SUFFICIENT DEGREE CONDITIONS FOR GRAPH EMBEDDINGS

BY

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DISSERTATION

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Abstract

In this dissertation, we focus on the sufficient conditions to guarantee one graph being the subgraph of another. In Chapter 2, we discuss *list packing*, a modification of the idea of graph packing. This is fitting one graph in the complement of another graph. Sauer and Spencer showed a sufficient bound involving maximum degrees, and this was further explored by Kaul and Kostochka to characterize all extremal cases. Bollobás and Eldridge (and independently Sauer and Spencer) developed edge sum bounds to guarantee packing. In Chapter 2, we introduce the new idea of list packing and use it to prove stronger versions of many existing theorems. Namely, for two graphs, if the product of the maximum degrees is small or if the total number of edges is small, then the graphs pack.

In Chapter 3, we discuss the problem of finding k vertex-disjoint cycles in a multigraph. This problem originated from a conjecture of Erdős and has led to many different results. Corrádi and Hajnal looked at a minimum degree condition. Enomoto and Wang independently looked at a minimum degree-sum condition. More recently, Kierstead, Kostochka, and Yeager characterized the extremal cases to improve these bounds. In Chapter 3, we improve on the multigraph degree-sum result. We characterize all multigraphs that have simple Ore-degree at least $4k-3$, but do not contain k vertex-disjoint cycles. Moreover, we provide a polynomial time algorithm for deciding if a graph contains k vertex-disjoint cycles.

Lastly, in Chapter 4, we consider the same problem but with chorded cycles. Finkel looked at the minimum degree condition while Chiba, Fujita, Gao, and Li addressed the degree-sum condition. More recently, Molla, Santana, and Yeager improved this degree-sum result, and in Chapter 4, we will improve on this further.

To Simone, my parents, my sister, and my dog. You all made sure that I lived a life outside of this dissertation.

Acknowledgments

I've done most of the things in my life by setting a big goal, and then I achieve it by keeping my head down and focusing on the small goal at hand. This has allowed me to complete the big goals of writing a dissertation and earning a doctorate, but it is only now that I truly am taking a moment to lift my head to look back at everything I've accomplished and how I got here. There are so many people that I have to thank that it could be another hundred pages on its own.

I want to thank all of my family. To my mom and dad, all of our lives have changed immensely over the years, but the one constant I could count on was your love and support. Failure is in the back of the minds of many graduate students, but I always knew that, if I fell, you would be there to catch me. To my sister, you are so smart, considerate, and adaptable. Your opinion on anything means a great deal to me, and I know that whenever you speak, I should listen. Your advice has helped me countless times along the way. To Simone, you are my heart. I truly do not know what to say, but I do know that when I failed you were there, when I succeeded you were there, and your love was constant throughout. This dissertation would have been impossible without that love.

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And now, I need to thank my dog, Indy. Unless she has been studying on her own, she will not be able to read this, but I figure the Acknowledgments section should acknowledge all those who contributed regardless of their ability to read it. You always knew when I needed to get out of bed, get out of the house, or just take a break to play around a little. I did most of the work for this dissertation with you at my side or at my feet. I will never forget your loyalty and all that you gave me.

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List of Symbols

$[n]$	Set of integers $\{1, \ldots, n\}$.
$V(G)$	Vertex set of a graph G.
$\lvert G\rvert$	Number of vertices in a graph G, i.e. order of G.
$E(G)$	Edge set of a graph G.
$\lVert G\rVert$	Number of edges in a graph G, i.e. size of G.
$d_G(v)$	Degree of v, i.e. the number of edges incident to v, denoted $d(v)$ when G is implied.
$N_G(v)$	Neighborhood of v, i.e. $\{u \in V(G) : uv \in E(G)\}$.
$\delta(G)$	Minimum degree of a vertex in a graph G.
$\Delta(G)$	Maximum degree of a vertex in graph G.
$\sigma_2(G)$	Minimum sum of degrees of two nonadjacent vertices, i.e. $\min\{d(x) + d(y) : xy \notin E(G)\}$
$\lVert S, T\rVert$	Number of edges with one endpoint in S and the other endpoint T.
$H \subseteq G$	H is a subgraph of a graph G.
$G[S]$	Subgraph of a graph G induced by the set $S \subseteq V(G)$.
$\alpha(G)$	Independence number of a graph G.
K_n	Complete graph on n vertices.
$K_{n,m}$	Complete bipartite graph with bipartite sets of size n and m.
$\overline{G}$	Complement of a graph G (requires G being simple).
$k\,G$	The disjoint union of k copies of a graph G.
C_n	Cycle on n vertices.

$G \cup H$ For multigraphs G and H, this denotes the multigraph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

$G \vee H$ For graphs G and H, this denotes $G \cup H$ together with all edges from $V(G)$ to $V(H)$.

$K(X)$ The complete graph with vertex set X. We use $K_t(X) = K(X)$ to indicate that $|X| = t$. If we only want to specify one vertex v of K_t we write $K_t(v)$.

$\underline{G}$ The underlying simple graph of G, i.e. the simple graph on $V(G)$ such that two vertices are adjacent in $\underline{G}$ if and only if they are adjacent in G.

$\mathcal{DO}_k$ The class of multigraphs G whose underlying simple graph $\underline{G}$ satisfies $\sigma_2(\underline{G}) \geq 4k - 3$.

$s_G(v)$ The simple degree, i.e. $s_G(v) = d_{\underline{G}}(v)$, also denoted $s(v)$ when G is clear.

$\mathcal{S}(G)$ The minimum simple degree, i.e. $\mathcal{S}(G) = \delta(\underline{G})$.

$\mathcal{S}O(G)$ The minimum simple Ore-degree, i.e. $\mathcal{S}O(G) = \sigma_2(\underline{G})$.

$c(G)$ The maximum number of disjoint cycles contained in G.

Chapter 1

Introduction

Ever since Euler, mathematicians have been looking to find graphs situated within other graphs. A graph H is a subgraph of a graph G, denoted $H \subseteq G$, simply if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Since this depends on labeling the vertices of a graph, we use the same notation to represent when H is isomorphic to a subgraph of G. For instance, C_n is a cycle on n vertices, and we can consider whether or not $C_n \subseteq G$.

One well-known property is whether or not a graph contains a *Hamiltonian cycle*. A Hamiltonian cycle is a cycle that passes through every vertex of a graph and uses each edge at most once. Therefore, finding a Hamiltonian cycle in a graph G on n vertices simply amounts to determining if $C_n \subseteq G$. There are many problems of a similar type. Mathematicians have looked at the number of triangles in a graph, paths of a given length in a graph, or even graphs with only large cycles. This is the main theme of this thesis, and we will be answering variants of the question:

For a graph H meeting some given set of conditions,

what conditions on a host graph G guarantee that $H \subseteq G$?

In Chapter 2, we discuss *list packing*. The original idea of graph packing can be stated as follows: Given two graphs G and H both on n vertices, we say that G and H pack if $H \subseteq \overline{G}$, i.e. H is a subgraph of the complement of G. Equivalently, can we place each graph on the same set of n vertices in an edge-disjoint manner? This is rather easy if G and H have few edges, but becomes more complicated as the number of edges increases. Another complication occurs when there are vertices of high degree. In Chapter 2, we introduce the new idea of list packing and use it to prove stronger versions of existing theorems by Bollobás and Eldridge and also Sauer and Spencer. Namely, for two graphs, if the product of the maximum degrees is small or if the total number of edges is small, then the graphs pack.

In Chapter 3, we discuss the problem of finding vertex-disjoint cycles in a multigraph. We do not restrict the size of the cycles so that the problem of finding k disjoint cycles in a multigraph G is the same as determining if there exist $a_1, \ldots, a_k \geq 1$ such that $C_{a_1} + \cdots + C_{a_k} \subseteq G$. Here, $G + H$ is the same as $G \cup H$ but with $V(G) \cap V(H) = \emptyset$. Note that here we allow 1- and 2-cycles so that indeed we are considering *multigraphs*. For this problem, large independent sets, i.e. vertex subsets with no edges among them, in the graph can form barriers to finding cycles. In Chapter 3, we characterize all multigraphs that have simple Ore-degree at least $4k - 3$ but do not contain k vertex-disjoint cycles for $k \geq 5$. Moreover, we provide a polynomial time algorithm for deciding if a graph contains k vertex-disjoint cycles or not.

Lastly, in Chapter 4, we consider the same problem but with chorded cycles. A *chorded* cycle is simply a cycle with at least one extra edge. A large independent set still serves as a barrier to chorded cycles. In Chapter 4, we will characterize the family of graphs for which the Ore-degree is at least $6k - 3$ but that do not contain k vertex-disjoint chorded cycles.

For the remainder of this chapter, we will provide foundation and background for the main results of this dissertation. In Subsection 1.1, we define many terms that will appear throughout the dissertation. In each of the following subsections, we provide background, and we conclude with the main theorem for the topic. For example, Subsection 1.2 provides some background and theorems, and then it concludes with the main theorem from Chapter 2. Thus, the introduction provides the reader with the context and results of my research, and the remainder of the dissertation will prove these results to be true.

1.1 Notation

We mostly use standard definitions in graph theory. Here, we present some of those definitions along with some of the more important definitions used in this thesis.

Definition 1.1. A *graph* G is an ordered pair (V, E) where $V = V(G)$ is the vertex set and

$E = E(G)$ is the edge set of G with each edge being a subset of the vertices in V of size one or two. A graph is a *simple graph* whenever the edge set does not contain loops or multiple edges. A *loop* is an edge consisting of a single vertex, and a *multiple edge* is an edge that appears multiple times in the same edge set. For any multigraph G, we still can refer to its underlying simple graph $\underline{G}$, i.e. the simple graph on $V(G)$ such that two vertices are adjacent in $\underline{G}$ if and only if they are adjacent in G.

Definition 1.2. For a loopless graph G and a vertex $v \in V(G)$, the *neighborhood* of v, denoted $N_G(v)$ is the set of vertices in $V(G)$ adjacent to v. The *degree* of v, $d_G(v)$ is the number of edges incident to v, i.e. $d_G(v) = |\{uv \in E(G) : u \in V(G)\}|$. To also consider loops, we make each loop edge contribute 2 to the degree. If $u \in N_G(v)$, we say that u is a *neighbor* of v. Similarly, for a set $S \subseteq V(G)$, we write $N_G(S)$ for the set of vertices in $V(G) - S$ with at least one neighbor in S. When the graph G is clear from context, we will write $N(v)$, $N(S)$, and $d(v)$ instead of $N_G(v)$, $N_G(S)$, and $d_G(v)$, respectively. For a multigraph, we use $s_G(v)$ to denote the simple degree, i.e. $s_G(v) = |N_G(v)| = d_{\underline{G}}(v)$, where we again use $s(v)$ when G is clear.

Definition 1.3. For a graph G, $\delta(G)$ is the minimum degree of a vertex in G and $\Delta(G)$ is maximum degree of a vertex in G. For a multigraph G, we use $\mathcal{S}(G)$ to denote the minimum simple degree, i.e. $\mathcal{S}(G) = \delta(\underline{G})$.

Definition 1.4. For disjoint sets $S, T \subseteq V(G)$, we write $\|S, T\|_G$ for the number of edges from S to T. If $S = \{u\}$, then we will write $\|u, T\|_G$ instead of $\|\{u\}, T\|_G$. When the graph G is clear from context, we will simplify the notation to $\|S, T\|$ and $\|u, T\|$, respectively.

Definition 1.5. For graphs H and G, we say that H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 1.6. Let G be a graph and $S \subseteq V(G)$. The *subgraph induced by* S, denoted $G[S]$, has vertex set S and $E(G[S]) = \{e \subseteq S : e \in E(G)\}$.

Definition 1.7. For a graph G, a set $S \subseteq V(G)$ is *independent* if $G[S]$ contains no edges. The *independence number* of G, denoted $\alpha(G)$ is the size of the largest independent set in G.

Definition 1.8. The *complete graph* on n vertices, denoted K_n, is a simple graph with n vertices and $E(K_n) = \binom{V(K_n)}{2}$, i.e. all possible edges.

Definition 1.9. The *complete bipartite graph* $K_{m,n}$ has vertex set $X \cup Y$, with $|X| = m$ and $|Y| = n$, and edge set $\{xy : x \in X \text{ and } y \in Y\}$.

Definition 1.10. The *complement* of a simple graph G, denoted $\overline{G}$, is a simple graph with vertex set $V(G)$ and edge set $\left\{e \in \binom{V(G)}{2} : e \notin E(G)\right\}$.

Definition 1.11. A *matching* M is an edge set where no two distinct edges share a common endpoint. A matching $M \subseteq E(G)$ is a *perfect matching* if every vertex in $V(G)$ is the endpoint of some edge in M.

Definition 1.12. A *cycle* on n vertices, denoted C_n, is a connected graph with $d(v) = 2$ for each $v \in V(C_n)$.

Definition 1.13. Two graphs are *disjoint* if they have no vertices in common. Hence, by "disjoint", we mean "vertex-disjoint" unless otherwise noted.

Definition 1.14. The *union* of two graphs G and H, denoted $G \cup H$, is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The disjoint union of G and H is denoted $G + H$, and for an integer k, k copies of G is denoted by $k\,G$.

Definition 1.15. The *join* of two graphs G and H, denoted $G \vee H$, is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G) \text{ and } y \in V(H)\}$.

Definition 1.16. A subset $W \subseteq V(G)$ of vertices in G is a *clique* if $G[W]$ is a complete graph.

Definition 1.17. For a vertex $v \in V(G)$, $G - v$ denotes the graph obtained from G by removing the vertex v and all edges incident to v.

Definition 1.18. Consider the graph triple (G_1, G_2, G_3) with $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_1 \cup V_2, E_3)$ where $|V_1| = |V_2| = n$. A *list packing* of the graph triple is a bijection $f : V_1 \to V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and for each $u \in V_1$, $uf(u) \notin E_3$. We say graphs G_1 and G_2 *pack* if (G_1, G_2, G_3) has a list packing where $G_3 = 2nK_1$, i.e. a graph with no edges..

1.2 List Packing (Chapter 2)

The notion of graph packing is a well-known concept in graph theory and combinatorics. Two graphs on n vertices are said to *pack* if there is an edge-disjoint placement of the graphs onto the same set of vertices. In 1978, two seminal papers, [1] and [2], on extremal problems on graph packing appeared in the same journal. In particular, Sauer and Spencer [1] proved sufficient conditions for packing two graphs with bounded product of maximum degrees.

Theorem 1.19. *(Sauer-Spencer [1]) Let G_1 and G_2 be two graphs of order n. If $2\Delta(G_1)\Delta(G_2) < n$, then G_1 and G_2 pack.*

This result is sharp and later Kaul and Kostochka [3] characterized all graphs for which Theorem 1.19 is sharp.

Theorem 1.20. *(Kaul-Kostochka [3]) Let G_1 and G_2 be two graphs of order n and $2\Delta(G_1)\Delta(G_2) \leq n$. Then G_1 and G_2 do not pack if and only if one of G_1 and G_2 is a perfect matching and the other either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.*

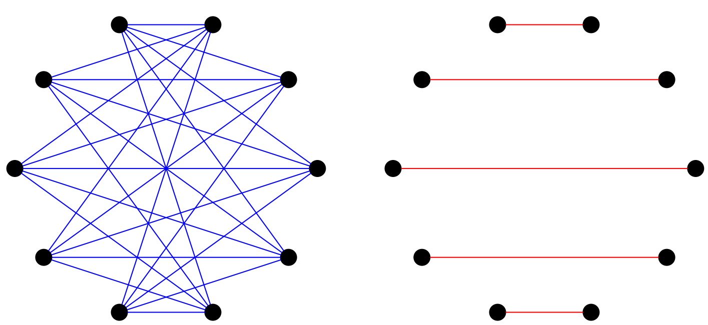


Figure 1.1: $K_{n/2,n/2}$ and $\frac{n}{2}K_2$ ($n = 10$ shown)

In the same paper, Sauer and Spencer gave sufficient conditions for packing two graphs with given total number of edges.

Theorem 1.21. *(Sauer-Spencer [1]) Let G_1 and G_2 be two graphs of order n. If $|E(G_1)| + |E(G_2)| \leq \frac{3}{2}n - 2$, then G_1 and G_2 pack.*

This result is best possible, since $G_1 = K_{1,n-1}$ and $G_2 = \frac{n}{2}K_2$ do not pack, see Figure 1.2. Independently, Bollobás and Eldridge [2] proved the stronger result that the bound of Theorem 1.21 can be significantly strengthened when $\Delta(G_1), \Delta(G_2) < n - 1$.

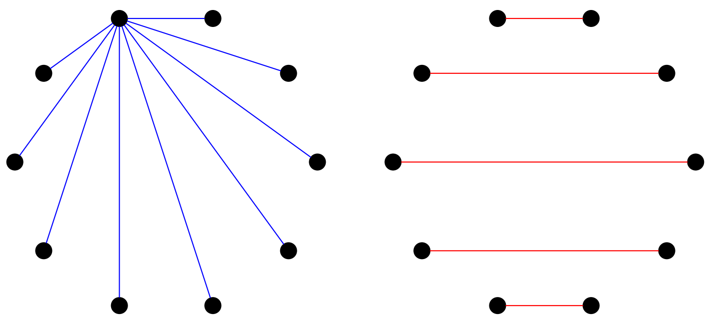


Figure 1.2: $K_{1,n-1}$ and $\frac{n}{2}K_2$ ($n = 10$ shown)

Theorem 1.22. *(Bollobás-Eldridge [2]) Let G_1 and G_2 be two graphs of order n. If $\Delta(G_1)$, $\Delta(G_2) \leq n-2$, $|E(G_1)| + |E(G_2)| \leq 2n-3$, and $\{G_1, G_2\}$ is not one of the following pairs: $\{2K_2, K_1 \cup K_3\}, \{\overline{K}_2 \cup K_3, K_2 \cup K_3\}, \{3K_2, \overline{K}_2 \cup K_4\}, \{\overline{K}_3 \cup K_3, 2K_3\}$, $\{2K_2 \cup K_3, \overline{K}_3 \cup K_4\}, \{\overline{K}_4 \cup K_4, K_2 \cup 2K_3\}, \{\overline{K}_5 \cup K_4, 3K_3\}$ (Figure 1.3), then G_1 and G_2 pack.*

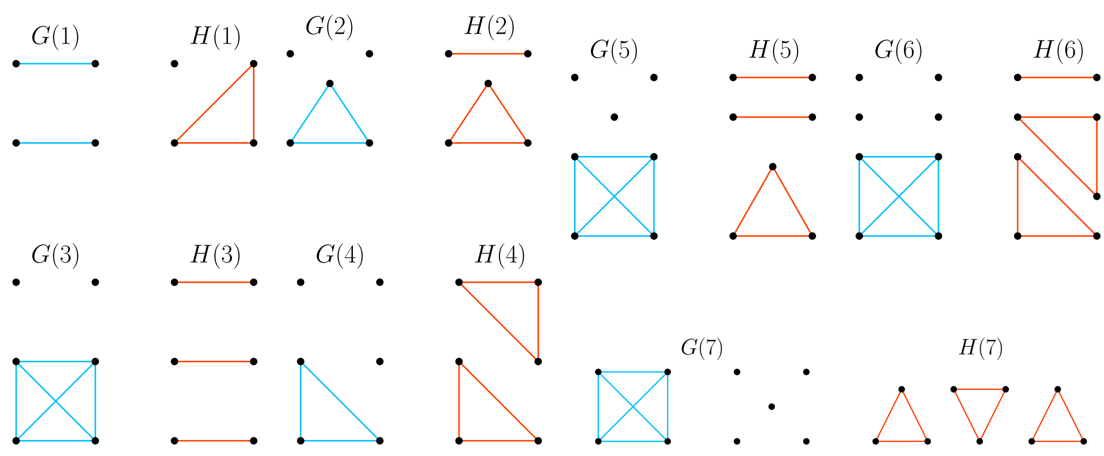


Figure 1.3: Bad pairs in Theorems 1.22 and 1.24.

This result is also sharp, since the graphs $G_1 = C_n$ and $G_2 = K_{1,n-2} \cup K_1$ satisfy the maximum degree conditions, have $2n-2$ edges, and do not pack. There are other extremal examples.

Variants of the packing problem have been studied and, in particular, restrictions of permissible packings arise both within proofs and are posed as independent questions. The notion of a bipartite packing was introduced by Catlin [4] and was later studied by Hajnal and Szegedy [5]. This variation of traditional packing involves two bipartite graphs $G_1 = (X_1 \cup Y_1, E_1)$ and $G_2 = (X_2 \cup Y_2, E_2)$ where permissible packings send X_1 onto X_2 and Y_1

onto Y_2. The problem of fixed-point-free embeddings, studied by Schuster in 1978, considers a different restriction to the original packing problem [6]. In this case, $G_1 = G$ is packed with $G_2 = G$ under the additional restraint that no vertex of G_1 is mapped to its copy in G_2. In [7], Schuster's result is used to prove a necessary condition for packing two graphs with given maximum and average degree bounds.

In Chapter 2, we will generalize the idea of graph packing to *list packing* and prove stronger theorems that will imply Theorems 1.19, 1.20, 1.21, and 1.22. Specifically, with Definition 1.18, we prove

Theorem 1.23. *Let* $\mathbf{G} = (G_1, G_2, G_3)$ *be a graph triple with* $|V_1| = |V_2| = n$. *If*

$$\Delta(G_1)\Delta(G_2) + \Delta(G_3) \leq n/2, \tag{1.1}$$

then $\mathbf{G}$ *does not pack if and only if* $\Delta(G_3) = 0$ *and one of* G_1 *or* G_2 *is a perfect matching and the other is* $K_{\frac{n}{2},\frac{n}{2}}$ *with* $\frac{n}{2}$ *odd or contains* $K_{\frac{n}{2}+1}$. *Consequently, if* $\Delta(G_1)\Delta(G_2)+\Delta(G_3) < n/2$, *then* $\mathbf{G}$ *packs.*

Theorem 1.24. *Let* $\mathbf{G} = (G_1, G_2, G_3)$ *be a graph triple with* $|V_1| = |V_2| = n$. *If* $\Delta(G_1), \Delta(G_2) \leq n-2$, $\Delta(G_3) \leq n-1$, $|E_1| + |E_2| + |E_3| \leq 2n-3$ *and the pair* $\{G_1, G_2\}$ *is none of the 7 pairs in Figure 1.3, then* $\mathbf{G}$ *packs.*

1.3 Cycles (Chapter 3)

The problem of finding the maximum number of disjoint cycles in a graph is NP-hard, since even a partial case of it is:

Theorem 1.25. *(Garey-Johnson [8]) Determining whether a* $3n$*-vertex graph has* n *disjoint triangles is an* NP*-complete problem.*

On the other hand, Bodlaender [9] and independently Downey and Fellows [10] showed that this problem is *fixed parameter tractable*:

Theorem 1.26. *(Bodlaender [9], Downey-Fellows [10]) For every fixed* k, *the question whether an* n*-vertex graph has* k *disjoint cycles can be resolved in linear (in* n*) time.*

Since the general problem is hard, it is natural to look for sufficient conditions that ensure the existence of "many" disjoint cycles in a graph. One well-known result of this type is the following theorem of Corrádi and Hajnal [11] from 1963:

Theorem 1.27. *(Corrádi-Hajnal [11]) Let $k \in \mathbb{Z}^+$. Every graph G with $|G| \geq 3k$ and $\delta(G) \geq 2k$ contains k disjoint cycles.*

The hypothesis $\delta(G) \geq 2k$ is best possible, as shown by the $3k$-vertex graph $H = \overline{K}_{k+1} \vee K_{2k-1}$, which has $\delta(H) = 2k - 1$ but does not contain k disjoint cycles. The proof yields a polynomial algorithm for finding k disjoint cycles in the graphs satisfying the conditions of the theorem.

Theorem 1.27 was refined and generalized in several directions. Enomoto [12] and Wang [13] generalized the Corrádi-Hajnal Theorem in terms of the minimum Ore-degree:

Theorem 1.28. *(Enomoto [12], Wang [13]) Let $k \in \mathbb{Z}^+$. Every graph G with $|G| \geq 3k$ and*

$$\sigma_2(G) \geq 4k - 1$$

contains k disjoint cycles.

Kierstead, Kostochka, and Yeager [14] refined Theorem 1.27 by characterizing all simple graphs that fulfill the weaker hypothesis $\delta(G) \geq 2k - 1$ and contain k disjoint cycles. This refinement depends on an extremal graph $\mathbf{Y_{k,k,k}}$ where $\mathbf{Y_{h,s,t}} = \overline{K_h} \vee (K_s \cup K_t)$ and $\mathbf{Y_{h,s,t}}(X_0, X_1, X_2) = \overline{K_h}(X_0) \vee (K_s(X_1) \cup K_t(X_2))$.

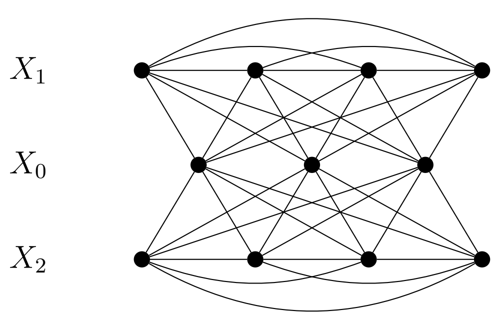


Figure 1.4: $\mathbf{Y_{h,t,s}}$, shown with $h = 3$ and $t = s = 4$.

Theorem 1.29. *(Kierstead-Kostochka-Yeager [14]) Let $k \geq 2$. Every simple graph G with $|G| \geq 3k$ and $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if:*

(i) $\alpha(G) \leq |G| - 2k$;

(ii) *if k is odd and $|G| = 3k$, then $G \neq \mathbf{Y_{k,k,k}}$; and*

(iii) *if $k = 2$ then G is not a wheel.*

Theorem 1.28 was refined in a similar way in [14] and [15] (see Theorem 3.10 in Chapter 3).

Dirac [16] described all 3-connected multigraphs that do not have two disjoint cycles and posed the following question:

Question 1.30 ([16]). Which $(2k-1)$-connected multigraphs[1] do not have k disjoint cycles?

Kierstead, Kostochka, and Yeager [17] used Theorem 1.29 to answer Question 1.30 (see Theorem 3.8).

A *loop* is an edge consisting of a single vertex, and a *strong edge* is an edge with multiplicity greater than one. For every multigraph G, let $V_1 = V_1(G)$ be the set of vertices in G incident to loops, and $V_2 = V_2(G)$ be the set of vertices in $G - V_1$ incident to strong edges. Let $F = F(G)$ be the simple graph with $V(F) = V_2$ and $E(F)$ consisting of the strong edges in $G - V_1$.

In Chapter 3, we will resolve the Ore-type version of Question 1.30 for multigraphs in an algorithmic way. In Theorem 3.12, we consider the class $\mathcal{DO}_k$ of multigraphs G whose underlying simple graph $\underline{G}$ satisfies $d_{\underline{G}}(x) + d_{\underline{G}}(y) \geq 4k-3$ for all nonadjacent vertices x and y, and describe all graphs in $\mathcal{DO}_k$ that do not have k disjoint cycles. Using this description we construct a polynomial time algorithm that, for every multigraph G in $\mathcal{DO}_k$, classifies whether G has k disjoint cycles or not. More explicitly, we prove the following where, for a matching M, we use $W(M)$ to denote its vertex set (Note, Gallai-Edmonds Decomposition will be defined in Chapter 3):

Theorem 1.31. *Let $k \geq 5$ and $n \geq k$ be integers. Let G be an n-vertex multigraph in $\mathcal{DO}_k$ with no loops. Set $F = F(G)$, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Let (D, A, C) be the*

[1] Dirac used the word *graphs*, but in [16] this appears to mean *multigraphs*.

Gallai-Edmonds decomposition of F and let $D' = V(G) - V(F)$. Then G does not contain k disjoint cycles if and only if one of the following holds:

(Q1) $n < 3k - \alpha'$;

(Q2) $n > 2k + 1$, $\alpha(G) = n - 2k + 1$, *and either*

(Q2a) some maximum independent set is not incident to any strong edge, or

(Q2b) for some two distinct maximum independent sets J and J', all strong edges intersecting $J \cup J'$ have a common vertex outside of $J \cup J'$;

(Q3) $k' \geq 5$ *and* $n = 3k - \alpha'$, *and G has a vertex $x \in D'$ of degree $k + \alpha' - 1$ such that for each maximum matching M in F, the set $N(x) - W(M)$ is independent;*

(Q4) $3k - \alpha' \leq n \leq 3k - \alpha' + 1$ *and* $k' \leq 4$ *and* $\underline{G} - W(M)$ *has no* $k - |M|$ *disjoint cycles for all (possibly nonmaximum) matchings M in F; or*

(Q5) $k' \geq 5$ *and* $n = 3k - \alpha'$, $|F| - 2\alpha' \in \{0, |D|-2, |D|-1\}$ *and for all maximum matchings M in F either* $\alpha(G - W(M)) = k' + 1$ *or* $G - W(M) \subseteq \mathbf{Y_{k',c,2k'-c}}$ *for some odd* $c \leq k'$.

Theorem 1.32. *There is a polynomial time algorithm that for every multigraph $G \in \mathcal{DO}_k$ decides whether G has k disjoint cycles or not.*

1.4 Chorded Cycles (Chapter 4)

A natural next step is to consider subgraphs other than cycles. In Chapter 4, we look to find k vertex-disjoint *chorded* cycles. Recall, in 1963, Corrádi and Hajnal verified a conjecture of Erdős.

Theorem 1.33. *(Corrádi–Hajnal [11]) Let $n, k \geq 1$ be integers such that $n \geq 3k$. If $\delta(G) \geq 2k$, then G contains k disjoint cycles.*

Enomoto and Wang then independently strengthened the result by replacing the minimum degree condition with a minimum Ore-degree condition.

Theorem 1.34. *(Enomoto [12], Wang [13]) Every graph G on $|G| \geq 3k$ vertices with $\sigma_2(G) \geq 4k-1$ contains k disjoint cycles.*

Finkel proved the chorded cycle analog of the Corrádi-Hajnal Theorem 1.33.

Theorem 1.35. *(Finkel [18]) Every graph G on $|G| \geq 4k$ vertices with $\delta(G) \geq 3k$ contains k disjoint chorded cycles.*

This was then expanded upon with a result of Chiba, Fujita, Gao, and Li. While looking at a combination of chorded and simple cycles, they were able to prove the following which appears as a corollary in their paper.

Theorem 1.36. *(Chiba–Fujita–Gao–Li [19]) Every graph G on $|G| \geq 4k$ vertices with $\sigma_2(G) \geq 6k-1$ contains a collection of k disjoint chorded cycles.*

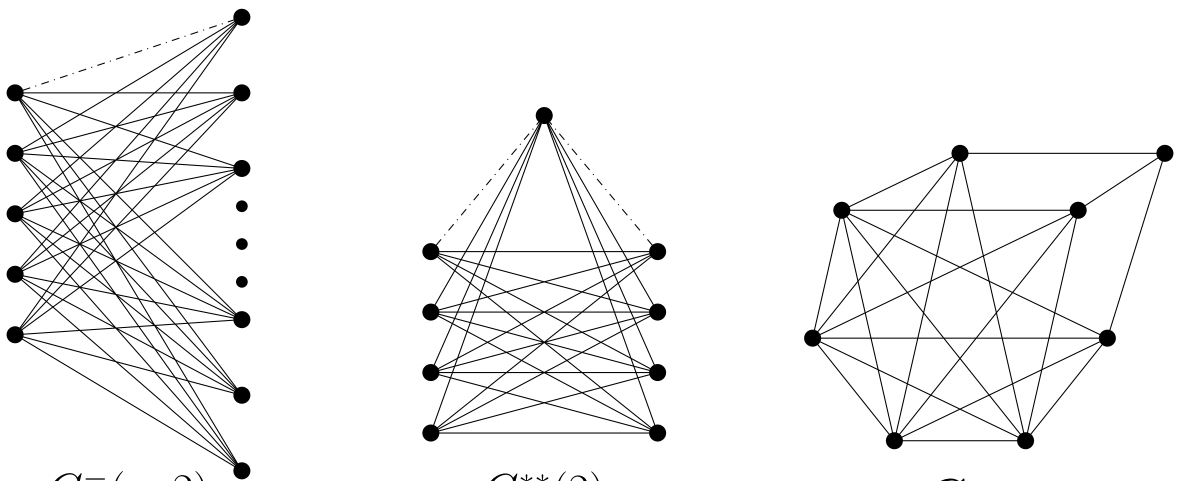
More recently, Molla, Santana, and Yeager characterized what occurs in the extremal case.

Theorem 1.37. *(Molla–Santana–Yeager [20]) For $k \geq 2$, let G be a graph with $n = |G| \geq 4k$ and $\sigma_2(G) \geq 6k-2$. Then G does not contain k disjoint chorded cycles if and only if $G \in \{G_1(n,k), G_2(k)\}$, where $G_1(n,k) = K_{3k-1,n-3k+1}$ for $n \geq 6k-2$ and $G_2(k) = K_{3k-2,3k-2,1}$ for $k \geq 2$.*

We prove that, with minor exceptions, these examples from the Molla, Santana, and Yeager result work when we reduce the lower bound by 1. Our exceptional graphs include $G_1(n,k) = K_{3k-1,n-3k+1}$, $G_2(k) = K_{3k-2,3k-2,1}$, and G_3 which is produced from a K_7 after removing the edges of a triangle T and adding a vertex whose neighborhood is $V(T)$ (see Figure 1.5). Now, if $G_2(k)$ has partite sets $\{v\}, A, B$, then we define $G_2^*(k) = G_2(k) - vx$ for any $x \in A \cup B$ and $G_2^{**}(k) = G_2(k) - vx - vy$ for any $x \in A, y \in B$. But also, or any edge $e \in E(G_1(n,k))$, we define $G_1^-(n,k) = G_1(n,k) - e$. Our exceptional graphs also include $G_1^-(n,k)$, $G_2^*(k)$, and $G_2^{**}(k)$.

Theorem 1.38. *Let $k \geq 2$, G be an n-vertex graph with $n \geq 4k$, and $\sigma_2(G) \geq 6k-3$. Then G does not contain k vertex-disjoint chorded cycles if and only if*

- $G_1^-(n,k) \subseteq G \subseteq G_1(n,k)$,
- $G_2^{**}(k) \subseteq G \subseteq G_2(k)$, *or*
- $G = G_3$.



$G_1^-(n,2)$ $G_2^{**}(2)$ $\mathbf{G_3}$

Figure 1.5: Graphs for Theorem 1.38 with $k = 2$. Dashed lines indicate missing edges.

Chapter 2

List Packing

2.1 Introduction

Recall, in Definition 1.18, we define a *list packing* of the graph triple (G_1, G_2, G_3) with $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$, and $G_3 = (V_1 \cup V_2, E_3)$. It is a bijection $f : V_1 \to V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and for each $u \in V_1$, $uf(u) \notin E_3$. Note that both G_1 and G_2 are graphs on n vertices so that G_3 has $2n$ vertices, and one can think of the edge set E_3 as a list of restrictions that must be avoided when packing G_1 and G_2.

This notion is closely related to Vizing's concept of list coloring [21]. Suppose we wish to color a graph G with the colors $\{1, \dots, k\}$. A *list assignment* L is a function on the vertex set $V(G)$ that returns a set of colors $L(v) \subseteq \{1, \dots, k\}$ permissible for v. A *list coloring*, more specifically an *L-coloring*, is a proper coloring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$. The problem of list coloring G can be stated within the framework of list packing. A proper L-coloring of a graph G is equivalent to a list packing where $G_1 = G$ along with an appropriate number of isolated vertices, G_2 is a disjoint union of K_n's each representing a color, and E_3 consists of all edges going between a vertex $v \in V_1$ and the copies of K_n corresponding to colors *not* in $L(v)$. Note the list $L(v)$ denotes permissible colors in a list coloring while $N_3(v)$ specifies forbidden vertices in a list packing.

Similarly, the variations of packing discussed above can be modeled using this framework. A bipartite packing is a packing of the triple (G_1, G_2, G_3) where E_3 consists of all edges between X_i and Y_{3-i} for $i = 1, 2$. A fixed-point-free embedding is a packing of the triple (G, G, G_3) where $E_3 = \{(v, v) : v \in V(G)\}$. Further, several important theorems on the ordinary packing can be stated in terms of list packing. The results of this paper prove natural generalizations of Theorems 1.19–1.22. In particular, we extend Theorem 1.19 and

Theorem 1.20 as follows.

Theorem 2.1. *Let* $\mathbf{G} = (G_1, G_2, G_3)$ *be a graph triple with* $|V_1| = |V_2| = n$*. If*

$$\Delta(G_1)\Delta(G_2) + \Delta(G_3) \le n/2, \tag{2.1}$$

then $\mathbf{G}$ *does not pack if and only if* $\Delta(G_3) = 0$ *and one of* G_1 *or* G_2 *is a perfect matching and the other is* $K_{\frac{n}{2},\frac{n}{2}}$ *with* $\frac{n}{2}$ *odd or contains* $K_{\frac{n}{2}+1}$*. Consequently, if* $\Delta(G_1)\Delta(G_2)+\Delta(G_3) < n/2$*, then* $\mathbf{G}$ *packs.*

The main result of this paper is the following list version of Theorem 1.22.

Theorem 2.2. *Let* $\mathbf{G} = (G_1, G_2, G_3)$ *be a graph triple with* $|V_1| = |V_2| = n$*. If* $\Delta(G_1), \Delta(G_2) \le n-2$*,* $\Delta(G_3) \le n-1$*,* $|E_1| + |E_2| + |E_3| \le 2n-3$ *and the pair* $\{G_1, G_2\}$ *is none of the 7 pairs in Figure 2.1, then* $\mathbf{G}$ *packs.*

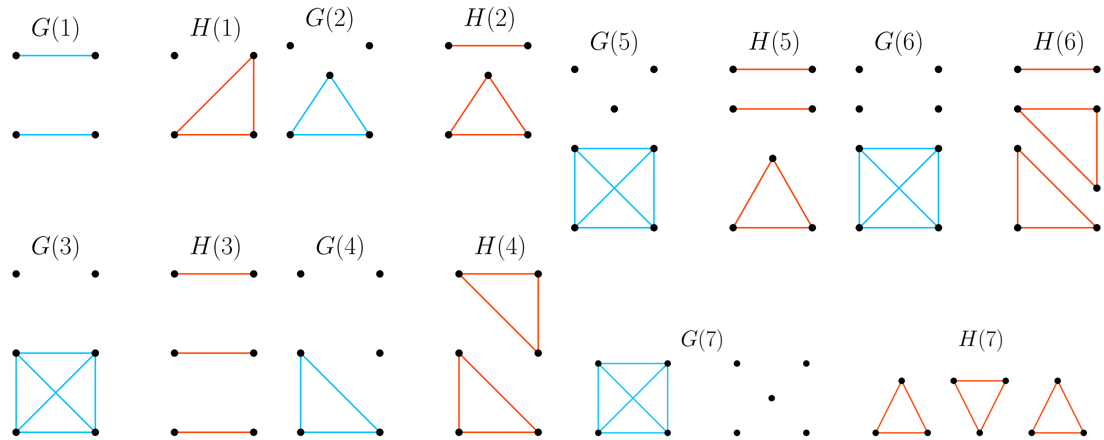


Figure 2.1: Bad pairs in Theorem 2.2.

Theorem 2.2 is sharp and has more sharpness examples than Theorem 1.22. Some of these are shown in Figure 2.2 where each left column of vertices corresponds to V_1 and each right column corresponds to V_2. First, the condition $\Delta(G_3) \le n-1$ cannot be removed, since a vertex in V_1 adjacent to all vertices in V_2 cannot be placed at all (Figure 2.2a). The restriction on $|E_1| + |E_2| + |E_3|$ is also sharp, as there are several examples of graphs with $|E_3| > 0$ and edge sum equal to $2n-2$ that do not pack.

