asymptotically $n_{k,r}$ is about $2^{r-1}k/r$. Then as a corollary of Theorem 114 we obtain the following result for r-graphs (note that this is a restatement of Theorem 15).

Theorem 116. *Let $n \ge n_{k,r} \ge k \ge 4r \ge 12$. If $\mathcal{H}$ is an n-vertex 2-connected r-graph with no Berge cycle of length k or longer, then $e(\mathcal{H}) \le f(n,k,r,\lfloor (k-1)/2 \rfloor) = f^*(n,k,r,\lfloor (k-1)/2 \rfloor)$.*

For n large, this bound is almost $2^{r-1}/r$ stronger than the (exact) bound with no restriction on connectivity. Again we have sharpness example $\mathcal{H}_{n,k,r,\lfloor (k-1)/2 \rfloor}$.

8.2.2 Connected hypergraphs without long Berge path

We also obtain a result for connected graphs with no Berge path of length k.

Theorem 117. *Let $n \ge k \ge r \ge 3$. If $\mathcal{H}$ is an n-vertex Sperner connected r^--graph with no Berge path of length k, then $e(\mathcal{H}) \le \max\{f(n,k,r,\lfloor (k-1)/2 \rfloor), f(n,k,r,1)\}$.*

For integers $k \geq r$, let $n'_{k,r}$ be the smallest positive integer n such that $f(n,k,r,\lfloor(k-1)/2\rfloor) \geq f(n,k,r,1)$. Then we obtain the following result for r-uniform graphs with no Berge path of length k as a corollary of Corollary 117. This improves Theorem 66.

Theorem 118. *Let $n \geq n'_{k,r} \geq k \geq 4r \geq 12$. If $\mathcal{H}$ is an n-vertex connected r-graph with no Berge path of length k, then $e(\mathcal{H}) \leq f(n,k,r,\lfloor(k-1)/2\rfloor) = f^*(n,k,r,\lfloor(k-1)/2\rfloor)$.*

The family $\mathcal{H}_{n,k,r,\lfloor(k-1)/2\rfloor}$ again shows sharpness of our bounds.

8.3 Proof outline

The basic idea of the proof is to consider instead of the family of r-graphs the larger family of Sperner r^--graphs. Then we can in some situations shrink some edges keeping the r^--graph Sperner.

We start with a dense Sperner r^--graph $\mathcal{H}$. By definition, each edge e in $\mathcal{H}$ yields a clique of order $|e|$ in the 2-shadow of $\mathcal{H}$. If $\mathcal{H}$ contains a long Berge cycle C, then $\partial_2\mathcal{H}$ contains a cycle of the same length. However, the converse is not always true. So, our first goal is to reduce $\mathcal{H}$ to a smaller dense Sperner r^--graph $\mathcal{H}'$ for which we know that the existence of a long cycle in $\partial_2\mathcal{H}'$ implies the existence of a long cycle in $\mathcal{H}'$ itself.

Our second goal is to give an upper bound on the maximum size of a Sperner family of cliques of order at most r in the shadow $\partial_2\mathcal{H}'$ that does not have long cycles. This automatically yields a bound on $|\mathcal{H}'|$.

We systematically consider incidence graphs of r^--graphs instead of the r^--graphs themselves, because we find the language of 2-connected bipartite graphs convenient for our goals.

In Section 4, we prove two results for the maximum number of cliques in graphs without long cycles or paths which will later be applied to the 2-shadows of r^--graphs. Specifically, we give upper bounds for the size of Sperner families of cliques of size at most r in graphs with bounded circumference and graphs that do not contain long paths between every pair of vertices.

In Sections 5 and 6, we prove that our hypergraphs have such a dense subhypergraph that we may reduce to, working in the language of incidence bigraphs in Section 5 and the language of hypergraphs in Section 6. In Section 7, we combine the results from Sections 4-6 to prove Theorem 114. Finally, in Section 8 we prove Theorem 117 for Berge paths in connected hypergraphs.

8.4 Sperner cliques in graphs

A set family H is called *Sperner* if no element of H is contained in another element of H. In particular, every uniform family is Sperner.
The classic proof of LYM Inequality yields also the following result.

Theorem 119. *Let H be a set of h elements. Let $\mathcal{C}$ be a Sperner family of subsets of H such that $|C| \le r$ for each $C \in \mathcal{C}$. Then $|\mathcal{C}| \le \binom{h}{\min\{r, \lfloor h/2 \rfloor\}}$.*

8.4.1 Cliques in graphs with bounded circumference

Recall

$$f(n,k,r,a) := \binom{k-a}{\min\{r, \lfloor \frac{k-a}{2} \rfloor\}} + (n-k+a)\binom{a}{\min\{r-1, \lfloor a/2 \rfloor\}}.$$

For fixed positive integers $n \ge k \ge r$, $f(n,k,r,a)$ is convex over integers a in $[0, \lfloor (k-1)/2 \rfloor]$ (see the appendix for a proof). Thus the value of $f(n,k,r,a)$ is maximized at one of the endpoints of the domain.
For a graph G and a positive integer r, let $N_{\mathrm{Sp}}(G,r)$ denote the maximum size of a Sperner family $\mathcal{C}$ of subsets of $V(G)$ such that for each $C \in \mathcal{C}$, $G[C]$ is a clique of size at most r.

Theorem 120. *Let n, k, r be positive integers with $n \ge k$. Let G be an n-vertex 2-connected graph with no cycle of length k or longer. Then*

$$N_{\mathrm{Sp}}(G,r) \le \max\{f(n,k,r,2), f(n,k,r,\lfloor (k-1)/2 \rfloor)\}.$$

To prove Theorem 120, we again use the following structural theorem by Kopylov for 2-connected graphs without long cycles.

Theorem 121 (Kopylov [Kop77])**.** *Let $n \ge k \ge 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. Suppose that G is a 2-connected n-vertex graph with no cycle of length at least k.*
Then either

(121.1) *the t-core $H(G,t)$ is empty, the graph G is t-disintegrable; or*

(121.2) *$|H(G,t)| = s$ for some $t+2 \le s \le k-2$, and $H(G,t) = H(G,k-s)$, i.e., the rest of the vertices can be removed by a $(k-s)$-disintegration.*

Proof of Theorem 120. Set $t := \lfloor (k-1)/2 \rfloor$. Let G be an n-vertex 2-connected graph with no cycle of length k or longer. Let $\mathcal{C}$ be a Sperner family of subsets of $V(G)$ that are cliques of size at most r with $|\mathcal{C}| = N_{\mathrm{Sp}}(G,r)$. Apply Theorem 121 to G. If (121.1) holds, then every vertex is deleted in the t-disintegration. At the time of its deletion, each vertex v has at most t neighbors and by Theorem 119, is contained in at most $\binom{t}{\min\{r-1, \lfloor t/2 \rfloor\}}$ cliques of

$\mathcal{C}$ (since each clique containing v has at most $r-1$ other vertices). After $n-k+t$ steps in the disintegration process, the remaining $k-t$ vertices contain at most $\binom{k-t}{\min\{\lfloor((k-t)/2)\rfloor, r\}}$ elements of $\mathcal{C}$. Therefore $|\mathcal{C}| \le N_{\mathrm{Sp}}(G,r) \le f(n,k,r,t)$.

Now suppose (121.2) holds. Then we consecutively delete vertices of degree at most $k-s$ until we arrive at the core $H(G,t)$ of size s. As in the previous case, when deleting a vertex v of degree at most $k-s$, we remove at most $\binom{k-s}{\min\{(k-s)/2, r-1\}}$ cliques of $\mathcal{C}$ containing v. Since $H(G,t)$ contains at most $\binom{s}{\min\{s/2,r\}} = \binom{k-(k-s)}{\min\{(k-(k-s))/2\rfloor, r\}}$ cliques in $\mathcal{C}$, we obtain

$$|\mathcal{C}| = N_{\mathrm{Sp}}(G,r) \le f(n,k,k-s) \le \max\{f(n,k,r,2), f(n,k,r,t)\}.$$

The last inequality holds by the convexity of f. $\square$

8.4.2 k-path connected graphs

A graph G is *ℓ-hamiltonian* if for each linear forest L with ℓ edges (and no isolated vertex) on the vertex set $V(G)$ there is a hamiltonian cycle in $G \cup L$ that contains L.

A graph G is *k-path connected* if for each pair of vertices $x, y \in V(G)$, G contains an x,y-path with k or more vertices. In particular, every n-vertex 1-hamiltonian graph is n-path connected. The following theorem will be helpful for us.

Theorem 122 (Enomoto [Eno84])**.** *Let G be a 3-connected graph on n vertices such that for every pair of vertices u,v such that $uv \notin E(G)$, $d(u)+d(v) \ge t$. Then G is k-path connected where $k = \min\{n, 2t-1\}$.*

Define the function

$$h_{\mathrm{Sp}}(n,\ell,r,d) := \binom{n-d+\ell}{\min\{r, \lfloor\frac{n-d+\ell}{2}\rfloor\}} + (d-\ell)\binom{d}{\min\{r-1, \lfloor d/2\rfloor\}}.$$

Note that $h_{\mathrm{Sp}}(n,\ell,r,d) = f(n,n+\ell,r,d)$. For given positive n, r, and $\ell \ge 0$, the function $h_{\mathrm{Sp}}(n,\ell,r,d)$ is convex for $\ell \le d \le n$.

Theorem 123. *Let n,d,r,ℓ be integers with $0 \le \ell < d \le \lfloor\frac{n+\ell-1}{2}\rfloor$. If G is an n-vertex graph with minimum degree $\delta(G) \ge d$, and G is not ℓ-hamiltonian, then*

$$N_{\mathrm{Sp}}(G,r) \le \max\left\{h_{\mathrm{Sp}}(n,\ell,r,d), h_{\mathrm{Sp}}(n,\ell,r,\lfloor\frac{n+\ell-1}{2}\rfloor)\right\}.$$

Proof. Let $\mathcal{C}$ be a Sperner family of cliques of size at most r in G. Suppose that $N_{\mathrm{Sp}}(G,K_r) > h_{\mathrm{Sp}}(n,\ell,r,\lfloor(n+\ell-1)/2\rfloor)$. By a generalization of Pósa's theorem, there exists some $\ell < k < \lfloor(n+\ell-1)/2\rfloor$ such that $V(G)$ contains a subset D of $k-\ell$ vertices with degree at most k (and so $k \ge \delta(G) \ge d$).

For each vertex $v \in D$, v is contained in at most $\binom{k}{\min\{k/2, r-1\}}$ cliques of $\mathcal{C}$, and $G - D$ contains at most $\binom{n-k+\ell}{\min\{\lfloor (n-k+\ell)/2 \rfloor, r\}}$ cliques of $\mathcal{C}$. Hence $|\mathcal{C}| \le N_{\mathrm{Sp}}(G, r) \le h_{\mathrm{Sp}}(n, \ell, r, k) \le h_{\mathrm{Sp}}(n, \ell, r, d)$. $\square$

Our new result is:

Theorem 124. *Let $n \ge 4$. Let G be an n-vertex 2-connected graph. If*

$$N_{\mathrm{Sp}}(G, r) > \frac{n-2}{k-3}\binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}}, \tag{8.1}$$

then G is k-path connected.

Proof of Theorem 124. We use induction on n. If $n \le k-1$, then by Theorem 119,

$$N_{\mathrm{Sp}}(G, r) \le \binom{n}{\min\{r, \lfloor n/2 \rfloor}} = \frac{n-2}{k-3}\left(\frac{k-3}{n-2}\binom{n}{\min\{r, \lfloor n/2 \rfloor\}}\right).$$

And for $n \le k-1$,

$$\frac{k-3}{n-2}\binom{n}{\min\{r, \lfloor n/2 \rfloor\}} \le \frac{k-3}{(k-1)-2}\binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} = \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}}.$$

Hence (8.1) does not hold.
If $n = k$, consider any $x, y \in V(G)$ such that there is no hamiltonian x, y-path in G. If $xy \in E(G)$, then G is not 1-hamiltonian, then by Theorem 123 with $d = 2$ (since G is 2-connected),

$$N_{\mathrm{Sp}}(G, r) \le \max\{h_{\mathrm{Sp}}(n, 1, r, 2), h_{\mathrm{Sp}}(n, 1, r, \lfloor n/2 \rfloor)) = h_{\mathrm{Sp}}(n, 1, r, 2)$$

$$= \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 2 < \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}}\frac{k-2}{k-3} = \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}}\frac{n-2}{k-3},$$

and (8.1) again does not hold. If $xy \notin E(G)$, then the graph $G' := G \cup xy$ satisfies $N_{\mathrm{Sp}}(G', r) \ge N_{\mathrm{Sp}}(G, r)$, and G' is not 1-hamiltonian. So again we obtain $N_{\mathrm{Sp}}(G, r) \le N_{\mathrm{Sp}}(G', r) \le \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}}\frac{n-2}{k-3}$.
Thus from now on we may assume $n \ge k+1$.

