ORDERABILITY OF HOMOLOGY SPHERES OBTAINED BY DEHN FILLING

BY

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DISSERTATION

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Abstract

In my thesis, I study left-orderability of $\mathbb{Q}$-homology spheres. I use $\widetilde{PSL_2}\mathbb{R}$ representations as a tool. First, I showed this tool has its limitations by constricting a series of $\mathbb{Z}$-homology spheres with potentially left-orderable fundamental groups but no non trivial $\widetilde{PSL_2}\mathbb{R}$ representations.

However, this tool is still useful in most cases. With $\widetilde{PSL_2}\mathbb{R}$ representations, I construct the holonomy extension locus of a $\mathbb{Q}$-homology solid torus which is an analog of its translation extension locus. Using extension loci, I study $\mathbb{Q}$-homology 3-spheres coming from Dehn fillings of $\mathbb{Q}$-homology solid tori and construct intervals of orderable Dehn fillings.

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Chapter 1

INTRODUCTION

1.1 The L-space conjecture and left-orderability

A nontrivial group is called left-orderable if there exists a strict total order on the set of group elements which is invariant under left multiplication. We will say that a closed 3-manifold is orderable when its fundamental group is left-orderable.

The reason why we care about left-orderability is that this property is conjectured to detect L-spaces. Recall an irreducible $\mathbb{Q}$-homology 3-sphere (abbr. $\mathbb{Q}$HS) Y is called an L-space if $\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$, i.e. it has minimal Heegaard Floer homology [34]. Boyer, Gordon, and Watson conjectured in [6] that a $\mathbb{Q}$HS is a non L-space if and only if its fundamental group is left-orderable. A stronger conjecture states that for an irreducible $\mathbb{Q}$-homology 3-sphere, being a non L-space, having left-orderable fundamental group and admitting a co-orientable taut foliation are the same (see e.g. Culler-Dunfield [11]). This conjecture is known as the L-space conjecture. It has been studied extensively in recent years and evidence has accumulated in favor of the conjecture [5, 21].

One of the main difficulties of proving the conjecture is to show left-orderability of a fundamental group. Various tools have been developed to study left-orderability. In particular, $\widetilde{PSL_2}\mathbb{R}$ representations have been proven very useful in studying left-orderability of 3-manifold groups [16, 11, 26, 39, 40]. Boyer, Rolfsen, and Wiest showed that a compact, connected, P^2-irreducible 3-manifold is left-orderable if and only if its fundamental group admits a non-trivial homomorphism to a left-orderable group [4]. The group $\widetilde{PSL_2}\mathbb{R}$ is left-orderable. So we can show orderability of a 3-manifold by constructing a non trivial homomorphism from its fundamental group to $\widetilde{PSL_2}\mathbb{R}$.

To study $\widetilde{PSL_2\mathbb{R}}$ representations, Culler and Dunfield introduced the idea of the translation extension locus of a compact 3-manifold M with torus boundary [11]. They gave several criteria implying whole intervals of Dehn fillings of M have left-orderable fundamental groups.

1.2 The translation extension locus

We follow the notation in [11]. Denote $PSL_2\mathbb{R}$ by G, and $\widetilde{PSL_2\mathbb{R}}$ by $\widetilde{G}$. Let $R_{\widetilde{G}}(M) = \mathrm{Hom}(\pi_1 M, \widetilde{G})$ be the variety of $\widetilde{G}$ representations of $\pi_1(M)$. For a precise definition of the representation variety, see Section 2.2.

The name translation extension locus comes from the fact that we need to use translation number in the definition. For an elements $\widetilde{g}$ in $\widetilde{G}$, define translation number to be

$$\mathrm{trans}(\widetilde{g}) = \lim_{n\to\infty} \frac{\widetilde{g}^n(x) - x}{n} \text{ for some } x \in \mathbb{R}.$$

Then trans: $R_{\widetilde{G}}(\partial M) \to H^1(\partial M; \mathbb{R})$ can be defined by taking $\widetilde{\rho}$ to trans$\circ\widetilde{\rho}$.

Let M be a knot complement in a $\mathbb{Q}$HS or equivalently a $\mathbb{Q}$-homology solid torus. To study $\widetilde{G}$ representations of M whose restrictions to $\pi_1(\partial M)$ are elliptic, Culler and Dunfield gave the following definition of translation extension locus.

Definition 1.1. *(See [11] Section 4) Let $PE_{\widetilde{G}}(M)$ be the subset of representations in $R_{\widetilde{G}}(M)$ whose restriction to $\pi_1(\partial M)$ are either elliptic, parabolic, or central. Consider composition*

$$PE_{\widetilde{G}}(M) \subset R_{\widetilde{G}}(M) \xrightarrow{\iota^*} R_{\widetilde{G}}(\partial M) \xrightarrow{trans} H^1(\partial M; \mathbb{R})$$

The closure in $H^1(\partial M; \mathbb{R})$ of the image of $PE_{\widetilde{G}}(M)$ under trans $\circ\, \iota^*$ is called translation extension locus and denoted $EL_{\widetilde{G}}(M)$.

They showed that translations extension locus of a knot complement in $\mathbb{Q}$HS satisfies the following properties.

Theorem 1.1. *[11, Theorem 4.3] The extension locus $EL_{\widetilde{G}}(M)$ is a locally finite union of analytic arcs and isolated points. It is invariant under $D_\infty(M)$ with quotient homeomorphic to a finite graph. The quotient contains finitely* *many points which are ideal or parabolic in the sense defined above. The locus* $EL_{\widetilde{G}}(M)$ *contains the horizontal axis* L_0, *which comes from representations to* $\widetilde{G}$ *with abelian image.*

They obtained the following results using translation extension loci.

Theorem 1.2. *[11, Theorem 7.1] Suppose that* M *is a longitudinally rigid irreducible* $\mathbb{Q}$*-homology solid torus and that the Alexander polynomial of* M *has a simple root* ξ *on the unit circle. When* M *is not a* $\mathbb{Z}$*-homology solid torus, further suppose that* $\xi^k \neq 1$ *where* $k > 0$ *is the order of the homological longitude* λ *in* $H_1(M; \mathbb{Z})$. *Then there exists* $a > 0$ *such that for every rational* $r \in (-a, 0) \cup (0, a)$ *the Dehn filling* $M(r)$ *is orderable.*

Theorem 1.3. *[11, Theorem 1.4] Let* K *be a hyperbolic knot in a* $\mathbb{Z}$*-homology* 3*-sphere* Y . *If the trace field of the knot exterior* M *has a real embedding then:*

(a) For all sufficiently large n, *the* n*-fold cyclic cover of* Y *branched over* K *is orderable.*

(b) There is an interval I *of the form* $(-\infty, a)$ *or* (a, ∞) *so that the Dehn filling* $M(r)$ *is orderable for all rational* $r \in I$.

(c) There exists $b > 0$ *so that for every rational* $r \in (-b, 0) \cup (0, b)$ *the Dehn filling* $M(r)$ *is orderable.*

Recently, Herald and Zhang [24] improved Theorem 1.2 in the case of M being a $\mathbb{Z}$-homology solid torus by removing the longitudinally rigid condition of M. Their result is stated as follows.

Theorem 1.4. *Let* M *be the exterior of a knot in an integral homology* 3*-sphere such that* M *is irreducible. If the Alexander polynomial* $\Delta(t)$ *of* M *has a simple root on the unit circle, then there exists a real number* $a > 0$ *such that, for every rational slope* $r \in (-a, 0) \cup (0, a)$, *the Dehn filling* $M(r)$ *has left-orderable fundamental group.*

I will construct holonomy extension locus which has similar properties to translations extension locus as described in Theorem 1.1 and prove theorems with similar conclusions to Theorem 1.2 and Theorem 1.3 but different hypotheses.

1.3 Holonomy extension locus

Let M be the complement of a knot in a $\mathbb{Q}$HS or equivalently a $\mathbb{Q}$-homology solid torus. In my thesis, to encode information about boundary-hyperbolic representations of $\pi_1(M)$, I construct the holonomy extension locus which is an analog of the translation extension locus. The exact definition of all the terminologies are described in Chapter 4.

Definition 4.3. *Let $PH_{\widetilde{G}}(M)$ be the subset of representations whose restriction to $\pi_1(\partial M)$ are either hyperbolic, parabolic, or central. Consider the composition*

$$PH_{\widetilde{G}}(M) \subset R^{aug}_{\widetilde{G}}(M) \xrightarrow{\iota^*} R^{aug}_{\widetilde{G}}(\partial M) \xrightarrow{EV} H^1(\partial M;\mathbb{R}) \times H^1(\partial M;\mathbb{Z})$$

The closure of $EV \circ \iota^*(PH_{\widetilde{G}}(M))$ in $H^1(\partial M;\mathbb{R})$ is called the holonomy extension locus and denoted $HL_{\widetilde{G}}(M)$.

The following theorem describes the structure of a holonomy extension locus.

Theorem 4.1. *The holonomy extension locus $HL_{\widetilde{G}}(M) = \bigsqcup_{i,j\in\mathbb{Z}} H_{i,j}(M)$, $-k_M \leq j \leq k_M$ is a locally finite union of analytic arcs and isolated points. It is invariant under the affine group $D_\infty(M)$ with quotient homeomorphic to a finite graph with finitely many points removed. Each component $H_{i,j}(M)$ contains at most one parabolic point and has finitely many ideal points locally.*

The locus $H_{0,0}(M)$ contains the horizontal axis L_0, which comes from representations to $\widetilde{G}$ with abelian image.

1.4 Main result of my thesis

I give examples where there are no irreducible $PSL_2(\mathbb{R})$ representations. Let $\mathcal{M}$ be the manifold $m137$ [8] and $\mathcal{M}(1,n)$ be the integral homology sphere obtained by $(1,n)$ Dehn fillings on $\mathcal{M}$. The main result states:

Theorem 3.1. *For all $n \ll 0$, the manifold $\mathcal{M}(1,n)$ is a hyperbolic integral homology 3-sphere where*

a) $\pi_1(\mathcal{M}(1,n))$ does not have a nontrivial $\widetilde{PSL_2}(\mathbb{R})$ representation.

b) $\mathcal{M}(1,n)$ *is not an L-space.*

This means that we can not produce an order on $\pi_1(\mathcal{M}(1,n))$ simply by pulling back the action of $\widetilde{PSL_2}(\mathbb{R})$ on $\mathbb{R}$.

Using holonomy extension loci, I study $\mathbb{Q}$HSs coming from Dehn fillings of $\mathbb{Q}$-homology solid tori and construct intervals of left-orderable Dehn fillings. The following are the main two applications. The first theorem was also proven independently by Steven Boyer.

Theorem 6.1. *Suppose* M *is the exterior of a knot in a* $\mathbb{Q}$*-homology* 3*-sphere that is longitudinal rigid. If the Alexander polynomial* Δ_M *of* M *has a simple positive real root* $\xi \neq 1$*, then there exists a nonempty interval* $(-a,0]$ *or* $[0,a)$ *such that for every rational* r *in the interval, Dehn filling* $M(r)$ *is orderable.*

Theorem 7.1. *Suppose* M *is a hyperbolic* $\mathbb{Z}$*-homology solid torus. Assume the longitudinal filling* $M(0)$ *is a hyperbolic mapping torus of a homeomorphism of a genus* 2 *orientable surface and its holonomy representation has trace field with a real embedding at which the associated quaternion algebra splits. Then every Dehn filling* $M(r)$ *with rational* r *in an interval* $(-a,0]$ *or* $[0,a)$ *is orderable.*

Chapter 2

BACKGROUND

In the L-space conjecture, we study $\mathbb{Q}$-homology/$\mathbb{Z}$-homology 3-spheres. They are Dehn fillings of $\mathbb{Q}$-homology/$\mathbb{Z}$-homology solid tori, where a $\mathbb{Q}$-homology/$\mathbb{Z}$-homology solid torus is a compact 3-manifold with a torus boundary whose rational/integral homology groups are the same as a solid torus.

2.1 Preliminaries in graph theory

To study holonomy extension locus, we need some basic definitions from graph theory. We call a graph finite if its edge set and vertex set are both finite. In fact, a holonomy extension locus is still slightly different from a finite graph. It is the union of a finite graph part and finitely many branches going to infinity. So we need some proper notion to describe it and we can use the notion finite graph with finitely many points removed.

2.2 Representation Variety and Character Variety

An affine algebraic set is defined to be the zeros of a set of polynomials. In my thesis, I also need real semialgebraic sets [1, Chapter 3], which are defined by polynomial inequalities. The dimension of a real semialgebraic set is equal to its topological dimension. An affine algebraic variety is an irreducible affine algebraic set.

With these notions, we can define representation and character variety of a 3-manifold M. We are interested in representations into Lie groups $PSL_2\mathbb{C} \simeq PGL_2\mathbb{C}$ and $PSL_2\mathbb{R}$. The set of $PSL_2\mathbb{C}$ representations, $\mathrm{Hom}(\pi_1(M), PSL_2\mathbb{C})$ is an affine algebraic set in some $\mathbb{C}^n$ equipped with Zariski topology. We call it the $PSL_2\mathbb{C}$ representation variety of M and denote it by $R(M)$. The group

$PSL_2\mathbb{C}$ acts on $R(M)$ by conjugation, so we can consider the geometric invariant theory quotient $R(M)//PSL_2\mathbb{C}$, which we denote by $X(M)$. It is called the $PSL_2\mathbb{C}$ character variety of M.

Recall $G = PSL_2\mathbb{R}$, $\widetilde{G} = \widetilde{PSL_2}\mathbb{R}$. Similarly we can consider G representation variety $R_G(M)$. Also we define the G character variety $X_G(M)$ to be the geometric invariant theory quotient $R_G(M)//PGL_2\mathbb{R}$. Both $R_G(M)$ and $X_G(M)$ are real algebraic varieties.

Let $f : \widehat{X}(M) \to X(M)$ be a birational map with $\widehat{X}(M)$ a smooth projective curve. Then $\widehat{X}(M)$ is called the smooth projectivization of $X(M)$. Points in $\widehat{X}(M) - f^{-1}(X(M))$ are called ideal points. To each ideal point, we can associate incompressible surfaces to it. See [9] for more details.

2.3 Augmented Representation Variety and Character Variety

We will also need the augmented representation variety and character variety. See [2, Section 10] for more details.

As a subgroup of $PSL_2\mathbb{C}$, G acts on $P^1(\mathbb{C})$ by the Möbius transformation as well as on $S^1 = P^1(\mathbb{R}) \subset P^1(\mathbb{C})$. Nontrivial abelian subgroups of G either have one (if the subgroup contains parabolic elements) or two fixed points(if the subgroup contains hyperbolic or elliptic elements) on $P^1(\mathbb{C})$.

Let $R_G^{\text{aug}}(M)$ be the subvariety of $R_G(M) \times P^1(\mathbb{C})$ consisting of pairs (ρ, z) with z is a fixed point of $\rho(\pi_1(\partial M))$. Let $X_G^{\text{aug}}(M)$ be the GIT quotient of $R_G^{\text{aug}}(M)$ under the diagonal action of G by conjugation and Möbius transformations. There is a natural regular map $\pi : X_G^{\text{aug}}(M) \to X_G(M)$ which forgets the second factor.

The reason why we need augmented character variety $X_G^{\text{aug}}(M)$ is that given $\gamma \in \pi_1(\partial M)$ there is a regular function e_γ which sends $[(\rho, z)]$ to the square of the eigenvalue of $\rho(\gamma)$ corresponding to z. In contrast, on $X_G(M)$ only the trace of $[\rho(\gamma)]$ is well-defined up to sign and we cannot specify which eigenvalue we want. In Chapter 4, I will need eigenvalues of images of hyperbolic and parabolic representations to define holonomy extension locus.

The fiber of $\pi : X_G^{\text{aug}}(M) \to X_G(M)$ contains 2 points except at $[\rho]$ with $\rho|_{\pi_1(\partial M)}$ parabolic (fiber is one point) or trivial (fiber isomorphic to $P^1(\mathbb{C})$).