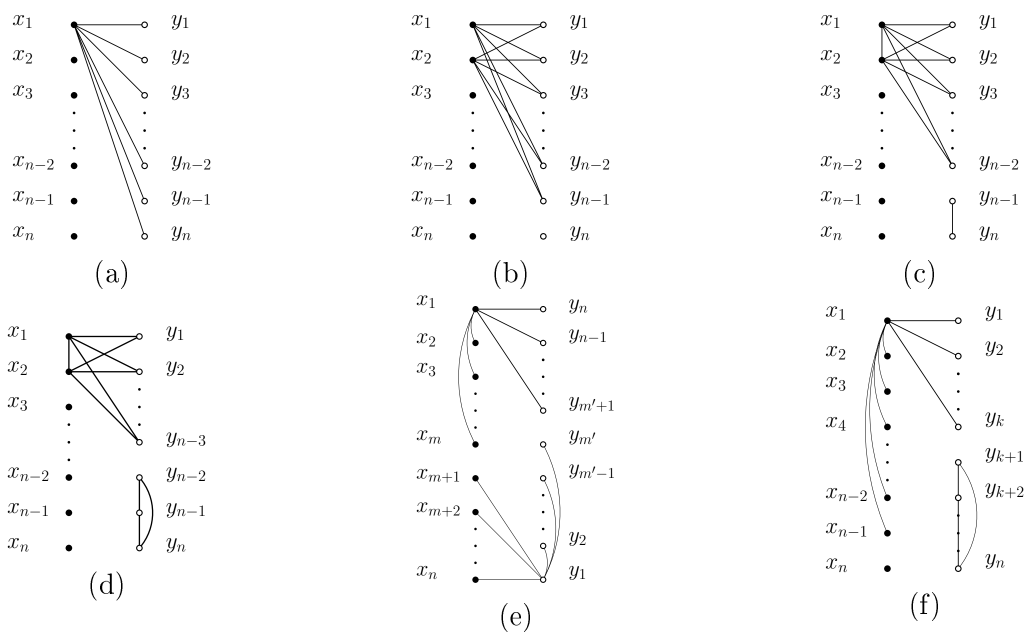


Figure 2.2: Sharpness examples for Theorem 2.2.

For example, let G_1 and G_2 be independent sets and $x_1, x_2 \in V_1$ be adjacent to the same $n-1$ vertices in V_2 (Figure 2.2b). In this case (G_1, G_2, G_3) does not pack. If E_1 consists of a single edge x_1x_2, E_2 consists of a single edge $y_{n-1}y_n$, and E_3 consists of all edges between $\{x_1, x_2\}$ and $V_2 - y_{n-1} - y_n$ (Figure 2.2c), then (G_1, G_2, G_3) also does not pack. Similarly, if E_1 contains a single edge, G_2 contains a triangle and $(n-3)$ isolated vertices, and G_3 consists of all edges between non-isolated vertices in G_1 and isolated vertices in G_2, then (G_1, G_2, G_3) does not pack (Figure 2.2d).

Alternatively, consider $G_1 = K_{1,m-1} \cup \overline{K}_{n-m}$, $G_2 = K_{1,m'-1} \cup \overline{K}_{n-m'}$ (for any choice of m, m' such that $m - 1 \neq n - m'$), and E_3 consisting of all edges between the center of the star in G_1 and isolated vertices in G_2 as well as between the center of the star in G_2 and isolated vertices in G_1 (Figure 2.2e). Indeed, since $m - 1 \neq n - m'$, mapping the center of the star in G_1 to the center of the star in G_2 will create a conflict. Then, since the center of the star in G_1 must be mapped to a leaf in G_2 and a leaf in G_1 must be mapped to the center of the star in G_2, (G_1, G_2, G_3) does not pack. Finally, consider $G_1 = K_{1,n-1} \cup K_1$, $G_2 = C_k \cup \overline{K}_{n-k}$ (for any choice of k), and let E_3 consist of all possible edges between the center of the star in G_1 and isolated vertices in G_2 (Figure 2.2f). In this case, (G_1, G_2, G_3) does not pack since the center of the star in G_1 is adjacent to $n-2$ vertices in G_1, but must be mapped to a vertex in the cycle in G_2.

The notion of list packing arose while my collaborators and I were working on a conjecture of Żak [7] on packing n-vertex graphs with given sizes and maximum degrees. In this situation, list packing provides a stronger inductive assumption that facilitates a proof. In [22], we heavily use Theorems 2.1 and 2.2 of this paper to get an approximate solution to Żak's conjecture.

2.1.1 Notation

For this chapter, we will define a *graph triple* $\mathbf{G} = (G_1, G_2, G_3)$ of order n to consist of a pair of n-vertex graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets together with a bipartite graph $G_3 = (V_1 \cup V_2, E_3)$. For $i \in \{1, 2, 3\}$, let $e_i = |E_i|$. Let $V = V(\mathbf{G}) = V_1 \cup V_2$. An edge in $E_1 \cup E_2$ is a *white* edge, while an edge in E_3 is a *yellow* edge. The edge set of $\mathbf{G}$ is $E(\mathbf{G}) = E_1 \cup E_2 \cup E_3$.

Let $i \in \{1, 2\}$ and $v \in V_i$. Then the *white neighborhood* of v, denoted $N_i(v)$, is the set of neighbors of v in G_i, and $d_i(v) = |N_i(v)|$. A vertex in $N_i(v)$ is a *white neighbor* of v. For convenience, we say that $N_{3-i}(v) = \emptyset$ (and hence $d_{3-i}(v) = 0$) since $v \notin V_{3-i}$. The *yellow neighborhood* of v, denoted $N_3(v)$, is the set of neighbors of v in G_3 and $d_3(v) = |N_3(v)|$. A vertex in $N_3(v)$ is a *yellow neighbor* of v. Furthermore, the *neighborhood* of v, denoted $N(v)$, is the disjoint union $N_i(v) \cup N_3(v)$ and vertices in the neighborhood are *neighbors*. The *degree* of v is $d_i(v) + d_3(v)$ and is denoted $d(v)$. For $i \in \{1, 2, 3\}$, define $\Delta_i = \Delta_i(\mathbf{G})$ to be $\max_{v \in V} d_i(v)$.

If $W \subseteq V$, then the triple induced by W is $\mathbf{G}[W] = (G_1[W], G_2[W], G_3[W])$, where $G_i[W]$ is the subgraph of G_i induced by the set W. Similarly, the triple $\mathbf{G} - W$ is $(G_1 - W, G_2 - W, G_3 - W)$. The *underlying graph* $\underline{\mathbf{G}}$ of a triple $\mathbf{G}$ is the graph with vertex set $V(\mathbf{G})$ and edge set $E(\mathbf{G})$. Finally, we say the graph triple $\mathbf{G}$ *packs* if the triple has a list packing.

2.2 Proof of Theorem 2.1

($\Leftarrow$) Suppose G_1 is a perfect matching and G_2 contains $K_{\frac{n}{2}+1}$ or $\frac{n}{2}$ is odd and $G_2 = K_{\frac{n}{2},\frac{n}{2}}$. In the first case, for any mapping $f : V_1 \to V_2$, some edge of G_1 will be mapped to an edge in the clique $K_{\frac{n}{2}+1}$. In the second case, since $\frac{n}{2}$ odd, some edge of a perfect matching on V_2 has one endpoint in each partite set of G_2. Thus, $\mathbf{G} = (G_1, G_2, G_3)$ cannot pack.

($\Rightarrow$) Assume that a graph triple $\mathbf{G}$ is a counterexample with the minimum $|E_3|$. By Theorem 1.19, $E_3 \neq \emptyset$. By the minimality of $|E_3|$, we may assume that there is a mapping f which has a conflict at only one edge $vw \in E_3$, i.e., $f(v) = w$. For each $a \in V_1 - v$, define the mapping f_a by $f_a(v) = f(a), f_a(a) = w$ and $f_a(x) = f(x)$ for all $x \in V_1 - a - v$. We claim that there is a mapping f_a that satisfies:

(i) $f_a(N_1(a)) \cap N_2(w) = \emptyset$,

(ii) $f_a(N_1(v)) \cap N_2(f(a)) = \emptyset$,

(iii) $f_a(a) \notin N_3(v)$, and

(iv) $w \notin N_3(a)$.

Indeed, $V_1 - v$ has at most $\Delta_1\Delta_2$ vertices that may violate (i), at most $\Delta_1\Delta_2$ vertices that may violate (ii), at most $\Delta_3 - 1$ vertices that may violate (iii) and at most $\Delta_3 - 1$ vertices that may violate (iv). Since $\mathbf{G}$ does not pack, $n-1 = |V_1 - v| \leq (\Delta_3 - 1) + (\Delta_3 - 1) + 2\Delta_1\Delta_2$. But this inequality yields $n + 1 \leq 2(\Delta_3 + \Delta_1\Delta_2)$, contradicting (2.1).

Thus some f_a satisfies (i)–(iv). Then under f_a there is no conflict along edge vw and no new conflicts are introduced. Since the only conflict in f was along vw, f_a is a packing, a contradiction to the choice of $\mathbf{G}$. $\square$

2.3 Preliminary facts

The following lemma is an extension of Theorem 1.21.

Lemma 2.3. *Let $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple with $|V_1| = |V_2| = n$. If $\Delta_3 \leq n - 1$ and $e_1 + e_2 + e_3 \leq \left\lfloor \frac{3}{2}n \right\rfloor - 2$, then the triple $\mathbf{G} = (G_1, G_2, G_3)$ packs.*

Proof. It is enough to prove the lemma in the case

$$e_1 + e_2 + e_3 = \left\lfloor \frac{3}{2}n \right\rfloor - 2. \tag{2.2}$$

The proof will proceed by induction on n. If $e_3 = 0$, then the result holds by Theorem 1.21. If $n = 2$, then $e_1+e_2+e_3 = 1$ and $\mathbf{G}$ packs. If $e_1 = 0$ or $e_2 = 0$, then the problem reduces to finding a perfect matching in $K_{n,n} - E_3$. By the König-Egerváry Theorem, if $K_{n,n} - E_3$ has no perfect matching, then it has a vertex cover C with $|C| = n-1$. This means that $G_3 - C$ is a complete bipartite graph with $n+1$ vertices, say $G_3 - C = K_{k,n+1-k}$. Since $\Delta_3 \leq n-1$, we have $2 \leq k \leq n-1$ and so $|E(G_3 - C)| = k(n+1-k) \geq 2n-2$, contradicting (2.2). Therefore, $e_1, e_2, e_3 > 0$ and so $n \geq 4$.

We now claim that

$$\Delta_3 \leq n - 2. \tag{2.3}$$

Otherwise, by symmetry, we may assume that $d_3(v) = n-1$ for some $v \in V_1$. Let $V_2 - N_3(v) = \{y\}$. Then at most $n/2 - 1$ edges in $\mathbf{G}$ are not adjacent to v. In particular, there is $u \in V_2$ that has no neighbors in $(V_1 \cup V_2) - v$. If $u = y$, then we pack $\mathbf{G} - v - y$ by induction and extend this packing by assigning v to y.

If $uv \in E_3$ and there is $w \in V_1 - v$ with $d(w) \geq 1$, then consider $\mathbf{G}' = \mathbf{G} - w - u$. The total number of edges decreases by at least 2, and v is incident with exactly $n-2$ yellow edges. So, since $\mathbf{G}'$ contains at most $\left\lfloor \frac{3}{2}n \right\rfloor - 4$ edges, $\Delta_3(\mathbf{G}') = n-2$. Thus $\mathbf{G}'$ packs by induction, and we can extend the packing by sending w to u. Finally, if $uv \in E_3$ and G_1 has no edges, it is enough to find an ordering $(v_1, \ldots, v_n)$ of V_1 and an ordering $(y_1, \ldots, y_n)$ of V_2 such that $v_i y_i \notin E_3$ for all i. We order V_1 so that $v_1 = v$ and $d_3(v_{i+1}) \leq d_3(v_i)$ for all $1 \leq i \leq n-1$ and find a nonneighbor y_i for v_i greedily one by one for $i = 1, \ldots n$. This is possible, since $G_3 - v_1$ has at most $n/2 - 1$ edges and so for $i \geq 2$, v_i has at most $\frac{n/2-1}{i-1}$ neighbors in $V_2 - \{y_1, \ldots, y_{i-1}\}$. This proves (2.3).

We now proceed in three cases.

CASE 1: There exists an $i \in \{1, 2\}$ and a vertex $x \in V_i$ such that $d_i(x) = 0$ and $d_3(x) > 0$. By symmetry, we may assume $i = 1$. If there exists $y \in V_2 - N_3(x)$ with $d_3(x) + d(y) \geq 2$, then by (2.3) the triple $\mathbf{G}-x-y$ satisfies the lemma. By induction, $\mathbf{G}-x-y$ has a packing and this packing can be extended to $\mathbf{G}$ by assigning x to y. Otherwise, we may assume $d(y)=0$ for every $y \in V_2 - N_3(x)$ and $d_3(x)=1$. Let $N_3(x)=\{z\}$. Since $\Delta_3 \leq n-1$, there exists a vertex $w \in V_1 - N_3(z)$ that can be mapped to z. As $d(y)=0$ for each $y \in V_2 - z$, any bijection from $V_1 - w$ onto $V_2 - z$ is a packing of $\mathbf{G}-w-z$. This packing extends to a packing of $\mathbf{G}$ by assigning w to z.

CASE 2: There exists an $i \in \{1,2\}$ and a vertex $x \in V_i$ such that $d_i(x)=d_3(x)=0$. Again, we may assume $i=1$. Similarly to Case 1, if there exists $y \in V_2$ with $d(y) \geq 2$, then the triple $\mathbf{G}-x-y$ satisfies the lemma. By induction $\mathbf{G}-x-y$ has a packing and this packing can be extended to $\mathbf{G}$ by assigning x to y. So we may assume that $d(y) \leq 1$ for all $y \in V_2$. Then since $e_3 > 0$, there is $y \in V_2$ with $d_3(y)=1$ and $d_2(y)=0$. But this means we now have Case 1.

CASE 3: For $i \in \{1,2\}$, $d_i(v) \geq 1$ for every $v \in V_i$. By (2.2), there is $x \in V_1 \cup V_2$ with $d(x) \leq 1$. By symmetry, we may assume $x \in V_1$. By the case assumption, $d_1(x)=1$, and so $d_3(x)=0$. Let $N_1(x)=\{z\}$. Since $e_3>0$, there is $y \in V_2$ incident with a yellow edge. Let $\mathbf{G}''$ be obtained from the triple $\mathbf{G}-x-y$ by joining z with an edge to each vertex in $N_2(y)$. Note that we have deleted $1+d_2(y)+d_3(y)$ edges and added only $d_2(y)$ edges. Since $d_3(y) \geq 1$, $|E(\mathbf{G}'')| \leq \left\lfloor \frac{3}{2}(n-1) \right\rfloor - 2$.

For $i \in \{1,2\}$, $d_i(v) \geq 1$ for each $v \in V_i$, so $e_1, e_2 \geq \frac{n}{2}$. Every vertex in $\mathbf{G}''$ is incident to at most Δ_3 yellow edges present in $\mathbf{G}$ and at most $d_2(y) \leq \Delta_2$ newly added yellow edges. Hence, each vertex in $\mathbf{G}''$ is incident to at most $e_2+e_3 \leq (\frac{3}{2}n-2)-e_1 \leq n-2$ yellow edges. Thus the triple $\mathbf{G}''$ satisfies the conditions of Lemma 2.3 and, by induction, $\mathbf{G}''$ packs. Due to the added yellow edges, z was sent to a vertex in $V_2 - N_2(y)$. Therefore, this packing extends to a packing of $\mathbf{G}$ by mapping x to y. □

Lemma 2.3 along with the following corollary will serve as a base case for a proof of Theorem 2.2.

Corollary 2.4. *Suppose* $\mathbf{G}$ *is a graph triple* (G_1, G_2, G_3) *of order* $n \geq 1$. *If* $e_1 + e_2 + e_3 \leq n$, *then either:*

(1) $\mathbf{G}$ *has a packing, or*

(2) $e_1 = e_2 = 0$ *and for some* $i \in \{1, 2\}$, *there exists* $v \in V_i$ *adjacent to all vertices in* V_{3-i}, *or*

(3) $n = 2$, $e_3 = 0$ *and* $G_1 \cong G_2 \cong K_2$.

Proof. For $n \geq 4$, the result follows from Lemma 2.3. If $n = 1$, then either there are no edges and so $\mathbf{G}$ packs, or there is a single edge in E_3, and (2) holds.

If $n = 2$ and $e_1 + e_2 = 2$, then (3) holds. If $n = 2$, $e_3 = 1$ and $e_1 + e_2 \leq 1$, then $\mathbf{G}$ has a packing. Finally, if $n = 2$ and $e_3 = 2$, then either $\mathbf{G}$ has a packing or (2) holds.

The last case is $n = 3$. If $e_3 = 0$, then in the worst case, $e_1 + e_2 = 3$. In this case, either $\{G_1, G_2\} \cong \{K_{1,2}, K_2 \cup K_1\}$ or $\{G_1, G_2\} \cong \{K_3, \overline{K}_3\}$ and so $\mathbf{G} = (G_1, G_2, \overline{K}_6)$ packs in all cases. Suppose now $e_1 = 0$. Then similarly to the proof of Lemma 2.3, $\mathbf{G}$ packs if $K_{3,3} - E_3$ has a perfect matching. If $K_{3,3} - E_3$ has no such matching, then by the König-Egerváry Theorem, G_3 has a complete bipartite subgraph with 4 vertices. Since $e_3 \leq 3$, the only possibility is that $G_3 \supseteq K_{1,3}$, i.e. (2) holds. Thus, $e_1, e_2, e_3 \geq 1$, which means $e_1 = e_2 = e_3 = 1$. Up to isomorphism, there are only 3 cases, and Figure 2.3 shows a packing in each case with the function $f : V_1 \to V_2$ defined by $f(x_i) = y_i$ for each $i \in \{1, 2, 3\}$. $\square$

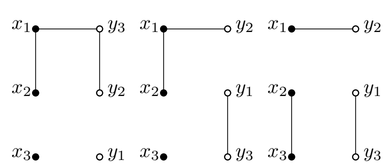


Figure 2.3: Graph triples of order $n = 3$ and $e_1 = e_2 = e_3 = 1$.

2.4 Proof of Theorem 2.2

Let $\mathbf{G} = (G_1, G_2, G_3)$ of order n be a counterexample to Theorem 2.2 with the smallest order. By Corollary 2.4, $n \geq 4$. Also, by Theorem 1.22, we may assume $E_3 \neq \emptyset$.

Lemma 2.5. *$\Delta_3 \le n-2$.*

Proof. Suppose that there exist $v \in V_1$ and $w \in V_2$ such that $N_3(v) = V_2 - w$. Since $|E(\mathbf{G} - v - w)| \le (2n-3) - (n-1) = n-2$, the triple $\mathbf{G} - v - w$ packs by Corollary 2.4. If $d_1(v) = 0$ or $d_2(w) = 0$, then additionally placing v on w is a packing of $\mathbf{G}$. So assume $d_1(v) \ge 1$ and $d_2(w) \ge 1$.

Let $\mathbf{G}' = (G_1', G_2', G_3')$ be obtained from $\mathbf{G}$ by deleting, in G_3, all $n-1$ edges connecting v with V_2 and all edges (maybe zero) connecting w with V_1. We now show that after mapping v to w, there are enough isolated vertices in either $V_2 - w$ or $V_1 - v$ to complete the packing.

First suppose v and w are in different components of the underlying graph $\underline{\mathbf{G}'}$. Define X and Y to be the vertex sets of the component of $\underline{\mathbf{G}'}$ containing v and w, respectively (possibly $X = Y$). Define $Z = X \cup Y$ and let $z = |Z|$. For $i \in \{1, 2\}$, let $Z_i = Z \cap V_i$, with size z_i. The graph $\underline{\mathbf{G}'} - Z$ has $2n - z$ vertices and at most $2n - 3 - (n-1) - (z-2)$ edges. So $\underline{\mathbf{G}'} - Z$ has a least $(2n - z) - (n - z) = n$ components, and at least z of them have no edges, i.e. are singletons. At least z_1 of the singletons are in V_2 or at least z_2 of them are in V_1. Suppose the former holds. In particular, there is a set $S \subseteq V_2 - w$ of singletons with $|S| = d_1(v)$.

Consider the triple $\mathbf{G}'' = \mathbf{G} - v - w - N_1(v) - S$. The triple $\mathbf{G}''$ has order $n - d_1(v) - 1$ and $|E(\mathbf{G}'')| \le 2n-3-(n-1)-d_1(v)-d_2(w) = n-2-d_1(v)-d_2(w)$. The number of edges in $\mathbf{G}''$ is less than the order of $\mathbf{G}''$, so by Corollary 2.4, $\mathbf{G}''$ packs. This packing, together with the placement of v and $N_1(v)$, gives a packing of $\mathbf{G}$, a contradiction. □

Lemma 2.6. *$\Delta_1, \Delta_2 \le n-3$.*

Proof. Suppose $\Delta_1 = n-2$, the other case is symmetric. Let $v, v' \in V_1$ and $N_1(v) = V_1 - v - v'$.

CASE 1: There is $w \in V_2 - N(v)$ with no neighbors in V_2. Consider the triple $\mathbf{G}' = \mathbf{G} - v - w$. Since $d(v) \ge n-2$, $|E(\mathbf{G}')| \le (2n-3) - (n-2) = n-1$. By Lemma 2.5, $\Delta_3(\mathbf{G}') \le n-2$, so $\mathbf{G}$ packs by Corollary 2.4. This packing can be extended to a packing of $\mathbf{G}$ by sending v to w.

CASE 2: Every $w \in V_2 - N(v)$ has a white neighbor. Let W' be the set of vertices in V_2 reachable from V_1 in the underlying graph $\underline{\mathbf{G}}$, and let $W = V_2 - W'$. Since $\underline{\mathbf{G}} - W$ has at least $(n-2)+|W'|$ edges, $|W'| \leq n-1$. So $W \neq \emptyset$ and if $d_1(v') = a$, then

$$|E(\mathbf{G}[W])| \leq (2n-3) - (n-2) - a - |W'| = |W| - 1 - a. \tag{2.4}$$

Let W_1 be the vertex set of a smallest tree component in $G_2[W]$. By the case assumption, every vertex in $G_2[W]$ has positive degree. Since there are no yellow edges incident to W, the degree of each vertex in $G_2[W]$ is equal to its degree in $\mathbf{G}$. Let $y \in W_1$ be a vertex of degree 1 in $G_2[W]$ and let $y' \in W_1$ be the neighbor of y. Suppose $d_2(y') = b$. Let $\mathbf{G}' = (G_1', G_2', G_3')$ be the triple obtained from $\mathbf{G} - \{v, v', y, y'\}$ by adding the $a(b-1)$ yellow edges connecting the white neighbors of v' with the (necessarily white) neighbors of y' distinct from y. The graph triple $\mathbf{G}'$ has $2(n-2)$ vertices and

$$|E(\mathbf{G}')| \leq 2n - 3 - (n-2) - a - b + a(b-1) = n - 1 - 2a + b(a-1). \tag{2.5}$$

If $\mathbf{G}'$ packs, then because of the added edges, this packing extends to a packing of $\mathbf{G}$ by sending v to y and v' to y'. Suppose it does not.

Case 2.1: $a \leq 1$. Then by (2.5),

$$\begin{aligned}&|E(\mathbf{G}')| \leq n-2 \textit{ with equality only if } a = 0,\ b = 1, \textit{ and the only edges in}\\ &E(\mathbf{G}) - E(\mathbf{G}') \textit{ are } yy' \textit{ and the } n-2 \textit{ white edges incident with } v.\end{aligned} \tag{2.6}$$

By Corollary 2.4, $|E(\mathbf{G}')| = n-2$ and $\mathbf{G}'$ either has no white edges or has no yellow edges, since $\mathbf{G}'$ does not pack. Then (2.6) yields $a = 0$, $b = 1$, and $E(\mathbf{G}) - E(\mathbf{G}')$ has no yellow edges. Since $e_3 > 0$, this implies $\mathbf{G}'$ has no white edges, but this contradicts the case conditions together with $b = 1$.

Case 2.2: $a \geq 2$. By (2.4), $G_2[W]$ has at least $a+1$ tree components. So by the choice of W_1,

$$2 \leq b + 1 \leq |W_1| \leq |W|/(a+1) \leq n/(a+1) \tag{2.7}$$

and thus $b \leq -1 + n/(a+1)$. Since $a \geq 2$, by (2.5),

$$\begin{aligned}|E(\mathbf{G}')| &\leq n - 1 - 2a + \left(\frac{n}{a+1} - 1\right)(a-1) \\ &= n - 3a + n\frac{a}{a+1} - \frac{n}{a+1} \\ &\leq n + n\frac{a}{a+1} - 3a - 3 \\ &\leq n + n\frac{a}{a+1} - 9 < 2(n-2) - 3\end{aligned}$$

Since $\mathbf{G}'$ does not pack and the last strict inequality ensures that the examples from Figure 2.1 do not appear as $\mathbf{G}'$, by induction, some vertex z in $\mathbf{G}'$ has yellow degree $n-2$ or white degree at least $n-3$. But since we deleted at least $n-2+a+b \geq n+1$ edges out of $2n-3$ in $\mathbf{G}$, the number of edges in $E(\mathbf{G}') \cap E(\mathbf{G})$ (and thus the total number of white edges in $E(\mathbf{G}')$) is at most $n-4$. It follows that the vertex z has yellow degree $n-2$ in $\mathbf{G}'$ and is incident to an added yellow edge. Since all added yellow edges connect W_1 with V_1, $z \in W_1 \cup V_1$.

If $z \in W_1$, then by the definition of W, all $n-2$ yellow edges incident to z are in $E(\mathbf{G}') - E(\mathbf{G})$. By the construction of $\mathbf{G}'$, this yields $a \geq n-2$, which contradicts (2.7) since $n \geq 4$. Thus $z \in V_1$ and is adjacent to each vertex in $V(G_2')$. But by the definition of W and $\mathbf{G}'$, the vertices in $W - W_1$ are not incident with yellow edges in $\mathbf{G}'$. This is a contradiction, since $W - W_1 \neq \emptyset$ by (2.4). □

Lemma 2.7. *Every vertex of* $\mathbf{G}$ *has a white neighbor.*

Proof. Suppose $v \in V$ has no white neighbor. Without loss of generality, assume $v \in V_1$.

CASE 1: $d_3(v) = 0$.

Case 1.1: Some $w \in V_2$ has degree at least 2. Then $\mathbf{G} - v - w$ contains at most $2(n-1)-3$ edges. By Lemmas 2.5 and 2.6, $\mathbf{G} - v - w$ satisfies the conditions of Theorem 2.2 for $n' = n-1$. Since any packing of $\mathbf{G} - v - w$ can be extended to a packing of $\mathbf{G}$ by sending v to w, it does not pack. So by the minimality of $\mathbf{G}$, $\mathbf{G} - v - w$ is one of the examples in Figure 2.1.

In particular, $G_3 - v - w$ has no yellow edges. This means all yellow edges in $\mathbf{G}$ are incident to w. Since each of the examples in Figure 2.1 has exactly $2(n-1)-3$ edges, $d(w) = 2$.

If both edges adjacent to w are yellow, since every graph in Figure 2.1 contains 3 vertices of positive degree, there is some $v' \in V_1 - N(w)$ with $d(v') \geq 1$. Then $\mathbf{G} - v' - w$ contains fewer than $2(n-1)-3$ edges and no yellow edges. By Theorem 1.22, $\mathbf{G} - v' - w$ packs and this packing can be extended to a packing of $\mathbf{G}$ by sending v' to w.

Since $e_3 > 0$, the remaining possibility is that w has exactly one neighbor $w' \in V_2$ and one neighbor in V_1. As above, we can choose some $v' \in V_1 - N_1(w)$ with positive degree. Create a new graph triple $\mathbf{G}'$ from $\mathbf{G}$ by removing v' and w and adding yellow edges from w' to $N_1(v')$. This triple $\mathbf{G}'$ has exactly $2(n-1)-3$ edges, and all yellow edges in $\mathbf{G}'$ are incident to w', since w was the only vertex in $\mathbf{G}$ incident to yellow edges. So $\Delta_3(\mathbf{G}') = d_1(v') \leq n-3$ by Lemma 2.6. Additionally, no white edges were added, so again by Lemma 2.6, $\Delta_1(\mathbf{G}'), \Delta_2(\mathbf{G}') \leq n-3$. Thus, $\mathbf{G}'$ satisfies the conditions of Theorem 2.2 and has at least one yellow edge. Hence $\mathbf{G}'$ is not one of the examples from Figure 2.1. By the minimality of $\mathbf{G}$, the triple $\mathbf{G}'$ packs, and this packing can be extended to a packing of $\mathbf{G}$ by sending v' to w.

Case 1.2: $d(w) \leq 1$ for each $w \in V_2$. If there exists $w \in V_2$ such that $d(w) = 0$, then in view of Case 1.1, each $u \in V_1$ has degree at most 1, and $\mathbf{G}$ packs by Corollary 2.4.

Thus, $d(w) = 1$ for each $w \in V_2$. Since $e_3 > 0$, there exists $w \in V_2$ such that $d_3(w) = d(w) = 1$. Let $N_3(w) = \{v'\}$. Fix $u \in V_1 - v'$ with $d(u)$ maximum. If $d(u) = 0$, then

$$\sum_{v \in V_1 \cup V_2} d(v) \leq d_3(v') + n \leq \Delta_3(\mathbf{G}) + n \leq 2n - 1.$$

In particular, $|E(\mathbf{G})| < n$. Corollary 2.4 and the strict inequality imply that $\mathbf{G}$ packs. So suppose $d(u) \geq 1$. Since $d(w) = 1$ and $d(u) \geq 1$, $|E(\mathbf{G} - u - w)| \leq 2(n-1) - 3$. By Lemmas 2.5 and 2.6, $\mathbf{G} - u - w$ satisfies the conditions of Theorem 2.2. If $d(u) = 1$, then v' is the only vertex in $V_1 \cup V_2$ with degree at least 2, and hence $\mathbf{G} - u - v$ is not one of the examples from Figure 2.1. Similarly, if $d(u) \geq 2$, then $|E(\mathbf{G} - u - w)| \leq 2(n-1) - 4$ and again $\mathbf{G} - u - w$ is not one of the examples from Figure 2.1. Therefore, there is a packing of $\mathbf{G} - u - w$, and sending u to w extends this packing to a packing of $\mathbf{G}$.

CASE 2: $d_3(v) \geq 1$. Among the vertices in $V_2 - N_3(v)$ with maximum degree, let w be a vertex that minimizes $d_3(w)$. By Case 1, $d(w) \geq 1$. Consider the triple $\mathbf{G}' := \mathbf{G} - v - w$. Since $d(v) + d(w) \geq 2$ and $vw \notin E(\mathbf{G})$, by Lemmas 2.5 and 2.6, $\mathbf{G}'$ satisfies the conditions of Theorem 2.2 for $n' = n - 1$. If $\mathbf{G}'$ packs, then the packing extends to a packing of $\mathbf{G}$ by sending v to w. Therefore by Theorem 2.2, $d(v) = d(w) = 1$, and $\mathbf{G}'$ is an example from Figure 2.1.

However, by the choice of w and the fact that $d(w) = 1$, all vertices in $V_2 - N_3(v)$ have degree at most 1 in $\mathbf{G}$ and, hence, in $G_2 - w$. By inspection, $G_2 - w$ is either $G_1(1)$ or $G_2(3)$ in Figure 2.1, as every other graph in Figure 2.1 has at least two vertices with degree at least 2. Since each of $G_2(1)$ and $G_1(3)$ has an isolated vertex, and by Case 1, $\mathbf{G}$ has no isolated vertices in V_1, we have removed an incident yellow edge when deleting w. It follows that $d_3(w) = d(w) = 1$. Each of $G_1(1)$ and $G_2(3)$ has at least 4 vertices incident to exactly one white edge. Since $d_2(w) = 0$, in the process of removing v and w from $\mathbf{G}$, we have removed only one edge incident to $V_2 - w$. Thus, G_2 contains a vertex with degree 1 incident to a white edge, contradicting our choice of w. $\square$

Proof of Theorem 2.2: Let $\mathbf{G}$ be our minimum counterexample. Since $e_3 > 0$, $\mathbf{G}$ has a yellow edge xy with $x \in V_1$ and $y \in V_2$. Since $|E(\mathbf{G})| \leq 2n - 3 < 2n$, there are vertices of degree at most 1. We may assume that $v \in V_1$ and $d(v) \leq 1$. By Lemma 2.7, v has a white neighbor, v'. Since $d(v) = 1$, $v \neq x$. We obtain the triple $\mathbf{G}'$ from $\mathbf{G}$ by removing v and y, and adding a yellow edge from v' to each vertex in $N_2(y)$. Then, $|E(\mathbf{G}')| \leq |E(\mathbf{G})| - 2 \leq 2(n-1) - 3$. The triple $\mathbf{G}'$ has at least one yellow edge (connecting v' with a white neighbor of y), so it is not an example from Figure 2.1. Since we have not added any white edges, by Lemma 2.6, $\Delta_1(\mathbf{G}'), \Delta_2(G') \leq n - 3$. If $\Delta_3(G') \leq n - 2$, then $\mathbf{G}'$ satisfies the conditions of Theorem 2.2 and so there exists a packing of $\mathbf{G}'$. This packing extends to a packing of $\mathbf{G}$ by sending v to y.

Thus, $\Delta_3(\mathbf{G}') = n - 1$. By Lemma 2.7, $e_1 + e_2 \geq n$, so $\Delta_3 \leq e_3 \leq n - 3$. Since v' is the only vertex whose degree in $\mathbf{G}'$ exceeds the degree in $\mathbf{G}$ by at least 2, it is the only vertex with yellow degree $n-1$ in $\mathbf{G}'$. In particular, by construction this implies that in $\mathbf{G}$, every vertex in $V_2 - y$ is either in $N_3(v')$ or in $N_2(y)$.

Since the underlying graph $\underline{\mathbf{G}}$ of $\mathbf{G}$ contains $2n$ vertices and at most $2n-3$ edges, it contains at least 3 tree components. Consider a tree component T that contains neither v' nor y. Since every vertex in $V_2 - y$ is adjacent to y or v',

$$T \text{ contains only vertices in } V_1 \text{ that do not have neighbors in } V_2. \tag{2.8}$$

By Lemma 2.7, T is not a single vertex. Let $u \in V_1$ be a leaf vertex, so $d(u) = 1$, and let $u' \in V_1$ be its neighbor.

Consider the triple $\mathbf{G}''$ formed from $\mathbf{G} - u - y$ by adding a yellow edge from u' to each vertex in $N_2(y)$. As with $\mathbf{G}'$, $|E(\mathbf{G}'')| \leq |E(\mathbf{G})| - 2 \leq 2(n-1) - 3$ and $\mathbf{G}''$ contains a yellow edge, so it is not an example from Figure 2.1. No white edges have been added, so by Theorem 2.6, $\Delta_1(\mathbf{G}''), \Delta_2(\mathbf{G}'') \leq n-2$. By (2.8), u' is incident to exactly $d_2(y) \leq \Delta_2 \leq n-3$ yellow edges and every other vertex in $\mathbf{G}''$ is incident to at most $\Delta_3 + 1 \leq n-2$ yellow edges. So $\mathbf{G}''$ satisfies the conditions of Theorem 2.2. Therefore, there exists a packing of $\mathbf{G}''$, and this packing extends to a packing of $\mathbf{G}$ by sending u to y. $\square$

Chapter 3

Cycles

3.1 Introduction

The goal of this chapter is to resolve the Ore-type version of Question 1.30 for multigraphs in an algorithmic way. In Theorem 3.12, we consider the class $\mathcal{DO}_k$ of multigraphs G whose underlying simple graph $\underline{G}$ satisfies $d_{\underline{G}}(x) + d_{\underline{G}}(y) \geq 4k - 3$ for all nonadjacent vertices x and y, and describe all graphs in $\mathcal{DO}_k$ that do not have k disjoint cycles. Using this description we construct a polynomial time algorithm that for every multigraph G in $\mathcal{DO}_k$ decides whether G has k disjoint cycles or not.

In the next section, we introduce/recall notation and discuss existing results to be used later on. In Section 3.3 we state our main results, Theorem 3.12 and Theorem 3.13. In Sections 3.4 and 3.5, we prove Theorem 3.12. Finally, we prove Theorem 3.13 in Section 3.6.

3.2 Preliminaries and known results

3.2.1 Notation

A *loop* is an edge consisting of a single vertex, and a *strong edge* is an edge with multiplicity greater than one. For every multigraph G, let $V_1 = V_1(G)$ be the set of vertices in G incident to loops, and $V_2 = V_2(G)$ be the set of vertices in $G - V_1$ incident to strong edges. Let $F = F(G)$ be the simple graph with $V(F) = V_2$ and $E(F)$ consisting of the strong edges in $G - V_1$. We define $\alpha' = \alpha'(F)$ to be the size of a maximum matching in F. Let $\underline{G}$ denote the *underlying simple graph of* G, i.e. the simple graph on $V(G)$ such that two vertices are adjacent in $\underline{G}$ if and only if they are adjacent in G. For $e \notin E(G)$, let $G + e$ denote the graph with $V(G+e)=V(G)$ and $E(G+e)=E(G)\cup\{e\}$. For a path P with $P\cap G=\emptyset$, let $\mathrm{sd}(G,e,P)$ be the result of subdividing e with P.

Recall that $K_t(X)=K(X)$ denotes the complete graph with vertex set X where $|X|=t$. Similarly, $K(Y,Z)$ is the complete Y,Z-bigraph. We also extend this notation to the case that Y is a graph. Then $K(Y,Z)$ is $K(V(Y),Z)\cup Y$.

For $S\subseteq V(G)$, let $\overline{S}=V(G)-S$ and let $N_G(S)=\bigcup_{v\in S}N_G(v)$. For a matching M, let $W=W(M)$ denote the set of vertices saturated by M, and $G'=G'(M)=G-W(M)$. If $|F|=2\alpha'$ then $G'(M)=G'(M')$ for all perfect matchings M and M' in F.

We define $\mathcal{D}_k$ to be the family of multigraphs G with $\mathcal{S}(G)\geq 2k-1$ and $\mathcal{DO}_k$ to be the family of multigraphs G with $\mathcal{SO}(G)\geq 4k-3$. For a graph $G\in\mathcal{DO}_k$, call a vertex $v\in V(G)$ *low* if $d_G(v)\leq 2k-2$. Let $\mathcal{B}_k=\{G\in\mathcal{D}_k : c(G)<k\}$, and let $\mathcal{BO}_k=\{G\in\mathcal{DO}_k : c(G)<k\}$, where $c(G)$ is the maximum number of cycles in G.