Claim 125. *G is 3-connected.*

Proof. Suppose $\{v_1, v_2\}$ is a separating set. Let C_1 be the vertex set of a component of $G - \{v_1, v_2\}$ and $C_2 = V(G) - C_1$. For $i = 1, 2$, let G_i be obtained from $G - C_{3-i}$ by adding edge $v_1 v_2$ if it is not in G. Let $n_i = |V(G_i)|$. By construction, each of G_1 and G_2 is

2-connected. Also,

$$n_1 + n_2 = n + 2 \quad \text{and} \quad N_{\mathrm{Sp}}(G, r) \leq N_{\mathrm{Sp}}(G_1, r) + N_{\mathrm{Sp}}(2, r). \tag{8.2}$$

By (8.2), some of G_i satisfies (8.1). By symmetry, suppose G_2 does. If $x, y \in V(G_2)$, then we are done by induction. Suppose neither of x and y is in $V(G_2)$. Then by induction, G_2 has a v_1, v_2-path P with at least k vertices. Also, the 2-connected graph G_1 has two disjoint paths P_1 and P_2 from $\{x, y\}$ to $\{v_1, v_2\}$. Then $P_1 \cup P \cup P_2$ forms a long x, y-path.
Finally, suppose $x \in V(G_2)$ and $y \notin V(G_2)$. Again by induction, G_2 has a v_1, x-path P with at least k vertices. Also, the 2-connected graph G_1 has a v_1, y-path P_1 that avoids v_2. Then $P \cup P_1$ is what we need. □

Claim 126. *$\delta(G) \geq \frac{k+1}{2}$.*

Proof. Suppose $v_1 \in V(G)$ and $d(v_1) \leq k/2$. Since G is 3-connected, we can choose a neighbor v_2 of v_1 so that $v_2 \notin \{x, y\}$. Let G' be obtained from G by contracting v_1 and v_2 into a new vertex that we again will call v_1. Since G was 3-connected, G' is 2-connected.
Let $\mathcal{S}_G$ be a maximum Sperner family of cliques of size at most r in G. We construct a family $\mathcal{S}'$ of cliques of size at most r in G' from $\mathcal{S}_G$ by
(a) deleting from $\mathcal{S}_G$ all cliques containing v_1; and
(b) replacing each clique $S \in \mathcal{S}_G$ with $v_2 \in S$ and $v_1 \notin S$ with the clique $S - v_2 + v_1$.
We claim that $\mathcal{S}'$ is Sperner. Indeed, suppose $S_1, S_2 \in \mathcal{S}'$ and $S_1 \subset S_2$. Since $\mathcal{S}_G$ was Sperner, $v_1 \in S_2 - S_1$. But then $S_2 - v_1 + v_2 \in \mathcal{S}_G$ and $S_1 \subset S_2 - v_1 + v_2$.
By construction and Theorem 119,

$$|\mathcal{S}_G| - |\mathcal{S}'| \leq \binom{d(v_1)}{\min\{r-1, \lfloor d(v_1)/2 \rfloor\}} \leq \binom{\lfloor k/2 \rfloor}{\min\{r, \lfloor k/4 \rfloor\}}.$$

But

$$\binom{\lfloor k/2 \rfloor}{\min\{r, \lfloor k/4 \rfloor\}} \leq \frac{1}{k-3} \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}},$$

and hence G' satisfies (8.1). So by the minimality of G, graph G' has a long x, y-path. But then G also does. □

Applying Theorem 122 completes the proof of our theorem. □

8.5 Constructing happy incidence bigraphs

8.5.1 Language of layered r^--bigraphs

A *layered bigraph* is a bigraph $G = (A, Y; E)$ in which parts A and Y are ordered.

An r^--*bigraph* is a layered bigraph $G = (A, Y; E)$ with $d(a) \leq r$ for each $a \in A$.
A layered bigraph $G = (A, Y; E)$ is *Sperner* if the family $\{N(a) : a \in A\}$ is Sperner. By definition, if $N(a) = \{v, u\}$ in a Sperner bigraph, then the codegree of the pair vu is 1.
In particular, the incidence graph $G_{\mathcal{H}}$ of an r^--graph $\mathcal{H}$ is a Sperner r^--bigraph if and only if $\mathcal{H}$ is Sperner.
A vertex $a \in A$ of a layered bigraph $G = (A, Y; E)$ is *happy*, if the the codegree $d(x, y)$ of each pair $\{x, y\} \subseteq N(a)$ is at least $d(a) - 1$ (and *unhappy* otherwise). A layered bigraph $G = (A, Y; E)$ is *happy* if every vertex $a \in A$ is happy.
A vertex $y \in Y$ of degree 2 in is *special*, if each of the two neighbors is either unhappy or also has degree 2.
Vertices $x, y \in Y$ and $a \in A$ form a *special triple* if x and y are special (in particular they have degree 2), $N(a) = \{x, y\}$, and the other neighbors of x and y are unhappy.
Given a layered bigraph $G = (A, Y; E)$, let *the shadow* $\partial(G)$ be the graph F with vertex set Y such that $xy \in E(F)$ iff there is $a \in A$ with $\{x, y\} \subseteq N(a)$.
For each graph H, the *circumference*, $c(H)$, is the length of a longest cycle in H.
We first prove a simple corollary of Hall's Theorem.

Lemma 127 (Folklore)**.** *Let $G = (A, B; E)$ be a bipartite graph with no isolated vertices such that for each $a \in A$ and every $b \in N(A)$, $d(a) \geq d(b)$. Then G has a matching covering A.*

Proof. Suppose that G has no matching covering A. By Hall's Theorem, there is $S \subseteq A$ with $|S| > |N(S)|$. Choose a minimum such S, say $S = \{a_1, \ldots, a_s\}$. By the minimality of S, G has a matching M covering $S' := S - a_s$, say $M = \{a_i b_i : 1 \leq i \leq s - 1\}$. Since $|N(S)| \leq s - 1$, we have $N(S) = \{b_1, \ldots, b_{s-1}\}$. So,

$$d(a_1) + \ldots + d(a_{s-1}) + d(a_s) = e(S, N(S)) = d_S(b_1) + \ldots + d_S(b_{s-1}) \leq d(a_1) + \ldots d(a_{s-1}),$$

a contradiction. □

Lemma 128. *Let $r \geq 3$. If $G = (A, Y; E)$ is a happy Sperner r^--bigraph and $\partial(G)$ contains a cycle of length $\ell \geq r$, then G contains a cycle of length 2ℓ.*

Proof. Let $C = x_1, \ldots, x_\ell$ be a cycle of length $\ell \geq r$ in $\partial(G)$. Let F be the bipartite graph with parts $Q = E(C)$ and A such that a pair $(x_i x_{i+1}, a)$ is an edge in F if and only if $\{x_i x_{i+1}\} \subseteq N(a)$. If $\ell \geq r + 1$, then since each $a \in A$ has degree less than ℓ, a is adjacent to at most $d(a) - 1$ pairs $x_i x_{i+1}$. On the other hand, for each edge $(x_i x_{i+1}, a)$ in F, $d_F(\{x_i x_{i+1}\}) \geq d(a) - 1$ since G is happy. So by the previous lemma, F has a matching that covers $E(C)$, say with $x_i x_{i+1}$ matched to $f(x_i x_{i+1}) \in A$. Then we obtain the cycle $x_1, f(x_1 x_2), x_2, f(x_2 x_3), \ldots, x_\ell, f(x_\ell x_1), x_1$ of length 2ℓ in G.

Now suppose $\ell = r$. If for every $a \in A$, $N_G(a) \neq \{x_1, \ldots, x_r\}$, then $d_F(a) \leq d(a) - 1$, and we are done as in the previous case. So suppose there exists an a such that $N_G(a) = \{x_1, \ldots, x_r\}$. Then because G is Sperner, each $a' \in A - a$ is adjacent to at most $r - 1$ vertices in $\{x_1, \ldots, x_r\}$, and hence $d_F(a') \leq (r-1) - 1$. Consider the graph $F - a$. For $a' \in A - a$,

$$d_{F-a}(a') = d_F(a') \leq \min\{r-2, d(a') - 1\}.$$

If some vertex $x_i x_{i+1}$ was adjacent to a in F, then $d_F(x_i x_{i+1}) \geq d(a) - 1 = r - 1$ and so $d_{F-a}(x_i x_{i+1}) \geq r - 2$. Otherwise, for each $x_i x_{i+1}$ not adjacent to a in F, and each $a' \in N_F(x_i, x_{i+1})$, $d_{F-a}(x_i x_{i+1}) = d_F(x_i x_{i+1}) \geq d(a') - 1$, so we are finished as in the first case. □

The same proof also yields the following Lemma for paths of any length.

Lemma 129. *Let $G = (A, Y; E)$ be a happy r^--bigraph. If $\partial(G)$ contains a path with ℓ vertices, then G contains a path with $2\ell - 1$ vertices with endpoints in Y.*

We will often use the following known property of 2-connected graphs.

Lemma 130. *Let G be a 2-connected graph, $xy \in E(G)$ and $S \subset V(G)$ with $|S| \leq |V(G)| - 2$.*

(1) *$G - xy$ is 2-connected iff $G - xy$ has a cycle containing x and y;*

(2) *the graph G/S obtained by gluing the vertices of S into one vertex s^* is 2-connected iff s^* is not a cut vertex of G/S.*

8.5.2 Unhappy r^--bigraphs

Definition 131. *Let $G = (A, Y; E)$ be a Sperner layered 2-connected r^--bigraph $G = (A, Y; E)$. A* **shrinking of** *G is one of the following operations:*

(1) *deleting an edge of G incident to an unhappy vertex,*

(2) *deleting a special vertex $y \in Y$ and all neighbors $b \in N(y)$ with $d(b) = 2$,*

(3) *deleting a special triple $x, y \in Y$ and $a \in A$, or*

(4) *gluing together all but one of the neighbors of some unhappy vertex $a \in X$.*

The goal of this subsection is to prove that unhappy Sperner layered 2-connected r^--bigraphs not admitting a shrinking have a special structure and high maximum average degree. The main result of the subsection is the following lemma.

Lemma 132. *Suppose $k \geq r \geq 3$ are integers. Let $G = (A, Y; E)$ be a Sperner layered 2-connected r^--bigraph with $c(G) < 2k$ that is not happy. Then either G admits a shrinking such that the resulting graph G' satisfies*

(S1) G' *is* 2*-connected;*

(S2) $|E'| \le |E|$, $|Y'| \le |Y|$, *and* $|E'| + |Y'| < |E| + |Y|$*;*

(S3) G' *is Sperner;*

(S4) $|A| - |A'| \le |Y| - |Y'|$*; and*

(S5) $c(G') < 2k$,

or for every unhappy vertex $a \in A$*, there exists three vertices* $y_1, y_2, y_3 \in N(a)$ *and three subgraphs* B_1, B_2, B_3 *of* G *such that for* $i \in \{1, 2, 3\}$

(B1) $y_i \in V(B_i)$, $a \notin V(B_i)$, *and* y_i *is the only neighbor of* a *in* B_i*;*

(B2) B_i *is* 2*-connected and Sperner;*

(B3) *there exists a* $x_i \in Y$ *such that* $\{a, x_i\}$ *separates* B_i *from* $G - B_i$*;*

(B4) $G - (B_i - x_i) - a$ *is Sperner and* 2*-connected; and*

(B5) *for* $j \in \{1, 2, 3\} - \{i\}$, $|V(B_i) \cap V(B_j)| \le 1$ *with equality if and only if* $x_i = x_j$*.*

Proof. Suppose, $G = (A, Y; E)$ is a Sperner layered 2-connected r^--bigraph with $c(G) < 2k$ that is not happy. Then it has an unhappy vertex $a \in A$. Let $N_G(a) = \{y_1, \ldots, y_t\}$. Since a is unhappy, $t \ge 3$. Assume that there are no G' satisfying the lemma. We derive a series of properties of such G.

A vertex $y_i \in N(a)$ is an *a-menace*, if there is a vertex $m(a, y_i) \in A - a$ such that $N(a) - y_i \subseteq N(m(a, y_i))$. Since G is Sperner,

$$G - ay_i \text{ is Sperner if and only if } y_i \text{ is not an } a\text{-menace.} \tag{8.3}$$

For brevity, we call pairs of vertices in Y of codegree 1 *thin* and of codegree at least 2 — *thick*.

Claim 133. $N(a)$ *contains a thin pair.*

Proof. Suppose that all pairs of $N(a)$ are thick pairs. For each $y_i \in N(a)$, the graph $G_i := G - ay_i$ trivially satisfies (S2), (S4), and (S5) in the definition of shrinking. We will show that G_i is also 2-connected, i.e., it satisfies (S1). Let $y_j, y_k \in N(a) - y_i$. Because every pair of $N(a)$ is thick, there exists distinct vertices $b_{ij}, b_{ik} \ne a$ such that $\{y_i, y_j\} \in N(b_{ij})$ and $\{y_i, y_k\} \in N(b_{ik})$. Applying Lemma 130 with the cycle $y_i b_{ij} y_j a y_k b_{ik} y_i$ certifies that G_i is 2-connected.

If for some $1 \le i \le t$, the graph G_i is Sperner, i.e., satisfies (S3), then we are done. Assume not. Because a is the only vertex with a changed neighborhood in G_i, for all i there exists a vertex b_i in G such that $\{y_1, \ldots, y_t\} - \{y_i\} \subset N(b_i)$. Furthermore, for $i \ne j$, $b_i \ne b_j$, otherwise some $N(b_i)$ contains $N(a)$, contradicting the fact that G is Sperner.