2.4 $\widetilde{PSL_2\mathbb{R}}$

Consider the Lie group $SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\}$. So we can parameterize $SU(1,1)$ by (γ, ω) where $\gamma = -\overline{\beta}/\alpha \in \mathbb{C}$ and $\omega = \arg\alpha$ is defined modulo 2π. Then $SL_2\mathbb{R} \simeq SU(1,1)$ can be described as $\{(\gamma,\omega) \mid |\gamma| < 1, -\pi \leq \omega < \pi\}$. As the universal cover of $SL_2\mathbb{R}$ and $G = PSL_2\mathbb{R}$, $\widetilde{G} = \widetilde{PSL_2\mathbb{R}}$ is also a Lie group and can be described as $\{(\gamma, \omega) \in \mathbb{C} \times \mathbb{R} \mid |\gamma| < 1, -\infty < \omega < \infty\}$ with group operation given by:

$$(\gamma,\omega)(\gamma',\omega') = \left((\gamma + \gamma' e^{-2i\omega})(1 + \bar{\gamma}\gamma' e^{-2i\omega})^{-1}, \omega + \omega' + \frac{1}{2i} \log{(1 + \bar{\gamma}\gamma' e^{-2i\omega})(1 + \gamma\bar{\gamma}' e^{2i\omega})^{-1}} \right) \tag{2.1}$$

So we have a copy of $\mathbb{R}$ sitting inside $\widetilde{G}$ as an abelian subgroup.

The following properties of $\widetilde{G}$ can be found in [28]. The universal cover of S^1 is $\mathbb{R}$, where S^1 can be viewed as lifting to unit length intervals. Being the universal cover of G which acts on $S^1 = P^1(\mathbb{R})$ by Möbius transformation, $\widetilde{G}$ acts on $\mathbb{R}$ so it is left-orderable. For elements in $\widetilde{G}$, define the translation number to be

$$\text{trans}(\widetilde{g}) = \lim_{n\to\infty} \frac{\widetilde{g}^n(x) - x}{n} \text{ for some } x \in \mathbb{R}.$$

It's independent of the choice of x.

Let $A \in SL_2\mathbb{R}$, $A \neq \pm Id$. Then A is called elliptic if $|\text{trace}(A)| < 2$ and in this case A is conjugate to a matrix of the form

$$\begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}, 0 \leq \alpha < 2\pi.$$

The matrix A is called parabolic if $|\text{trace}(A)| = 2$ and it is conjugate to a matrix of the form

$$\pm \begin{bmatrix} 1 & 2u \\ 0 & 1 \end{bmatrix}, -\infty < u < \infty.$$

The matrix A is called hyperbolic if $|\text{trace}(A)| > 2$ and in this case it is conjugate to a matrix of the form

$$\pm \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, a \neq 0.$$

Elements of $SU(1,1)$ are classified in the same way via the identification $SU(1,1) \simeq SL_2\mathbb{R}$. We then call an element of $\widetilde{G}$ elliptic, parabolic or hyperbolic if it covers an element of the corresponding type in $SU(1,1)$. By Lemma 2.1 in [28], conjugacy classes in $\widetilde{G}$ can be presented as

- elliptic: $(0, \alpha)$, with $-\infty < \alpha/2\pi < \infty$ the translation number of elements in the conjugacy class.
- parabolic: $(\frac{iu}{1+iu}, \tan^{-1}(u) + k\pi)$, with $u \in \mathbb{R}$ and $k \in \mathbb{Z}$ the translation number of elements in the conjugacy class.
- hyperbolic: $(\frac{a-a^{-1}}{a+a^{-1}}, k\pi)$ with $a \in \mathbb{R}$ and $k \in \mathbb{Z}$ the translation number.

In particular, if $\widetilde{g}$ is conjugate to $(0, k\pi)$, then $\widetilde{g}$ is called central, with $k \in \mathbb{Z}$ the translation number.

Representative of conjugacy class of hyperbolic elements may not be unique. To solve this problem, we need fixed points of each element as extra information and define augmented $\widetilde{G}$ representations.

2.5 Augmented $\widetilde{PSL_2}\mathbb{R}$ Representations

As a subgroup of $PSL_2\mathbb{C}$, G acts on $P^1(\mathbb{C})$. There is a natural action of $\widetilde{G}$ on $P^1(\mathbb{C})$ by projecting to G. Hyperbolic and elliptic elements have two fixed points and parabolic elements have one fixed point on $P^1(\mathbb{C})$. Consider the following subset of $\widetilde{G} \times P^1(\mathbb{C})$,

$$\mathrm{Aug}(\widetilde{G}) = \{(\widetilde{A}, v) | \widetilde{A} \in \widetilde{G}, v \in P^1(\mathbb{C}) \text{ is a fixed point of } \widetilde{A}\}.$$

Denote by $A \in G$ the projection of $\widetilde{A} \in \widetilde{G}$. Notice that v is in fact a fixed point of A on $P^1(\mathbb{C})$. Then for any element $(\widetilde{A}, v)$ in $\mathrm{Aug}(\widetilde{G})$ with $\widetilde{A}$ hyperbolic, we can use $(\frac{a-a^{-1}}{a+a^{-1}}, k\pi)$ as the representative of the conjugacy class of $\widetilde{A}$ in $\widetilde{G}$, where a is any of the square root of the derivative of A at v. And it doesn't matter which root we choose as $\frac{a-a^{-1}}{a+a^{-1}}$ is an even function.

We can now construct the augmented $\widetilde{G}$ representation variety of M. Let $R^{\text{aug}}_{\widetilde{G}}(M)$ be the subvariety of $R_{\widetilde{G}}(M) \times P^1(\mathbb{C})$ consisting of pairs $(\widetilde{\rho}, z)$ with z a fixed point of $\rho(\pi_1(\partial M))$, where ρ is the projection of $\widetilde{\rho}$. A subgroup of G may not have a fixed point on $P^1(\mathbb{C})$. However, an abelian subgroup has at least one fixed point on $P^1(\mathbb{C})$. So $\rho(\pi_1(\partial M))$ has at least one fixed point. Similarly, let $R^{\text{aug}}_{\widetilde{G}}(\partial M)$ be the subvariety of $R_{\widetilde{G}}(\partial M) \times P^1(\mathbb{C})$ consisting of pairs $(\widetilde{\rho}, z)$ with z a fixed point of $\rho(\pi_1(\partial M))$.

There is a natural projection from $R^{\text{aug}}_{\widetilde{G}}(-)$ to $R_{\widetilde{G}}(-)$ forgetting the second factor. We call a representation in $R_{\widetilde{G}}(\partial M)$ hyperbolic/elliptic/parabolic if its image in $\widetilde{G}$ contains an element of the corresponding type and call it central if its image contains only central elements. We call a representation in $R^{\text{aug}}_{\widetilde{G}}(\partial M)$ hyperbolic/elliptic/parabolic/central if its projection to $R_{\widetilde{G}}(\partial M)$ is of the corresponding type.

Chapter 3

A COUNTER EXAMPLE

3.1 Introduction

As stated in Chapter 2, to show the fundamental group $\pi_1(Y)$ of a 3-manifold Y is orderable, it is most common to consider $\widetilde{PSL_2}(\mathbb{R})$ representations of $\pi_1(Y)$. In fact in many cases, $\widetilde{PSL_2}(\mathbb{R})$ representations are sufficient to define an order on $\pi_1(Y)$ [11]. However, Theorem 3.1 shows that, even in the case of non L-space integral homology spheres, orders coming from $\widetilde{PSL_2}(\mathbb{R})$ are not enough to prove the L-space conjecture.

It is conjectured that any integer homology 3-sphere different from the 3-sphere admits an irreducible representation in $SU_2(\mathbb{C})$ (see e.g. Kirby's problem list [29, Problem 3.105]). Zentner showed that if one enlarges the target group to $SL_2(\mathbb{C})$, then every such integral homology 3 sphere has an irreducible representation [43]. In contrast, I will give examples where there are no irreducible $PSL_2(\mathbb{R})$ representations. Let $\mathcal{M}$ be the manifold $m137$ [8] and $\mathcal{M}(1, n)$ be the integral homology sphere obtained by $(1, n)$ Dehn fillings on $\mathcal{M}$. The main result of this chapter states:

Theorem 3.1. *For all $n \ll 0$, the manifold $\mathcal{M}(1, n)$ is a hyperbolic integral homology 3-sphere where*

a) $\pi_1(\mathcal{M}(1, n))$ does not have a nontrivial $\widetilde{PSL_2}(\mathbb{R})$ representation.

b) $\mathcal{M}(1, n)$ is not an L-space.

This means that we can not produce an order on $\pi_1(\mathcal{M}(1, n))$ simply by pulling back the action of $\widetilde{PSL_2}(\mathbb{R})$ on $\mathbb{R}$.

Section 2 of this chapter is devoted to proving part (a) of Theorem 3.1. Let $X_0(\mathcal{M})$ be the component of the $SL_2(\mathbb{C})$ character variety of $\mathcal{M}$ containing the character of an irreducible representation (see Culler-Shalen [13] for definition). Here is an outline of the approach. Let $X_{0,\mathbb{R}}(\mathcal{M})$ be the real points of $X_0(M)$. Define $[\rho] \in X_{0,\mathbb{R}}(\mathcal{M})$ and denote by s the trace of $\rho(\lambda)$ where λ is the homological longitude of $\mathcal{M}$. The proof is divided into two parts. In the first part, I show that points on the $|s| < 2$ components of $X_{0,\mathbb{R}}(\mathcal{M})$ all correspond to $SU_2(\mathbb{C})$ representations while points on the $|s| > 2$ components correspond to $SL_2(\mathbb{R})$ representations. In the second part, I show that $SL_2(\mathbb{R})$ representations of $\pi_1(\mathcal{M})$ give rise to no $SL_2(\mathbb{R})$ representations of $\pi_1(\mathcal{M}(1,n))$ when $n \ll 0$. This part of the proof is basically analysing real solutions to the A-polynomial of $\mathcal{M}$ under the relation $\mu\lambda^n = 1$ given by $(1,n)$ Dehn filling , where μ is a choice of meridian of $\partial\mathcal{M}$.

In Section 3, by applying techniques in the paper by Rasmussen, Rasmussen [37] and Gillespie [19], I show that none of the $(1,n)$ Dehn fillings on $m137$ is an L-space, completing the proof of Theorem 3.1.

3.2 $\widetilde{PSL_2}(\mathbb{R})$ representations

I will prove Theorem 3.1 (a) in this section.

SnapPy [12] gives us the following presentation of the fundamental group of $\mathcal{M} = m137$:

$$\pi_1(\mathcal{M}) = \langle \alpha, \beta \mid \alpha^3\beta^2\alpha^{-1}\beta^{-3}\alpha^{-1}\beta^2 \rangle.$$

The peripheral system of $\mathcal{M}$ can be represented as:

$$\{\mu, \lambda\} = \{\alpha^{-1}\beta^2\alpha^4\beta^2, \alpha^{-1}\beta^{-1}\} = \{\beta^2\lambda^{-1}\beta^{-3}\lambda^{-1}\beta^2, \lambda\}$$

where λ is the homological longitude and μ is a choice of meridian. Then we can rewrite the fundamental group as:

$$\pi_1(\mathcal{M}) = \langle \lambda, \beta \mid \beta^{-1}\lambda^{-1}\beta^{-1}\lambda^{-1}\beta^2\lambda = \lambda\beta^{-2}\lambda^{-1}\beta^2 \rangle, \tag{3.1}$$

and the meridian becomes $\mu = \beta^2\lambda^{-1}\beta^{-3}\lambda^{-1}\beta^2$ under this presentation.

Remark. *The triangulation of* $m137$ *we used (included in [17]) to get these presentations is different from SnapPy's default triangulation. We got it by performing random Pachner moves on the default triangulation in SnapPy. In particular, our notations for longitude and meridian in the peripheral system are meridian and longitude respectively in SnapPy's default notations.*

We will first look at irreducible $SL_2(\mathbb{C})$ representations of the fundamental group of $\mathcal{M}$ before we look at those of Dehn fillings of $\mathcal{M}$. Denote by $X(\mathcal{M})$ the $SL_2(\mathbb{C})$ character variety of $\mathcal{M}$, that is the Geometric Invariant Theory quotient $\mathrm{Hom}(\pi_1(\mathcal{M}), SL_2(\mathbb{C}))//SL_2(\mathbb{C})$. It is an affine variety [13]. Suppose $\rho : \pi_1(\mathcal{M}) \longrightarrow SL_2(\mathbb{C})$ is a representation of the fundamental group of $\mathcal{M}$. Recall that a representation ρ of G in $SL_2(\mathbb{C})$ is irreducible if the only subspaces of $\mathbb{C}^2$ invariant under $\rho(G)$ are $\{0\}$ and $\mathbb{C}^2$ [13]. This is equivalent to saying that ρ can't be conjugated to a representation by upper triangular matrices. Otherwise ρ is called reducible. We will call a character irreducible (reducible) if the corresponding representation is irreducible (reducible).

First, I determine which components of $X(\mathcal{M})$ contain characters of irreducible representations. Computation with SnapPy [12] shows that the Alexander polynomial $\Delta_{\mathcal{M}}$ of $m137$ is 1, which has no root. So there are no reducible non-abelian representations [9, Section 6.1]. Therefore all the reducible representations are abelian. Since $H_1(\mathcal{M}) = \mathbb{Z}$, there is only one such component and it is parameterized by the image of β and is isomorphic to $\mathrm{Hom}(\mathbb{Z}, SL_2(\mathbb{C}))//SL_2(\mathbb{C}) \simeq \mathbb{C}$. Moreover, it is disjoint from any component of $X(\mathcal{M})$ containing the character of an irreducible representation [9, Section 6.2]. For more details, we refer the readers to Tillmann's note [38] where he studied $m137$ as an example.

An abelian representation of $\pi_1(\mathcal{M})$ that induces an abelian representation of $\pi_1(\mathcal{M}(1,n))$ factors through the abelianization $ab(\pi_1(\mathcal{M}(1,n))) = 1$. So they correspond to trivial $SL_2(\mathbb{C})$ representations and we don't need to worry about them.

Now we consider components of $X(\mathcal{M})$ that contain the character of an irreducible representation. We have:

Lemma 3.1. *There is a single component $X_0(\mathcal{M})$ of $X(\mathcal{M})$ containing an irreducible character. The functions $s = tr\rho(\lambda) = tr\rho(\alpha^{-1}\beta^{-1}) = tr\rho(\alpha\beta)$ and $t = tr\rho(\beta)$ give complete coordinates on $X_0(\mathcal{M})$, which is the curve in $\mathbb{C}^2$ cut out by*

$$(-2 - 3s + s^3)t^4 + (4 + 4s - s^2 - s^3)t^2 - 1 = 0$$

Moreover, $w := tr\rho(\lambda\beta) = tr\rho((\lambda\beta)^{-1}) = t - \frac{1}{t(s+1)}$.

Proof of Lemma 3.1. Let $X_0(\mathcal{M})$ be $X(\mathcal{M}) - \{\text{reducible characters}\}$. From the discussion above, we know that all the reducible characters form a single component of $X(\mathcal{M})$ and this component is disjoint from any other component of $X(\mathcal{M})$. So $X_0(\mathcal{M})$ is Zariski Closed. We will show later that $X_0(\mathcal{M})$ is actually an irreducible algebraic variety, as claimed in the lemma.

Suppose $[\rho] \in X_0(\mathcal{M})$. So ρ is an irreducible representation. By conjugating ρ if necessary, we can assume that ρ has the form

$$\rho(\lambda) = \begin{pmatrix} z & 1 \\ 0 & 1/z \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} x & 0 \\ y & 1/x \end{pmatrix}.$$

From the relator of $\pi_1(\mathcal{M})$ in (3.1) we have $\rho(\beta)^{-1}\rho(\lambda)^{-1}\beta^{-1}\rho(\lambda)^{-1}\rho(\beta)^2\rho(\lambda) = \rho(\lambda)\rho(\beta)^{-2}\rho(\lambda)^{-1}\rho(\beta)^2$. Comparing the entries of the matrices on both sides, we get four equations. These four equations together with $s = z + 1/z$, $t = x+1/x$ and $w = zx + z^{-1}x^{-1} + y$ form a system $\mathcal{S}$ which defines $X_0(\mathcal{M})$. By computing a Gröbner basis of this system, SageMath [14] gives the following generators of the radical ideal $I = I(X_0(\mathcal{M}))$:

$$stw - t^2 - w^2 - s + 2 \tag{3.2}$$
$$t^3 - w^3 + st - sw - 2t + w \tag{3.3}$$
$$st^2 - tw - w^2 - s + 1 \tag{3.4}$$
$$sw^3 - s^2t + s^2w - t^2w - tw^2 + st - sw + t \tag{3.5}$$

Subtracting (3.4) from (3.2), we get:

$$w = t - \frac{1}{t(s+1)}. \tag{3.6}$$

Eliminating w, we get a defining equation for $X_0(\mathcal{M})$:

$$\begin{aligned} 0 &= (-2 - 3s + s^3)t^4 + (4 + 4s - s^2 - s^3)t^2 - 1 \\ &= (s-2)(s+1)^2t^4 - (s-2)(s+2)(s+1)t^2 - 1. \end{aligned} \tag{3.7}$$

Thus, we can think of $X_0(\mathcal{M})$ as living in $\mathbb{C}^2$.

