

# Collateral Constraints and Asset Prices\*

Georgy Chabakauri  
London School of Economics

Brandon Yueyang Han  
University of Maryland

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## Abstract

We study the effects of collateral constraints in an economy populated by investors with nonpledgeable labor incomes and heterogeneous preferences and beliefs. We show that these constraints inflate stock prices, generate spikes and crashes in price-dividend ratios and volatilities, clustering of volatilities, and leverage cycles. They also lead to substantial decreases in interest rates and increases in Sharpe ratios when investors are anxious about hitting constraints due to production crises in the economy. Furthermore, stock prices have large collateral premiums over nonpledgeable incomes. We derive asset prices and stationary distributions of the investors' consumption shares in closed form.

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# 1. Introduction

Financial markets play a key role in facilitating risk sharing and efficient allocation of assets among investors. However, trading in financial assets often entails moral hazard due to investors' incentives to default on their risky positions. The moral hazard can be alleviated by collateralized trades whereby risky positions are backed by financial capital that can be seized in the event of default. The latter arrangement restores the functionality of financial markets at a cost of restricting risk sharing among investors. In this paper, we develop a parsimonious model that sheds new light on the economic effects of such restrictions. In particular, we show how collateralization inflates asset prices, generates repeated booms and busts in the stock market, and leads to spikes, crashes, and clustering of stock return volatilities, persistent periods of loose and binding constraints, and cycles of high and low leverage. Our analysis is facilitated by closed-form solutions and the stationarity of equilibrium processes.

We consider a pure exchange economy with one consumption good produced by a tree with i.i.d. shocks, similar to Lucas (1978). The economy is populated by two representative investors with heterogeneous constant relative risk aversion (CRRA) preferences over consumption and heterogeneous beliefs about the output growth rate. Each investor receives a fraction of the tree's output as labor income and invests total wealth in financial assets such as bonds and stocks. The investors have limited liability and can re-enter the financial market following defaults on risky positions in financial assets. In the event of default, the financial assets can be seized by counterparties but labor income cannot be expropriated. The arising moral hazard problem is resolved by requiring risky positions to be backed by collateral in such a way that each investor's total financial wealth stays positive at all times, and hence, investors can always pay back to counterparties. We also allow the aggregate consumption to experience rare large negative shocks. These shocks help us explore how mere anxiety about the possibility of a production crisis affects the economy by tightening collateral constraints. Our closed-form solutions allow us to prove some of the results for general model parameters rather than for particular calibrations.

First, we show that collateral constraints increase the prices of all tradable assets with positive cash flows relative to a frictionless economy. Moreover, these increases in prices are larger when investors are closer to their default boundaries. In particular, the stock price-dividend ratio spikes upwards in response to small economic shocks near the default

boundaries of investors, giving rise to repeated periods of high and low stock prices.

The intuition for the latter results is as follows. In a frictionless economy, the investors' consumption shares gradually approach zero or one, and hence the economic impact of one of the investors vanishes in the long-run (e.g., Blume and Easley, 2006; Yan, 2008; Chabakauri, 2015). The collateral requirements restrict financial losses and protect investors from losing their consumption shares. As a result, the consumption shares are bounded away from zero and one. Moreover, the constraints never bind simultaneously for both investors, and at each moment one of the investors is unconstrained. The unconstrained investor's marginal utility of consumption is proportional to the prices of Arrow-Debreu securities. This marginal utility is expected to be higher in the economy with constraints because the unconstrained investor's consumption is expected to be lower than in the unconstrained economy due to the upper bound on the consumption share, discussed above. Consequently, the prices of Arrow-Debreu securities, and hence, the prices of all assets with positive cash flows, are higher in the constrained economy.

The dynamics of the price-dividend ratio determines the effect of constraints on volatilities. We show that collateral constraints dampen volatilities in bad times, when the aggregate consumption is low, and amplify them in good times, when the aggregate consumption is high. The latter effect makes collateral constraints a useful tool for curbing excessive volatility in bad times. The explanation is that the price-dividend ratio spikes up both in good and bad times because in good (bad) times the pessimistic (optimistic) investors in our economy lose wealth and may bind on their constraints. As a result, the price-dividend ratio is procyclical in good times and countercyclical in bad times. Consequently, the price-dividend ratio and the dividend move in the same direction in good times and in opposite directions in bad times. Because the stock price is the product of the price-dividend ratio and the dividend, the stock return volatility increases in good times and decreases in bad times. The volatility experiences spikes and crashes due to the sensitivity of price-dividend ratios to small shocks when investors are close to hitting their constraints. Moreover, the periods of high and low volatility are persistent because of the persistence of periods when constraints are likely to bind, as discussed below, which gives rise to the clustering of volatilities.

We also derive the distributions of investors' consumption shares in analytic form and show that they are stationary and nondegenerate (i.e., their support is a closed interval rather than a single point). The analysis of these distributions yields three economic

insights. First, there is nontrivial time-variation of asset prices in the long run. Second, periods of binding collateral constraints are persistent. That is, the economy stays close to default boundaries for some time because hitting a constraint makes likely hitting it again in the near future due to the slow accumulation of wealth over time. Third, we show that all investors, including those with incorrect beliefs, survive in the long run and can have a large economic impact in equilibrium because the constraints prevent investors from losing their consumption shares, similar to the related literature (e.g., Blume and Easley, 2006; Cao, 2018). We note that the nondegeneracy of consumption share distributions and the persistence of the periods of binding constraints are more difficult to demonstrate than survival, and, to our best knowledge, these results are new to the literature.

Next, we show that the mere possibility of a large (albeit unpredictable) drop in the aggregate output next period decreases interest rates and increases Sharpe ratios in the current period when the irrational optimist is close to hitting the collateral constraint. The latter effect only occurs when production crises and collateral constraints are jointly present in the economy. Hence, the collateral constraints amplify the spillover of the production crisis to the financial market. The amplification effect arises because investors “fly to quality” by buying riskless bonds when there is a possibility of hitting the collateral constraint next period. We note that lower interest rates and higher Sharpe ratios can be generated by alternative mechanisms and constraints, discussed in the literature review below. However, the amplification mechanism, to our best knowledge, has not been studied before. We also show that investor heterogeneity and the stationarity of equilibrium give rise to cycles of high and low leverage. In particular, the leverage is high when investors are far away from their default boundaries, and drops to zero when investors hit their constraints.

Finally, we measure the collateral liquidity premium of the stock versus labor income. This premium arises because dividends and labor incomes are collinear but incomes are nonpledgeable. First, we derive shadow prices of claims to labor incomes such that exchanging marginal units of these claims for the consumption good at shadow prices does not affect investors’ welfare. Then, we construct portfolios of stocks that replicate labor incomes. We define the collateral liquidity premium as the percentage difference in the value of the replicating portfolio and the shadow price. The premium from the view of a particular investor widens close to that investor’s default boundary and ranges from 0% to 40% in our calibration, which demonstrates the economic significance of collateralization.

Moreover, the nontradability of labor income does not contribute to this premium because in economies with pledgeable labor income investors circumvent nontradability by taking short positions in the stock.

The paper develops a new methodology for studying the effects of collateralization. This methodology allows us to obtain closed-form equilibrium processes and prove their properties, which previously could only be studied numerically. For example, we prove that our constraints increase price-dividend ratios and generate spikes in asset prices, and lead to nondegeneracy and stationarity of consumption share distributions. The paper also introduces a tractable discrete-time setting that makes exposition less technical and permits taking continuous-time limits. The tractability and stationarity make our model a convenient benchmark for asset pricing research that can be extended in various directions.

**Related Literature.** We contribute to the literature by uncovering several new economic effects of collateral constraints. In particular, we show that these constraints give rise to U-shaped price-dividend ratios that spike when constraints are likely to bind, cause sharp increases and crashes of stock return volatility, magnify volatility in good states and dampen it in bad states, and amplify the effects of crises on interest rates and Sharpe ratios. Our paper is the first to derive the stationary distribution of an investor's consumption share in an economy with collateral constraints in closed form, and to use it to demonstrate the persistence of periods of binding constraints and nondegeneracy of the long-run equilibrium. The nondegeneracy of equilibrium implies time variation of asset prices and cycles of high and low leverage in the long run. The paper also sheds new light on the effects of constraints on interest rates, Sharpe ratios, and collateral premiums that have been previewed in the related literature. In particular, similar to this literature, the constraints decrease interest rates, increase Sharpe ratios, and give rise to collateral premiums for stocks. In contrast to the previous studies, the effects of constraints on interest rates and Sharpe ratios occur only during periods of anxious economy, when the mere possibility of a production crisis tightens the constraints. We also develop a new approach to capturing collateral premiums and evaluate their economic significance.

Closest to us are papers that study economies in which investors have limited liability and face solvency constraints. Deaton (1991) considers a partial equilibrium model in which investors trade in a riskless asset with an exogenous interest rate and face a nonnegativity constraint on their financial wealth. Detemple and Serrat (2003) also study

an economy with a nonnegative wealth in which investors have heterogeneous beliefs and identical risk aversions. They show that this constraint introduces a singularity component into interest rates when the constraint binds while stock risk premiums have the same structure as in unconstrained economies. In contrast to our work, they do not compute price-dividend ratios, volatilities, and consumption share distributions, and do not study the effects of rare production crises and heterogeneity in preferences.

Chien and Lustig (2010) study a similar constraint in an economy with a continuum of ex-ante identical investors that receive nonpledgeable labor incomes affected by idiosyncratic shocks. They develop similar methods using multipliers in a discrete-time setting, but they do not have closed-form solutions and do not consider differences in beliefs about fundamentals. Lustig and Van Nieuwerburgh (2005) study the role of housing collateral when labor income is nonpledgeable. The main difference of our paper from the latter two papers is that our investors are ex-ante heterogeneous and are not affected by idiosyncratic shocks to labor income. The economic effects of heterogeneity in preferences and beliefs are different from the effects of ex-post heterogeneity in realized idiosyncratic income shocks in the above literature. For example, Krueger and Lustig (2010) show the irrelevance of market incompleteness induced by these income shocks for the risk premiums.

Cao (2018) proves the survival of investors with incorrect beliefs in the long run in economies with general collateral constraints and stationary endowment processes bounded away from zero, and shows similar results numerically in an example with nonstationary endowments. Blume et al (2018) explore potential benefits from imposing trading restrictions, such as natural borrowing constraints, in economies with bounded endowments and investors with heterogeneous beliefs. In contrast to these works, our results do not rely on bounded endowments. Moreover, we derive consumption share distributions in closed form, and establish their bimodality, stationarity, and nondegeneracy (i.e., their support is a closed interval rather than a single point). Kubler and Schmedders (2003) prove the existence of stationary equilibria in dynamic economies with general collateral constraints. Rampini and Viswanathan (2018) study household insurance in an economy with collateral constraints with limited enforcement and deep-pocket risk-neutral lenders who provide state-contingent claims to households at zero risk premium. Our model is different from the latter paper in that all investors in our economy are risk averse, and risk premiums are endogenous and time-varying. Gromb and Vayanos (2002, 2010, 2018) and Brunnermeier and Pedersen (2009) study economies with CARA investors subject to margin constraints,

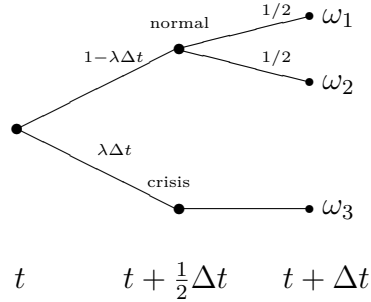
which have similarities with our constraints. In contrast to their models, in our model all investors have CRRA preferences and interest rates are endogenous.

Geanakoplos (2003, 2009), Fostel and Geanakoplos (2008, 2014), and Geanakoplos and Zame (2014) develop the theory of collateral constraints in two- and three-period economies. Our constraint prevents investors from defaulting in the worst-case scenario as in Geanakoplos (2003, 2009), and leads to higher asset prices as in Fostel and Geanakoplos (2008). Simsek (2013) studies a two-period economy with a continuum of states and shows that collateral constraints have asymmetric disciplining effects, depending on investor's beliefs, and also shows how defaultable debt endogenously emerges in equilibrium. Biais, Hombert, and Weill (2018) study a two-period economy with multiple trees and imperfect collateral pledgeability. In contrast to this literature, we focus on the nonpledgeability of labor income rather than the imperfect pledgeability of assets.

Kehoe and Levine (1993), Kocherlakota (1996), Tsyrennikov (2012), and Osambela (2015) study economies in which investors are weakly better off not defaulting and are permanently excluded from securities markets if they default. Alvarez and Jermann (2000) show that such constraints can be implemented by imposing certain "not too tight" solvency portfolio constraints. Alvarez and Jermann (2001) find that such constraints help explain equity premiums in the U.S. economy. They solve a simple example in closed form and develop a numerical method for the general case. In contrast to this literature, our investors have limited liability and can re-enter the market after a default.

Our paper is related to the literature on the economic effects of borrowing, margin, short-sale, and position limit constraints (e.g., Harrison and Kreps, 1978; Detemple and Murthy, 1997; Basak and Cuoco, 1998; Basak and Croitoru, 2000, 2006; Pavlova and Rigobon, 2008; Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015; Rytchkov, 2014; Brumm et al, 2015; Buss et al, 2016), portfolio insurance (e.g., Basak, 1995) and VaR constraints (e.g., Basak and Shapiro, 2001). Our economic results are different from the results in this literature. First, the latter constraints can increase or decrease stock prices depending on whether the investors' risk aversions are greater or less than one (e.g., Chabakauri, 2015), whereas our collateral constraints always increase stock prices irrespective of risk aversions and beliefs. Second, these constraints typically dampen stock return volatility whereas our collateral constraints amplify them in some states of the economy.

The paper is also related to macro-finance, financial intermediation, and banking literatures that study economies with frictions (Kiyotaki and Moore, 1997; Krusell and Smith,



**Figure 1**  
**States of the Economy**

After time  $t$  the economy moves to a normal state with probability  $1 - \lambda\Delta t$  and to a crisis state with probability  $\lambda\Delta t$ . Conditional on being in a normal state the economy moves to either  $\omega_1$  or  $\omega_2$  with equal probabilities.

1998; Brunnermeier and Sannikov, 2014; Klimenko, Pfeil, Rochet, and De Nicolo, 2016; Kondor and Vayanos, 2019) and to the literature on frictionless economies with heterogeneous investors (e.g., Chan and Kogan, 2002; Basak, 2005; Yan, 2008; Bhamra and Uppal, 2014; Atmaz and Basak, 2018; Borovička, 2019; Massari, 2019, among others).

## 2. Economic setup

We consider a pure-exchange infinite-horizon economy with one consumption good produced by an exogenous Lucas (1978) tree. The economy is populated by two representative heterogeneous investors  $A$  and  $B$  that hold shares in the tree and receive labor income each period. To facilitate the exposition, we start with a discrete-time economy with dates  $t = 0, \Delta t, 2\Delta t, \dots$ , and later take a continuous-time limit.

At each point of time  $t = 0, \Delta t, 2\Delta t, \dots$  the economy is in one of the three states:  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$ . With probability  $1 - \lambda\Delta t$  the economy is either in state  $\omega_1$  or state  $\omega_2$ , which we call *normal states*, and with probability  $\lambda\Delta t$  in state  $\omega_3$ , which we call the *crisis state*. Parameter  $\lambda > 0$  is the *crisis intensity*. States  $\omega_1$  and  $\omega_2$  have probabilities  $1/2$  conditional on the economy being in a normal state. Figure 1 depicts the structure of uncertainty.

### 2.1. Output, financial markets, and investor heterogeneity

At date  $t$  the tree produces  $D_t\Delta t$  units of aggregate output, where  $D_t$  follows a process

$$\Delta D_t = D_t[\mu_D\Delta t + \sigma_D\Delta w_t + J_D\Delta j_t], \quad (1)$$



where  $\mu_D \geq 0$ ,  $\sigma_D > 0$ , and  $J_D \leq 0$  are output growth mean, volatility, and drop during a crisis, respectively, and  $\Delta D_t = D_{t+\Delta t} - D_t$  is the change in output. Processes  $w_t$  and  $j_t$  are discrete-time analogues of a Brownian motion and Poisson processes, respectively. These processes follow dynamics  $w_{t+\Delta t} = w_t + \Delta w_t$  and  $j_{t+\Delta t} = j_t + \Delta j_t$ , where increments  $\Delta w_t$  and  $\Delta j_t$  are i.i.d. random variables given by:

$$\Delta w_t = \begin{cases} +\sqrt{\Delta t}, & \text{in state } \omega_1, \\ -\sqrt{\Delta t}, & \text{in state } \omega_2, \\ 0, & \text{in state } \omega_3, \end{cases} \quad \Delta j_t = \begin{cases} 0, & \text{in state } \omega_1, \\ 0, & \text{in state } \omega_2, \\ 1, & \text{in state } \omega_3. \end{cases} \quad (2)$$

It can be easily verified that  $\mathbb{E}_t[\Delta w_t | \text{normal}] = 0$  and  $\text{var}_t[\Delta w_t | \text{normal}] = \Delta t$ , similar to a Brownian motion, where  $\mathbb{E}_t[\cdot]$  and  $\text{var}_t[\cdot]$  are expectation and variance conditional on time- $t$  information. Parameters  $\mu_D$ ,  $\sigma_D$ , and  $J_D$  are such that  $D_t > 0$  for all  $t$ . Chabakauri (2014) shows that process (1) converges to a continuous-time Lévy process as  $\Delta t \rightarrow 0$ .