In particular, each pair in $N(a)$ belongs in the neighborhoods of a and $d(a) - 2$ additional vertices, contradicting that a is unhappy. $\square$

Claim 134. *All distinct thick pairs in $N(a)$ are disjoint.*

Proof. Suppose not. First we show that there exist some thick pairs $\{y_{i^*}, y_{j^*}\}, \{y_{i^*}, y_{k^*}\}$ and a thin pair $\{y_{s^*}, y_{t^*}\}$ such that $s^*, t^* \neq i^*$. Let $\{y_i, y_j\}, \{y_i, y_k\}$ and $\{y_s, y_t\}$ be any intersecting thick pairs of $N(a)$ and a thin pair respectively where without loss of generality, $y_s \notin \{y_i, y_j\}$. If $y_t \neq y_i$ then we are done. If not then consider instead the pair $\{y_s, y_j\}$. If it is thin, then we take this pair instead of $\{y_s, y_t\}$. If it is thick, then we let $\{y_i, y_j\}, \{y_s, y_j\}$ be our intersecting thick pairs with y_j playing the role of y_{i^*} and $\{y_s, y_t\} = \{y_s, y_i\}$ be the thin pair.
Now consider the graph $G - ay_{i^*}$. As in the previous claim, it satisfies (S2), (S4), and (S5) as well as (S1) in the definition of shrinking where we define vertices $b_{i^*j^*}, b_{i^*k^*}$ similarly. Since no other vertex contains the pair $\{y_{s^*}, y_{t^*}\}$ in its neighborhood, $G - ay_{i^*}$ is Sperner. □

Claim 135. *The codegree of each pair in $N(a)$ is at most 2.*

Proof. Suppose there exist distinct vertices $b_1, b_2 \neq a$ both adjacent to y_1 and y_2. Since $\{y_1, y_2\}$ is a thick pair, $\{y_1, y_3\}$ and $\{y_2, y_3\}$ are thin by the previous claim. Let P be a shortest path in $G - a$ from y_3 to $\{y_1, y_2\}$. Note that if P contains b_1 or b_2, then by the minimality of $|P|$, either y_1 or y_2 follows directly after. Therefore we may assume by symmetry that $y_1 \in P$ and $b_2 \notin P$. Consider the graph $G - ay_1$. Trivially it satisfies (S2), (S4), and (S5). Because $\{y_2, y_3\}$ is thin, it also satisfies (S3). Finally, the cycle $y_3Py_1b_2y_2ay_3$ certifies that (S1) is satisfied. □

Claim 136. *If a proper subset S of $N(a)$ is a separating set in G, then S contains an a-menace.*

Proof. If the claim does not hold, choose a smallest separating subset $S = \{y_1, \ldots, y_s\}$ of $N(a)$ not containing a-menaces. Since S is a proper subset of $N(a)$, $s < t$. Let D_1 and D_2 be components of $G - S$, where D_1 contains a. By the minimality of S,

$$\textit{each } y_i \in S \textit{ has a neighbor in } D_2. \tag{8.4}$$

Since G is 2-connected, there are two v_t, S-paths P_1 and P_2 sharing only v_t. By symmetry we may assume that P_1 avoids a. Let y_1 be the end of P_1 in S. By (8.4), there is a y_1, y_2-path P_3 all whose internal vertices are in D_2.
Consider $G' = G - ay_1$. Properties (S2), (S4) and (S5) in the claim of the lemma hold for G' by definition. Since y_1 is not an a-menace, by (8.3), G' is Sperner, i.e. (S3) holds. Cycle $y_2av_tP_1y_1P_3y_2$ together with Lemma 130 show that G' is 2-connected. Thus, G' satisfies the lemma. □

Claim 137. *$N(a)$ has no thick pairs.*

Proof. Suppose pair y_1y_2 is thick. By Claims 134 and 135, $d(y_1y_2) = 2$ and the common neighbor $b \in A - a$ of y_1 and y_2 has no other neighbors in $N(a)$. Let $N(b) = \{y_1, y_2, z_1, \ldots, z_s\}$. Since G is Sperner, $s \geq 1$.

By Claim 134, neither of y_1 and y_2 is an a-menace. So, by Claim 136, $G - y_1 - y_2$ contains an a,b-path P_1. We may assume that v_t is the second and z_1 is the second to last vertices of P_1. Since $d(y_1y_2) = 2$, by Claim 5, $z_1 \notin N(a)$. So $v_t \neq z_1$.

Case 1: $d(y_1) = 2$. Then $d(y_1z_1) = 1$ and hence b is unhappy. So, since $d(y_1y_2) = 2$, by Claim 134, $d(y_2z_1) = 1$. Consider $G' = G - y_1$. As in the proof of Claim 136, (S2), (S4) and (S5) hold for G' by definition. Cycle $y_2ay_2P_1by_2$ together with Lemma 130 sertify that G' is 2-connected, i.e., (S1) holds. Only the neighborhoods of a and b in A' are distinct from those in A. So the fact that $d(y_2z_1) = d(y_2y_t) = 1$ shows that G' is Sperner. This proves Case 1.

Case 2: $d(y_1) \geq 3$. Let $c \in N(y_1) - a - b$, where if possible we choose c to be adjacent to z_1. Since G is 2-connected, $G - y_1$ has a shortest path P_2 from c to $V(P_1) \cup \{y_2\}$. Let x be the end of P_2 in $V(P_1) \cup \{y_2\}$.

Case 2.1: $x \neq b$. Consider $G' = G - ay_1$. As above, (S2), (S4) and (S5) trivially hold for G'. Since only the neighborhood of a in A' is distinct from those in A and $d(y_2y_t) = 1$, G' is Sperner. We need now only to show that G' is 2-connected. If $x = y_2$, then cycle $cP_2y_2aP_1by_1c$ certifies this. If $x \in V(P_1) - b$, then our certificate is cycle $cy_1by_2aP_1(a,x)xP_2c$, where $P_1(a,x)$ denotes the subpath of P_1 from a to x.

Case 2.2: $x = b$. Note that because $x \neq z_1$, by the choice of c and the choice of P_2, $z_1 \notin N(c)$ for any $c \in N(y_1) - a - b$. In particular, $d(y_1z_1) = 1$, and so b is unhappy. The second to last vertex of P_2 is none of z_1, y_1, y_2, so we may assume it is z_2. Consider $G' = G - by_1$. Cycle $cP_2by_2ay_1c$ shows that G' is 2-connected. As above, (S2), (S4) and (S5) trivially hold for G'. Thus if G' is Sperner, then the claim is proved. If G' is not Sperner, then y_1 is a b-menace, and there is a vertex $g \in A - b$ such that $N(g) \supset \{y_2, z_1, z_2\}$. Since $z_1a \notin E$, $g \neq a$. But then instead of the path P_2, we can consider the path $P_2(c, z_2)z_2gz_1$, and will have Case 2.1. $\square$

$$\textit{Let } G' \textit{ be obtained from } G \textit{ by gluing all vertices in } N(a) - y_t \textit{ into one vertex } y^*. \tag{8.5}$$

(S2) holds for G' trivially. When gluing the vertices, we lose edges only if some pair $y_i, y_j \in N(a)$ have a common neighbor. But because $\{y_i, y_j\}$ is thin, they have no common neighbors other than a. Hence $|E'| = |E| - (t-2)$ and $|Y'| = |Y| - (t-2)$ so (S4) holds. Property (S5) is less clear but still is true: If G' has a cycle C of length at least $2k$, then it must go through y^*. Furthermore, if C does not go through a, then either C is present in G with y^* replaced by some y_i, or it can be extended through a connecting some y_i and y_j. If C does through a, then it uses edges ay_t and ay^*; we can modify C in G to a cycle of the same length. Thus, (S5) also holds.

Since all pairs in $N(a)$ are thin, none of y_i is an a-menace. So by Claim 136 and Lemma 130, G' is 2-connected. Again, since all pairs in $N(a)$ are thin, $N_{G'}(a)$ is not contained in any other neighborhood. Hence, in order the lemma to fail, by symmetry there are $b_1, b_2 \in A - a$ such that $N_G(b_2) - y_2 \subset N_G(b_1)$ and $y_1 b_1 \in E$. Note that b_1 and b_2 each contain exactly one vertex in $N(a)$ (y_1 and y_2 respectively), and there is $x \in N(b_1) \cap N(b_2)$ such that $x \notin N(a)$.

Claim 138. $d(b_2) = 2$.

Proof. Suppose $N(b_2) \supseteq \{y_2, x_1, x_2\}$. Then by the definition of b_1, $N(b_1) \supseteq \{y_1, x_1, x_2\}$. So by Claim 137 applied to b_1 and b_2, because the pair $\{x_1, x_2\}$ is thick, both b_1 and b_2 are happy . Since G is 2-connected, $G - a$ has a shortest path P from v_t to $Z = \{y_1, y_2, b_1, b_2, x_1, x_2\}$. Let z be the last vertex of P. By symmetry, we may assume $z \in \{y_2, b_2, x_2\}$. Consider $G' = G - ay_2$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. If $z = y_2$, then the cycle $aPy_2b_2x_2b_1y_1a$ shows that G' is 2-connected.

So suppose $z \in \{b_2, x_2\}$. Since b_2 is happy, there is another b_3 adjacent to y_2 and x_2. By definition, it is distinct from b_1 and a. So if $z = x_2$ and P does not pass through b_3, then we have cycle $aPx_2b_3y_2b_2x_2b_1y_1a$. Similarly, if $z = b_2$ and P does not pass through b_3, then we have cycle $aPb_2y_2b_3x_1b_1y_1a$. Finally, if P passes through b_3, then we have cycle $aP(a, b_3)b_3y_2b_2x_1b_1y_1a$. □

Claim 139. $d(y_2) \geq 3$.

Proof. Recall $x = N(b_1) \cap N(b_2)$. Assume $N(y_2) = \{a, b_2\}$. By Claim 136, $G - y_1 - y_2$ has an a, x-path P. We can choose a shortest such path. Let c be the second to last vertex in P.

Case 1: $c \neq b_1$. Consider $G' = G - b_2 - y_2$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. The cycle $aPxb_1y_1a$ shows that G' is 2-connected.

Case 2: $c = b_1$. Let z be the previous to c vertex of P. Since all pairs in $N(a)$ are thin, $z \neq v_t$. If b_1 is happy, then there exists a vertex $b_3 \neq b_1$ with $\{y_1, x\} \subseteq N(b_3)$. Then b_3 can play the role of b_1 in the definition of b_1 and b_2. In this case, we get Case 1 and are done. Thus, b_1 is unhappy. Hence all pairs in $N(b_1)$ are thin.

If $d(x) = 2$, consider $G' = G - b_2 - y_2 - x$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ and in $N(b_1)$ are thin, G' is Sperner. The cycle aPb_1y_1a shows that G' is 2-connected.

So suppose $b_4 \in N(x) - b_1 - b_2$. Since G is 2-connected, $G - x$ has an b_4, a-path P_1. If P_1 does not intersect $\{b_1, y_1\}$, then we have Case 1 with $P = aP_1b_4x$. So, suppose u is the first vertex in $\{b_1, y_1\}$ that is hit by P_1. Note that if P_1 meets $P - u$ before u, then we can modify it to avoid intersecting with $\{b_1, y_1\}$. Thus we assume below that this is not the case.

If $u = y_1$, consider $G' = G - ay_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. The cycle $aPb_1y_1P_1(y_1, b_4)xb_2y_2a$ shows that G' is 2-connected. Finally, if $u = b_1$, consider $G' = G - b_1x$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ are thin, G' is Sperner. The cycle $aPb_1P_1(b_1, b_4)xb_2y_2a$ shows that G' is 2-connected. $\square$

Claim 140. *Set $\{x, y_1, y_2\}$ separates a from b_1.*

Proof. Suppose not. Then $G - \{x, y_1, y_2\}$ has an a, b_1-path P. Note that $b_2 \notin P$ since $N(b_2) = \{x, y_2\}$. Let the second vertex of P be v_t.

If b_1 is happy, then there is $b_3 \in A - b_1$ with $N(b_3) \supseteq \{y_1, x\}$. Consider $G' = G - ay_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. We need to show that G' is 2-connected. If $b_3 \in P$, then the cycle $aP(a, b_3)b_3y_1b_1xb_2y_2a$ certifies this. Otherwise, the cycle $aPb_1y_1b_3xb_2y_2a$ certifies this.

So, b_1 is unhappy, and all pairs in $N(b_1)$ are thin. If $d(y_1) = 2$, consider $G' = G - y_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ and in $N(a)$ are thin, G' is Sperner. The cycle $aPb_1xb_2y_2a$ shows that G' is 2-connected.

Thus, $d(y_1) \geq 3$. Let $c \in N(y_1) - a - b_1$. Let P_1 be a shortest path in $G - y_1$ from c to $V(P) \cup \{x, y_2\}$. Let z be the last vertex of P_1. If $z \in V(P) - b_1$, consider $G' = G - ay_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. The cycle $aP(a, z)zP_1cy_1b_1xb_2y_2a$ certifies that G' is 2-connected.