To prove the lemma, we must show that $X_0(M)$ is irreducible or equivalently the polynomial $P(s,t) := (s-2)(s+1)^2t^4 - (s-2)(s+2)(s+1)t^2 - 1$ in (3.7) does not factor in $\mathbb{C}[s,t]$. Assume $P(s,t)$ factors. Suppose it factors as

$$(at^2+bt+c)(dt^2+et-1/c) = adt^4+(ae+bd)t^3+(cd-a/c+be)t^2+(ce-b/c)t-1,$$

where $a, b, d, e \in \mathbb{C}[s]$ and $c \in \mathbb{C} - \{0\}$. Setting the coefficients of t and t^3 to be 0, we get $b = c^2e$ and $ae = -c^2de$. If $e \neq 0$, then $a = -c^2d$. But this is impossible as $ad = (s-2)(s+1)^2$ is a polynomial in s of odd degree. So $e = 0$ and it follows that $b = 0$. Comparing the coefficients of t^2 and t^4, we get

$$ad = (s-2)(s+1)^2 \tag{3.8}$$

and

$$cd - a/c = -(s-2)(s+2)(s+1). \tag{3.9}$$

So $\text{degree}(a) + \text{degree}(d) = 3$ and $\max\{\text{degree}(a), \text{degree}(d)\} \geq 3$, which implies exactly one of a and d has degree 3 and the other has degree 0. Without loss of generality, we can assume that $\text{degree}(a) = 3$ and $\text{degree}(d) = 0$. Multiply both sides of (3.9) by c, we get $a = c^2d + c(s-2)(s+2)(s+1)$. So the coefficient of s^3 in a is c. Comparing with the coefficient of s^3 in (3.8), we know that $d = 1/c$. Eliminating a and d gives us an equality $1 + (s-2)(s+2)(s+1) = (s-2)(s+1)^2$, which does not hold.

Else suppose $P(s,t)$ factors as

$$(at+c)(bt^3+dt^2+et-1/c) = abt^4+(ad+cb)t^3+(cd+ae)t^2+(ce-a/c)t-1,$$

where $a, b, d, e \in \mathbb{C}[s]$ and $c \in \mathbb{C} - \{0\}$. Setting the coefficients of t and t^3 to be 0, we get $a = c^2e$ and $b = ced$. Comparing the coefficients of t^2 and t^4, we get

$$c^3de^2 = (s-2)(s+1)^2 \tag{3.10}$$

and

$$cd + c^2e^2 = -(s-2)(s+2)(s+1). \tag{3.11}$$

So $\text{degree}(d) + 2\text{degree}(e) = 3$ and $\max\{\text{degree}(d), 2\text{degree}(e)\} \geq 3$ which implies $\text{degree}(d) = 3$ and $\text{degree}(e) = 0$. Comparing the coefficients of s^3 in (3.10) and (3.11), we know that $c^2e^2 = -1$. Plugging into (3.10), we get $cd = (s-2)(s+1)^2$, which when plugging into (3.11) implies $c^2e^2 = -(s+1)(s-2)$, a contradiction. So $P(s,t)$ is irreducible over $\mathbb{C}$. Therefore $X_0(\mathcal{M})$ has only one component. $\square$

To find irreducible $SL_2(\mathbb{R})$ representations of $\pi_1(\mathcal{M})$, we need to check all real points on $X_0(\mathcal{M})$, which correspond to real solutions of (3.7). Notice that equation (3.7) has no solutions when $s = -1$ or 2, so (3.7) is a quadratic equation in t^2. In order for t to be real, t^2 has to be real and nonnegative. Then first we need the discriminant to be nonnegative. That is:

$$\Delta_1 = (s+1)^2(s-2)(s^3 + 2s^2 - 4s - 4) \geq 0.$$

So $s \in U := (-\infty, p_1] \cup [p_2, p_3] \cup (2, \infty)$, where $p_1 \approx -2.9032$, $p_2 \approx -0.8061$ and $p_3 \approx 1.7093$ are three roots of cubic polynomial $s^3 + 2s^2 - 4s - 4$.

The following lemma will help us determine when a $SL_2(\mathbb{C})$ representation of $\pi_1(\mathcal{M})$ can be conjugated into $SL_2(\mathbb{R})$ by simply checking where it lies on the character variety.

Lemma 3.2. *The real points $X_{0,\mathbb{R}}(\mathcal{M}) = X_0(\mathcal{M}) \cap \mathbb{R}^2$ of $X_0(\mathcal{M})$ has 6 connected components:*

Points on the two components with $|s| < 2$ correspond to $SU_2(\mathbb{C})$ representations.

Points on the four components with $|s| > 2$ correspond to $SL_2(\mathbb{R})$ representations.

Remark. *The above lemma shows that in our case, the absolute value of one character being smaller than 2 implies that the representation is $SU_2(\mathbb{C})$. But in general, this is not true.*

To prove this lemma, we need to determine when $[\rho] \in X_{0,\mathbb{R}}(\mathcal{M})$ corresponds to $\rho \in SU_2(\mathbb{C})$ and when it corresponds to $\rho \in SL_2(\mathbb{R})$. It can't be in both because otherwise it would be reducible [11, Lemma 2.10] and we know $X_0(\mathcal{M})$ contains only irreducible characters. The tool we use is a reformulation of Proposition 3.1 in [27] which states that given three angles $\theta_i \in [0, \pi]$, $i = 1, 2, 3$, there exist three $SU_2(\mathbb{C})$ matrices C_i, satisfying $C_1C_2C_3 = I$ with eigenvalues $\exp(\pm i\theta_i)$ respectively if and only if these angles satisfy:

$$|\theta_1 - \theta_2| \leq \theta_3 \leq \min\{\theta_1 + \theta_2, 2\pi - (\theta_1 + \theta_2)\}. \tag{3.12}$$

We want to rewrite the above inequality in terms of traces of C_1, C_2 and C_3. We have the following lemma:

Lemma 3.3. *Suppose $t_1, t_2, t_3 \in (-2, 2)$ are the traces of three matrices $C_1, C_2, C_3 \in SL_2(\mathbb{C})$ satisfying $C_1C_2C_3 = I$. Then C_1, C_2, C_3 are simultaneously conjugate into $SU_2(\mathbb{C})$ if and only if*

$$(2t_3 - t_1t_2)^2 \leq (4 - t_1^2)(4 - t_2^2).$$

Proof. Suppose $t_1 = 2\cos(\theta_1)$, $t_2 = 2\cos(\theta_2)$ and $t_3 = 2\cos(\theta_3)$ where $\theta_1, \theta_2, \theta_3 \in [0, \pi]$.

If $0 \leq \theta_1 + \theta_2 \leq \pi$, then the inequality (3.12) becomes $|\theta_1 - \theta_2| \leq \theta_3 \leq \theta_1 + \theta_2$. Taking cosine, we get $\cos(\theta_1 + \theta_2) \leq \cos(\theta_3) \leq \cos(\theta_1 - \theta_2)$.

If $\pi \leq \theta_1 + \theta_2 \leq 2\pi$, then the inequality becomes $|\theta_1 - \theta_2| \leq \theta_3 \leq 2\pi - (\theta_1 + \theta_2)$. Taking cosine, we also get $\cos(\theta_1 + \theta_2) \leq \cos(\theta_3) \leq \cos(\theta_1 - \theta_2)$.

Use the relations $t_1 = 2\cos(\theta_1)$, $t_2 = 2\cos(\theta_2)$, and $t_3 = 2\cos(\theta_3)$, we get in both cases that:

$$\frac{t_1t_2}{4} - \sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)} \leq \frac{t_3}{2} \leq \frac{t_1t_2}{4} + \sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)}.$$

Then

$$-\sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)} \leq \frac{t_3}{2} - \frac{t_1t_2}{4} \leq \sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)}.$$

So we have:

$$\left|\frac{t_3}{2} - \frac{t_1t_2}{4}\right| \leq \sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)}.$$

Squaring both sides and simplifying, we get

$$(2t_3 - t_1t_2)^2 \leq (4 - t_1^2)(4 - t^2),$$

as desired. □

With the criterion of Lemma 3.3 in hand, we now can prove Lemma 4.8.

Proof of Lemma 4.8. The six components correspond to $s \in (-\infty, p_1] \cup [p_2, p_3] \cup (2, \infty)$ and $t \in (-\infty, 0) \cup (0, \infty)$.

Set $C_1 = \rho(\lambda)$, $C_2 = \rho(\beta)$ and $C_3 = \rho(\beta^{-1}\lambda^{-1}) = \rho((\lambda\beta)^{-1})$. Then $t_1 = s$,

$t_2 = t$ and $t_3 = w$. Applying Lemma 3.3 we have:

$$(2w - st)^2 \leq (4 - s^2)(4 - t^2). \tag{3.13}$$

Plugging (3.6) into (3.13) and simplifying:

$$(s-2)^2 t^2 + \frac{4(s-2)}{s+1} + \frac{4}{t^2(s+1)^2} \leq (4 - s^2)(4 - t^2).$$

Multiplying both sides by $t^2(s+1)^2$, we get:

$$(s+1)^2(s-2)^2 t^4 + 4(s-2)(s+1)t^2 + 4 \leq (4 - s^2)(s+1)^2(4 - t^2)t^2.$$

which simplifies to:

$$-(s+1)^2(s-2)t^4 + (s^2 + 3s + 3)(s-2)(s+1)t^2 + 1 \leq 0.$$

Plugging in (3.7), we get

$$(s+1)^3(s-2)t^2 \leq 0,$$

which always holds when $s \in (p_2 \approx -0.8061, p_3 \approx 1.7093) \subset (-2, 2)$.

So, points on $X_{0,\mathbb{R}}(\mathcal{M})$ correspond to $SU_2(\mathbb{C})$ representations if and only if $|s| < 2$ and correspond to $SL_2(\mathbb{R})$ representations if and only if $|s| > 2$. $\square$

Proof of Theorem 3.1 (a). Lemma 2 tells us a $SL_2(\mathbb{C})$ representation ρ of $m137$ is real if and only if eigenvalues of $\rho(\lambda)$ are real. Moreover, the condition $\mu\lambda^n = 1$ forces the eigenvalues of $\rho(\mu)$ to also be real in this case. So we could restrict our attention to $|s| > 2$ and look at the A-polynomial instead (see e.g. [9] for definition of A-polynomial). Recall that z is an eigenvalue of $\rho(\lambda)$. Denote by m the eigenvalue of $\rho(\mu)$ which shares the same eigenvector with z. The A-polynomial of $m137$ is computed by SAGE [14] as:

$$\begin{aligned}&(z^4 + 2z^5 + 3z^6 + z^7 - z^8 - 3z^9 - 2z^{10} - z^{11}) + m^2(-1 - 3z - 2z^2 - z^3 \\ &+ 2z^4 + 4z^5 + z^6 + 4z^7 + z^8 + 4z^9 + 2z^{10} - z^{11} - 2z^{12} - 3z^{13} - z^{14}) \\ &+ m^4(-z^3 - 2z^4 - 3z^5 - z^6 + z^7 + 3z^8 + 2z^9 + z^{10}).\end{aligned}$$

Denote by $A = -1-2z-3z^2-z^3+z^4+3z^5+2z^6+z^7 = (z-1)(z^2+z+1)^3$ and $B = 1+3z+2z^2+z^3-2z^4-4z^5-z^6-4z^7-z^8-4z^9-2z^{10}+z^{11}+2z^{12}+3z^{13}+z^{14}$.

So the A-polynomial could be simplified as $-z^4A - Bm^2 + z^3Am^4$. We are interested in the real solutions of

$$-z^4A - Bm^2 + z^3Am^4 = 0. \tag{3.14}$$

Now consider the $(1, n)$ Dehn filling on $m137$. Then we are adding an extra relation $\mu\lambda^n = 1$, which is $\rho(\mu)\rho(\lambda)^n = I$ under the representation ρ, i.e.

$$\rho(\mu) = \rho(\lambda)^{-n} = \begin{pmatrix} z^{-n} & * \\ 0 & z^n \end{pmatrix}.$$

Restricting to $\partial\mathcal{M}$ gives us the relation $m = z^{-n}$.

When n is negative, we shall denote $n' = -n$. So we have $m = z^{n'}$. Plugging into (3.14) and dividing both sides by z^4, we get

$$-A - Bz^{2n'-4} + Az^{4n'-1} = 0. \tag{3.15}$$

We will show the following lemma is true, completing the proof of Theorem 3.1 (a).

Lemma 3.4. *Equation* (3.15) *has no real solutions when n' is large enough.*

Proof of Lemma 3.4. Define $F(z) = A(z^{4n'-1} - 1) - Bz^{2n'-4}$. We'll show $F(z) > 0$.

First notice that $A = 0$ only when $z = 1$. And $A > 0$ when $z > 1$ while $A < 0$ when $z < 1$. The polynomial B has 6 real roots which are all simple: -2.3396, -1.4121, -0.7082, -0.4274, 0.8684, 1.1516 (rounded to the fourth digit).

As we saw earlier, the domain for s is $U := (-\infty, p_1 \approx -2.9032] \cup [p_2 \approx -0.8061, p_3 \approx 1.7093] \cup (2, \infty)$. So the $|s| > 2$ condition restricts s to $(-\infty, p_1 \approx -2.9032] \cup (2, \infty)$. Then $z \in V := (-\infty, -2.5038] \cup [-0.3994, 0) \cup (0, 1) \cup (1, \infty)$. Notice that $z^7A(1/z) = -A(z)$ and $z^{14}B(1/z) = B(z)$. Interchange z with $1/z$ in $F(z)$ gives us $F(1/z) = A(1/z)(z^{-(4n'-1)} - 1) - B(1/z)z^{-(2n'-4)} = F(z)/z^{4n'+6}$. So we can assume $|z| < 1$.

case 1: $0.8684 \leq z < 1$

In this case, we have $A(z) < 0$, $B(z) \leq 0$ and $z^{4n'-1} - 1 < 0$. So $F(z) > 0$.

case 2: $-0.3994 \leq z < 0.8684$ and $z \neq 0$

In this case, we have $A(z) < C_5 < 0$ and $C_6 > B(z) > 0$ for some constants C_5 and C_6. When n' is large enough, we have $|C_5| \times |(z^{4n'-1} - 1)| > C_6 z^{2n'-4}$. So $A(z^{4n'-1} - 1) = |A| \times |(z^{4n'-1} - 1)| > Bz^{2n'-4}$ and it follows that $F(z) > 0$.

Therefore when $n' = -n$ is large enough, we always have $F(z) > 0$ on the domain V. So equation (3.15) has no real solution when $n' \gg 0$. □

It follows from the above lemma that equation (3.14) has no real solution when $n \ll 0$ and thus equality $\rho(\mu)\rho(\lambda)^n = I$ does not hold for $n \ll 0$.

From all the discussion above, we can now conclude that $\mathcal{M}(1, n)$ has no nontrivial $SL_2(\mathbb{R})$ representation and thus no nontrivial $PSL_2(\mathbb{R})$ representation for $n \ll 0$. Since the first Betti number of $\mathcal{M}(1, n)$ is 0, the lift of a trivial $PSL_2(\mathbb{R})$ representation of $\pi_1(\mathcal{M}(1, n))$ into $\widetilde{PSL_2}(\mathbb{R})$ will be trivial. So all representations of $\pi_1(\mathcal{M}(1, n))$ into $\widetilde{PSL_2}(\mathbb{R})$ are trivial for $n \ll 0$, proving Theorem 3.1 (a). □

In contrast, when n is positive there are examples of non trivial $SL_2(\mathbb{R})$ representations.

Plugging $m = z^{-n}$ into (3.14) and multiplying both sides by z^{4n-3}, we get

$$-A + Bz^{2n-3} + Az^{4n+1} = 0.$$

Similarity, define $G(z) = A(z^{4n+1} - 1) + Bz^{2n-3}$. Since $G(1) = -4$, $G(0.8684) > 0$, $G(z)$ must have at least one root in $[0.8684, 1)$. So $\pi_1(\mathcal{M}(1, n))$ has at least one nontrivial $SL_2(\mathbb{R})$ representation for any $n > 0$. They lift to a $\widetilde{PSL_2}(\mathbb{R})$ representations, since the Euler number of any representation of an integral homology sphere vanishes [18, Section 6].