The economy is populated by two representative investors  $A$  and  $B$ . Each investor stands for a continuum of identical investors of unit mass. Fractions  $l_A$  and  $l_B$  of the aggregate output  $D_t \Delta t$  are paid to investors  $A$  and  $B$  as their labor incomes, respectively. Labor incomes are nontradable. Fractions  $l_A$  and  $l_B$  can also be interpreted as nontradable shares in the aggregate output such as holdings of illiquid assets. The remaining fraction  $1 - l_A - l_B$  is paid as a dividend to the shareholders.

The investors can trade three securities at each date  $t$ : 1) a riskless bond in zero net supply, which pays one unit of consumption at date  $t + \Delta t$ ; 2) one stock in net supply of one unit, which is a claim to the stream of dividends  $(1 - l_A - l_B)D_t \Delta t$ ; 3) a one-period insurance contract in zero net supply, which pays one unit of consumption in the crisis state  $\omega_3$  and zero otherwise. Absent any frictions the market is complete. Market completeness and the absence of idiosyncratic shocks to labor income are required for tractability, and allow us to solve the model in closed form. Bond, stock, and insurance prices  $B_t$ ,  $S_t$ , and  $P_t$ , respectively, are determined in equilibrium.

## 2.2. Investor heterogeneity and optimization problems

The investors have heterogeneous CRRA preferences over consumption, given by

$$u_i(c) = \begin{cases} \frac{c^{1-\gamma_i}}{1-\gamma_i}, & \text{if } \gamma_i \neq 1, \\ \ln(c), & \text{if } \gamma_i = 1, \end{cases} \quad (3)$$

where  $i = A, B$ . The investors agree on time- $t$  asset prices and the aggregate output but disagree on the probabilities of states. Both investors have subjective beliefs given by

$$\pi_i(\omega_1) = \frac{1 - \lambda_i \Delta t}{2} (1 + \delta_i \sqrt{\Delta t}), \quad \pi_i(\omega_2) = \frac{1 - \lambda_i \Delta t}{2} (1 - \delta_i \sqrt{\Delta t}), \quad \pi_i(\omega_3) = \lambda_i \Delta t, \quad (4)$$

where crisis intensities  $\lambda_i$  and disagreement parameters  $\delta_i$  are such that probabilities (4) are positive. It is immediate to verify that  $\pi_i(\omega_1) + \pi_i(\omega_2) + \pi_i(\omega_3) = 1$ , and hence,  $\pi_i(\omega)$  is a probability measure. Throughout the paper,  $\mathbb{E}_t^i[\cdot]$  and  $\text{var}_t^i[\cdot]$  denote conditional expectations and variances under the probability measure of investor  $i$ .

It can be easily verified that time- $t$  conditional expected output growth rate in normal times under the beliefs of investor  $i$  is given by:

$$\mathbb{E}_t^i \left[ \frac{\Delta D_t}{D_t} \middle| \text{normal} \right] = (\mu_D + \delta_i \sigma_D) \Delta t. \quad (5)$$

Therefore, parameter  $\delta_i$  measures the extent of investor  $i$ 's pessimism (when  $\delta_i < 0$ ) or optimism (when  $\delta_i > 0$ ) relative to the objective probability measure. For tractability, we assume that investors do not update probabilities over time. We also assume that investor  $B$  is weakly less risk averse and more optimistic than investor  $A$ :  $\gamma_A \geq \gamma_B$ ,  $\lambda_A \geq \lambda_B$  and  $\delta_B \geq \delta_A$ . The latter parametric restriction is imposed to simplify the exposition and does not affect the qualitative results in the paper.<sup>1</sup>

At date 0 the investors have certain endowments of financial assets. The total time- $t$  disposable wealth of investor  $i$  is given by  $W_{it} + l_i D_t \Delta t$ , where  $W_{it}$  is the financial wealth, defined as the time- $t$  value of all positions in financial assets acquired at the previous date, and  $l_i D_t \Delta t$  is the labor income. At date  $t$ , investor  $i$  allocates wealth to  $c_{it} \Delta t$  units of consumption,  $b_{it}$  units of bond, and a portfolio of risky assets  $n_{it} = (n_{i,st}, n_{i,pt})$ , where  $n_{i,st}$  and  $n_{i,pt}$  are units of stock and insurance, respectively.

In a frictionless economy, the financial wealth  $W_{it}$  can become negative when investors take risky positions backed by their future labor income. However, in our economy only financial assets are pledgeable whereas labor incomes are not. Moreover, the investors have limited liability. That is, they can default when their financial wealth becomes negative

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<sup>1</sup>This assumption makes it easier to see that the consumption share of investor  $A$ ,  $s_t = c_{At}^*/D_t$  (introduced in Section 2.3 below), is countercyclical because in good (bad) times less risk averse optimists have more (less) wealth and consumption than more risk averse pessimists. If this assumption is relaxed, all the qualitative results in this paper remain the same. Section IA.2 of the Internet Appendix provides an example of equilibrium processes in an economy where the less risk averse investor is more pessimistic than the more risk averse investor, and also presents the exact condition for the countercyclicity or procyclicality of the state variable  $s_t$ .

and then re-enter the market, which gives rise to a moral hazard problem, similar to the related literature (e.g., Chien and Lustig, 2010; Geanakoplos, 2009). This problem is addressed here by requiring the investors to keep their next-period financial wealth  $W_{i,t+\Delta t}$  positive at all times, so that their pledgeable capital is sufficient to cover all liabilities such as debt and short positions. Intuitively, constraint  $W_{i,t+\Delta t} \geq 0$  requires investors to cross-collateralize their pledgeable financial assets in such a way that losses on one position are always offset by gains on the other positions.

Investor  $i = A, B$  maximizes expected discounted utility with time discount  $\rho$

$$\max_{c_{it}, b_{it}, n_{it}} \mathbb{E}_t^i \left[ \sum_{\tau=t}^{\infty} e^{-\rho\tau} u_i(c_{i\tau}) \Delta t \right], \quad (6)$$

subject to the self-financing budget constraints, given by

$$W_{it} + l_i D_t \Delta t = c_{it} \Delta t + b_{it} B_t + n_{it} (S_t, P_t)^\top, \quad (7)$$

$$W_{i,t+\Delta t} = b_{it} + n_{it} \left( S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t, \mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}} \right)^\top. \quad (8)$$

and the collateral constraint:

$$W_{i,t+\Delta t} \geq 0, \quad (9)$$

where  $W_{i,t+\Delta t}$  is the financial wealth at date  $t + \Delta t$  given by equation (8).

To provide further intuition for the constraint (9), following Gromb and Vayanos (2018), we observe that it is equivalent to the following collateral constraint:

$$W_{it} + (l_i D_t - c_{it}) \Delta t \geq \max_{\omega_{t+\Delta t}} \left\{ n_{i,st} \left( S_t - \frac{S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t}{1 + r_t \Delta t} \right) + n_{i,pt} \left( P_t - \frac{\mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}}}{1 + r_t \Delta t} \right) \right\}. \quad (10)$$

The constraint (10) is obtained by substituting bond holding  $b_{it}$  from equation (7) into equation (8) for wealth  $W_{i,t+\Delta t}$ , and then rearranging term in the inequality  $W_{i,t+\Delta t} \geq 0$ . The expression on the right-hand side of the constraint (10) represents the largest possible loss of a risky position evaluated in present value terms. Therefore, this constraint indicates that the investors are allowed to invest in portfolios of assets using these portfolios as collateral, but are required to put up a sufficient amount of their own capital to cover the losses in the worst-case scenario. The coefficients multiplying asset holdings  $n_{i,st}$  and  $n_{i,pt}$  in (10) and evaluated at the worst-case state  $\omega_{t+\Delta t}$  are endogenous margin requirements that show the investors' own capital invested per unit of asset.

The constraint (10) is similar to collateral constraints in Brunnermeier and Pedersen (2009) and Gromb and Vayanos (2018) with the difference being that we allow investors

to “cross-margin” their positions so that one risky asset can be used to cover margins on the other. Brunnermeier and Pedersen (2009) discuss the institutional features of such constraints and point out that it is increasingly possible to “cross-margin”.

**Remark 1 (Partially pledgeable labor income).** Our model can be easily extended to economies where fraction  $k_i \in [0, 1]$  of investor  $i$ 's labor income can be pledged. The requirement to keep the next-period pledgeable wealth nonnegative is then given by:

$$W_{i,t+\Delta t} + \underbrace{\frac{k_i l_i}{1 - l_A - l_B} (S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t)}_{\text{measure of pledgeable labor income}} \geq 0. \quad (11)$$

The second term in constraint (11) measures the value of the pledgeable income. Let  $k_i l_i D_t \Delta t$  be the pledgeable income of investor  $i$ . This income is proportional to stock dividends  $(1 - l_A - l_B) D_t \Delta t$ , and hence, can be replicated by a portfolio of  $\hat{n}_i = k_i l_i / (1 - l_A - l_B)$  units of stock with cum-dividend value  $\hat{n}_i (S_t + (1 - l_A - l_B) D_t \Delta t)$ . The investors can circumvent the nontradability of pledgeable income by shorting stocks against this income. Hence, the claims to pledgeable income are, effectively, tradable and have the same value as the replicating portfolio. The requirement to have positive pledgeable wealth then becomes  $W_{i,t+\Delta t} + \hat{n}_i (S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t) \geq 0$ , which is equivalent to constraint (11). Lemma A.1 in the Appendix shows that models with  $k_i \neq 0$  reduce to models with  $k_i = 0$  by a change of variable. Hence, the economic implications of our baseline model with constraint (9) and the model with a more general constraint (11) are the same.

### 2.3. Equilibrium

**Definition.** An equilibrium is a set of asset prices  $\{B_t, S_t, P_t\}$  and of consumption and portfolio policies  $\{c_{it}^*, b_{it}^*, n_{it}^*\}_{i \in \{A, B\}}$  that solve optimization problem (6) for each investor, given processes  $\{B_t, S_t, P_t\}$ , and consumption and securities markets clear:

$$c_{At}^* + c_{Bt}^* = D_t, \quad b_{At}^* + b_{Bt}^* = 0, \quad n_{A,St}^* + n_{B,St}^* = 1, \quad n_{A,Pt}^* + n_{B,Pt}^* = 0. \quad (12)$$

In addition to asset prices, we derive price-dividend and wealth-consumption ratios  $\Psi = S / ((1 - l_A - l_B) D)$  and  $\Phi_i = W_i^* / c_i^*$ , respectively. We also derive the annualized  $\Delta t$ -period riskless interest rate  $r_t$ , stock mean-return  $\mu_t$  and volatility  $\sigma_t$  in normal times, and the percentage change of the stock price in the crisis state, denoted by  $J_t$ .

We derive the equilibrium in terms of state variable  $v_t$  given by the log-ratio of marginal utilities of investors evaluated at their shares of the aggregate consumption  $c_{it}^*/D_t$ :

$$v_t = \ln \left( \frac{(c_{At}^*/D_t)^{-\gamma_A}}{(c_{Bt}^*/D_t)^{-\gamma_B}} \right). \quad (13)$$

Substituting consumption shares of investors  $A$  and  $B$ , denoted by  $s_t = c_{At}^*/D_t$  and  $1 - s_t = c_{Bt}^*/D_t$ , into equation (13), we express  $v_t$  as a function of  $s_t$ :

$$v_t = \gamma_B \ln(1 - s_t) - \gamma_A \ln(s_t). \quad (14)$$

Variable  $v_t$  is a decreasing function  $s_t$ , and hence,  $s_t$  is an alternative state variable.

We assume that the exogenous model parameters are such that

$$\mathbb{E}_t^i \left[ e^{-\rho\Delta t} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_i} \right] < 1, \quad i = A, B. \quad (15)$$

Condition (15) is necessary and sufficient for the existence of equilibrium in homogeneous-agent economies populated only by investor  $A$  or investor  $B$ .

### 3. Characterization of equilibrium

In this section, we provide the characterization of equilibrium. First, we explain our methodology using a simple example with logarithmic investors and no production crises. Then, we extend the methodology to the general economy described in Section 2. Finally, we derive asset prices and the distributions of investor  $A$ 's consumption share  $s_t$  in closed form in the continuous-time limit of the economy.

#### 3.1. Simple example and discussion of methodology

We sketch our methodology in a simple economy with two logarithmic investors and without production crises ( $\lambda_A = \lambda_B = 0$ ). The investors disagree about the probabilities of states  $\omega_1$  and  $\omega_2$ , given by equations (4), and have disagreement parameters  $\delta_B = \delta/2$  and  $\delta_A = -\delta/2$ , where  $\delta > 0$ . We keep the derivation of equilibrium as close as possible to the derivation in the frictionless economy and clearly outline the sources of tractability of our model.

First, we show that the optimal consumptions satisfy the first order conditions (FOCs)

$$\frac{\xi_{i,t+\Delta t}}{\xi_{it}} = e^{-\rho\Delta t} \frac{(c_{i,t+\Delta t}^*)^{-1} + \ell_{i,t+\Delta t}}{(c_{it}^*)^{-1}}, \quad i = A, B, \quad (16)$$

where  $\xi_{it}$  are the investors' state price densities (SPDs) and  $\ell_{i,t+\Delta t} \geq 0$  are the Lagrange multipliers for the collateral constraints  $W_{i,t+\Delta t} \geq 0$ . Conditions (16) are derived in Lemma 1 below using the standard method of Lagrange multipliers by maximizing the expected discounted utility (6) subject to the collateral constraint  $W_{i,t+\Delta t} \geq 0$  and the budget constraint rewritten in the present-value form as follows:

$$c_{it}\Delta t + \mathbb{E}_t^i \left[ \frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t} \right] = W_{it} + l_i D_t \Delta t. \quad (17)$$

The budget constraint (17) simply states that the sum of the current consumption and the present value of the next-period financial wealth should be equal to the sum of the current financial wealth and the labor income.

Next, we provide a heuristic derivation of the dynamics of the state variable (13), which in the economy with log-investors is given by  $v_t = \ln(c_{Bt}^*/c_{At}^*)$ . Consider first the unconstrained region of the state space where the Lagrange multipliers  $\ell_{i,t+\Delta t}$  vanish. Then, the FOCs (16) are the same as in the frictionless economy. Consequently, the dynamics of  $v_t$  in that region is the same as in the frictionless economy, and is given by  $v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t$ , where  $\mu_v = 0$  and  $\sigma_v = \delta + o(\Delta t)$ .<sup>2</sup>

Let  $\bar{v}$  and  $\underline{v}$  denote the values of the state variable when the constraints of investors  $A$  and  $B$  bind, respectively. We argue below that variable  $v_t$  follows dynamics

$$v_{t+\Delta t} = \max\{\underline{v}; \min\{\bar{v}; v_t + \mu_v \Delta t + \sigma_v \Delta w_t\}\}, \quad (18)$$

where  $\mu_v = 0$  and  $\sigma_v = \delta + o(\Delta t)$ . The intuition for the dynamics (18) is that when the investors hit their constraints they consume a fraction of labor income and cannot take risky positions that could lead to further financial losses. Hence, the constraints protect the investors' consumption from falling below a certain limit. More formally, suppose investor  $A$  may hit the collateral constraint next period. Similar to the derivation of the frictionless dynamics of variable  $v_t$ , we find that  $v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t - \ln(1 + \ell_{A,t+\Delta t} c_{A,t+\Delta t}^*)$ , where  $\ell_{A,t+\Delta t} \geq 0$  is the Lagrange multiplier. Hence,  $v_{t+\Delta t} = \bar{v} < v_t + \mu_v \Delta t + \sigma_v \Delta w_t$  if the constraint binds, and  $v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t$  otherwise. The latter two cases imply that  $v_{t+\Delta t} = \min\{\bar{v}, v_t + \mu_v \Delta t + \sigma_v \Delta w_t\}$ . A similar analysis for investor  $B$  gives rise to the lower bound  $\underline{v}$ . Combining the results, we obtain the dynamics (18).

<sup>2</sup>When constraints do not bind and  $\ell_{i,t+\Delta t} = 0$ , FOCs (16) imply that  $v_{t+\Delta t} - v_t = \ln(c_{B,t+\Delta t}^*/c_{Bt}^*) - \ln(c_{A,t+\Delta t}^*/c_{At}^*) = \ln(\xi_{A,t+\Delta t}/\xi_{At}) - \ln(\xi_{B,t+\Delta t}/\xi_{Bt})$ . Moreover, the SPDs are ratios of risk-neutral and physical probabilities so that  $\xi_{i,t+\Delta t}/\xi_{it} = \pi^{RN}(\omega_{t+\Delta t})/\pi_i(\omega_{t+\Delta t})$ . Consequently,  $v_{t+\Delta t} - v_t = \ln(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t}))$ . It can be directly verified that  $\ln(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})) = \mu_v \Delta t + \sigma_v \Delta w_t$ , where  $\mu_v = 0$  and  $\sigma_v = (\ln(1 + 0.5\delta\sqrt{\Delta t}) - \ln(1 - 0.5\delta\sqrt{\Delta t}))/\sqrt{\Delta t} = \delta + o(\Delta t)$ .