If $z \in \{b_1, x, y_2\}$, consider $G' = G - b_1y_1$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ are thin, G' is Sperner. Let P_2 denote the path $ay_2b_2xb_1$. Then the cycle $ay_1cP_1zP_2(z, b_1)b_1Pa$ certifies that G' is 2-connected. $\square$

Claim 141. *Set $\{x, a\}$ separates y_2 from $N(a) - y_2$.*

Proof. Suppose not. Let P be a shortest a, x-path in $G - y_1 - y_2$. By Claim 140, P does not go through b_1. Let the second vertex of P be v_t. Let P_1 be a shortest path in $G - a - x$ from y_2 to $(N(a) - y_2) \cup V(P)$. Let z be the last vertex of P_1. If $b_1 \in V(P)$, then we can take $z = y_1$. Consider $G' = G - ay_2$. As before, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. If $z \in N(a) - v_t$ then the cycle $y_2P_1zaPxb_2y_2$ certifies that G' is 2-connected. Otherwise, the cycle $y_2P_1zP(z, a)ay_1b_1xb_2y_2$ does it. $\square$

Let C_2 be the vertex set of the component of $G - a - x$ containing y_2 and let $G_2 = G[C_2 \cup \{a, x\}]$. By Claim 141, $C_2 \cap N(a) = \{y_2\}$. If x has no neighbors in $C_2 - b_2$, then by Claim 140, y_2 would be a cut vertex, a contradiction. Thus, in view of b_2, no vertex in

$G_2 - a$ separates x from y_2. Since no vertex in $G_2 - a$ may separate $\{y_2, x\}$ from any other vertex, we conclude

$$G_2 - a \textit{ is 2-connected and the unique neighbor of } a \textit{ in } C_2 \textit{ is } y_2. \tag{8.6}$$

Claim 142. *Set $\{x, a\}$ separates y_1 from $N(a) - y_1$.*

Proof. Suppose not. If $d(b_1) = 2$, then by symmetry of b_1 and b_2 and the previous claim, we are done. So $d(b_1) \geq 3$. Let $x' \in N(b_1) - y_1 - x$. Let P be a shortest a, x-path in $G - y_1 - y_2$. By Claim 141, P does not go through b_2. Let the second vertex of P be v_t.
Let P_1 be a shortest path in $G - a - x$ from $\{y_1, b_1\}$ to $V(P) \cup (N(a) - y_1 - y_2)$. Let z_1 be the first vertex of P_1 and z_2 — the last. If $z_1 = y_1$, consider $G' = G - ay_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. If $z_2 \in N(a) - v_t$ then the cycle $y_1P_1z_2ay_2b_2xb_1y_1$ certifies that G' is 2-connected. Otherwise, the cycle $y_1P_1z_2P(z_2, a)ay_2b_2xb_1y_1$ does it.
So suppose $z_1 = b_1$.
Case 1: b_1 is unhappy. If $z_2 \in V(P)$, then we consider $G' = G - xb_1$. As above, (S2),(S4) and (S5) hold for G'. Since b_1 is unhappy, all pairs in $N(b_1)$ are thin, and hence G' is Sperner. The cycle $b_1P_1z_2P(z_2, x)xb_2y_2ay_1b_1$ certifies that G' is 2-connected. So below we assume $z_2 = y_3$ and $t \geq 4$.
If $d(y_1) = 2$, then we consider $G' = G - y_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ and $N(b_1)$ are thin, G' is Sperner. The cycle $b_1P_1y_3ay_2b_2xb_1$ certifies that G' is 2-connected.
Thus there is $b_0 \in N(y_1) - a - b_1$. If $G - b_1 - y_1$ has a path from b_0 to $N(a) - y_1$, then we would have the case $z_1 = b_1$ above. Hence there is no such path. But then $G - V(P) - N(a)$ has a b_0, b_1-path P_2. In this case, we consider $G' = G - y_1b_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ are thin, G' is Sperner. The cycle $y_1b_0P_2b_1xb_2y_2ay_1$ certifies that G' is 2-connected.
Case 2: b_1 is happy. Then there is another common neighbor b'_1 of x and y_1. Again, consider $G' = G - ay_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(a)$ are thin, G' is Sperner. If $b'_1 \notin P_1$ and $z_2 \in N(a) - v_t$ then the cycle $b_1P_1z_2ay_2b_2xb'_1y_1b_1$ certifies that G' is 2-connected. If $b'_1 \notin P_1$ and $z_2 \in V(P)$ then the cycle $b_1P_1z_2P(z_2, a)ay_2b_2xb'_1y_1b_1$ does it. If $b'_1 \in P_1$, then we switch the roles of b_1 and b'_1: consider the path $P'_1 = P_1(b'_1, z_2)$. □

Claim 143. *Vertex a has only one neighbor (namely, y_1) in the component C_1 of $G - x - a$ containing y_1 and b_1.*

Proof. Otherwise, $\{x, a\}$ would not separate y_1 from $N(a) - y_1$, a contradiction to Claim 142. □

Similarly to the definition of G_2, let C_1 be the vertex set of the component of $G - a - x$ containing y_1 and let $G_1 = G[C_1 \cup \{a, x\}]$. By Claim 143, $C_1 \cap N(a) = \{y_1\}$.

Claim 144. *$G_1 - a$ is 2-connected.*

Proof. **Case 1:** $G - a - b_1$ has an x, y_1-path P. Then $P + b_1$ forms a cycle in $G_1 - a$ containing x and y_1. Since G is 2-connected and $\{y_1, x\}$ is a separating set in G_1, this finishes the case.
Case 2: $d(b_1) = 2$. Then y_1 can play the role of y_2, and we are done by (8.6).
Case 3: Vertex b_1 separates y_1 from x in $G_1 - a$, and b_1 has a neighbor $y' \notin \{x, y_1\}$. If b_1 were happy, there would be $b' \neq b_1$ adjacent to x and y_1 and we would have Case 1. So, b_1 is unhappy. Let P_1 be a shortest path from y' to $\{a, x\}$ in $G - b_1$. and z be the last vertex on P_1.
Suppose first that $z = a$. Then by Claim 143, the second to last vertex of P_1 is y_1. Consider $G' = G - y_1b_1$. As above, (S2),(S4) and (S5) hold for G'. Since b_1 is unhappy, all pairs in $N(b_1)$ are thin. Thus G' is Sperner. The cycle $y'P_1ay_2b_2xb_1y'$ certifies that G' is 2-connected.
Suppose now that $z = x$. Since Case 1 does not hold, $y_1 \notin P_1$. Consider $G' = G - xb_1$. As above, (S2),(S4) and (S5) hold for G'. Since all pairs in $N(b_1)$ are thin, G' is Sperner. The cycle $y'P_1xb_2y_2ay_1b_1y'$ certifies that G' is 2-connected. □

Claim 145. *$G - C_1$ and $G - C_2$ are 2-connected Sperner r^--graphs.*

Proof. Let P be a shortest y_3, x-path in $G - a$. By Claim 141 and 142, P avoids $C_1 \cup C_2$. For $i = 1, 2$, the cycle $y_3Pxb_{3-i}y_{3-i}ay_3$ certifies that $G - C_i$ is 2-connected. Since the degrees of the vertices in $G - C_1$ and $G - C_2$ are dominated by those in G, $G - C_1$ and $G - C_2$ are r^--graphs. Since a is the only vertex in $A \cap V(G - C_i)$ whose degree decreased w.r.t. G and all pairs in $N(a)$ are thin, $G - C_1$ and $G - C_2$ are Sperner. □
Now set $B_1 = G_1 - a$, $B_2 = G_2 - a$, and $x_1 = x_2 = x$. Note that the choice of y_t in (8.5) was arbitrary. So we may repeat the proof instead taking G'' to be the graph obtained by gluing $N(a) - y_1$ into a single vertex y^{**}. If G'' satisfies (S1) - (S5), then we are done. Otherwise we find some vertices $y_1', y_2' \in N(a) - y_1$ which play the role of y_1 and y_2. We may assume that $y_1' \notin \{y_1, y_2\}$ and it is coupled with some vertex x' which plays the role of x.
Again, repeating the previous proofs for Claims 138-145 with y_1' and y_2', we obtain that either G admits a shrinking, or we can define G_1' similarly to play the role of G_1 (defined after Claim 143) for y_1'. Let $B_3 = G_1' - a$, $y_3 = y_1'$, and $x_3 = x'$. We now show that (B1) - (B5) hold.
(B1) and (B3) are trivial. Since G was Sperner each vertex of $A \cap V(B_i)$ has the sane neighborhood in B_i, B_i is also Sperner. Hence together with (8.6) and Claim (144), we get

(B2). Claim 145 proves (B4). Claims 141 and 142 imply that $V(B_1) \cap V(B_2) = \{x\}$, and $y'_1(= y_3)$ is contained in a component of $G - \{a, x\}$ not containing y_1 and y_2. In particular, B_3 is disjoint from B_1 and B_2 except possibly at x' if $x' = x$. This proves (B5) and thus the Lemma 132. $\square$

8.5.3 Consequences of Lemma 132

This technical lemma implies the following more applicable fact.

Lemma 146. *Suppose $k \geq 5$, $r \geq 3$ are integers with $k \geq r$. Set $t = \lfloor (k-1)/2 \rfloor$. Let $G = (A, Y; E)$ be a Sperner layered 2-connected r^--bigraph with $c(G) < 2k$ that is not happy. Then either G admits a shrinking such that the resulting graph satisfies (S1) - (S5), or there exists an unhappy vertex $a^* \in A$ and some block B^* satisfying the hypothesis of Lemma 2.4 such that B^* is happy and $|A \cap B^*| \leq \binom{t}{\min\{r-1, \lfloor t/2 \rfloor\}}(|Y \cap B^*| - 2)$.*

Proof. Suppose G does not admit any shrinking. By Lemma 132, for each unhappy vertex a we obtain some $\{y_i, x_i, B_i\}$ for $i \in \{1, 2, 3\}$ satisfying (B1) - (B5).

Claim 147. *For each unhappy a, at most one B_i has a (x_i, y_i)-path of length k or longer.*

Proof. Suppose without loss of generality that for $i \in \{1, 2\}$, there exists a (y_i, x_i)-path P_i in B_i of length at least k. Recall that $y_1, y_2 \in N(a)$. Let P_3 be a (x_1, x_2)-path internally disjoint from $V(B_1) \cup V(B_2)$ (where P_3 may be a singleton). Then $P_1 \cup P_3 \cup P_2 \cup a$ is a cycle of length at least $2k - 1$, i.e., length at least $2k$. $\square$

Among all vertices in A that are not happy, choose a and a corresponding 2-connected graph B_1 from Lemma 2.4 so that (a) B_1 does not have a (x_i, y_i)-path of length k or longer, and (b) subject to (a), $|V(B_1)|$ is minimized.
Suppose first that B_1 contains an unhappy vertex a'. By Lemma 2.4, there exists $\{x'_i, y'_i, B'_i\}$ for $i \in \{1, 2, 3\}$ satisfying (B1)-(B5) with a'.

Claim 148. *At most one $j \in \{1, 2, 3\}$ satisfies $V(B'_j) \not\subseteq V(B_1)$.*

Proof. Suppose without loss of generality $V(B'_2) \not\subseteq V(B_1)$ and $V(B'_3) \not\subseteq V(B_1)$. Then since $\{x_1, a\}$ separates B_1 from $G - (B_1 - x) - a$, and B'_2 and B'_3 are 2-connected, $\{x_1, a\} \subseteq V(B'_2)$ and $\{x_1, a\} \subseteq V(B'_3)$. But this violates (B5). $\square$

Therefore we may assume $V(B'_1), V(B'_2) \subseteq V(B_1)$. By Claim 147, we can also assume that $V(B'_1)$ has no (x'_1, y'_1)-path of length k or longer. Furthermore, since $a' \in V(B_1) - V(B'_1)$, $|V(B'_1)| < |V(B_1)|$. But this contradicts the choice of a and B_1. Thus B_1 cannot have any unhappy vertices, i.e., B_1 is happy.

Consider the shadow $\partial(B_1)$ of B_1. By Lemma 129, $\partial(B_1)$ is not $\lceil (k+1)/2\rceil$-path connected, otherwise B_1 would contain an (x_1, y_1)-path of length at least $2\lceil (k+1)/2\rceil - 1 \geq k$, a contradiction.

Let $\alpha = \lceil (k-1)/2\rceil$, $\beta = \lfloor (k-1)/2\rfloor$.

Claim 149. $\frac{1}{\alpha-2}\binom{\alpha}{\min\{r,\lfloor \alpha/2\rfloor\}} \leq \binom{\beta}{\min\{r-1,\lfloor \beta/2\rfloor\}}$.