3.3 No L-space fillings

In this section, I will prove Theorem 3.1 (b) using results from Gillespie's paper [19], which is based on Rasmussen and Rasmussen's paper [37]. In fact, I will show that none of the non-longitudinal fillings of $m137$ is an L-space. The homology groups in this section are all homology with integral coefficients.

Suppose Y is a compact connected 3-manifold with a single torus as boundary. I will follow Gillespie's [37] notation. Define the set of slopes on ∂Y as:

$$\mathcal{S}l(Y) = \{a \in H_1(\partial Y)|\ a \text{ is primitive}\}/\pm 1.$$

Define the set of L-space filling slopes of Y:

$$\mathcal{L}(Y) = \{a \in \mathcal{S}l(Y)|\ Y(a) \text{ is an L-space}\}.$$

Moreover, Y is said to have genus 0 if $H_2(Y, \partial Y)$ is generated by a surface of genus 0.

We will use Theorem 1.2 from Gillespie's paper [19] which is stated as:

Theorem 3.2. *The following are equivalent*

1) $\mathcal{L}(Y) = Sl(Y) - \{l\}$.

2) Y *has genus* 0 *and has an L-space filling.*

Proof of Theorem 3.1 (b). Let $l \in Sl(\mathcal{M})$ be the homological longitude. In our case l can be taken to be $[\lambda]$. I will show that none of the $(1, n)$ fillings to $\mathcal{M}$ is an L-space.

I will find one non L-space filling first. Snappy [12] shows that $(1, -1)$ filling on the knot 8_{20} complement with homological framing is homeomorphic to $m011(2, 3)$, which is also homeomorphic to $\mathcal{M}(1, -3)$. Ozsváth and Szabó showed that if some $(1, p)$ Dehn filling of a knot complement in $S^3$with homological framing is an L-space, then the Alexander polynomial of the knot has coefficients ± 1 [34, Corollary 1.3]. We can compute with SnapPy [12] that the Alexander Polynomial of 8_{20} is $x^4 - 2x^3 + 3x^2 - 2x + 1$. So $\mathcal{M}(1, -3)$ is not an L-space. Therefore

$$-3l + [\mu] \notin \mathcal{L}(\mathcal{M}) \neq \mathcal{S}l(\mathcal{M}) - \{l\} \ni -3l + [\mu],$$

By Theorem 3.2, either $\mathcal{M}$ has no L-space fillings or $\mathcal{M}$ has positive genus.

The manifold $\mathcal{M}$ can be viewed as the complement of a knot K in $S^2 \times S^1$ [15]. This knot K intersects each S^2 three times. So $[K] \neq 0$ in $H_1(S^2 \times S^1; \mathbb{Z})$. It follows that $H_2(\mathcal{M}, \partial\mathcal{M})$ is generated by genus 0 surface $(S^2 \times \{P\}) \cap \mathcal{M}$ for generic point P on K. So $\mathcal{M}$ has genus 0, which forces $\mathcal{M}$

to have no L-space filling. Therefore none of the integral homology spheres $\mathcal{M}(1, n)$ is an L-space. $\square$

Chapter 4

HOLONOMY EXTENSION LOCUS

In the next four chapters, I will use $\widetilde{PSL_2\mathbb{R}}$ representations to prove some results about left-orderability.

In this chapter, I define the holonomy extension locus, show its structure and explain how it works.

4.1 Definition of holonomy extension locus

Definition 4.1. *For hyperbolic element $\widetilde{g} \in \widetilde{G}$, take $v \in P^1(\mathbb{C})$ to be a fixed point of $\widetilde{g}$. Define $ev : Aug(\widetilde{G}) \longrightarrow \mathbb{R} \times \mathbb{Z}$, $(\widetilde{g}, v) \mapsto (\ln(|a|), trans(\widetilde{g}))$, with a any of the square root of the derivative of g (projection of $\widetilde{g}$ in G) at v.*

For parabolic elements, define $ev : Aug(\widetilde{G}) \longrightarrow \mathbb{R}\times\mathbb{Z}$, taking $\widetilde{g}$ to $(0, trans(\widetilde{g}))$.

Lemma 4.1. *The map $ev(-, v)$ preserves group structure of hyperbolic or parabolic abelian subgroup of $\widetilde{G}$ with v any fixed point of the subgroup. As a consequence, $ev((\widetilde{\rho}(-), v)) : \pi_1(\partial M) \to \mathbb{R} \times \mathbb{Z}$ is a group homomorphism for $\widetilde{\rho}$ hyperbolic or parabolic, where v is a fixed point of $\widetilde{\rho}(\pi_1(\partial M))$.*

Proof. Any nontrivial hyperbolic/parabolic abelian subgroup of $\widetilde{G}$ has at least one fixed point in $P^1(\mathbb{C})$ and let v be any one of them. Consider the stabilizer group $\mathrm{Stab}(v) \subset SL_2\mathbb{R}$ of v. We can define a homomorphism eig: $\mathrm{Stab}(v) \longrightarrow \mathbb{R}^\times$ which takes $g \in \mathrm{Stab}(v)$ to $|a|$ where $gv = av$. Since $\pm I$ is the kernel, this homomorphism descends to a homomorphism from the stabilizer group of v in G to $\mathbb{R}^\times$ which we will still call eig. As trans is also a homomorphism and $\mathrm{ev}(\widetilde{g}, v) = (\ln(\mathrm{eig}(g)), \mathrm{trans}(\widetilde{g}))$ for any $\widetilde{g} \in \widetilde{G}$ where $g \in G$ is the projection, it follows that $\mathrm{ev}(-, v)$ preserves group structure of hyperbolic or parabolic abelian subgroup of $\widetilde{G}$.

When $\widetilde{\rho}$ is hyperbolic/parabolic, $\widetilde{\rho}(\pi_1(\partial M))$ becomes an abelian hyperbolic/parabolic subgroup of $\widetilde{G}$, with v a fixed point. So being the composite of two homomorphisms $\widetilde{\rho}$ and $\mathrm{ev}(-,v)$, $\mathrm{ev}((\widetilde{\rho}(-),v)) : \pi_1(\partial M) \to \mathbb{R} \times \mathbb{Z}$ is also a group homomorphism.

□

Identifying $\mathrm{Hom}(\pi_1(\partial M), \mathbb{R} \times \mathbb{Z})$ with $H^1(\partial M;\mathbb{R}) \times H^1(\partial M;\mathbb{Z})$, we can view $\mathrm{ev}((\widetilde{\rho}(-),v))$ as living in $H^1(\partial M;\mathbb{R}) \times H^1(\partial M;\mathbb{Z})$. Let M be an irreducible $\mathbb{Q}$-homology solid torus, and let $\iota : \partial M \to M$ be the inclusion map. With the above lemma, we can now define:

Definition 4.2. *Let $PH_{\widetilde{G}}(M)$ be the subset of representations whose restriction to $\pi_1(\partial M)$ are either hyperbolic, parabolic, or central. Define EV : $R^{aug}_{\widetilde{G}}(\partial M) \longrightarrow H^1(\partial M;\mathbb{R}) \times H^1(\partial M;\mathbb{Z})$ by $(\widetilde{\rho}, v) \mapsto ev((\widetilde{\rho}(-),v))$ on $\iota^*(PH_{\widetilde{G}}(M))$, where ι^* is the restriction $R^{aug}_{\widetilde{G}}(M) \longrightarrow R^{aug}_{\widetilde{G}}(\partial M)$ of representations of $\pi_1(M)$ to $\pi_1(\partial M)$.*

Lemma 4.2. *Fix $v \in P^1(\mathbb{C})$. Let H_v be the set of hyperbolic elements of $\widetilde{G}$ that fix v. Then any two elements of H_v with the same image under $ev(-,v)$ are conjugate in $\widetilde{G}$.*

Proof. We will use the homomorphism eig as in 4.1 and the property that $\mathrm{ev}(\widetilde{g},v) = (\ln(\mathrm{eig}(g)), \mathrm{trans}(\widetilde{g}))$ for any $\widetilde{g} \in \widetilde{G}$ where $g \in G$ is the projection.

Two elements $\widetilde{g}$ and $\widetilde{g'}$ in H_v are conjugate if and only if $gv = g'v$ and $\mathrm{trans}(\widetilde{g}) = \mathrm{trans}(\widetilde{g'})$. So if $\mathrm{ev}(g,v) = \mathrm{ev}(g',v)$, then $\mathrm{eig}(g) = \mathrm{eig}(g')$ and $\mathrm{trans}(\widetilde{g}) = \mathrm{trans}(\widetilde{g'})$, implying that g is conjugate to g'. □

Definition 4.3. *Consider the composition*

$$PH_{\widetilde{G}}(M) \subset R^{aug}_{\widetilde{G}}(M) \xrightarrow{\iota^*} R^{aug}_{\widetilde{G}}(\partial M) \xrightarrow{EV} H^1(\partial M;\mathbb{R}) \times H^1(\partial M;\mathbb{Z})$$

The closure of $EV \circ \iota^*(PH_{\widetilde{G}}(M))$ in $H^1(\partial M;\mathbb{R}) \times H^1(\partial M;\mathbb{Z})$ is called the holonomy extension locus of M and denoted $HL_{\widetilde{G}}(M)$.

We will call a point in $HL_{\widetilde{G}}(M)$ a hyperbolic/parabolic/central point if it comes from a representation $\widetilde{\rho} \in PH_{\widetilde{G}}(M)$ such that $\widetilde{\rho}|_{\pi_1(\partial M)}$ is hyperbolic/parabolic/central.

Definition 4.4. *We call a point in $HL_{\widetilde{G}}(M)$ an ideal point if it only lies in the closure $\overline{EV \circ \iota^*(PH_{\widetilde{G}}(M))}$ but not in $EV \circ \iota^*(PH_{\widetilde{G}}(M))$.*

Lemma 4.3. *Suppose $(\widetilde{\rho}, v) \in R^{aug}_{\widetilde{G}}(\partial M)$ is hyperbolic or central. If $EV(\widetilde{\rho}, v)(\gamma) = (0,0)$ for some $\gamma \in \pi_1(\partial M)$, then $\widetilde{\rho}(\gamma) = 1$.*

Proof. It follows from Lemma 4.2 that $ev(\widetilde{\rho}(\gamma), v) = EV(\widetilde{\rho}, v)(\gamma) = (0,0)$ implies $\widetilde{\rho}(\gamma)$ is conjugate to the identity element of $\widetilde{G}$. So $\widetilde{\rho}(\gamma) = 1$. $\square$

Suppose λ is the homological longitude of M. Define

$$k_M = \min\{-\chi(S) | S \text{ is a connected incompressible surface of } M \text{ that bounds } \lambda\}.$$

We will use Milnor-Wood inequality in the form of Proposition 6.5 from [11].

Proposition 4.1. *Suppose S is a compact orientable surface with one boundary component. For all $\widetilde{\rho} : \pi_1(S) \to \widetilde{G}$ one has*

$$|trans(\widetilde{\rho}(\delta))| \leq max(-\chi(S), 0) \quad \textit{where } \delta \textit{ is a generator of } \pi_1(\partial S).$$

Applying this proposition, we see immediately that $|\text{trans}(\widetilde{\rho}(\lambda))| \leq k_M$.

In the next theorem, we will show that $HL_{\widetilde{G}}(M) = \bigsqcup_{i,j\in\mathbb{Z}} H_{i,j}(M)$, $-k_M \leq j \leq k_M$. Each $H_{i,j}(M) := HL_{\widetilde{G}}(M) \cap (\mathbb{R}^2 \times \{i\} \times \{j\}) \subset \mathbb{R}^2$ is a finite union of analytic arcs and isolated points. Denote the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ by $D_\infty(M)$. Then $D_\infty(M)$ acts on $\mathbb{R}^2 \times \mathbb{Z}^2$ by translating (x, y, i_1, j) to (x, y, i_2, j) for any $i_1, i_2, j \in \mathbb{Z}$ and taking (x, y, i, j) to $(-x, -y, -i, -j)$ by reflecting about $(0, 0, 0, 0)$. We will show $HL_{\widetilde{G}}(M)$ is invariant under the action of $D_\infty(M)$.

Define L_r to be line of slope $-r$ going through the origin in $\mathbb{R}^2$. Then L_0 is the x-axis. Now we can state the theorem.

Theorem 4.1. *The holonomy extension locus $HL_{\widetilde{G}}(M) = \bigsqcup_{i,j\in\mathbb{Z}} H_{i,j}(M)$, $-k_M \leq j \leq k_M$ is a locally finite union of analytic arcs and isolated points. It is invariant under the affine group $D_\infty(M)$ with quotient homeomorphic to a finite graph with finitely many points removed. Each component $H_{i,j}(M)$ contains at most one parabolic point and has finitely many ideal points locally.*

The locus $H_{0,0}(M)$ contains the horizontal axis L_0, which comes from representations to $\widetilde{G}$ with abelian image.

Remark. *If we assume the manifold M is small, i.e. it has no closed essential surface, then there is no ideal point in $HL_{\widetilde{G}}(M)$. The proof is similar to [11, Lemma 6.8]. See Lemma 4.7.*

4.2 Properties of holonomy extension locus

Lemma 4.4. *The holonomy extension locus $HL_{\widetilde{G}}(M)$ is invariant under $D_\infty(M)$.*

Proof. We will show the image I of $PH_{\widetilde{G}}(M)$ under $\mathrm{EV}\circ\iota^*$ is invariant under $D_\infty(M)$. Take $(\widetilde{\rho}, v) \in PH_{\widetilde{G}}(M)$ and let $t = \mathrm{EV}\circ\iota^*(\widetilde{\rho}, v)$ be the corresponding point in I. Let s be the generator of the center of $\widetilde{G}$ which is isomorphic to $\mathbb{Z}$ and take any $\varphi \in H^1(M;\mathbb{Z})$. Then $PH_{\widetilde{G}}(M) \ni \varphi\cdot\widetilde{\rho} : \gamma \mapsto \widetilde{\rho}(\gamma)s^{\varphi(\gamma)}$ is another lift of $\pi\circ\widetilde{\rho}$, where $\pi : \widetilde{G} \to G$ is the projection. It's easy to see that $\widetilde{\rho}(\pi_1(\partial M))$ and $\varphi\cdot\widetilde{\rho}(\pi_1(\partial M))$ share the same fixed point v. We can check that for any $\gamma \in \pi_1(M)$, we have $\mathrm{ev}(\varphi\cdot\widetilde{\rho}(\gamma), v) = \mathrm{ev}(\widetilde{\rho}(\gamma)s^{\varphi(\gamma)}, v) = \mathrm{ev}(\widetilde{\rho}(\gamma), v) + (0, \varphi(\gamma))$. So $\mathrm{EV}\circ\iota^*(\varphi\cdot\widetilde{\rho}, v) = \mathrm{EV}\circ\iota^*(\widetilde{\rho}s^{\varphi}, v) = \mathrm{EV}\circ\iota^*(\widetilde{\rho}, v) + (0, \varphi)$. It follows that I is invariant under translation by elements of $\iota^*(H^1(M;\mathbb{Z})) \subset H^1(\partial M;\mathbb{R})$.

Next, we will show $HL_{\widetilde{G}}(M)$ is invariant under reflection about the origin in $\mathbb{R}^2\times\mathbb{Z}^2$. Define f to be the element in $\mathrm{Homeo}(\mathbb{R})$ taking $x \in \mathbb{R}$ to $-x$, and consider the conjugate action of f on $\widetilde{G}$. The group $\widetilde{G}$ is preserved under this conjugation because $\pi(f\widetilde{g}f^{-1})$ has the same action as $\pi(\widetilde{g}^{-1})$ on S^1 for any $\widetilde{g} \in \widetilde{G}$. Suppose a is a square root of the derivative of $\pi(g)$ at v, then a^{-1} is a square root of the derivative of $\pi(\widetilde{g}^{-1})$ at v and a^{-1} is a square root of the derivative of $\pi(f\widetilde{g}f^{-1})$ at $-v$. Moreover we can check that

$$\mathrm{trans}(f\widetilde{g}f^{-1}) = \lim_{n\to\infty}\frac{(f\widetilde{g}f^{-1})^n(0) - 0}{n} = \lim_{n\to\infty}\frac{f\widetilde{g}^n(-0) - 0}{n} = -\mathrm{trans}(\widetilde{g}).$$

This shows that $\mathrm{ev}(\widetilde{\rho}(\gamma), v) = -\mathrm{ev}(f\widetilde{\rho}f^{-1}(\gamma), -v)$ and it follows that $\mathrm{EV}\circ\iota^*(\widetilde{\rho}, v) = -\mathrm{EV}(f\widetilde{\rho}f^{-1}, -v)$. Given such an f, the image of $(f\widetilde{\rho}f^{-1}, -v)$ in I is $-t$, proving invariance. $\square$

As a consequence of Lemma 4.4, we can now look at the quotient $PL_{\widetilde{G}}(M) = HL_{\widetilde{G}}(M)/D_\infty(M)$. In fact $PL_{\widetilde{G}}(M) = \sqcup_{-k_M\le j\le k_M} H_{0,j}(M)/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts on the disjoint union by taking $(x,y) \in H_{0,j}(M)$ to $(-x,-y) \in H_{0,-j}(M)$. In particular, $\mathbb{Z}/2\mathbb{Z}$ acts on $H_{0,0}(M)$ via reflection about the origin.