The fact that the dynamics (18) of variable  $v_t$  is a capped version of the unconstrained dynamics is the first source of the tractability of our model. The second source of tractability is, as we show next, that the multipliers  $\ell_{i,t+\Delta t}$  cancel out in the expressions for wealths  $W_{it}$ . In particular, substituting SPDs in equation (16) into the budget constraint (17), we notice that the multiplier  $\ell_{i,t+\Delta t}$  cancels out due to the complementary slackness condition  $\ell_{i,t+\Delta t}W_{i,t+\Delta t} = 0$  for the constraint (9), and the budget constraint becomes

$$W_{it} + l_i D_t \Delta t = c_{it} \Delta t + \mathbb{E}_t^i \left[ e^{-\rho \Delta t} \frac{(c_{i,t+\Delta t}^*)^{-1}}{(c_{it}^*)^{-1}} W_{i,t+\Delta t} \right]. \quad (19)$$

From the market clearing condition  $c_{At}^* + c_{Bt}^* = D_t$  and the expression  $v_t = \ln(c_{Bt}^*/c_{At}^*)$ , we find that  $c_{At}^* = D_t/(1 + e^{v_t})$  and  $c_{Bt}^* = D_t e^{v_t}/(1 + e^{v_t})$ . We also rewrite the financial wealths as  $W_{it} = \Phi_i(v_t) c_{it}^*$ , where  $\Phi_i(v_t)$  is the wealth-consumption ratio of investor  $i$ , and substitute into the budget constraint (19). In particular, for investor  $A$ , we find that

$$\Phi_A(v_t) = \mathbb{E}_t^A \left[ e^{-\rho \Delta t} \Phi_A(v_{t+\Delta t}) \right] + (1 - l_A (1 + e^{v_t})) \Delta t. \quad (20)$$

We observe that when  $v_{t+\Delta t} \in (\underline{v}, \bar{v})$ , the dynamics of variable  $v_t$  and the equation (20) are the same as in the frictionless economy. Hence, taking limit  $\Delta t \rightarrow 0$  and proceeding as in the frictionless case, by applying Itô's Lemma or Taylor expansions for small  $\Delta t$ , we find that ratio  $\Phi_A(v_t)$  satisfies the following equation inside the interval  $(\underline{v}, \bar{v})$ :

$$\frac{\delta^2}{2} \Phi_A''(v) - \frac{\delta^2}{2} \Phi_A'(v) - \rho \Phi_A(v) + 1 - l_A (1 + e^v) = 0. \quad (21)$$

The solution of equation (21) is given by

$$\Phi_A(v) = C_{A-} e^{\varphi_- v} + C_{A+} e^{\varphi_+ v} + \frac{1 - l_A}{\rho} - \frac{l_A}{\rho} e^v, \quad (22)$$

where  $\varphi_{\pm} = 0.5 \pm \sqrt{1 + 8\rho/\delta^2}$ , and constants  $C_{A+}$  and  $C_{A-}$  can be found from the boundary conditions  $\Phi_A'(\underline{v}) = \Phi_A'(\bar{v}) = 0$ . The latter boundary conditions can be derived using Taylor expansions of equation (20) near the boundaries. Similarly, for investor  $B$ ,

$$\Phi_B(v) = \left( C_{B-} e^{\varphi_- v} + C_{B+} e^{\varphi_+ v} + \frac{1 - l_B}{\rho} e^v - \frac{l_B}{\rho} \right) e^{-v}. \quad (23)$$

The boundaries are found from the conditions  $W_{it} = 0$  or, equivalently,  $\Phi_A(\bar{v}) = \Phi_B(\underline{v}) = 0$ .

Finally, we find the price-dividend ratio  $\Psi$ . From the market clearing conditions,  $S_t = W_{At} + W_{Bt}$ . Hence, ratio  $\Psi$  can be expressed as  $\Psi_t = (\Phi_{At} s_t + \Phi_{Bt} (1 - s_t)) / (1 - l_A - l_B)$ . After some algebra, using wealth-consumption ratios (22) and (23), we find that

$$\Psi(v) = \frac{C_- e^{\varphi_- v} + C_+ e^{\varphi_+ v}}{1 + e^v} + \frac{1}{\rho}, \quad (24)$$

where constants  $C_{\pm}$  are found in Corollary 1 below from the boundary conditions.

### 3.2. Equilibrium in the general case

First, we derive the investors' state price densities (SPD)  $\xi_{At}$  and  $\xi_{Bt}$  defined as processes such that asset prices can be expressed as follows (Duffie, 2001, p.23):

$$B_t = \mathbb{E}_t^i \left[ \frac{\xi_{i,t+\Delta t}}{\xi_{it}} \right], \quad (25)$$

$$S_t = \mathbb{E}_t^i \left[ \frac{\xi_{i,t+\Delta t}}{\xi_{it}} \left( S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t \right) \right], \quad (26)$$

$$P_t = \mathbb{E}_t^i \left[ \frac{\xi_{i,t+\Delta t}}{\xi_{it}} \mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}} \right], \quad (27)$$

where  $i = A, B$ . The state price density  $\xi_{it}$  exists for each investor  $i$  due to the absence of arbitrage (Duffie, 2001, p.4). There is no arbitrage in the economy because zero-investment strategies with nonnegative payoffs are feasible under constraints (7)–(9). The SPDs  $\xi_{At}$  and  $\xi_{Bt}$  differ due to differences in beliefs and are linked by the change of measure equation<sup>3</sup>

$$\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{\pi_A(\omega_{t+\Delta t})}{\pi_B(\omega_{t+\Delta t})}. \quad (28)$$

We find the SPDs from the investor's first order conditions, reported in Lemma 1.

#### Lemma 1 (The first order condition).

The SPDs  $\xi_{it}$  and optimal consumptions  $c_{it}^*$  satisfy the first order conditions

$$\frac{\xi_{i,t+\Delta t}}{\xi_{it}} = e^{-\rho \Delta t} \frac{(c_{i,t+\Delta t}^*)^{-\gamma_i} + \ell_{i,t+\Delta t}}{(c_{it}^*)^{-\gamma_i}}, \quad (29)$$

where  $\ell_{i,t+\Delta t} \geq 0$  is the Lagrange multiplier for collateral constraint (9) satisfying the complementary slackness condition  $\ell_{i,t+\Delta t} W_{i,t+\Delta t}^* = 0$ .

We use Lemma 1 to derive the dynamics of state variable  $v_t$ , similar to Section 3.1. When constraints do not bind, the Lagrange multipliers  $\ell_{i,t+\Delta t}$  vanish and the conditions (29) are the same as in a frictionless economy. Consequently, the dynamics of the variable  $v_t$  in the unconstrained region of the state-space is the same as in the frictionless economy. Next, let  $\bar{v}$  and  $\underline{v}$  be the values of the state variable  $v_t$  when constraints (9) of investors  $A$  and  $B$  bind, respectively. We show that state variable  $v_t$  stays within boundaries  $\underline{v} \leq v_t \leq \bar{v}$  because collateral constraints limit the investors' losses of wealth and consumption. The boundaries  $\underline{v}$  and  $\bar{v}$  are found from the conditions  $W_{it} = 0$ , which are equivalent to

$$\Phi_A(\bar{v}) = 0, \quad \Phi_B(\underline{v}) = 0, \quad (30)$$

<sup>3</sup>Equations (25)–(27) can be rewritten a system equations for three unknowns  $\pi_i(\omega_k) \xi_{i,t+\Delta t}(\omega_k) / \xi_{it}$ , where  $k = 1, 2, 3$  for a fixed  $i$ . The solution of this system is unique when the matrix of asset payoffs is invertible, and hence,  $\pi_B(\omega_{t+\Delta t}) \xi_{B,t+\Delta t} / \xi_{Bt} = \pi_A(\omega_{t+\Delta t}) \xi_{A,t+\Delta t} / \xi_{At}$  for all states.



where  $\Phi_i(v_t)$  are wealth-consumption ratios given by equations (A26) and (A27) in the Appendix. Proposition 1 below reports the dynamics of  $v_t$ .

**Proposition 1 (Closed-form dynamics of the state variable  $v_t$ ).**

Given the boundaries  $\bar{v}$  and  $\underline{v}$ , the dynamics of the state variable  $v_t$  is given by:

$$v_{t+\Delta t} = \max\left\{\underline{v}; \min\{\bar{v}; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\}\right\}, \quad (31)$$

where drift  $\mu_v$ , volatility  $\sigma_v$ , and jump  $J_v$  are given in closed form by equations (A23)–(A25) in the Appendix. Suppose,  $\gamma_A \neq \gamma_B$  and/or  $\delta_A \neq \delta_B$ . Then, for a sufficiently small  $\Delta t$  the boundaries  $\bar{v}$  and  $\underline{v}$  are reflecting; that is,  $v_t$  does not stay at the boundaries forever.

Dynamics (31) reveals that the constraint does not alter the process for the state variable when the constraint does not bind, and the effects of constraints are captured by the bounds on process  $v_t$ . This property of state variable  $v_t$  plays an important role in establishing the clustering of volatilities and other results in Section 4 below, and is difficult to see using numerical methods. Proposition IA.1 in the Internet Appendix proves the existence of finite time-independent bounds  $\bar{v}$  and  $\underline{v}$  satisfying equations (30). We use the closed-form dynamics (31) to prove the existence and stationarity of equilibrium, derive the SPDs, and study the effects of collateralization on asset prices. Proposition 2 below reports the SPD, asset prices, and their properties.

**Proposition 2 (State price density and the effects on asset prices).**

1) The state price density under the beliefs of investor  $A$  is given by:

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t} \left( \frac{s(v_{t+\Delta t})}{s(v_t)} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} \exp\left(\max\{0; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \bar{v}\}\right), \quad (32)$$

where investor  $A$ 's time- $t$  consumption share  $s(v_t)$  solves equation (14).

2) The price-dividend ratio  $\Psi(v_t)$  is uniformly bounded, the stock price  $S_t$  is given by

$$S_t = (1 - l_A - l_B) D_t \mathbb{E}_t^A \left[ \sum_{\tau=t+\Delta t}^{+\infty} \frac{\xi_{A\tau}}{\xi_{At}} \frac{D_\tau}{D_t} \right], \quad (33)$$

and the prices of the bond and the insurance contract are given by  $B_t = \mathbb{E}_t^A[\xi_{A,t+\Delta t}]/\xi_{At}$  and  $P_t = \mathbb{E}_t^A[\xi_{A,t+\Delta t} 1_{\{\omega_{t+\Delta t}=\omega_3\}}]/\xi_{At}$ , respectively.

3) The prices of bond, stock, and the insurance contract are higher in the economy with collateral constraints than in the frictionless economy, conditional on both economies having the same current output  $D_t$  and the state variable  $v_t$ .

Equation (32) decomposes the SPD into two terms. The first term is the ratio of marginal utilities of investor  $A$  at dates  $t+\Delta t$  and  $t$ ,  $e^{-\rho\Delta t}(s(v_{t+\Delta t})D_{t+\Delta t})^{-\gamma_A}/(s(v_t)D_t)^{-\gamma_A}$ , and is the same as in the frictionless economy. The second term captures the effect of the friction on the SPD, and is only activated when the constraint of investor  $A$  is binding. The SPD of investor  $B$  can be obtained along the same lines.

Proposition 2 also demonstrates that collateralization inflates asset prices. This is because the SPD in the constrained economy exceeds its counterpart in the frictionless economy due to the positive Lagrange multiplier  $\ell_{i,t+\Delta t}$  in the first order condition (29). We discuss the intuition in Section 4.1. Proposition IA.2 in the Internet Appendix provides the verification theorem for the optimality of investors' optimal strategies.

### 3.3. Closed-form solution in a continuous-time limit

Next, we take continuous-time limit  $\Delta t \rightarrow 0$  and derive the equilibrium in closed form. Taking the limit allows rewriting equations for the price-dividend and wealth-consumption ratios as differential-difference equations. For tractability, we derive these ratios in terms of a transformed ratio  $\widehat{\Psi}(v; \theta)$ , which satisfies a simpler equation reported in Lemma 2.

**Lemma 2 (Differential-difference equation).** *In the limit  $\Delta t \rightarrow 0$ , the price-dividend ratio  $\Psi$  and wealth-consumption ratios  $\Phi_i$  are given by:*

$$\Psi(v) = \widehat{\Psi}(v; -\gamma_A)s(v)^{\gamma_A}, \quad (34)$$

$$\Phi_i(v) = \frac{(\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}})\widehat{\Psi}(v; 1 - \gamma_A) + (\mathbf{1}_{\{i=B\}} - l_i)\widehat{\Psi}(v; -\gamma_A)}{\mathbf{1}_{\{i=A\}}s(v) + \mathbf{1}_{\{i=B\}}(1 - s(v))}s(v)^{\gamma_A}, \quad (35)$$

where  $s(v)$  solves equation (14) and  $\widehat{\Psi}(v; \theta)$  satisfies a differential-difference equation

$$\begin{aligned} & \frac{\widehat{\sigma}_v^2}{2}\widehat{\Psi}''(v; \theta) + (\widehat{\mu}_v + \delta_A\widehat{\sigma}_v + (1 - \gamma_A)\sigma_D\widehat{\sigma}_v)\widehat{\Psi}'(v; \theta) \\ & - \left( \lambda_A + \rho - (1 - \gamma_A)(\mu_D + \delta_A\sigma_D) + \frac{(1 - \gamma_A)\gamma_A}{2}\sigma_D^2 \right) \widehat{\Psi}(v; \theta) \\ & + \lambda_A(1 + J_D)^{1-\gamma_A}\widehat{\Psi}(\max\{v; v + \widehat{J}_v\}; \theta) + s(v)^\theta = 0, \end{aligned} \quad (36)$$

subject to the reflecting boundary conditions

$$\widehat{\Psi}(\underline{v}; \theta) = 0, \quad \widehat{\Psi}'(\bar{v}; \theta) - \widehat{\Psi}(\bar{v}; \theta) = 0, \quad (37)$$

where  $\widehat{\mu}_v, \widehat{\sigma}_v \geq 0$ , and  $\widehat{J}_v \leq 0$  are constants given by:

$$\widehat{\mu}_v = (\gamma_A - \gamma_B) \left( \mu_D - \frac{\sigma_D^2}{2} \right) + \lambda_A - \lambda_B + \frac{\delta_A^2 - \delta_B^2}{2}, \quad (38)$$

$$\hat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta_B - \delta_A, \quad (39)$$

$$\hat{J}_v = (\gamma_A - \gamma_B)\ln(1 + J_D) + \ln\left(\frac{\lambda_B}{\lambda_A}\right). \quad (40)$$

The boundaries  $\bar{v}$  and  $\underline{v}$  solve equations

$$\frac{\hat{\Psi}(\bar{v}; 1 - \gamma_A)}{\hat{\Psi}(\bar{v}; -\gamma_A)} = l_A, \quad \frac{\hat{\Psi}(\underline{v}; 1 - \gamma_A)}{\hat{\Psi}(\underline{v}; -\gamma_A)} = 1 - l_B. \quad (41)$$

We observe that equation (36) is linear, in contrast to economies with constraints directly imposed on trading strategies of investors (e.g., Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015; Rytchkov, 2014), and is a differential-difference equation with a “delayed” argument  $v + \hat{J}_v$  in the fourth term, where  $\hat{J}_v \leq 0$ . This term is further complicated by the fact that the argument is restricted to stay above the boundary  $\underline{v}$ , which gives rise to the term with a peculiar argument  $\max\{\underline{v}; v + \hat{J}_v\}$ . This term captures the investors’ decisions in anticipation of hitting their constraint.

Before deriving the equilibrium in the general case, we provide analytical price-dividend ratios for a special case in Corollary 1 below.

**Corollary 1 (Analytical P/D ratios).** *Suppose  $\lambda_A = \lambda_B = \lambda > 0$  and  $\gamma_A = \gamma_B = \gamma$ , where  $\gamma$  is an integer. Then, price-dividend ratio  $\Psi(v)$  is given by:*

$$\Psi(v) = \frac{1}{(1 + e^{v/\gamma})^\gamma} \left( C_- e^{\varphi_- v} + C_+ e^{\varphi_+ v} + \sum_{k=0}^{\gamma} \binom{\gamma}{k} \frac{e^{kv/\gamma}}{h(k/\gamma)} \right), \quad (42)$$

where  $h(\varphi)$  is a characteristic polynomial of equation (36) given by

$$h(\varphi) = \rho - (1 - \gamma)(\mu_D + \delta_A \sigma_D) + \frac{(1 - \gamma)\gamma}{2} \sigma_D^2 + \lambda(1 - (1 + J_D)^{1-\gamma}) - (\hat{\mu}_v + \delta_A \hat{\sigma}_v + (1 - \gamma)\sigma_D \hat{\sigma}_v)\varphi - \frac{\hat{\sigma}_v^2 \varphi^2}{2}, \quad (43)$$

$\varphi_-$  and  $\varphi_+$  are a negative and positive solutions of equation  $h(\varphi) = 0$ , and constants  $C_\pm$  are given by equations (A59) and (A60) in the Appendix, respectively.

In Section 4 below, we argue that the analytical ratio (42) captures some important properties of price-dividend ratio, which also hold in the general case with arbitrary risk aversions and crises intensities. Hence, this special case can be used as a tractable benchmark in asset pricing research. Proposition IA.3 in the Internet Appendix presents the closed-form price-dividend ratio for general CRRA risk aversions and beliefs. Although

the general closed-form solution is complex, it provides a constructive proof for the existence of price-dividend ratios. We also solve equations (36)–(37) using the method of finite differences, and double-check that the numerical and closed-form solutions coincide.

We call the interval  $v \in [\underline{v}, \underline{v} - \widehat{J}_v]$  in the state-space a period of *anxious economy*, similar to Fostel and Geanakoplos (2008), albeit the investor disagreement does not increase during these periods as in the latter paper. When the economy falls into this state, even a small possibility of a crisis renders the collateral constraint binding and leads to deleveraging. To explore the economic effects of the anxious economy, using the SPD (32), we derive the interest rates  $r_t$  and stock risk premium in normal times  $\mu_t - r_t$  in Proposition 3 below.