Proof. First suppose $\alpha = \beta$, i.e., k is odd. Then the case $\min\{r, \lfloor \alpha/2\rfloor\} = \alpha/2$ is trivial. Otherwise $\frac{1}{\alpha-2}\binom{\alpha}{r} = \frac{1}{\alpha-2}\frac{\alpha-r+1}{r}\binom{\beta}{r-1} \leq \binom{\beta}{r-1}$. So assume $\alpha = \beta + 1$. If $\min\{r, \lfloor \alpha/2\rfloor\} = r$ (so $\min\{r-1, \lfloor \beta/2\rfloor \ = r-1$), then we have $\frac{1}{\alpha-2}\binom{\alpha}{r} = \frac{1}{\beta-1}\frac{\beta+1}{r}\binom{\beta}{r-1} \leq \binom{\beta}{r-1}$. Otherwise if $\lfloor \alpha/2\rfloor < r$, then $\lfloor \beta/2\rfloor \leq r-1$, and $\frac{1}{\alpha-2}\binom{\alpha}{\lfloor \alpha/2\rfloor} = \frac{1}{\beta-1}\binom{\beta+1}{\lfloor (\beta+1)/2\rfloor} = \frac{1}{\beta-1}\frac{\beta+1}{\lfloor (\beta+1)/2\rfloor}\binom{\beta}{\lfloor (\beta+1)/2\rfloor-1} \leq \binom{\beta}{\lfloor \beta/2\rfloor}$. ☐

Therefore because $\partial(B_1)$ is not $(\alpha+1)$-path connected, by Theorem 124 and the previous claim,

$$\begin{aligned} |A \cap B_1| \leq N_{\mathrm{Sp}}(\partial(B_1), r) &\leq \frac{|Y \cap B_1| - 2}{\alpha - 2}\binom{\alpha}{\min\{r, \lfloor \alpha/2\rfloor\}} \\ &\leq (|Y \cap B_1| - 2)\binom{\beta}{\min\{r-1, \lfloor \beta/2\rfloor\}}. \end{aligned}$$

☐

8.6 Constructing happy r^--graphs

In this section, we translate Lemma 132 into the language of r^--graphs. We also refine it.

8.6.1 Unhappy r^--graphs

A Sperner r^--graph $\mathcal{H}$ is *happy* if its layered incidence bigraph $I(\mathcal{H})$ is happy, and is *unhappy* otherwise. The happy and unhappy vertices in $I(\mathcal{H})$ correspond to happy and unhappy edges in $\mathcal{H}$.

For an unhappy edge e in an unhappy r^--graph $\mathcal{H}$ and a vertex $v \in e$, let $F(\mathcal{H}, e, v)$ denote the r^--graph obtained from $\mathcal{H}$ by replacing e with $e - v$.

A vertex v of degree 2 in an unhappy r^--graph $\mathcal{H}$ is *special* if each of the two incident edges, say e_1 and e_2, is either unhappy or a graph edge (i.e., contains exactly two vertices). If v is special and incident with e_1 and e_2, then $F(\mathcal{H}, v, e_1, e_2)$ is the r^--graph obtained from $\mathcal{H}$ by deleting v and for $i = 1, 2$ deleting e_i if $|e_i| = 2$ and replacing e_i with $e_i - v$ otherwise.

A graph edge vu in an unhappy r^--graph $\mathcal{H}$ is *special* if both v and u are special, and both adjacent to vu edges are unhappy. If vu is special and adjacent to e_1 and e_2, then $F(\mathcal{H}, vu)$

is the r^--graph obtained from $\mathcal{H}$ by deleting v and u, replacing e_1 with $e_1 - v$, and replacing e_2 with $e_2 - u$.

A 2-*block* in a 2-connected $\mathcal{H}$ is a 2-connected $\mathcal{H}' \subset \mathcal{H}$ such that only two vertices of $\mathcal{H}'$ have neighbors outside of $\mathcal{H}'$. These two vertices will be called *outer vertices of* $\mathcal{H}'$.

A 2-block $\mathcal{H}'$ with outer vertices x and y in an unhappy Sperner r^--graph $\mathcal{H}$ is *special* if $\mathcal{H}'$ is happy and there is exactly one edge, say a, in $G - E(\mathcal{H}')$ containing y, and this edge does not contain x.

Given a special 2-block $\mathcal{H}'$ with outer vertices x and y in an unhappy Sperner r^--graph $\mathcal{H}$, the r^--graph $F(\mathcal{H}, \mathcal{H}', x, y)$ is obtained from $\mathcal{H}$ by deleting all vertices of $\mathcal{H}' - x - y$ together with the edges containing them and adding edge $\{x, y\}$ if it is not in $\mathcal{H}$.

Translating from the language of incidence bipartite graphs to hypergraphs, we obtain the following versions of Lemmas 128 and 129 about Berge cycles and Berge paths.

Lemma 150. *Let $r \geq 3$. Let $\mathcal{H}$ be a happy r^--graph. If the 2-shadow $\partial_2\mathcal{H}$ contains a cycle of length $\ell \geq r+1$, then $\mathcal{H}$ contains a Berge cycle of length ℓ on the same base vertices. Furthermore, if $\partial_2\mathcal{H}$ contains a path, then $\mathcal{H}$ contains a Berge path with the same base vertices.*

For simplicity, for an r^--graph $\mathcal{H}$, denote $\sum|E(\mathcal{H})| := \sum_{e\in E(\mathcal{H})}|e|$. For example, if $\mathcal{H}$ is r-uniform, then $\sum|E(\mathcal{H})| = r|E(\mathcal{H})|$. We also obtain the following as a corollary of Lemma 146.

Lemma 151. *Suppose $k \geq r \geq 3$ are integers, and set $t = \lfloor (k-1)/2 \rfloor$. Let $\mathcal{H}$ be a Sperner 2-connected r^--graph with $c(\mathcal{H}) < k$ that is not happy. Then we can obtain a Sperner 2-connected r^--graph $\mathcal{H}'$ such that*

(i) $\sum|E(\mathcal{H}')| \leq \sum|E(\mathcal{H})|$, $|V(\mathcal{H}')| \leq |V(\mathcal{H})|$, *and* $\sum|E(\mathcal{H}')| + |V(\mathcal{H})| < \sum|E(\mathcal{H})| + |V(\mathcal{H}')|$;

(ii) $|E(\mathcal{H})| - |E(\mathcal{H}')| \leq \binom{t}{\min\{r-1, \lfloor t/2 \rfloor\}}(|V(\mathcal{H})| - |V(\mathcal{H}')|)$; *and*

(iii) $c(\mathcal{H}') < k$

using one of the following transformations:

(T1) *for an unhappy edge e and $v \in e$, replacing H with $F(H, e, v)$;*

(T2) *for a special vertex v with incident edges e_1 and e_2, replace H with $F(H, v, e_1, e_2)$;*

(T3) *for a special edge vu, replace H with $F(H, vu)$;*

(T4) *glue together all but one vertices of an unhappy edge;*

(T5) *for a special 2-block H' with outer vertices say x, y, replace H with $F(H, H', x, y)$.*

Furthermore, if (T5) *is not applied, then instead of* (ii)*, we obtain* $|E(\mathcal{H})| - |E(\mathcal{H}')| \leq (|V(\mathcal{H})| - |V(\mathcal{H}')|)$.

8.6.2 A refinement of Lemma 151

Suppose we start from a Sperner 2-connected unhappy r^--graph $\mathcal{H}$ with at least k vertices and $c(\mathcal{H}) < k$. Lemma 151 provides that we can obtain from $\mathcal{H}$ a happy Sperner 2-connected r^--graph in several steps using the following rule at each step:

$$\textit{if possible, apply (T1); if not then try (T2), then (T3) and so on.} \tag{8.7}$$

We may think that we have started from $\mathcal{H} = \mathcal{H}_0$ and after Step i obtain $\mathcal{H}_i$ from $\mathcal{H}_{i-1}$ using one of (T1)–(T5).
Claims 2.7–2.8 in the proof of Lemma 151 yield that following (8.7), at each Step i,

if (T1) *is not applied on Step $i+1$, then in each unhappy edge a of $\mathcal{H}_i$, thick pairs form a matching,* (8.8)

and

if neither (T1) *nor* (T2) *is applied on Step $i+1$, then all pairs of vertices in each unhappy edge a of $\mathcal{H}_i$ are thin.* (8.9)

Claim 152. *If* (T2) *was applied on Step i, then* (T1) *cannot be applied on Step $i+1$.*

Proof. Suppose $\mathcal{H}_i = F(\mathcal{H}_{i-1}, v, e_1, e_2)$ and $\mathcal{H}_{i+1} = F(\mathcal{H}_i, e_0, w)$.

Case 1: Edge e_0 is neither $e_1 - v$ nor $e_2 - v$. We want to show that in this case, e_0 is unhappy in $\mathcal{H}_{i-1}$ and $\mathcal{H}' = F(\mathcal{H}_{i-1}, e_0, w)$ is a Sperner 2-connected r^--graph satisfying (i)–(iii) with $\mathcal{H}_{i-1}$ in place of $\mathcal{H}$. That would contradict Rule (8.7).
To prove the first part (that e_0 is unhappy in $\mathcal{H}_{i-1}$), recall that e_0 is unhappy in $\mathcal{H}_i$. But the codegree in $\mathcal{H}_i$ of each pair in $V(\mathcal{H}_i)$ is the same as in $\mathcal{H}_{i-1}$.
To prove the second part, we use the fact that $\mathcal{H}'$ can be obtained from $\mathcal{H}_{i+1}$ by adding back vertex v and for $j = 1, 2$ constructing e_j either by adding v to $e_j - v \in \mathcal{H}_{i+1}$ when $|e_j| \geq 3$ or adding edge e_j when $|e_j| = 2$. Since the incidence graph $I(\mathcal{H}_{i+1})$ is 2-connected and this operation corresponds to adding a vertex of degree 2 or an ear to $I(\mathcal{H}_{i+1})$, $I(\mathcal{H}')$ also is 2-connected. Since $\mathcal{H}_{i+1}$ is Sperner, and $\mathcal{H}'$ differs from it only e_1, e_2 and v, H' is also Sperner: new edges are not contained in any old edge because of v, and no old edge can be contained in e_j, since otherwise it would be contained in $e_j - v$ in $\mathcal{H}_{i+1}$. Properties (i)–(iii) are trivial.

Case 2: $e_0 = e_1 - v$. In this case, we know that e_1 is unhappy in $\mathcal{H}_{i-1}$ and want to show that $\mathcal{H}' = F(\mathcal{H}_{i-1}, e_1, w)$ is a Sperner 2-connected r^--graph satisfying (i)–(iii) with $\mathcal{H}_{i-1}$ in place of $\mathcal{H}$. Now $\mathcal{H}'$ can be obtained from $\mathcal{H}_{i+1}$ by adding back vertex v, adding v to

$e_0 - w$ and constructing e_2 either by adding v to $e_2 - v \in \mathcal{H}_{i+1}$ when $|e_2| \geq 3$ or adding edge e_2 when $|e_2| = 2$. The rest is as in Case 1. $\square$

Practically the same proof yields the following similar claim.

Claim 153. *If* (T3) *was applied on Step i, then* (T1) *cannot be applied on Step $i+1$.* $\square$

The proof of the next claim is somewhat different.

Claim 154. *If* (T4) *was applied on Step i, then* (T1) *cannot be applied on Step $i+1$.*

Proof. Suppose $\mathcal{H}_{i-1}$ has an unhappy edge $a = \{y_1, \ldots, y_t\}$ such that $\mathcal{H}_i$ is obtained from H_{i-1} by gluing $\{y_1, \ldots, y_{t-1}\}$ into a new vertex y^*, and $\mathcal{H}_{i+1} = F(\mathcal{H}_i, e, w)$. By (8.9),

all pairs of vertices in each unhappy edge of $\mathcal{H}_{i-1}$ are thin. In particular, the size of each edge in $\mathcal{H}_i$ apart from the edge $y^* y_t$ is the same as in $\mathcal{H}_{i-1}$. (8.10)

Case 1: $w \neq y^*$. By (8.10), in $\mathcal{H}_{i-1}$, $|e \cap a| \leq 1$. So, since e is unhappy in $\mathcal{H}_i$, it is also unhappy in $\mathcal{H}_{i-1}$. We want to show that $\mathcal{H}' = F(\mathcal{H}_{i-1}, e, w)$ is a Sperner 2-connected r^--graph satisfying (i)–(iii). Since each pair in e is thin, $\mathcal{H}'$ is Sperner. Properties (i)–(iii) are evident, so we need to check that $\mathcal{H}'$ is 2-connected.

By construction, $\mathcal{H}'$ can be obtained from the 2-connected $\mathcal{H}_{i+1}$ by blowing up vertex y^* into vertices $y_1, \ldots, y_{t-1}$ (each of a positive degree) and replacing edge $y^* y_t$ with a. In terms of the incidence graphs, in the 2-connected $I(\mathcal{H}_{i+1})$, we split y^* into $t-1$ vertices of degree at least 1, delete vertex y^* and add vertex a adjacent to $y_1, \ldots, y_t$. It is easy to check that the new graph is 2-connected.