Lemma 4.5. *$PL_{\widetilde{G}}(M)$ has finitely many connected components. In particular, each $H_{i,j}(M)$ has finitely many connected components.*

Proof. The proof works similarly as Lemma 6.2 of [11].

Let $\Pi : R_{\widetilde{G}}(M) \to R_G(M)$ be the map between representation varieties induced by $\pi : \widetilde{G} \to G$. Let $PH_G(M)$ be the subset of $R_G(M)$ consisting of representations whose restrictions to $\pi_1(\partial M)$ consist only of hyperbolic, parabolic and trivial elements. The set $PH_G(M)$ is a subset of the real algebraic set $R_G(M)$ cut out by polynomial inequalities. It follows that $PH_G(M)$ is a real semialgebraic set.

Let $PH_G^{\text{lift}}(M) \subset PH_G(M)$ be the image of $PH_{\widetilde{G}}(M)$ under Π. By continuity of the translation number, $PH_G^{\text{lift}}(M)$ is a union of connected components of $PH_G(M)$. Moreover $PH_G^{\text{lift}}(M) \subset PH_G(M)$ is the quotient of $PH_{\widetilde{G}}(M)$ under the action of $H^1(M, \mathbb{Z})$ and Π is the covering map. So it is also a real semialgebraic set and thus has finitely many connected components.

The action of $H^1(M, \mathbb{Z})$ on $PH_{\widetilde{G}}(M)$ induces an action of $\mathbb{Z} \leq D_\infty(M)$ on $HL_{\widetilde{G}}(M)$. Let $\Pi^{-1}(PH_G^{\text{lift}}(M))$ be any sheet in the covering of $PH_G^{\text{lift}}(M)$. So $PL_{\widetilde{G}}(M) = \overline{EV \circ \iota^*(\Pi^{-1}(PH_G^{\text{lift}}(M)))}/(\mathbb{Z}/2\mathbb{Z})$, and thus has finitely many components. Let $PH_G^j(M)$ be the subset of $PH_G^{\text{lift}}(M)$ consisting of representations with translation number of the homological longitude being j. Then $PH_G^{\text{j}}(M)$ is a finite union of connected components of $PH_G(M)$. It follows that $H_{i,j}(M) = \overline{EV \circ \iota^*(\Pi^{-1}(PH_G^{\text{j}}(M)))}$ has finitely many components, where $\Pi^{-1}(PH_G^{\text{j}}(M))$ is any sheet in the covering of $PH_G^{\text{j}}(M)$. $\square$

Proof of Theorem 4.1. First notice that the index j is bounded, which follows from Proposition 4.1.

Define $c : H^1(\partial M; \mathbb{R}) \times H^1(\partial M; \mathbb{Z}) \to X_G(\partial M), (f_1, f_2) \mapsto$ character of ρ, where ρ is given by $\rho(\mu) = \begin{bmatrix} e^{f_1(\mu)} & 0 \\ 0 & e^{-f_1(\mu)} \end{bmatrix}, \rho(\lambda) = \begin{bmatrix} e^{f_1(\lambda)} & 0 \\ 0 & e^{-f_1(\lambda)} \end{bmatrix}$.

Consider the dual basis $\{\mu^*, \lambda^*, m^*, l^*\}$ for $H^1(\partial M; \mathbb{R}) \times H^1(\partial M; \mathbb{Z})$, where $\mu^*(p\mu + q\lambda) = p$, $\lambda^*(p\mu + q\lambda) = q$, $m^*(p\mu + q\lambda) = p$ and $l^*(p\mu + q\lambda) = q$ for any $p\mu + q\lambda \in \pi_1(\partial M)$. Take $(x, y, i, j) \in HL_{\widetilde{G}}(M)$. If we use trace-squared coordinates on $X_G(\partial M)$, we get

$$c(x, y, i, j) = (e^{2x} + e^{-2x} + 2, e^{2y} + e^{-2y} + 2, e^{2x+2y} + e^{-2x-2y} + 2).$$

It is easy to check that $c(-x, -y, -i, -j) = c(x, y, i, j)$ and $c(x, y, i + n_1, j + n_2) = c(x, y, i, j)$, where n_1 and n_2 are integers.

Consider the diagram

$$\begin{tikzcd}
PH_{\widetilde{G}}(M) \arrow[r, "\mathrm{EV}\circ\iota^*"] \arrow[d] & H^1(\partial M;\mathbb{R}) \times H^1(\partial M;\mathbb{Z}) \arrow[d, "c"] \\
X_G(M) \arrow[r, "\iota^*"] & X_G(\partial M)
\end{tikzcd}$$

The vertical map c maps $HL_{\widetilde{G}}(M)$ into $\overline{\iota^*(X_G(M))}$. Being the image of a real algebraic set under a polynomial map, $X_G(M)$ is a real semialgebraic subset of $X_{\mathbb{R}}(M)$. Since $\iota^*(X(M)) \subset X(\partial M)$ has complex dimension at most 1 [11, Lemma 2.4], then the real semialgebraic set $\overline{\iota^*(X_G(M))}$ has real dimension at most 1. Moreover $\iota^*(X_G(M))$ is a locally finite graph as $X_G(M)$ is. Thus, its preimage under c is a locally finite graph with analytic edges that is invariant under $D_\infty(M)$. So each $H_{i,j}(M)$ and thus $PL_{\widetilde{G}}(M)$ is a locally finite graph and by Lemma 4.5 it has finitely many connected components. Therefore $PL_{\widetilde{G}}(M)$ is homeomorphic to a finite graph with finitely many points removed.

Suppose D is a closed disc in $H^1(\partial M;\mathbb{R})$, then $D \cap H_{i,j}(M)$ lives in a finite graph. Since by Lemma 4.5 $H_{i,j}(M)$ has finitely many components, then $D \cap H_{i,j}(M)$ also has finitely many components and thus is a finite graph. So $D \cap H_{i,j}(M)$ is the closure of a set of finitely many components in a finite graph and thus contains finitely many ideal points.

Parabolic points can only occur at origin of each $H_{i,j}(M)$, so there can be at most one parabolic point in each component $H_{i,j}(M)$.

Recall from Section 2.4 that there is an abelian subgroup of $\widetilde{G}$ that is isomorphic to $\mathbb{R}$. Consider diagonal representations in G. They lift to a one parameter family of abelian representations $\pi_1(M) \to \widetilde{G}$ by sending the generator of $H_1(M;Z)_{\text{free}} \cong H_1(M;Z)/(\text{torsion}) \cong \mathbb{Z}$ to a given element in $\mathbb{R}$. Since the longitude λ of ∂M is 0 in $H_1(M;\mathbb{Z})_{\text{free}}$, this one parameter family of abelian representations give rise to the line L_0 in $H_{0,0}(M)$.

□

The following lemma describes some other properties of $HL_{\widetilde{G}}(M)$.

Lemma 4.6 (structure of $H_{i,j}(M)$)**.** *Suppose for some i, j, $H_{i,j}(M)$ contains an arc that goes to infinity. Then this arc approaches asymptotes $y = -rx$ in $\mathbb{R}^2$ as it goes to infinity, where r is the boundary slope of the associated incompressible surface to some ideal point of $\widehat{X}(M)$.*

Proof. The vertical map c in the proof of Theorem 4.1 maps $HL_{\widetilde{G}}(M)$ into $\overline{\iota^*(X_G(M))}$. Let $\widehat{X}(M)$ be the smooth projectivization of $X(M)$.

Suppose $H_{i,j}(M)$ contains an arc A that goes to infinity, then there is a an ideal point x of $\widehat{X}(M)$ that is the limit of a sequence of characters $\{[\rho_k]\}$ in $X(M)$ of hyperbolic representations $\{\rho_k\}$ such that images of lifts $\{\widetilde{\rho_k}\}$ under $\mathrm{EV}\circ\iota^*$ are contained in A. To show this, suppose images of $\{\widetilde{\rho_k}\}$ under $\mathrm{EV}\circ\iota^*$ go to infinity in $HL_{\widetilde{G}}(M)$. Then $\{[\rho_k]\}$ march off to infinity in $X(M)$ as eigenvalues of either $\{\rho_k(\mu)\}$ or $\{\rho_k(\lambda)\}$ go to infinity. Thus by passing to a subsequence $\{[\rho_k]\}$ converge to an ideal point x of $\widehat{X}(M)$. Notice that traces of elliptic and parabolic elements of G are bounded, by passing to a subsequence, we can assume that $\widetilde{\rho_k}|_{\pi_1(\partial M)}$ are hyperbolic. Moreover can choose a sequence of points $\{v_k\}$ where $v_k \in P^1(\mathbb{C})$ is a common fixed point of $\rho_k(\pi_1(\partial M))$ acting on $P^1(\mathbb{C})$. And by passing to a subsequence, we can assume $\{v_k\}$ limits to $v \in P^1(\mathbb{C})$.

By the result in [9, Section 5.7], there exists $\beta \in \pi_1(\partial M)$ such that $\mathrm{tr}^2_\beta(x) = b^2 + b^{-2} + 2$ is finite and $\beta = p\mu + q\lambda$, where $r = p/q$ is the boundary slope of the associated incompressible surface to the ideal point x. Then $\lim_{k\to\infty} \mathrm{tr}^2_\beta([\rho_k]) = b^2 + b^{-2} + 2$ as $[\rho_k] \to x$, where b^2 is a positive real number as it is the limit of eigenvalue square of hyperbolic G matrices. Moreover, b has to be a root of unity by [9, Section 5.7]. It follows that $b^2 = 1$, which implies $\lim_{k\to\infty} \rho_k(\beta) = I$. It follows that $\lim_{k\to\infty} \widetilde{\rho_k}(\beta) = \widetilde{I}$, where $\widetilde{I}$ is a lift of I with translation number $\lim_{k\to\infty} \mathrm{trans}(\widetilde{\rho_k}(\beta)) = p\lim_{k\to\infty} \mathrm{trans}(\widetilde{\rho_k}(\mu)) + q\lim_{k\to\infty} \mathrm{trans}(\widetilde{\rho_k}(\lambda)) = pi + qj$. Then we can check slope of the asymptote of the arc containing $\{\mathrm{EV}(\widetilde{\rho_k}, v_k)\}$ in $HL_{\widetilde{G}}(M)$. We have $p\lim_{k\to\infty} \mathrm{EV}(\widetilde{\rho_k}, v_k)(\mu) + q\lim_{k\to\infty} \mathrm{EV}(\widetilde{\rho_k}, v_k)(\lambda) = p\lim_{k\to\infty} \mathrm{ev}(\widetilde{\rho_k}(\mu), v_k) + q\lim_{k\to\infty} \mathrm{ev}(\widetilde{\rho_k}(\lambda), v_k) = \lim_{k\to\infty} \mathrm{ev}(\widetilde{\rho_k}(p\mu + q\lambda), v_k) = \mathrm{ev}(\widetilde{I}, v) = (\ln(|b|) = 0, pi + qj)$. So $\lim_{k\to\infty} \mathrm{slope}[\rho_k] = -r$ and thus the curve A is asymptotic to the line of slope $-r$ going through the origin.

□

Holonomy extension locus is related to the A-polynomial which was first introduced in [9]. To explain this relation, we will start with the definition of eigenvalue variety [38, Section 7].

Let $R^{\mathrm{aug}}_U(M)$ be the subvariety of $R^{\mathrm{aug}}(M)$ defined by two equations which specify that the lower left entries in $\rho(M)$ and $\rho(L)$ are equal to zero. Consider the eigenvalue map,

$$R_U^{\mathrm{aug}}(M) \to (\mathbb{C} - 0)^2$$

Taking closure of image of this map and discarding zero dimensional components, we get the eigenvalue variety $\mathfrak{E}(M)$ of M, which is defined by a principal ideal. A generator for the radical of this ideal is called the A-polynomial. We will call points that are only in the closure but not in the image ideal points.

We are only interested in the intersection of $\mathfrak{E}(M)$ with $\mathbb{R}^2$ as those points come from boundary parabolic/hyperbolic/trivial representations. The composition $R_G^{\mathrm{aug}} \to R_U^{\mathrm{aug}}(M) \to \mathbb{R}^2 \cap \mathfrak{E}(M)$ gives a map from boundary hyperbolic representations to eigenvalues of the meridian and longitude of the boundary, which is similar to but not entirely the same as $\mathrm{EV} \circ \iota^*$ defined in 4.3.

Recall that M is called a small manifold if it contains no closed essential surface. We will prove the following lemma.

Lemma 4.7. *If M is small, then there is no ideal point in $HL_{\widetilde{G}}(M)$ or $(\mathbb{R}^2 - \mathbf{0}) \cap \mathfrak{E}(M)$.*

Proof. The proof works the same way as in [11, Lemma 6.8]. Suppose t_0 is an ideal point in $HL_{\widetilde{G}}(M)$ (resp. $(\mathbb{R}^2 - \mathbf{0}) \cap \mathfrak{E}(M)$) and $\{\widetilde{\rho_i}\} \subset PH_{\widetilde{G}}(M)$ is a sequence of $\widetilde{G}$ representations whose images in $HL_{\widetilde{G}}(M)$ (resp. $(\mathbb{R}^2 - \mathbf{0}) \cap \mathfrak{E}(M)$) converge to t_0. Suppose $\{[\rho_i]\}$ is the sequence of corresponding characters in $X_G(M)$. A similar argument shows that by passing to a subsequence, $[\rho_i]$ lies in a single irreducible component X' of $X(M)$ and either $[\rho_i]$ limit to a character χ in $X_G(M)$ or the $[\rho_i]$ march off to infinity in the noncompact curve X'. In the latter case, as both $|\mathrm{tr}(\rho_i(\mu))|$ and $|\mathrm{tr}(\rho_i(\lambda)|$ are bounded above, $|\mathrm{tr}(\rho_i(\gamma))|$ is bounded above for any $\gamma \in \pi_1(\partial M)$. The argument of [9, Section 2.4] produces a closed essential surface associated to a certain ideal point of X', contradicting our hypothesis that M is small.

In the case when the $[\rho_i]$ limit to χ in $X_G(M)$, a similar argument shows that t_0 is not actually an ideal point, proving the lemma.

□

Finally, we use the following lemma to construct order. Recall that L_r is a line through origin in $\mathbb{R}^2$ with slope $-r$.

Lemma 4.8. *If L_r intersects $H_{0,0}(M)$ component of $HL_{\widetilde{G}}(M)$ at non parabolic or ideal points, and assume $M(r)$ is irreducible, then $M(r)$ is left-orderable.*

Proof. Let $f = (x_1, y_1)$ be a point in $L_r \cap H_{0,0}(M)$ that is different from the origin as f is not parabolic by assumption and further assume that it is not ideal. Then there exists a preimage $\widetilde{\rho} \in R_{\widetilde{G}}(M)$ of f which is hyperbolic when restricting to $\pi_1(\partial M)$. Suppose $\gamma \in \pi_1(\partial M)$ realizes slope $r = j/k$, i.e. $\gamma = \lambda^k \mu^j$. By definition of $L_r : y = -rx$, we have $f(\gamma) = \mathrm{EV}(\widetilde{\rho})(\gamma) = \mathrm{ev} \circ \widetilde{\rho}(\gamma) = (ky_1 + jx_1, k \cdot \mathrm{trans}(\lambda) + j \cdot \mathrm{trans}(\mu)) = (k(-jx_1/k) + jx, k0 + j0) = (0, 0)$. It follows from Lemma 4.3 that $\widetilde{\rho}(\gamma) = 1$, so we get an induced representation $\overline{\rho} : \pi_1(M(r)) \to \widetilde{G}$. As f is different from the origin, then we can always find an element $\eta \in \pi_1(\partial M)$ with slope different from r such that $\overline{\rho}(\eta) \neq 0$, which implies that $\overline{\rho}$ is nontrivial. Since $M(r)$ is irreducible, it follows from [4, Theorem 3.2] that $\pi_1(M(r))$ is left-orderable.