**Proposition 3 (Interest rates and risk premiums in the limit).** *For a sufficiently small interval  $\Delta t$ , the interest rate  $r_t$  and the risk premium  $\mu_t - r_t$  in normal times are given by:*

$$r_t = \begin{cases} \tilde{r}_t + \lambda_A - \lambda_A(1 + J_D)^{-\gamma_A} \left( \frac{s(\max\{\underline{v}; v_t + \widehat{J}_v\})}{s_t} \right)^{-\gamma_A} + O(\Delta t), & \text{for } \underline{v} < v_t < \bar{v}, \\ - \left( \frac{s_t \mathbf{1}_{\{v_t = \bar{v}\}}}{\gamma_A} + \frac{(1 - s_t) \mathbf{1}_{\{v_t = \underline{v}\}}}{\gamma_B} \right) \frac{|\widehat{\sigma}_v| \Gamma_t}{2\sqrt{\Delta t}} + O(1), & \text{for } v = \underline{v} \text{ or } v = \bar{v}, \end{cases} \quad (44)$$

$$\begin{aligned} \mu_t - r_t = & \Gamma_t \left( \sigma_D - \frac{s_t \delta_A}{\gamma_A} - \frac{1 - s_t}{\gamma_B} \delta_B + \frac{|\widehat{\sigma}_v|}{2} \left( \frac{(1 - s_t) \mathbf{1}_{\{v = \underline{v}\}}}{\gamma_B} - \frac{s_t \mathbf{1}_{\{v = \bar{v}\}}}{\gamma_A} \right) \right) \sigma_t \\ & - \lambda_A(1 + J_D)^{-\gamma_A} J_t \left( \frac{s(\max\{\underline{v}; v_t + \widehat{J}_v\})}{s_t} \right)^{-\gamma_A} + O(\sqrt{\Delta t}), \end{aligned} \quad (45)$$

where  $\tilde{r}_t$  is the interest rate in the unconstrained economy without crisis risk, given by:

$$\begin{aligned} \tilde{r}_t = & \rho + \gamma_A(\mu_D + \delta_A \sigma_D) - \frac{\gamma_A(1 + \gamma_A)}{2} \sigma_D^2 + \left( \frac{\gamma_A \sigma_D \widehat{\sigma}_v - (\widehat{\mu}_v + \delta_A \widehat{\sigma}_v)}{\gamma_B} \right) (1 - s_t) \Gamma_t \\ & - \widehat{\sigma}_v^2 \left( \frac{1}{2\gamma_B^2} (1 - s_t)^2 \Gamma_t^2 + \frac{1}{2\gamma_A^2 \gamma_B^2} s_t (1 - s_t) \Gamma_t^3 \right), \end{aligned} \quad (46)$$

drift  $\widehat{\mu}_v$ , volatility  $\widehat{\sigma}_v$ , and  $\widehat{J}_v$  of variable  $v$  are given by equations (38)–(40), volatility  $\sigma_t$  in normal times and jump size  $J_t$  are given by equations (A72)–(A73) in the Appendix, and  $\Gamma_t \equiv \gamma_A \gamma_B / (\gamma_A(1 - s_t) + \gamma_B s_t)$  is the risk aversion of a representative investor.

The effects of collateral constraints on interest rates and risk premiums arise due to the investors' concern that a potential crisis can render the constraint binding next period when the economy is close to boundary  $\underline{v}$ . The third term in the first equation in (44) for

the interest rate quantifies the impact of collateralization on precautionary savings due to a downward jump in the aggregate consumption, which we further discuss in Section 4.

Equations (44) and (45) also feature terms with indicator functions  $\mathbf{1}_{\{v=\underline{v}\}}$  and  $\mathbf{1}_{\{v=\bar{v}\}}$ , which are nonzero only at the boundaries  $\underline{v}$  and  $\bar{v}$ . For the interest rate  $r_t$  these terms have the order of magnitude proportional to  $1/\sqrt{\Delta t}$ , and hence, the interest rate has singularities at the boundaries  $\underline{v}$  and  $\bar{v}$  when  $\Delta t \rightarrow 0$ . The intuition is that near the boundaries  $\underline{v}$  and  $\bar{v}$  even a small shock  $\Delta w_t$  can lead to a default. Thus, when the investor's constraint binds at time  $t$ , this investor allocates a larger fraction of income to the riskless asset than in the interior region  $\underline{v} < v_t < \bar{v}$ , which decreases interest rates.

Similar singularities arise in a continuous-time model of Detemple and Serrat (2003). Our discrete-time analysis sheds new light on these singularities by uncovering their order of magnitude  $1/\sqrt{\Delta t}$ . Consequently, the per-period rate  $r_t \Delta t$  is finite and has an order of magnitude  $O(\sqrt{\Delta t})$ . Moreover, in contrast to the latter paper, due to production crises, the collateralization in our model affects the interest rates and risk premiums not only at the boundaries but also for the whole period of anxious economy.

### 3.4. Stationary distribution of consumption share

Absent any frictions, the state variable  $v$  follows an arithmetic Brownian motion with a jump. This process is nonstationary and induces nonstationarity in the unconstrained equilibrium where one of the investor's consumption share gradually converges to zero. As a result, with the exception of some knife-edge parameter combinations, only one of the investors has a significant impact on asset prices in the frictionless economy in the long run (e.g., Blume and Easley, 2006; Yan, 2008; Chabakauri, 2015).

Collateral constraints (9) help both investors survive in the long run by protecting them against losing their shares of aggregate consumption beyond certain limits, similar to the previous literature on survival (e.g., Blume and Easley, 2006; Cao, 2018, among others). Our contribution is that we derive the probability density function (PDF) of consumption share  $s$  in closed form, show that this PDF is stationary and nondegenerate, and find parameters that determine its shape. The latter result is important because it implies nontrivial time-variation of asset prices in the long run. For tractability, we assume that there are no production crises so that  $\lambda_A = \lambda_B = 0$ . Proposition 4 reports the results.

**Proposition 4 (Stationary distribution of consumption share).** *Suppose,  $\lambda_A =$*

$\lambda_B = 0$ . Then, the PDF  $f(s, \tau; s_t; t)$  of consumption share  $s$  at time  $\tau$  conditional on observing share  $s_t$  at time  $t$  is given in closed form by expression (A84) in the Appendix. Furthermore, the stationary PDF of consumption share  $s$  is given by:

$$f(s) = \begin{cases} \frac{2\hat{\mu}_v}{\hat{\sigma}_v^2} \left( \frac{\gamma_A}{s} + \frac{\gamma_B}{1-s} \right) \frac{\left( (1-s)^{\gamma_B} / s^{\gamma_A} \right)^{2\hat{\mu}_v / \hat{\sigma}_v^2} \mathbf{1}_{\{\underline{s} \leq s \leq \bar{s}\}}}{\left( (1-\underline{s})^{\gamma_B} / \underline{s}^{\gamma_A} \right)^{2\hat{\mu}_v / \hat{\sigma}_v^2} - \left( (1-\bar{s})^{\gamma_B} / \bar{s}^{\gamma_A} \right)^{2\hat{\mu}_v / \hat{\sigma}_v^2}}, & \text{if } \hat{\mu}_v \neq 0, \\ \left( \frac{\gamma_A}{s} + \frac{\gamma_B}{1-s} \right) \frac{\mathbf{1}_{\{\underline{s} \leq s \leq \bar{s}\}}}{\gamma_B \ln \left( \frac{1-\underline{s}}{1-\bar{s}} \right) + \gamma_A \ln \left( \frac{\bar{s}}{\underline{s}} \right)}, & \text{if } \hat{\mu}_v = 0, \end{cases} \quad (47)$$

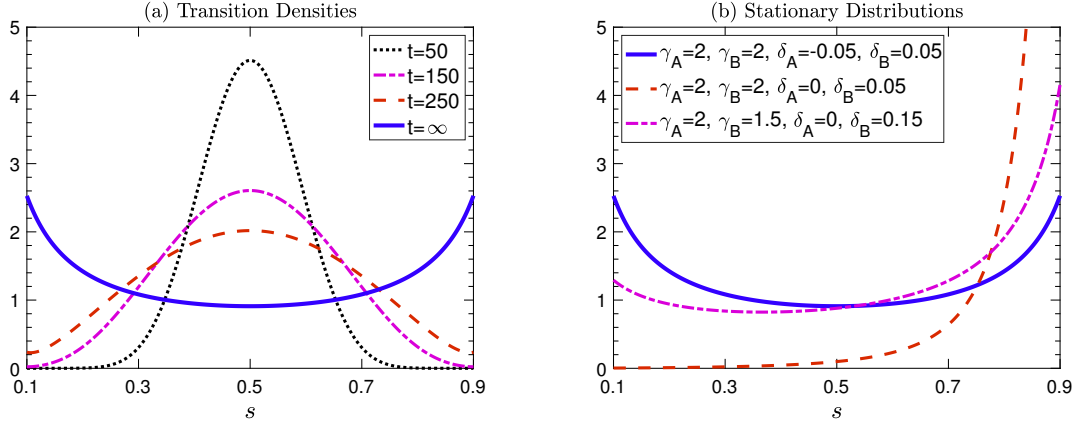
where  $\hat{\mu}_v = (\gamma_A - \gamma_B)(\mu_D - \sigma_D^2/2) + (\delta_A^2 - \delta_B^2)/2$ ,  $\hat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta_B - \delta_A$ ,  $\mathbf{1}_{\{\underline{s} \leq s \leq \bar{s}\}}$  is an indicator function, and  $\underline{s}$  and  $\bar{s}$  are the bounds on the consumption share  $s$ , which solve equation (14) for  $\bar{v}$  and  $\underline{v}$ , respectively.

Proposition 4 confirms that both investors survive in the long run, and that consumption share  $s$  has a well-defined stationary distribution. The beliefs enter PDF (47) via the ratio of the drift and variance of process  $v_t$ , given by  $\hat{\mu}_v / \hat{\sigma}_v^2$ . This ratio determines the relative dominance of investors in the economy. In particular, for bounds  $\underline{s}$  and  $\bar{s}$  that are symmetric around 0.5, the PDF is concentrated around  $\underline{s}$  if  $\hat{\mu}_v > 0$  and around  $\bar{s}$  if  $\hat{\mu}_v < 0$ . We note that the drift of state variable  $v$  can be rewritten as  $\hat{\mu}_v = (\gamma_A - \gamma_B)(\mu_D - \sigma_D^2/2) + (\delta_A - \delta_B)(\delta_A + \delta_B)/2$ , and hence, the drift is influenced both by the dispersion of beliefs  $\delta_A - \delta_B$  and the average bias  $(\delta_A + \delta_B)/2$ . In the special case  $\gamma_A = \gamma_B$  the relative dominance of investors is determined by a simple ratio of the average bias and the dispersion of beliefs:

$$\frac{\hat{\mu}_v}{\hat{\sigma}_v^2} = \frac{(\delta_A + \delta_B)/2}{\delta_B - \delta_A}. \quad (48)$$

In another special case where risk aversions are different but beliefs are correct on average, that is,  $(\delta_A + \delta_B)/2 = 0$ , the drift  $\hat{\mu}_v$  is determined by differences in risk aversions.

Figure 2a shows the stationary PDF and transition densities  $f(s, t; s_0, 0)$  when investors have the same risk aversions  $\gamma_A = \gamma_B = 2$  and opposite beliefs  $\delta_A = -\delta_B = -0.05$ . Other model parameters are described in the legend of the figure. From equation (5) for expected dividend growth rate, we observe that the latter disagreement parameters  $\delta_i$  imply that investor  $A$  ( $B$ ) believes that expected output growth is approximately 10% lower (higher) than under the true probabilities. The stationary PDF is symmetric and bimodal, so that both investors occasionally have large consumption share. Figure 2b shows two additional



**Figure 2**

**Convergence to stationary distribution of consumption share  $s_t = c_{A,t}^*/D_t$**

Figure 2a shows transition densities  $f(s, t; s_0, 0)$  for the starting point  $s_0 = 0.5$  and the stationary distribution  $f(s)$  (i.e., density for  $t = \infty$ ). We set  $\gamma_A = 2$ ,  $\gamma_B = 2$ ,  $\mu_D = 0.018$ ,  $\sigma_D = 0.032$ ,  $\lambda_A = \lambda_B = 0$ ,  $\rho = 0.02$ ,  $\delta_A = -0.05$  (i.e.,  $\mathbb{E}^A[\Delta D_t/D_t|\text{normal}] \approx 0.9\mu_D$ ),  $\delta_B = 0.05$  (i.e.,  $\mathbb{E}^B[\Delta D_t/D_t|\text{normal}] \approx 1.1\mu_D$ ),  $\underline{s} = 0.1$ ,  $\bar{s} = 0.9$ ,  $l_A = 0.1257$ , and  $l_B = 0.1178$ . Figure 2b shows stationary distributions for different sets of model parameters.

distributions when investor  $A$  has correct beliefs ( $\delta_A = 0$ ) and investor  $B$  is optimistic ( $\delta_A > 0$ ). In the case  $\gamma_A = \gamma_B = 2$ ,  $\delta_A = 0$ ,  $\delta_B = 0.05$  the stationary distribution peaks at  $s = 0.9$  which means that the rational investor  $A$  has large consumption share in the economy. In the case  $\gamma_A = 2$ ,  $\gamma_B = 1.5$ ,  $\delta_A = 0$ ,  $\delta_B = 0.15$  the stationary distribution is bimodal so that both rational and overly optimistic investors occasionally have large consumption shares in the economy.

The economic implication of the bimodality of the stationary PDF under some model parameters is that the periods of binding constraints are likely to be persistent. The closed-form dynamics (31) for the state variable  $v$  helps explain the bimodality of the PDF. From this dynamics, we observe that after hitting a boundary the process  $v_t$  remains in its vicinity for some time. Hence, because variable  $v$  follows an arithmetic Brownian motion in the interval  $(\underline{v}, \bar{v})$ , the probability of hitting the same boundary again is high.

## 4. Analysis of Equilibrium

In this section, we demonstrate the economic implications of our model. In Section 4.1, we show that capital constraints amplify the effect of rare crises on generating lower interest rates and higher Sharpe ratios, lead to spikes and crashes of stock prices and stock return

volatilities, amplify volatility in good times and decrease it in bad times, and generate volatility clusters. Section 4.2 measures the economic significance of collateralization by quantifying the collateral premium of the stock.

We study the equilibrium for calibrated parameters. We set the parameters of the aggregate consumption process to  $\mu_D = 0.018$ ,  $\sigma_D = 0.032$ ,  $J_D = -0.25$ , and the crisis intensities of investors  $A$  and  $B$  to  $\lambda_A = 0.02$  and  $\lambda_B = 0.01$ , respectively. The risk aversions are  $\gamma_A = \gamma_B = 2$ , and the time discount is  $\rho = 0.02$ . The disagreement parameters are  $\delta_A = -\delta_B = -0.05$ , and they correspond to the case in which investor  $A$  ( $B$ ) believes that the mean growth rate given by (5) is approximately equal to  $0.9\mu_D$  ( $1.1\mu_D$ ). The shares of labor income  $l_A = 0.1257$  and  $l_B = 0.1178$  are chosen to generate symmetric bounds on investor  $A$ 's consumption share:  $\underline{s} = 0.1$  and  $\bar{s} = 0.9$ .<sup>4</sup> We note that all our qualitative results on interest rates, market prices of risk, price-dividend ratios, and stock return volatilities remain the same when investors have different risk aversions and identical beliefs or when both risk aversions and beliefs are different. Moreover, the results do not depend on whether the more risk averse investor is more pessimistic or optimistic than the less risk averse investor.<sup>5</sup>

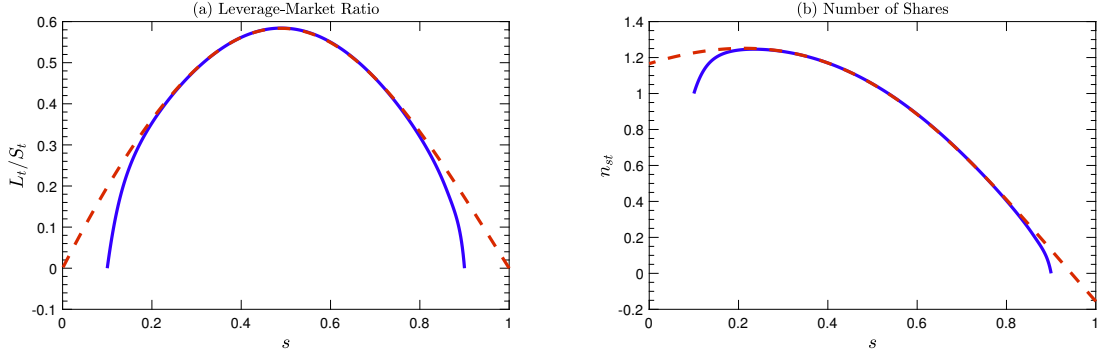
We plot the equilibrium processes as functions of consumption share  $s_t = c_{At}^*/D_t$  because  $s$  conveniently lies in the interval  $(0, 1)$  and is more intuitive than variable  $v$ . We observe that consumption share  $s$  is countercyclical in the sense that  $\text{corr}_t(ds_t, dD_t) < 0$ . Intuitively, the aggregate wealth and consumption shift to (away from) investor  $A$  following negative (positive) shocks to output because this investor is more pessimistic than investor  $B$ . We call a process procyclical (countercyclical) if that process is a decreasing (increasing) function of  $s$ . We interpret periods of low (high)  $s_t$  as good (bad) times in the economy, because during these periods the output  $D_t$  is high (low).

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<sup>4</sup>Drift  $\mu_D$  and volatility  $\sigma_D$  are within the ranges considered in the literature (e.g., Basak and Cuoco, 1998; Chan and Kogan, 2002; Rytchkov, 2014). To avoid finding bounds  $\underline{s}$  and  $\bar{s}$  numerically, we set them exogenously to  $\underline{s} = 0.1$  and  $\bar{s} = 0.9$  and then recover the shares of labor incomes  $l_A = 0.1257$  and  $l_B = 0.1178$ , which imply these bounds in equilibrium. First, we find  $\underline{v}$  and  $\bar{v}$  from equation (14) for state variable  $v$ , and then find  $l_A$  and  $l_B$  from equations (41).