If $G\in\mathcal{DO}_k$ is an n-vertex multigraph and $\alpha(G)\geq n-2k+2$, then for any distinct v_1,v_2 in a maximum independent set I, $s(v_1)+s(v_2)\leq(2k-2)+(2k-2)<4k-3$. Thus $\alpha(G)\leq n-2k+1$ for every n-vertex $G\in\mathcal{DO}_k$; so we call $G\in\mathcal{DO}_k$ *extremal* if $\alpha(G)=n-2k+1$. If $G\in\mathcal{DO}_k$ is extremal, and v_1 and v_2 are distinct vertices in a maximum independent set I, then $s(v_1)+s(v_2)\leq(2k-1)+(2k-1)=4k-2$. Since $\mathcal{SO}(G)\geq 4k-3$, this means that for some $v\in\{v_1,v_2\}$ we have $s(v)=2k-1$ and I is exactly $V(G)-N(v)$. Thus to check whether G is extremal it is enough to check for every $v\in V(G)$ with $s(v)=2k-1$ whether the set $V(G)-N(v)$ is independent. If I is a maximum independent set in an extremal $G\in\mathcal{DO}_k$, then since $\mathcal{SO}(G)\geq 4k-3$,

at most one vertex in I has nonneighbors in $V(G)-I$, and any such vertex has at most one nonneighbor in $V(G)-I$. (3.1)

We call all such maximum independent sets in an extremal graph *big sets*. On the other hand, if x is a common vertex of big sets I and J, then $s(x)\leq|G|-|I\cup J|\leq 2k-1-|J-I|$. Hence for every $y\in I-x$, $s(x)+s(y)\leq 4k-2-|J-I|$, and so $|J-I|\leq 1$. Furthermore, if $|J-I|=1$ and there is $x'\in J\cap I-x$, then $s(x)+s(x')\leq 2(n-\alpha(G)-1)=4k-4$, a contradiction. Thus in this case $\alpha(G) = 2$. This yields the following.

Let G be extremal. If $|G| > 2k + 1$ then every two distinct big sets in G are disjoint. If $|G| = 2k+1$, sets $I, J \subset V(G)$ are big and $x \in I \cap J$, then $s(x) = 2k - 2$. (3.2)

3.2.2 Gallai-Edmonds Theorem

We will use the classical Gallai-Edmonds Theorem on the structure of graphs without perfect matchings. Recall that a graph H is *odd* if $|H|$ is odd, and that $o(H)$ denotes the number of odd components of H. For a matching M and $uv \in M$ we say that u is the *M-mate* of v. For a graph H and $S \subseteq V(H)$, the *deficiency* $\mathrm{def}(S)$ is $o(H - S) - |S|$. Next, $\mathrm{def}(H) := \max\{\mathrm{def}(S) : S \subseteq V(H)\}$. For each graph H, $\mathrm{def}(F) \geq 0$, since $\mathrm{def}(\emptyset) = o(H) \geq 0$.

Theorem 3.1. *(Gallai-Edmonds) Let H be a graph and D be the set of $v \in V(H)$ such that there is a maximum matching in H not covering v. Let A be the set of the vertices in $V(H) - D$ that have neighbors in D, and let $C = V(H) - D - A$. Let $H_1, \ldots, H_k$ be the components of $H[D]$. If M is a maximum matching in H, then all of the following hold:*

(a) $C \cup A \subseteq W(M)$ and the M-mates of A are in distinct components of $H[D]$.

(b) For each H_i and every $v \in V(H_i)$, $H_i - v$ has a perfect matching.

(c) If $\emptyset \neq S \subseteq A$, then $N(S)$ intersects at least $|S| + 1$ components of $H[D]$.

(d) $\mathrm{def}(H) = \mathrm{def}(A) = k - |A|$.

We refer to (D, A, C) as the Gallai-Edmonds decomposition (GE-decomposition) of H.

3.2.3 Results for $\mathcal{D}_k$

Since every cycle in a simple graph has at least 3 vertices, the condition $|G| \geq 3k$ is necessary in Theorem 1.27. However, it is not necessary for multigraphs, since loops and multiple edges form cycles with fewer than three vertices. Theorem 1.27 can easily be extended to multigraphs, although the statement is no longer as simple:

Theorem 3.2. *For $k \in \mathbb{Z}^+$, let G be a multigraph with $\mathcal{S}(G) \geq 2k$, and set $F = F(G), V_1 = V_1(G)$, and $\alpha' = \alpha'(F)$. Then G has no k disjoint cycles if and only if*

$$|V(G)| - |V_1| - 2\alpha' < 3(k - |V_1| - \alpha'), \tag{3.3}$$

i.e., $|V(G)| + 2|V_1| + \alpha' < 3k$.

Proof. If (3.3) holds, then G does not have enough vertices to contain k disjoint cycles. If (3.3) fails, then we choose $|V_1|$ cycles of length one and α' cycles of length two from $V_1 \cup V(F)$. By Theorem 1.27, the remaining (simple) graph contains $k - |V_1| - \alpha'$ disjoint cycles. □

Theorem 3.2 yields the following.

Corollary 3.3. *Let G be a multigraph with $\mathcal{S}(G) \geq 2k - 1$ for some integer $k \geq 2$, and set $F = F(G), V_1 = V_1(G)$, and $\alpha' = \alpha'(F)$. Suppose G contains at least one loop. Then G has no k disjoint cycles if and only if $|V(G)| + 2|V_1| + \alpha' < 3k$.*

Recall $\mathbf{Y_{h,t,s}}$ from Figure 1.4 is $K_{h,t,s}$ where the $s-$ and $t-$sets are replaced with cliques. Since acyclic graphs are exactly forests, Theorem 1.29 can be restated as follows:

Theorem 3.4. *(Kierstead-Kostochka-Yeager [14]) For $k \in \mathbb{Z}^+$, let G be a simple graph in $\mathcal{D}_k$. Then G has no k disjoint cycles if and only if one of the following holds:*

(α) $|G| \leq 3k - 1$;

(β) $k = 1$ and G is a forest with no isolated vertices;

(γ) $k = 2$ and G is a wheel;

(δ) $\alpha(G) = n - 2k + 1$; or

(ϵ) $k > 1$ is odd and $G = \mathbf{Y_{k,k,k}}$.

Dirac [16] described all 3-connected multigraphs that do not have two disjoint cycles:

Theorem 3.5. *(Dirac [16]) Let G be a 3-connected multigraph. Then G has no two disjoint cycles if and only if one of the following holds:*

(A) $\underline{G} = K_4$ and the strong edges in G form either a star (possibly empty) or a 3-cycle;

(B) $G = K_5$;

(C) $\underline{G} = K_5 - e$ *and the strong edges in* G *are not incident to the ends of* e*;*

(D) $\underline{G}$ *is a wheel, where some spokes could be strong edges; or*

(E) G *is obtained from* $K_{3,|G|-3}$ *by adding non-loop edges between the vertices of the (first) 3-class.*

Going further, Lovász [23] described *all* multigraphs with no two disjoint cycles. To state his result, let a *bud* be a vertex incident to at most one edge. Also, let $W_n = K_1 \vee C_n$ be the wheel with $n+1$ vertices and $\mathbf{W_n^+}$ be obtained from W_n by replacing each spoke with a strong edge. Similarly, let $\mathbf{K_{3,n-3}^+}$ be the n-vertex multigraph obtained from $K_{3,n-3}$ by adding strong edges connecting all pairs of the vertices of the (first) 3-class. Then, each multigraph described by Theorem 3.5(A) above is contained either in $\mathbf{W_3^+}$ or in $\mathbf{K_{3,1}^+}$.

Lovász [23] observed that any connected multigraph can be transformed into a multigraph with minimum degree at least 3 or a multigraph with exactly one vertex without affecting the maximum number of disjoint cycles in it by using a sequence of operations of the following two types: (i) deleting a bud; (ii) replacing a vertex v of degree 2 that has neighbors x and y (where $v \notin \{x,y\}$ but possibly $x = y$) by a new (possibly parallel) edge connecting x and y.

He also proved the following:

Theorem 3.6. *(Lovász [23]) Let* H *be a multigraph with* $\delta(H) \geq 3$*. Then* H *has no two disjoint cycles if and only if :*

$(L1)$ $H = K_5$*;*

$(L2)$ $H \subseteq \mathbf{W_{|H|-1}^+}$*;*

$(L3)$ $H \subseteq \mathbf{K_{3,|H|-3}^+}$*; or*

$(L4)$ H *is obtained from a forest* T *and vertex* x *with possibly some loops at* x *by adding edges linking* x *to* T*.*

Say that a multigraph G *has the* 2*-property* if the vertices of degree at most 2 form a clique $Q(G)$ (possibly with some multiple edges). Let $G \in \mathcal{DO}_2$ with no two disjoint cycles. Then G has the 2-property. By Lovász's observation above, G can be transformed to a multigraph

H that has exactly one vertex or is of type (L1)–(L4) by a sequence of deleting buds and/or contracting edges. Note that if a multigraph G' has the 2-property, then the multigraph obtained from G' by deleting a bud or contracting an edge also has the 2-property. Thus, H and all the intermediate multigraphs have the 2-property. Reversing this transformation, G can be obtained from H by adding buds and subdividing edges. If H has exactly one vertex and at most one edge, then any multigraph with the 2-property that can be obtained from H this way has maximum degree at most 2 or is a path with a single loop at one end. Hence, G is a K_i for $i \leq 3$, is a path on at most 3 vertices with a loop at an endpoint or forms a strong edge. If $\delta(H) \geq 3$, then the clique $Q := Q(G)$ cannot have more than 2 vertices: by the definition of $Q(G)$, $|Q| \leq 3$, and if $|Q| = 3$ then Q induces a K_3-component of G and $\delta(G - Q) \geq 3$; thus $G - Q$ has another cycle. Let $Q' := V(G) - V(H)$. By above, $Q \subseteq Q'$. If $Q' \neq Q$, then Q consists of a single leaf in G with a neighbor of degree 3, so G is obtained from H by subdividing an edge and adding a leaf to the vertex of degree 2. If $Q' = Q$, then Q is a component of G, or $G = H + Q + e$ for some edge $e \in E(H, Q)$, or at least one vertex of Q subdivides an edge $e \in E(H)$. In the last case, when $|Q| = 2$, e is subdivided twice by Q.

In case (L4), because $\delta(H) \geq 3$, either T has at least two buds, each linked to x by multiple edges, or T has one bud linked to x by an edge of multiplicity at least 3. So this case cannot arise from G. Also, $\delta(H) = 3$, unless $H = K_5$, in which case $\delta(H) = 4$. So Q is not an isolated vertex, lest deleting Q leave H with $\delta(H) \geq 5 > 4$; and if Q has a vertex of degree 1 then $H = K_5$. Else all vertices of Q have degree 2, and Q consists of the subdivision vertices of one edge of H. This yields the following characterization of multigraphs in $G \in \mathcal{D}O_2$ with no two disjoint cycles.

Set $Z_t = \{z_1, \ldots, z_t\}$, and define $\mathbf{S_3} = K(Z_5) \cup z_1xy$, $\mathbf{S_4} = \mathrm{sd}(K(Z_5), z_1z_2, x) \cup xy$, and $\mathbf{S_5} = \mathrm{sd}(K(Z_5), z_1z_2, xy)$ (See Figure 3.1).

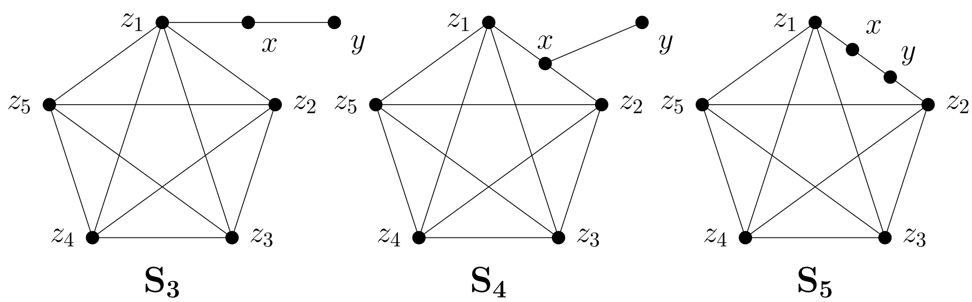


Figure 3.1: Graphs $\mathbf{S_3}$, $\mathbf{S_4}$, and $\mathbf{S_5}$

Corollary 3.7. *All $G \in \mathcal{DO}_2$ with $|G| \geq 4$ and no 2 disjoint cycles satisfy one of:*

(Y1) $G \subseteq \mathbf{S_3}$*;*

(Y2) $G \subseteq \mathbf{S_4}$*;*

(Y3) $G = \mathbf{S_5}$*;*

(Y4) $G \in \{H, \mathrm{sd}(H,e,x), \mathrm{sd}(H,e,xy)\}$*, where* $W_{|H|-1} \subseteq H \subseteq \mathbf{W^+_{|H|-1}}$*;*

(Y5) $G \in \{H, \mathrm{sd}(H,e,x), \mathrm{sd}(H,e,xy)\}$*, where* $H \subseteq \mathbf{K^+_{3,|H|-3}}$ *and H contains $K_{3,|H|-3}$ minus an edge.*

By Corollary 3.3, in order to describe the multigraphs in $\mathcal{D}_k$ not containing k disjoint cycles, it is enough to describe such multigraphs with no loops. Recently, Kierstead, Kostochka, and Yeager [17] proved the following:

Theorem 3.8. *(Kierstead-Kostochka-Yeager [17]) Let $k \geq 2$ and $n \geq k$ be integers. Let G be an n-vertex graph in $\mathcal{D}_k$ with no loops. Set $F = F(G)$, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Then G does not contain k disjoint cycles if and only if one of the following holds:*

(a) $n + \alpha' < 3k$*;*

(b) $|F| = 2\alpha'$ *(i.e., F has a perfect matching) and either*
(i) k' is odd and $G - F = \mathbf{Y_{k',k',k'}}$*, or*
(ii) $k' = 2 < k$ *and* $G - F = W_5$*;*

(c) G *is extremal and either*

(i) some big set is not incident to any strong edge, or

(ii) for some two distinct big sets I_j *and* $I_{j'}$*, all strong edges intersecting* $I_j \cup I_{j'}$ *have a common vertex outside of* $I_j \cup I_{j'}$ *and if* $v \in I_j \cap I_{j'}$ *(this may happen only if* $k' = 2$*), then* v *is not incident with a strong edge;*

(d) $n = 2\alpha' + 3k'$*,* k' *is odd, and there is* $S = \{v_0, \dots, v_s\} \subseteq V(F)$ *such that* $F[S]$ *is a star with center* v_0*,* $F - S$ *has a perfect matching and either*

(i) $G - (F - S + v_0) = \mathbf{Y_{k'+1,k',k'}}$*, or*

(ii) $s = 2$*,* $v_1v_2 \in E(G)$*,* $G - F = \mathbf{Y_{k'-1,k',k'}}$ *and* G *has no edges between* $\{v_1, v_2\}$ *and the set* X_0 *in* $G - F$*;*

(e) $k = 2$ *and* $W_{n-1} \subseteq G \subseteq \mathbf{W^+_{n-1}}$*;*

(f) $k' = 2$*,* $|F| = 2\alpha' + 1 = n - 5$*, and* $G - F = C_5$*.*

3.2.4 Results for $\mathcal{DO}_k$

Theorem 1.28 can be restated as follows.

Theorem 3.9. *(Enomoto [12], Wang [13]) For* $k \in \mathbb{Z}^+$*, let* G *be a simple graph with* $\mathcal{SO}(G) \geq 4k - 1$ *and* $|G| \geq 3k$*. Then* G *has* k *disjoint cycles.*

Theorem 3.6 implies a description of graphs in $\mathcal{DO}_2$ with no two disjoint cycles (see Corollary 3.7).

The next theorem summarizes the results of [14] and [15].

Theorem 3.10. *For* $k, n \in \mathbb{Z}^+$ *with* $n \geq 3k$*, let* G *be an* n*-vertex simple graph in* $\mathcal{DO}_k$*. Then* G *has no* k *disjoint cycles if and only if one of the following holds:*

(S1) $k = 1$ *and* G *is a forest with at most one isolated vertex;*

(S2) $k = 2$ *and* G *satisfies the conditions of Corollary 3.7;*

(S3) $\alpha(G) = n - 2k + 1$*;*

(S4) $k = 3$ *and* $G = \mathbf{F_1}$ *(see Fig. 3.2);*

(S5) $k = 3$ *and* $G = \mathbf{F_2} = \{t\} \vee \overline{O}_5$ *where* $\mathbf{O_5}$ *is the* 5*-chromatic graph in Fig. 3.3;*

(S6) $k = 3$ *and* G *is the graph* $\mathbf{F_3}$ *in Fig. 3.4;*

(S7) $k \geq 3$, $n = 3k$, $\alpha(G) \leq k$, *and* $\chi(\overline{G}) > k$;

(S8) $k \geq 3, n = 3k$, *and* $G \subseteq \mathbf{Y_{k,c,2k-c}}$ *for some odd* $1 \leq c \leq 2k-1$;

(S9) $k \geq 3, n = 3k$, *and* $G = \mathbf{Y_{k-1,1,2k}}$.

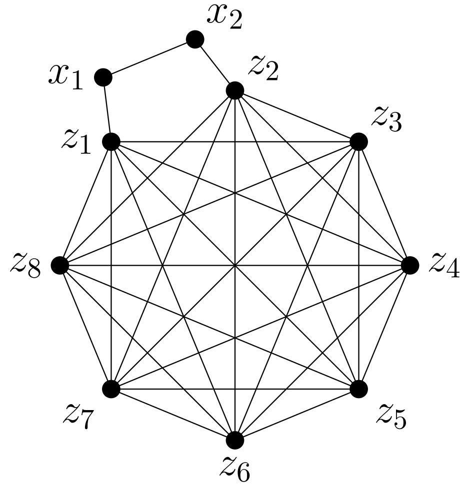


Figure 3.2: Graph $\mathbf{F_1}$

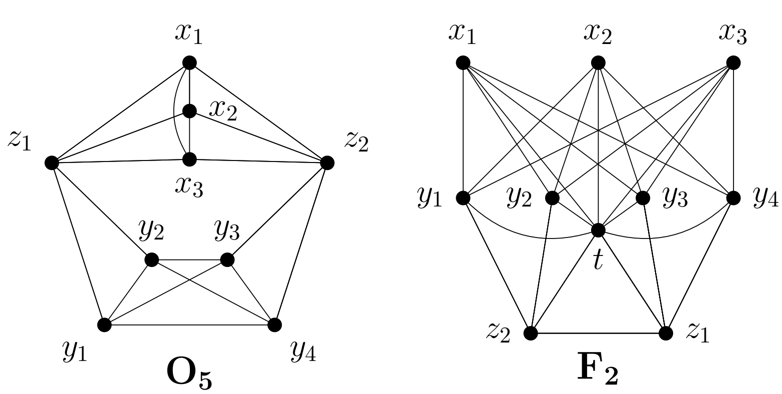


Figure 3.3: Graphs $\mathbf{O_5}$ and $\mathbf{F_2}$

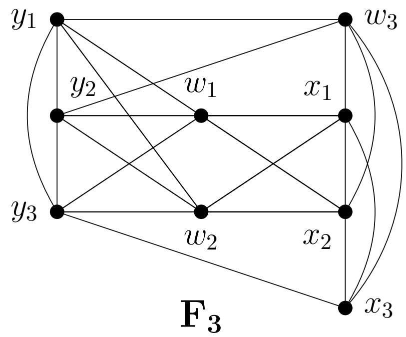


Figure 3.4: Graph $\mathbf{F_3}$

Remark 3.11. *The result of Rabern [24] (see also [25, 26]) implies that if (S7) holds then $k \leq 4$.*

3.3 Main results

In this section we state our main results. We call a graph $G \in \mathcal{DO}_k$ a counterexample if it does not have k disjoint cycles. We believe that it is possible to give an explicit list of all counterexamples in the style of previous results, but the list would be quite long and complicated. Here we are content to give broad categories of counterexamples together with a poly-time algorithm that determines membership.

The first theorem supports the algorithmic problem by proving that for $k \geq 5$ the loopless multigraphs in $\mathcal{DO}_k$ are counterexamples if and only if they belong to at least one of five categories. The second theorem gives a poly-time algorithm that detects if $G \in \mathcal{DO}_k$ is a counterexample by describing, for each of the five categories, a poly-time algorithm that determines membership.

Theorem 3.12. *Let $k \geq 5$ and $n \geq k$ be integers. Let G be an n-vertex multigraph in $\mathcal{DO}_k$ with no loops. Set $F = F(G)$, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Let (D, A, C) be the GE-decomposition of F and let $D' = V(G) - V(F)$. Then G does not contain k disjoint cycles if and only if one of the following holds:*

(Q1) $n < 3k - \alpha'$;

(Q2) $n > 2k+1$, *G is extremal and either*

- *(Q2a)* *some big set is not incident to any strong edge, or*
- *(Q2b)* *for some two distinct big sets J and J', all strong edges intersecting $J \cup J'$ have a common vertex outside of $J \cup J'$;*

(Q3) $k' \geq 5$ *and* $n = 3k - \alpha'$, *and G has a vertex $x \in D'$ of degree $k + \alpha' - 1$ such that for each maximum matching M in F, the set $N(x) - W(M)$ is independent;*

(Q4) $3k - \alpha' \leq n \leq 3k - \alpha' + 1$ *and* $k' \leq 4$ *and* $\underline{G} - W(M)$ *has no $k - |M|$ disjoint cycles for all (possibly nonmaximum) matchings M in F; or*

(Q5) $k' \geq 5$ *and* $n = 3k - \alpha'$, $|F| - 2\alpha' \in \{0, |D|-2, |D|-1\}$ *and for all maximum matchings M in F either $\alpha(G'(M)) = k' + 1$ or $G'(M) \subseteq \mathbf{Y_{k',c,2k'-c}}$ for some odd $c \leq k'$.*

With this theorem, we can also prove the multigraph analogue of Theorem 1.32.

Theorem 3.13. *There is a polynomial time algorithm that for every multigraph $G \in \mathcal{DO}_k$ decides whether G has k disjoint cycles or not.*

3.4 Proof of Theorem 3.12: Sufficiency

We will prove that if G contains a set $\mathcal{C} = \{C_1, \ldots, C_k\}$ of k disjoint cycles, then all of the conditions (Q1)–(Q5) fail. Given such $\mathcal{C}$, let $M \subseteq \mathcal{C}$ be the set of cycles in $\mathcal{C}$ that are strong edges, $m = |M|$ and $\mathcal{C}' = \mathcal{C} - M$. Since $m \leq \alpha'$ and each cycle that is not a strong edge has at least 3 vertices, $n \geq 2m + 3(k-m) = 3k - m \geq 3k - \alpha'$; so (Q1) does not hold.

If (Q2) holds, then G is extremal. Every big set J satisfies $|V(G) - J| < 2k$. So some cycle $C_J \in \mathcal{C}$ has at most one vertex in $V(G) - J$. Since J is independent, C_J has at most one vertex in J. Thus C_J is a strong edge and (Q2a) fails. Suppose there are big sets J and J' satisfying (Q2b). Then, $|W(M) \cap (J \cup J')| \leq 1$ since $n > 2k+1$ implies $J \cap J' = \emptyset$ by (3.2), and so for some $I \in \{J, J'\}$, $I \subseteq V - W(M)$. By this fact and independence, each cycle in $\mathcal{C}$ has at least two vertices outside of I, and so $|I| \leq n - 2k$ contradicting the definition of a big set. So (Q2b) also fails.

Note that the $k - |M|$ cycles in $\mathcal{C}'$ correspond to disjoint cycles in $\underline{G} - W(M)$, so (Q4) does not hold.

If $n = 3k - \alpha'$, then $m = \alpha'$, every cycle in $\mathcal{C}'$ is a triangle and every vertex in $G - W(M)$ belongs to exactly one triangle in $\mathcal{C}'$. Therefore, (Q3) and (Q5) do not hold.

3.5 Proof of Theorem 3.12: Necessity

Suppose G does not have k disjoint cycles and that none of the conditions (Q1)–(Q5) hold. Because (Q4) does not hold, either $n \geq 3k - \alpha' + 2 = 2k + k' + 2$, or $k' \geq 5$; the latter implies that $n \geq 3k - \alpha' = 2k + k' \geq 2k + 5$. Therefore,

$$n \geq 2k + 3. \tag{3.4}$$

Among the maximum matchings in F, choose a matching M such that

$$\alpha(G - W) \text{ is minimum, where } W = W(M). \tag{3.5}$$

Then $|M| = \alpha'$, $G' = G - W$ is simple, and $\mathcal{SO}(G') \geq 4k - 3 - 4\alpha' = 4k' - 3$. So $G' \in \mathcal{DO}_{k'}$. Let $n' := |V(G')| = n - 2\alpha'$. Since (Q1) does not hold,

$$n' \geq 3k'. \tag{3.6}$$

If $n' = 3k'$, then G' is quite dense, so sometimes it will be convenient to consider the complement of $\underline{G}$. For $v \in V(G)$, let $\overline{N}[v] = V(G) - N[v]$ and $\overline{s}(v) = |\overline{N}[v]| = n - 1 - s(v)$. When $n' = 3k'$, we have $n = 2k + k'$ and thus the inequality $s(v) + s(u) \geq 4k - 3$ can be written as

$$\overline{s}(v) + \overline{s}(u) \leq 2k' + 1 \quad \text{for all } vu \notin E(G). \tag{3.7}$$

Since G' has no k' disjoint cycles and $n' \geq 3k'$, one of (S1)–(S9) in Theorem 3.10 hold for G' with k' in place of k. We will now show that each of (S1)–(S9) will lead to one of

(Q1)–(Q5) holding or some other contradiction.

CASE 1: (S4), (S5), or (S6) hold for G'.

Then, $k' \leq 3$ and $3k' \leq |G'| \leq 3k'+1$, so (Q4) holds, because G has no k disjoint cycles.

CASE 2: (S3) holds for G'.

Then $n \geq 3k' + 2\alpha' > 2k+1$ and G' is extremal. Let J be a big set in G'. Then $|J| = n' - 2k' + 1 = n - 2k + 1$. So G is extremal and J is a big set in G. Since (Q2a) fails, some $w \in J$ has a strong neighbor v. Let vu be the edge in M containing v. In F, consider the maximum matching $M' = M - vu + wv$, and set $G'' = G - W(M')$. By (3.5), G'' contains a big set J', and J' is big in G. Since $w \notin J'$, $J' \neq J$. So by (3.2), $J' \cap J = \emptyset$ (possibly, $u \in J'$). Since (Q2a) fails, some $w' \in J'$ has a strong neighbor v'. Possibly, $v' = v$, but then since (Q2b) fails, some $w'' \in J \cup J'$ has a strong neighbor $v'' \neq v$. Thus we can choose notation so that $v' \neq v$. As M' is maximum, there is an edge $v'u' \in M'$. Set $M'' = M' + w'v' - v'u'$ and $G^* := G - W(M'')$. Again by (3.5), G^* contains a big set J''. Since $w, w' \notin J''$, we have $J'' \notin \{J, J'\}$. So by (3.2), $J'' \cap (J \cup J') = \emptyset$. Thus, since $V(G^*) \supseteq (J - w) \cup (J' - w') \cup J''$,

$$n' \geq 3|J| - 2 = 3(n' - 2k' + 1) - 2 = 3n' - 6k' + 1,$$

which yields $2n' \leq 6k' - 1$, a contradiction to $n' \geq 3k'$. Hence (Q2) holds.

CASE 3: (S7) holds for G'.

So $k' \geq 3$, $|G'| = 3k'$, $\alpha(G') \leq k'$ and $\chi(\overline{G'}) > k'$. Since $|G'| = 3k'$, (3.7) must hold. Since $\chi(\overline{G'}) > k'$, G' contains an induced subgraph G_0 such that $\overline{G_0}$ is a vertex-$(k'+1)$-critical graph. By (3.7),

for every $xy \in E(\overline{G_0})$, the sum of the degrees of x and y in $\overline{G_0}$ is at most $2k'+1$. (3.8)

The $(k'+1)$-critical graphs satisfying (3.8) were studied recently. If $k' \geq 5$, then by results in [24] and [25], $\overline{G_0} = K_{k'+1}$, which means $\alpha(G') \geq k' + 1$, a contradiction to the case. If $k' \leq 4$, then (Q4) holds.

CASE 4: (S1) holds for G'.

So $k' = 1$ and G' is a forest with at most one isolated vertex. Since $k \geq 5$, $|M| \geq 4$. Let $xz, x'z', x''z''$ be three strong edges in M.

Case 4.1: G' has at least two non-singleton components, say H_1 and H_2. Then $n' \geq 4$. For $i = 1, 2$, let P_i be a longest path in H_i, and let u_i and w_i be the ends of P_i. As $\mathcal{SO}(G) \geq 4k - 3$, at most two edges between $W = W(M)$ and $\{u_1, u_2, w_1, w_2\}$ are missing in G. So we may assume that at most one edge between $\{x, z\}$ and $\{u_1, u_2, w_1, w_2\}$ is missing in G. By symmetry, we assume that among these edges only xu_1 could be missing in G. Then the $\alpha' - 1$ strong edges of $M - xz$ and the cycles $xu_2P_2w_2x$ and $zu_1P_1w_1z$ form k disjoint cycles in G, a contradiction.

Case 4.2: G' has a unique non-singleton component H, and this H is not a star. Let $P = y_1 \dots y_t$ be a longest path in H. Since H is not a star, $t \geq 4$. Then y_1 is a leaf in G', and either $d_{G'}(y_2) = 2$ or y_2 is adjacent to a leaf $l \neq y_1$. Let $y_1' = y_2$ if $d_H(y_2) = 2$ and $y_1' = l$ otherwise. Similarly, either $d_{G'}(y_{t-1}) = 2$ or y_{t-1} is adjacent to a leaf $l' \neq y_t$ since P is maximal. Let $y_t' = y_{t-1}$ if $d_H(y_{t-1}) = 2$ and $y_t' = l'$ otherwise. Since $y_1y_t', y_1'y_t \notin E(G)$ and $G \in \mathcal{DO}_k$,

> *the number of missing edges between $\{y_1, y_1', y_t, y_t'\}$ and W in G is at most $q + r$, where $q = |\{y_1', y_t'\} \cap \{y_2, y_{t-1}\}|$ and r is the number of low vertices in $\{y_1, y_1', y_t, y_t'\}$.* (3.9)

Since $q \leq 2$, $r \leq 2$ and $|M| \geq 3$, we can assume that at most one edge between $\{x, z\}$ and $\{y_1, y_1', y_t, y_t'\}$ is missing in G. So we get a contradiction as at the end of Case 4.1.

Case 4.3: The unique non-singleton component H of G' is a star. The leaves of the star, along with the isolated vertex if it exists, form an independent set of size $n' - 1 = n' - 2k' + 1$. By (3.4), we are in CASE 2.

Remark. The proof of the next case works even if (3.5) does not hold, and we will use this in CASE 6.

CASE 5: (S9) holds for G'.

So $n' = 3k'$ and $G' \subseteq \mathbf{Y_{k'-1,1,2k'}}(Y, \{x\}, Z)$. If $k' \leq 4$, then (Q4) holds. So below we assume

$$k' \geq 5. \tag{3.10}$$

Since $n' = 3k'$, we will often use (3.7). Since each $y \in Y$ has $k'-2$ nonneighbors in Y, (3.7) yields

$$|\overline{N}[y] - Y| + |\overline{N}[y'] - Y| \leq 5 \quad \text{for all distinct } y, y' \in Y. \tag{3.11}$$

Since x is not adjacent to any of the $2k'$ vertices in Z, by (3.7)

$$N(x) = V(G) - Z - x \text{ and } N(z) = V(G) - x - z \text{ for each } z \in Z. \tag{3.12}$$

If x has a strong neighbor v_0 with the M-mate u_0, then we construct k disjoint cycles in G as follows. First, take the α' strong edges in $M - v_0u_0 + v_0x$. By (3.12), $G[Z] = K_{2k'}$ and each $y \in Y + u_0$ is adjacent to all of Z. So, we take k' 3-cycles each of which contains one vertex in $Y + u_0$ and two vertices in Z. This contradiction shows that $x \in D'$.

Since $x \in D'$ and $d(x) = k + \alpha' - 1$, if (Q3) does not hold, then F has a maximum matching M' such that

$$\text{there are } u_1, u_2 \in V(G) - W(M') - Z \text{ with } u_1u_2 \in E(G). \tag{3.13}$$

For $i = 1, 2$ the symmetric difference $M \triangle M'$ contains a path P_i of an even length an end of which is u_i. Since the other end w_i of P_i is not covered by M, $w_i \in V(G') \cap D$. Also by definition, none of the vertices in G' is an internal vertex in P_i. In particular, $x \notin V(P_i)$. Let M'' be the maximum matching in F such that $M \triangle M'' = P_1 \cup P_2$. Then $V(G) - W(M'') = V(G') - \{w_1, w_2\} \cup \{u_1, u_2\}$. If $|\{w_1, w_2\} \cap Z| = \ell_Z$ and $|\{w_1, w_2\} \cap Y| = \ell_Y$, then we can renumber the vertices in $Z - \{w_1, w_2\}$ and $Y - \{w_1, w_2\}$ as $z_1, \ldots, z_{2k'-\ell_Z}$,

$y_1, \ldots, y_{k'-1-\ell_Y}$ and construct k disjoint cycles in G as follows. Take the $k-k'$ strong edges in M'', then take the cycle xu_1u_2x and for $j = 1, \ldots, k'-1-\ell_Y$ take the cycle (y_j, z_{2j-1}, z_{2j}). Finally, if $\ell_Y \geq 1$, then $|Z - \{z_1, \ldots, z_{2(k'-1-\ell_Y)}, w_1, w_2\}| = 3\ell_Y$, then we simply take ℓ_Y triangles in the remaining complete graph $G[Z - \{z_1, \ldots, z_{2(k'-1-\ell_Y)}, w_1, w_2\}]$. Hence (Q3) holds.

CASE 6: (S8) holds for G'.

So $n' = 3k'$ and $G' \subseteq \mathbf{Y_{k',c,2k'-c}}(Y, X, Z)$ for $k' \geq 3$ and some odd $1 \leq c \leq k'$. If $k' \leq 4$, then (Q4) holds. So as in the previous case, we assume $k' \geq 5$.

Let M' be an arbitrary maximum matching in F. Since G has no k disjoint cycles, $G'(M')$ does not have k' disjoint triangles. Therefore, by (3.6), (3.10) and Theorem 3.10 (with Remark 3.11), we have that one of (S3), (S8), or (S9) hold in $G'(M')$. By the remark before CASE 5,

$$\textit{if (S9) holds in } G'(M'), \textit{ then (Q3) holds.} \tag{3.14}$$

If (S3) holds in $G'(M')$, then $\alpha(G'(M')) = n' - 2k' + 1 = k' + 1$. Therefore, if we assume (Q3) does not hold, to show that (Q5) holds, we only need to show that $|F| - 2\alpha' \in \{0, |D| - 2, |D| - 1\}$ which is true when $|W| \leq 2 + |A| + |C|$.

Since $n' = 3k'$, we will often use (3.7). Since each $y \in Y$ has $k' - 1$ nonneighbors in Y, (3.7) yields

$$|\overline{N}[y] - Y| + |\overline{N}[y'] - Y| \leq 3 \quad \text{for all } y, y' \in Y. \tag{3.15}$$

By (3.15),

$$\text{there is } y_0 \in Y \text{ such that } |\overline{N}[y] - Y| \leq 1 \text{ for every } y \in Y - y_0. \tag{3.16}$$

Since each $x \in X$ has $2k' - c$ nonneighbors in Z, if x has a nonneighbor $y \in Y$, then by (3.7),

$$2k' + 1 \geq \overline{s}(x) + \overline{s}(y) \geq (2k' - c + 1) + (k' - 1 + 1) = 3k' - c + 1,$$

which yields $c = k'$. Moreover, if in this case some $z \in Z$ also has a nonneighbor $y' \in Y$, then again by (3.7), $2k'+1 \geq \overline{s}(x)+\overline{s}(z) \geq (k'+1)+(k'+1) = 2k'+2$, a contradiction. Thus, we may assume (by possibly switching the roles of X and Z when $c = k'$) that

$$|\overline{N}[x] \cap W| \leq 1 \text{ and } \overline{N}[x] \cap \overline{W} = Z \text{ for each } x \in X, \tag{3.17}$$

and

$$|\overline{N}[z] - X| \leq 1 \text{ for each } z \in Z, \text{ and if } c = k' \text{ then } G[Z] = K_c. \tag{3.18}$$

We will need the following lemma.

Lemma 3.14. *Let $t \geq 2$ and $\epsilon \in \{0,1\}$. Let H be a graph with a partition $V(H) = R \cup Q$ such that $|R| = 2t+\epsilon$, $|Q| = 3t - |R| = t - \epsilon$, and let $y_0 \in Q$. If*

1. *each $u \in R$ has at most one nonneighbor in H and*
2. *each $y \in Q - y_0$ has at most $1+\epsilon$ nonneighbors in R and*
3. *y_0 has at most 2 nonneighbors in R and has only $1+\epsilon$ nonneighbors if $t = 2$.*

then H contains t vertex-disjoint triangles.

Proof. Using induction, note the lemma holds for $t = 2$. If $t \geq 3$ then H has a triangle $T = y_0z_1z_2y_0$ with $z_1, z_2 \in R$. By induction $H' := H - T$ has $t-1$ disjoint triangles. □

Claim 3.15. *Let $G' \subseteq \mathbf{Y}_{\mathbf{k',c,2k'-c}}(Y, X, Z)$ for $k' \geq 4$ and an odd $c \leq k'$. Suppose there are $w \in V(G')$ and $u \in W$ such that F has an M-alternating u,w-path P.*

(A) If $w \in Y \cup Z$, then u has no neighbor in $Y - w$ or no neighbor in X.

(B) If $w \in X$, then u has no neighbor in Y or no neighbor in Z.

Proof. Let M' be the matching obtained from M by switching edges on P. Then $W(M') = W(M) - w + u$. Set $t = (2k'-c-1)/2$. Since $1 \leq c \leq k'$ and is odd, by (3.10),

$$|Z| = 2k' - c \geq 5 \text{ and } k'-1 \geq t \geq 2. \tag{3.19}$$

Arguing by contradiction, we assume the lemma fails and construct k disjoint cycles.

CASE 1: $w \in Y \cup Z$. Since (A) does not hold, u has neighbors $x \in X$ and $y \in Y - w$.

Pick $y \in N(u) \cap Y - w$ with $s(y)$ minimum. Then for y_0 defined in (3.16), we have

$$\textit{if } y_0 \in Y - w - y, \textit{ then } y_0 u \notin E(G), \textit{ and so by } (3.15), |\overline{N}[y_0] \cap Z| \leq 2. \tag{3.20}$$

By (3.17), $T := uxyu \subseteq G$. Set $\epsilon := 0$ if $w \in Z$; else $\epsilon := 1$. Partition $Y - y - w$ as $\{Q, \overline{Q}\}$ so that $|Q| = t - \epsilon$, $|\overline{Q}| = \frac{c-1}{2}$, and $y_0 \in \overline{Q} \cup \{w, y\}$ if $c > 1$. So $t \geq 3$, if $y_0 \in Q$. Regardless, by (3.16), (3.18) and (3.20), Q and $R := Z - w$ satisfy the conditions of Lemma 3.14. Thus $Q \cup R$ contains t disjoint triangles. By (3.17), $(X - x) \cup \overline{Q}$ contains $\frac{c-1}{2}$ disjoint triangles. Counting these $k' - 1$ triangles, T, and $k - k'$ strong edges of M' gives k disjoint cycles.

CASE 2: $w \in X$. Since (B) fails, there are $z \in N(u) \cap Z$ and $y \in N(u) \cap Y$. Our first goal is to show there is an edge with ends in $N(u) \cap Y$ and $N(u) \cap Z$. If $N(u) \cap N(z) \cap Y \neq \emptyset$ then we are done. Else, by (3.18), $N(z) \cap Y = Y - y = \overline{N}[u] \cap Y$. Let $y' \in Y - y$. By (3.15) applied to y and y', $|\overline{N}[y] \cap Z| \leq 2$. By (3.7) applied to u and y', $|\overline{N}[u] \cap Z| \leq 2$. By (3.19), $|Z| \geq 5$, so there is $z' \in Z \cap N(u) \cap N(y)$, and we are done.