Case 2: $w = y^*$. By (8.10), there is a unique $v_1 \in a - y_t$ such that $e' = e - y^* + v_1 \in \mathcal{H}_{i-1}$. Since e is unhappy in $\mathcal{H}_i$, it has a pair xy of codegree at most $|e| - 2$. If $y^* \notin \{x, y\}$, then the codegree of xy in $\mathcal{H}_{i-1}$ also is at most $|e| - 2$. And if $y^* = y$, then the codegree of $y_1 x$ in $\mathcal{H}_{i-1}$ is at most $|e| - 2$. Thus e' is unhappy in $\mathcal{H}_{i-1}$. The rest is as in Case 1. $\square$

8.6.3 Stopping at $k-1$ vertices

Lemma 155. *Suppose $r \geq 3$ and $k \geq r$ are integers. Let $\mathcal{H}$ be a Sperner 2-connected r^--graph with $c(\mathcal{H}) < k$ and at least k vertices that is not happy. Suppose $\mathcal{H} = \mathcal{H}_0, \ldots, \mathcal{H}_i, \mathcal{H}_{i+1}$ is a sequence of r^--graphs obtained by iteratively applying Lemma 151 following Rule* (8.7) *to $\mathcal{H}$ until $\mathcal{H}_{i+1}$ is happy. If* (T5) *was never applied and $|V(\mathcal{H}_{i+1})| = k-1$, then $|E(\mathcal{H}_{i+1})| \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2$.*

Proof. Since (T1) does not change the number of vertices and $\mathcal{H}_0$ has at least k vertices, one of (T2), (T3), or (T4) was applied. Moreover, by Claims 152–154, one of (T2), (T3), or (T4) was applied to $\mathcal{H}_i$ to obtain the happy r^--graph $\mathcal{H}_{i+1}$. For short, denote $\mathcal{H}' = \mathcal{H}_{i+1}$.

If $\mathcal{H}'$ has a vertex of degree at most 3, then the number of edges in $\mathcal{H}'$ is at most $\binom{k-3}{\min\{r,\lfloor (k-3)/2\rfloor\}}+\binom{3}{\min\{r-1,1\}}$, and we are done. Hence

$$\delta(\mathcal{H}') \geq 3. \tag{8.11}$$

In the following, for any r^--graph $\mathcal{A}$ and any vertex $v \in V(\mathcal{A})$, we use $\mathcal{A}-v$ to denote the r^--graph obtained by removing vertex v and shrinking any edge e that contains v to the edge $e-v$, unless $|e|=2$, in which case we simply delete e in $\mathcal{A}-v$. Note that $\mathcal{A}-v$ need not be Sperner, even if $\mathcal{A}$ is Sperner.

Case 1: (T4) was the last applied operation. Let $a=\{y_1,\dots,y_t\}$ be the unhappy edge such that $\mathcal{H}'$ is obtained from H_i by gluing $\{y_1,\dots,y_{t-1}\}$ into a new vertex y^*. Since $\mathcal{H}'$ is happy, $\mathcal{H}_i - a$ is happy. The r^--graph $F(\mathcal{H}_i,a,y_t)$ satisfies (i)-(iii) and is Sperner by (8.10). So if $F(\mathcal{H}',a,y_t)$ is 2-connected, then we would have applied (T1) to $\mathcal{H}_i$ instead of (T4), a contradiction to Rule (8.7). Therefore

$$\textit{the incidence graph } I(\mathcal{H}_i - a) \textit{ has a vertex } x_t \textit{ separating } y_t \textit{ from } \{y_1,\dots,y_{t-1}\}. \tag{8.12}$$

If x_t corresponds to an edge b in $\mathcal{H}_i - a$, then some pair of its vertices is thin. So, since $\mathcal{H}_i - a$ is happy, $|b|=2$. Then instead of x_t, we can choose as a vertex x'_t separating y_t from $\{y_1,\dots,y_{t-1}\}$ the neighbor of x_t that is farther from y_t. Thus we may assume that x_t corresponds to a vertex in $\mathcal{H}_i - a$.

If $x_t \notin \{y_1,\dots,y_{t-1}\}$, then y_t and y^* are also separated by x_t in $\mathcal{H}'-y^*y_t$. Since there are at least 2 components in $\mathcal{H}'-y^*y_t-x_t$, the largest block of $\mathcal{H}'-y^*y_t$ has at most $|V(\mathcal{H}')-1| = k-2$ vertices.

We have that

$$|E(\mathcal{H}')| = |E(\mathcal{H}'-y^*y_t)|+1 \leq \binom{k-2}{\min\{r,\lfloor (k-2)/2\rfloor\}}+1+1 = \binom{k-2}{\min\{r,\lfloor (k-2)/2\rfloor\}}+2.$$

If $x_t \in \{y_1,\dots,y_{t-1}\}$, then let $\mathcal{C}$ be a component of $(\mathcal{H}_i-a)-x_t$ which does not contain y_t. Then $\mathcal{C}$ contains a vertex $y \notin \{y_1,\dots,y_{t-1}\}$, otherwise every edge of $\mathcal{C}+x_t$ in $\mathcal{H}_i$ would be a subset of the edge a, contradicting that $\mathcal{H}_i$ is Sperner. Thus in $\mathcal{H}'-y^*y_t$, y and y_t are in different blocks. Hence we again get $|E(\mathcal{H}')| \leq \binom{k-2}{\min\{r,\lfloor (k-2)/2\rfloor\}}+2$.

Case 2: $\mathcal{H}_{i+1} = F(\mathcal{H}_i,v,e_1,e_2)$ for some special vertex v. By (8.8), if $|e_1| \geq 4$, then some pair in $e_1 - v$ is thin, and hence $e_1 - v$ is unhappy in $\mathcal{H}_{i+1}$, a contradiction the happiness of $\mathcal{H}_{i+1}$. Thus $|e_1|,|e_2| \leq 3$. Since $\mathcal{H}_i$ was unhappy, we may assume that $|e_1|=3$, say $e_1=\{v,v',v''\}$. By (8.8), either vv' or vv'' is a thin pair in $\mathcal{H}_i$. Suppose vv'' is thin. Consider $\mathcal{H}''=F(\mathcal{H}_i,e_1,v')$. Since vv'' is thin, $\mathcal{H}''$ is Sperner. If $\mathcal{H}''$ is 2-connected, we get a contradiction to Rule (8.7). Thus the incidence graph $I(\mathcal{H}'')$ has a cut vertex x separating v' from $\{v, v''\}$. We claim that

$$\textit{we can choose } x \textit{ corresponding to a vertex in } \mathcal{H}'' \textit{ distinct from } v. \textit{ (We allow } x = v''.\textit{)} \tag{8.13}$$

Indeed, if v separates v' from v'' in $I(\mathcal{H}'')$, then vertex e_1 in the incidence graph $I(\mathcal{H}_i)$ separates v' from v'', a contradiction to the 2-connectedness of $\mathcal{H}_i$. If x corresponds to an edge in $I(\mathcal{H}_i)$, then again x contains thin pairs. If $|x| \geq 3$. Then x is unhappy. By the choice $\mathcal{H}_{i+1}$, the only unhappy edge in $\mathcal{H}''$ could be e_2. Recall that in this case, $|e_2| = 3$, say $x = e_2 = \{v, w, w'\}$. But in this case, one of v, w and w' also separates v' from v'', and we know that it is not v. Recall that vv'' is a thin pair, and so $v'' \notin \{w, w'\}$. Otherwise if $|x| = 2$, then both of its vertices are cut vertices. This proves (8.13).
Recall that $|V(\mathcal{H}'')| = |V(\mathcal{H}_i)| = k$ and $e(\mathcal{H}'') = e(\mathcal{H}_i) \leq e(\mathcal{H}_{i+1}) + 1$. Suppose first that each component of $\mathcal{H}'' - x$ has at least 3 vertices. Since $\mathcal{H}'' - x$ has $k - 1$ vertices and at least 2 connected components, $k \geq 7$, and the largest component of $\mathcal{H}'' - x$ has at most $k - 4$ vertices. Therefore we obtain

$$e(\mathcal{H}_{i+1}) \leq e(\mathcal{H}'') \leq \binom{k-3}{\min\{r, \lfloor (k-3)/2 \rfloor\}} + \binom{4}{2} \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2.$$

Now suppose that some component $\mathcal{C}$ of $\mathcal{H}'' - x$ contains at most 2 vertices. By (8.11), $|\mathcal{C}| = 2$ and each of the two vertices in $\mathcal{C}$ either has degree in $\mathcal{H}''$ less than in $\mathcal{H}_{i+1}$ or is v. But the only vertex having degree in $\mathcal{H}''$ less than in $\mathcal{H}_{i+1}$ is v', and the vertices v and v' are in distinct components of $\mathcal{H}'' - x$.

Case 3: $\mathcal{H}_{i+1} = F(\mathcal{H}_i, vu)$ for some special edge vu. Let e_1 be the unhappy edge incident to v and e_2 be the unhappy edge incident to u. By (8.9), all pairs in e_1 and e_2 are thin. So since $\mathcal{H}_{i+1}$ is happy, $|e_1| = |e_2| = 3$. Let $e_1 = \{v, v', v''\}$ and $e_2 = \{u, u', u''\}$, where possibly $v' = u'$. As in Case 2, consider $\mathcal{H}'' = F(\mathcal{H}_i, e_1, v')$. Since vv'' is thin, $\mathcal{H}''$ is Sperner. If $\mathcal{H}''$ is 2-connected, we get a contradiction to Rule (8.7). Thus the incidence graph $I(\mathcal{H}'')$ has a cut vertex x separating v' from $\{v, v''\}$.
Similarly to the proof of (8.13), we derive

we can choose x corresponding to a vertex in $\mathcal{H}''$ distinct from v and u. (We allow $x = v''$.) (8.14)

Furthermore, $x \notin \{u', u''\}$. Now $|V(\mathcal{H}'')| = |V(\mathcal{H}_i)| = k+1$ and $e(\mathcal{H}'') = e(\mathcal{H}_i) = e(\mathcal{H}_{i+1}) + 1$.
Note that there cannot be any isolated vertices in $\mathcal{H}'' - x$ since by (8.11), $\delta(\mathcal{H}'') \geq 3$. Also, as in the previous case, there cannot be a component of $\mathcal{H}'' - x$ with exactly 2 vertices. So

we may assume that each component of $\mathcal{H}'' - x$ has at least 3 vertices.
Let $\mathcal{C}$ be the component of $\mathcal{H}'' - x$ that contains v. Then $\mathcal{C}$ must also contain u and at least two of the vertices in $\{v'', u', u''\}$. Therefore $|\mathcal{C}| \geq 4$. In particular, since $\mathcal{H}'' - x$ contains exactly k vertices and at least 2 connected components, $k \geq |\mathcal{C}| + 3 \geq 7$.
As in Case 2, if the largest component of $\mathcal{H}'' - x$ has at most $k - 4$ vertices (so $k \geq 8$ since $|\mathcal{C}| \geq 4$), then

$$e(\mathcal{H}_{i+1}) \leq e(\mathcal{H}'') \leq \binom{k-3}{\min\{r, \lfloor (k-3)/2 \rfloor\}} + \binom{5}{2} \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2,$$

a contradiction.
Now suppose a component $\mathcal{C}'$ of $\mathcal{H}'' - x$ has $k - 3$ or $k - 2$ vertices. If $\mathcal{C}'$ contains v, (i.e., $\mathcal{C}' = \mathcal{C}$), then since $\mathcal{C}$ contains u as well, and u and v are incident to exactly 3 edges (vu, e_1, and e_2),

$$e(\mathcal{H}''[\mathcal{C} + x]) \leq \binom{|\mathcal{C}'| - 2 + 1}{\min\{r, \lfloor (|\mathcal{C}'| - 2 + 1)/2 \rfloor\}} + 3.$$

For $|\mathcal{C}'| = k - 3$ we get

$$e(\mathcal{H}'') \leq \binom{k-4}{\min\{r, \lfloor (k-3)/2 \rfloor\}} + 3 + \binom{4}{2} \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2,$$

and for $|\mathcal{C}'| = k - 2$ we get

$$e(\mathcal{H}'') \leq \binom{k-3}{\min\{r, \lfloor (k-3)/2 \rfloor\}} + 3 + \binom{3}{2} \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2.$$

So $\mathcal{C}' \neq \mathcal{C}$. But since $|\mathcal{C}| \geq 4$, we have $|V(\mathcal{H}'')| \geq |\mathcal{C}'| + |\mathcal{C}| + 1 \geq 4 + (k-3) + 1 = k + 2$, a contradiction. $\square$

8.7 Proof of Theorem 114

Proof. Apply Lemma 151 repeatedly to $\mathcal{H}$ following Rule (8.7) to obtain an r^--hypergraph $\mathcal{H}'$ that is happy. By Lemma 150, $\partial_2 \mathcal{H}'$ has no cycle of length k or longer.
Let n_S and m_S be the number of vertices and r^--edges respectively that were deleted going from $\mathcal{H}$ to $\mathcal{H}'$ by applying operations (T1)-(T4), and let n_B and m_B be the number of vertices and r^--edges respectively that were deleted from applying operation (T5). So $n = |V(\mathcal{H}')| + n_S + n_B$ and $|E(\mathcal{H})| \leq N_{\mathrm{Sp}}(\partial_2 \mathcal{H}', r) + m_S + m_B$. If $|V(\mathcal{H}')| \geq k$, then by Theorem 120 (applied to $\partial_2 \mathcal{H}'$) and Lemma 151, we have

$$|E(\mathcal{H}')| \leq N_{\mathrm{Sp}}(\partial_2 \mathcal{H}', r) + m_S + m_S$$

$$\leq \max\{f(|V(\mathcal{H}')|, k, r, 2), f(|V(\mathcal{H}')|, k, r, t)\} + n_S + \binom{t}{\min\{r-1, \lfloor t/2 \rfloor\}} n_B \tag{8.15}$$

First suppose that $n_B = 0$, i.e., (T5) was never applied. Examining the coefficient of n_S we see $1 \leq \min\{2, \binom{t}{\min\{r-1,\lfloor t/2\rfloor\}}\}$. So in the case $|V(\mathcal{H}')| \geq k$, from (8.15), we get $|E(\mathcal{H}')| \leq \max\{f(n, r, k, 2), f(n, r, k, t)\}$, as desired. Otherwise, if $|V(\mathcal{H}')| \leq k-1$, then either

$$|E(\mathcal{H}')| \leq \binom{k-2}{\min\{r, \lfloor (k-2)/2 \rfloor\}} + 2 = f(k-1, k, r, 2)$$

by Lemma 155, or $|V(\mathcal{H}')| \leq k-2$ and

$$|E(\mathcal{H}')| \leq \binom{|V(\mathcal{H}')|}{\min\{r, \lfloor |V(\mathcal{H}')|/2 \rfloor\}} \leq f(|V(\mathcal{H}')|, k, r, 2).$$

Either way we obtain $|E(\mathcal{H})| \leq f(n, k, r, 2)$.