□

Chapter 5

EXAMPLES

In this chapter, I will show some examples of holonomy extension loci.

Our first example is the figure eight knot 4_1, whose Alexander polynomial is $t^2 - 3t + 1$.

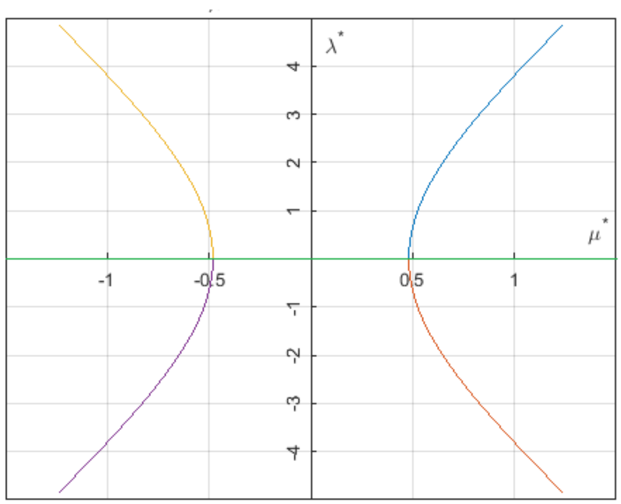

Figure 5.1: Holonomy Extension Locus $HL_{\widetilde{G}}(4_1)$

There is nothing interesting going on in the translation extension locus of the figure-eight knot complement as it contains only the x-axis $y = 0$ coming from abelian representations. The above figures shows its holonomy extension locus which has no other copies except $H_{0,0}(M)$ since the translation extension locus has no component other than the x axis. The figure-eight knot complement has genus 1, so the $2g-1$ bound for translation number j of the longitude is not sharp. There are two asymptotes of the graph with slopes ± 4. So fillings of figure-eight knot complement with slope lying in the interval $(-4, 4)$ are orderable. This phenomenon was first noticed by Steven Boyer.

Our next example is the $(7, 3)$ two-bridge knot 5_2. Complements of two-bridge knots are small [23, Theorem 1(a)]. So holonomy extension loci of two-bridge knots do not have ideal points by Lemma 4.7.

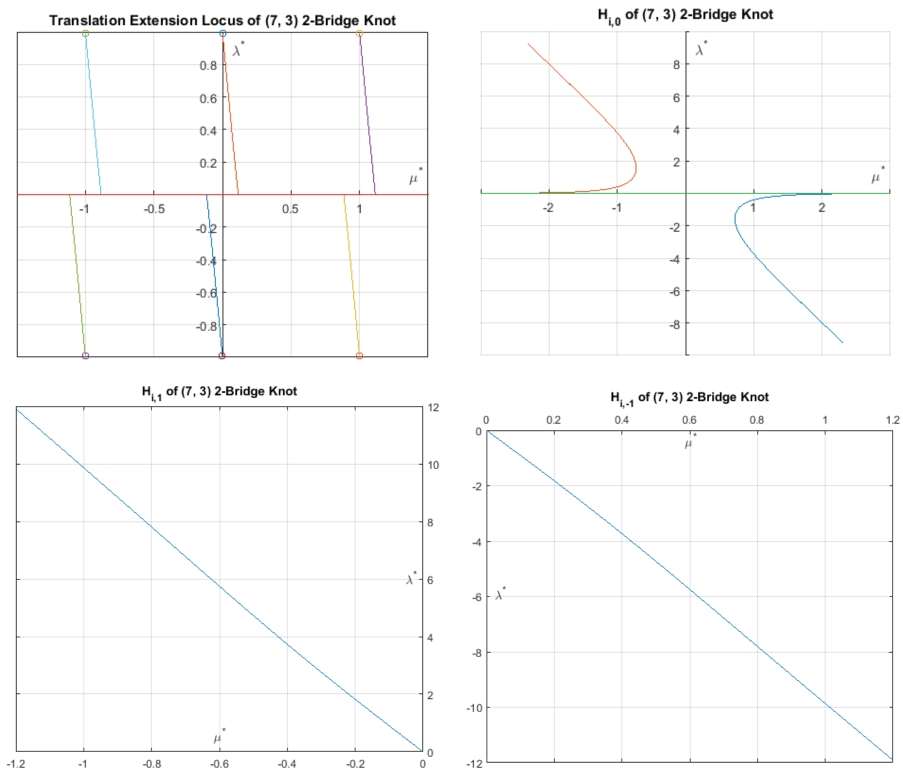

Figure 5.2: Holonomy and Translation Extension Locus of $(7, 3)$ 2-bridge Knot

The top left figure is the translation extension locus of the $(7, 3)$ two-bridge knot, where the six circles are parabolic points. The translation extension locus tells us $(-\infty, 1)$ fillings are orderable.
The top right figure is the $H_{0,0}(M)$ component of its holonomy extension locus. There are two asymptotes with slope -4 and 0. The interval of left-orderable Dehn fillings we can read off from the holonomy extension locus is [0,4). So compared to translation extension locus, the holonomy extension locus does tell us something more.
The two figures on the bottom are $H_{0,1}$ and $H_{0,-1}$. Notice that asymptotes in $H_{0,\pm 1}$ both have slope -10. Actually, boundary slopes associated to ideal points of the character variety of the $(7, 3)$ two-bridge knot complement are 0, 4, 10. This result confirms Lemma 4.6.

The $(7, 3)$ two-bridge knot, whose genus is 1, is a twist knot of three half twist. So its Alexander polynomial is not monic and it follows that it is not fibered [36]. Moreover, it cannot be an L-space knot [33, Corollary 1.3]. In [11, Section 9, Question (4)], it is observed that for fibered knots, the $2g - 1$ bound for translation number of the longitude is never sharp. However we can see from this example that for non fibered knots, this bound can be sharp.

For the above examples, we actually computed equations of the graphs. Next we show some more complicated pictures produced by programs [10] written by Culler and Dunfield under SageMath [14]. We will only show the quotient $PL_{\widetilde{G}}(M)$ of $HL_{\widetilde{G}}(M)$ under the action of $D_\infty(M)$, where we identify $H_{0,j}$ with $H_{0,-j}$ when $j \neq 0$ and quotient $H_{0,0}$ down by reflection about the origin.

Our first example is $t03632$, which has a loop in its holonomy extension locus.

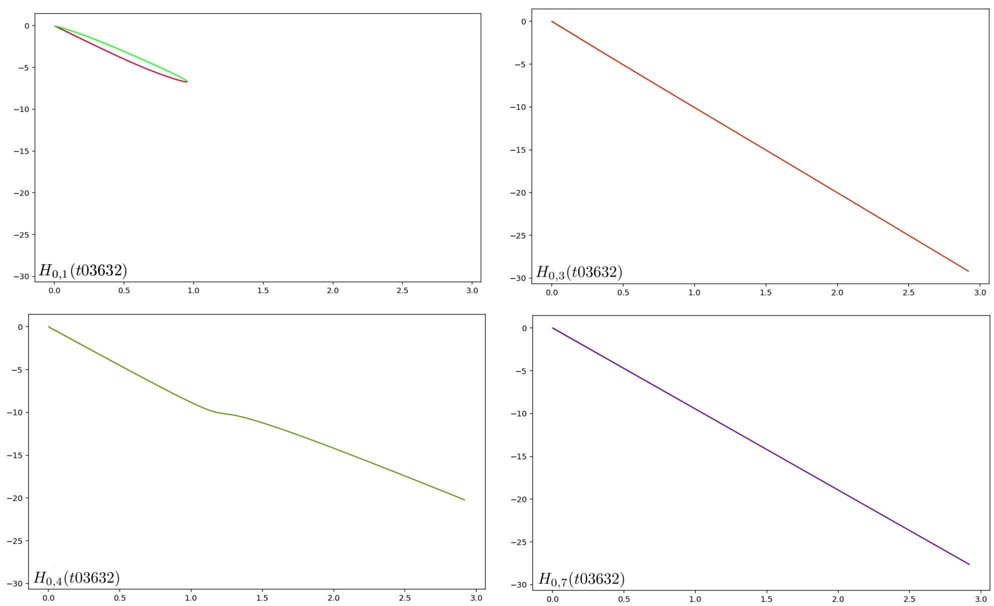

Figure 5.3: $PL_{\widetilde{G}}(t03632)$

Top left figure is $H_{0,1}$ of $t03632$, where we see a small loop based at the origin (parabolic point). The Alexander polynomial of $t03632$ has no positive real root. The locus $H_{0,0}$ contains nothing other than the horizontal line representing abelian representations so we will not show it here.

Our next example is 7_3 which has more interesting $H_{0,0}$.

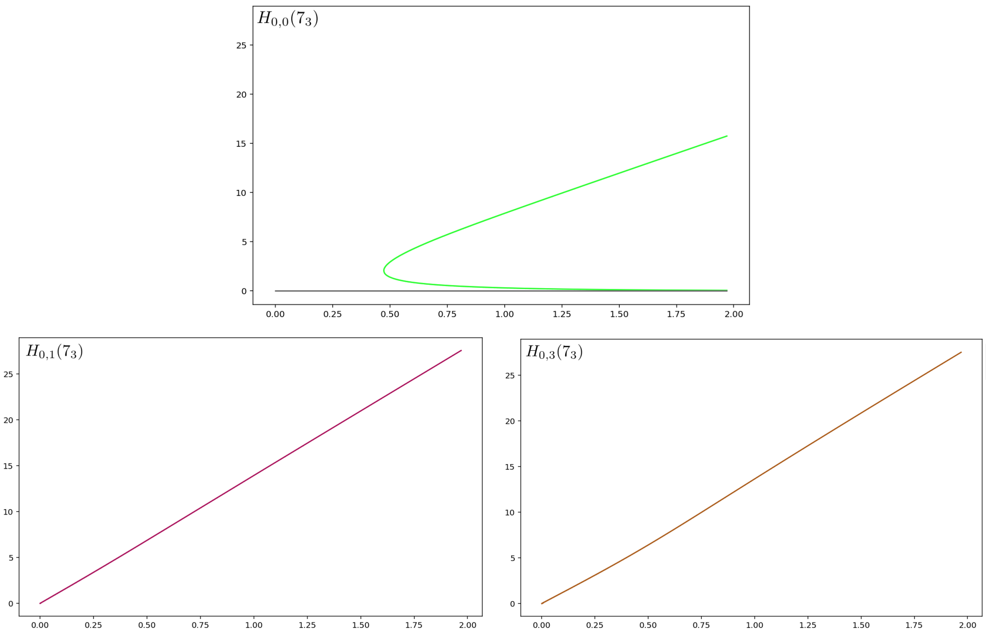

Figure 5.4: $PL_{\widetilde{G}}(7_3)$

The Alexander polynomial of 7_3 is $2t^4 - 3t^3 + 3t^2 - 3t + 2$, which has no real root. But we can see $H_{0,0}$ (figure on top) contains an arc that is different from the x-axis, even though this arc does not intersect the x-axis.

5.1 Simple Roots of the Alexander Polynomial

When the Alexander polynomial Δ_M of M has a positive root ξ, we can draw a point $(\ln(\xi)/2, 0)$ on the x-axis and call it an Alexander point. When ξ is a simple root, Lemma 6.1 predicts that there is an arc coming out of the Alexander point $(\ln(\xi)/2, 0)$. Moreover, this Alexander point corresponds to the abelian representation associated to the root ξ of Δ_M, e.g. ρ_α as constructed in proof of Lemma 6.1. We use large dots to indicate Alexander points in our figures.

In addition to the example of the figure eight knot shown in Figure 5.1, we will show more holonomy extension loci with Alexander points.

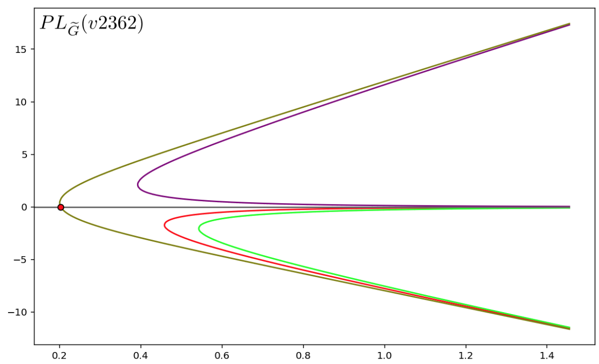

Figure 5.5: $PL_{\widetilde{G}}(v2362)$

This figure is $PL_{\widetilde{G}}(v2362)$, the quotient of the holonomy extension locus of $v2362$. The Alexander polynomial of $v2362$ is $6t^2 - 13t + 6$ which has two simple real roots $2/3$ and $3/2$. So we can expect to see the Alexander point $(\frac{1}{2}\ln(\frac{3}{2}), 0)$. (The other point $(\frac{1}{2}\ln(\frac{2}{3}) = -\frac{1}{2}\ln(\frac{3}{2}), 0)$ is mapped to the same point under the quotient.) We can see in this figure that the arc that goes through the Alexander point is not tangent to the x-axis at the Alexander point.

5.2 Multiple Roots of the Alexander Polynomial

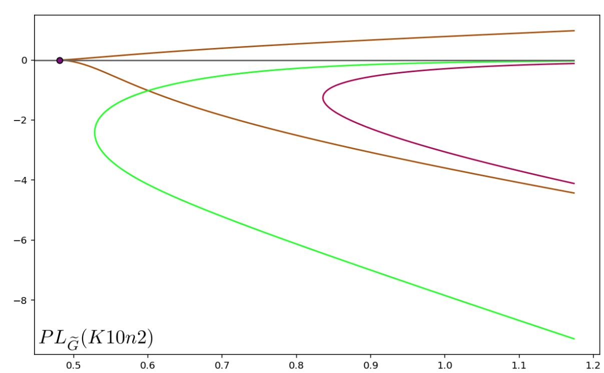

Figure 5.6: $PL_{\widetilde{G}}(K10n2)$

This figure is $PL_{\widetilde{G}}(K10n2)$, the quotient of the holonomy extension locus of $K10n2$. It only contains the quotient locus $H_{0,0}/(\mathbb{Z}/2\mathbb{Z})$. The Alexander polynomial of $K10n2$ has two positive real double roots that are reciprocals of each other. We can see that the two arcs are tangent to the x-axis at the Alexander point.

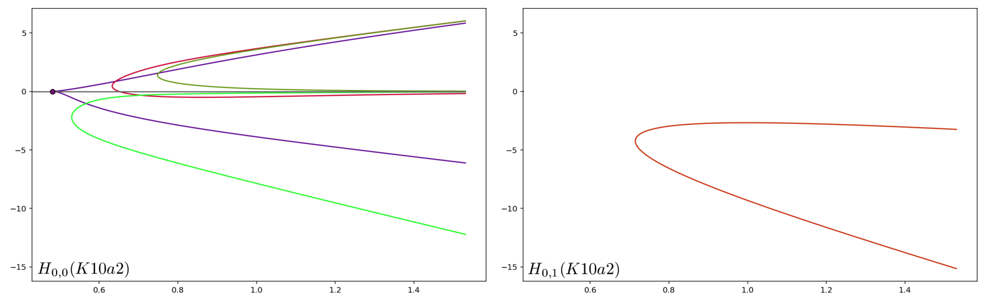

Figure 5.7: $PL_{\widetilde{G}}(K10a2)$

The Alexander polynomial of $K10a2$ has two positive real double roots that are reciprocals of each other. We can see that two arcs in $H_{0,0}(K10a2)$ in the left figure are tangent to the x-axis at the Alexander point.

The above examples $K10n2$ and $K10a2$ have typical patterns for multiple roots. They all have arcs tangent to the x-axis at Alexander points.

The manifold $K9a37$ in our next example also has Alexander polynomial with double roots. However the local picture of its holonomy extension locus at the Alexander point is quite different from Figure 5.6, 5.7.

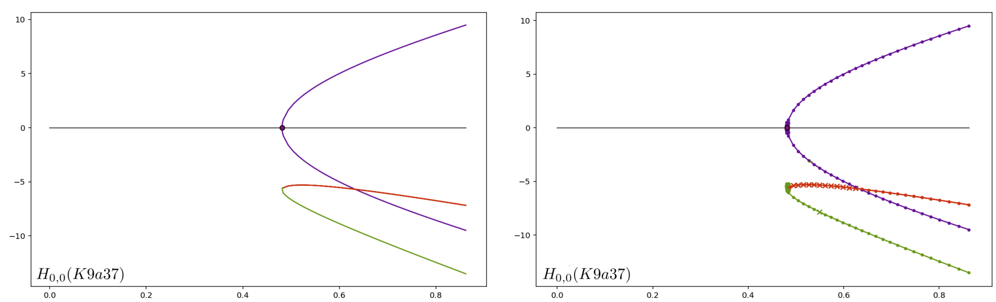

Figure 5.8: $H_{0,0}(K9a37)$

The Alexander polynomial of $K9a37$ has two positive real double roots. The figure on the left is $H_{0,0}$ of he holonomy extension locus of $K9a37$. (To be precise, we still need to remove a small segment of arc on the red curve to get the actual $H_{0,0}(K9a37)$.) We can see that there is an arc coming out of the Alexander point in both directions but not tangent to the x-axis.