Pick $yz \in E$ with $y \in N(u) \cap Y$ and $z \in N(u) \cap Z$ so that $s(y)$ is minimum and let $T := uzyu$. Then for y_0 defined in (3.16), using (3.15),

$$\text{if } y_0 \in Y - y \text{ then } |\overline{N}[y_0] \cap (Z - z)| \leq 2, \tag{3.21}$$

since $y_0 u \notin E(G)$ or $y_0 z \notin E(G)$.

Partition $Y - y$ as $\{Q, \overline{Q}\}$ so that $|Q| = t$, $|\overline{Q}| = \frac{c-1}{2}$, and $y_0 \in \overline{Q} + y$ if $c > 1$. So $t \geq 3$, if $y_0 \in Q$. Regardless, by (3.16), (3.18) and (3.21), Q and $R := Z - z$ satisfy the conditions of Lemma 3.14. Thus $Q \cup R$ contains t disjoint triangles. By (3.17), $(X - w) \cup \overline{Q}$ contains $\frac{c-1}{2}$ disjoint triangles. Counting these $k' - 1$ triangles, T, and $k - k'$ strong edges of M' gives k disjoint cycles. □

Claim 3.16. *Let $G' \subseteq \mathbf{Y_{k',c,2k'-c}}(Y, X, Z)$ for $k' \geq 4$ and an odd $c \leq k'$. Then, $|D \cap W| \leq 2$.*

Proof. Suppose $u \in D \cap W$. Then there is a matching M' and vertex $w_u \in V(G')$ such that $W(M') = W(M) + w_u - u$ and there is an M, M'-alternating path from u to w_u. By Claim 3.15, u has no neighbors in $Y - w_u$ or in X or in Z.

By degree condition (3.7), there is at most one $u \in D \cap W$ with no neighbor in X or no neighbor in Z; otherwise for any $x \in X$ and $z \in Z$ we have the contradiction

$$\|\{x, z\}, W\| \leq 4\alpha' - 2 \text{ and so } s(x) + s(z) \leq 4k' - 2 + 4\alpha' - 2 \leq 4k - 4.$$

Similarly, there is at most one $u \in D \cap W$ with at most one neighbor in Y; otherwise, as $k' \geq 4$, there are distinct $y, y' \in Y$ with

$$\|\{y, y'\}, W\| \leq 4\alpha' - 4 \text{ and so } s(y) + s(y') \leq 4k' + 4\alpha' - 4 \leq 4k - 4.$$

Thus $|D \cap W| \leq 2$. □

Claim 3.16 yields that $|W| \leq 2 + |A| + |C|$. Thus (Q5) holds.

CASE 7: (S2) holds for G'.

So $n' \geq 3k'$ and $k' = 2$ and G' satisfies one of (Y1)–(Y5) from Corollary 3.7. If $n' \leq 7$ then (Q4) holds, so assume $n' \geq 8$. This implies that G' satisfies either (Y4) or (Y5). As $k \geq 5$, $|M| = \alpha' = k - k' \geq 3$.

Define a vertex $v \in \overline{W}$ to be *i-acceptable* if $|N(v) \cap W| \geq 2\alpha' - i$, *acceptable* if it is 1-acceptable, and *good* if it is 0-acceptable. Let $u, v \in \overline{W}$ with $uv \notin E$. If i and j are minimum natural numbers such that u is i-acceptable and v is j-acceptable, then

$$i + j \leq d_{G'}(u) + d_{G'}(v) - 5. \tag{3.22}$$

Case 7.1: G' satisfies (Y4), i.e., $G' \in \{H, \mathrm{sd}(H, e, x), \mathrm{sd}(H, e, xy)\}$, where $W_{|H|-1} \subseteq H \subseteq \mathbf{W}^+_{|\mathbf{H}|-\mathbf{1}}$. Set $t = |H| - 1$. Let H have center v_0 and rim $v_1 \ldots v_t v_1$, and let $\mathbf{W'_t}$ be the result of adding a parallel edge between v_0 and v_1 in W_t. Since G' is simple, we may assume $H \in \{W_t, \mathbf{W'_t}\}$. If $G' \neq H$ then we may assume that the subdivided edge e is incident to v_1. As $n' \geq 8$, $t \geq 5$.

Case 7.1.1: $t = 5$. If the subdividing vertex x exists, by (3.22), the subdividing vertices and v_3, v_4, v_5 are all good, v_2 is acceptable, and v_1 is 2-acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ with $av_1, bv_2 \in E$. If there is no subdividing vertex, we can use symmetry to maintain the same traits. Then there are k disjoint cycles $v_0v_4v_5v_0$, av_1xa, bv_2v_3b, and $|M - ab|$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 7.1.2: $t \geq 6$. By (3.22), the rim vertices v_3, v_4, v_5, v_6 are all acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ such that av_3v_4a and bv_5v_6b are cycles. Let C be the smallest cycle containing v_0, v_1, v_2 (and any subdividing vertices). Then there are k disjoint cycles C, av_3v_4a, bv_5v_6b and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 7.2: G' satisfies (Y5), i.e., $G' \in \{H, \mathrm{sd}(H, e, x), \mathrm{sd}(H, e, xy)\}$, where

$$K_{3,|H|-3}(Y, Z_t) - e' \subseteq H \subseteq \mathbf{K}^+_{\mathbf{3,|H|-3}}(Y, Z_t)$$
$$\text{with } Y = \{y_1, y_2, y_3\}, Z_t = \{z_1, \cdots, z_t\}.$$

As $n' \geq 8$, $t \geq 3$. If $\alpha(G') \geq n' - 2k' + 1$ then (Q2) holds and so (S3) holds which falls under CASE 2. So assume the subdividing vertex x exists in G'.

Case 7.2.1: $e = y_hy_i$, where $\{h, i, j\} = [3]$. Since $\alpha(G') \leq n' - 2k'$ and $Z + x$ is independent, e is subdivided twice. As $d_{G'}(x) = 2$, every vertex of Z is adjacent to every vertex of Y (and no other vertex of G'). Thus $G' = \mathrm{sd}(H, e, xy)$ and the vertices of $Z + x + y$ are all good.

Suppose $t = 3$. Then $d_{G'}(y_j) \leq 5$. By (3.22), y_j is 2-acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ with $ay_j \in E$. Thus there are k disjoint cycles ay_jz_1a, $bxyb$, $z_2y_hz_3y_iz_2$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Otherwise $t \geq 4$. Then, for every $ab \in M$, there are k disjoint cycles $axya$, $bz_1y_1z_2b$, $z_3y_2z_4y_3z_3$, and $\alpha' - 1$ other strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 7.2.2: $e \in E(Y, Z_t)$. Now H is simple. Say $e = y_1z_1$ and $e' = y'z'$. If $e' \notin E(H)$ then $y' \neq y_1$. By degree conditions $xz' \in E$, so $z' = z_1$. As $xz_i, z_1z_i \notin E$ for $i \geq 2$, (3.22) implies all vertices of $Z - z_1$ and all subdividing vertices are good, z_1 is acceptable, and z_1 is good if $e' \notin H$.

Case 7.2.2.1: $t \geq 4$. Let $ab \in M$ with $a \in N(z_1)$. If $t \geq 5$ then there are k disjoint cycles, $az_1xa, bz_2y_1z_3b, z_4y_2z_5y_3z_4$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$. Else

$t = 4$. Since $d_{G'}(y_2) \le 6$ and $xy_2 \notin E$, (3.22) implies y_2 is 3-acceptable. As z_1 is acceptable and $|M| \ge 3$, there is an edge $ab \in M$ with $az_1, by_2 \in E$. As x and z_2 are good, this yields k disjoint cycles az_1xa, by_2z_2b, $z_3y_1z_4y_3z_3$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 7.2.2.2: $t = 3$ and z_1y_1 is subdivided twice with $z_1x, yy_1 \in E$. Then x and y are both good. Since $d_{G'}(y_1) \le 5$ and $xy_1 \notin E$, y_1 is 2-acceptable. As z_1 is acceptable, there is an edge $ab \in M$ with $az_1, by_1 \in E$. Thus, there are k disjoint cycles az_1xa, by_1yb, $y_2z_2y_3z_3y_2$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 7.2.2.3: $t = 3$ and z_1y_1 is subdivided once. Suppose there is an edge $y_iy_j \in E$, where $[3] = \{i, j, h\}$. Then $d_{G'}(y_h) \le 5$ and either $y_hx \notin E$ or $y_hz_1 \notin E$. By (3.22), y_h is 3-acceptable. As $|M| \ge 3$, there is an edge $ab \in M$ with $az_1, by_h \in M$. Thus there are k disjoint cycles az_1xa, by_hz_2b, $y_iz_3y_jz_3$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$. So assume $\|G[Y]\| = 0$.

If $|F| = 2\alpha'$ then (Q4) holds. Else there are edges $ab, a'b' \in M$ and a vertex $u \in \overline{W}$ with $au \in E(F)$. All vertices of G' are good except one of y_1, z_1 might only be acceptable. Choose notation so that $\{b, a', b'\} = \{c_1, c_2, c_3\}$ and $|N(c_1) \cap \overline{W}| \ge 6$ and $|N(c_2) \cap \overline{W}|, |N(c_3) \cap \overline{W}| \ge 7$. By inspection $G' - u$ contains a perfect matching $\{e_1, e_2, e_3\}$ with $e_1 \subseteq N(c_1)$. Thus G contains k disjoint cycles, $c_1e_1c_1$, $c_2e_2c_2$, $c_3e_3c_3$, aua and $\alpha' - 2$ other strong edges, contradicting $G \in \mathcal{BO}_k$. $\square$

3.6 Proof of Theorem 3.13

To construct the algorithm, we first describe several subroutines.

Lemma 3.17. *Let $k \ge 4$ and $n = 3k$. There is a subroutine that for any simple graph $G = (V, E) \in \mathcal{DO}_k$ with $|G| = n$ checks whether $G \subseteq \mathbf{Y_{k,s,2k-s}}$ for some $s \le k$, and in this case constructs the representation $G \subseteq \mathbf{Y_{k,s,2k-s}}(Y, X, Z)$, all in $O(n^3)$ time.*

Proof. Note that if $G \subseteq \mathbf{Y_{k,s,2k-s}}(Y, X, Z)$ for some $s \le k$, then we can choose notation so that every $x \in X$ is adjacent to every other vertex in $X \cup Y$ by (3.17). Search for a vertex x such that $d(x) \le 2k - 1$ and $N[x]$ can be partitioned as $\{Q, R\}$ so that $Q = \{v \in N[x] :$

$N[v] = N[x]\}$ and R is independent with $|R| = k$. This takes $O(n^3)$ time. If we find such an x then $G \subseteq \mathbf{Y_{k,s,2k-s}}(R, Q, Z)$, $Q = X$, and $R = Y$. Otherwise, $G \nsubseteq \mathbf{Y_{k,s,2k-s}}$ for any $s \leq k$. □

Lemma 3.18. *There are subroutines that for any simple graph F with $|F| = n$ construct:*

1. *a maximum matching of F in $O(n^{2.5})$ time;*

2. *a GE-decomposition (D, A, C) of F in $O(n^{3.5})$ time.*

Proof. For (1) see [27]. For (2), using (1), find the sizes of a maximum matching in F and in all the graphs $(F - v)$ with $v \in V(F)$. This can be done in $O(n^{3.5})$ time. Set $D = \{v \in V(F) : \alpha'(F - v) = \alpha'(F)\}$, $A = N_F(D) - D$ and $C = V(F) - D - A$. □

Now, we are ready to define our algorithm.

Setup. We are given a positive integer k and a multigraph $G \in \mathcal{DO}_k$. By Corollary 3.3 we may assume G is loopless. Construct the simple graph F induced by the strong edges of G and the GE-decomposition (D, A, C) of F in $O(n^{3.5})$ time. Set $|G| = n$, $D' = V(G) - V(F)$, $\alpha' = \alpha'(F)$ and $k' = k - \alpha'$.

If $k \leq 4$, then construct G_1 from G by subdividing each edge. Then G_1 is a simple graph with $n + \|G\|$ vertices and $2\|G\|$ edges, and the number of disjoint cycles in G_1 equals that in G. By Theorem 1.26, we can determine whether G_1 has k disjoint cycles in linear time in $n + \|G\|$. So, in total this step takes $O(n^2)$ time. Thus below we assume $k \geq 5$ and apply Theorem 3.12 to G. Checking (Q1) is trivial, so it remains to show how to check (Q2)–(Q5).

Check (Q2). First check whether $n > 2k + 1$ and G is extremal: $\mathcal{DO}_k$. As observed in Subsection 2.1, every big set $J \subseteq G$ has the form $J = V(G) - N(v)$ for some vertex v with $s(v) = 2k - 1$. We can find all such sets in $O(n^3)$ time by checking whether $V(G) - N(v)$ is independent for each $v \in V(G)$ with $s(v) = 2k - 1$. If $n \leq 2k + 1$ or there are no such sets, then (Q2) fails. Otherwise, let $I_1, \dots, I_q$ be the big sets in G. As $n > 2k + 1$, (3.2) implies they are disjoint, so $q < n$. For each $j \in [q]$, check whether I_j has no strong neighbors or has a unique strong neighbor $w(j)$. This takes $O(n^2)$ time. If at least one I_j has no strong neighbors or $w(j) = w(j')$ for some distinct $j, j' \in [q]$, then (Q2) holds; otherwise, (Q2) does not hold.

Check (Q3). First confirm that $k' \geq 5$ and $n = 3k - \alpha'$. Next construct the set U of vertices $v \in D'$ with $s(v) = k + \alpha' - 1$. Then test each $v \in U$ to see if

(*) for some adjacent pair $\{x, y\} \in N(v)$ there is an α'-matching contained in $F - x - y$.

This uses $O(n^{5.5})$ steps. Now (Q3) holds if and only if (*) fails.

Check (Q4). First confirm that $3k - \alpha' \leq n \leq 3k - \alpha' + 1$ and $k' \leq 4$. If so then we still need to check whether $\underline{G} - W(M)$ has no $k - |M|$ disjoint cycles for all matchings M in F. If $|\underline{G} - W(M)| \leq 3(k - |M|) - 1$ then $\underline{G} - W(M)$ does not have enough vertices to have $k - |M|$ disjoint cycles. So it suffices to check for every $W \subseteq V(G)$ with $2(\alpha' - 1) \leq |W| \leq 2\alpha'$ whether (i) $F[W]$ has a perfect matching and (ii) $\underline{G} - W$ has no $k - |W|/2$ disjoint cycles. Then (Q4) holds if and only if (i) implies (ii) for all such W. As $n - |W| \leq 3(k - \alpha' + 1) \leq 15$, there are $O(n^{15})$ sets to test. Testing (i) takes $O(n^{2.5})$ time and testing (ii) takes $O(1)$ time. So altogether we use $O(n^{17.5})$ time.

Check (Q5). First confirm that $k' \geq 5$, $n = 3k - \alpha'$ and $|F| - 2\alpha' \in \{0, |D| - 2, |D| - 1\}$. If so then we still need to check that for all maximum matchings M either (i) $\alpha(G - W(M)) = k' + 1$ or (ii) $G - W(M) \subseteq \mathbf{Y_{k',c,2k'-c}}$ for some odd $c \leq k'$. We do this by checking certain subsets $W \subseteq V(G)$ to see if $F[W]$ has a perfect matching M satisfying (i) and (ii). If $|F| - 2\alpha' = 0$ then $W := W(M) = V(F)$; else, using $|F| - 2\alpha' \in \{|D| - 2, |D| - 1\}$, $V(F) = A \cup C \cup D$ and $W = A \cup C \cup (W \cap D)$, we have

$$|D - W| = |F| - 2\alpha' \geq |D| - 2 = |D - W| + |W \cap D| - 2,$$

so $|W \cap D| \leq 2$. Thus we only need to check $O(n^2)$ sets W. By Lemma 3.17 and the argument in Check (Q2), each check takes $O(n^3)$ time, so all together we use (n^5) time.

This completes our description of the algorithm. $\square$

Chapter 4

Chorded Cycles

4.1 Introduction

Recall, our main result for this chapter is

Theorem 4.1. *Let $k \geq 2$, G be an n-vertex graph with $n \geq 4k$, and $\sigma_2(G) \geq 6k-3$. Then G does not contain k vertex-disjoint chorded cycles if and only if*

- $G_1^-(n,k) \subseteq G \subseteq G_1(n,k)$,
- $G_2^{**}(k) \subseteq G \subseteq G_2(k)$, *or*
- $G = G_3$.

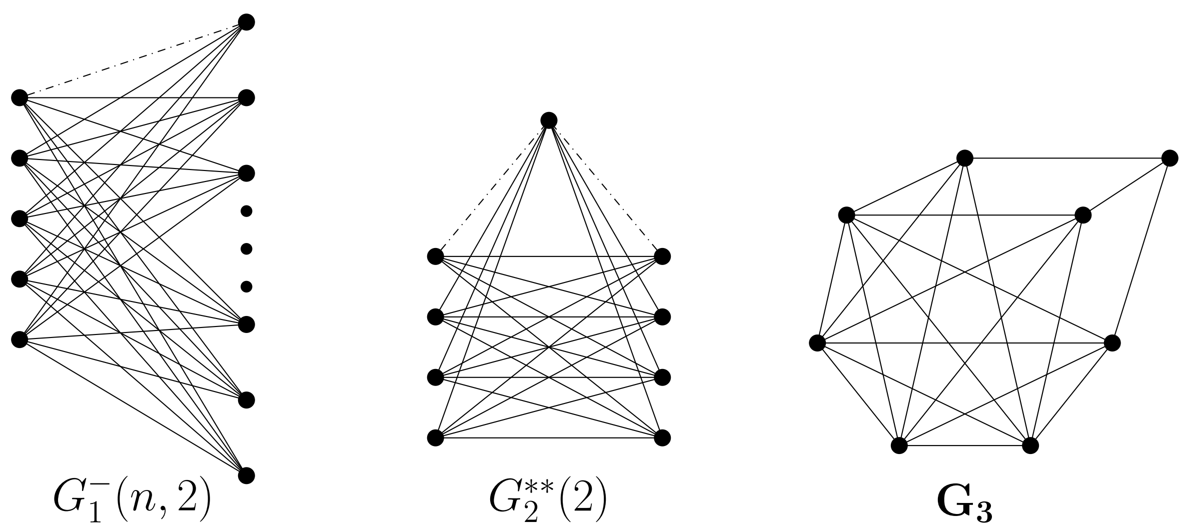


Figure 4.1: Graphs for Theorem 4.1 with $k=2$. Dashed lines indicate missing edges.

Observe that if we relax the bound by 1 more to $\sigma_2(G) \geq 6k-4$, then we run into significantly more exceptional graphs. For example, we would have to consider all graphs

$G_1(n,k) - M$ where M is a matching or exactly two edges incident to the same vertex in the independent set of size $n-(3k-1)$. Or, we could have $G_2(k) - M$ where M is a matching, or M consists of up to $3k-1$ edges incident to the dominating vertex or M consists of two edges incident to the same vertex. Or, we could even have G consisting of a K_3 and a K_7 with one vertex in each identified together, i.e. K_7 with a pendant triangle. This graph has $\sigma_2(G) = 6(2) - 4 = 8$, but every chorded cycle must exist in the K_7. This results in at most one chorded cycle. This list is by no means exhaustive, but it does show the significant jump in complexity beyond the few graphs in Theorem 4.1.

4.1.1 Notation

Let $H \subseteq G$ be a cycle. If $|H| = n$, we say it is an n-cycle. A *chord* of H is an edge $e \in E(G) \setminus E(H)$ with both endpoints in $V(H)$. Hence, a chorded cycle is a cycle with a chord.

For $H \subseteq G$, we will use $G[H]$ to denote the graph G induced by $V(H)$, i.e. $G[V(H)]$. Likewise, we will use $=$ to denote graph isomorphism. Hence, we will frequently use $G[P] = K_4$ instead of $G[V(P)] \cong K_4$. Also, when there is no ambiguity, H^- will denote the isomorphism class of H with a single removed edge and H^+ will denote the isomorphism class of H with a single added edge. We will make frequent use of the graphs K_4^- and C_5^+ as they are both simple chorded cycles. Also, the *paw* is the 4-vertex graph obtained from a copy K of K_3 by adding one vertex adjacent to exactly one vertex in K, i.e. $K_{1,3}^+$.

4.1.2 Outline

In Section 4.2, we set up our proof and present some known proofs to aid in the main proof. We then break up our main proof into Sections 4.3–4.5. The strategy for our proof is to take an optimal collection of chorded cycles, and then consider the set R of the remaining vertices. In Section 4.3, we handle the case where $G[R]$ does not have a spanning cycle. Then, we address the case when $G[R]$ does have a spanning cycle but $k \geq 3$ in Section 4.4. Lastly, we handle the pesky case when $G[R]$ has a spanning cycle and $k = 2$ in Section 4.5.

We tie all this together and conclude the proof in Section 4.6.

4.2 Setup and Preliminaries

4.2.1 Setup

Consider the minimum k such that Theorem 4.1 fails. For a given $n \geq 4k$, let G be an edge-maximal n-vertex graph satisfying the Ore condition

$$d(y) + d(z) \geq 6k - 3 \quad \textit{for every distinct} \quad y, z \in V(G) \quad \textit{with} \quad yz \notin E(G) \tag{4.1}$$

such that G has at most $k-1$ disjoint chorded cycles.

For such a graph G, let $\mathcal{F}$ be a collection of disjoint chorded cycles chosen by the following conditions:

(O1) the number of chorded 4-cycles is maximum,

(O2) subject to the preceding, the number of K_4 is maximum,

(O3) subject to the preceding, the k–tuple $(F_1, \ldots, F_k)$ has $(|F_1|, \ldots, |F_k|)$ least lexicographically, where $|F_i| = \infty$ for nonexistent F_i.

(O4) subject to the preceding, the k-tuple $(||F_1||, \ldots, ||F_k)||$ is greatest lexicographically, where we use a similar convention of $||F_i|| = 0$ for nonexistent F_i.

(O5) subject to the preceding, the total number of chords in the cycles of $\mathcal{F}$ is maximum,

(O6) subject to the preceding, the length of a longest path P in $R := V(G) - V(\mathcal{F}) = \{v_1, \ldots, v_r\}$ is maximum. If $|P| = |R|$, then the number of Hamiltonian cycles in $G[R]$ is maximum, unless $|R| = 4$ in which case we maximize the copies of $K_{1,3}^{+}$ in $G[R]$,

(O7) subject to the preceding, $|E(G[R])|$ is maximum, and

(O8) subject to the preceding, $\sum_{v \in R} d_G(v)$ is maximum.

Let $R = \{v_1, \ldots, v_r\}$ and $P = v_1 v_2 \cdots v_p$. Let $\mathcal{F} = \{F_1, F_2, \ldots\}$ where F_i is the chorded cycle $x_{i,1} x_{i,2} \cdots x_{i,s_i}$. When the cycle F is unambiguous, we will use $\{x_1, x_2, \ldots\}$ to denote $V(F)$.

A vertex v is *low* if $d(v) \leq 3k-2$ and *high* otherwise. By (4.1), the set of low vertices in G forms a clique. We will heavily use the following lemma from [20]. We omit the proof as it is identical to that found in [20], albeit with differently labeled conditions.

Lemma 4.2. *([20]) Let G be a graph, $\mathcal{F}$ an optimal collection, $v \in R$, and $F \in \mathcal{F}$.*

(1) If $||v, F|| \geq 4$, then $||v, F|| = 4 = |F|$, and $F = K_4$.

(2) If $||v, F|| = 3$, then $|F| \in \{4, 5, 6\}$. Moreover,

(a) if $|F| = 4$, then F has a chord incident to the nonneighbor of v;

(b) if $|F| = 5$, then F is singly chorded, and the endpoints of the chord are disjoint from the neighbors of v;

(c) if $|F| = 6$, then F has three chords, with $F = K_{3,3}$, and $G[F + v] = K_{3,4}$.

Since we will often refer to Lemma 4.2, unless noted otherwise, we will use the following conventions:

- if $F = K_4$, then $V(F) = \{x_1, x_2, x_3, x_4\}$,
- if $F = K_4^-$, then $V(F) = \{x_1, x_2, x_3, x_4\}$ with missing edge $x_1 x_3$,
- if $F = C_5^+$, then $V(F) = \{x_1, x_2, x_3, x_4, x_5\}$ with unique chord $x_3 x_5$, and
- if $F = K_{3,3}$, then $V(F) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ with parts $A = \{x_1, x_3, x_5\}$ and $B = \{x_2, x_4, x_6\}$.

4.2.2 Preliminaries

In view of Lemma 4.2, the following observations will be useful.

Lemma 4.3. *If $k \geq 2$ and $F = K_{3,3}$, then* *(i) for every $v \in R$, either $N(v) \cap V(F) \subseteq A$ or $N(v) \cap V(F) \subseteq B$, and*

(ii) for every $u, v \in R$ with $uv \in E(G)$, if there is $x \in N(v) \cap N(u) \cap V(F)$, then $N(v) \cap V(F) = N(u) \cap V(F) = \{x\}$.

Proof. If some $v \in R$ has neighbors in both A and B, say $vx_1, vx_2 \in E(G)$, then $vx_1x_4x_3x_2v$ is a 5-cycle with chord x_1x_2, contradicting (O3). This proves (i).

Suppose now that $u, v \in R$, $uv \in E(G)$, $x_1 \in N(v) \cap N(u) \cap V(F)$, and there is $x_i \in N(v) \cap V(F) - x_1$. By Part (i), $x_i \in A$. Then vx_1ux_iv is a 4-cycle with chord uv, contradicting (O1). □

Lemma 4.4. *For all $x \in V(F)$, $d_F(x) \leq 3$.*

Proof. Suppose that F induces a chorded cycle $x_1x_2 \cdots x_tx_1$ with $t \geq 5$ and that some vertex, say x_1, has $N(x_1) \supseteq \{x_2, x_i, x_j, x_t\}$ for some $2 < i < j < t$. Then, $F' = x_1x_2 \cdots x_jx_1$ is a shorter cycle with chord x_1x_i, contradicting (O3). □

An easy observation is that

$$\text{\textit{If a vertex v in a graph G has 3 neighbors on a path P' disjoint from v, then $G[P'+v]$ has a chorded cycle.}} \tag{4.2}$$

This yields:

Lemma 4.5. *For every path $P' \subseteq G[R]$ and a vertex $v \in R - V(P')$, $||v, P'|| \leq 2$. In particular, if $v_1v_2 \ldots v_s$ is a path in $G[R]$, then $||v_1, \{v_2, \ldots, v_s\}|| \leq 2$.*

We first prove some lemmas that apply to all k.

Lemma 4.6. *For $k \geq 2$, $|R| \geq 4$.*

Proof. Suppose that $|R| \leq 3$ (possibly, $R = \emptyset$). Let $F_t = x_1x_2 \ldots x_sx_1$ be the last existing chorded cycle in the k-tuple from Rule (O3) and $|F_t| = s$. Since $t < k$ and $n \geq 4k$, $s \geq 5$. We break up the remainder of the proof using the following claims.

Claim 4.7. *For all $1 \leq i \leq t-1$, $||F_t, F_i|| \leq 3s$.*

Proof. Suppose that for some $i < t$, $||F_t, F_i|| \geq 3s+1$. Let F_i be the chorded cycle $y_1 \dots y_{s_i} y_1$. Then there is an $x_k \in V(F_t)$ with $||x_k, F_i|| \geq 4$.

If $s_i \geq 5$, then there is $1 \leq j \leq s_i - 1$ such that x_k has at least 3 neighbors on a path in $F_i - y_j - y_{j+1}$. In this case, $G[F_i - y_j - y_{j+1} + x_k]$ contains a chorded cycle F_i' shorter than F_i, and hence the family $\mathcal{F}' = \{F_1, \dots, F_{i-1}, F_i'\}$ is better than $\mathcal{F}$ by (O3), a contradiction. Thus, we only need to consider the case $s_i = 4$. In this case, $V(F_i) \subseteq N(x_k)$.

If $F_i = K_4^-$, then there is $y \in F_i$ such that $G[F_i - y + x_k] = K_4$. Since $s \geq 5$, this means that the family $\mathcal{F}' = \mathcal{F} - F_i - F_t + (F_i - y + x_k)$ has more K_4 than $\mathcal{F}$, contradicting (O2). Thus, $F_i = K_4$.

If there exists a vertex $y \in V(F_i)$ with $||y, F_t|| \geq 5$, then $G[F_t - x_{k-1} - x_k + y]$ contains a chorded cycle and $G[F_i - y + x_k] = K_4$, contradicting (O3). Thus,

$$3s + 1 \leq ||F_i, F_t|| \leq 4|F_i| = 16. \tag{4.3}$$

This means $s = 5$ and each $y \in V(F_i)$ has exactly 4 neighbors in F_t. So, if any $y \in F_i$ is not adjacent to x_{k+1}, then $G[F_i - y + x_k] = K_4$ and $G[F_t - x_k - x_{k+1} + y]$ contains a chorded cycle F_t' that is shorter than F_t. This contradicts (O3). Thus each $y \in V(F_i)$ is adjacent to x_{k+1}. Considering x_{k+1} in place of x_k, we get that each $y \in F_i$ is adjacent to x_{k+2}, and so on. Then each $y \in F_i$ is adjacent to each $x \in F_t$, contradicting (4.3). This proves the claim. □

Claim 4.8. $s = 5$.

Proof. Suppose that $s \geq 6$. First, recall that for all $v \in V(F_t)$, $d_{F_t}(v) \leq 3$ by Lemma 4.4. Then, by the previous claim,

$$\sum_{v \in V(F_t)} d_{G-R}(v) \leq 3s(t-1) + 3s = 3st \leq 3s(k-1).$$

Also, since $s \geq 6$, the biggest clique of F_t is order at most 3. Since low vertices appear in a clique, each low vertex of F_t can be paired with a nonadjacent high vertex in F_t, which implies that

$$\sum_{v \in V(F_t)} d_G(v) \geq (6k-3) \cdot \frac{s}{2}. \tag{4.4}$$

Hence,

$$||V(F_t), R|| \geq \left(3k - \frac{3}{2}\right) s - 3(k-1)s = \frac{3}{2}s \geq 9.$$

By the assumption $|R| \leq 3$, if $||V(F_t), R|| \geq 10$, then some vertex in R has at least 4 neighbors in F_t which implies $F_t = K_4$ by Lemma 4.2, a contradiction to $s \geq 6$. Therefore,

$$||F_t, R|| = 9, \ s = 6, \text{ and } F_t = K_{3,3}. \tag{4.5}$$

This implies that F_t has at most 2 low vertices, each of which we can pair with a nonadjacent high vertex, and there will remain at least two unpaired high vertices. So, we can improve (4.4) to

$$\sum_{v \in V(F_t)} d_G(v) \geq 2(6k-3) + 2(3k-1) = 18k - 8,$$

and so $||F_t, R|| \geq (18k - 8) - 3(k-1) \cdot 6 = 10$, contradicting (4.5). $\square$

Claim 4.9. *F_t has at least 2 low vertices.*

Proof. Suppose that F_t has at most one low vertex, and a has the smallest among the vertices in F_t. Let a' be a non-neighbor of a in F_t. Then all vertices in $F_t - a - a'$ are high vertices, and

$$\sum_{v \in V(F_t)} d_G(v) \geq (d_G(a) + d_G(a')) + 3(3k-1) \leq (6k-3) + 9k - 3 = 15k - 6.$$

Also, by Claim 4.7,

$$\sum_{v \in V(F_t)} d_{G-R}(v) \leq 3st \leq 15t \leq 15(k-1).$$

So, $||F_t, R|| \geq (15k-6) - (15k-15) = 9$. Since $|F_t| = 5$, if $v \in R$ has $||v, F_t|| \geq 4$, then $G[F_t + v]$ contains a shorter chorded cycle, a contradiction. Then by pigeonhole and $|R| \leq 3$, we have $|R| = 3$, say $R = \{z_1, z_2, z_3\}$, and each $z_i \in R$ has exactly 3 neighbors in F_t. By Lemma 4.2, if we consider F_t as a 5-cycle $x_1x_2x_3x_4x_5x_1$ with the unique chord x_3x_5, then $N(z_1) \cap N(z_2) \cap V(F_t) = \{x_1, x_2, x_4\}$. Hence $G[R \cup F_t]$ has a 4-cycle $z_1x_1z_2x_2z_1$ with chord x_1x_2, contradicting (O1). $\square$

Now, if we were to have two chorded 5-cycles $F, F' \in \mathcal{F}$, then we can choose $x, y \in V(F)$ and $u, v \in V(F')$. Then, $G[\{x, y, u, v\}] = K_4$ since the set of low vertices forms a clique. This contradicts (O1). Thus F_t is the only 5-cycle in $\mathcal{F}$, and hence $|V(\mathcal{F})| = 4(t-1) + 5 = 4t + 1 \leq 4k - 3$. Since $n \geq 4k$, we have $|R| \geq 4$ unless $n = 4k$ and $\mathcal{F}$ consists of $(k-2)$ 4-cycles and one 5-cycle.

We now handle the case $n = 4k$. Since K_{4k} contains k disjoint K_4, G has a non-edge xy. Since G is an edge-maximal counterexample, $G + xy$ contains k disjoint chorded cycles $F_1, \ldots, F_k$. Since $n = 4k$, these are all 4-cycles. By the choice of G, after deleting xy from $G + xy$, we ruin one of the F_j and thus produce $|R| = 4$. □

Lemma 4.10. *If $k = 2$, $\delta(G) \geq 3$.*

Proof. Let $v \in V(G)$ be a vertex with $d_G(v) = \delta(G)$ and $G' := G - N[v]$. If $d_G(v) \leq 1$, then there are $n - 2$ nonneighbors, all of degree at least 8. Then G' is a graph on $n - 2$ vertices with $\delta(G') \geq 7$; in particular, $|G'| \geq 8$. So by Theorem 1.35, G' has 2 disjoint chorded cycles.

Now, suppose v is a vertex with $d_G(v) = 2$, say $N(v) = \{v_1, v_2\}$. For each $u \in V(G')$, $uv \notin E(G)$ so $d_G(u) \geq (6k-3) - 2 = 7$ and hence $d_{G'}(u) \geq 5$.

CASE 1: $v_1v_2 \in E(G)$. If there exists $u \in N_G(v_1) \cap N_G(v_2) - v$, then uv_1vv_2u is a 4-cycle with chord v_1v_2. Since $\delta(G' - u) \geq 5 - 1 = 4$, Theorem 1.35 provides a second chorded cycle unless $|G' - u| = n - 4 < 4$, which is not the case. Therefore, we have two disjoint chorded cycles.

Hence, for every $u \in V(G')$, $||u, \{v, v_1, v_2\}|| \leq 1$ and so $\delta(G') \geq 6$. Again, Theorem 1.35 provides that G' has two disjoint chorded cycles unless $|G'| < 8$. But then, $G' \cong K_7$. Thus, $d_{G'}(u) = 6$ but $d_G(u) \geq 7$ for each $u \in V(G')$. Then, $||\{v_1, v_2\}, G'|| \geq 7$ and so $||v_i, G'|| \geq 4$ for some $i \in \{1, 2\}$. Hence, for $u_1, u_2, u_3 \in N_{G'}(v_i)$, $G[\{v_i, u_1, u_2, u_3\}] = K_4$ and $G' - \{u_1, u_2, u_3\} = K_4$, a contradiction.

CASE 2: $v_1v_2 \notin E(G)$. By symmetry, we may assume $d_G(v_1) \geq d_G(v_2)$, so that the Ore condition (4.1) yields $d_G(v_1) \geq 5$. Consider $G'' := G - v - v_2$. Since v is not adjacent to the vertices in $V(G'') - v_1$,

$$\textit{for all } u \in V(G'' - v_1),\ d_G(u) \geq 7 \textit{ and hence, } d_{G''}(u) \geq 6. \tag{4.6}$$

Since $d_{G''}(v_1) \geq 4$, $\sigma_2(G'') \geq 10$ and so by Theorem 1.37, either $n'' := |G''| = 7$, or $G'' = G_1(n'', 2)$, or $G'' = G_2(2)$.

Case 2.1: $n'' = 7$. Then $G'' = K_7$ and since each $u \in V(G'') - v_1$ has $d_G(u) \geq 7$, $uv_2 \in E(G)$ for all such u. Hence, $\delta(G - v) \geq 6$ and $|V(G) - v| = 8$ so that $G - v$ has 2 disjoint chorded cycles by Theorem 1.35.

Case 2.2: $G'' = G_1(n'', 2)$. Call the partite sets A, B with $|A| = 5$ and $|B| = n'' - 5$. For each $u \in B$, $d_{G''}(u) = 5$ so $d_G(u) \leq 6$. For each $u \in A$, $d_G(u) = d_{G''}(u) + ||u, \{v, v_2\}|| \leq n'' - 5 + 1$. By the Ore condition, we have $n'' \geq 9$ so that there exists $u \in B - v_1$. Then, we contradict the Ore condition of G since $uv \notin E(G)$ but $d_G(u) + d_G(v) \leq 6 + 2 = 8 < 9$.

Case 2.3: $G'' = G_2(2)$. Then, only one vertex in $u \in V(G'')$ has $d_{G''}(u) \geq 6$ and so u is the only vertex in G with degree at least 7 since $d_G(v), d_G(v_2) \leq 6$. This contradicts (4.6). □

Lemma 4.11. *If $k = 2$, G is 3-connected.*

Proof. If G is disconnected, then by Lemma 4.10 we can use Theorem 1.36 componentwise to yield two disjoint chorded cycles, a contradiction.

Suppose G has a cut-vertex x. Let Y be a component of $G - x$ disjoint from $\mathcal{F}$. By Lemma 4.10, $\delta(G) \geq 3$ so that $\delta(G[Y]) \geq 2$. But the set $S := \{v \in Y : d_Y(v) = 2\}$ induces a clique since for all $s, t \in S$, $d_G(s) + d_G(t) \leq 6$. Hence, $\sigma_2(G[Y]) \geq 5$ and so $G[Y]$ contains a chorded cycle by Theorem 1.36, so we have 2 disjoint chorded cycles, a contradiction.

Now, suppose that there exists a separating set S with $|S| = 2$. Say $S = \{u, v\}$ and $G - S$ has components A and B.

CASE 1: For all $w \in V(G) - S$, $d_G(w) \geq 4$. Let $G_A = G[A + u]$ and $G_B = G[B + v]$. If $d_{G_A}(u) \geq 2$, then $\sigma_2(G_A) \geq 5$, otherwise $\sigma_2(G_A - u) \geq 5$. In both cases, G_A contains a chorded cycle. Similarly, G_B has a chorded cycle, a contradiction.

CASE 2: There exists a vertex $y \in V - S$ such that $d_G(y) = 3$, say $y \in A$. By (4.1),

$$d_G(w) \geq 6 \text{ for all } w \in B. \tag{4.7}$$

Let P' be a shortest u, v–path in $G[B \cup S]$.

Case 2.1: For some $w \in B - V(P')$, $||w, P'|| \geq 4$. Then $||w, P' - v|| \geq 3$ and so $G[P' - v + w] \subseteq G[B + u]$ contains a chorded cycle. Take a maximal path $Q := v_0 v_1 \cdots v_s$ in $G[A + v]$ with $v_0 = v$ and $v_i \in A$ for $i > 0$. If $d_{G[A+v]}(v_s) \geq 3$, then $||v_s, Q|| \geq 3$ and so $G[Q]$ contains a chorded cycle. Otherwise, by Lemma 4.10, $d_{G[A+v]}(v_s) = 2$ and $d_G(v_s) = 3$. Since Q is maximal, $N_{G[A+v]}(v_s) = \{v_{s-1}, v_i\}$ for some $0 \leq i \leq s - 2$ and $\{u, v\}$ being a separating set means that $N_G(v_s) = \{v_{s-1}, v_i, u\}$.