So we may assume that at least one application of (T5) was required to obtain $\mathcal{H}'$.

Denote $H' := \partial_2 \mathcal{H}'$ and let Q be the t-core of H' (that is, the resulting graph from applying t-disintegration to H'). If H' is t-disintegrable, i.e., Q is empty, then $N_{\mathrm{Sp}}(H', r) \leq f(|V(H')|, k, r, t)$ and so by (8.15), we get $|E(\mathcal{H})| \leq f(n, k, r, t)$. So we may assume that Q is non-empty. In particular, since $\delta(Q) \geq t+1$, $|V(Q)| \geq t+2$.

Claim 156. *The graph Q is 1-hamiltonian.*

Proof. First note that $|V(Q)| \leq k-1$: the case for $|V(H')| \leq k-1$ is trivial, and if $|V(H')| \geq k$, then by applying Kopylov's Theorem, we obtain $|V(Q)| \leq k-2$.

Next, we claim that Q is 3-connected. If not, then there exists a cut set $\{x, y\} \subset V(Q)$ and at least two components in $H' - \{x, y\}$. Since $\delta(Q) \geq t+1$, for each of these components C, $|C \cup \{x, y\}| \geq t+2$. Hence $|V(Q)| \geq 2(t+2) - 2 \geq k$, a contradiction to $|V(Q)| \leq k-1$. Therefore Q is 3-connected. By Enomoto's Theorem (Theorem 122), Q is s-path connected where $s = \min\{|V(Q)|, 2(t+1)\} = |V(Q)|$. I.e., Q is 1-hamiltonian. □

Let $q := |V(Q)|$. Let $\mathcal{B}$ be a special (in particular, happy) block that was removed in some application of (T5), and set $B = \partial_2 \mathcal{B}$. Let x_B and a_B be the vertex-edge cut pair corresponding to $\mathcal{B}$, where some vertex $y_B \in V(\mathcal{B}) \setminus V(\mathcal{H}')$ is contained in a_B.

Claim 157. *Suppose H' is s-path connected. There does not exist a (x_B, y_B)-path of length at least $k - s + 1$ in B.*

Proof. Since $\mathcal{H}$ is 2-connected, its incidence bigraph contains two shortest disjoint paths P_1, P_2 from $\{x_B, a_B\}$ to $V(\mathcal{H}')$ (where possibly $|V(P_1)$ or $V(P_2) = 1$). Note that these paths are internally disjoint from $V(\mathcal{H}') \cup V(\mathcal{B})$. In $\mathcal{H}$, P_1 and P_2 yield Berge paths $\mathcal{P}_1$ and $a \cup \mathcal{P}_2$ from x_B to $V(\mathcal{H}')$ and y_B to $V(\mathcal{H}')$ respectively. Say P_i has endpoint $v_i \in V(\mathcal{H}')$.

Now suppose there exists a path of length at least $k-s+1$ from x_B to y_B. This yields a Berge path $\mathcal{P}_3$ from x_B to y_B with at least $k-s+1$ base vertices such that all edges of $\mathcal{P}_3$ are contained in $V(\mathcal{B})$. Similarly, we find a Berge path $\mathcal{P}_4$ from v_1 to v_2 with at least s base vertices such that all edges of $\mathcal{P}_4$ are contained in $V(\mathcal{H}')$.
Then $\mathcal{P}_1 \cup \mathcal{P}_3 \cup a \cup \mathcal{P}_2 \cup \mathcal{P}_4$ is a Berge cycle of length at least $(k-s+1)+s-1=k$, a contradiction.

□

Claim 158. *If H' contains a subgraph S that is s-path connected, then H' is also s-path connected.*

Proof. Let $\{x,y\} \subset V(H')$. We will show that there exists an (x,y)-path in H' with at least s vertices. Let P_x, P_y be two disjoint shortest paths from $\{x,y\}$ to $V(S)$, say with endpoints v_x and v_y respectively (where possibly one or both paths are singletons). Such paths exist because H' is 2-connected. Let P_S be a (v_x,v_y)-path in S of length at least S. Then $P_x \cup P_S \cup P_y$ has length at least s. □

Therefore the previous claim shows that H' is q-path connected. Applying Claim 157 and Theorem 124, we get

$$e(\mathcal{B}) \le N_{\mathrm{Sp}}(B,r) \le \frac{|V(B)|-2}{k-q-2}\binom{k-q}{\min\{r,\lfloor (k-q)/2\rfloor\}}. \tag{8.16}$$

Summing up over all blocks deleted via big cuts, we obtain

$$m_B \le n_B\left(\frac{1}{k-q-2}\binom{k-q}{\min\{r,\lfloor (k-q)/2\rfloor\}}\right) \tag{8.17}$$

Claim 159. *For each integer $s \ge 3$, $\frac{1}{s-2}\binom{s}{\min\{r,\lfloor s/2\rfloor\}} \le \binom{s}{\min\{r-1,\lfloor s/2\rfloor\}}$.*

Proof. The case for $\min\{r,\lfloor s/2\rfloor\} = \lfloor s/2\rfloor$ is trivial. So we may assume $s \ge 2r+2$. We have $\frac{1}{s-2}\binom{s}{r} = \frac{1}{s-2}\frac{s-r+1}{r}\binom{s}{r-1} \le \binom{s}{r-1}$. □

So first suppose that $|V(\mathcal{H}')| \ge k$. By Kopylov's theorem, $t+2 \le q \le k-2$, and $V(H') - V(Q)$ can be removed via $(k-s)$-disintegration. Therefore

$$e(\mathcal{H}') \le \binom{q}{\min\{r,\lfloor q/2\rfloor\}} + (|V(\mathcal{H}')|-q)\binom{k-q}{\min\{r-1,\lfloor (k-q)/2\rfloor\}},$$

and hence by (8.17) and the previous claim,

$$e(\mathcal{H}) = e(\mathcal{H}') + m_B + m_S \le$$

$$\leq \binom{q}{\min\{r, \lfloor q/2 \rfloor\}} + (|V(\mathcal{H}')| - q)\binom{k-q}{\min\{r-1, \lfloor (k-q)/2 \rfloor\}}$$
$$+n_B \left(\frac{1}{k-q-2} \binom{k-q}{\min\{r, \lfloor (k-q)/2 \rfloor\}} \right) + n_S$$
$$\leq \binom{q}{\min\{r, \lfloor q/2 \rfloor\}} + (n-q)\binom{k-q}{\min\{r-1, \lfloor (k-q)/2 \rfloor\}} \leq \max\{f(n,k,r,t), f(n,k,r,2)\},$$

where the last inequality follows from the convexity of the function f. So from now on we may assume $|V(H')| \leq k-1$.

Claim 160. *Let S be a 1-hamiltonian subgraph of H' with $s := |V(S)|$ and $t+2 \leq s \leq k-2$. Let S' be the result of $(k-s)$-disintegration applied to H'. Then S' is also 1-hamiltonian.*

Proof. We will show a stronger statement: S' is $(k - |V(S')|)$-hamiltonian. Suppose not. Set $s' := |V(S')|$. Applying Theorem 123 with $d = k-s$ (so $d \leq 2t+2-(t+2) = t$) and $\ell = k - s'$, we get

$$N_{\mathrm{Sp}}(S', r) \leq \max\{h_{\mathrm{Sp}}(s', k-s', r, k-s), h_{\mathrm{Sp}}(s', k-s', r, \lfloor s'/2 \rfloor,)\}.$$

If $h_{\mathrm{Sp}}(q', k-s', r, k-s) \geq h_{\mathrm{Sp}}(s', k-s', r, \lfloor s'/2 \rfloor)$, then

$$\begin{aligned} N_{\mathrm{Sp}}(S', r) &\leq h_{\mathrm{Sp}}(s', k-s', r, k-s) \\ &= \binom{s}{\min\{r, \lfloor s/2 \rfloor\}} + (s'-s)\binom{k-s}{\min\{r-1, \lfloor (k-s)/2 \rfloor\}} \\ &= f(s', k, r, k-s), \end{aligned}$$

Recall that since S is 1-hamiltonian, H' is s-path connected. Hence for each $\mathcal{B}$ deleted in an application of (T5), $\partial_2 \mathcal{B}$ is not $(k-s+1)$-path connected.
It follows that

$$e(\mathcal{H}) \leq N_{\mathrm{Sp}}(H', r) + m_B + m_S$$
$$\leq f(s', k, r, k-s) + (|V(H')| - s' + n_B)\binom{k-s}{\min\{r-1, \lfloor (k-s)/2 \rfloor\}} + n_S \leq f(n, k, r, k-s).$$

So by the convexity of the function f, we are done.
Next suppose $h_{\mathrm{Sp}}(s', k-s', r, k-s) \leq h_{\mathrm{Sp}}(s', k-s', r, \lfloor s'/2 \rfloor)$. For simplicity, let $a := \lfloor s'/2 \rfloor$. We have that $2 \leq a \leq \lfloor (k-1)/2 \rfloor = t$.

$$
\begin{aligned}
N_{\mathrm{Sp}}(S',r) &\leq h_{\mathrm{Sp}}(s',k-s',r,a)\\
&= \binom{s'-(a-k+s')}{\min\{r,\lfloor (s'-(a-k+s'))/2\rfloor\}} + (a-k+s')\binom{a}{\min\{r-1,\lfloor a/2\rfloor\}}\\
&= \binom{k-a}{\min\{r,\lfloor (k-a)/2\rfloor\}} + (s'-(k-a))\binom{a}{\min\{r-1,\lfloor a/2\rfloor\}}\\
&\leq f(s',k,r,a) \leq f(s',k,r,t).
\end{aligned}
$$

Therefore

$$
e(\mathcal{H}) \leq f(s',k,r,t) + (|V(H')| - s' + n_B)\binom{k-s}{\min\{r-1,\lfloor (k-s)/2\rfloor\}} + n_S \leq f(n,k,r,t).
$$

□

Starting from the 1-hamiltonian subgraph Q of H', we obtain a sequence of graphs $Q = Q_0 \subset Q_1 \subset \ldots \subset Q_q$ such that Q_i is the resulting 1-hamiltonian subgraph obtained from $(k-|V(Q_{i-1})|)$-disintegration applied to H'. The sequence ends when either the graph Q_{q+1} resulting from the $(k-|V(Q_q)|)$-disintegration of H' is exactly Q_q, or $|V(Q_q)| = k-1$. In the former case, we have that $|V(Q_{q+1})| = |V(Q_q)| =: q'$. Then

$$
e(\mathcal{H}) \leq N_{\mathrm{Sp}}(H',r) + m_B + m_S
$$
$$
\leq f(q',k,r,k-q') + (|V(H')| - q' + n_B)\binom{k-q'}{\min\{r-1,\lfloor (k-q')/2\rfloor\}} + n_S \leq f(n,k,r,k-q').
$$

Finally suppose that $|V(Q_q)| = k-1$. Then H' is $(k-1)$-path connected. Because $\mathcal{H}'$ is 2-connected, we can complete a Berge path in $\mathcal{H}'$ with at least $k-1$ vertices to a Berge cycle of length at least k. This proves the theorem.

□

8.8 Proof of Theorem 117 for paths

Proof. Let $\mathcal{H}$ be a counterexample of Theorem 117 with minimum $\sum_{e\in E(\mathcal{H})}|e|$ on at least $k+1$ vertices. If $\mathcal{H}$ contains a Berge cycle of length $k+1$ or longer, then removing any edge from this Berge cycle yields a Berge path with at least $k+1$ base vertices, a contradiction. If $\mathcal{H}$ contains a Berge cycle of length exactly k, then we use the following Lemma which contradicts that $n := |V(\mathcal{H})| \geq k+1$.