<sup>5</sup>Section IA.2 of the Internet Appendix presents two additional examples of equilibria. The first example shows the equilibrium processes in the economy where investors have different risk aversions but identical beliefs. The second example shows these processes when both risk aversions and beliefs are different, and the less risk averse investor is also more pessimistic than the more risk averse investor. The equilibrium processes in these economies have the same features as the processes in our baseline calibration in Section 4.1. In particular, interest rates are lower and Sharpe ratios are higher during anxious times, the price dividend ratios are U-shaped and sensitive to small shocks near the boundaries  $\underline{s}$  and  $\bar{s}$ , and stock return volatilities are higher in good times and lower in bad times.





**Figure 3**

**Leverage and stock holdings of optimistic and less risk averse investor  $B$**

Panels (a) and (b) depict optimistic investor  $B$ 's leverage/market price ratio  $L_t/S_t$  and the number of shares  $n_{st}$ , respectively, as functions of consumption share  $s_t = c_{At}^*/D_t$ . The solid and dashed lines correspond to constrained and unconstrained economies, respectively. The model parameters are  $\gamma_A = 2$ ,  $\gamma_B = 2$ ,  $\rho = 0.02$ ,  $\mu_D = 0.018$ ,  $\sigma_D = 0.032$ ,  $J_D = -0.25$ ,  $\lambda_A = 0.02$ ,  $\lambda_B = 0.01$ ,  $\delta_A = -0.05$ ,  $\delta_B = 0.05$ ,  $\underline{s} = 0.1$ ,  $\bar{s} = 0.9$ ,  $l_A = 0.1257$ , and  $l_B = 0.1178$ .

### 4.1. Equilibrium processes

Figure 3 depicts investor  $B$ 's leverage/market ratio  $L_t/S_t$  and stock holdings  $n_{Bt}$  in the constrained (solid line) and unconstrained (dashed line) economies. Figure 3a demonstrates the cyclicity of leverage. The leverage is lowest when either investor  $A$  or investor  $B$  bind on their constraints. Intuitively, when  $s = \bar{s}$ , investor  $B$ 's financial wealth is zero, and hence,  $B$  lacks collateral and cannot borrow. When  $s = \underline{s}$ , investor  $A$ 's financial wealth is zero and the labor income  $l_A D_t \Delta t$  is infinitesimally small in the continuous-time limit. The liquidity dries up because investor  $A$  cannot supply credit. The leverage cycles are present only in the constrained economy, and do not occur in the unconstrained economy where the state variable  $s$  is nonstationary and gradually converges to 0 or 1.

Figure 3b presents the number of stocks held by investor  $B$ . Consider first the unconstrained economy with pledgeable labor income. From Figure 3b, we observe that the optimistic investor  $B$  shorts stocks in the unconstrained economy when consumption share  $s$  is close to 1. The intuition is that in bad times, following a sequence of negative shocks to output, investor  $B$  shorts stocks to finance consumption and backs short positions by the pledgeable labor income. The stream of labor income  $l_B D_t \Delta t$  is equivalent to dividends from holding  $\hat{n}_B = l_B / (1 - l_A - l_B)$  units of nontradable shares in the Lucas tree. Short-selling allows to circumvent the nontradability of labor income and freely adjust the effective share  $\hat{n}_B + n_{B,st}$  in the Lucas tree. Overcoming the nontradability of labor incomes

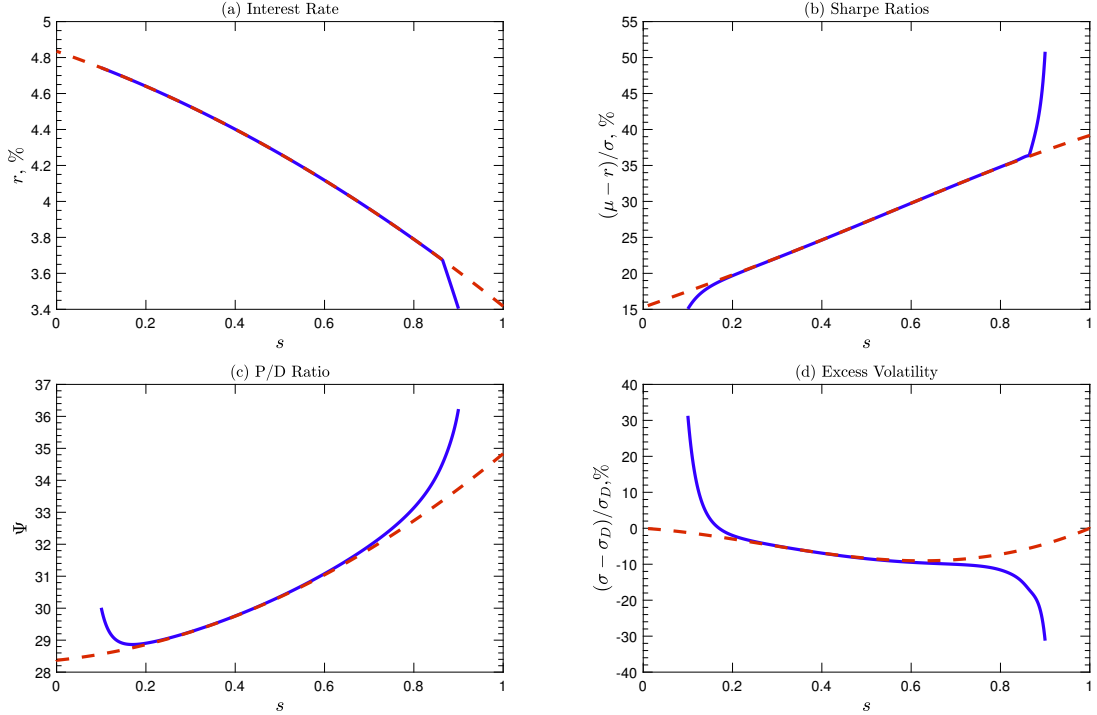
makes this economy similar to the nonstationary unconstrained economy. The financial wealth can then become negative. In the constrained economy, the nonnegative wealth constraint precludes investor  $B$  from shorting. The trading strategy of investor  $A$  equals  $1 - n_{Bt}^*$  in equilibrium and can be analyzed similarly. Investor  $A$  also has an additional motive to short stocks due to being more pessimistic than investor  $B$ .

Figure 4 depicts the interest rate  $r_t$ , Sharpe ratio  $(\mu_t - r_t)/\sigma_t$ , price-dividend ratio  $\Psi$ , and excess stock return volatility  $(\sigma_t - \sigma_D)/\sigma_D$  in the constrained (solid line) and unconstrained (dashed line) economies. Figure 4a shows the interest rate  $r_t$ .<sup>6</sup> The interest rate declines sharply when the economy enters into an anxious state close to the boundary  $\bar{s}$  where even a small possibility of a crisis next period makes the constraint of investor  $B$  binding. The intuition is as follows. In the unconstrained economy, a crisis around state  $\bar{s}$  generates wealth transfer to the pessimistic investor  $A$  and increases her consumption share  $s$  above  $\bar{s}$ . In the constrained economy, consumption share  $s$  is capped by  $\bar{s}$ . Consequently, following a crisis, investor  $A$ 's marginal utility  $(c_A^*)^{-\gamma_A}$  is higher in the constrained than in the unconstrained economy. As a result, investor  $A$  is more willing to smooth consumption in the constrained economy, and hence, the interest rate declines due to the precautionary savings motive. In particular, the investor buys more bonds, which drives interest rates down. Figure 4b shows that the Sharpe ratio increases to compensate investor  $A$  for buying risky assets from investor  $B$  in bad times when consumption share  $s$  is large, and slightly decreases in good times due to high volatility (as discussed below). We note that the decreases in interest rates and increases in Sharpe ratios during anxious times arise only when both the crises and the constraints are simultaneously present, and hence, the crises and constraints reinforce the effects of each other. Equation (44) for the interest rate and equation (45) for the risk premium show that absent any crises ( $\lambda_A = \lambda_B = 0$ ), the constraints affect  $r_t$  and  $\mu_t - r_t$  only at the boundaries of the state-space.

From Figure 4c, we observe that the collateral constraints increase price-dividend ratio  $\Psi$  relative to the unconstrained economy, in line with Proposition 2. The price-dividend ratio is also a decreasing (increasing) function of consumption share  $s$  near the boundary  $\underline{s}$  ( $\bar{s}$ ), which makes it U-shaped and sensitive to small shocks when the constraints are likely to bind. Proposition IA.5 in the Internet Appendix shows that the latter property holds for all model parameters under which the equilibrium exists. We further note that price-

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<sup>6</sup>We exclude the singularities in the dynamics of  $r_t$  and focus on the dynamics in the unconstrained region because the economy spends an infinitesimal amount of time at the boundaries.



**Figure 4**  
**Equilibrium processes**

Panels (a)–(d) show interest rate  $r_t$ , Sharpe ratio  $(\mu_t - r_t)/\sigma_t$ , price-dividend ratio  $\Psi_t$ , and excess volatility  $(\sigma_t - \sigma_D)/\sigma_D$  as functions of  $s_t = c_{At}^*/D_t$  for the constrained (solid lines) and unconstrained (dashed lines) economies. The model parameters are the same as in Figure 3.

dividend ratio  $\Psi$  in Figure 4c is a convex function of consumption share  $s$ , which reinforces its resemblance to a U-shape. Proposition IA.5 also establishes sufficient conditions for the convexity of the price-dividend ratio in the unconstrained economy  $\Psi^{unc}$  and the difference  $\Psi - \Psi^{unc}$ . Ratio  $\Psi$  is then convex as the sum of the latter two convex components. In particular, Proposition IA.5 shows that ratio  $\Psi^{unc}$  is convex when  $\gamma_A = \gamma_B \geq 1$  and the difference  $\Psi - \Psi^{unc}$  is convex under an additional restriction on crises intensities  $\lambda_A = \lambda_B > 0$  (or when  $\lambda_A = \lambda_B = 0$  and  $\gamma_A \geq \gamma_B \geq 1$ ).<sup>7</sup> We note that the latter conditions are satisfied for the analytical price-dividend ratio (42) in Corollary 1.

Next, we discuss the intuition for the fact that price-dividend ratios in the constrained economy increase more at the boundaries than in the interior of the interval  $(\underline{s}, \bar{s})$  and are decreasing (increasing) functions of consumption share  $s$  near  $\underline{s}$  ( $\bar{s}$ ). Suppose, consumption share  $s$  is close to the boundary  $\bar{s}$ , where investor  $B$ 's constraint is binding but investor

<sup>7</sup>A nonconvex ratio  $\Psi$  emerges, for example, when  $\gamma_A = \gamma_B < 1$  and  $\underline{s} = 0.01$  and  $\bar{s} = 0.99$  because the corresponding unconstrained price-dividend ratio  $\Psi^{unc}$  is concave, and ratio  $\Psi$  closely follows its contour inside the interval  $(\underline{s}, \bar{s})$  and then spikes up at the boundaries giving rise to nonconvexities.

$A$  is unconstrained. Because investor  $A$ 's constraint is loose the state price density  $\xi_{At}$  is proportional to investor  $A$ 's marginal utility  $(c_{At}^*)^{-\gamma_A}$ . In the constrained economy the consumption share of investor  $A$  is capped at  $\bar{s} < 1$  whereas in the unconstrained economy it can increase above  $\bar{s}$ . Therefore, the marginal utility of investor  $A$  and, hence, the state price density are expected to be higher in the constrained than in the unconstrained economy. Hence, stocks are more valuable in the constrained economy around the boundary  $\bar{s}$ . The intuition around  $\underline{s}$  can be analyzed in a similar way. An additional economic force (explored in Section 4.2 below) contributing to a higher stock price is that the stock can be used as collateral that helps relax the constraint, which gives rise to a premium.

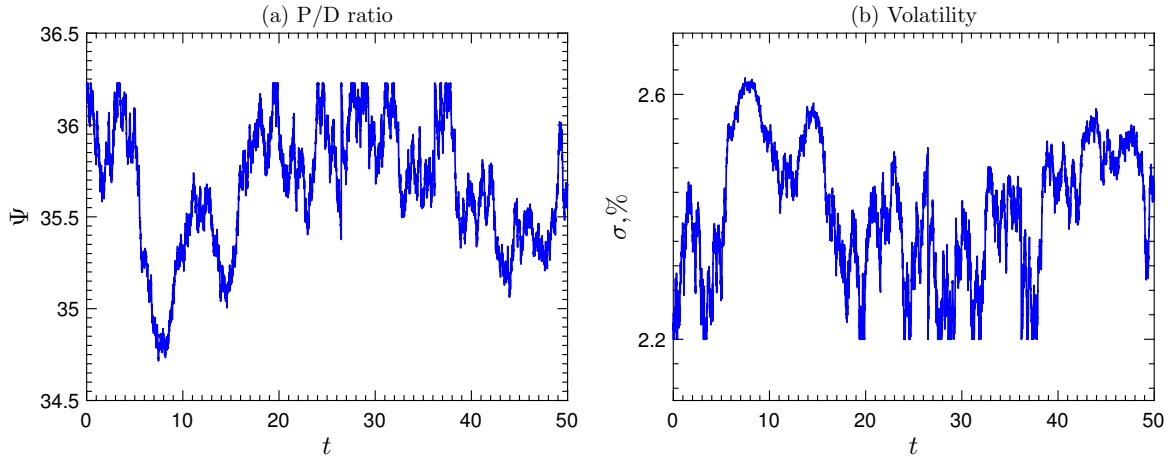
The results in Figure 4d demonstrate that the constraint makes volatility more procyclical, reducing it in bad times (around  $\bar{s}$ ) and increasing it in good times (around  $\underline{s}$ ). This is because U-shaped price-dividend ratio in the constrained economy is more procyclical in good times (i.e., around  $\underline{s}$ ) and more countercyclical in bad times (i.e., around  $\bar{s}$ ) than in the unconstrained economy. Stock price  $S_t = \Psi_t D_t$  is more volatile in good times (around  $\underline{s}$ ) because both  $\Psi_t$  and  $D_t$  change in the same direction, and is less volatile in bad times (around  $\bar{s}$ ) because  $\Psi_t$  and  $D_t$  change in opposite directions and partially offset the effects of each other. Lower volatility in bad times is in line with the previous literature on the effects of portfolio constraints on asset prices (e.g., Chabakauri, 2013, 2015; Brunnermeier and Sannikov, 2014, among others). The empirical literature finds that volatility tends to be higher in bad times (e.g., Schwert, 1989). However, high volatility can be explained by high uncertainty about the economic growth and learning effects in bad times (e.g., Veronesi, 1999), which we do not study in this paper to focus on the effects of collateral constraints that are not confounded by other effects.

Boundary conditions (37) allow us to explore volatility  $\sigma_t$  near the boundaries  $\underline{s}$  and  $\bar{s}$  using closed-form expressions in Corollary 2 below.

**Corollary 2 (Stock return volatility at the boundaries).** *Stock return volatility in normal times  $\sigma_t$  satisfies the following boundary conditions:*

$$\sigma(\underline{s}) = \sigma_D + \frac{\gamma_B \underline{s} \hat{\sigma}_v}{\gamma_A(1 - \underline{s}) + \gamma_B \underline{s}} > \sigma_D, \quad \sigma(\bar{s}) = \sigma_D - \frac{\gamma_A(1 - \bar{s}) \hat{\sigma}_v}{\gamma_A(1 - \bar{s}) + \gamma_B \bar{s}} < \sigma_D. \quad (49)$$

By continuity, inequalities (49) also hold in the vicinity of the boundaries. Figure 4d shows that volatility  $\sigma_t$  is steep at the boundaries: it spikes close to  $\underline{s}$  and crashes close to  $\bar{s}$ , consistent with Corollary 2. It also evolves in three regimes of low, medium, and



**Figure 5**

**Simulated P/D ratio  $\Psi$  and stock return volatility  $\sigma$  over time**

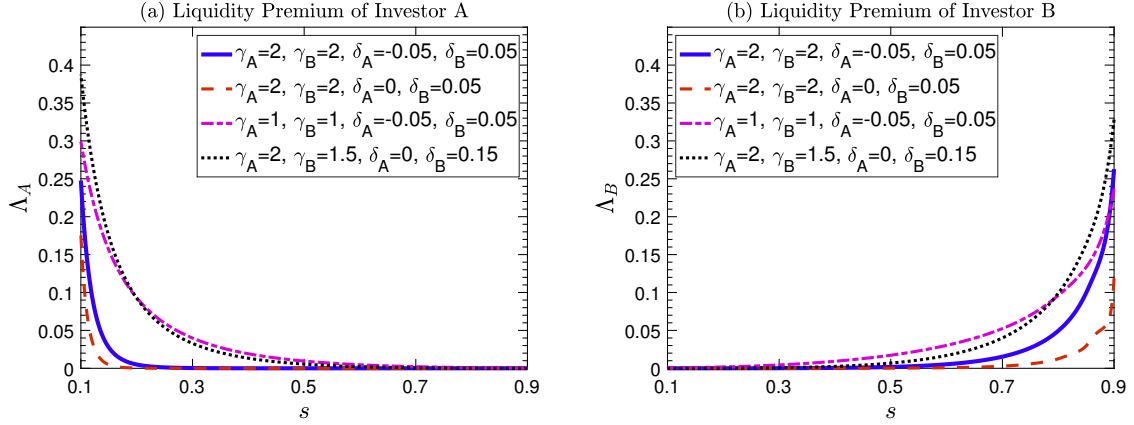
Panels (a) and (b) show the spikes and crashes of simulated P/D ratio and volatility  $\sigma$ , and clustering of volatility  $\sigma$  over the period of 50 years. The model parameters are the same as in Figure 3.

high volatility, which resembles volatility clustering documented in the empirical literature (e.g., Bollerslev, 1987). The distribution of consumption share  $s$  in Figure 2 implies that the economy persists in these clusters for some time.