Consider v_{i+1}. Path $v_0 v_1 \cdots v_i v_s v_{s-1} \cdots v_{i+1}$ has the same length as Q, so $d_{G[A+v]}(v_{i+1}) = 2$ and $d_G(v_{i+1}) = 3$. Hence by (4.1), $v_{i+1} v_s \in E(G)$ and so $i + 1 = s - 1$. Also, $N_G(v_{s-1}) = \{v_{s-2}, v_s, u\}$ so that $u v_{s-1} v_{s-2} v_s u$ is a 4-cycle with chord $v_{s-1} v_s$. By the subcase, $||w, P' - u|| \geq 3$ so that $G[P' - u + w] \subseteq G[B + v]$ contains a chorded cycle. Thus we found 2 disjoint chorded cycles. This completes the subcase.

Case 2.2: For all $w \in B - P'$, $||w, P'|| \leq 3$. First, $B - P' \neq \emptyset$ as otherwise $x \in V(P')$ and $xy \notin E(G)$ implies that $d_G(x) \geq 9 - d_G(y) = 6$ so that P' has a chord, contradicting it being a shortest u, v–path. Then, $\delta(G[B - P']) \geq 6 - 3 = 3$ and so $G[B - P']$ contains a chorded cycle by Theorem 1.36. Now consider $G[A + P'] \supseteq G[A + S]$. All maximum paths Q' in $G[A + P']$ have endpoints of degree 2 in $G[A + P']$ or else we have our chorded cycle. Therefore,by Lemma 4.10 all such maximum paths are u, v–paths. Consider one such path $Q' := v'_0 v'_1 \cdots v'_t$ where $u = v'_0, v = v'_t$. Note, $t = 1$ implies A is already disconnected in G and $t = 2$ implies $N(v'_1) = \{u, v\}$, contradicting $\delta(G) \geq 3$, so we assume $t \geq 3$. By Lemma 4.10, $d_{G[A+P']}(v'_{t-1}) \geq 3$.

If $x v'_{t-1}$ for some $x \in A - Q'$, then $v'_0 \cdots v'_{t-1} x$ is a part of a different maximum path, hence must have v as an endpoint, i.e. there exists $0 < i < t - 1$ and vertices $x_{i+1}, \cdots, x_{t-2} \in A - Q'$ where $x = x_{t-2}$ such that $Q'' := v'_0 \cdots v'_i x_{i+1} \cdots x_{t-2} v'_{t-1} v'_t$ is maximum. In other words, $G' := G[P' \cup Q' \cup \{x_{i+1}, \ldots, x_{t-2}\}]$ is a Θ-graph with 3-vertices v'_i and v'_{t-1}, and so $x = x_{t-2}$

cannot have a third neighbor in G'. So, there exists $y \in A - V(Q') - V(Q'')$ such that $xy \in E(G)$. But then $v'_0 \cdots v'_{t-1}xy$ is longer than Q' contradicting Q' being maximum.

Thus, $N_G(v'_{t-1}) \subseteq Q'$ and so $v'_j v'_{t-1}$ for some $0 \leq j \leq t-3$. But then we have a cycle $vP'uQ'v$ with chord $v'_j v'_{t-1}$. Hence, each of $G[A+P']$ and $G[B-P']$ contains a chorded cycle thus providing 2 disjoint chorded cycles, a contradiction. $\square$

Lemma 4.12. *Suppose $k \geq 2$, $R = \{v_1, v_2, v_3, v_4\}$, $E(G[R]) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$, and $F \in \mathcal{F}$ is such that $F = K_4$. If $||R, F|| \geq 11$, then*
(a) $||R, F|| = 11$ and
(b) $\sum_{j=1}^{4} d_G(v_j) \geq 12k - 5$.

Proof. Suppose $||R, F|| \geq 11$. We proceed in a series of claims. First, we show that

$$||x_i, R|| \leq 3 \quad \text{for each } 1 \leq i \leq 4. \tag{4.8}$$

Indeed, assume $x_1v_j \in E(G)$ for each $1 \leq j \leq 4$. Since $||R, F|| \geq 11$, we may assume $||v_1, F|| \geq \lceil 11/4 \rceil = 3$. Then $G[F - x_1 + v_1] \supseteq K_4^-$ and $G[R - v_1 + x_1] = K_4^-$, a contradiction to Rule (O1). This proves (4.8).

By (4.8), we may assume that $||x_i, R|| = 3$ for $1 \leq i \leq 3$ and $||x_4, R|| \geq 2$. For $1 \leq i \leq 3$, let $j(i)$ be such that $v_{j(i)}x_i \notin E(G)$. Our next claim is:

$$||v_{j(i)}, F|| \leq 1 \quad \text{for each } 1 \leq i \leq 3. \tag{4.9}$$

Indeed, suppose that for example $j(1) = 4$ and $||v_4, F|| \geq 2$. Since $v_4x_1 \notin E(G)$, this means $G[F - x_1 + v_4]$ contains K_4^-. But by the choice of x_1, also $G[R - v_4 + x_1] = K_4^-$, a contradiction to Rule (O1). This proves (4.9).

If there exist i_1, i_2 such that $j(i_1) \neq j(i_2)$, then by (4.9), $||R, F|| = ||\{v_{j(i_1)}, v_{j(i_2)}\}, F|| + ||R - \{v_{j(i_1)}, v_{j(i_2)}\}, F|| \leq 2 \cdot 1 + 2 \cdot 4 = 10$, a contradiction. Thus, we may assume

$$j(1) = j(2) = j(3) = 4. \tag{4.10}$$

We can now say more:

$$x_4v_1, x_4v_3 \notin E(G). \tag{4.11}$$

If (4.11) does not hold, then by symmetry we may assume $x_4v_1 \in E(G)$. In this case, by (4.10), $G[F - x_1 + v_1] = K_4$ and $G[R - v_1 + x_1] = K_{1,3}^+$. But a $K_{1,3}^+$ is "better" than a C_4 by (O6), a contradiction.

Since $||R, F|| \geq 11$, (4.8) and (4.11) together imply $||R, F|| = 11$ and $N(x_4) \cap R = \{v_2, v_4\}$. So, if one of v_1 or v_3, say v_3, is low, then x_4 is high, and replacing F with $G[F - x_4 + v_3]$ yields another K_4, and $R - v_3 + x_4$ again induces C_4. But it contradicts Rule (O8), since $d(x_4) > d(v_3)$. Hence, both v_1 and v_3 are high, and $\sum_{j=1}^{4} d_G(v_j) = d_G(v_2) + d_G(v_4) + 2(3k-1) \geq (6k-3) + 2(3k-1) = 12k - 5$. This proves the lemma. □

Lemma 4.13. *For $k \geq 3$, $K_4 \in \mathcal{F}$ or $\{K_4^-, K_4^-\} \subseteq \mathcal{F}$.*

Proof. Suppose that $K_4 \notin \mathcal{F}$ and $\{K_4^-, K_4^-\} \not\subseteq \mathcal{F}$. Let $G' = G - V(F_1)$. We first show that

$$\sigma(G') \geq 6(k-1) - 3. \tag{4.12}$$

Otherwise, there exist $x, y \in V(G')$ such that $d_{G'}(x) + d_{G'}(y) \leq 6(k-1) - 4 = 6k - 10$. Then, $||\{x, y\}, F_1|| \geq 7$ and so by symmetry we may assume $||x, F_1|| \geq 4$. If $|F_1| \geq 5$, then $G[F_1 + x]$ contains a chorded cycle shorter than F_1, contradicting (O3). Otherwise, $F_1 = K_4^-$ and $G[F_1 + x] \supseteq K_4$, contradicting (O2). This proves (4.12).

By the minimality of G, either $n - |F_1| < 4(k-1)$ or G' has $k-2$ disjoint cycles, and G' is an excluded graph. Note that $G' \neq G_3$ since G_3 contains K_4. If $||F_1, x|| \leq 2$ for all $x \in V(G')$, then $\sigma(G') \geq 6k - 3 - 4 = 6(k-1) - 1$. By Theorem 1.36, G' has $k-1$ disjoint cycles, so G has k disjoint chorded cycles, a contradiction. Then, either $n - |F_1| < 4(k-1)$, or $G' \supseteq G_1^-(n - |F_1|, k-1)$, or $G' \supseteq G_2^{**}(k-1)$, and $||x, F_1|| = 3$ for some $x \in V(G')$ so that by Lemma 4.2, we have the following cases:

CASE 1: $F_1 = K_4^-$. In this case, $n - |F_1| \geq 4(k-1)$, so G' is an excluded graph. Since $G' \supseteq G_2^{**}(k-1)$ contains K_4^-, we have $G' \supseteq G_1^-(n-4, k-1)$ with partite sets A and B where $|A| = (n-4) - 3(k-1) + 1 = n - 3k$ and $|B| = 3(k-1) - 1 = 3k - 4$.

There exists a vertex $y_0 \in A$ such that for each $y \in A - y_0$, $||y, F_1|| = 3$, $N_{G'}(y) = B$, and $||y_0, F_1|| \geq 2$ or else we violate (4.12). For the same reason, $|A| \geq |B|$ and so $|A| \geq |B| \geq 5$. Note, for each $y \in A - y_0$, we have $x_1, x_3 \in N(y)$. So, take some $y_1 \in A - y_0$ and suppose by symmetry that $N(y_1) \cap F_1 = \{x_1, x_2, x_3\}$. We have the chorded cycle F_1' comprised of $y_1x_2x_4x_3y_1$ with chord x_2x_3. Then, the remaining graph $G'' := G - \{y_1, x_2, x_3, x_4\}$ contains $K_{|A|-1,|B|+1} - x_1y_0$ since x_1 is adjacent to all of $A - y_0$. G'' has $k-1$ disjoint chorded cycles unless $|A - y_1| = n - 3k < 3k - 3$. But then, (4.12) implies $n - 3k = 3k - 4$ and so A and B are interchangeable. Namely, there exists $z_0 \in B$ such that each $z \in B - z_0$ has $||z, F_1|| = 3$ and $N_{G'}(z) = A$. Moreover, if any $y \in A - y_0$ and $z \in B - z_0$ share the same neighborhood in F_1, say $N(y) \cap N(z) \cap V(F_1) = \{x_1, x_3, x_2\}$, then $G[\{x_1, x_2, y, z\}] = K_4$, a contradiction. So, we can take any $y_1, y_2 \in A - y_0$ and $z_1, z_2 \in B - z_0$ where $N(y_1) \cap V(F_1) = N(y_2) \cap V(F_1) = \{x_1, x_3, x_2\}$ and $N(z_1) \cap V(F_1) = N(z_2) \cap V(F_1) = \{x_1, x_3, x_4\}$ to get 4-cycles $y_1z_2x_1x_2y_1$ and $y_2z_1x_4x_3y_2$ with chords y_1x_1 and z_1x_3, respectively, a contradiction.

CASE 2: $F_1 = C_5^+$ with unique chord x_3x_5. If $n - 5 < 4(k-1)$, then $n \geq 4k$ implies $n = 4k$. Now, $\{x_1, x_3\}$ and $\{x_2, x_4\}$ form independent sets, so for $W := \{x_1, x_2, x_3, x_4\}$, we have $||W, V(G')|| \geq 2(6k - 3 - 5) = 12k - 16$. For each $y \in V(G')$, $||y, F_1|| \leq 3$ and so we have $||W, V(G')|| \leq 12k - 15$. Hence, there exists a $y_0 \in V(G')$ such that for each $y \in V(G') - y_0$, $||y, W|| = 3$. For such a y, if $yx_4 \notin E(G)$, then $G[\{y, x_1, x_2, x_3\}] = K_4$, a contradiction.

Thus, $n-5 < 4(k-1)$ and hence G' is an excluded graph. Whether $G' \supseteq G_1^-(n-5, k-1)$ or $G' \supseteq G_2^{**}(k-1)$, there exists an independent set A of size $3k - 5 \geq 3$. Moreover, there is a vertex $y_0 \in A$ such that for all vertices $y_1, y_2 \in A - y_0$, we have $d_{G'}(y_1), d_{G'}(y_2) = 3k - 4$, $y_1y_2 \notin E(G)$, and so $||y_1, F_1||, ||y_2, F_1|| = 3$ by the Ore condition. Hence, $N(y_1) \cap N(y_2) \supseteq \{x_1, x_2\}$ in which case we have the 4-cycle $y_1x_1y_2x_2y_1$ with chord x_1x_2, contradicting (O1).

CASE 3: $F_1 = K_{3,3}$ with partite sets $A' = \{x_1, x_3, x_5\}$ and $B' = \{x_2, x_4, x_6\}$. Then $G_2^{**}(k-1), G_3 \not\subseteq G'$ as each of them has smaller chorded cycles than $K_{3,3}$. Suppose $n - 6 < 4(k-1)$. Then, since each partite set of F_1 has at most one low vertex, we can by symmetry assume x_1, x_2 are high vertices, in which case $||\{x_1, x_2\}, V(G')|| \geq 2(3k - 1 - 3) = 6k - 8 \geq$

$4(k-1) > n-6$. So, there is a vertex $y \in N(x_1) \cap N(x_2) \cap V(G')$, and hence $yx_1x_4x_3x_2y$ is a 5-cycle with chord x_1x_2, contradicting (O3).

So, $n-6 \geq 4(k-1)$ and $G' \supseteq G_1^-(n-6, k-1)$ with partite sets A and B where $|A| = (n-6) - 3(k-1) + 1 = n - 3k - 2$ and $|B| = 3(k-1) - 1 = 3k - 4$.

By (4.1), there exists a vertex $y_0 \in A$ such that for each $y \in A - y_0$, $||y, F_1|| = 3$, $N_{G'}(y) = B$, and $||y_0, F_1|| \geq 2$. By Lemma 4.2, for each such y, either $N(y) \cap V(F_1) = A'$ or $N(y) \cap V(F_1) = B'$.

Case 3.1: All $y \in A - y_0$ have neighbors in one partite set, say A'. Then no vertex $z \in V(B)$ has a neighbor in A' or else we have a chorded 5-cycle, contradicting (O3). Then, for each pair $x, x' \in B'$, we have $||\{x, x'\}, B|| \geq 6k - 9$. This implies that there exists some vertex $z_0 \in B$, such that for all $z \in B - z_0$, $N(z) \cap V(F_1) = B'$. Hence, our graph G is $K_{n-3k+1,3k-1} - y_0z_0$ or $K_{n-3k+1,3k-1}$, i.e. $G_1^-(n, k-1)$ or $G_1(n, k-1)$, a contradiction.

Case 3.2: $A - y_0$ is partitioned into two non-empty sets A_1, A_2 where for all $y \in A_1$, $N(y) \cap V(F_1) = A'$ and for all $y \in A_2$, $N(y) \cap V(F_1) = B'$. Again, we cannot have any $z \in V(B)$ with $||z, F_1|| \neq 0$, as otherwise we contain a C_5^+. Hence, for each $x \in B'$, $N(x) \subseteq A' \cup A_2 + y_0$ and for each $y \in A'$, $N(y) \subseteq B' \cup A_1 + y_0$. If say $|A_1| \leq |A_2|$, then for each $x, x' \in A'$,

$$d(x) + d(x') \leq 2 \cdot 3 + 2|A_1| + ||y_0, \{x, x'\}|| \leq 8 + 2\left\lfloor \frac{|A| - 1}{2} \right\rfloor \leq n - 3k + 5. \tag{4.13}$$

Therefore, $n \geq 9k - 8$. If $|A_1|, |A_2| \geq 3$, then take any $A_1' \subseteq A_1, A_2' \subseteq A_2$ each with $|A_1'|, |A_2'| = 3$ and so $G[A_1', A'], G[A_2', B'] = K_{3,3}$. Then, $G - \{A_1', A', A_2', B'\} \supseteq K_{n-3k-8,3k-4}$. If $n > 9k - 8$ or $k \geq 4$, then this remaining graph contains $k - 2$ disjoint chorded cycles because $n - 3k - 8 \geq 3(k-2)$, a contradiction. But if $k = 3$ and $n = 9k - 8 = 19$, then $n - 3k + 5 = 6k - 3$, and so we need equality in (4.13). However, $|A| = n - 3k - 2 = 8$ is even and so this is impossible.

Otherwise, we may assume $|A_2| \leq 2$. Since B' is independent, it contains two high vertices, say $x, x' \in B'$. For $k \geq 3$, $d_G(x) + d_G(x') \leq 2|A_2 + y_0| + 2 \cdot 3 \leq 12$. Hence, $||\{x, x'\}, B|| \geq 6k - 3 - 12 \geq 3$ and so there exist $z, z' \in B \cap (N(x) \cup N(x'))$. Since

$|B \cup B'| = 4k-1$, y_0 can have at most one nonneighbor in $B \cup B'$. So we can assume $y_0z, y_0x \in E(G)$. But then for each $z'' \in N(y_0) \cap (B-z)$ and $y \in A - y_0$, $xy_0z''yzx$ is a 5-cycle with chord y_0z, contradicting (O3). □

4.3 Case: $G[R]$ does not have a Hamiltonian path

Suppose that our collection $\mathcal{F}$ utilizing Rules (O1)–(O8), yields a remainder vertex set R where $G[R]$ does not contain a Hamiltonian path, i.e. $G[R]$ does not contain a spanning path of length $|R|$.

Lemma 4.14. *For $k \geq 2$ and all $z \in R - V(P)$, $d_{G[R]}(z) \geq 2$.*

Proof. Suppose there is a vertex $z \in R - V(P)$ such that $d_{G[R]}(z) \leq 1$. By Lemma 4.5 and since P is maximum, $d_P(v_1) \leq 2$ and $||v_1, R - V(P)|| = 0$ and symmetrically for v_p. Therefore $d_{G[R]}(v_1), d_{G[R]}(v_p) \leq 2$. Since $v_1z, v_pz \notin E(G)$, $2d(z) + d(v_1) + d(v_p) \geq 12k - 6$. By the case and Lemma 4.5, $2d_{G[R]}(z) + d_{G[R]}(v_1) + d_{G[R]}(v_p) \leq 6$, hence

$$||\{v_1, v_p\}, \mathcal{F}|| + 2||z, \mathcal{F}|| \geq 12k - 6 - 6 = 12(k-1). \tag{4.14}$$

Claim 4.15. *For all $F \in \mathcal{F}$, if $||\{v_1, v_p\}, F|| + 2||z, F|| \geq 12$, then*

(a) $||\{v_1, v_p\}, F|| + 2||z, F|| = 12$,

(b) $|\mathcal{F}| = k - 1$,

(c) $||v, F|| = 3$ *for all* $v \in \{v_1, v_p, z\}$, *and*

(d) $d_{G[R]}(z) = 1, d_{G[R]}(v_1) = 2$, *and* $d_{G[R]}(v_p) = 2$.

Proof. Suppose that there exists $F \in \mathcal{F}$ such that $||\{v_1, v_p\}, F|| + 2||z, F|| \geq 13$. Since $||\{v_1, v_p\}, F|| \leq 8$, $||z, F|| \geq 3$. Moreover, there is some $v \in \{v_1, v_p, z\}$ such that $||v, F|| = 4$, so $F = K_4$ by Lemma 4.2. Since some vertex has 4 neighbors in F, if possible choose $x \in [(N(v_1) \cup N(v_p)) \setminus N(z)] \cap V(F)$, otherwise just choose $x \in [N(v_1) \cup N(v_p)] \cap V(F)$.

Then $G[F-x+z]=K_4$ and $G[V(P)+x]$ contains a path longer than P, contradicting (O6). This proves (a).

Hence, by (4.14), $||\{v_1,v_p\},F||+2||z,F||=12$ for all $F\in\mathcal{F}$ and thus $|\mathcal{F}|=k-1$, $d_{G[R]}(v_1)=d_{G[R]}(v_p)=2$, and $d_{G[R]}(z)=1$. This proves (b) and (d). If $||v,F||=4$ for some $v\in\{v_1,v_p,z\}$, then again $F=K_4$. If possible choose $x\in[(N(v_1)\cup N(v_p))\setminus N(z)]\cap V(F)$, otherwise just choose $x\in[N(v_1)\cup N(v_p)]\cap V(F)$. Either $G[F-x+z]\supseteq K_4$ and $G[P+x]$ contains a path longer than P, or $||z,F||=2$ in which case $G[F-x+z]=K_4^-$ and $xv_1\cdots v_px$ is a cycle with chord incident to v_1 since $d_{G[R]}(v_1)=d_P(v_1)=2$. In any case, we have a contradiction thus proving (c), and this completes the claim. $\square$

By Lemma 4.2, we have four cases:

CASE 1: $F=K_4$. There exists $x\in N(v_1)\cap N(v_p)\cap V(F)$ and so $G[F-x+z]\supseteq K_4^-$ and $G[P+x]$ has a cycle with chord incident to v_1 since $d_P(v_1)=2$. Thus, we have k disjoint chorded cycles, a contradiction.

CASE 2: $F=K_4^-$. By Lemma 4.2, $\{x_1,x_3\}\subseteq N(z)\cap N(v_1)\cap N(v_p)\cap V(F)$. Then $G[F-x_1+z]=K_4^-$ and $G[P+x_1]$ has a cycle with chord incident to v_1 since $d_P(v_1)=2$. Thus, we have k disjoint chorded cycles, a contradiction.

CASE 3: $F=C_5^+$. By Lemma 4.2, $\{x_1,x_2,x_4\}\subseteq N(z)\cap N(v_1)\cap N(v_p)\cap V(F)$. Then $G[\{z,x_1,x_2,v_1\}]\supseteq K_4^-$, contradicting (O1).

CASE 4: $F=K_{3,3}$. We may assume $N(z)\cap V(F)=A=\{x_1,x_3,x_5\}$. If there exists $v\in\{v_1,v_p\}$ with $N(v)\cap V(F)=B=\{x_2,x_4,x_6\}$, then $G[F-x_2+z]=K_{3,3}$ and $G[P+x_2]$ contains a path longer than P, contradicting (O6). Hence, $N(z)\cap V(F)=N(v_1)\cap V(F)=N(v_p)\cap V(F)=A$. Moreover, since this is the last case, each $F_i\in\mathcal{F}$ also is a $K_{3,3}$. Then, $d_G(z)=1+3(k-1)=3k-2$, so z is a low vertex. Since $zx_2\notin E(G)$, x_2 is not low. However, $G[F-x_2+z]=K_{3,3}$ and P is still a path in $G[R-z+x_2]$, so that either there is a longer path and we contradict (O6), or the degree sum in R has increased, contradicting (O8). This proves Lemma 4.14. $\square$

Lemma 4.16. *Let $k \geq 2$ and $P_1 = z_1 z_2 \cdots z_s$ be a maximal path in $G[R - V(P)]$. Then, $d_{G[R]}(z_1) = 2$ or $d_{G[R]}(z_s) = 2$.*

Proof. By Lemma 4.14, $d_{G[R]}(z_1) \geq 2$ and $d_{G[R]}(z_s) \geq 2$ so suppose that $d_{G[R]}(z_1), d_{G[R]}(z_s) \geq 3$. If $s = 1$, then $||z_1, P|| \geq 3$, contradicting Lemma 4.5. Therefore $z_1 \neq z_s$. If $d_{P_1}(z_1) = d_{P_1}(z_s) = 1$, then each vertex has at least two neighbors on P, say $z_1 v_i, z_1 v_j, z_s v_m, z_s v_\ell \in E(G)$ with $i < j$ and $m < \ell$. By symmetry assume $i < m$. Then $z_1 \cdots z_s v_\ell v_{\ell-1} \cdots v_i z_1$ is a cycle with chord $z_s v_m$, so we have k disjoint chorded cycles, a contradiction. Otherwise, some endpoint of P_1 has a chord on its own maximal path, say $z_1 z_i \in E(G)$ for some $3 \leq i \leq s$. Still $z_1 v_m, z_s v_j \in E(G)$ for some $1 \leq m, j \leq p$. Then $z_1 \cdots z_s v_j \cdots v_m z_1$ is a cycle with chord $z_1 z_i$, so we have k disjoint chorded cycles, a contradiction. □

Lemma 4.17. *If $k \geq 2$, then $G[V(P)]$ is not a cycle.*

Proof. Suppose $G[V(P)]$ is the cycle $v_1 v_2 \ldots v_p v_1$. For notation, let $P_1 = P$, and let $P_2 = w_1 w_2 \ldots w_q$ be a longest path in $G[R] - V(P_1)$. By the maximality of P_1 and P_2, neither of w_1 nor w_q has a neighbor in $R - V(P_1) - V(P_2)$ and by Lemma 4.14, $d_{G[R]}(w_1) \geq 2$ and $d_{G[R]}(w_q) \geq 2$, so we conclude that

$$q \geq 3,\ d_{G[R]}(w_1) = d_{G[R]}(w_q) = 2, \textit{ and } w_1 \textit{ and } w_q \textit{ are incident to chord(s) of } P_2. \tag{4.15}$$

Let $W_1 = \{v_1, v_2\}$, $W_2 = \{w_1, w_q\}$ and $W = W_1 \cup W_2$. Since $v_1 w_1, v_2 w_q \notin E(G)$,

$$||W, V(G) - R|| \geq \sum_{w \in W} d_G(w) - \sum_{w \in W} d_{G[R]}(w) \geq 2(6k - 3) - 4(2) = 12(k - 1) - 2. \tag{4.16}$$

Hence if $|\mathcal{F}| \leq k - 2$, then there is $F \in \mathcal{F}$ with $||W, F|| \geq 13$. So, there is $w' \in W$ with $||w', F|| \geq 4$. Then by Lemma 4.2 , $F = K_4$. If $w' \in W_j$, then to have $||W, F|| \geq 13$, there is $w'' \in W_{3-j}$ with $||w'', F|| \geq 3$. Therefore, we have $w \in W_2$ and $v \in W_1$ with $||w, F||, ||v, F|| \geq 3$. Let $x \in [N(v) \setminus N(w)] \cap V(F)$ if possible, otherwise simply choose $x \in N(v) \cap V(F)$. Thus $G[F - x + w] = K_4$ and $G[P + x]$ has a path longer than P. This contradicts (O6). Therefore,

$$|\mathcal{F}| = k - 1. \tag{4.17}$$

By (4.16), there is $F \in \mathcal{F}$ with

$$||W, F|| \geq 10. \tag{4.18}$$

By symmetry, we may assume

$$||v_1, F|| \geq ||v_2, F|| \textit{ and } ||w_1, F|| \geq ||w_q, F||. \tag{4.19}$$

Let $||W_i, F|| \geq 5$ with $W_i := \{v'_1, v'_2\}$ and $W_{3-i} := \{w'_1, w'_q\}$. Assume first that $F = K_{3,3}$. By Lemma 4.2, we may assume that $N(v'_1) \cap V(F) = A$ and $\{x_2, x_4\} \subseteq N(v'_2) \cap V(F)$. Since $||W_{3-i}, F|| \geq 4$, both w'_1, w'_q have neighbors on F, say $x \in N(w'_1) \cap V(F)$ and $x' \in N(w'_q) \cap V(F)$, respectively, where we may assume $N(w'_1) \cap V(F) \subseteq A$ and $N(w'_q) \cap V(F) \subseteq B$ by Lemma 4.3. Then both $G[V(P_{3-i}) \cup \{x, x'\}]$ and $G[V(P_i) \cup (F - \{x, x'\})]$ contain chorded cycles, thus producing k disjoint chorded cycles.

Now assume that $F \neq K_{3,3}$. Then there exists $x \in N_G(v'_1) \cap N_G(v'_2) \cap V(F)$. Note, this holds even if $F = C_5^+$, as if $v' \in W$ has $N(v') \cap V(F) = \{x_3, x_5\}$, i.e. the endpoints of the chord, then we have $G[\{v', x_3, x_4, x_5\}] = K_4^-$, contradicting (O1). Note that $G[V(P_i) \cup \{x\}]$ contains a chorded cycle. It follows that each of w'_1, w'_q has at most two neighbors on $F - x$, and w'_1 and w'_q cannot both have neighbors on $F - x$, for otherwise, $G[V(P_{3-i}) \cup (F - x)]$ contains a chorded cycle and we obtain k disjoint chorded cycles. Therefore, $||W_{3-i}, F - x|| \leq 2$.

If x is adjacent to both w'_1 and w'_q, then by the same argument, $||W_i, F - x|| \leq 2$, a contradiction to $||W_i, F|| \geq 5$. So $||W_{3-i}, F|| \leq 3$. It follows that $||W_i, F|| \geq 7$, and thus for some $v' \in W_i$, $||v', F|| \geq 4$, so $F = K_4$, and v'_1, v'_2 have at least three common neighbors on F.

Since $||W_i, F|| \leq 8$, $||W_{3-i}, F|| \geq 2$. So we may assume that either w'_1 has two neighbors on F, or w'_1, w'_q both have neighbors on F. In the former case, let $x \in N_G(v'_1) \cap N_G(v'_2) \cap V(F) \setminus N_G(w'_1)$, then $G[\{w'_1\} \cup (F - x)] \supseteq K_4^-$ and $G[V(P_i) \cup \{x\}]$ contains a chorded cycle. In the latter case, let $x_1, x_q \in V(F)$ where $x_1w_1', x_qw_q' \in E(G)$, respectively and $x \in N_G(v_1') \cap N_G(v_2') - x_1 - x_q$. Then $G[V(P_i) \cup \{x\}]$ and $G[V(P_{3-i}) \cup \{x_1, x_q\}]$ contain chorded cycles. $\square$

Lemma 4.18. *Suppose $k \geq 2$, $V(P) \neq R$, and $z \in R - V(P)$ is the endpoint of a maximal path in $R - V(P)$ with $d_{G[R]}(z) = 2$. Let $W = \{v_1, v_p, z\}$. Then,*

(a) For all $F \in \mathcal{F}$, $||W, F|| \leq 9$,

(b) $|\mathcal{F}| = k - 1$, and

(c) P has a chord with an endpoint in $\{v_1, v_p\}$.

Proof. Suppose that $||W, F|| \geq 10$ for some $F \in \mathcal{F}$. Then, either $||z, F|| = 4$ or $||v, F|| = 4$ for some $v \in \{v_1, v_p\}$. If $||z, F|| = 4$, then some $x \in V(F)$ has $xv_1 \in E(G)$. So $G[F - x + z] = K_4$ but $xv_1 \cdots v_p$ is longer than P, contradicting (O6). If say $||v_1, F|| = 4$, then $||\{z, v_p\}, F|| \geq 6$ and so there exists some $x \in N(z) \cap N(v_p) \cap V(F)$. So $G[F - x + v_1] = K_4$ and $v_2 \cdots v_p xz$ is longer than P contradicting (O6). This proves Lemma 4.18(a).

Due to P being maximum and $G[P]$ not a cycle by Lemma 4.17, W is an independent set so $d(v_1) + d(v_p) + d(z) \geq 9k - 4$. Note, at most one of these vertices is low. Then,

$$||W, \mathcal{F}|| \geq 9k - 4 - 6 = 9(k-1) - 1 \tag{4.20}$$

By Lemma 4.18(a), if $|\mathcal{F}| \leq k - 2$, then $||W, \mathcal{F}|| \leq 9(k-2) < 9(k-1) - 1$, a contradiction. This proves Lemma 4.18(b)

For Lemma 4.18(c), if P had no chord at an endpoint, then $d_{G[R]}(v_1) = d_{G[R]}(v_p) = 1$ and so we could improve (4.20) to $||W, \mathcal{F}|| \geq 9k - 4 - 4 = 9(k-1) + 1$, a contradiction to Lemma 4.18(a) or Lemma 4.18(b). This completes the proof. $\square$

Below we continue to use the special set $W = \{v_1, v_p, z\}$ where $z \in R - V(P)$ is the endpoint of a maximal path in $R - V(P)$ with $d_{G[R]}(z) = 2$.

Lemma 4.19. *If $k \geq 2$ and $F \in \mathcal{F}$ with $F = K_4^-$ or $F = C_5^+$, then $||W, F|| \leq 7$. Hence, $K_4^-, C_5^+ \notin \mathcal{F}$.*

Proof. Suppose that there is some $F \in \mathcal{F}$ such that $||W, F|| \geq 8$. If $F = C_5^+$, then by Lemma 4.2, there are two vertices $w, w' \in W$ such that $N(w) \cap V(F) = N(w') \cap V(F)$. Namely, $G[\{w, w'\} \cup (N(w) \cap V(F))] \supseteq K_4^-$, a contradiction to (O1). Hence, $F = K_4^-$.

If $||z, F|| = 3$, then for each $x \in \{x_1, x_3\}$, $G[F - x + z] = K_4^-$ and $G[P + x]$ contains a longer path than P since there is some $v \in \{v_1, v_p\}$ with $||v, F|| = 3$ as well.

Otherwise, $||z, F|| = 2$ and $||v, F|| = 3$ for every $v \in \{v_1, v_p\}$. If there is $x \in N(z) \cap \{x_1, x_3\}$, then again $G[F - x + v_1] = K_4^-$ and $G[P + x]$ contains a path longer than P, contradicting (O6). Hence, $N(z) \cap V(F) = \{x_2, x_4\}$. But then, $G[F - x_1 + z] = K_4^-$ and $x_1 v_1 \cdots v_p x_1$ is a cycle with chord from Lemma 4.18(c). This gives us k disjoint chorded cycles, a contradiction. This proves the first part of the claim.

Since W is an independent set, we have $||W, \mathcal{F}|| \geq 9k - 4 - 6 = 9(k-1) - 1$. By Lemma 4.18(a) and the first claim, if there exists a single cycle of type K_4^- or C_5^+, then we have $||W, \mathcal{F}|| \leq 7 + 9(k-2) < 9(k-1) - 1$, a contradiction. This completes the proof. □

Lemma 4.20. *Suppose that z is the same from Lemma 4.18 and $||\{v_1, v_p, z\}, F|| \geq 8$. If $k = 2$, then $F = K_{3,3}$. If $k \geq 3$, then $F = K_{3,3}$, or $F = K_4$ and $||z, F|| = 0$.*

Proof. Suppose that $F \neq K_{3,3}$, then by Lemma 4.19, $F = K_4$. Then some $x \in V(F)$ is adjacent to v_1 or v_p.

If $||z, F|| = 4$, then $G[F - x + z] = K_4$ and $G[P + x]$ contains a path longer than P, a contradiction.

If $||z, F|| = 3$, then $||\{v_1, v_p\}, F|| \geq 5$, and some $x' \in F$ is adjacent to both v_1 and v_p. In this case, $G[F - x' + z]$ contains K_4^- and $G[P + x']$ contains a cycle with chord due to Lemma 4.17.

If $||z, F|| = 1$, then $||\{v_1, v_p\}, F|| \geq 7$. We may assume that $||v_1, F|| = 4$ and $||v_p, F|| \geq 3$. If v_p and z have a common neighbor, say x, on F, then $G[F - x + v_1] = K_4$ and $G[P - v_1 + x + z]$ contains a path longer than P. If v_p and z have no common neighbor on F, let $x \in F$ be the neighbor of z, then $G[F - x + v_p] = K_4$ and $G[P - v_p + x + z]$ contains a path longer than P.

Finally, let $||z, F|| = 2$. If say $||v_1, F|| = 4$, then v_p and z have no common neighbors $x \in V(F)$ (otherwise, we get $G[F - x + v_1] = K_4$ and $G[P - v_1 + x + z]$ contains a path longer than P); hence, there exists $x \in N(v_1) \cap N(v_p) \cap V(F) \setminus N(z)$ so that $G[F - x + z] = K_4^-$ and $G[P + x]$ contains a chorded cycle by Lemma 4.18(c). So $||v_1, F|| = ||v_p, F|| = 3$. If $N(v_1) \cap V(F) = N(v_p) \cap V(F)$, then some $x \in N(v_1) \cap V(F)$ is not a neighbor of z, thus $G[P + x]$ and $G[F - x + z]$ both contain chorded cycles. If $N(v_1) \cap V(F) \neq N(v_p) \cap V(F)$, then z has no neighbors in $N(v_1) \cap F - N(v_p)$ and $N(v_p) \cap F - N(v_1)$ as we can contradict (O6) again, so $N(z) \cap F = N(v_1) \cap N(v_p) \cap F$. Let $\{x_1, x_2\} = V(F) - N(z)$ and $x_1 \in N(v_1)$ (thus $x_2 \in N(v_p)$). Let z' be the other endpoint of P_1. If $z = z'$, then $||z, P|| = 2$. Let $N(z) \cap V(P) = \{v_i, v_j\}$ with $i < j$. Then $v_1 v_2 \cdots v_j z x_3 v_1$ is a chorded cycle and $G[F - x_3 + v_p] = K_4^-$. So $z' \neq z$. Then similar to z, $N(z') \cap V(F) = N(z) \cap V(F) = \{x_3, x_4\}$. Now $G[P + x_1 + x_2]$ contain a chorded cycle and $G[\{x_3, x_4, z, z'\}] \supseteq K_4^-$.

Therefore, $||z, F|| = 0$. Now let $k = 2$. Note that $d_G(v_1) \leq 6$, and by Lemma 4.16, $d_G(z) = d_{G[R]}(z) = 2$, a contradiction to (4.1). □

By Lemma 4.20, each copy of $F = K_{3,3} \in \mathcal{F}$, has $||w, F|| = 3$ for all $w \in W$. Moreover, each

$$V(F) \text{ has partite sets } A_F, B_F \text{ such that } ||W, A_F|| = 0.$$

Otherwise, if v_1 and v_p share a neighbor $x \in V(F)$, then $G[V(P) + x]$ contains a chorded cycle and $G[F - x + z] = K_{3,3}$. If instead z and say v_1 share a neighbor $x \in V(F)$, then $G[F - x + v_p] = K_{3,3}$ and $z x v_1 \cdots v_{p-1}$ is a path longer than P, contradicting (O6).

Now, choose $z' \in A_F$. If $||z', R|| \geq 3$, then $G[F - z' + z] = K_{3,3}$, P is still a path in $G[R - z + z']$, so we either contradict (O6) or (O7) since $d_{R'}(z') \geq 3 > 2 \geq d_{G[R]}(z)$. If $||z', R|| = 2$, then the same process yields a contradiction to (O8) since z is low but $zz' \notin E(G)$ implies z' is not low. Hence, $||z', R|| \leq 1$. So, $d(z') \leq 3(k-2) + 4 + 1 = 3k - 1$, but z' not low implies that we have $||z', F_1|| = 4$ where F_1 is still our unique copy of $K_4 \in \mathcal{F}$. Hence, for each $x \in V(F_1) \cap (N(v_1) \cup N(v_p))$, $G[F - z' + z] = K_{3,3}$ and $G[F_1 - x + z'] = K_4$ and $G[V(P) + x]$ contains a path longer than P, contradicting (O6). Thus, this completes the case if $k \geq 3$.

All that remains is to handle the base case of $k = 2$.

Lemma 4.21. *If $k = 2$, $G - V(P)$ forms a subgraph of $K_{3,m}$, where $m = n - 3 - |V(P)|$, such that v_1, v_p have only neighbors in the partite set of size 3.*

Proof. Let $P_1 = z_1 \dots z_s$ be the longest path in $R - V(P)$. Consider $W = \{v_1, v_p, z_1\}$, where by Lemma 4.16 we assume that $d_R(z_1) = 2$. Clearly, W is an independent set. So at most one vertex in W is low. Then $d_G(v_1) + d_G(v_p) + d_G(z_1) \geq 9 + 5 = 14$. As $d_{G[R]}(x) \leq 2$ for each $x \in W$, we have $||W, F|| \geq 14 - 6 = 8$. By Lemma 4.20, $F = K_{3,3}$. Let A_F, B_F be the two parts of F. Let $A_F = \{x_1, x_3, x_5\}$ and $B_F = \{x_2, x_4, x_6\}$.

We claim that v_1, v_p and z_1 have only neighbors in one part of F, say A_F. By symmetry, assume that $N(v_1) \cap V(F) = A_F$. Suppose $N(v_p) \cap F \subseteq A_F$. If $N(z_1) \cap V(F) \subseteq B_F$, let $x_1 \in N(v_1) \cap N(v_p) \cap V(F)$, then $G[F - x_1 + z_1] = K_{3,3}^-$ and $G[P + x_1]$ contains another chorded cycle.