Remark: There is an arc A_0 in $H_{0,2}(K9a37)$ (not shown since it does not belong to $H_{0,0}(K9a37)$) that is tangent to the bottom arc (green) shown in the above pictures at some point and our current graphing program is unable to separate these two tangent curves automatically. So we have to remove A_0 from the pictures above by hand.

The holonomy extension locus of $K9a37$ has some interesting phenomena, which are shown in Figure 5.8 on the right. The 'x's on the red curve (second curve from the bottom) mean that this point comes from a $PSL_2\mathbb{C}$ representation ρ that is not $PSL_2\mathbb{R}$ even though $\rho|_{\partial M}$ is a $PSL_2\mathbb{R}$ representation. So these points do not belong to the holonomy extension locus. (The small dots on the curves simply mean this point comes from a $PSL_2\mathbb{R}$ representation.) From this example, we can see that an arc in a holonomy extension locus can end at a point that is not the infinity, Alexander point or parabolic point. We guess such a point could be a Tillmann point (see [11] end of Section 5 for definition).

The statement of Lemma 6.1 requires the root of the Alexander polynomial to be simple. When we have a root that is not simple, we expect to see an example where there is no arc coming out of the corresponding Alexander point at all, as this is what happened in the translation extension locus in Figure 10 of Section 5 of [11]. However, we were not able to find such an example at this moment as the graphing program is still unfinished and we only have very limited number of samples.

Remark. *In addition to issues with graphing like unseparated curves and Tillmann points as mentioned above, we also spotted missing components. In the above example* $K9a37$*, we know a curve in* $H_{0,2}(K9a37)$ *is missing from our figure. In their graphing program, Culler and Dunfield use gluing varieties rather than character varieties to simplify computation. Some of the graphing issues might be caused by this. Check the end of Section 5 of [11] for more details about computation and graphing issues.*

Chapter 6

Alexander polynomials and orderability

In this chapter, we prove Theorem 6.1. To state the theorem, we will need some definitions from [11]. We say a compact 3-manifold Y has few characters if each positive dimensional component of the $PSL_2\mathbb{C}$ character variety of Y consists entirely of characters of reducible representations. An irreducible $\mathbb{Q}$-homology solid torus M is called longitudinally rigid when its Dehn filling $M(0)$ along the homological longitude has few characters.

The following result was also proven independently by Steven Boyer.

Theorem 6.1. *Suppose M is the exterior of a knot in a $\mathbb{Q}$-homology 3-sphere that is longitudinal rigid. If the Alexander polynomial Δ_M of M has a simple positive real root $\xi \neq 1$, then there exists a nonempty interval $(-a, 0]$ or $[0, a)$ such that for every rational r in the interval, Dehn filling $M(r)$ is orderable.*

The following lemma is key to proving Theorem 6.1.

Lemma 6.1. *Suppose M is an irreducible $\mathbb{Q}$-homology solid torus. If $\xi \neq 1$ is a simple positive real root of the Alexander polynomial, then there exists an analytic path $\rho_t : [-1, 1] \to R_G(M)$ where:*

(a) The representations ρ_t are irreducible over $PSL_2\mathbb{C}$ for $t \neq 0$.

(b) The corresponding path $[\rho_t]$ of characters in $X_G(M)$ is also a nonconstant analytic path.

(c) $tr^2_\gamma(\rho_t)$ is nonconstant in t for some $\gamma \in \pi_1(\partial M)$.

Proof. First I prove (a) and (b).

As in Proposition 10.2 of [25], let $\alpha : \pi_1(M) \to \mathbb{R}_+ = (\mathbb{R} > 0)$ be a representation such that α factors through $H_1(M; \mathbb{Z})_{\text{free}} \cong \mathbb{Z}$ and takes a generator of $H_1(M; \mathbb{Z})_{\text{free}}$ to ξ. Let $\rho_\alpha : \pi_1(M) \to PSL_2\mathbb{R}$ be the associated diagonal representation given by

$$\rho_\alpha = \pm \begin{bmatrix} \alpha^{1/2}(\gamma) & 0 \\ 0 & \alpha^{-1/2}(\gamma) \end{bmatrix}, \text{ where } \alpha^{1/2}(\gamma) \text{ is the positive square root of } \alpha(\gamma).$$

Then $\chi_\alpha = \mathrm{tr}^2(\rho_\alpha)$ is real valued, as $\alpha(\gamma)+1/\alpha(\gamma)+2 \in \mathbb{R}\ \forall \gamma \in \pi_1(M)$. Since $\mathrm{Im}(\alpha)$ is contained in $\mathbb{R}_+$ but not in $\{\pm 1\}$, $\mathrm{Im}(\rho_\alpha)$ is contained in $PGL_2(\mathbb{R})$ and in fact in $PSL_2\mathbb{R}$. Next, we carry out the computation of obstruction in the real setting. Let $\mathfrak{sl}_2(\mathbb{C})$ be the complexification of $\mathfrak{sl}_2(\mathbb{R})$, we have the corresponding isomorphism of cohomology groups.

$$H^*(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\alpha) = H^*(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\alpha) \otimes_{\mathbb{R}} \mathbb{C}.$$

So computations with complex variety $X(M)$ in the proof of [25, Theorem 1.3] can be carried out in the real case. It follows that the tangent space to $X_G(M)$ at χ_α is $H^*(\pi_1(M); \mathbb{R}_+ \oplus \mathbb{R}_-)//\mathbb{R}^* \cong \mathbb{R}$ and thus χ_α is a smooth point. Carrying out the computation of obstructions in the real setting, we are able to show that $d_+ + d_- \in H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_{\rho_\alpha})$ can be integrated to an analytic path $\rho_t : [-1, 1] \to R_G(M)$ with $\rho_0 = \rho_\alpha$ and ρ_t irreducible over $PSL_2\mathbb{C}$ for $t \neq 0$. So χ_α is contained in a curve containing characters of irreducible $PSL_2\mathbb{R}$ representations, which gives (a).

The path $[\rho_t] \subset X_G(M)$ is nonconstant because ρ_t is irreducible whenever $t \neq 0$ and thus cannot have same character as the reducible representation ρ_0, proving (b).

Next, we will prove (c). In fact the existence of $\gamma \in \pi_1(\partial M)$ such that $\mathrm{tr}^2_\gamma(\rho_t)$ is nonconstant in t is proved similarly as [11, Lemma 7.3 (4)]. We first construct nonabelian representation $\rho^+ \in R_G(M)$ which corresponds to $[\rho_\alpha]$ in $X_G(M)$. Then the Zariski tangent space of $X_G(M)$ at $[\rho_\alpha]$ can be identified with $H^1(M; \mathfrak{sl}_2(\mathbb{R})_{\rho^+})$ while the Zariski tangent space of $X_G(\partial M)$ at $[\rho^+ \circ \iota]$ can be identified with $H^1(\partial M; \mathfrak{sl}_2(\mathbb{R})_{\rho^+})$. So the proof of (c) boils down to showing the injectivity of $\iota^* : H^1(M; \mathfrak{sl}_2(\mathbb{R})_{\rho^+}) \to H^1(\partial M; \mathfrak{sl}_2(\mathbb{R})_{\rho^+})$. See [11, Lemma 7.3 (4)] for more details.

□

We will also need the following property of closed 3 manifolds with few characters.

Lemma 6.2. *Suppose Y is a closed 3 manifold with $H_1(Y,\mathbb{Q}) = \mathbb{Q}$. If Y has few characters, then Y is irreducible.*

Proof. Prove by contradiction. If Y is reducible, then we can decompose it as a connected sum $Y_1 \sharp Y_2$, where $H_1(Y_1,\mathbb{Q}) = \mathbb{Q}$ and Y_2 is a $\mathbb{Q}$HS. So $\pi_1(Y) = \pi_1(Y_1) * \pi_1(Y_2)$. We want to use $PSL_2\mathbb{C}$ representations of Y_1 and Y_2 to construct a dimension one component of $PSL_2\mathbb{C}$ character variety of Y containing an irreducible representation so that it contradicts the assumption that Y has few characters. As $H_1(Y_1,\mathbb{Z}) = \mathbb{Z} \oplus$ (possible torsion), we can construct a nontrivial abelian $PSL_2\mathbb{C}$ representation ρ_1 of Y_1 by composing $\pi_1(Y_1) \twoheadrightarrow \mathbb{Z}$ and $\mathbb{Z} \hookrightarrow PSL_2\mathbb{C}$. For Y_2, there are two cases. If $H_1(Y_2,\mathbb{Z})$ contains a cyclic subgroup H, then similarly we can construct a nontrivial abelian $PSL_2\mathbb{C}$ representation ρ_2 of Y_2 by composing $\pi_1(Y_2) \twoheadrightarrow H$ and $H \hookrightarrow PSL_2\mathbb{C}$. If Y_2 is actually a $\mathbb{Z}$HS, then by Theorem 9.4 of [43], there is an irreducible $SL_2\mathbb{C}$ representation ρ_2 of $\pi_1(Y_2)$. Moreover we can make ρ_2 an irreducible $PSL_2\mathbb{C}$ representation by simply projecting to $PSL_2\mathbb{C}$. So we can construct a set of $PSL_2\mathbb{C}$ representations $\rho_P = \rho_1 * P\rho_2P^{-1}$ of Y, where P is any matrix in $PSL_2\mathbb{C}$. These representations are not conjugate to each other as long as they have different P and at least one of them is irreducible as we can vary P so that ρ_1 and $P\rho_2P^{-1}$ are not upper triangular at the same time. □

Now we can prove Theorem 6.1.

Proof of Theorem 6.1. Let ρ_t be the associated path in $R_G(M)$ given by Lemma 6.1. As ρ_0 factors through $H_1(M;\mathbb{Z})_{\text{free}} \cong \mathbb{Z}$, we can lift it and its lift $\widetilde{\rho_0}$ also factors through $H_1(M;\mathbb{Z})_{\text{free}}$. Hence $\text{trans}(\widetilde{\rho_0}(\lambda)) = 0$. And $\widetilde{\rho_0}$ is mapped to a point on the horizontal axis of $H_{0,0}(M)$ as $\rho_0(\lambda) = I$. The x coordinate of $\widetilde{\rho_0}$, $\ln(|\xi|)$ is nonzero as $\xi \neq \pm 1$.

As ρ_0 lifts, we can extend this lift to a continuous path $\widetilde{\rho_t}$ in $R_{\widetilde{G}}(M)$. Moreover, we can assume $\widetilde{\rho_t}$ is actually in $R^{\text{aug}}_{\widetilde{G}}(M)$, as fixed points of $\widetilde{\rho_t}(\pi_1(\partial M))$ also vary continuously with t.

Let k be the index of $\langle \iota_*(\mu)\rangle$ in $H_1(M,\mathbb{Z})_{\text{free}}$, where $\iota : \partial M \to M$ is the inclusion. By construction $\text{tr}^2_\mu(\widetilde{\rho_0}) = \xi^k + 2 + \xi^{-k} > 4$, so there exists $\varepsilon > 0$ such that $\text{tr}^2_\mu(\widetilde{\rho_t}) \geq 4$ for $t \in [-\varepsilon, \varepsilon]$. As $\rho_t(\mu)$ is hyperbolic, $\rho_t(\lambda)$ is also hyperbolic. Therefore ρ_t is a path in $PH_G(M)$ and $\widetilde{\rho_t}$ is a path in $PH_{\widetilde{G}}(M)$.

Then we can build path A by composing $\widetilde{\rho}_t$ with $\mathrm{EV}\circ\iota^* : PH_{\widetilde{G}}(M) \to HL_{\widetilde{G}}(M)$. That the path A is nonconstant follows from Lemma 6.1. Moreover, it is not contained in x-axis L_0. If it is contained in the x-axis, then $\rho_t(\lambda) = I$ as $\rho_t(\lambda)$ is always hyperbolic or trivial. So each ρ_t factors through representations of the 0 filling $M(0)$. Therefore $[\rho_t]$ must lie in a component of $X(M(0))$ of dimension at least 1, contradicting the assumption that M is longitudinally rigid.

Since all points in A come from actual $\widetilde{G}$ representations, there is no ideal point in A. As all but at most three Dehn fillings of a knot complement are irreducible [22, Theorem 1.2], we can shrink A if necessary so that none of the Dehn fillings involved is reducible. The only parabolic point in $H_{0,0}(M)$ is the origin so A contains no parabolic point. Applying Lemma 4.8, we get interval $(0, a)$ or $(-a, 0)$ of orderable Dehn fillings.

Finally, we show $M(0)$ is orderable. The first Betti number of $M(0)$ is 1 as rational homology groups of $M(0)$ are the same as $S^2 \times S^1$. The irreducibility of $M(0)$ follows from Lemma 6.2. So we can apply Theorem 1.1 of [4] and show that $\pi_1(M(0))$ is left-orderable, completing the proof of the theorem.

$\square$

Chapter 7

Real embeddings of trace fields and orderability

In this chapter, we use a different assumption for the manifolds we study, and prove Theorem 7.1.

Let Y be a closed hyperbolic 3-manifold with fundamental group Γ. Let $\rho_{hyp} : \Gamma \to PSL_2\mathbb{C}$ be the holonomy representation of Y. The trace field $K = \mathbb{Q}(\mathrm{tr}\Gamma)$ of ρ_{hyp} is the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by the traces of lifts to $SL_2\mathbb{C}$ of all elements in $\rho_{hyp}(\Gamma)$. It is a number field by [30, Theorem 3.1.2]. Assume we have a real embedding σ of the trace field K into $\mathbb{R}$.

Define the associated quaternion algebra to be $D = \{\Sigma a_i\gamma_i | a_i \in K, \gamma_i \in \rho_{hyp}(\Gamma)\}$. To say D splits at the real embedding σ means $D \otimes_\sigma \mathbb{R} \cong M_2(\mathbb{R})$, which implies that we can conjugate Γ into $PSL_2\mathbb{R}$. So we get a Galois conjugate representation $\overline{\rho} : \Gamma \to PSL_2\mathbb{R}$. See Section 2.1 and 2.7 of [30] for more details.

The following conjecture is due to Dunfield.

Conjecture 1. *Suppose M is a hyperbolic $\mathbb{Z}$ homology solid torus. Assume the longitudinal filling $M(0)$ is hyperbolic and its holonomy representation has trace field with a real embedding at which the associated quaternion algebra splits. Then every Dehn filling $M(r)$ with rational r in an interval $(-a, a)$ is orderable.*

By adding some extra conditions, I am able to prove the following result.

Theorem 7.1. *Suppose M is a hyperbolic $\mathbb{Z}$-homology solid torus. Assume the longitudinal filling $M(0)$ is a hyperbolic mapping torus of a homeomorphism of a genus 2 orientable surface and its holonomy representation has trace field with a real embedding at which the associated quaternion algebra splits. Then every Dehn filling $M(r)$ with rational r in an interval $(-a, 0]$ or $[0, a)$ is orderable.*

First let us fix some notations. Denote the holonomy representation of hyperbolic manifold $M(0)$ by $\rho_{hyp} : \pi_1(M(0)) \longrightarrow PSL_2\mathbb{R}$ and the projection map $p : \pi_1(M) \to \pi_1(M(0))$. The composition $\rho_M = p \circ \rho_{hyp}$ has kernel normally generated by the longitude λ. The Galois conjugate of ρ_M is denoted by ρ_0. It is also the Galois conjugate of ρ_{hyp} composed with p. Denote $\rho_V : \pi_1(V) \longrightarrow PSL_2\mathbb{R}$ the induced representation of ρ_{hyp} on $V = S^1 \times D^2 \subset M(0)$, $\rho_{T^2} : \pi_1(T^2) \longrightarrow PSL_2\mathbb{R}$ the induced representation of ρ_{hyp} on $\partial M = T^2$.

Let Γ be a group and let $\rho : \Gamma \to PSL_2\mathbb{C}$ be a representation. Then we can turn the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ into a Γ module via the adjoint representation, which means taking conjugation $g \cdot a := \rho(g)a\rho(g)^{-1}$. Denote this Γ module by $\mathfrak{sl}_2(\mathbb{C})_\rho$.