Figure 5 plots the simulated dynamics of the P/D ratio and stock return volatility over a period of 50 years. Consistent with our discussion above, the dynamics of P/D ratio in Figure 4a exhibits intervals of booms and busts around the times when the collateral constraints become binding. These intervals resemble periods of inflating and deflating bubbles in the economy. The volatility  $\sigma$  in Figure 4b evolves in clusters of high and low volatility, as explained above.

## 4.2. Collateral liquidity premium

In this section, we measure the liquidity premium of stocks over labor income arising because stocks can be used as collateral. We consider a marginal representative investor  $i$  that does not affect asset prices and characterize this investor's shadow indifference price  $\hat{S}_{it}$  of labor income. We define  $\hat{S}_{it}$  as the price such that exchanging marginal  $\Delta l_i$  claims to the stream of labor income for  $\hat{S}_{it}\Delta l_i$  units of wealth leaves the investor's utility unchanged. Consider the investor  $i$ 's value function  $V_i(W_{it}, v_t; l_i)$ . Price  $\hat{S}_{it}$  is the solution of equation



**Figure 6**

**Collateral liquidity premiums from the view of investors  $A$  and  $B$**

The Figure shows the collateral liquidity premiums (51) of stocks over nonpledgeable labor incomes from the view of investors  $A$  and  $B$  for different sets of risk aversions and beliefs. All other model parameters are the same as in Figure 3.

$V_i(W_{it}^*, v_t; l_i) = V_i(W_{it}^* + \widehat{S}_{it}\Delta l_i, v_t; l_i - \Delta l_i)$ . In the limit  $\Delta l_i \rightarrow 0$ , we find that

$$\widehat{S}_{it} = \frac{\partial V_i(W_{it}^*, v_t; l_i)/\partial l_i}{\partial V_i(W_{it}^*, v_t; l_i)/\partial W_{it}}. \quad (50)$$

The definition of shadow indifference price  $\widehat{S}_{it}$  comes from the literature on the valuation of derivative securities in incomplete markets (e.g., Davis, 1997).

The labor incomes  $l_i D_t \Delta t$  are proportional to dividends  $(1 - l_A - l_B) D_t \Delta t$ . Therefore, if claims on labor incomes were tradable and pledgeable, shadow price  $\widehat{S}_{it}$  would have been equal to  $S_t/(1 - l_A - l_B)$ . However, labor incomes are nontradable and nonpledgeable. Hence, from the view of investor  $i$ , the stock enjoys a liquidity premium, defined as

$$\Lambda_{it} = \frac{S_t/(1 - l_A - l_B) - \widehat{S}_{it}}{S_t/(1 - l_A - l_B)}. \quad (51)$$

We find derivatives in equation (50) using the envelope theorem. Then, we derive prices  $\widehat{S}_{it}$  and show that premiums (51) are positive and large. Proposition 5 reports our results.

**Proposition 5 (Shadow prices and the liquidity premium).** *In the limit  $\Delta t \rightarrow 0$ , investor  $i$ 's shadow price of a unit of labor income is given by:*

$$\widehat{S}_{it} = \widehat{\Psi}_i(v; -\gamma_A) s(v)^{\gamma_A} D_t, \quad i = A, B, \quad (52)$$

where  $\widehat{\Psi}_i(v; \theta)$  satisfies differential-difference equation (36) subject to the following boundary conditions for investors  $A$  and  $B$

$$\widehat{\Psi}'_A(\underline{v}; \theta) = 0, \quad \widehat{\Psi}'_A(\bar{v}; \theta) = 0, \quad (53)$$

$$\widehat{\Psi}'_B(\underline{v}; \theta) = \widehat{\Psi}_B(\underline{v}; \theta), \quad \widehat{\Psi}'_B(\bar{v}; \theta) = \widehat{\Psi}_B(\bar{v}; \theta). \quad (54)$$

The investors' liquidity premiums for stocks  $\Lambda_A$  and  $\Lambda_B$  are positive, and hence,

$$S_t/(1 - l_A - l_B) > \widehat{S}_{At}, \quad S_t/(1 - l_A - l_B) > \widehat{S}_{Bt}. \quad (55)$$

The premium  $\Lambda_{it} > 0$  arises because the stock can be used as collateral whereas the labor income cannot. We note that this premium is zero in the frictionless economy, and hence, the nontradability of labor income and the possibility of shorting stocks do not contribute to the premium. This is because, as discussed in Section 4.1, in an unconstrained economy with fully pledgeable labor income the investors can circumvent the nontradability of labor income by shorting stocks. We further remark that the shadow prices and liquidity premiums can be found in closed form, similar to stock prices in Section 3, but we do not present them for brevity.

Figures 6a and 6b plot the liquidity premiums (51) for investors  $A$  and  $B$ , respectively, for different values of risk aversions and beliefs, with other model parameters being the same as in Section 4.1. We observe that investors  $A$  and  $B$  have different valuations of their labor incomes due to differences in preferences and beliefs. Their premiums  $\Lambda_i$  are close to zero when the investors are far away from the boundaries where their respective constraints become binding. The premiums increase up to 40% close to the boundaries where the stock is more valuable for the purposes of relaxing the constraints. Large premiums  $\Lambda_{it}$  imply the economic significance of stock pledgeability. The premiums are smallest for the case  $\gamma_A = \gamma_B = 2$ ,  $\delta_A = 0$  and  $\delta_B = 0.05$  because the economy in this case is closer to the homogeneous-agent economies than in the other cases depicted in Figure 6.

## 5. Conclusion

We develop a parsimonious and tractable theory of asset pricing under collateral constraints. We show that requiring investors to collateralize their trades gives rise to rich dynamics of asset prices and their moments. The constraints lead to booms and busts in stock prices, cause spikes, crashes, and clustering of volatilities, amplify volatilities in good states and dampen them in bad states, decrease interest rates and increase Sharpe ratios when optimistic investors are close to default boundaries, and induce cycles of high and low leverage. The tractability of our model allows us to obtain asset prices and the distributions of consumption shares in closed form.

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## Appendix: Proofs

**Lemma A.1 (Change of variable).** *Let  $\hat{n}_i = k_i l_i / (1 - l_A - l_B)$ . The maximization of expected discounted utility (6) subject to budget constraints (7) and (8), and constraint (11) is equivalent to maximizing (6) with respect to  $c_{it}$ ,  $b_{it}$  and  $\tilde{n}_{it}$  subject to the following set of constraints:*

$$\widetilde{W}_{it} + l_i D_t \Delta t = c_{it} \Delta t + b_{it} B_t + \tilde{n}_{it} (S_t, P_t)^\top, \quad (\text{A1})$$

$$\widetilde{W}_{i,t+\Delta t} = b_{it} + \tilde{n}_{it} \left( S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t, \mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}} \right)^\top, \quad (\text{A2})$$

$$\widetilde{W}_{i,t+\Delta t} \geq 0, \quad (\text{A3})$$

where  $\widetilde{W}_{it} = W_{it} + \hat{n}_i S_t$  and  $\widetilde{W}_{i,t+\Delta t} = W_{i,t+\Delta t} + \hat{n}_i (S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t})$ .

**Proof of Lemma A.1.** Substituting  $n_{it} = \tilde{n}_{it} - (\hat{n}_i, 0)$  into (7) and (8), we obtain constraints (A1) and (A2). Rewriting constraint (11) in terms of variable  $\widetilde{W}_{i,t+\Delta t}$ , we obtain (A3). Finally, we note that  $\widetilde{W}_{it} = W_{it} + \hat{n}_i S_t$  is worth  $\widetilde{W}_{i,t+\Delta t}$  next period. Hence, (A1) and (A2) can be seen as self-financing budget constraints. ■

### Lemma A.2 (Concavity of the value function).

1) Let  $V_i(W_{it}, v_t; l_i)$  denote the value function of investor  $i$ , where  $v_t$  is the state variable. Then, the value function solves the following equation of dynamic programming:

$$V_i(W_{it}, v_t; l_i) = \max_{c_{it}} \left\{ u_i(c_{it}) \Delta t + e^{-\rho \Delta t} \mathbb{E}_t^i [V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)] \right\}, \quad (\text{A4})$$

subject to the static budget and collateral constraints:

$$W_{it} + l_i D_t \Delta t = c_{it} \Delta t + \mathbb{E}_t^i \left[ \frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t} \right], \quad (\text{A5})$$

$$W_{i,t+\Delta t} \geq 0. \quad (\text{A6})$$

2) Value function  $V_i(W_{it}, v_t; l_i)$  is a concave function of wealth  $W_{it}$ .

### Proof of Lemma A.2.

1) We start by demonstrating the equivalence of the dynamic and static budget constraints (7)–(8) and (A5), respectively. Multiplying equation (8) by  $\xi_{i,t+\Delta t} / \xi_{it}$ , taking expectation operator  $\mathbb{E}_t^i[\cdot]$  on both sides, and using equations (25)–(27) for asset prices, we obtain:

$$\mathbb{E}_t^i \left[ \frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t} \right] = b_{it} B_t + n_{it} (S_t, P_t)^\top. \quad (\text{A7})$$

From the budget constraint (7), we observe that the right-hand side of (A7) equals  $W_{it} + l_i D_t \Delta t$ , and hence, we obtain the static budget constraint (A5). Conversely, if there exists  $W_{i,t+\Delta t}$  satisfying constraints (A5) and (A6) there exist trading strategies  $b_{it}$  and  $n_{it}$  that replicate  $W_{i,t+\Delta t}$  because the underlying market is effectively complete (i.e., the payoff matrix is invertible). Finally, the dynamic programming equation (A4) is obtained by rewriting the optimization problem (6) in a recursive form.

2) Consider wealth levels  $W_{it}$  and  $\widehat{W}_{it}$ . Let  $\{c_{it}^*, b_{it}^*, n_{it}^*\}$  and  $\{\widehat{c}_{it}^*, \widehat{b}_{it}^*, \widehat{n}_{it}^*\}$  be optimal consumptions and portfolios that correspond to  $W_{it}$  and  $\widehat{W}_{it}$ , respectively, and satisfy constraints (7)–(9). For any  $\alpha \in [0, 1]$ , policies  $\{\alpha \widehat{c}_{it}^* + (1-\alpha)c_{it}^*, \alpha \widehat{b}_{it}^* + (1-\alpha)b_{it}^*, \alpha \widehat{n}_{it}^* + (1-\alpha)n_{it}^*\}$  are admissible for wealth  $\alpha W_{it} + (1-\alpha)\widehat{W}_{it}$ . By concavity of CRRA utilities:

$$\begin{aligned} V_i(\alpha W_{it} + (1-\alpha)\widehat{W}_{it}, v_t; l_i) &\geq \sum_{\tau=t}^{\infty} u_i(\alpha \widehat{c}_{it}^* + (1-\alpha)c_{it}^*) \\ &\geq \sum_{\tau=t}^{\infty} (\alpha u_i(\widehat{c}_{it}^*) + (1-\alpha)u_i(c_{it}^*)) \\ &= \alpha V_i(W_{it}, v_t; l_i) + (1-\alpha)V_i(\widehat{W}_{it}, v_t; l_i). \end{aligned} \quad (\text{A8})$$

Therefore,  $V_i(W_{it}, v_t; l_i)$  is a concave function of wealth. ■

**Proof of Lemma 1.** Consider the Lagrangian for the optimization problem (A4) subject to constraints (A5) and (A6):

$$\begin{aligned} \mathcal{L} &= u_i(c_{it})\Delta t + e^{-\rho\Delta t} \mathbb{E}_t^i [V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)] \\ &+ \eta_{it} \left( W_{it} + l_i D_t \Delta t - c_{it} \Delta t - \mathbb{E}_t^i \left[ \frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t} \right] \right) + \mathbb{E}_t^i [e^{-\rho\Delta t} \ell_{i,t+\Delta t} W_{i,t+\Delta t}], \end{aligned} \quad (\text{A9})$$

where the multiplier  $\ell_{i,t+\Delta t} \geq 0$  satisfies the complementary slackness condition  $\ell_{i,t+\Delta t} W_{i,t+\Delta t} = 0$ . Differentiating the Lagrangian (A9) with respect to  $c_{it}$  and  $W_{i,t+\Delta t}$ , we obtain:

$$u'_i(c_{it}^*) = \eta_{it}, \quad (\text{A10})$$

$$e^{-\rho\Delta t} \left( \frac{\partial V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)}{\partial W} + \ell_{i,t+\Delta t} \right) = \eta_{it} \frac{\xi_{i,t+\Delta t}}{\xi_{it}}. \quad (\text{A11})$$

By the envelope theorem (e.g, Back (2010, p.162)):

$$\frac{\partial V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)}{\partial W} = u'_i(c_{i,t+\Delta t}^*). \quad (\text{A12})$$

Substituting the partial derivative of the value function (A12) and the marginal utility (A10) into equation (A11), and then dividing both sides of the equation by  $u'_i(c_{it}^*)$ , we obtain the SPD (29). ■

**Proof of Proposition 1.**

**Step 1.** Consider the case in which constraints do not bind, and hence,  $\ell_{i,t+\Delta t} = 0$ . Then, using equation (13) for the state variable  $v_t$  and the first order conditions (29), we obtain:

$$\begin{aligned} v_{t+\Delta t} - v_t &= \ln \left( \frac{(c_{A,t+\Delta t}^*/c_{At}^*)^{-\gamma_A}}{(c_{B,t+\Delta t}^*/c_{Bt}^*)^{-\gamma_B}} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right) \\ &= \ln \left( \frac{\xi_{A,t+\Delta t}/\xi_{At}}{\xi_{B,t+\Delta t}/\xi_{Bt}} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right). \end{aligned} \quad (\text{A13})$$

From the above equation and the change of measure equation (28), which relates SPDs  $\xi_{A,t+\Delta t}$  and  $\xi_{B,t+\Delta t}$ , we obtain the dynamics of  $v_t$  when constraints do not bind:

$$v_{t+\Delta t} - v_t = \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right). \quad (\text{A14})$$

Let  $\bar{v}$  and  $\underline{v}$  be the boundaries satisfying Equations (30), at which the constraints of investors  $A$  and  $B$  bind, respectively. Let investor  $A$ 's constraint be binding so that  $v_{t+\Delta t} = \bar{v}$ , and hence,  $\ell_{A,t+\Delta t} \geq 0$ . Using Equation (13) for  $v_t$ , first order conditions (29), and  $\ell_{A,t+\Delta t} \geq 0$ , we obtain:

$$\begin{aligned} \bar{v} - v_t &\leq \ln \left( \frac{((c_{A,t+\Delta t}^*)^{-\gamma_A} + \ell_{A,t+\Delta t}) / (c_{At}^*)^{-\gamma_A}}{(c_{B,t+\Delta t}^*/c_{Bt}^*)^{-\gamma_B}} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right) \\ &= \ln \left( \frac{\xi_{A,t+\Delta t}/\xi_{At}}{\xi_{B,t+\Delta t}/\xi_{Bt}} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right) = \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right). \end{aligned} \quad (\text{A15})$$

Similarly, for  $v_{t+\Delta t} = \underline{v}$  we find that  $\underline{v} - v_t \geq \ln(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t}))(D_{t+\Delta t}/D_t)^{\gamma_A - \gamma_B}$ . The latter two inequalities imply that when the constraint binds  $v_{t+\Delta t}$  is given by:

$$v_{t+\Delta t} = \max \left\{ \underline{v}; \min \left\{ \bar{v}; v_t + \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right) \right\} \right\}. \quad (\text{A16})$$

We observe that (A16) is also satisfied in the unconstrained case in which  $\underline{v} < v_{t+\Delta t} < \bar{v}$ . It remains to prove that  $v_t$  does not escape  $[\underline{v}, \bar{v}]$  interval. Consider a marginal investor of type  $A$ . We conjecture that  $v_t$  follows dynamics (A16) and verify that the consumption choice of investor  $A$  indeed implies this dynamics. The analysis for investor  $B$  is similar.

We have shown above that  $v_t$  satisfies inequality (A15) when investor  $A$  is constrained. Now, we show the opposite: investor  $A$  is constrained when  $v_t$  satisfies (A15). Hence,  $v_{t+\Delta t}$  cannot exceed  $\bar{v}$ . Consider  $v_t$  such that  $v_t + \ln(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t}))(D_{t+\Delta t}/D_t)^{\gamma_A - \gamma_B} > \bar{v}$  for some  $\omega_{t+\Delta t}$  and  $v_t \in (\underline{v}, \bar{v})$ . Because  $\underline{v} < v_t < \bar{v}$ , investor  $A$  consumes  $c_{At}^* = s(v_t)D_t$ , as

shown above. We show that the constraint of investor  $A$  binds and  $c_{A,t+\Delta t}^* = s(\bar{v})D_{t+\Delta t}$ . This consumption level confirms that  $v_{t+\Delta t} = \bar{v}$  is indeed an equilibrium outcome.

Consider the constraint of investor  $A$  at date  $t$  in the state  $\omega_{t+\Delta t}$  where  $v_{t+\Delta t} = \bar{v}$ :

$$W_{A,t+\Delta t} \geq 0 \equiv \Phi_A(\bar{v})s(\bar{v})D_{t+\Delta t}, \quad (\text{A17})$$

where the last equality holds by the definition of  $\bar{v}$ . Using the concavity of the value function, proven in Lemma 1, and condition (A12) from the envelope theorem, we obtain:

$$u'_A(c_{A,t+\Delta t}^*) = \frac{\partial V_A(W_{A,t+\Delta t}, \bar{v}; l_A)}{\partial W} \leq \frac{\partial V_A(\Phi_A(\bar{v})s(\bar{v})D_{t+\Delta t}, \bar{v}; l_A)}{\partial W} = u'_A(s(\bar{v})D_{t+\Delta t}). \quad (\text{A18})$$

Because  $u'_i(c)$  is a decreasing function, we find that  $c_{A,t+\Delta t}^*/D_{t+\Delta t} \geq s(\bar{v})$ .