Now suppose $N(v_p) \cap V(F) \subseteq B_F$. If $N(z_1) \cap F \subseteq B_F$, then v_p and z_1 have a common neighbor, say x_2, thus $G[F - x_2 + v_1] = K_{3,3}$ and $z x_2 v_p v_{p-1} \cdots v_2$ is a path longer than P, contradicting (O6). If $N(z_1) \cap V(F) \subseteq A_F$, then when $||z_1, F|| = 3$, let $x_2 \in N(v_p)$, then $G[F - x_2 + z_1] = K_{3,3}$ and $G[P + x_2]$ contains a path longer than P, and when $||v_p, F|| = 3$, let $x_1 \in N(z_1)$, then $G[F - x_1 + v_p] = K_{3,3}$ and $G[P - v_p + x_1 + z]$ contains a path longer than P.

By symmetry, z_s has only neighbors in A_F as well. Let $x_1, x_3 \in N(z_1) \cap V(F)$ and $z_s x_3 \in E(G)$ and $x_5 \in N(v_1) \cap N(v_p) \cap V(F)$. Then $G[P + x_5]$ and $G[P_1 \cup \{x_1, x_3, x_2\}]$ both contain chorded cycles since P has a chord by Lemma 4.18. So $P_1 = \{z_1\}$. It follows that $R - P$ consists of isolated vertices and each of the vertices in $R - P$ has neighbors in A_F. $\square$

Lemma 4.22. *If $k = 2$ and $G[R]$ has no hamiltonian path, then G contains 2 disjoint chorded cycles.*

Proof. By Lemma 4.21, we may assume that G consists of a path P and a subgraph H of $K_{3,m}$ with parts $A = \{x_1, x_3, x_5\}$ and $B = \{y_1, \dots, y_m\}$ such that v_1, v_p are all not adjacent to B. As B is an independent set, at most one vertex in B is low, so we may assume that $d_G(y_i) \geq 5$ for each $1 \leq i \leq m - 1$.

Also, at most one of v_1, v_p is low, and $d(v_1), d(v_p) \leq 5$. So we may assume that $d(v_1) = 5, d(v_p) \geq 4$ and $d(y_m) \geq 4$. It follows that v_1 is adjacent to all vertices of A and v_p is adjacent to at least two vertices in A. Likewise, for each $i \in [m-1]$, since y_i has only three neighbors in H, it has at least two neighbors in P. As $R - P \neq \emptyset$, $m - 1 \geq 3$. We may assume that $v_p x_3 \in E(G)$ and $y_m x_3, y_m x_5 \in E(G)$.

CASE 1: There exists $1 \le i < j \le p$ and a vertex in $B - y_m$, say y_1, with $y_1 v_i, y_1 v_j \in E(G)$, and a vertex, say y_2 with $y_2 v_s \in E(G)$, such that $i < j < s$ or $s < i < j$. If $i < j < s$, then $v_1 P v_j y_1 x_1 v_1$ contains chord $y_1 v_i$, and $v_s P v_p x_3 y_3 x_5 y_2 v_s$ contains chord $x_3 y_2$. Similarly, if $s < i < j$, then $v_p P v_i y_1 x_3 v_p$ contains chord $y_1 v_j$, and $v_1 P v_s y_2 x_1 y_3 x_5 v_1$ contains chord $x_5 y_2$. This proves the case.

For CASE 1 to not apply for all vertices in $B - y_m$, we need $N(y_s) \cap V(P) = \{v_i, v_j\}$. We now handle that case.

CASE 2: For each $s \in [m-1]$, $N(y_s) \cap V(P) = \{v_i, v_j\}$, where $i < j$. Let $v_1 v_t \in E(G)$. If $t \le i$, then $v_1 P v_i y_1 x_1 v_1$ contains chord $v_1 v_t$, and $v_j P v_p x_3 y_2 x_5 y_3 v_j$ contains chord $x_3 y_3$.

If $i < j < t$, then $v_1 v_t P v_p x_3 y_4 x_5 v_1$ contains chord $v_1 x_3$ and $G[x_1, y_1, y_2, y_3, v_i, v_j] = K_{3,3}$.

So $v_1 v_t \in E(G)$ with $i < t \le j$. Since $y_m v_p \notin E(G)$, either $d_{G[R]}(v_p) = 2$ or $N(y_m) = \{x_1, x_3, x_5, v_i, v_j\}$.

Case 2.1: We don't have a chord incident to v_p and $N(y_m) = \{x_1, x_3, x_5, v_i, v_j\}$. Now, $j < p-1$ as otherwise $P' = v_1 \cdots v_{p-1} y_1$ is a maximum path and $G[V(H) - y_1 + v_p] \supseteq K_{3,3}$, contradicting (O7) since $||y_1, P' - y_1|| = 2$ but $||v_p, P - v_p|| = 1$. Note that also $d_{G[V(P)]}(v_{p-1}) \le 3$ so $||v_{p-1}, H|| \ge 1$. By the case, a neighbor must be in $\{x_1, x_3, x_5\}$, say $v_{p-1} x_1 \in E(G)$. Hence, $x_1 v_1 v_t v_{t+1} \cdots v_p x_1$ is a cycle with chord $x_1 v_{p-1}$ and $G[v_i, x_3, x_5, y_1, y_2, y_3] = K_{3,3}$. This produces two disjoint chorded cycles and ends the subcase.

Case 2.2: $d_{G[R]}(v_p) = 2$. By symmetry with v_1, $v_p v_q \in E(G)$ for some $i \le q < j$. Now $G[x_1, x_5, v_i, y_1, y_2, y_3] = K_{3,3}$.

If $i < t \le q \le j$, then $v_1 v_t P v_p x_3 v_1$ contains chord $v_p v_q$. Otherwise $q < t$, and so $G[V(P)]$ is a Θ-graph.

If $i < q < t < j$, then $v_1 v_t v_{t-1} \cdots v_q v_p v_{p-1} v_j y_4 x_3 v_1$ contains chord $v_p x_3$ unless $m = 4$ and $v_j y_4 \notin E(G)$. But then $N(y_m) \supseteq A$ by (4.1) and so we can swap the roles of y_1 and y_4, i.e. $G[x_1, x_5, v_i, y_2, y_3, y_4] = K_{3,3}$ and $v_1 v_t v_{t-1} \cdots v_q v_p v_{p-1} v_j y_1 x_3 v_1$ contains chord $v_p x_3$.

If $i < q < t = j$, then $v_1 v_q \notin E(G)$ and $d_{G[P]}(v_q) = 3$ implies $||v_q, A|| \ge 9 - 5 - 3 = 1$, say $x_1 v_q \in E(G)$. Then $x_1 v_1 v_t v_{t-1} \cdots v_q v_p x_1$ is a cycle with chord $x_1 v_q$.

If $i = q < t < j$, a similar argument holds with $G[x_1, x_5, v_j, y_2, y_3, y_4] = K_{3,3}$ and $x_1v_pv_qv_{q+1}\cdots v_tv_1x_1$ is a cycle with chord x_1v_t.

Now it remains to handle when $i = q$ and $j = t$ so $i = q < t = j$. First, we claim

$$i = j - 2.$$

Otherwise, $||v, H|| \geq 2$ for $v \in \{v_{i+1}, v_{i+2}\}$. By the case, we have $N(v) \cap V(H) \subseteq A$. Hence, there exists, say $x_1 \in A \cap N(v_{i+1}) \cap N(v_{i+2})$. Then, $x_1v_1\cdots v_{i+1}x_1$ is a cycle with chord x_1v_i and $G[V(H) - x_1 + v_j] \supseteq K_{3,3}$, producing two disjoint chorded cycles.

Next, we claim $i = 2$. Note, $G[P]$ is a Θ-graph so $d_{G[P]}(v_2) = 2$, as otherwise $G[P]$ contains a chorded cycle. So, $||v_2, H|| \geq 2$ since $v_2v_p \notin E(G)$. If $v_2y_k \in E(G)$ for some $k \in [m-1]$, then $y_kv_2\cdots v_jy_k$ is a cycle with chord y_kv_i. Since $N(v_2) \cap V(H)$ is contained in either A or B, by symmetry assume $x_1v_2, x_3v_2 \in E(G)$. Then $v_1v_2x_1v_px_3v_1$ is a 5-cycle with chord v_1x_1, contradicting (O3).

By symmetry, we also have $j = p-1$. This together with $i = 2$ and $i = j-2$ yields $i = q = 2$, $j = t = 4$, and $p = 5$ so that $G[V(P)] = K_{2,3}$. By the symmetry of $K_{2,3}$, $\{v_1, v_3, v_5, y_m\}$ is an independent set so at most one vertex has degree 4. Thus, $G \in \{G_1(n,2), G_1^-(n,2)\}$ with parts $\{x_1, x_3, x_5, v_2, v_4\}$ and $B \cup \{v_1, v_3, v_5\}$, a contradiction.

□

This contradicts our initial assumption that G is a counterexample to Theorem 4.1. This completes the $k = 2$ case, and hence the case where $G[R]$ does not have a Hamiltonian path.

4.4 Case: $G[R]$ has a Hamiltonian path and $k \geq 3$

In this section we consider the case that $G[R]$ does have a Hamiltonian path and $k \geq 3$. We first prove a stronger version of Lemma 4.6.

Lemma 4.23. *For $k \geq 3$, $|R| \geq 5$.*

Proof. Suppose that $|R| \leq 4$. Then by Lemma 4.6, $|R| = 4$, say $R = \{v_1, v_2, v_3, v_4\}$.

CASE 1: $G[R] \subseteq C_4$. First we show that

$$||R, F|| \leq 12 \text{ for every } F \in \mathcal{F}. \tag{4.21}$$

Indeed, suppose that $||R, F|| > 12$ for some $F \in \mathcal{F}$. By Lemma 4.2, $F = K_4$. Since $||R, F|| \geq 13$, there is $v_i \in R$ with $||v_i, F|| = 4$ and there is $x \in V(F)$ with $||x, R|| = 4$. So, $G[V(F) - x + v_i] = K_4$ and $G[R - v_i + x] \supset G[R]$. This contradicts either (O1) or (O8). This shows (4.21).

Our next claim is

$$\text{if } F \in \mathcal{F} \text{ and } F = C_5^+, \text{ then } ||R, F|| \leq 9. \tag{4.22}$$

Indeed, suppose $F \in \mathcal{F}$ is the cycle $x_1x_2x_3x_4x_5x_1$ with chord x_3x_5 and $||R, F|| \geq 10$. By Lemma 4.2, for each $1 \leq i \leq 4$, $||v_i, F|| \leq 3$ and if $||v_i, F|| = 3$, then $N(v_i) \cap F = \{x_1, x_2, x_4\}$. So, there are $v, v' \in R$ with $N(v) \cap F = N(v') \cap F = \{x_1, x_2, x_4\}$. In this case, replacing F in $\mathcal{F}$ by the chorded 4-cycle $G[\{v, v', x_1, x_2\}]$ we obtain a family $\mathcal{F}'$ that is better than $\mathcal{F}$ by (O1). This proves (4.22).

Next we show that

$$\text{if } F \in \mathcal{F} \text{ and } F = K_4^-, \text{ then } ||R, F|| \leq 10. \tag{4.23}$$

Indeed, suppose $F \in \mathcal{F}$ is the cycle $x_1x_2x_3x_4x_1$ with chord x_2x_4 and $||R, F|| \geq 11$. By Lemma 4.2, for each $1 \leq i \leq 4$, $||v_i, F|| \leq 3$ and if $||v_i, F|| = 3$, then $N(v_i) \supseteq \{x_1, x_3\}$. This means that there is $1 \leq j \leq 4$ such that each $v \in R - v_j$ has 3 neighbors in F and v_j has at least 2 neighbors in F. If $x_1v_j \in E(G)$, then for each $j' \in [4] - j$, $G[F - x_1 + v_{j'}] = K_4^-$ and $G[R - v_{j'} + x_1] \supset G[R]$, which contradicts either (O1) or (O8). Thus, $x_1v_j \notin E(G)$ and symmetrically $x_3v_j \notin E(G)$, implying $N(v_j) \cap V(F) = \{x_2, x_4\}$. Then, $G[V(F) - x_1 + v_j] = K_4^-$ and $G[R - v_j + x_1] \supset G[R]$, again contradicting either (O1) or (O8). This proves (4.23).

Since R has two disjoint pairs of nonadjacent vertices,

$$||R, \mathcal{F}|| \geq 2(6k - 3) - 2||G[R]|| \geq 12(k - 1) - 2 = 12(k - 2) + 10. \tag{4.24}$$

So, by (4.21),

$$\begin{aligned}&\mathcal{F}\ \textit{has}\ k-1\ \textit{chorded cycles and}\\&||G[R]||\geq 3,\ \textit{i.e.}\ G[R]\ \textit{is a cycle}\ v_1v_2v_3v_4v_1\ \textit{or a path}\ v_1v_2v_3v_4.\end{aligned}\tag{4.25}$$

Now we show

$$K_4 \in \mathcal{F}. \tag{4.26}$$

Indeed, otherwise by Lemma 4.13, $\mathcal{F}$ contains two cycles F_1 and F_2 inducing K_4^- in G. Hence by (4.21) and (4.23), $||R,\mathcal{F}|| \leq 2\cdot 10 + 12(t-2) = 12(k-1)-4$, a contradiction to (4.24).

We also need

$$\textit{if } F \in \mathcal{F} \textit{ and } F = K_4, \textit{ then } ||R,F|| \leq 11. \tag{4.27}$$

If $G[R] = C_4$, then this follows from Lemma 4.12; so by (4.25) we may assume $G[R]$ is a path $v_1v_2v_3v_4$. Suppose $||R,F|| \geq 12$. If $||v_1,F|| = 4$, then for each $x \in V(F)$, $||x, R-v_1|| \leq 1$, since otherwise $G[F-x+v_1] = K_4$ and $G[R-v_1+x]$ has a Hamiltonian path and more edges than $G[R]$, which contradicts (O7). But then $||R,F|| \leq 8$. If $||v_1,F|| = 3$, then each $x \in F$ has at most 2 neighbors in $R-v_1$, since otherwise $G[F-x+v_1] \supseteq K_4^-$ and $G[R-v_1+x] = K_4^-$, which contradicts (O1). But in this case $||R,F|| \leq 3+2\cdot 4 = 11$, as claimed. Thus, $||v_1,F|| \leq 2$ and symmetrically $||v_4,F|| \leq 2$. Since $||R,F|| \geq 12$, this yields $||v_1,F|| = ||v_4,F|| = 2$ and $||v_2,F|| = ||v_3,F|| = 4$. Then, let $x_1 \in N(v_1)\cap V(F)$, $x_4 \in N(v_4)\cap V(F) - x_1$, and x_2, x_3 be the remaining vertices in F. In this notation, $G[\{v_1,v_2,x_1,x_2\}] \supseteq K_4^-$ and $G[\{v_3,v_4,x_3,x_4\}] \supseteq K_4^-$, a contradiction to (O1).

Our last claim is:

$$\textit{if } F \in \mathcal{F},\ F = K_{3,3} \textit{ and } R \textit{ contains } \ell \textit{ low vertices, then } ||R,F|| \leq 12-\ell. \tag{4.28}$$

Indeed, since low vertices form a clique in G and $G[R]$ is bipartite, $\ell \leq 2$. So, there is nothing to prove if $||R,F|| \leq 10$. Also by (4.21) we may assume $\ell \geq 1$. Suppose $||R,F|| \geq 13-\ell \geq 11$. By Lemma 4.2, there is $1 \leq j \leq 4$ such that each vertex in $R-v_j$ has 3 neighbors in F and v_j has at least $||R,F||-9 \geq 2$ neighbors in F. By the same lemma, if the partite sets of F are A and B, then we may assume that $(N(v_1) \cup N(v_3)) \cap V(F) \subseteq A$ and $(N(v_2) \cup N(v_4)) \cap V(F) \subseteq B$. By above, at least $||R, F|| + \ell - 12 \geq 1$ low vertices in R have 3 neighbors in F; so let v_i be such a vertex. By symmetry, we may assume i is odd. Since $||\{v_2, v_4\}, B|| \geq 5$, there is $b \in B$ adjacent to both, v_2 and v_4. As $bv_i \notin E(G)$, b is high. Then $G[F - b + v_i] = K_{3,3}$, $G[R - v_i + b] \subseteq G[R]$, and $\sum_{w \in R - v_i + b} d(w) > \sum_{v \in R} d(v)$. This is a contradiction to (O8) which proves (4.28).

By (4.26), we may assume $F_1 = K_4$. So by (4.24), (4.21), (4.23), (4.22), and (4.27), $\mathcal{F}$ does not contain copies of K_4^- and C_5^+. In other words,

$$F_1 = K_4 \textit{ and each chorded cycle in } \mathcal{F} \textit{ is a } K_4 \textit{ or a } K_{3,3}. \tag{4.29}$$

By (4.25), we have two subcases.

Case 1.1: $G[R]$ is a path $v_1v_2v_3v_4$. Since $||G[R]|| = 3$, (4.24) together with (4.21) imply that $||R, F|| = 12$ for all $F \in \mathcal{F}$. But this is impossible by (4.26) and (4.27).

Case 1.2: $G[R]$ is a cycle $v_1v_2v_3v_4v_1$. By Lemma 4.12, either $||R, F_1|| \leq 10$ or $||R, F_1|| = 11$ and $\sum_{j=1}^{4} d_G(v_j) \geq 12k - 5$. In both cases, by (4.24),

$$||R, \mathcal{F} - F_1|| \geq 12(k-2).$$

In view of (4.21), this means for each $F \in \mathcal{F} - F_1$, $||F, R|| = 12$. Hence by (4.29) and (4.27), each such chorded cycle, and in particular, F_2 induces $K_{3,3}$. Then by (4.28), all vertices in R are high. So, instead of (4.24) we have

$$||R, \mathcal{F}|| \geq 4(3k-1) - 2||G[R]|| = 12(k-1).$$

This together with (4.27) and (4.29) contradicts (4.21). This proves the case.

CASE 2: $G[R] \nsubseteq C_4$. If $G[R]$ does not have a degree 3 vertex, then $G[R] = K_3 + K_1$, say v_1 is the isolated vertex, and $v_2v_3v_4v_2$ is a 3-cycle. Then, $d_G(v_1) + d_G(v_2) \geq 6k - 3$ and so $||\{v_1, v_2\}, \mathcal{F}|| \geq 6k - 5 = 6(k-1) + 1$. Hence, there exists some $F \in \mathcal{F}$ such that

$||\{v_1, v_2\}, F|| \geq 7$. By Lemma 4.2, $F = K_4$. Whether $||v_1, F||$ is 3 or 4, there exists some $x \in V(F)$ such that $G[F - x + v_1] = K_4$ and $xv_2 \in E(G)$. But then $G[R - v_1 + x] \supseteq K^+_{1,3}$, contradicting (O6).

So suppose $G[R]$ does have degree 3 vertex v_1. If the set $\{v_2, v_3, v_4\}$ is independent, then it has at most one low vertex and so $||\{v_2, v_3, v_4\}, \mathcal{F}|| \geq 9k-4-3 = 9(k-1)+2$. Hence, there exists some $F \in \mathcal{F}$ such that $||\{v_2, v_3, v_4\}, F|| \geq 10$ and so a vertex, say v_2, has 4 neighbors in F. By Lemma 4.2, $F = K_4$. By pigeonhole, there exists $x \in N(v_3) \cap N(v_4) \cap V(F)$. Then $G[F - x + v_2] = K_4$ and $G[R - v_2 + x] \supseteq C_4$, contradicting (O6).

The last possibility is that $G[R] = K^+_{1,3}$, say v_2v_3 is an edge. Consider

$$f(R, F) := 2 \cdot ||v_4, F|| + ||v_2, F|| + ||v_3, F||. \tag{4.30}$$

Our main claim is that for each $F \in \mathcal{F}$

$$f(R, F) \leq 12 \textit{ and if } f(R, F) = 12 \textit{ then } F \textit{ is a } K_4. \tag{4.31}$$

Indeed, suppose $f(R, F) \geq 13$. Then for some $2 \leq j \leq 4$, $||v_j, F|| = 4$. So by Lemma 4.2, $F = K_4$. Since $|F| = 4$, to have $f(R, F) \geq 13$, we need $||\{v_2, v_3\}, F|| \geq 13 - 8 = 5$ and $||v_4, F|| \geq \lceil (13-8)/2 \rceil = 3$. Then there is $x \in N(v_2) \cap N(v_3) \cap V(F)$. Hence $G[F - x + v_4] \supseteq K^-_4$ and $G[R - v_4 + x] \supseteq K^-_4$, contradicting (O1).

Suppose now $f(R, F) = 12$ and $F \neq K_4$. By Lemma 4.2, to have $f(R, F) = 12$ we need

$$||v_2, F|| = ||v_3, F|| = ||v_4, F|| = 3. \tag{4.32}$$

If F induces a cycle $x_1x_2x_3x_4x_1$ with chord x_2x_4, then $\{x_1, x_3\} \subseteq N(v_2) \cap N(v_3) \cap N(v_4) \cap V(F)$. Hence $G[R - v_4 + x_1] \supseteq K^-_4$ and $G[F - x_1 + v_4] \supseteq K^-_4$, contradicting (O1).

If F induces a cycle $x_1x_2x_3x_4x_5x_1$ with chord x_3x_5, then by (4.32) and Lemma 4.2, $\{x_1, x_2, x_4\} \subseteq N(v_2) \cap N(v_3) \cap N(v_4) \cap V(F)$. So, $G[\{v_2, v_3, x_2, x_1\}] = K_4$, contradicting (O1).

If F induces a $K_{3,3}$ with parts A and B, then by (4.32) and Lemma 4.2, $N(v_i) \cap V(F) \in \{A, B\}$ for all $2 \leq i \leq 4$. By Lemma 4.3, we may assume $N(v_2) \cap V(F) = A$, $N(v_3) \cap V(F) =$

B, and by symmetry, $N(v_4) \cap V(F) = A$. Then for each $a \in A$, $v_1v_2v_3av_4v_1$ is a chorded 5-cycle contradicting (O3). This proves (4.31).

Observe that

$$\sum_{F \in \mathcal{F}} f(R,F) = 2 \cdot d(v_4) + d(v_2) + d(v_3) - 6$$
$$\geq 2(6k-3) - 6 = 12k - 12 = 12(k-1).$$

By (4.31), this means $\mathcal{F}$ has $k-1$ disjoint chorded cycles and $f(R,F) = 12$ for each $F \in \mathcal{F}$. Also by (4.31),

$$\textit{each } F \in \mathcal{F} \textit{ induces a } K_4. \textit{ In particular, } |V(G)| = 4k. \tag{4.33}$$

Now we show that

$$\textit{for each } F \in \mathcal{F},\ ||v_4, F|| = 4, \textit{ and each } x \in F \textit{ has exactly one neighbor in } \{v_2, v_3\}. \tag{4.34}$$

Indeed, let $V(F) = \{x_1, x_2, x_3, x_4\}$. To have $f(R,F) = 12$, we need $||v_4, F|| \geq 2$. Furthermore, if $N(v_4) \cap V(F) = \{x_1, x_2\}$, then $||\{v_2, v_3\}, F|| = 8$. Then for $x \in V(F) - N(v_4)$, $G[F - x + v_4] \supseteq K_4^-$ and $G[R - v_4 + x] \supseteq K_4^-$, contradicting (O1). If there exists $x \in N(v_2) \cap N(v_3) \cap V(F)$ with $|N(v_4) \cap V(F) - x| \geq 2$, then we reach the same contradiction. So $||\{v_2, v_3\}, F|| \leq 5$. Then, for $f(R,F) = 12$, we have $||v_4, F|| = 4$ but also $||\{v_2, v_3\}, F|| = 4$. Now if there exists $x \in N(v_2) \cap N(v_3) \cap V(F)$, we again have $||N(v_4) \cap V(F) - x| \geq 2$. So we have $N(v_2) \cap N(v_3) \cap V(F) = \emptyset$. This proves (4.34).

Our last claim is that

$$\textit{for each } F \in \mathcal{F},\ ||v_1, F|| = 0. \tag{4.35}$$

Suppose now that for some $F \in \mathcal{F}$ and $x \in F$, $v_1x \in E(G)$. By (4.34), $G[F - x + v_4] = K_4$ and $G[R - v_4 + x] = K_4^-$, contradicting (O1).

By (4.35), $||v_1, \mathcal{F}|| = 0$ and so $d_G(v_1) = 3$. Let $x \in V(F_1)$. By (4.33), (4.34), and (4.35),

$$d_G(x) \leq |V(G) - x - v_1| - 1 \leq 4k - 3$$

Hence $d_G(x) + d_G(v_1) \leq 4k$, contradicting the condition $d_G(v_1) + d_G(x) \geq 6k - 3$. This completes the case and the claim. □

A *broken* Θ*-graph* is obtained from a Θ-graph by deleting a vertex of degree 2. We will view each broken Θ-graph as a triple $(F_{u,v}, P_u, P_v)$ where $F_{u,v}$ is a cycle with two special vertices u and v, a path P_u starts at u, finishes at u' with $V(F_{u,v}) \cap P_u = \{u\}$ and a path P_v starts at v, finishes at v' with $[V(F_{u,v}) \cup V(P_u)] \cap P_v = \{v\}$.

Observation 4.24. *If F is a broken Θ-graph $(F_{u,v}, P_u, P_v)$, then for each triple T of vertices, F contains a path P_T passing through all vertices of T.*

Proof. Since the graph $F - (P_u - u)$ has a Hamiltonian path P', if $T \cap (V(P_u) - u) = \emptyset$, then we may take $P_T = P'$. Thus it is enough to consider the case that T has a vertex $t_1 \in P_u - u$, and similarly, a vertex $t_2 \in P_v - v$. Then for any choice of t_3, F has a $v'u'$-path containing t_3. Since any such path contains $P_v \cup P_u$, it contains all T. □

Lemma 4.25. *Suppose $k \geq 3$, $R = \{v_1, \ldots, v_r\}$, and $G[R]$ is a Θ-graph with branching vertices v_0 and v_0'. Let $F \in \mathcal{F}$ with $F = K_4$. Then*

(a) $||R, F|| \leq 3r$, and

(b) if $|\mathcal{F}| = k - 1$, then $||R, F|| \leq 2r + 2$.

Proof. By Lemma 4.23, $|R| \geq 5$, and $|R| = 5$ implies $G[R] = K_{2,3}$. Suppose the lemma does not hold for G so $G[R]$ is a spanning Θ_{j_1,j_2,j_3}-graph consisting of three paths $P_i = v_0 v_{i,1} v_{i,2} \cdots v_{i,j_i} v_0'$ connecting v_0 with v_0' for $i = 1, 2, 3$ where $j_i \geq 1$ since otherwise it is itself a chorded cycle. We proceed in a series of claims. Our first claim is:

$$||v_{i,j}, F|| \leq 3 \quad \text{for each } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq j_i. \tag{4.36}$$

Indeed, assume that for some $v_{i,j} \in R - v_0 - v_0'$, $x_\ell v_{i,j} \in E(G)$ for each $1 \le \ell \le 4$. If for at least one ℓ, $||x_\ell, R - v_{i,j}|| \ge 3$, then we consider $F' = F - x_\ell + v_{i,j}$ and $R' = R - v_{i,j} + x_\ell$. By the case, $F' = K_4$, and (since $R - v_{i,j}$ is a broken Θ-graph) by Observation 4.24, $G[R']$ contains a chorded cycle. This contradicts (O3), so $||x_\ell, R - v_{i,j}|| \le 2$ for all $1 \le i \le 4$. It follows that $||F, R|| \le 4 + 2 \cdot 4 = 12$. This satisfies both (a) and (b), contradicting our assumption. This proves (4.36).

Next we show

$$||\{v_0, v_0'\}, F|| \le 6. \tag{4.37}$$

Indeed, suppose (4.37) does not hold. Then by symmetry, we may assume $||v_0, F|| = 4$ and $N(v_0') \supseteq \{x_2, x_3, x_4\}$. Hence, for any two vertices $v, v' \in R - v_0'$, graph $G[R - v_0']$ has a path passing through v, v' and v_0. Thus if x_1 has two neighbors, say $v, v' \in R - v_0 - v_0'$, then $G[R - v_0' + x_1]$ contains a chorded cycle. Since $G[F - x_1 + v_r] = K_4$, this contradicts (O3). Therefore, $||x_1, R - v_0 - v_0'|| \le 1$. Similarly, since $N(v_0') \supseteq \{x_2, x_3, x_4\}$, if $||x_i, R - v_0 - v_0'|| \ge 2$ for some $2 \le i \le 4$, then $G[R - v_0 + x_i]$ contains a chorded cycle and $G[F - x_i + v_0] = K_4$. Again, this contradicts (O3). This proves (4.37).

Together, (4.36) and (4.37) imply (a).

Suppose now $|\mathcal{F}| = k - 1$. Then we can strengthen (4.36) to

$$||v_{i,j}, F|| \le 2 \quad \text{for each } 1 \le i \le 3 \text{ and } 1 \le j \le j_i. \tag{4.38}$$

Indeed, assume there exists $v_{i,j} \in R - v_0 - v_0'$ with $N(v_{i,j}) \cap V(F) = \{x_2, x_3, x_4\}$. If for at least one ℓ, $||x_\ell, R - v_{i,j}|| \ge 3$, then $G[F - x_\ell + v_{i,j}] \supseteq K_4^-$, and by Observation 4.24, $G[R - v_{i,j} + x_\ell]$ contains a chorded cycle. Since $|\mathcal{F}| = k - 1$, we have obtained k disjoint chorded cycles, a contradiction. Therefore $||x_\ell, R - v_{i,j}|| \le 2$ for all $1 \le \ell \le 4$, and hence $||F, R|| \le 3 + 2 \cdot 4 = 11$, as claimed. This satisfies both (a) and (b), contradicting our assumption. This proves (4.38). Together with (4.37), this implies

$$||F, R|| \le 2 \cdot (r - 2) + 6 = 2r + 2. \qquad \square$$

Lemma 4.26. *For $k \geq 3$, $G[R]$ does not contain a spanning Θ-graph.*

Proof. Suppose $G[R]$ contains a spanning Θ-subgraph $G' = \Theta_{r_1,r_2,r_3}$. Since $G[R]$ has no chorded cycles, $r_1, r_2, r_3 \geq 1$. Furthermore, since adding an edge to a Θ-graph creates a chorded cycle, $G[R] = G'$. Let ℓ be the number of low vertices in $G[R]$. Since $G[R]$ is triangle-free and low vertices form cliques, $\ell \leq 2$. Also, $\sum_{v \in V(R)} d_G(v) \geq (3k-1)r - \ell$. Then,

$$||R, \mathcal{F}|| \geq (3k-1)r - \ell - 2(r+1) = 3(k-1)r - \ell - 2 \tag{4.39}$$

so there exists some cycle $F \in \mathcal{F}$ such that $||R, F|| \geq 3r - \frac{\ell+2}{k-1} \geq 3r - 2$. We proceed with a series of claims.

Claim 4.27. *If $F = K_{3,3}$, then $||R, F|| \leq 3r - 2$ or we have the exception $G[R] = K_{2,3}$ and $\ell = 0$, in which case $||R, F|| \leq 3r$.*

Proof. Suppose that $||R, F|| \geq 3r - 1$. By Lemma 4.2, there exists some vertex $w \in V(R)$ such that for all $v \in V(R) - v_0$, $||v, F|| = 3$ and $||v_0, F|| \geq 2$. Also by Lemma 4.2, for all $v \in V(R) - w$, $N(v) \cap V(F) \in \{A, B\}$. Moreover, by Lemma 4.3 if $v, w \in V(R)$ with $vw \in E(G)$, then $N(v) \cap N(w) \cap V(F) = \emptyset$. Hence, $G[R \cup F]$ is bipartite with parts $A \cup R_A$ and $B \cup R_B$ where $R = R_A \cup R_B$. Note, $|A \cup R_A|, |B \cup R_B| \geq 5$ since $r \geq 5$ and each of R_A and R_B contain at most one low vertex since each is an independent set. So, we can partition $A \cup R_A$ into $A' \cup R'_A$ and $B \cup R_B$ into $B' \cup R'_B$ so that $R'_A \cup R'_B$ contain no low vertices and $G[A' \cup B'] = K_{3,3}$. This contradicts (O8) unless $\ell = 0$.

Suppose $\ell = 0$ but say $r_1 > 1$. Specifically, this means we can find three vertices $v', v, v'' \in R$ such that $v'vv''$ is a path in $G[R]$, the middle vertex v is a degree 2 vertex in the Θ-graph, and $v \neq w$. By symmetry we may assume $N(v') \cap V(F), N(v'') \cap V(F) \subseteq A$ and $N(v) \cap V(F) = B$. Also, some other $u \in R - v - v' - v''$ has $N(u) \cap V(F) \subseteq A$. Take $x \in N(u) \cap N(v') \cap N(v'') \cap V(F) \subseteq B$. But then, $G[F - x + v] = K_{3,3}$, and $G[R - v + x] \supseteq \Theta^+_{r_1,r_2,r_3}$ and thus also contains a chorded cycle. This contradicts (O3). Hence, we are left with $G[R] = K_{2,3}$, and this proves the claim. $\square$

Claim 4.28. *If $F = C_5^+$ then $||R, F|| \leq 2r + 1$. Moreover, since $r \geq 5$, $||R, F|| \leq 3r - 4$.*

Proof. Suppose $||R, F|| \geq 2r + 2$. Then, by Lemma 4.2, there exist two vertices $v, v' \in V(R)$ with $||v, F|| = 3$. Specifically, Lemma 4.2 gives $N(v) \cap N(v') \cap V(F) = \{x_1, x_2, x_4\}$, but then $vx_1v'x_2v$ forms a cycle with chord x_1x_2, contradicting (O3). This proves the claim. □

Claim 4.29. *If $F = K_4^-$, then $||R, F|| \leq 3r - 3$.*

Proof. Suppose that $||R, F|| \geq 3r - 2$. Let v_0, v_0' be the branching vertices in $G[R]$. There exists a $v' \in V(R) - v_0 - v_0'$ such that $||v', F|| = 3$. Moreover, all but possibly two vertices in R have neighbors x_1 and x_3 such that $||x, R|| \geq r - 2 \geq 3$ for every $x \in \{x_1, x_3\}$. If there exist such a $v' \in V(R) - v_0 - v_0'$ and $x \in \{x_1, x_3\}$ with $||v', F|| = 3$ and $||x, R - v'|| \geq 3$, then by Observation 4.24, $G[R - v' + x]$ contains a chorded cycle and $G[F - x + v'] \supseteq K_4^-$. Hence, $||\{x_1, x_3\}, R - v'|| \leq 4$ and so

$$||\{x_2, x_4\}, R|| \geq ||F, R|| - ||\{x_1, x_3\}, R|| \geq 3r - 2 - 6 = 3r - 8.$$

Then, there exists $v'' \in R$ with $||\{x_2, x_4\}, v''|| \geq \frac{1}{r}(3r - 8) = 3 - \frac{8}{r} > 1$ and so $v''x_2, v''x_4 \in E(G)$. Hence, $G[R - v'']$ is a broken Θ-graph so $G[R - v'' + x_1]$ contains a chorded cycle by Observation 4.24 and $G[F - x_1 + v''] = K_4^-$. □

Now, if $|\mathcal{F}| < k - 1$, then by Lemma 4.25,

$$||R, \mathcal{F}|| \geq 3(k-2)r + 3r - \ell - 2 \geq 3(k-2)r + 11$$

gives that any type of F would yield a contradiction to (4.39). Hence $|\mathcal{F}| = k - 1$, and by Lemma 4.13, either $\{K_4^-, K_4^-\} \subseteq \mathcal{F}$ or $K_4 \in \mathcal{F}$. If $\{K_4^-, K_4^-\} \subseteq \mathcal{F}$, then Claim 4.29 contradicts (4.39). If $F_1 = K_4$, then by Lemma 4.25, $||R, F_1|| \leq 2r + 2 \leq 3r - 3$, and so all other cycles F have $||R, F|| \geq 3r - 1$. This forces the exceptional case of Claim 4.27 in which $\ell = 0$ which again contradicts (4.39). This completes the proof of the lemma. □

Lemma 4.30. *If $k \geq 3$ and $r \geq 5$, then $G[R]$ has no Hamiltonian cycle.*

Proof. Suppose $G[R]$ has a Hamiltonian cycle $v_1v_2\cdots v_rv_1$. Then, since $G[R]$ has no chorded cycles, $G[R]$ is this cycle. Since R has at most two low vertices and by the conditions on G,

$$\sum_{i=1}^{r} d_G(v_i) \geq (r-4)(3k-1) + 2(6k-3) = (3k-1)r - 2.$$

Since, $d_{G[R]}(v_i) = 2$ for each i, we get $||R, V(G) - R|| \geq (3k-3)r - 2$. Hence, if $|\mathcal{F}| \leq k-2$, then

$$\text{there is } F \in \mathcal{F} \text{ such that } ||R, F|| \geq \left\lceil \tfrac{(3k-3)r-2}{|\mathcal{F}|} \right\rceil \geq 3r + \tfrac{3r-2}{k-2} > 3r. \tag{4.40}$$

Then there is $v_i \in R$, say v_1 with $||v_1, F|| \geq 4$. By Lemma 4.2, $F = K_4$, $||v_1, F|| = 4$, and there is $x \in V(F)$ such that $||x, R|| \geq ||F, R||/4 \geq (3r+1)/4 \geq 16/4 = 4$. Then $G[V(F) - x + v_1] = K_4$ and $G[R - v_1 + x]$ has a vertex x adjacent to at least 3 vertices on the path $v_2, \ldots, v_r$, a contradiction to (O3). Thus $|\mathcal{F}| = k-1$.

Similarly to (4.40),

$$\text{there is } F \in \mathcal{F} \text{ such that } ||R, F|| \geq \left\lceil \tfrac{(3k-3)r-2}{|\mathcal{F}|} \right\rceil = 3r - \tfrac{2}{k-1} \geq 3r - 1 > 2r. \tag{4.41}$$

By Lemma 4.2, we have the following cases.

CASE 1: $F = K_4$. By symmetry, we may assume $||v_1, F|| \geq 3$. Also, there is $x \in V(F)$ with

$$||x, R|| \geq \left\lceil \frac{3r-1}{4} \right\rceil \geq \left\lceil \frac{14}{4} \right\rceil = 4.$$

Then $G[F - x + v_1]$ contains a K_4^- and $G[R - v_1 + x]$ has a vertex x adjacent to at least 3 vertices on the path $v_2, \ldots, v_r$. Thus G has k disjoint chorded cycles, a contradiction. This completes the case.

For the remaining cases, since no vertex in R may have 4 neighbors in F, by (4.41) we may assume that

$$||v_i, F|| = 3 \text{ for } 1 \leq i \leq r-1, \text{ and } ||v_r, F|| \geq 2. \tag{4.42}$$

CASE 2: $F = K_4^-$. We may assume that x_1x_3 is the missing edge. By (4.42), for $1 \le i \le r-1$, vertex v_i is adjacent to x_1 and x_3. Thus $G[F - x_1 + v_1] = K_4^-$ and $G[\{x_1, v_2, v_3, v_4\}] = K_4^-$, thus giving k chorded cycles, a contradiction.

CASE 3: $F = C_5^+$. By Lemma 4.2 and (4.42), v_2 and v_3 are adjacent to x_1 and x_2, i.e., $G[\{v_2, v_3, x_1, x_2\}] = K_4$, contradicting (O1).