To study smoothness of a point on the character variety, we need to study the Zariski tangent space at that point.

Definition 7.1. *[35, 3.1.3] Suppose V is an affine algebraic variety in $\mathbb{C}^n$. Let $I(V) = \{f \in \mathbb{C}[x_1, \ldots, x_n] | f(x) = 0 \ \forall x \in V\}$ be the ideal of V. Define the Zariski tangent space to V at p to be the vector space of derivatives of polynomials.*

$$T_p^{Zar}(V) = \{\frac{d\gamma}{dt}|_{t=0} \in \mathbb{C}^n | \gamma \in (\mathbb{C}[t])^n, \gamma(0) = p \ s.t. \ f \circ \gamma \in t^2\mathbb{C}[t] \ \forall f \in I(V)\}$$

A point p on V is called smooth if the dimension of $T_p^{\mathrm{Zar}}(V)$ is equal to the dimension of the component of V which p lies on.

Weil's infinitesimal rigidity in the compact case, which is stated as follows, is key to the proof of Theorem 7.1.

Theorem 7.2. *Let M be a compact 3-manifold with torus boundary whose interior admits a hyperbolic structure with finite volume, then $H^1(M(0), \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) = 0$. [41](See also [35, Section 3.3.3][31])*

The reference [35] works with $SL_2\mathbb{C}$ rather than $PSL_2\mathbb{C}$ character varieties. So to apply argument in [35], we will lift $PSL_2\mathbb{R}$ representations to $SL_2\mathbb{R}$ when necessary. That they always lift is guaranteed by [13, Proposition 3.1.1].

The proof of Theorem 7.1 relies on the following lemma whose proof is based on Weil's theorem.

Lemma 7.1. *Suppose ρ_0 is defined as above. Then there exists an arc c in $R_G(M)$ such that* *(a) $c \ni \rho_0$ is a smooth point of $R_G(M)$.*

(b) tr_γ^2 is the local parameter of arc c near ρ_0, where $\gamma \in \pi_1(\partial M)$ is some primitive element different from the longitude λ.

Proof. (a) First, let us prove ρ_0 is a smooth point of $R_G(M)$. We compute the Mayer-Vietoris sequence for cohomology with local coefficient, associated to decomposition $M(0) = M \cup_{\partial M} V$.

$$
\begin{aligned}
\cdots \to\ & H^1(M(0); \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) \\
& \to H^1(V; \mathfrak{sl}_2(\mathbb{C})_{\rho_V}) \oplus H^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho_M}) \to H^1(T^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_{T^2}}) \to \\
& \to H^2(M(0); \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) \to \cdots
\end{aligned}
$$

It follows from Weil's infinitesimal rigidity 7.2 that $H^1(M(0); \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) = 0$. So $H^1(V; \mathfrak{sl}_2(\mathbb{C})_{\rho_V}) \oplus H^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho_M}) \to H^1(T^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_{T^2}})$ is an injection. To see that it is actually an isomorphism, note that by Poincare duality $H^2(M(0); \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) \cong H^1(M(0); \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) = 0$.

Let $X_c(M)$ be the component of $X(M)$ containing $[\rho_M]$. As ρ_V and ρ_{T^2} are nontrivial, by [3, Theorem 1.1 (i)], we get $\dim_{\mathbb{C}} H^1(V; \mathfrak{sl}_2(\mathbb{C})_{\rho_V}) = 1$ and $\dim_{\mathbb{C}} H^1(T^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_{T^2}}) = 2$. So $\dim_{\mathbb{C}} H^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho_M}) = 1$. By [35, Proposition 3.5], we have inclusion of the Zariski tangent space $T^{\mathrm{Zar}}_{\rho_M}(X_c(M)) \hookrightarrow H^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho_M})$. So $\dim_{\mathbb{C}} T^{\mathrm{Zar}}_{\rho_M}(X_c(M)) \leq \dim_{\mathbb{C}} H^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho_M}) = 1$.

Following from Thurston's result [13, Proposition 3.2.1], $\dim X_c(M) \geq 1$ as $\rho_M(\mathrm{im}(\pi_1(\partial M) \to \pi_1(M))) = \mathbb{Z}$. Since $\dim_{\mathbb{C}} X_c(M) \leq \dim_{\mathbb{C}} T^{\mathrm{Zar}}_{\rho_M}(X_c(M))$, then $\dim X_c(M) = \dim T^{\mathrm{Zar}}_{\rho_M}(X_c(M)) = \dim_{\mathbb{C}} H^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho_M}) = 1$. Therefore $[\rho_M]$ is a smooth point of $X(M)$.

To show the Galois conjugate ρ_0 of ρ_M is also a smooth point, we use the same argument as in the proof of [11, Lemma 8.3]. Construct X_1 by taking the $\mathbb{C}$-irreducible component X_0 of $X(M)$ containing $[\rho_M]$, which must be defined over some number field, and then take the union of the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit of X_0. Then X_1 is the unique $\mathbb{Q}$-irreducible component of $X(M)$ that contains $[\rho_M]$. Since X_1 is invariant under the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-action, it contains $[\rho_0]$ as well as $[\rho_M]$. As by definition, $T^{\mathrm{Zar}}_{\rho_M}(X(M))$ is defined by derivatives of a set of polynomials. Then $T^{\mathrm{Zar}}_{\rho_0}(X(M))$ is defined by derivatives of Galois conjugates of this set of polynomials and thus should have dimension 1, same as $T^{\mathrm{Zar}}_{\rho_M}(X(M))$. Any component of X_1 has the same dimension as $X_c(M)$, which is 1. So $[\rho_0]$ is a smooth point of X_1 and thus of $X(M)$.

Moreover, By Théorème 3.15 of [35], $[\rho_M]$ is γ-regular (see [35, Definition 3.21] for definition) for some simple closed curve $\gamma \subset \partial M$. So tr_γ is a local parameter $X(M)$ at $[\rho_M]$. Since $[\rho_M]$ is not λ-regular as $\rho_M(\lambda) = I$, γ must be a curve different from λ. Locally the sign of tr_γ does not change, so we could make tr^2_γ the local parameter. Whether a regular function is a local parameter at a smooth point on the curve X_1 can be expressed purely algebraically and hence is $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. It follows that $[\rho_0]$ is also a smooth point of X_1 with local parameter tr^2_γ.

Applying [11, Proposition 2.8], we get a smooth arc $\overline{c}$ of real points in $X_{\mathbb{R}}(M)$ containing $[\rho_0]$, locally defined by tr^2_γ being real. By restricting ϵ if necessary, we can assume that every character in $\overline{c}$ comes from an irreducible $PSL_2\mathbb{C}$ representation. Since $[\rho_0] \in X_{PSL_2\mathbb{R}}(M)$ is irreducible, we can restrict ϵ so that $\overline{c}$ is actually contained in $X_{PSL_2\mathbb{R}}(M)$ as both $X_{PSL_2\mathbb{R}}(M)$ and $X_{SU_2(\mathbb{C})}(M)$ are closed in $X(M)$[11, Lemma 2.12]. Then by [11, Lemma 2.11] we can lift $\overline{c}$ to $c \in R_{PSL_2\mathbb{R}}(M)$ and c is still parametrized by tr^2_γ. $\square$

Lemma 7.2. *trans*$(\widetilde{\rho_0}(\lambda))$ *is an even integer.*

Proof. When mapping down to $SL_2\mathbb{R}$, the image of $\widetilde{\rho_0}(\lambda) \in \widetilde{PSL_2\mathbb{R}}$ is I. It follows from [11, Claim 8.5] that $\mathrm{trans}(\widetilde{\rho_0}(\lambda))$ is an even integer. $\square$

Now we are ready to prove Theorem 7.1.

Proof of Theorem 7.1. First we lift the arc $c \subset R_G(M)$ as constructed in Lemma 7.1 to $\tilde{c} \in R_{\widetilde{G}}(M)$. In the case of hyperbolic integer solid torus M, $H^2(\pi_1(M); \mathbb{Z}) \cong H^2(M; \mathbb{Z}) = 0$, so we can always lift.

Since $M(0)$ admits a complete hyperbolic structure, elements in $\pi_1(M(0))$ are mapped to loxodromic elements in $PSL_2\mathbb{C}$ by ρ_{hyp}. So $\lambda \in \pi_1(M(0))$ mapped to either hyperbolic or elliptic under the Galois conjugate ρ_0. Therefore we divide our proof in two cases according to the image of the longitude λ.

Remark. *We do not consider the case that λ is mapped to parabolic because $\rho_{hyp}(\lambda)$ is hyperbolic and Galois conjugate cannot take norm greater than* 2 *to* 2.

Case 1: λ is mapped to an elliptic element.

At $\widetilde{\rho_0}$, the local parameter $s = \mathrm{tr}^2(\widetilde{\rho_0}(\gamma)) < 4$. As $\tilde{c}$ is parameterized near $\widetilde{\rho_0}$ by $\mathrm{tr}^2_\gamma \in [s-\epsilon, s+\epsilon]$, we can require $s+\epsilon < 4$ so that $\tilde{c} \subset PE_{\widetilde{G}}(M)$. Then we map $\tilde{c}$ down to arc $A \subset EL_{\widetilde{G}}(M)$ which is locally parameterized by tr^2_γ on some small interval $[0, \delta]$.

To obtain an interval of orderable Dehn fillings, we want to apply Lemma 8.4 of [11] which works similarly as Lemma 4.8. So we need to show that A is not contained in the horizontal axis L_0 of $EL_{\widetilde{G}}(M) \subset \mathbb{R}^2$. If it is contained in L_0, suppose $\mathrm{trans}(\widetilde{\rho_0}(\lambda))$ equals integer k, which implies every representation $\rho_t \in c$ satisfies $\mathrm{trans}(\widetilde{\rho_t}(\lambda)) = k$. Then $\rho_t(\lambda) = \pm I$ since $\rho_t(\lambda)$ is either elliptic or trivial. So all ρ_t factor through $\pi_1(M(0))$ and it follows that $[\rho_t]$ lie in an irreducible component of $X(M(0))$ with complex dimension at least one. But we have seen that $H^1(M(0); \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) = 0$, so $1 \leq \dim T^{\mathrm{Zar}}_{\rho_0}(X(M(0))) = \dim T^{\mathrm{Zar}}_{\rho_{hyp}}(X(M(0))) \leq \dim_{\mathbb{C}} H^1(M(0); \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) = 0$, which is a contradiction.

Now we can draw arc A inside the translation extension locus $EL_{\widetilde{G}}(M)$ near $\widetilde{\rho_0}$. It contains no ideal point as all points on A come from $\widetilde{G}$ representations. Applying Lemma 8.4 of [11], we get $a > 0$ so that L_r meets $EL_{\widetilde{G}}(M)$ for all r in interval $(-a, a)$. Invoking [22, Theorem 1.2p], we can shrink a to make $M(r)$ irreducible. Then we can apply Lemma 4.4 of [11].

Case 2: λ is mapped to a hyperbolic element.

This case is similar to Case 1 except we start with s=$\mathrm{tr}^2(\widetilde{\rho_0}(\gamma)) > 4$. As $\tilde{c}$ is parameterized by $\mathrm{tr}^2_\gamma \in [s-\epsilon, s+\epsilon]$, we can require $s - \epsilon > 4$ so that $\tilde{c} \subset PH_{\widetilde{G}}(M)$. Again map $\tilde{c}$ down to arc $A \subset HL_{\widetilde{G}}(M)$ which is locally parameterized by tr^2_γ on some small interval $[-\delta, \delta]$.

To show $A \subset H_{0,0}(M)$, we compute $\mathrm{trans}(\widetilde{\rho_0}(\lambda))$ and show it is 0. By assumption, $M(0)$ is a mapping torus of a homeomorphism of a genus 2 surface S. Then $M(0) = M_\phi$ where ϕ is a pseudo-anosov map of S since $M(0)$ is hyperbolic. Suppose there is a G representation ρ_0 of $\pi_1(M(0))$, then it restricts to a G representation $\rho_0|_S$ of $\pi_1(S)$. Let $\mathrm{eu}(\rho_0|_S)$ be the Euler number of $\rho_0|_S$ as defined in [20] (or equivalently in [32, 42]). It is equal to $\mathrm{trans}(\widetilde{\rho_0}([a_1, b_1][a_2, b_2]))$ with a_1, b_1, a_2, b_2 the standard generators of $\pi_1(S)$ and is thus equal to $\mathrm{trans}(\widetilde{\rho_0}(\lambda))$. We claim that $|\mathrm{eu}(\rho_0|_S)| \neq 2$. Otherwise $\rho_0|_S$ would determines a hyperbolic structure on S (Milnor-Wood inequality [32, 42]) which is invariant under ϕ, implying that ϕ has finite order which contradicts that ϕ is pseudo-anosov. So $|\mathrm{trans}(\widetilde{\rho_0}(\lambda))| = |\mathrm{eu}(\rho_0|_S)| \neq 2$. By Lemma 7.2 and Proposition 4.1, we must have $\mathrm{trans}(\widetilde{\rho_0}(\lambda)) = 0$.

Claim that A is not contained in the horizontal axis L_0 of $H_{0,0} \subset \mathbb{R}^2$. If it is contained in the horizontal axis, then $\rho_t(\lambda) = \pm I$ since $\rho_t(\lambda)$ is either hyperbolic or trivial. So all ρ_t factor through $\pi_1(M(0))$ and it follows that $[\rho_t]$ lie in an irreducible component of $X(M(0))$ with complex dimension at least one. But we have seen that $H^1(M(0); \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) = 0$, so $1 \leq \dim T^{\mathrm{Zar}}_{\rho_0}(X(M(0))) = \dim T^{\mathrm{Zar}}_{\rho_{hyp}}(X(M(0))) \leq \dim_{\mathbb{C}} H^1(M(0); \mathfrak{sl}_2(\mathbb{C})_{\rho_{hyp}}) = 0$, which is a contradiction.

So we have constructed arc $A \subset H_{0,0}(M)$ that is not contained in L_0 near $\widetilde{\rho_0}$. Then we can find $a > 0$ such that L_r meets $H_{0,0}(M)$ at points that are not parabolic or ideal and $M(r)$ irreducible for all r in interval $(0, a)$ or $(-a, 0)$. Applying Lemma 4.8 then tells us $M(r)$ is orderable for r in $(0, a)$ or $(-a, 0)$.

Finally, we show $M(0)$ is orderable. The first Betti number of $M(0)$ is 1 as the integral homology groups of $M(0)$ are the same as those of $S^2 \times S^1$. The irreducibility of $M(0)$ follows from the assumption that it is hyperbolic. So we can apply Theorem 1.1 of [4] and show that $\pi_1(M(0))$ is left-orderable, completing the proof of the theorem.

$\square$

Remark. *The assumption that $M(0)$ being a mapping torus of genus 2 is used to show trans$(\lambda) = 0$. It is a very strong. However, when λ is mapped to elliptic, $M(0)$ being a mapping torus in not needed at all. When λ is mapped to hyperbolic, the author does not know how to weaken this assumption.*

Using the method of Calegari [7, Section 3.5], we are able to prove the following result.

Lemma 7.3. *Suppose M is a mapping torus of closed surface S of genus at least 2 and $\pi_1(M)$ has no torsion. If M has a faithful G representation ρ. Then $\rho|_S$ can never be discrete.*

Proof. First notice that ρ is indiscrete, as otherwise $\rho(\pi_1(M)) \leq G$ acts on $\mathbb{H}^2$ with quotient a hyperbolic surface, which is impossible as M is a closed 3 manifold.

Now suppose $\rho|_S$ is discrete, then $\rho|_S$ determines some hyperbolic structure on S as it is faithful. So $\rho(\pi_1(S))$ consists of hyperbolic elements only. Moreover, any isometry of S is of finite order as it has to preserve the hyperbolic structure. Let $\pi_1(M) = \langle t \rangle \ltimes \pi_1(S)$. Then $\rho(t)$ acts on $\rho(\pi_1(S))$ by conjugation and normalizes $\rho(\pi_1(S))$. Since $\mathrm{Isom}^+(S)$ is of finite order, the action of $\rho(t)$ on $\rho(\pi_1(S))$ by conjugation is of finite order. To show that actually $\rho(t)$ is a finite order element in G, notice that $\rho(\pi_1(S))$ has at least two hyperbolic elements of different axes. But this contradicts the fact that ρ is a faithful representation as $\pi_1(M)$ has no torsion. So $\rho|_S$ could not be discrete. $\square$

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