Investor  $B$  is unconstrained when  $v_{t+\Delta t} = \bar{v}$ , and hence, has SPD

$$\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = e^{-\rho\Delta t} \left( \frac{c_{B,t+\Delta t}^*}{c_{Bt}^*} \right)^{-\gamma_B} = e^{-\rho\Delta t} \left( \frac{(1-s(\bar{v}))D_{t+\Delta t}}{(1-s(v_t))D_t} \right)^{-\gamma_B}. \quad (\text{A19})$$

From the change of measure equation (28) and the FOC (29), the SPD of investor  $A$  is

$$\begin{aligned} \frac{\xi_{A,t+\Delta t}}{\xi_{At}} &= \frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \\ &= e^{-\rho\Delta t} \frac{(c_{A,t+\Delta t}^*)^{-\gamma_A} + \ell_{A,t+\Delta t}}{(c_{At}^*)^{-\gamma_A}}. \end{aligned} \quad (\text{A20})$$

From (A20) and (A19), we find the Lagrange multiplier:

$$\begin{aligned} \frac{l_{A,t+\Delta t}}{(c_{A,t+\Delta t}^*)^{-\gamma_A}} &= \left( \frac{c_{A,t+\Delta t}^*}{c_{At}^*} \right)^{\gamma_A} \left( \frac{(1-s(\bar{v}))D_{t+\Delta t}}{(1-s(v_t))D_t} \right)^{-\gamma_B} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} - 1 \\ &\geq \left( \frac{s(\bar{v})D_{t+\Delta t}}{s(v_t)D_t} \right)^{\gamma_A} \left( \frac{(1-s(\bar{v}))D_{t+\Delta t}}{(1-s(v_t))D_t} \right)^{-\gamma_B} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} - 1 \\ &= \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right) e^{v_t - \bar{v}} - 1 > 0. \end{aligned} \quad (\text{A21})$$

The first inequality follows from the fact that  $c_{A,t+\Delta t}^* \geq s(\bar{v})D_{t+\Delta t}$ , which we proved above. The second equality holds by the definition of state variable (13). The second inequality comes from the assumption that  $v_t + \ln\left(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})\right) \left(D_{t+\Delta t}/D_t\right)^{\gamma_A - \gamma_B} > \bar{v}$ . Hence, the Lagrange multiplier  $l_{A,t+\Delta t}$  is strictly positive. From the complementary slackness condition, the constraint (A17) must be binding. Therefore, inequality (A18) becomes an equality, and hence,  $c_{A,t+\Delta t}^* = s(\bar{v})D_{t+\Delta t}$ .



**Step 2.** We now look for coefficients  $\mu_v$ ,  $\sigma_v$  and  $J_v$  such that:

$$\begin{aligned} \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t &= \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \right) \\ &= \ln \left( \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \right) + (\gamma_A - \gamma_B) \ln(1 + \mu_D \Delta t + \sigma_D \Delta w_t + J_v \Delta j_t). \end{aligned} \quad (\text{A22})$$

We write identity (A22) in each of the states  $\omega_{t+\Delta t} \in \{\omega_1, \omega_2, \omega_3\}$  and obtain a system of three linear equations with three unknowns  $\mu_v$ ,  $\sigma_v$  and  $J_v$ . Solving this system, we find

$$\mu_v = \frac{1}{2\Delta t} \left( (\gamma_A - \gamma_B) \ln[(1 + \mu_D \Delta t)^2 - \sigma_D^2 \Delta t] + \ln \left( \frac{1 - \lambda_B \Delta t}{1 - \lambda_A \Delta t} \right)^2 + \ln \left( \frac{1 - \delta_B^2 \Delta t}{1 - \delta_A^2 \Delta t} \right) \right), \quad (\text{A23})$$

$$\sigma_v = \frac{1}{2\sqrt{\Delta t}} \left( (\gamma_A - \gamma_B) \ln \left( \frac{1 + \mu_D \Delta t + \sigma_D \sqrt{\Delta t}}{1 + \mu_D \Delta t - \sigma_D \sqrt{\Delta t}} \right) + \ln \left( \frac{(1 + \delta_B \sqrt{\Delta t})(1 - \delta_A \sqrt{\Delta t})}{(1 - \delta_B \sqrt{\Delta t})(1 + \delta_A \sqrt{\Delta t})} \right) \right), \quad (\text{A24})$$

$$J_v = (\gamma_A - \gamma_B) \ln(1 + \mu_D \Delta t + J_D) + \ln \left( \frac{\lambda_B}{\lambda_A} \right) - \mu_v \Delta t. \quad (\text{A25})$$

**Step 3.** Finally, we show that the boundaries are reflecting for a sufficiently small  $\Delta t$ . Assume that  $\sigma_v > 0$ ; the case  $\sigma_v < 0$  is considered analogously. Then, for a sufficiently small  $\Delta t$ , we observe that  $\mu_v \Delta t - \sigma_v \sqrt{\Delta t} < 0$  and  $\mu_v \Delta t + \sigma_v \sqrt{\Delta t} > 0$  because  $\sqrt{\Delta t}$ -terms dominate  $\Delta t$ -terms, and as  $\Delta t \rightarrow 0$ ,  $\mu_v \rightarrow \hat{\mu}_v$  and  $\sigma_v \rightarrow \hat{\sigma}_v$ , where  $\hat{\mu}_v$  and  $\hat{\sigma}_v$  are given by equations (38) and (39), respectively. Moreover,  $\hat{\sigma}_v > 0$  because  $\gamma_A \geq \gamma_B$  and  $\delta_B \geq \delta_A$ , and at least one of the latter inequalities is strict. Then, the boundaries are reflecting because 1) if  $v_t = \bar{v}$ , then  $v_{t+\Delta t} = \bar{v} + \mu_v \Delta t - \sigma_v \sqrt{\Delta t} < \bar{v}$  with positive probability; 2) if  $v_t = \underline{v}$ , then  $v_{t+\Delta t} = \underline{v} + \mu_v \Delta t + \sigma_v \sqrt{\Delta t} > \underline{v}$  with positive probability. ■

**Lemma A.3 (Wealth-consumption ratios).** *The investors' wealth-consumption ratios  $\Phi_i$  are uniformly bounded and given by:*

$$\Phi_A(v_t) = \mathbb{E}_t^A \left[ \sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} \left( \frac{D_\tau}{D_t} \right)^{1-\gamma_A} \left( \frac{s(v_\tau)}{s(v_t)} \right)^{1-\gamma_A} \left( 1 - \frac{l_A}{s(v_\tau)} \right) \Delta t \right], \quad (\text{A26})$$

$$\Phi_B(v_t) = \mathbb{E}_t^B \left[ \sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} \left( \frac{D_\tau}{D_t} \right)^{1-\gamma_B} \left( \frac{1-s(v_\tau)}{1-s(v_t)} \right)^{1-\gamma_B} \left( 1 - \frac{l_B}{1-s(v_\tau)} \right) \Delta t \right]. \quad (\text{A27})$$

**Proof of Lemma A.3.** Substituting FOC (29) into the budget constraint (A5) and using the complementary slackness condition  $\ell_{i,t+\Delta t} W_{i,t+\Delta t}^* = 0$ , we obtain:

$$W_{At}^* = \mathbb{E}_t^A \left[ e^{-\rho \Delta t} \left( \frac{c_{A,t+\Delta t}^*}{c_{At}^*} \right)^{-\gamma_A} W_{A,t+\Delta t}^* \right] + (c_{At}^* - l_A D_t) \Delta t. \quad (\text{A28})$$

Substituting  $W_{At}^* = \Phi_{At} c_{At}^*$  and  $c_{At}^* = s(v_t)D_t$  into equation (A28) and iterating, we obtain equation (A26). Let  $\bar{s} = s(\underline{v})$   $\underline{s} = s(\bar{v})$ , where  $s(v)$  is given by equation (14). Then,  $\bar{s} \geq s \geq \underline{s} > 0$ . Using the bounds on  $s_t$ , we obtain the following uniform bound on  $\Phi_A$ :

$$\Phi_A(v_t) \leq Const \times \mathbb{E}_t^A \left[ \sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} \left( \frac{D_\tau}{D_t} \right)^{1-\gamma_A} \Delta t \right]. \quad (\text{A29})$$

The series on the right-hand side of the latter inequality is convergent due to condition (15) on model parameters. Equation (A27) is obtained along the same lines. ■

**Proof of Proposition 2.** 1) First, we derive the SPD  $\xi_{At}$  under the correct beliefs of investor  $A$ . When investor  $A$ 's constraint does not bind, substituting  $c_{At}^* = s(v_t)D_t$  into the first order condition (29) we find that

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t} \left( \frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A}. \quad (\text{A30})$$

Equation (A30) is consistent with SPD (32) because when the constraint does not bind  $v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t < \bar{v}$ , and hence the exponential term in (32) vanishes.

When the constraint of investor  $A$  binds, the constraint of investor  $B$  is loose: the constraints cannot bind simultaneously because the stock market would not clear otherwise. Therefore, the ratio  $\xi_{B,t+\Delta t}/\xi_{Bt}$  is given by FOC (29) for investor  $B$  with  $\ell_B = 0$ . Using equation (28), we rewrite the latter SPD under the correct beliefs of investor  $A$ :

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t} \left( \frac{1 - s(v_{t+\Delta t})}{1 - s(v_t)} \right)^{-\gamma_B} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_B} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}. \quad (\text{A31})$$

Next, from equation (14) for consumption share  $s$  we find that  $(1 - s_t)^{-\gamma_B} = e^{-v_t} s_t^{-\gamma_A}$ . Substituting the latter equality into equation (A31), and also using equation (A22) for the increment  $v_{t+\Delta t} - v_t$ , we obtain:

$$\begin{aligned} \frac{\xi_{A,t+\Delta t}}{\xi_{At}} &= e^{-\rho\Delta t} \left( \frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} e^{v_t - v_{t+\Delta t}} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \\ &= e^{-\rho\Delta t} \left( \frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} \exp\{v_t - v_{t+\Delta t} + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\}. \end{aligned} \quad (\text{A32})$$

The fact that the constraint of investor  $A$  is binding means that  $v_{t+\Delta t} = \bar{v}$  and  $v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \geq \bar{v}$  (because otherwise  $v_{t+\Delta t} < \bar{v}$ , and hence, the constraint does not bind). Therefore, the exponential term in equation (A32) can be replaced with  $\exp(\max\{0, v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \bar{v}\})$ . The latter term vanishes when the constraint of investor  $A$

does not bind, and we obtain equation (A30). Therefore, both equations (A30) and (A32) are summarized by equation (32) for  $\xi_{A,t+\Delta t}/\xi_{At}$ .

2) From the market clearing condition,  $S_t = W_{At} + W_{Bt}$ . Dividing by  $D_t$  and then rewriting in terms of wealth-consumption ratios, we obtain that  $(1 - l_A - l_B)\Psi(v_t) = \Phi_A(v_t)s(v_t) + \Phi_B(v_t)(1 - s(v_t))$ . Hence,  $\Psi(v_t)$  is uniformly bounded because  $\Phi_i(v_t)$  are uniformly bounded by Lemma A.3. The fact that stock price  $S_t$  is given by (33) can be verified by substituting  $S_t$  in (33) into the recursive equation (26).

3) In the unconstrained economy, the state variable  $v_t^{unc}$  follows dynamics:

$$v_t^{unc} = \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t. \quad (\text{A33})$$

Let  $U_{t+\Delta t} = U_t + \Delta U_t$  and  $V_{t+\Delta t} = V_t + \Delta V_t$ , where the increments are given by:

$$\Delta U_t = \max\{0; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \bar{v}\}, \quad (\text{A34})$$

$$\Delta V_t = \max\{0; \underline{v} - v_t - \mu_v \Delta t - \sigma_v \Delta w_t - J_v \Delta j_t\}. \quad (\text{A35})$$

The process for the state variable in the constrained economy can be rewritten as

$$v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t + \Delta V_t - \Delta U_t. \quad (\text{A36})$$

If the state variables have the same value at time 0, i.e.,  $v_0 = v_0^{unc}$ , we obtain:

$$v_t = v_t^{unc} + V_t - U_t. \quad (\text{A37})$$

Next, we prove that the SPD is higher in the constrained economy.

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t} \left( \frac{s(v_{t+\Delta t})}{s(v_t)} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} \exp(\Delta U_t), \quad (\text{A38})$$

$$\frac{\xi_{A,t+\Delta t}^{unc}}{\xi_{At}^{unc}} = e^{-\rho \Delta t} \left( \frac{s(v_{t+\Delta t}^{unc})}{s(v_t^{unc})} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A}. \quad (\text{A39})$$

Iterating the above equations, we obtain:

$$\frac{\xi_{At}}{\xi_{A0}} = e^{-\rho t} \left( \frac{s(v_t)}{s(v_0)} \frac{D_t}{D_0} \right)^{-\gamma_A} \exp(U_t), \quad (\text{A40})$$

$$\frac{\xi_{At}^{unc}}{\xi_{A0}^{unc}} = e^{-\rho t} \left( \frac{s(v_t^{unc})}{s(v_0)} \frac{D_t}{D_0} \right)^{-\gamma_A}. \quad (\text{A41})$$

By the definition of  $s(v)$  in equation (14), we have  $e^v = (1 - s(v))^{\gamma_B} \cdot s(v)^{-\gamma_A}$ . Hence,

$$\begin{aligned} \frac{\xi_{At}/\xi_{A0}}{\xi_{At}^{unc}/\xi_{A0}^{unc}} &= \left( \frac{s(v_t)}{s(v_t^{unc})} \right)^{-\gamma_A} \exp(U_t) = \left( \frac{s(v_t^{unc} + V_t - U_t)}{s(v_t^{unc})} \right)^{-\gamma_A} e^{v_t^{unc}} e^{-(v_t^{unc} - U_t)} \\ &\geq s(v_t^{unc} - U_t)^{-\gamma_A} e^{-(v_t^{unc} - U_t)} \cdot s(v_t^{unc})^{\gamma_A} e^{v_t^{unc}} \\ &= (1 - s(v_t^{unc} - U_t))^{-\gamma_B} \cdot (1 - s(v_t^{unc}))^{\gamma_B} \geq 1. \end{aligned} \quad (\text{A42})$$

Therefore, we conclude that  $\xi_{At}/\xi_{A0} \geq \xi_{At}^{unc}/\xi_{A0}^{unc}$ . The latter inequality and the equations for asset prices (25)–(27) imply that prices are higher in the constrained economy. ■

**Proof of Lemma 2.** The price-dividend ratio  $\Psi$  and wealth-output ratio  $\widehat{\Phi}_{it} \equiv W_{it}/D_t$  are functions of the state variable  $v$ , and satisfy equations:

$$\Psi(v_t) = \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{D_{t+\Delta t}}{D_t} (\Psi(v_{t+\Delta t}) + \Delta t) \right], \quad (\text{A43})$$

$$\widehat{\Phi}_i(v_t) = \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{D_{t+\Delta t}}{D_t} \widehat{\Phi}_i(v_{t+\Delta t}) \right] + (\mathbf{1}_{\{i=A\}} s(v_t) + \mathbf{1}_{\{i=B\}} (1 - s(v_t)) - l_i) \Delta t. \quad (\text{A44})$$

These equations are obtained by substituting  $S_t = (1 - l_A - l_B) D_t \Psi(v_t)$  into equation (26) for the stock price, and  $\widehat{\Phi}_{it} = D_t W_{it}$  into static budget constraints (A5). Define the following function in discrete time:

$$\widehat{\Psi}(v_t; \theta) = \mathbb{E}_t^A \left[ e^{-\rho \Delta t + \Delta U_t} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \widehat{\Psi}(v_{t+\Delta t}; \theta) \right] + s(v_t)^\theta \Delta t, \quad (\text{A45})$$

where  $\Delta U_t$  is given by equation (A34).

Comparing equation (A45) with equations (A43) and (A44) for  $\Psi$  and  $\widehat{\Phi}_i$  and using the linearity of equation (A45), it is easy to observe that  $\Psi(v_t)$  and  $\widehat{\Phi}_i(v_t)$  are given by the following equations in terms of  $\widehat{\Psi}(v_t; \theta)$ :

$$\Psi(v_t) = \widehat{\Psi}(v_t, -\gamma_A) s(v_t)^{\gamma_A} - \Delta t, \quad (\text{A46})$$

$$\widehat{\Phi}_i(v_t) = \left( (\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}}) \widehat{\Psi}(v; 1 - \gamma_A) + (\mathbf{1}_{\{i=B\}} - l_i) \widehat{\Psi}(v; -\gamma_A) \right) s(v)^{\gamma_A}. \quad (\text{A47})$$

Taking limit  $\Delta t \rightarrow 0$ , and noting that  $\Phi_i(v_t) = \widehat{\Phi}_i(v_t) / (\mathbf{1}_{\{i=A\}} s(v_t) + \mathbf{1}_{\{i=B\}} (1 - s(v_t)))$ , we obtain equations (34) and (35) for  $\Psi(v_t)$  and  $\Phi_i(v_t)$ .