CASE 4: $F = K_{3,3}$. By Lemma 4.2 and the symmetry, we may assume that for every odd $1 \le i \le r-1$, $N(v_i) \cap F = A$ and for every even $2 \le i \le r-1$, $N(v_i) \cap F = B$. In particular, r is even, $r \ge 6$, and $N(v_r) \cap F \subseteq B$. Then $G[F - x_1 + v_2] = K_{3,3}$, and x_1 has 3 neighbors on the path $v_3v_4 \cdots v_rv_1$ thus producing another chorded cycle, a contradiction.

□

Let $W_1 := \{v_1, v_2\}$, $W_2 := \{v_{r-1}, v_r\}$ and $W := W_1 \cup W_2 = \{v_1, v_2, v_{r-1}, v_r\}$. We will use ℓ to denote the number of low vertices in W.

Recall that by Lemma 4.30, $v_1v_r \notin E(G)$. If $v_2v_{r-1} \in E(G)$ and any of the edges v_1v_{r-1}, v_2v_r is in $E(G)$, then $G[R]$ has a cycle with the chord v_2v_{r-1}. If both edges v_1v_{r-1} and v_2v_r are in $E(G)$, then $G[R]$ has a spanning Θ-subgraph, a contradiction to Lemma 4.26. Thus,

$$||W_1, W_2|| \le 1. \textit{ In particular, } \sum_{v \in W} d_G(v) \ge 2(6k-3) = 12k - 6. \tag{4.43}$$

Lemma 4.31. *Suppose $k \ge 3$. Then for all $F \in \mathcal{F}$, $||W, F|| \le 12$; in particular, $|\mathcal{F}| = k-1$.*

Proof. Suppose that $||W, F|| \ge 13$. By symmetry we may assume $||W_1, F|| \ge 7$. Then, there exists $w \in W_1$ with $||w, F|| \ge 4$. So, by Lemma 4.2, $F = K_4$ and $||w, F|| = 4$. By the pigeonhole principle, there is $x \in V(F)$ with $||x, W|| = 4$. Then $G[F - x + w] = K_4$ and $||x, R - w|| \ge 3$. So if $G[R] - w$ has a Hamiltonian path (or even just a path containing $W - w$), then x has 3 neighbors on this path and hence $G[R - v_1 + x]$ contains a chorded cycle, contradicting (O3). Otherwise, by the definition of W, $w = v_2$ and $d_{G[R]}(v_1) = 1$. In particular, $||v_1, F|| = 3$ and $||v_2, F|| = 4$. Let x' be the non-neighbor on v_1 in F. If $||x', R|| \ge 2$, then $G[F - x' + v_1] = K_4$ and $G[R - v_1 + x']$ has Hamiltonian path and has more edges than $G[R]$, contradicting (O6). Thus,$N_G(x') \cap R = \{v_2\}$ and for all $y \in W - v_2$, $N_G(y) \cap R = W - v_2$. Hence, $G[W_2 \cup (V(F) - \{x, x'\})] = K_4$ and $G[W_1 \cup \{x, x'\}] = K_4^-$, contradicting (O1). Therefore, $||W, F|| \leq 12$.

By (4.43), $\sum_{w \in W} d_G(w) \geq 12k - 6$. By the definition of W,

$$||W, \mathcal{F}|| \geq 12k - 6 - 10 = 12(k-2) + 8.$$

Since $||W, F|| \leq 12$ for all F_i, this implies $|\mathcal{F}| \geq k - 1$. So $|\mathcal{F}| = k - 1$. □

Lemma 4.32. *For all $F \in \mathcal{F}$ with $F = K_4^-$, $||W, F|| \leq 10$. Moreover, if $d_{G[R]}(v_1) = d_{G[R]}(v_r) = 2$, then $||W, F|| \leq 9$.*

Proof. Suppose that $||W, F|| \geq 11$. Then by pigeonhole and symmetry we may assume $||W_1, F|| \geq 6$. By Lemma 4.2, $||v_1, F|| = ||v_2, F|| = 3$. By the same argument, we can choose $w \in W_2$ such that $||w, F|| = 3$ and for $w' \in W_2 - w$, we have $||w', F|| \geq 2$. Again by Lemma 4.2, $\{x_1, x_3\} \subseteq N(v_1) \cap N(v_2) \cap N(w)$. So if $\{x_1, x_3\} \cap N(w') \neq \emptyset$, say $x_1 w' \in E(G)$, then $G[F - x_1 + v_1] \supseteq K_4^-$, and x_1 has 3 neighbors on the path $P - v_1$, implying $G[R - v_1 + x_1]$ contains a chorded cycle. This contradicts (O3).

Hence, $N(w') \cap F = \{x_2, x_4\}$. Then $G[F - x_1 + w'] = K_4^-$, and x_1 has 3 neighbors in $R - w'$. So if $G[R - w']$ has a Hamiltonian path, then $G[R - w' + x_1]$ contains a chorded cycle, a contradiction to (O3). Otherwise, $w = v_r$, $w' = v_{r-1}$, and $d_{G[R]}(v_r) = 1$. In this case $G[F - x_1 + v_r] = K_4^-$, $G[R - v_r + x_1]$ has a Hamiltonian path and has more edges than $G[R]$, contradicting (O6). This proves the main claim of the lemma.

Now suppose $||W, F|| \geq 10$ and $d_{G[R]}(v_1) = d_{G[R]}(v_r) = 2$. There are $j \geq 3$ and $q \leq r - 2$ such that $v_1 v_j, v_q v_r \in E(G)$. By Lemma 4.26, $3 \leq j \leq q \leq r - 2$. First we show:

$$\textit{If } x \in \{x_1, x_3\} \textit{ and } i \in \{1, 2\}, \textit{ then } ||x, W_i|| \leq 1. \tag{4.44}$$

Indeed, assume say $x_1 v_1, x_1 v_2 \in E(G)$. If $||W_2, F - x_1|| \geq 3$, then $G[W_2 \cup F - x_1]$ contains a chorded 4-cycle with exactly one vertex in W_2, and $v_2 \ldots v_j v_1 x_1 v_2$ is a cycle with chord $v_1 v_2$, contradicting (O3). Thus $||W_2, F - x_1|| \leq 2$. If $||x_1, W_2|| = 2$, then symmetrically $||W_1, F - x_1|| \leq 2$ and hence $||F, W|| \leq ||x_1, W|| + 2 \cdot 2 = 4 + 4 = 8$, contradicting the assumption $||W, F|| \geq 10$. Hence,

$$||F, W_2|| \leq 2 + 1 = 3 \text{ and so } ||F, W_1|| \geq 10 - 3 = 7.$$

But this contradicts Lemma 4.2. This proves (4.44).

Recall that by Lemma 4.2, inequality $||W, F|| \geq 10$ implies that at least two vertices in R have 3 neighbors in F, and each vertex in R with 3 neighbors in F is adjacent to both x_1 and x_3. Together with (4.44), this yields that for each of $i \in \{1, 2\}$,

(a) exactly one vertex $w_i \in W_i$ has 3 neighbors in F (hence $w_i x_1, w_i x_3 \in E(G)$), and

(b) the other vertex $w_i' \in W_i$ has exactly two neighbors in F, and these neighbors are x_2 and x_4.

We know that one of x_2 and x_4, say x_2, is adjacent to w_2. Then $G[\{x_2, x_3\} \cup W_2] = K_4^-$ and $G[\{v_1, \ldots, v_j, x_1, x_4\}]$ contains a Hamiltonian cycle that is the union of the paths $v_2 \ldots v_j v_1$ and $w_1 x_1 x_2 w_1'$ which has chord $v_1 v_2$. This contradicts (O3) and proves the lemma. □

Lemma 4.33. *For all $F \in \mathcal{F}$ with $F = C_5^+$, $||W, F|| \leq 9$.*

Proof. Suppose $||W, F|| \geq 10$. By Lemma 4.2, there exist $v, v' \in W$ with $N(v) = N(v') = \{x_1, x_2, x_4\}$. Then, $G[\{v, v', x_1, x_2\}] \supseteq K_4^-$, contradicting (O1). □

Lemma 4.34. *For all $F \in \mathcal{F}$ with $F = K_{3,3}$, $||W, F|| \leq 11$. Moreover, if $||W, F|| = 11$ and $\sum_{w \in W} d_R(w) \geq 8$, then $v_2 v_r \in E(G)$ or $v_1 v_{r-1} \in E(G)$, $||\{v_1, v_r\}, F|| = 5$, and $\sum_{w \in W} d_R(w) = 8$.*

Proof. By Lemma 4.2, for each $w \in W$, $N(w) \cap F$ is contained in one part of F. So $||W, F|| \leq 12$. Assume that $||W, F|| \geq 11$. Then by Lemma 4.2, we may assume that $N(v_1) \cap V(F) = A = \{x_1, x_3, x_5\}$ and $N(v_2) \cap V(F) = B = \{x_2, x_4, x_6\}$.

We claim that $N(v_r) \cap V(F) \subseteq A$. For otherwise, $N(v_r) \cap V(F) \subseteq B$. Then by Lemma 4.2, $N(v_{r-1}) \cap V(F) \subseteq A$. Let x_2 be a neighbor of v_r, then $G[F - x_2 + v_1] = K_{3,3}$ and $G[R - v_1 + x_2]$ contains a spanning cycle $x_2 v_2 \cdots v_r x_2$, a contradiction to (O6) or Lemma 4.30. Thus $N(v_r) \cap V(F) \subseteq A$.

Now by Lemma 4.2, $N(v_{r-1}) \cap V(F) \subseteq B$. In order for $||W, F|| \geq 11$, we need $||v, F|| \geq 2$ for $v \in \{v_{r-1}, v_r\}$. Let $x_2, x_4 \in N(v_{r-1})$ and $x_1, x_3 \in N(v_r)$. Then $G[F - x_2 - x_4 + v_1 + v_r] = K_{3,3}$ and $G[P - v_1 - v_r + x_2 + x_4]$ is a spanning Θ-graph of the remainder, a contradiction to (O7) or Lemma 4.26, unless $d_{G[R]}(v_1) = d_{G[R]}(v_r) = 2$ where $v_1v_j, v_rv_q \in E(G)$ with $3 \leq j \leq q \leq r-2$. But then, $G[F - x_1 - x_2 + v_1 + v_2] = K_{3,3}$ and $x_2x_1v_rvqv_{q+1} \cdots v_{r-1}x_2$ is a cycle with chord $v_{r-1}v_r$, producing k disjoint chorded cycles. This proves the first claim of the lemma.

Moreover, we only needed the path chord v_qv_r with $q > 2$. A very similar argument would work if our path chord were v_1v_j with $j < r-1$. Hence, assume for the secondary claim that $||W, F|| = 11$ and $\sum_{w \in W} d_R(w) \geq 8$. If $v_1v_{r-1}, v_2v_r \notin E(G)$ or if $\sum_{w \in W} d_{G[R]}(w) > 8$ then $v_2v_i \in E(G)$ for some $3 < i \leq r-1$ (or $v_jv_{r-1} \in E(G)$ for some $2 \leq j < r-2$), but $G[F - x_2 + v_1] = K_{3,3}$ and $x_2v_2 \cdots v_{r-1}x_2$ is a cycle with chord v_2v_i (or v_jv_{r-1}). So $v_2v_r \in E(G)$ or $v_1v_{r-1} \in E(G)$, and $\sum_{w \in W} d_R(w) = 8$. If $||\{v_1, v_r\}, F|| = 6$, then $G[F - x_2 - x_4 + v_1 + v_r] = K_{3,3}$ and $G[R - v_1 - v_r + x_2 + x_4]$ is a Θ-graph, contradicting either (O7) or Lemma 4.26. This proves the secondary claim. $\square$

Lemma 4.35. *For all $F \in \mathcal{F}$ with $F = K_4$, $||W, F|| \leq 11$. Moreover, if $||W, F|| = 11$, then v_2v_r or $v_1v_{r-1} \in E(G)$, $||\{v_1, v_r\}, F|| \leq 5$, and $\sum_{w \in W} d_G(w) = 8$.*

Proof. Suppose that $||W, F|| \geq 12$. We need some extra notation. For each $x_i \in V(F)$, let $a_{i,j} := ||x_i, W_j||$. Hence, $\sum_{i,j} a_{i,j} = ||W, F|| = 12$.

CASE 1: $||W_1, F|| = 8$ and $||W_2, F|| \geq 4$. If there exist j_1, j_2 such that $a_{j_1,2} + a_{j_2,2} \geq 3$, then $G[W_2 + x_{j_1} + x_{j_2}] \supseteq K_4^-$ and $G[W_1 \cup V(F) - x_{j_1} - x_{j_2}] = K_4$, contradicting (O1). Otherwise $a_{j,2} = 1$ for all j.

If $||v_r, F|| = 4$, then $G[R] = C_r$ as otherwise, for each $x \in V(F)$, $G[V(F) - x + v_1] = K_4$ but $G[R - v_1 + x] \supseteq C_r$, contradicting (O6). By Lemma 4.30, we have $|R| \leq 4$ which contradicts Lemma 4.23.

If $||v_{r-1}, F|| = 4$, then $d_{G[R]}(v_1) = 2$ as otherwise, for each $x \in V(F)$, $G[F - x + v_1] = K_4$ but $\sum\{d_{G[R]}(v) : v \in R - v_1 + x\} \geq \sum\{d_{G[R]}(v) : v \in R\} + 1$, contradicting (O7). Moreover, if $v_1v_{r-1} \notin E(G)$, then for each $x \in V(F)$, $G[R - W_2 + x]$ contains a chorded cycle and $G[F - x + v_{r-1}] = K_4$, so we have k disjoint chorded cycles, a contradiction. Hence $v_1v_{r-1} \in$

$E(G)$. By symmetry, $||v_{r-2}, F|| = 4$ which gives $G[F - x + v_1] = K_4$ and $G[R - v_1 + x]$ containing a chorded cycle unless $r - 2 = 2$, i.e. $|R| = 4$ which contradicts Lemma 4.23.

If $||v_r, F|| \in \{1, 2, 3\}$, say $v_r x \in E(G)$, then $G[F - x + v_1] = K_4$ and $G[R - v_1 + x] = C_r$, contradicting Lemma 4.30 unless $|R| = 4$, in which case, we contradict Lemma 4.23.

CASE 2: $||W_1, F|| = 7$ and $||W_2, F|| \geq 5$. Choose j_1, j_2 such that $a_{j_1,2} + a_{j_2,2} \geq 3$. Then $G[W_2 + x_{j_1} + x_{j_2}] \supseteq K_4^-$ and $G[W_1 \cup V(F) - x_{j_1} - x_{j_2}] \supseteq K_4^-$, contradicting (O1).

CASE 3: $||W_1, F|| = 6$ and $||W_2, F|| = 6$. Then for $j \in \{1, 2\}$, $\{a_{1,j}, a_{2,j}, a_{3,j}, a_{4,j}\} \in \{\{2, 2, 2, 0\}, \{2, 2, 1, 1\}\}$. If there exists an i such that $(a_{i,1}, a_{i,2}) = (0, 0)$, then for each $i' \neq i$, $G[R - v_1 + x_{i'}]$ contains a chorded cycle due to $||x_{i'}, W - v_1|| = 3$, but also $G[F - x_{i'} + v_1] = K_4^-$, so we have k disjoint chorded cycles, a contradiction. Otherwise, each pair $(a_{i,1}, a_{i,2})$ has a nonzero entry.

Let $a_{i',1} = \min_i a_{i,1}$ and $a_{i'',2} = \min_i a_{i,2}$ and $k', k'' \in \{1, 2, 3, 4\} - i' - i''$ distinct. If $a_{i',1} = a_{i'',2} = 0$ and $i' = i''$, then $G[R - v_1 + x_{k'}]$ contains a chorded cycle due to $||x_{k'}, W - v_1|| = 3$, but also $G[F - x_{k'} + v_1] = K_4^-$, so we have k disjoint chorded cycles, a contradiction.

If $i' \neq i''$ or one of the minima is nonzero, then $G[W_1 + x_{i''} + x_{k''}] \supseteq K_4^-$ and $G[W_2 + x_{i'} + x_{k'}] \supseteq K_4^-$, contradicting (O1). This proves the main claim. We now prove the secondary statement in a series of claims.

Claim 4.36. *For $F = K_4$, , if $d_{G[R]}(v_1) = d_{G[R]}(v_r) = 2$, then $||W, F|| \leq 9$.*

Proof. Suppose that $v_1 v_i, v_q v_r \in E(G)$ for some $3 \leq i \leq q \leq r - 2$. For convenience, let $R_1 = \{v_1, \ldots, v_i\}$ and $R_2 = \{v_q, \ldots, v_r\}$ where $R \supseteq R_1 \cup R_2$. Suppose that $||W, F|| \geq 10$. By symmetry we may assume $||W_1, F|| \geq 5$ and so $||W_2, F|| \geq 2$. Hence, there exists $x \in V(F)$ such that $xv_1, xv_2 \in E(G)$. Then, if $||F - x, W_2|| \geq 2$, $G[F - x + \{v_r, \ldots, v_q\}]$ contains a chorded cycle and $xv_2 \cdots v_i v_1 x$ is another cycle with chord $v_1 v_2$ unless $i = q$. We postpone handling this exception to first address the case $||F - x, W_2|| \leq 1$. Therefore, $||x, W_2|| \geq 1$ and $||W_1, F|| \geq 7$. Specifically, there exists $x' \in V(F) - x$ such that $||\{x, x'\}, W_2|| \geq 2$. If these neighbors in W_2 are distinct, then $G[R_2 + x + x']$ contains a cycle with chord $v_{r-1} v_r$

and $||W_1, F - x - x'|| \geq 7 - 4 = 3$ so that $G[W_1 \cup V(F) - x - x'] \supseteq K_4^-$. If instead these neighbors in W_2 are not distinct, we can choose $y \in V(F) - x - x'$ such that $||y, W_1|| = 2$ and so $v_1 y v_2 \cdots v_i v_1$ is a cycle with chord $v_1 v_2$ and $G[F - y \cup W_2] \supseteq K_4^-$. Thus, we have k disjoint chorded cycles.

We now handle this exception where $i = q$. Recall, $||x, W_1|| = 2$. So, if $||F - x, W_2|| \geq 3$, $G[R_1 + x]$ contains a chorded cycle and $G[W_2 + F - x]$ contains a chorded cycle. Hence, $||F - x, W_2|| \leq 2$. If $||x, W_2|| = 2$, then $G[R_2 + x]$ contains a chorded cycle and $||W_1, F - x|| \geq 10 - ||W_2, F|| - ||x, W_1|| \geq 10 - 4 - 2 = 4$. So $G[F - x + W_1]$ contains a chorded cycle thus producing k disjoint chorded cycles. Therefore, $||x, W_2|| \leq 1$ and so $||F, W_2|| \leq 3$. Thus, $||F, W_1|| \geq 7$. Choose $x'', x''' \in V(F)$ such that $||\{x'', x'''\}, W_2|| \geq 2$. Then $G[R_2 + x'' + x''']$ contains a chorded cycle and $G[W_1 + F - \{x'', x'''\}]$ contains a chorded 4-cycle since $||W_1, F - x'' - x'''|| \geq 3$. This completes the case where $i = q$ and the claim. $\square$

Claim 4.37. *For $F = K_4$, if $d_{G[R]}(v_1) = 1$, $d_{G[R]}(v_r) = 2$, and $v_2 v_r \notin E(G)$, then $||W, F|| \leq 10$.*

Proof. Suppose $||W, F|| \geq 11$. First consider if $||v_1, F|| = 4$. Then, there exists $x \in V(F)$ such that $||x, W - v_1|| \geq 2$. If $\{v_2, v_r\} \supseteq N(x)$ or $\{v_{r-1}, v_r\} \supseteq N(x)$, then $G[R - v_1 + x]$ contains a chorded cycle with $G[F - x + v_1] = K_4$, producing k chorded cycles. If instead $\{v_2, v_{r-1}\} \supseteq N(x)$, $G[F - x + v_1] = K_4$, but $G[R - v_1 + x]$ contains a Hamiltonian path $x v_2 \cdots v_r$ with $|E(G[R - v_1 + x])| > |E(G[R])|$, contradicting (O7).

If $||v_1, F|| = 3$, say $x \in V(F) - N(v_1)$. Just as in the $||v_1, F|| = 4$ case, we have that $||x, W|| \leq 1$. Hence, $||W - v_1, F - x|| \geq 11 - 3 - 1 = 7$. Specifically, there exists $x' \in V(F) - x$ such that $||x', W - v_1|| = 3$. Therefore, $G[F - x' + v_1] = K_4^-$ and $G[R - v_1 + x']$ contains a chorded cycle since $||x', W - v_1|| = 3$. Again, this yields k chorded cycles.

Now, suppose $||v_1, F|| = 2$ where $N(v_1) \cap V(F) = \{x, x'\}$ and $\{y, y'\} = N(v_1) \cap V(F) - \{x, x'\}$. If $||W_2, F|| \leq 3$, then $||W_1, F|| = 8$ and $||W_2, F|| = 3$. Now, if $||\{y, y'\}, W_1|| \geq 1$ (say $||y, W_1|| \geq 1$), then $G[\{W_1 + x + y\}] \supseteq K_4^-$ and $G[R_2 + x']$ contains a chorded cycle, producing k chorded cycles. Hence, $||\{y, y'\}, W_1|| = 0$. If $||y, W_2|| = 2$ or $y v_{r-1}, y' v_r \in E(G)$ then $G[R_2 + y + y']$ contains a chorded cycle and $G[W_1 + x + x'] = K_4$, producing k disjoint chorded cycles. Thus, we are left with the subcase where there exists $v \in W_2$ such that

$vy, vy' \in E(G)$. Then, $G[F - x + v] \supseteq K_4^-$ and $||x, W - v|| = 3$ so that $G[R - v + x]$ contains a chorded cycle since there exists a path in $G[R]$ covering $W - v$. We are now left with the subcase where $||\{y, y'\}, W|| \geq 4$. In fact, the previous argument still works unless $||y, W - v_1||, ||y', W - v_1|| = 2$. Moreover, $N(y) \cap W = N(y') \cap W = \{v_2, v_{r-1}\}$. But then $||W, \{x, x'\}|| \geq 7$, so we may assume $||x, W|| = 4$. Then, $||W_1, F - x|| \geq 3$ and so $G[W_1 + c - x]$ contains a chorded cycle with $xv_rv_q \cdots v_{r-1}x$ is a cycle with chord $v_{r-1}v_r$. This again yields k disjoint chorded cycles.

Lastly, if $||v_1, F|| \leq 1$, then $||W_2, F|| \geq 11 - 5 = 6$ and so there exist at least 2 vertices $y, y' \in V(F)$ such that $||y, W_2|| = ||y', W_2|| = 2$. Moreover, we can choose y so that $||y, W_2|| = 2$ and $||W_1, F - y|| \geq 3$. Thus, $yv_{r-1} \cdots v_qv_ry$ is a cycle with chord $v_{r-1}v_r$ and $G[F - y + W_1]$ contains a chorded cycle, producing k disjoint chorded cycles. This completes the claim. □

Claim 4.38. *For $F = K_4$, if $d_{G[R]}(v_1) = d_{G[R]}(v_r) = 1$, then $||W, F|| \leq 10$.*

Proof. Suppose that $||W, F|| = 11$ since we have that $||W, F|| \leq 11$ by the main statement of the lemma and Lemmas 4.32-4.34. If $\sum_W d_{G[R]}(w) \leq 7$ and ℓ is the number of low vertices in W, then $||W, \mathcal{F}|| \geq 4(3k - 1) - \ell - \sum_W d_{G[R]}(w) \geq 12(k - 1) - 1$, a contradiction since $||W, F|| \leq 11(k - 1) < 12(k - 1) - 1$ for $k \geq 3$. So $\sum_W d_{G[R]}(w) \geq 8$.

CASE 1: There exists $x \in V(F)$ such that $||x, W|| = 4$. If $v \in \{v_1, v_r\}$ has $||v, F|| \geq 3$, then $G[F - x + v] \supseteq K_4^-$ and $G[R - v + x]$ contains a chorded cycle since $||x, R - v|| \geq 3$ and $G[R - v]$ contains a Hamiltonian path. Thus, we have k chorded cycles, a contradiction, and so $||v, F|| \leq 2$ for all $v \in \{v_1, v_r\}$. Since $||W, F|| = 11$, by symmetry we may assume $||W_1, F|| \geq 6$, and so there exist distinct $y, y' \in V(F) \cap N(v_1) \cap N(v_2)$. Since $||v_1, F|| \leq 2$, then $||v_1, F|| = 2$ and $||v_2, F|| = 4$. Hence, $||W_2, F|| = 5$ and so there exists $x \in V(F) \cap N(v_{r-1}) \cap N(v_r)$. Say $x \neq y$ and choose $x' \in V(F) - x - y$ with $||x', W_2|| \geq 1$. This choice is possible due to $||W_2, F|| = 5$ implying that $||v_{r-1}, F|| \geq 3$. From this, $G[W_2 + x + x'] \supseteq K_4^-$ and $G[W_1 \cup V(F) - x - x'] \supseteq K_4^-$ thus giving us k chorded cycles, a contradiction. This completes the case.

CASE 2: For all $x \in V(F)$, $||x, W|| \leq 3$. Since $\sum_W d_{G[R]}(w) \geq 8$, we have at least one chord on our path. First, consider the case that this chord e uses vertex v_2, v_{r-1}. By the case, this chord is not v_2v_r nor $v_{r-1}v_1$. In fact, if the chord is not v_2v_{r-1}, then we have two chords: v_2v_i and v_jv_{r-1} for some i, j. Due to this symmetry, we may assume $||W_1, F|| \geq 6$. Again, we will use the notation that for each $x_i \in F$, $a_{i,j} := ||x_i, W_j||$. Hence, $\sum_{i,j} a_{i,j} = ||W, F|| = 11$. If three vertices in $V(F)$ have two neighbors on one side, then we have a contradiction. Namely, if $a_{1,j} = a_{2,j} = a_{3,j} = 2$ and $i' := \arg\max_{1 \leq i \leq 3}\{a_{i,3-j}\}$, then $a_{i',3-j} \geq 1$. So for $j = 1$, $G[F - x_{i'} + v_1] \supseteq K_4^-$ and $G[R - v_1 + x_{i'}]$ contains a cycle with chord e, giving us k chorded cycles. For $j = 2$, just replace v_1 with v_r. Since at most two vertices in $V(F)$ have two neighbors on one side, assume $a_{1,1} = a_{2,1} = 2$, $a_{3,1} = 1$, and $a_{4,1} \leq 1$. By the case, $a_{1,2}, a_{2,2} \leq 1$ and so $a_{3,2} + a_{4,2} = ||\{x_3, x_4\}, W_2|| \geq 3$. Hence, $G[W_2 + x_3 + x_4] \supseteq K_4^-$ and $G[W_1 + v_1 + v_2] = K_4$, and so we have k chorded cycles, a contradiction. This completes the case and thus the claim. □

Claim 4.39. *Suppose $r \geq 5$, $G[R]$ contains a Hamiltonian path $v_1v_2 \ldots v_r$, and v_2v_r or $v_1v_{r-1} \in E(G)$. Let $F \in \mathcal{F}$ and $F = K_4$. If*

$$||\{v_1, v_r\}, F|| \geq 6, \tag{4.45}$$

then $||W, F|| \leq 10$.

Proof. Suppose (4.45) holds but $||W, F|| \geq 11$. By (4.45), $||v_1, F|| \geq 2$. Thus we have three cases.

CASE 1: $||v_1, F|| = 4$. Since $||W, F|| \geq 11$, there is $x \in F$ with $||x, W|| \geq 3$. Then $G[F - x + v_1] = K_4$, $G[R - v_1 + x]$ has a Hamiltonian path starting from x, and x has at least 2 neighbors in $G[R - v_1 + x]$, contradicting (O7).

CASE 2: $||v_1, F|| = 3$. Then by (4.45), $||v_r, F|| \geq 3$. If there is $x \in V(F) \cap N(v_r)$ adjacent to some $v \in \{v_2, v_{r-1}\}$, then $G[R - v_1 + x]$ contains a cycle passing through edges xv and xv_r with chord v_rv, and $G[V(F) - x + v_1] \supseteq K_4^-$, producing k disjoint chorded cycles. Otherwise, $||\{v_2, v_{r-1}, v_r\}, F|| \leq 3 \cdot 1 + 1 \cdot 2 = 5$, and hence $||W, F|| \leq 5 + 3 = 8$, a contradiction.

CASE 3: $||v_1, F|| = 2$. Then by (4.45), $||v_r, F|| = 4$. Suppose $N(v_1) \cap F = \{x_1, x_2\}$. If at least one $x \in \{x_3, x_4\}$ is adjacent to some $v \in \{v_2, v_{r-1}\}$, then again $G[R - v_1 + x]$ contains a cycle passing through edges xv and xv_r with chord $v_r v$, and $G[F - x + v_1] \supseteq K_4^-$, a contradiction since then we have k disjoint chorded cycles. Otherwise, $||\{v_2, v_{r-1}\}, F|| \leq 2 \cdot 2 = 4$, and hence $||W, F|| \leq 4 + 6 = 10$, as claimed. □

By these claims, we can only have $||W, F|| = 11$ if $v_2 v_r$ or $v_1 v_{r-1} \in E(G)$ but the condition of Claim 4.39 is not met, i.e. $||\{v_1, v_r\}, F|| \leq 5$, as claimed. Or, if $v_1 v_i \in E(G)$ or $v_j v_r \in E(G)$ for some $3 \leq i, j \leq r-2$, either $G[R]$ contains a chorded cycle or is a spanning Θ-graph, contradicting Lemma 4.26. Therefore, $d_{G[R]}(v_1) = 1$ and $d_{G[R]}(v_{r-1}) = d_{G[R]}(v_r) = 2$, and $d_{G[R]}(v_2) = 3$ so that $\sum_{w \in W} d_G(w) = 8$. This proves Lemma 4.35. □

We now finish the proof of the main result of this section. Recall, $P = v_1 \ldots v_r$ is a Hamiltonian path in $G[R]$, and $W = \{v_1, v_2, v_{r-1}, v_r\}$ has ℓ low vertices. Then,

$$||W, \mathcal{F}|| \geq 4(3k-1) - \ell - \sum_{w \in W} d_{G[R]}(w).$$

On the other hand, by Lemmas 4.32-4.35, for each $F \in \mathcal{F}$, $||W, F|| \leq 11$ and $|\mathcal{F}| = k - 1$. So $||W, \mathcal{F}|| \leq 11(k-1)$. It follows that $4(3k-1) - \ell - \sum_{w \in W} d_R(w) \leq 11(k-1)$, and we obtain

$$\sum_{w \in W} d_R(w) \geq k + 7 - \ell \geq 3 + 7 - 2 = 8. \tag{4.46}$$

If $\sum_{w \in W} d_R(w) > 8$, then by Lemmas 4.34 and 4.35 , $||W, F|| \leq 10$ for each $F \in \mathcal{F}$. Then

$$4(3k-1) - \ell - \sum_{w \in W} d_R(w) \leq ||W, \mathcal{F}|| \leq 10(k-1),$$

and we obtain $\sum_{w \in W} d_{G[R]}(w) \geq 2k + 6 - \ell \geq 2 \cdot 3 + 6 - 2 = 10$. As $d_{G[R]}(v_1), d_{G[R]}(v_r) \leq 2$ and $d_{G[R]}(v_2), d_{G[R]}(v_{r-1}) \leq 3$, we have $\ell = 2$, and $d_{G[R]}(v_1) = d_{G[R]}(v_r) = 2$ and $d_{G[R]}(v_2) = d_{G[R]}(v_{r-1}) = 3$, and $||W, F|| = 10$ for each F. By Lemma 4.32 and Claim 4.36, $F \notin \{K_4^-, K_4\}$, a contradiction to Lemma 4.13.

So $\sum_{w\in W} d_R(w) = 8$. It follows that $k = 3$ and $\ell = 2$, and $||W, F|| = 11$ for each $F \in \mathcal{F}$. By Lemmas 4.32-4.35, we may assume $\mathcal{F} = \{K_4, K_4\}$, $v_2v_r \in E(G)$, and $||\{v_1, v_r\}, F|| \leq 5$ and $||W, F| = 11$. Since $v_2v_r \in E(G)$, we have $d_{G[R]}(v_1) = 1$ by Lemma 4.26. So,

$$||\{v_1, v_r\}, \mathcal{F}|| \geq d_G(v_1) + d_G(v_r) - (d_{G[R]}(v_1) + d_{G[R]}(v_r)) \geq 6k - 3 - 3 = 12.$$

This contradicts $||\{v_1, v_r\}, F|| \leq 5$ for both F. This proves the case where $G[R]$ has a Hamiltonian path and $k \geq 3$.

4.5 Case: $G[R]$ has a Hamiltonian path and $k = 2$

Lastly, we handle the case where $k = 2$ and $G[R]$ has a Hamiltonian path. The sequence of the cases is similar to the previous section, but the proofs are different. Some things are harder because we now have only one chorded cycle in $\mathcal{F}$ with which to work, and some other things are easier because $G[R \cup C]$ is the entirety of G.

Lemma 4.40. *For $k = 2$, $G[R]$ is not a Hamiltonian cycle.*

Proof. Suppose that $G[R]$ is a cycle $v_1v_2 \dots v_rv_1$. By (4.1), and since the set of low vertices is a clique, at least $r - 2$ vertices in R have degree at least 5 but only degree 2 in $G[R]$. By Lemma 4.6 and since low vertices form a clique, we may assume

$$||R, F|| \geq (r-2)(3k-1) + 2(3k-2) - 2r = 3r - 2 \quad \text{and} \quad d_G(v_1), \dots, d_G(v_{r-2}) \geq 5. \tag{4.47}$$

By Lemma 4.2, we have four cases.

CASE 1: $F = K_4$. By (4.47), $3 \leq ||v_1, F|| \leq 4$. So

$$G[F - x_j + v_1] \quad \text{contains a } K_4^- \text{ for each } 1 \leq j \leq 4. \tag{4.48}$$

If $r \geq 5$, then by (4.47), some x_j has at least

$$\left\lceil \frac{3r - 2 - ||v_1, F||}{r - 1} \right\rceil \geq \left\lceil \frac{3r - 6}{r - 1} \right\rceil = 3 - \left\lfloor \frac{3}{r - 1} \right\rfloor = 3$$

neighbors in the path $G[R-v_1]$. But then $G[R-v_1+x_j]$ contains a chorded cycle. Together with (4.48), we have two disjoint chorded cycles, a contradiction.

If instead $r=4$, then by (4.47), we may assume that $||x_1, R|| \geq \lceil \frac{10}{4} \rceil = 3$. If $N(x_1) \supset R-v_1$, then $G[R-v_1+x_1] = K_4^-$ and by (4.47), v_1 has at least two neighbors in $V(F)-x_1$. So $G[F-x_1+v_1]$ also is a chorded cycle, a contradiction. Thus x_1 has exactly one nonneighbor in $R-v_1$, say v_i, $i \neq 1$.

If v_i has at least two neighbors in F (and hence in $F-x_1$), then again G has disjoint chorded cycles: $G[R-v_i+x_1]$ and $G[F-x_1+v_i]$, a contradiction. So v_i has at most one neighbor in F. Since $\delta(G) \geq 3$ by Lemma 4.10, we may assume that $N(v_i) \cap V(F) = \{x_4\}$. Then for each $z \in \{x_1, x_2, x_3, v_{i-2}\}$, $d_G(z) \geq 9 - d_G(v_i) = 6$, where we interpret v_{i-2} with modularity e.g. $v_{-1} = v_{r-1}$. So each such z is not adjacent only to v_i, and so $G \supseteq G_3(2)$. If $v_{i-1}x_4$ or $v_{i+1}x_4 \in E(G)$, say $v_{i+1}x_4 \in E(G)$, then we have two disjoint chorded cycles $G[\{v_i, v_{i+1}, x_3, x_4\}] = K_4^-$ and $G[R-v_i-v_{i+1}+x_1+x_2] = K_4$. Hence, $G = G_3(2)$. This proves the case.

CASE 2: $F = K_4^-$. By Lemma 4.2 and (4.47), for each $1 \leq i \leq r-2$ there exists $j_i \in \{2,4\}$ such that $N(v_i) \cap V(F) = \{x_{j_i}, x_1, x_3\}$. Note, $j_i \neq j_{i+1}$ as otherwise $G[v_i, v_{i+1}, x_1, x_{j_i}\}] = K_4$, contradicting (O2). By (4.1),

$$d(v_{r-1}) \geq 4 \text{ and } d(v_r) \geq 4. \tag{4.49}$$

We claim that

$$||\{v_{r-1}, v_r\}, \{x_1, x_3\}|| = 0. \tag{4.50}$$

Indeed, suppose for example $v_r x_3 \in E(G)$. If $r \geq 5$, then $G[F-x_3+v_1] = K_4^-$, and $G[R-v_1+x_3]$ contains a chorded cycle since $||R-v_1, x_3|| \geq 3$, producing two disjoint chorded cycles. So, let $r=4$, and then $G[\{v_4, v_1, x_{j_1}, x_3\}] \supseteq K_4^-$. Moreover, if v_3 has a neighbor in $\{x_{j_2}, x_1\}$, then also $G[\{v_3, v_2, x_{j_2}, x_1\}] \supseteq K_4^-$, producing two disjoint chorded cycles. So we may assume v_3 has no neighbor in $\{x_{j_2}, x_1\}$, and hence by (4.49), $x_3v_3 \in E(G)$. In this case, $G[\{v_1, v_4, v_3, x_3\}] = K_4^-$ and $G[\{v_2, x_1, x_2, x_4\}] = K_4^-$, a contradiction. This proves (4.50).

By (4.50), $N(v_r) \cap V(F) = \{x_2, x_4\} = N(v_{r-1}) \cap V(F)$. So $G[\{v_r, v_{r-1}, x_2, x_4\}] = K_4$, contradicting (O2).

CASE 3: $F = C_5^+$. By (4.47) and Lemma 4.2, $N(v_1) \cap V(F) = N(v_2) \cap V(F) = \{x_1, x_2, x_4\}$. Thus, $G[\{v_1, v_2, x_1, x_2\}| = K_4$, contradicting (O1).

CASE 4: $F = K_{3,3}$ with parts $A = \{x_1, x_3, x_5\}$ and $B = \{x_2, x_4, x_6\}$. By Lemma 4.3,

$$N(v_i) \cap F \subseteq A \text{ or } N(v_i) \cap F \subseteq B \text{ for every } 1 \leq i \leq r. \tag{4.51}$$

Together with (4.47), this implies $d_G(v_1) = d_G(v_2) = 5$ and so $d_G(v_{r-1}), d_G(v_r) \geq 4$. By symmetry, we may assume $N(v_1) \cap V(F) = A$ and then this forces $N(v_2) \cap V(F) = B$. By Lemma 4.3, $N(v_r) \cap V(F) \subseteq B$, say $N(v_r) \cap V(F) \supseteq \{x_2, x_4\}$. Similarly $v_2v_3 \in E(G)$ forces $N(v_3) \cap F \subseteq A$, and so on such that r is even. If $r = 4$, then G is a bipartite graph with parts $A \cup \{v_2, v_4\}$ and $B \cup \{v_1, v_3\}$, i.e. it is a subgraph of $K_{5,5}$. But after deleting any edge yz from a $K_{5,5}$, (4.1) fails for y and z, i.e. $d_G(y) + d_G(z) \leq 4 + 4 < 9$. Hence $G = K_{5,5} = G_1(10, 2)$, contradicting the assumption. So $r \geq 6$. Then $N(v_4) \cap V(F) = B$ and $N(v_5) \cap V(F) \subseteq A$. So $G[A \cup \{v_1, v_3, v_5\}]$ and $G[B \cup \{v_2, v_4, v_r\}]$ induce two disjoint chorded 6-cycles, a contradiction. $\square$

Lemma 4.41. *For $k = 2$, $G[R] \neq K_{2,3}$.*

Proof. Suppose $G[R]$ is a $K_{2,3}$ with parts $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, v_5\}$. By (4.1), and since the set of low vertices is a clique, similarly to (4.47), we may assume

$$||V_1, F|| \geq 3, \; ||V_2, F|| \geq 8. \tag{4.52}$$

By (4.52), $||R, F|| \geq 11$. By (4.52) and Lemma 4.2, we have four cases.