Lemma 161 (Győri, Katona, and Lemons [GKL16]). *Let $\mathcal{H}$ be a connected hypergraph with* *no Berge path of length k. If there is a Berge cycle of length k on the vertices $v_1, \ldots, v_k$ then these vertices constitute a component of $\mathcal{H}$.*

Therefore $\mathcal{H}$ contains no Berge cycle of length k or longer. If $\mathcal{H}$ is 2-connected, then by Theorem 114, $e(\mathcal{H}) \leq \max\{f(n,k,r,2), f(n,k,r,\lfloor (k-1)/2 \rfloor)\}$, and we are done.
Now suppose $\mathcal{H}$ is not 2-connected. Then the incidence bigraph $I_{\mathcal{H}}$ of $\mathcal{H}$ contains a set of cut vertices. If a cut vertex x of $I_{\mathcal{H}}$ corresponds to an edge in $\mathcal{H}$, then we say x is a cut edge of $\mathcal{H}$. Otherwise, we say x is a cut vertex of $\mathcal{H}$.
Suppose $\mathcal{H}$ has an cut-edge e. We claim that for each component $\mathcal{C}$ of $\mathcal{H} \setminus e$,

$$|V(\mathcal{C}) \cap e| \leq 1. \tag{8.18}$$

Indeed suppose that some component $\mathcal{C}$ of $\mathcal{H} \setminus e$ contains at least 2 vertices in e. Let $\mathcal{H}'$ be the r^--graph obtained by shrinking e to remove all but one vertex in $\mathcal{C}$ from e. Then $\mathcal{H}'$ is still connected and Sperner (since e is a cut edge of $\mathcal{H}$). Furthermore, after this operation, the length of a longest path cannot increase. This contradicts the choice of $\mathcal{H}$.
Now suppose $\mathcal{H}$ contains a cut edge e. By (8.18), e intersects every component of $\mathcal{H} \setminus e$ in at most one vertex. Let $\mathcal{H}'$ be the r^--graph obtained by contracting two vertices of e into a single vertex (and then deleting e if it now contains only one vertex). The new r^--graph $\mathcal{H}'$ is Sperner, contains no Berge P_k, and is connected. If $|V(\mathcal{H}')| \geq k+2$, we obtain that $\mathcal{H}'$ contradicts the choice of $\mathcal{H}$ (note that $e(\mathcal{H}') \geq e(\mathcal{H}) - 1 \geq \max\{f(n,k,r,1), f(n,k,r,\lfloor (k-1)/2 \rfloor)\} - 1 \geq \max\{f(n-1,k,r,1), f(n-1,k,r,\lfloor (k-1)/2 \rfloor)\}$).
Iterating this process, we may assume that $\mathcal{H}$ contains no cut edges unless $n = k+1$.
Case 1: $\mathcal{H}$ does not have a cut edge.
Any block $\mathcal{B}$ of $\mathcal{H}$ is a subhypergraph of $\mathcal{H}$. In particular, $\mathcal{B}$ is a Sperner 2-connected r^--graph. Let $\mathcal{B}_1, \ldots, \mathcal{B}_p$ be the blocks of $\mathcal{H}$. For each i, let s_i be the length of a longest Berge cycle in $\mathcal{B}_i$. Without loss of generality, we may assume $s_1 \geq \ldots \geq s_p$.

Claim 162. *For all $i \geq 2$, $s_1 + s_i \leq k+1$.*

In particular, $s_i \leq (k+1)/2$ for all $i \geq 2$.

Proof. Suppose $s_1 + s_i \geq k+2$. Let C_1 be a Berge cycle of $\mathcal{B}_1$ of length s_1 and let C_i be a Berge cycle of $\mathcal{B}_i$ of length s_i. Let P be a shortest Berge path from $V(\mathcal{B}_1)$ to $V(\mathcal{B}_i)$. Note that P contains at most one edge from each Berge cycle. Then removing an edge from each Berge cycle, we obtain together with P a Berge path whose base vertices cover $V(C_1) \cup V(C_i)$. Since $|V(C_1) \cap V(C_i)| \leq 1$, this path has at least $s_1 + s_i - 1 \geq k+1$ base vertices. □

For each block $\mathcal{B}_i$, let $n_i := |V(\mathcal{B}_i)|$. If $n_i = s_i$, then

$$e(\mathcal{B}_i) \leq \binom{s_i}{\min\{r, \lfloor s_i/2 \rfloor\}} \leq (n_i - 1)\binom{s_i - 1}{\min\{r-1, \lfloor (s_i-1)/2 \rfloor\}}.$$

If $n_i \geq s_i + 1$, then we apply Theorem 114 to $\mathcal{B}_i$ with cycle length $s_i + 1$. We obtain

$$e(\mathcal{B}_i) \leq \max\{f(n_i, s_i+1, r, 2), f(n_i, s_i+1, \lfloor s_i/2 \rfloor\}.$$

Furthermore,

$$\begin{aligned}
f(n_i, s_i+1, r, 2) &= \binom{s_i - 1}{\min\{r, \lfloor (s_i-1)/2 \rfloor\}} + 2(n_i - s_i + 1) \\
&\leq (s_i - 1)\binom{s_i - 2}{\min\{r-1, \lfloor (s_i-2)/2 \rfloor\}} + (n_i - s_i)\binom{s_i - 2}{\min\{r-1, \lfloor (s_i-2)/2 \rfloor\}} \\
&= (n_i - 1)\binom{s_i - 2}{\min\{r-1, \lfloor (s_i-2)/2 \rfloor\}}.
\end{aligned}$$

And $f(n_i, s_i+1, r, \lfloor s_i/2 \rfloor) \leq (n_i - 1)\binom{s_i - 1}{\min\{r-1, \lfloor (s_i-1)/2 \rfloor\}}$.
In all cases we get

$$e(\mathcal{B}_i) \leq (n_i - 1)\binom{s_i - 1}{\min\{r-1, \lfloor (s_i-1)/2 \rfloor\}}. \tag{8.19}$$

For $\mathcal{B}_1$, if $n_1 = s_1$ then $e(\mathcal{B}_1) \leq \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}}$ and so by (8.19),

$$e(\mathcal{H}) \leq \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + \sum_{i=2}^{p}(n_i - 1)\binom{s_i - 1}{\min\{r-1, \lfloor (s_i-2)/2 \rfloor\}}. \tag{8.20}$$

If $s_1 \geq \lceil (k+1)/2 \rceil$, then from (8.20) we obtain

$$e(\mathcal{H}) \leq \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + \sum_{i=2}^{p}(n_i - 1)\binom{k - s_1}{\min\{r-1, \lfloor (k-s_1)/2 \rfloor\}} \leq f(n, k, r, k - s_1)$$

$$\leq \max\{f(n, k, r, 1), f(n, k, r, \lfloor (k-1)/2 \rfloor\}).$$

Otherwise,

$$e(\mathcal{H}) \leq \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + \sum_{i=2}^{p}(n_i - 1)\binom{s_1 - 1}{\min\{r-1, \lfloor (s_1-1)/2 \rfloor\}} \leq f(n, k, r, \lfloor (k-1)/2 \rfloor).$$

If $n_1 \geq s_1 + 1$, then we get

$$e(\mathcal{B}_1) \le \max\{f(n_1, s_1+1, r, 2), f(n_1, s_1+1, r, \lfloor s_1/2 \rfloor\}).$$

If $f(n_1, s_1+1, r, \lfloor s_1/2 \rfloor) \ge f(n_1, s_1+1, r, 2)$, then together with (8.19), we get

$$e(\mathcal{H}) \le f(n_1, s_1+1, r, \lfloor s_1/2 \rfloor) + \sum_{i=2}^{p} (n_i - 1) \binom{\lfloor \frac{k-1}{2} \rfloor}{\min\{r-1, \lfloor \frac{k-1}{4} \rfloor\}} \le f(n, k, r, \lfloor (k-1)/2 \rfloor).$$

If $f(n_1, s_1+1, r, \lfloor s_1/2 \rfloor) < f(n_1, s_1+1, r, 2)$, then

$$f(n_1, s_1+1, r, 2) = \binom{s_1 - 1}{\min\{r, \lfloor (s_1-1)/2 \rfloor\}} + 2(n_1 - s_1 + 1) \le \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + 2(n_1 - s_1).$$

Thus we obtain

$$e(\mathcal{H}) \le \binom{s_1}{\min\{r, \lfloor s_1/2 \rfloor\}} + 2(n_1 - s_1) + \sum_{i=2}^{p} (n_i - 1) \binom{s_i - 1}{\min\{r-1, \lfloor (s_i-1)/2 \rfloor\}},$$

and we are done as in the the case for (8.20).

Case 2: $n = k+1$ and $\mathcal{H}$ contains a cut edge.

Let e be a cut edge of $\mathcal{H}$. By (8.18), each component $\mathcal{C}$ of $\mathcal{H} \setminus e$ contains only at most one vertex of e. If $|e| \ge 3$, then $e(\mathcal{H} \setminus e) \le \binom{k+1-2}{\min\{r, \lfloor (k+1-2)/2 \rfloor\}}$. Hence $e(\mathcal{H}) \le \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 1 < f(n, k, r, 1)$.

So we may assume $|e| = 2$. Suppose first that $\mathcal{H} \setminus e$ contains a component $\mathcal{C}$ with $2 \le |V(\mathcal{C})| \le k-1$.

Then

$$\begin{aligned} e(\mathcal{H}) &\le 1 + \binom{|V(\mathcal{C})|}{\min\{r, \lfloor |V(\mathcal{C})|/2 \rfloor\}} + \binom{(k+1) - |V(\mathcal{C})|}{\min\{r, \lfloor ((k+1) - |V(\mathcal{C})|)/2 \rfloor\}} \\ &\le 1 + \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 1 \\ &= f(n, r, k, 1). \end{aligned}$$

Thus $\mathcal{H} \setminus e$ must consist of one component of size k and one of size 1. The same also holds for every other cut edge e' of $\mathcal{H}$. This together with (8.18) implies that if $\mathcal{H}$ has two cut edges e, e', then e' is a cut edge of $\mathcal{H} \setminus e$, and vice versa. Therefore $e(\mathcal{H}) \le \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 2 = f(n, k, r, 1)$.

So we may assume that e is the only cut edge of $\mathcal{H}$. Let $\mathcal{C}$ be the component of $\mathcal{H}$ of size k. This component cannot contain a Berge cycle of length k, otherwise with e we would obtain Berge path with of length k.

If $\mathcal{C}$ is 2-connected, then by Theorem 114,

$$e(\mathcal{H}) = e(\mathcal{C}) + 1 \leq \max\{f(k,k,r,2), f(k,k,r,\lfloor (k-1)/2 \rfloor)\} < f(n,k,r,1).$$

Otherwise $\mathcal{C}$ has a cut vertex v and a block $\mathcal{B}$ with $2 \leq |V(\mathcal{B})| \leq k-1$. Therefore

$$e(\mathcal{C}) \leq \binom{|V(\mathcal{B})|}{\min\{r, \lfloor |V(\mathcal{B})|/2 \rfloor\}} + \binom{k-|V(\mathcal{B})|+1}{\min\{r, \lfloor (k-|V(\mathcal{B})|+1)/2 \rfloor\}} \leq \binom{k-1}{\min\{r, \lfloor (k-1)/2 \rfloor\}} + 1,$$

so we get $e(\mathcal{H}) = e(\mathcal{C}) + 1 \leq f(n,k,r,1)$. This proves the theorem. □

8.9 Concluding remarks

1. As it is mentioned in Theorem 116, if $k \geq 4r$ and n is asymptotically larger than $\frac{2^{r-1}}{r}k$, then our bound is also exact for r-graphs: a sharpness example is $\mathcal{H}_{n,k,r,\lfloor (k-1)/2 \rfloor}$. We think that for smaller n, our bound for r-graphs is not exact. It would be interesting and challenging to find exact bounds for the number of edges in n-vertex 2-connected r-graphs with no cycles of length k or longer for $k > r$ and $k \leq n < \frac{2^{r-1}}{r}k$.

2. When r is large, $k \geq 4r$ and n is polynomial in k, then $\mathcal{H}_{n,k,r,2}$ has not much more than $\binom{k-2}{r}$ edges. Also $\mathcal{H}_{n,k,r,2}$ is not uniform whenever $r \geq 4$. The following construction of 2-connected r-uniform hypergraphs also has more than $\binom{k-2}{r}$ edges in this case, although fewer edges than $\mathcal{H}_{n,k,r,2}$ has (and it works only for n such that $n-k+2$ is divisible by $r-1$).

   Construction 163. *Fix $k \geq 4r \geq 12$, $s \geq 1$, $n = k-2+s(r-1)$. Define the n-vertex r-graph $F_{n,k,r,s}$ as follows. The vertex set of $F_{n,k,r,s}$ is partitioned into $s+1$ sets $A_1, \ldots, A_s, C$ such that $|C| = k-2$ and $|A_i| = r-1$ for all $i \in [s]$. We fix two special vertices $c_1, c_2 \in C$. The edge set of $F_{n,k,r,s}$ consists of all edges contained in C and of the $2(r-1)$ edges of the form $A_i \cup \{c_j\}$ for $i \in [s]$ and $j \in [2]$.*

   We do not currently know of any *uniform* hypergraphs with more edges and no Berge cycles of length k or longer.

3. Note that here we use r^--graphs to prove a bound for r-graphs when $k > r$ and in Chapter 7 we used r^+-graphs (i.e. hypergraphs with the lower rank at least r) in the case $k < r$.

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