CASE 1: $F = K_4$. First, we show that

$$\text{for each } x \in V(F), |E(G[N(x) \cap R])| = 0. \tag{4.53}$$

Suppose that the neighborhood in R does have an edge, say $N(x) \supseteq \{v_1, v_5\}$. Then, at least one of $\{v_3, v_4\}$ is high, say v_3 is high so that $||v_3, F - x|| \geq 2$. Therefore, $G[R - v_3 + x] = C_5^+$ and $G[F - x + v_3] \supseteq K_4^-$, producing two disjoint chorded cycles. This proves (4.53).

Hence, for all $x \in V(F)$, $N(x) \cap R \subseteq V_1$ or $N(x) \cap R \subseteq V_2$. Since $||R, F|| \geq 11$, there is at most one vertex $x' \in V(F)$ with $N(x') \cap R \subseteq V_1$ and so $d_G(v_0) + d_G(v_0') \leq 8$, a contradiction to (4.1).

CASE 2: $F = K_4^-$. Similarly to the previous case, we show that

$$\text{for each } x \in \{x_1, x_3\}, |E(G[N(x) \cap R])| = 0. \tag{4.54}$$

Suppose that the neighborhood in R does have an edge, say $N(x) \supseteq \{v_1, v_5\}$. Then, at least one of $\{v_3, v_4\}$ is high, say v_3 is high so that $||v_3, F - x|| \geq 2$. Therefore, $G[R - v_3 + x] = C_5^+$ and $G[F - x + v_3] \supseteq K_4^-$, producing two disjoint chorded cycles. This proves (4.54).

So by (4.52), we may assume that the vertices in $H := \{v_1, v_3, v_4\}$ are high, and so we can say that for every $i \in \{3, 4\}$, there exists $j_i \in \{2, 4\}$ such that $N(v_i) \cap V(F) = \{x_1, x_3, x_{j_i}\}$. Also, by (4.54), $N(v_1) = \{x_2, x_4\}$ and $N(v_2) \subseteq \{x_2, x_4\}$ with $||v_2, \{x_2, x_4\}|| \geq 1$ by (4.1). Choose $j_2 \in \{2, 4\}$ such that $v_2 x_{j_2} \in E(G)$.

If $j_3 \neq j_4$, and say $j_2 = j_3$, then $v_2 x_{j_3} x_3 v_3 v_2$ and $v_1 x_{j_4} x_1 v_4 v_1$ are cycles with chords $x_{j_3} v_3$ and $x_{j_4} v_4$, respectively. Hence $j_3 = j_4$.

Suppose $v_5 x_{6-j_3} \in E(G)$. Then, $G[R - v_3 + x_{6-j_3}] \supseteq C_5^+$ and $G[F - x_{6-j_3} + v_3] = K_4^-$, producing two disjoint chorded cycles. Hence, $v_5 x_{6-j_3} \notin E(G)$.

Now that the graph is fairly defined, we may assume $j_3 = 2$. Then $v_3 x_2, v_4 x_2 \in E(G)$ and $x_{6-j_3} = x_4$. Moreover, $A := \{v_3, v_4, v_5, x_4\}$ and $B := \{v_1, v_2, x_1, x_3\}$ are independent sets. Then, $G[A \cup B] \subseteq K_{4,4}$ so that if for some $a \in A$ and $b \in B$, $ab \notin E(G)$, then $d_G(a) + d_G(b) \leq 8$ contradicting (4.1). Thus $G - x_2 = K_{4,4}$. If $v_2 x_4 \notin E(G)$, then $d_G(v_2) + d_G(x_4) \leq 8$, contradicting (4.1). Thus, we have $G \subseteq G_2(2)$ and $G \supseteq G_2(2) - x_2 v_2 - x_2 v_5$. In other words, $G_2^{**} \subseteq G \subseteq G_2$, and so G is an exceptional graph, a contradiction.

CASE 3: $F = C_5^+$. By (4.52) and again Lemma 4.2, there are two distinct vertices $v, v' \in$

$\{v_3, v_4, v_5\}$ which are high and so are adjacent to each of x_1, x_2. Thus $G[\{v, v', x_1, x_2\}] = K_4^-$, a contradiction to (O1).

CASE 4: $F = K_{3,3}$. By (O3) ,

$$N(v) \cap F \subseteq A \text{ or } N(v) \cap V(F) \subseteq B \text{ for every } v \in R. \tag{4.55}$$

Together with (4.52), we can assume that $\{v_1, v_3, v_4\}$ are high and so $d_G(v_1) = d_G(v_3) = d_G(v_4) = 5$ with $d_G(v_5), d_G(v_2) \geq 4$. By symmetry, we may assume $N(v_3) \cap V(F) = A$. Then by Lemma 4.3, $N(v_1) \cap V(F) \subseteq B$ and $N(v_2) \cap V(F) \subseteq B$. Since by (4.52), $d_G(v_1) \geq 5$, we may assume $N(v_1) \cap V(F) \supseteq \{x_2, x_4\}$. So again by Lemma 4.3, $N(v_4) \cap V(F) = A$. If $N(v_5) \cap V(F) \subseteq B$, then v_5 has at least two neighbors in B and hence $N(v_5) \cap B \cap N(v_1) \neq \emptyset$, say $x_2 \in N(v_5) \cap B \cap N(v_1)$. But then G has a 5-cycle $v_1v_5x_2x_1x_4v_1$ with chord v_1x_2, a contradiction to (O3). Therefore, $N(v_5) \cap V(F) \subseteq A$, and G is a bipartite graph with parts $A' := A \cup \{v_1, v_2\}$ and $B' := B \cup \{v_3, v_4, v_5\}$. Furthermore, if there exist $y, y' \in A'$ and $z, z' \in B'$ such that $yz \neq y'z'$ and $yz, y'z' \notin E(G)$, then (4.1) is violated. This occurs when $z \neq z'$ since $d_G(z) + d_G(z') \leq 8$, and otherwise $z = z'$ so if we let $z'' \in B - z$, then $d_G(z) + d_G(z'') \leq 3 + 5 = 8$. Thus, at most one edge can be missing so that $G_1^-(11, 2) \subseteq G \subseteq G_1(11, 2)$, a contradiction. □

Lemma 4.42. *For $k = 2$, $G[R]$ is not a Θ-graph.*

Proof. Suppose $G[R]$ is a Θ_{j_1,j_2,j_3}-graph formed by three internally disjoint paths $P_i = v_0v_{i,1}v_{i,2}\dots v_{i,j_i}v_0'$ for $i \in \{1, 2, 3\}$. Since $G[R]$ contains no chorded cycles, $j_i \geq 1$ for all i. If $j_1 = j_2 = j_3 = 1$, then $G[R] = K_{2,3}$, a contradiction to Lemma 4.41. So we may assume

$$j_3 = \max\{j_1, j_2, j_3\} \geq 2. \tag{4.56}$$

Let $R' := \{v_0, v_0', v_{1,1}, v_{2,1}, v_{3,1}, v_{3,2}\}$. Since $G[R]$ has no triangles, R contains at most 2 low vertices and so

$$||R', F|| \geq 2 \cdot 9 + 2 \cdot 5 - 14 = 14. \tag{4.57}$$

Moreover, we can assume by symmetry that the low vertices are contained in $\{v_{2,1}, v_0, v_{3,1}, v_{3,2}\}$ so that we can state

$$d_G(v_0'), d_G(v_{1,1}) \geq 5. \tag{4.58}$$

By (4.58) and Lemma 4.2, we have four cases:

CASE 1: $F = K_4$. By (4.57), there is $x_j \in V(F)$ with $||R', x_j|| \geq \lceil \frac{14}{4} \rceil = 4$. By (4.58), $G[F - x_j + v_{1,1}] \supseteq K_4^-$. By the choice, x_j has at least 3 neighbors in the path $P' := v_{1,2}v_{1,3}\ldots v_0'v_{2,j_2}\ldots v_0v_{3,1}v_{3,2}\ldots v_{3,j_3}$. Lemma 4.5 yields that $G[R - v_{1,1} + x_j]$ also contains a chorded cycle. So, we have two disjoint chorded cycles, a contradiction.

CASE 2: $F = K_4^-$. Since $v_{1,1}$ is high, we may assume $N(v_{1,1}) \cap V(F) = \{x_1, x_2, x_3\}$. If some $x_j \in V(F) - x_2$ has $||x_j, R - v_{1,1}|| \geq 3$, then as in CASE 1, $G[F - x_j + v_{1,1}] \supseteq K_4^-$ and $||x_j, P'|| \geq 3$ so that $G[R - v_{1,1} + x_j]$ contains a chorded cycle by Lemma 4.5. This is a contradiction since then we have two disjoint chorded cycles. So $||F - x_2, R|| = ||F - x_2, R - v_{1,1}|| + 2 \leq 6 + 2 = 8$. Since $|R'| = 6$, this together with (4.57) yields $||x_2, R'|| = 6$ and $||F - x_2, R'|| = ||F - x_2, R|| = 8$. Hence, with $N(v_{1,1}) \cap V(F) = \{x_1, x_2, x_3\}$, we have

$$N(v) \cap V(F) = \{x_1, x_2, x_3\} \text{ for all } v \in R \text{ with } ||v, F|| \geq 3. \tag{4.59}$$

If $d_G(v_{2,1}) \leq 4$ then by (4.1), $d_G(v_{3,1}) \geq 5$ and $d_G(v_{3,2}) \geq 5$. In this case, by (4.59), $G[\{v_{3,1}, v_{3,2}, x_1, x_2\}] = K_4$, contradicting (O2). So $d_G(v_{2,1}) \geq 5$ and by (4.59), $V(F) - N(v_{1,1}) = V(F) - N(v_{2,1}) = \{x_4\}$. Hence, G has cycle $x_1x_4x_3v_{1,1}v_0v_{2,1}x_1$ with chord $x_1v_{1,1}$ and, because $N(x_2) \supseteq R'$, cycle $x_2v_{3,1}v_{3,2}\ldots v_0'x_2$ with chord $x_2v_{3,2}$. So, we have two disjoint chorded cycles, a contradiction.

CASE 3: $F = C_5^+$. By Lemma 4.2, each $v \in R$ with $||v, F|| \geq 3$ is adjacent to $\{x_1, x_2, x_4\}$. By (4.1), at least two vertices $v, v' \in \{v_{1,1}, v_{2,1}, v_{3,1}\}$ have this property. Thus, $G[\{v, v', x_4, x_5\}] = K_4^-$, a contradiction to (O1).

CASE 4: $F = K_{3,3}$. Each $v \in R' - v_0 - v_0'$ has a nonneighbor in $\{v_{1,1}, v_{3,2}\}$ and $||v_{1,1}, F||, ||v_{3,2}, F|| \leq 3$. Hence, $d_G(v) \geq 4$ for each $v \in R - v_0 - v_0'$. In particular,

$||v_{3,1}, F|| \geq 2$ and $||v_{3,2}, F|| \geq 2$. By (O3) , $N(v_{3,1}) \cap N(v_{3,2}) \cap V(F) = \emptyset$. By symmetry, we may assume that $N(v_{3,1}) \cap V(F) \subseteq A$ and $N(v_{3,2}) \cap V(F) \subseteq B$. By (4.57), there is $x_j \in V(F)$ with $||R', x_j|| \geq \lceil \frac{14}{6} \rceil = 3$. If $x_j \in A$, then $G[F - x_j + v_{3,2}] \supseteq K_{3,3}^-$ (and so contains a chorded 6-cycle), and x_j has 3 neighbors on the path $v_{3,1}v_0v_{2,1} \ldots v_0'v_{1,j_1} \ldots v_{1,1}$. Together with Lemma 4.5, we have two disjoint chorded cycles, a contradiction. Similarly, if $x_j \in B$, then $G[F - x_j + v_{3,1}] \supseteq K_{3,3}^-$, and x_j has 3 neighbors on the path $v_{3,2}v_{3,3} \ldots v_0'v_{2,j_2} \ldots v_{2,1}v_0v_{1,1}$, a contradiction again. □

Lemma 4.43. *If $k = 2$, then $v_1v_3, v_rv_{r-2} \notin E(G)$.*

Proof. Suppose that $v_1v_3 \in E(G)$. Let $W_1 = \{v_1, v_2\}$, $W_2 = \{v_{r-1}, v_r\}$, and $R_1 = \{v_1, v_2, v_3\}$. If $v_rv_j \in E(G)$ for $j < r - 1$, then let $R_2 = \{v_r, v_{r-1}, \ldots, v_j\}$. Since $v_1v_r \notin E(G)$, $d_G(v_1) + d_G(v_r) \geq 9$ and so there exists $v \in \{v_1, v_r\}$ with $||v, F|| \geq 3$. We then have the four cases of Lemma 4.2.

CASE 1: $F = K_4$. We first claim

$$||v_r, F|| \leq 3. \tag{4.60}$$

Otherwise, $||v_r, F|| = 4$. If there exists $x \in N(v_1) \cap N(v_2) \cap V(F)$, then $G[R_1 + x]$ is a chorded cycle and $G[F - x + v_r] = K_4$, producing two disjoint chorded cycles. Hence, $N(v_1) \cap N(v_2) \cap V(F) = \emptyset$ but also $||W_1, F|| \leq 4$.

Now, consider if $d_{G[R]}(v_r) = 1$. Then, $||v, F|| = 2$ for $v \in W_1$. For each $x \in V(F)$, $||x, \{v_1, v_2, v_r\}|| \leq 2$ but also $d_{G[R]}(v_{r-1}) \leq 3$ so that either $v_iv_{r-1} \in E(G)$ for some $3 \leq i \leq r - 3$ or $xv_k \in E(G)$ for some $3 \leq k \leq r - 1$. If say $x \in N(v_2) \cap V(F)$, then $xv_2 \cdots v_rx$ is a cycle either with chord v_iv_{r-1} or xv_k. Subsequently, $x \in N(v_2) \cap V(F)$ implies $x \notin N(v_1)$ so that $G[F - x + v_1] = K_4^-$, producing two disjoint chorded cycles.

So, now consider if $d_{G[R]}(v_r) = 2$. Since $||W_1, F|| \leq 4$, there exists $v \in W_1$ such that $d_G(v) \leq 2 + 2 = 4$ and so $vv_{r-1} \notin E(G)$ implies $d_G(v_{r-1}) \geq 5$ so that $||v_{r-1}, F|| \geq 2$, say $x, x' \in N(v_{r-1}) \cap N(v_r) \cap V(F)$. If there exist $x_1, x_2 \in V(F)$ where $N(v_1) \cup N(v_2) \supseteq \{x_1, x_2\}$ such that $\{x_1, x_2\} \neq \{x, x'\}$, say $x' \notin \{x_1, x_2\}$, then $G[R_2 + x']$ and $G[W_1 + F - x']$ contain a chorded cycle. Hence, $||W_1, F|| \leq 1$ so that $d_G(v) = 2$ for some $v \in W_1$, contradicting Lemma 4.10. This proves (4.60).

We again consider when $d_{G[R]}(v_r) = 1$. By (4.60), $||v_1, F||, ||v_2, F|| \geq 9 - 4 - 2 = 3$. So, there exists $x, x' \in N(v_1) \cap N(v_2) \cap V(F)$. If possible, choose $x \in V(F)$ such that $||v_r, F - x|| \geq 2$. Then, $G[F - x + v_r] \supseteq K_4^-$ and $G[R_1 + x] = K_4^-$. Therefore, $||v_r, F - x||, ||v_r, F - x'|| \leq 1$ so that $||v_r, F|| = 2$ with $N(v_r) \cap V(F) = N(v_1) \cap N(v_2) \cap V(F)$ since $||v_r, F|| \geq 2$ by (4.1). By (4.1), $||v, F|| = 4$ for $v \in W_1$. But then, there exists $x'' \in N(v_1) \cap N(v_2) \cap V(F) - x - x'$ so that $G[R_1 + x''] = K_4^-$ and $G[F - x'' + v_r] = K_4^-$, producing two disjoint chorded cycles.

Suppose now $d_{G[R]}(v_r) = 2$. By (4.60) and the Ore condition, $d_G(v_1), d_G(v_2) \geq 9 - d_G(v_r) \geq 4$, say $v_1x_1, v_2x_2 \in E(G)$ with $x_1 \neq x_2$.

If $||W_2, F|| \geq 7$, then there exist $x \in N(v_{r-1}) \cap N(v_r) \cap V(F) - x_1 - x_2$. Hence, $G[R_2 + x]$ contains a chorded cycle and $G[W_1 + F - x] \supseteq C_5^+$, producing two disjoint chorded cycles.

If $||W_2, F|| = 6$, then there exist $x, x' \in N(v_{r-1}) \cap N(v_r)$ and $||W_1, F|| \geq 3$. If there exists $x'' \in \{x, x'\}$ such that $|(N(v_1) \cup N(v_2)) \cap (V(F) - x'')| \geq 2$, then $G[W_1 + F - x'']$ and $G[R_2 + x'']$ each contain a chorded cycle. Hence, $N(v_1) \cap V(F), N(v_2) \cap V(F) = \{x, x'\}$. Since $||W_2, F|| \geq 5$, choose $v \in W_2$ with $||v, F|| \geq 3$. Then, $G[R_1 + x] = K_4^-$ and $G[W_2 + F - x]$ contain chorded cycles since $||W_2, F - x|| \geq ||W_2, F|| - 2 \geq 4$.

Then, we handle the case when $||W_2, F|| = 5$ and $||W_1, F|| = 4$. Again, there exists $x \in N(v_{r-1}) \cap N(v_r) \cap V(F)$. If $|(N(v_1) \cup N(v_2)) \cap (V(F) - x)| \geq 2$, then $G[W_1 + F - x]$ and $G[R_2 + x]$ contain chorded cycles. So, $N(x) \supseteq W_1 \cup W_2$, and then $G[R_1 + x] = K_4^-$ and $G[W_2 + F - x]$ contain chorded cycles since $||W_2, F - x|| \geq 3$.

Lastly, if $||W_2, F|| \leq 4$, then some vertex in W_2 is low and so $||v, F|| \geq 3$ for all $v \in W_1$. Also, by Lemma 4.10, there exists $x \in N(v_r) \cap V(F)$. If say $x \in N(v_1) \cap V(F)$, then $xv_1v_3 \cdots v_rx$ is a cycle with chord v_rv_j and $G[F - x + v_2] \supseteq K_4^-$, producing two disjoint chorded cycles. Hence, $N(v_1) \cap V(F) = N(v_2) \cap V(F) = V(F) - x$. But then $d_G(v_1) + d_G(v_r) \leq (3 + 2) + (1 + 2) = 8 < 9$, contradicting (4.1) This completes the case.

CASE 2: $F = K_4^-$. If $d_{G[R]}(v_r) = 1$, then, by (4.1), we immediately have $||v, F|| = 3$ for all $v \in \{v_1, v_2, v_r\}$. Hence, $G[R_1 + x_1] = K_4^-$ and $G[F - x_1 + v_r] = K_4^-$, producing two disjoint chorded cycles. Now, suppose $d_{G[R]}(v_r) = 2$. We now have either W_1 or W_2 containing all low vertices in W.

If W_2, contains the low vertices, then for all $v \in W_1$, $||v, F|| = 3$. If also $|(N(v_{r-1}) \cup N(v_r)) \cap (V(F) - x)| \geq 2$ for some $x \in \{x_1, x_3\}$, then $G[F - x + W_2]$ and $G[R_1 + x]$ contain chorded cycles. Therefore, there exists $x' \in V(F)$ such that $N(x') \supseteq W_2$ and so $\{x_1, x_3\} = \{x, x'\}$. But then $G[R_2 + x']$ and $G[W_1 + F - x']$ contain chorded cycles.

Then, we have that W_1 contains the low vertices. Hence, $||W_2, F|| \geq 5$ and there exists $x \in N(v_{r-1}) \cap N(v_r) \cap V(F)$. If $||W_1, F - x|| \geq 3$, then $G[F - x + W_1]$ and $G[R_2 + x]$ contain chorded cycles. So, we have $x \in N(v_1) \cap N(v_2) \cap V(F)$ since $d_G(v_r) \leq 5$. But then, $G[R_1 + x] = K_4^-$ and $G[W_2 + F - x]$ contains a chorded cycle since $||W_2, F - x|| \geq 5 - 2 = 3$.

CASE 3: $F = C_5^+$. If $d_{G[R]}(v_r) = 1$, then, by (4.1), we immediately have $G[\{v_1, v_2, x_1, x_2\}] = K_4$, contradicting (O2). Now, suppose $d_{G[R]}(v_r) = 2$, then for some i, $||W_i, F|| \geq 5$. Moreover, for all $v \in W_i$, $||v, \{x_3, x_4, x_5\}|| \leq 1$ as otherwise $G[\{v, x_3, x_4, x_5\}] \supseteq K_4^-$, contradicting (O1). But then, $||W_i, \{x_1, x_2\}|| \geq 3$ forcing $G[W_i + x_1 + x_2] \supseteq K_4^-$, again contradicting (O1).

CASE 4: $F = K_{3,3}$. By Lemma 4.3, we may assume $N(v_1) \cap V(F) \subseteq A$ and $N(v_2) \cap V(F) \subseteq B$. Also, since $d_G(v_r) \leq 5$, $||v_1, A||, ||v_2, B|| \geq 2$, say $x_1, x_3 \in N(v_1)$ and $x_2, x_4 \in N(v_2)$. But then, $v_3 v_1 x_1 x_2 v_2 v_3$ is a 5-cycle with chord $v_1 v_2$, contradicting (O3). This completes the proof. □

Lemma 4.44. *For $k = 2$, $d_{G[R]}(v_1) = 1$ or $d_{G[R]}(v_r) = 1$.*

Proof. Suppose that $d_{G[R]}(v_1) = d_{G[R]}(v_r) = 2$. Let $v_1 v_s \in E(G)$ and $v_t v_r \in E(G)$. If $s - 1 = t$, then $G[R]$ is a chorded cycle, producing two disjoint chorded cycles. If $t < s - 1$, then $G[R]$ contains a spanning Θ-subgraph H consisting of 3 paths connecting v_t with v_s. In this case, adding any edge to this subgraph creates a chorded cycle. Thus $G[R] = H$, a contradiction to Lemma 4.42. Hence $t \geq s$. By Lemma 4.43, $s \geq 4$ and $t \leq r - 3$. So the set $S = \{v_1, v_{s-1}, v_{t+1}, v_r\}$ is independent. Therefore, by symmetry, we may assume that v_r

is the only vertex in S that can be low, i.e.

$$\textit{for every } v \in S - v_r,\ d_G(v) \geq 5 \textit{ and } d_G(v) + d_G(v_r) \geq 9. \tag{4.61}$$

By (4.61), we also have

$$||S, F|| \geq 11,\ \textit{and if } d_G(v_r) = 3,\ \textit{then}\ \ ||S, F|| \geq 13. \tag{4.62}$$

We claim that

$$\textit{for } q \in \{s, t\},\ \textit{there is at most one edge } v_i v_\ell \textit{ with } i < q < \ell. \tag{4.63}$$

Indeed, suppose there are edges $v_{i_1}v_{\ell_1}$ and $v_{i_2}v_{\ell_2}$ with $i_1, i_2 < q$ and $\ell_1, \ell_2 > q$. By symmetry, we may assume that $\ell_2 \geq \ell_1$ and $i_2 \leq i_1$. Then the cycle $v_{i_2}v_{i_2+1}\dots v_{\ell_2}v_{i_2}$ has chord $v_{i_1}v_{\ell_1}$. This proves (4.63).

By (4.61) and Lemma 4.2, we have four cases.

CASE 1: $F = K_4$. If there is $x \in V(F)$ with $||x, S|| = 4$, then $G[F - x + v_1] \supseteq K_4^-$ and x has 3 neighbors on the path $v_2 \cdots v_r$ so that $G[\{x, v_2, \dots, v_r\}]$ contains a chorded cycle by Lemma 4.5, a contradiction to (O3). Otherwise, $||S, F|| \leq 12$ and by (4.62), $d_G(v_r) \geq 4$. In this case, again by (4.62), there is $x \in V(F)$ with $||x, S|| = 3$. Let $v \in S - N(x)$. Then, $G[F - x + v] \supseteq K_4^-$ and x has 3 neighbors on any Hamiltonian path in $G[R - v]$, producing two disjoint chorded cycles.

CASE 2: $F = K_4^-$. Since G does not contain K_4, $||v, \{x_2, x_4\}|| = 1$ for all $v \in R$ with $||v, F|| \geq 3$. Therefore, $d_G(v_1) \leq 5$ and so $d_G(v_r) \geq 4$. In particular, $|N(v_r) \cap V(F)| \geq 2$. If $N(v_r) \cap V(F) = \{x_2, x_4\}$, then $G[F - x_1 + v_r] = K_4^-$ and x_1 is adjacent to the 3 vertices in $S - v_r$ on the path $v_1 v_2 \cdots v_{r-1}$. Otherwise, by the symmetry between x_1 and x_3, we may assume $x_1 v_r \in E(G)$. In this case, $G[F - x_1 + v_1] = K_4^-$ and x_1 is adjacent to the 3 vertices in $S - v_1$ on the path $v_2 v_3 \cdots v_r$, producing two disjoint chorded cycles.

CASE 3: $F = C_5^+$. By Lemma 4.2, $N(v) \cap V(F) = \{x_1, x_2, x_4\}$ for $v \in S - v_r$. Hence,

$G[\{v_1, v_{s-1}, x_1, x_2\}] = K_4^-$, contradicting (O1).

CASE 4: $F = K_{3,3}$. Let $V_1 := \{v_1, ..., v_{s-1}\}$ contain ℓ_1 low vertices and $V_2 := \{v_{t+1}, ..., v_r\}$ contain ℓ_2 low vertices. Also, let $W_1 = \{v_1, v_2\}$ and $W_2 = \{v_r, v_{r-1}\}$. If $\ell_1 > 1$, then by (4.63), $\ell_2 = 0$. So by switching the roles of V_1 and V_2 if needed, we can assume $\ell_1 \leq \ell_2$; in particular, $\ell_1 \leq 1$. For this assumption, we will no longer follow the symmetrical assumption of (4.61), i.e. some vertex in $S - v_r$ could be low instead of v_r.

By Lemma 4.2, for each vertex $v \in S + v_2$, either $N(v) \cap V(F) \subseteq A$ or $N(v) \cap V(F) \subseteq B$. Let $S_A = \{v \in S + v_2 : N(v) \cap V(F) \subseteq A\}$ and similarly for S_B. Since $S + v_2 = S_A \cup S_B$, we may assume that $|S_A| \geq \lceil 5/2 \rceil = 3$. By (4.61), there is $x \in A$ with $||x, S_A|| = |S_A|$. We now claim

$$\{v_1, v_2\} \not\subseteq S_A \tag{4.64}$$

Suppose $\{v_1, v_2\} \subseteq S_A$. If $|N(v_1) \cap N(v_2) \cap V(F)| \geq 2$, then $G[A + v_1 + v_2] \supseteq K_4^-$, contradicting (O3). So, $||W_1, A|| \leq 4$ and at least one of $\{v_1, v_2\}$ is low. Since V_1 has at most one low vertex and $d_G(v_r) \leq 5$, there is a unique $x' \in N(v_1) \cap N(v_2) \cap V(F)$. Say, $x' = x_1$. Since $||\{v_1, v_2\}, \{v_{t+1}, v_r\}|| = 0$, v_{t+1} and v_r are both high. If $v \in \{v_{t+1}, v_r\}$ is such that $v \in S_B$, then $x_1 v_1 v_s v_{s-1} \cdots v_2 x_1$ is a cycle with chord $v_1 v_2$ and $G[F - x_1 + v] = K_{3,3}$, producing two disjoint chorded cycles. Hence $v_{t+1}, v_r \in S_A$. Then, $x_1 v_1 v_s \cdots v_2 x_1$ is a cycle with chord $v_1 v_2$ and $x_3 v_{t+1} \cdots v_r x_5 x_4 x_3$ is a cycle with chord $x_3 v_r$, producing two disjoint chorded cycles. This proves (4.64).

Hence, $S_B \neq \emptyset$. If there is $v \in S_B$ such that $||v, B|| \geq 2$, then $G[F - x + v]$ has a chorded 6-cycle, and $G[R - v + x]$ contains a chorded cycle since $||x, R - v|| \geq |S_A| \geq 3$ and $G[R - v]$ contains a spanning path. Hence, $||v, B|| \leq 1$, in fact $||v, B|| = 1$ by (4.1), and $v \notin S$. This forces $d_G(v) \leq 4$, $v = v_2$ and so all $w \in S$ are high and $S_B = \{v_2\}$. Moreover, we have $v_2 v_i \in E(G)$ for some $4 \leq i \leq r - 1$ by (4.1). Since $\ell_1 \leq \ell_2$, v_i is low, $i > t + 1$, and all vertices $v \in R - v_2 - v_i$ are high with $||v, B|| \leq 1$.

If $v_{s-2} \neq v_2$, then $d_{G[R]}(v_{s-2}) \leq 3$ and v_{s-2} high implies $||v_{s-2}, A|| \geq 2$. Hence, $G[A + v_{s-2} + v_{s-1}] \supseteq K_4^-$, contradicting (O1). Therefore, $v_{s-2} = v_2$. By symmetry, $v_{t+2} = v_{r-1}$ and so $s = 4, t = r - 3$. Hence, $i = r - 1$.

Now, since $v_i = v_{r-1} \in V_2$ and $v_2v_{r-1} \in E(G)$, $d_{G[R]}(v_{r-1}) = 3$, so $||v_{r-1}, F|| \geq 1$. So $v_{r-1}x \in E(G)$ for some $x \in V(F)$. If $x \in A$, $N(x) \supseteq \{v_{r-2}, v_{r-1}, v_r\}$ so that $G[\{x, v_{r-2}, v_{r-1}, v_r\}] = K_4$, contradicting (O2). Otherwise, $x \in B$. If $x = x_2$, then $x_2v_2v_3v_4 \cdots v_{r-1}x_2$ is a cycle with chord v_2v_i and $G[F - x_2 + v_1] = K_{3,3}$, producing two disjoint chorded cycles. Therefore $x_2v_{r-1} \notin E(G)$, but say $x_4v_{r-1} \in E(G)$. Now, $d_{G[R]}(v_4) = 3$ as otherwise $v_4v_j \in E(G)$ for some $6 \leq j \leq r-3$, and then $v_2 \cdots v_{r-1}v_2$ is a cycle with chord v_4v_j, contradicting (O3). So, v_4 being high implies $||v_4, F|| \geq 2$. If $N(v_4) \cap V(F) \subseteq A$, then there exists $x' \in A$ such that $N(x') \supseteq \{v_1, v_3, v_4\}$ so that $G[\{x', v_1, v_3, v_4\}] \supseteq K_4^-$ contradicting (O1). So, $N(v_4) \cap V(F) \subseteq B$. Then, $G[F - x_1 + v_4] \supseteq K_{3,3}^-$ and $x_1v_1v_2v_{r-1}v_rv_{r-3}v_{r-2}x_1$ is a cycle with chord x_1v_r, producing two disjoint chorded cycles. This completes the case and the proof. □

4.6 Proof of Theorem 4.1

Recall, we consider the minimum k such that Theorem 4.1 fails. Now, let G be an edge-maximal counterexample such that $\mathcal{F}$ contains less than k chorded cycles. For any such graph G, let $\mathcal{F}$ be a collection of disjoint chorded cycles chosen by our rules (O1)–(O8). By Sections 4.3 and 4.4, we have $k = 2$ and $G[R]$ containing a Hamiltonian path. By the result of Section 4.5 and Lemma 4.44, $d_{G[R]}(v_1) = 1$ or $d_{G[R]}(v_r) = 1$. Suppose $d_{G[R]}(v_1) = 1$. By Lemmas 4.6 and 4.40, $|R| \geq 4$ and $G[R]$ is not a Hamiltonian cycle and so $v_1v_r \notin E(G)$. Moreover, by Lemma 4.43, $v_1v_3, v_rv_{r-2} \notin E(G)$.

CASE 1. $d_{G[R]}(v_r) = 1$. Then $||\{v_1, v_r\}, F|| \geq 7$. By symmetry, assume $||v_1, F|| \geq 4$. By Lemma 4.2, $F = K_4$ and so $d_G(v_1) = 5$ and $d_G(v_r) \geq 4$. Since $n \geq 8$, $r \geq 4$. Also, for each $v \in S := \{v_2, v_3, v_{r-2}, v_{r-1}\}$, $d_{G[R]}(v) \leq 3$. Note, $2 \leq |S| \leq 4$ and any vertex $v \in S - v_2$ has $||v, F|| \geq 1$ since $v_1v \notin E(G)$, but similarly $||v_2, F|| \geq 1$ since $v_2v_r \notin E(G)$. If there exists $x \in N(v_2) \cap N(v_{r-1}) \cap V(F)$, then $G[R - v_r + x]$ contains a cycle with chord xv_2 and $G[F - x + v_r] \supseteq K_4^-$. So, we have two disjoint chorded cycles, a contradiction. Hence, let $V(F) = \{x_1, x_2, x_3, x_4\}$ with $N(v_r) \cap V(F) \supseteq \{x_2, x_3, x_4\}$ and $v_{r-1}x, v_2x' \in E(G)$ for distinct

$x, x' \in V(F)$. If $x \in N(v_r) \cap V(F)$ then we can choose $x'' \in N(v_r) \cap V(F) - \{x, x'\}$ and get cycles $v_r x'' x v_{r-1} v_r$ and $v_1 v_2 x' x_1 v_1$ with chords $v_r x$ and $v_1 x'$, resp. Therefore, we are left to handle when $N(v_r) \cap V(F) = \{x_2, x_3, x_4\}$ and $N(v_{r-1}) \cap V(F) = \{x_1\}$, i.e. $x = x_1$. If v_2 or v_{r-1} has a chord on P, then $G[R - v_r + x_1]$ and $G[F - x_1 + v_r]$ each have a chorded cycle. Thus, $d_{G[R]}(v_2) = d_{G[R]}(v_{r-1}) = 2$ which forces $||v_2, F|| \geq 3$ since $v_2 v_r \notin E(G)$ and $||v_{r-1}, F|| \geq 2$ by (4.1), contradicting $N(v_2) \cap N(v_{r-1}) \cap V(F) = \emptyset$.

CASE 2. $d_{G[R]}(v_r) = 2$. Say $v_k v_r \in E(G)$ for exactly one $2 \leq k \leq r-3$. By symmetry, $d_{G[R]}(v_{k+1}) = 2$.

First, we claim

$$||v_1, F|| = 3 \tag{4.65}$$

First, suppose that $||v_1, F|| = 4$. By Lemma 4.2, $F = K_4$. Note also $d_{G[R]}(v_{r-1}) \leq 3$ and so $||\{v_{k+1}, v_{r-1}, v_r\}, F|| \geq 5$ by (4.1). Hence, there exists $x \in V(F)$ such that $||\{v_{k+1}, v_{r-1}, v_r\}, x|| \geq 2$. Thus, $G[F - x + v_1] = K_4$ and $G[P - v_1 + x]$ also has a path of length $|P|$ but with $|E(G[P - v_1 + x])| \geq |E(G[P])| + 1$ due to v_1 essentially being replaced by x, contradicting (O7) or contradicting (O6) if $v_2 x \in E(G)$.

Otherwise, $||v_1, F|| \leq 2$. By Lemma 4.10, $||v_1, F|| = 2$. Then $d_G(v) \geq 6$ for all $v \in \{v_{k+1}, v_{r-1}, v_r\}$. Specifically, $||v_{k+1}, F|| = ||v_r, F|| = 4$ and $||v_{r-1}, F|| \geq 3$. Hence, $F = K_4$, and there exists $x \in V(F) \cap N(v_{r-1}) \cap N(v_r) - N(v_1)$. Then, $G[F - x + v_1] = K_4^-$ and $x v_r v_k \cdots v_{r-1} x$ is a cycle with chord $v_{r-1} v_r$, thus producing two disjoint chorded cycles. This proves (4.65) so that now we have the four cases of Lemma 4.2.

Subcase 2.1: $F = K_4$. Then v_{r-1} and v_r have a common neighbor $x \in V(F)$ and so $G[F - x + v_1] \supseteq K_4^-$ but $x v_r v_k v_{k+1} \cdots v_{r-1} x$ is a cycle with chord $v_{r-1} v_r$, so we have two disjoint chorded cycles, a contradiction.

Subcase 2.2: $F = K_4^-$. If $x \in \{x_1, x_3\}$ is such that $x \in N(v_{r-1}) \cap N(v_r)$, then $x v_{r-1} \cdots v_k v_r x$ is a cycle with chord $v_{r-1} v_r$ and $G[F - x + v_1] \supseteq K_4^-$, producing two disjoint chorded cycles. Hence, $N(v_{r-1}) \cap V(F) = \{x_2, x_4\}$ and we may assume $N(v_r) = \{x_1, x_2, x_3\}$. If $v \in \{v_1, v_{k+1}\}$ is such that $N(v) \supseteq \{x_1, x_4, x_3\}$, then $G[\{v_1, \ldots v_{k+1}, x_1, x_4\}]$ contains a chorded cycle since $||\{x_1, x_4\}, \{v_1, v_{k+1}\}|| \geq 3$ and $x_2 v_{r-1} v_r x_3 x_2$ is a cycle with chord $x_2 v_r$. Hence, for all $v \in \{v_1, v_{k+1}, v_r\}$, $N(v) \cap V(F) = \{x_1, x_2, x_3\}$. Then, $G[F - x_1 + v_1] = K_4^-$

and $G[R-v_1+x_1]$ contains path $x_1v_r\cdots v_2$ as long as $|P|$. However, $|E(G[R-v_1+x_1])| \geq |E(G[R])|+1$, contradicting (O7). This completes the case.

Subcase 2.3: $F = C_5^+$. Since $d_G(v_1) = 4$, $||v_r, F|| \geq 3$. By Lemma 4.2, $N(v_1) \cap V(F) = N(v_r) \cap V(F) = \{x_1, x_2, x_4\}$. Hence, $v_1x_1v_rx_2v_1$ is a 4-cycle with chord x_1x_2, contradicting (O1).

Subcase 2.4: $F = K_{3,3}$. For definiteness, let $N(v_1) \cap V(F) = A$. Again, $d_G(v_1) = 4$ implies $d_G(v_r) \geq 5$ and so, by the case, $d_G(v_r) = 5$.

Suppose v_2 has a neighbor $z \in F$. If $z \in A$, say $z = x_1$, then G has a 5-cycle $v_1v_2x_1x_2x_3v_1$ with chord v_1x_1, contradicting (O3). Thus we may assume $z = x_2$. Since $v_1x_2 \notin E(G)$, $d_G(x_2) \geq 9 - d_G(v_1) = 5$. Then $G[F - x_2 + v_1] = K_{3,3}$ and for $R' = R - v_1 + x_2$, $G[R']$ has r-vertex path $x_2v_2\cdots v_r$ where $d_{G[R']}(x_2) \geq 2$, contradicting (O7) at least. Hence, $||v_2, F|| = 0$.

Then $d_G(v_2) \leq 3$. Since $d_G(v_r) = 5 < 9 - d_G(v_2)$, $v_2v_r \in E(G)$, i.e. $k = 2$. Since $G[R]$ has no chorded cycles, path P has no chords aside from v_2v_r. Hence, $v_2v_{r-1} \notin E(G)$ and $d_{G[R]}(v_{r-1}) = 2$. Therefore, $d_G(v_{r-1}) \geq 6$ by (4.1) and $||v_{r-1}, F|| = 4$. By Lemma 4.2, $F = K_4$, contradicting the case.

This completes the proof of Theorem 4.1.

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