First, we derive the equation for  $\widehat{\Psi}(v_t; \theta)$  when  $v_t$  belongs to the interior  $(\underline{v}, \bar{v})$ . For a sufficiently small  $\Delta t$  we have  $\Delta U_t = 0$ , where  $\Delta U_t$  is given by (A34). Then, we rewrite

the expectation of  $(D_{t+\Delta t}/D_t)^{1-\gamma_A}\widehat{\Psi}(v_t; \theta)$  as follows:

$$\begin{aligned} \mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \widehat{\Psi}(v_{t+\Delta t}; \theta) \right] &= (1 - \lambda_A \Delta t) \mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \widehat{\Psi}(v_{t+\Delta t}; \theta) \Big| \text{normal} \right] \\ &+ \lambda_A \Delta t \mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \widehat{\Psi}(v_{t+\Delta t}; \theta) \Big| \text{crisis} \right]. \end{aligned} \quad (\text{A48})$$

Noting that in the crisis  $D_{t+\Delta t}/D_t = 1 + \mu_v \Delta t + J_D$  and  $v_{t+\Delta t} = \max\{\underline{v}; v_t + \mu_v \Delta t + J_v\}$ , and in the normal state  $D_{t+\Delta t}/D_t = 1 + \mu_D \Delta t + \sigma_D \Delta w_t$  and  $v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t$ , using Taylor expansions for  $(D_{t+\Delta t}/D_t)^{1-\gamma_A}$  and  $\widehat{\Psi}(v_{t+\Delta t}; \theta)$ , we find:

$$\mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \widehat{\Psi}(v_{t+\Delta t}; \theta) \Big| \text{crisis} \right] = (1 + J_D)^{1-\gamma_A} \widehat{\Psi}(\max\{\underline{v}; v_t + J_v\}; \theta). \quad (\text{A49})$$

$$\begin{aligned} \mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \widehat{\Psi}(v_{t+\Delta t}; \theta) \Big| \text{normal} \right] &= \left[ 1 + \left( (1 - \gamma_A)(\mu_D + \delta_A \sigma_D) - \frac{(1 - \gamma_A)\gamma_A \sigma_D^2}{2} \right) \Delta t \right] \\ &\times \widehat{\Psi}(v_t; \theta) + (\mu_v + \delta_A \sigma_v + (1 - \gamma_A)\sigma_D \sigma_v) \times \widehat{\Psi}'(v_t; \theta) \Delta t + \frac{\sigma_v^2}{2} \widehat{\Psi}''(v_t; \theta) \Delta t + o(\Delta t). \end{aligned} \quad (\text{A50})$$

Substituting (A49)-(A50) into (A45), we obtain:

$$\begin{aligned} \widehat{\Psi}(v_t; \theta) &= \left[ 1 - \left( \lambda_A + \rho - (1 - \gamma_A)(\mu_D + \delta_A \sigma_D) + \frac{(1 - \gamma_A)\gamma_A \sigma_D^2}{2} \right) \Delta t \right] \widehat{\Psi}(v_t; \theta) \\ &+ (\mu_v + \delta_A \sigma_v + (1 - \gamma_A)\sigma_D \sigma_v) \widehat{\Psi}'(v_t; \theta) \Delta t + \frac{\sigma_v^2}{2} \widehat{\Psi}''(v_t; \theta) \Delta t \\ &+ \lambda_A (1 + J_D)^{1-\gamma_A} \widehat{\Psi}(\max\{\underline{v}; v_t + J_v\}; \theta) \Delta t + s(v)^\theta \Delta t + o(\Delta t). \end{aligned} \quad (\text{A51})$$

Canceling similar terms, dividing by  $\Delta t$ , taking limit  $\Delta t \rightarrow 0$ , and noting that  $\mu_v$ ,  $\sigma_v$  and  $J_v$  converge to  $\widehat{\mu}_v$ ,  $\widehat{\sigma}_v$  and  $\widehat{J}_v$  given by (38)-(40), we obtain equation (36) for  $\widehat{\Psi}(v_t; \theta)$ .

Next, we derive the boundary conditions for  $\widehat{\Psi}(v_t; \theta)$ . From equation (31), the state variable dynamics at lower bound is  $v_{t+\Delta t} = \underline{v} + \max\{0, \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\}$ . Let  $\Delta v_t = v_{t+\Delta t} - v_t$ . Then,

$$\Delta v_t = \max\{0, \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\}. \quad (\text{A52})$$

For sufficiently small  $\Delta t$  the increment  $\Delta v_t$  is positive only in a state in which  $\sigma_v \Delta w_t > 0$ . In such state,  $\Delta v_t = \mu_v \Delta t + |\sigma_v| \sqrt{\Delta t}$ . Therefore, the order of  $\mathbb{E}_t^A [\Delta v_t]$  is  $\sqrt{\Delta t}$ , but second order terms involving  $\Delta v_t$  have lower order:

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{\mathbb{E}_t^A [\Delta v_t]}{\sqrt{\Delta t}} &= \frac{|\widehat{\sigma}_v|}{2}, \\ \lim_{\Delta t \rightarrow 0} \frac{\mathbb{E}_t^A [(\Delta v_t)^2]}{\sqrt{\Delta t}} &= \lim_{\Delta t \rightarrow 0} \frac{\mathbb{E}_t^A [\Delta v_t \Delta t]}{\sqrt{\Delta t}} = \lim_{\Delta t \rightarrow 0} \frac{\mathbb{E}_t^A [\Delta v_t \Delta w_t]}{\sqrt{\Delta t}} = \lim_{\Delta t \rightarrow 0} \frac{\mathbb{E}_t^A [\Delta v_t \Delta j_t]}{\sqrt{\Delta t}} = 0. \end{aligned} \quad (\text{A53})$$

Taylor expansion of  $\widehat{\Psi}(v_{t+\Delta t}; \theta)$  at  $v_t = \underline{v}$  is given by

$$\widehat{\Psi}(v_{t+\Delta t}; \theta) = \widehat{\Psi}(\underline{v}; \theta) + \widehat{\Psi}'(\underline{v}; \theta)\Delta v_t + \frac{1}{2}\widehat{\Psi}''(\underline{v}; \theta)\Delta v_t^2 + o(\sqrt{\Delta t}). \quad (\text{A54})$$

In subsequent calculations, we keep terms with order of  $\sqrt{\Delta t}$ . Using the above results, we obtain the following expansion:

$$\begin{aligned} & \mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \widehat{\Psi}(v_{t+\Delta t}; \theta) \right] \\ &= \mathbb{E}_t^A \left[ (1 + \mu_D \Delta t + \sigma_D \Delta w_t + J_v \Delta j_t)^{1-\gamma_A} \left( \widehat{\Psi}(\underline{v}; \theta) + \widehat{\Psi}'(\underline{v}; \theta)\Delta v_t + \frac{1}{2}\widehat{\Psi}''(\underline{v}; \theta)\Delta v_t^2 \right) \right] \quad (\text{A55}) \\ &= \widehat{\Psi}(\underline{v}; \theta) + \widehat{\Psi}'(\underline{v}; \theta)\mathbb{E}_t^A[\Delta v_t] + o(\sqrt{\Delta t}). \end{aligned}$$

Substituting (A55) into (A45), taking into account that  $\Delta U_t = 0$  at  $v_t = \underline{v}$ , and canceling  $\widehat{\Psi}(\underline{v}; \theta)$  on both sides, we obtain the first boundary condition  $\widehat{\Psi}'(\underline{v}; \theta) = 0$ .

At the upper bound  $v_t = \bar{v}$  investor  $A$  is constrained, and hence,  $\Delta U_t$  in (A34) is positive. From (31) the state variable at the upper bound is

$$v_{t+\Delta t} = \min\{\bar{v}, v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \Delta U_t. \quad (\text{A56})$$

The order of  $\mathbb{E}_t^A[\Delta U_t]$  is  $\sqrt{\Delta t}$ , but second order terms involving  $\Delta U_t$  have order  $o(\sqrt{\Delta t})$ . Proceeding in the same way as (A53)-(A55), we arrive at

$$\widehat{\Psi}(\bar{v}; \theta) = \widehat{\Psi}(\bar{v}; \theta) + \left[ \widehat{\Psi}(\bar{v}; \theta) - \widehat{\Psi}'(\bar{v}; \theta) \right] \mathbb{E}_t^A[\Delta U_t] + o(\sqrt{\Delta t}). \quad (\text{A57})$$

Canceling similar terms, taking limit  $\Delta t \rightarrow 0$ , we obtain condition  $\widehat{\Psi}(\bar{v}; \theta) - \widehat{\Psi}'(\bar{v}; \theta) = 0$ .

Finally, we derive the equations for  $\bar{v}$  and  $\underline{v}$ . Substituting  $\Phi_i(v)$  from (35) into the equations  $\Phi_A(\bar{v}) = 0$ ,  $\Phi_B(\underline{v}) = 0$ , after some algebra, we obtain equations (41). ■

**Proof of Corollary 1.** For  $\lambda_A = \lambda_B$  and  $\gamma_A = \gamma_B = \gamma$  the differential-difference equation (36) becomes an ODE because  $\widehat{J}_v = 0$ . From equation (14) for consumption share  $s$  we find that  $s(v) = 1/(1 + e^{v/\gamma})$ . Substituting  $s(v)$  into (36) and setting  $\theta = -\gamma$  we obtain an equation for  $\widehat{\Psi}(v; -\gamma)$ . It can be directly verified that the solution of (36) satisfying boundary conditions (37) is given by  $\widehat{\Psi}(v; -\gamma) = C_- e^{\varphi-v} + C_+ e^{\varphi+v} + \widehat{\Psi}^{unc}(v)$ , where

$$\widehat{\Psi}^{unc}(v) = \sum_{k=0}^{\gamma} \binom{\gamma}{k} \frac{e^{kv/\gamma}}{h(k/\gamma)}, \quad (\text{A58})$$

and the coefficients are given by

$$C_+ = \frac{(1 - \varphi_-)e^{\varphi_- \bar{v}}(\widehat{\Psi}^{unc})'(\underline{v}) - \varphi_- e^{\varphi_- \underline{v}}((\widehat{\Psi}^{unc})(\bar{v}) - (\widehat{\Psi}^{unc})'(\bar{v}))}{\varphi_+(\varphi_- - 1)e^{\varphi_- \bar{v} + \varphi_+ \underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+ \bar{v} + \varphi_- \underline{v}}}, \quad (\text{A59})$$

$$C_- = \frac{(\varphi_+ - 1)e^{\varphi_+ \bar{v}}(\widehat{\Psi}^{unc})'(\underline{v}) + \varphi_+ e^{\varphi_+ \underline{v}}((\widehat{\Psi}^{unc})(\bar{v}) - (\widehat{\Psi}^{unc})'(\bar{v}))}{\varphi_+(\varphi_- - 1)e^{\varphi_- \bar{v} + \varphi_+ \underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+ \bar{v} + \varphi_- \underline{v}}}. \quad (\text{A60})$$

The P/D ratio is then given by (34), which takes form  $\widehat{\Psi}(v; -\gamma)/(1 + e^{v/\gamma})^\gamma$ . ■

**Proof of Proposition 3.** From equation (25) for the bond price and the fact that  $1 = B_t(1 + r_t \Delta t)$  we find that the riskless interest rate  $r_t$  is given by:

$$\begin{aligned} r_t &= \frac{1 - \mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}]}{\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}]\Delta t} \\ &= \left( \frac{1}{(1 - \lambda_A \Delta t)\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}] + \lambda_A \Delta t \mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}] - 1} - 1 \right) \frac{1}{\Delta t}, \end{aligned} \quad (\text{A61})$$

where  $\xi_{A,t+\Delta t}/\xi_{At}$  is given by equation (32). We separately calculate  $\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$  and  $\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}]$ , and then take the limit  $\Delta t \rightarrow 0$ .

We start with the derivation of  $\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$  when  $\underline{v} < v_t < \bar{v}$ , and hence, by continuity, for a sufficiently small  $\Delta t$  the economy is unconstrained next period, so that  $\underline{v} < v_{t+\Delta t} < \bar{v}$ . In the unconstrained region, we have  $\Delta v_t = \mu_v \Delta t + \sigma_v \Delta w_t$  and the SPD is given by (A30). From the expression for the SPD, using expansions (A74) and (A76) from Lemma A.4 below, we obtain:

$$\begin{aligned} \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \middle| \text{normal} \right] &= \\ &= \mathbb{E}_t^A \left[ \left( (1 + a_t \Delta v_t + b_t (\Delta v_t)^2) (1 - r_A \Delta t - \kappa_A \Delta w_t) \right) \middle| \text{normal} \right] + o(\Delta t) \\ &= \mathbb{E}_t^A \left[ 1 + a_t \Delta v_t + b_t (\Delta v_t)^2 - r_A \Delta t - \kappa_A \Delta w_t - \kappa_A a_t \Delta v_t \Delta w_t \middle| \text{normal} \right] + o(\Delta t) \\ &= 1 + \left( a_t (\widehat{\mu}_v + \delta_A \sigma_D) + b_t \widehat{\sigma}_v^2 - r_A - \kappa_A \delta_A - \kappa_A a_t \widehat{\sigma}_v \right) \Delta t + o(\Delta t). \end{aligned} \quad (\text{A62})$$

Conditioning on the crisis state, we have:

$$\begin{aligned} \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \middle| \text{crisis} \right] &= (1 - \rho \Delta t) (1 + \mu_D \Delta t + J_D)^{-\gamma_A} \left( \frac{s(\max\{\underline{v}, v_t + \mu_v \Delta t + J_v\})}{s(v_t)} \right)^{-\gamma_A} \\ &= (1 + J_D)^{-\gamma_A} \left( \frac{s(\max\{\underline{v}, v_t + \widehat{J}_v\})}{s(v_t)} \right)^{-\gamma_A} + o(\Delta t). \end{aligned} \quad (\text{A63})$$

Substituting  $a_t$  and  $b_t$  from (A75) into equation (A62), and then substituting (A62) and (A63) into equation (A61), after simple algebra, we obtain  $r_t$  in (44) for the case  $\underline{v} < v_t < \bar{v}$ .

Now, we derive  $r_t$  at the boundaries  $\underline{v}$  and  $\bar{v}$ . The SPD is given by (32). Using expansions (A74) and (A76), we obtain the following expansion:

$$\begin{aligned} \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \middle| \text{normal} \right] &= \mathbb{E}_t^A \left[ \left( (1 + a_t \Delta v_t + b_t (\Delta v_t)^2) (1 - r_A \Delta t - \kappa_A \Delta w_t) \right. \right. \\ &\quad \left. \left. \times (1 + \Delta U_t + 0.5 (\Delta U_t)^2) \middle| \text{normal} \right] + o(\Delta t) \\ &= \mathbb{E}_t^A \left[ 1 + a_t \Delta v_t + b_t (\Delta v_t)^2 - r_A \Delta t - \kappa_A \Delta w_t - \kappa_A a_t \Delta v_t \Delta w_t \right. \\ &\quad \left. + \Delta U_t - \kappa_A \Delta w_t \Delta U_t + a_t \Delta U_t \Delta v_t + 0.5 (\Delta U_t)^2 \middle| \text{normal} \right] + O(\Delta t), \end{aligned} \quad (\text{A64})$$

where  $\Delta U_t$  is given by equation (A34). Using equation (31) for the process  $v_t$  and equation (A34) for  $\Delta U_t$ , for a fixed  $v_t$  and sufficiently small  $\Delta t$ , we find that  $\Delta v_t$  and  $\Delta U_t$  at the boundaries are given by:

$$\Delta v_t = \begin{cases} \min(0, \mu_v \Delta t + \sigma_v \Delta w_t), & \text{if } v_t = \bar{v}, \\ \max(0, \mu_v \Delta t + \sigma_v \Delta w_t), & \text{if } v_t = \underline{v}, \end{cases} \quad (\text{A65})$$

$$\Delta U_t = \begin{cases} 0, & \text{if } v_t < \bar{v}, \\ \max(0, \mu_v \Delta t + \sigma_v \Delta w_t), & \text{if } v_t = \bar{v}, \end{cases} \quad (\text{A66})$$

We note that for a sufficiently small  $\Delta t$  the sign of  $\mu_v \Delta t + \sigma_v \Delta w_t$  is solely determined by the second term  $\sigma_v \Delta w_t$  because it has the order of magnitude  $\sqrt{\Delta t}$ . Using the latter observation, substituting equations (A65) and (A66) into equation (A64) and computing the expectation, we obtain:

$$\mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \middle| \text{normal} \right] = 1 + \begin{cases} \frac{|\sigma_v|(1-a_t)}{2} \sqrt{\Delta t} + O(\Delta t), & \text{if } v_t = \bar{v}, \\ \frac{|\sigma_v|a_t}{2} \sqrt{\Delta t} + O(\Delta t), & \text{if } v_t = \underline{v}. \end{cases} \quad (\text{A67})$$

Substituting (A67) and (A63) into equation (A61) for the interest rate  $r_t$ , we obtain  $r_t$  in (44) for the case in which  $v_t$  is at the boundary.

To obtain the risk premium, we first decompose stock returns as follows:

$$\frac{\Delta S_t + (1 - l_A - l_B) D_{t+\Delta t} \Delta t}{S_t} = \mu_t \Delta t + \sigma_t \Delta w_t + J_t \Delta j_t. \quad (\text{A68})$$



Multiplying both sides of (A68) by  $\xi_{A,t+\Delta t}/\xi_{At}$  and taking expectations, we obtain:

$$\mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{\Delta S_t + (1 - l_A - l_B)D_{t+\Delta t}\Delta t}{S_t} \right] = \mu_t \Delta t \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \right] + \sigma_t \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \Delta w_t \right] + J_t \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \Delta j_t \right]. \quad (\text{A69})$$

On the other hand, from the equation for stock price (26), we find that:

$$\mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{\Delta S_t + (1 - l_A - l_B)D_{t+\Delta t}\Delta t}{S_t} \right] = 1 - \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \right]. \quad (\text{A70})$$

Combining the last two equations and the equation (A61) for the interest rate, we obtain:

$$\mu_t - r_t = - \left( \sigma_t \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \Delta w_t \right] + J_t \mathbb{E}_t^A \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \Delta j_t \right] \right) \frac{1 + r_t \Delta t}{\Delta t}. \quad (\text{A71})$$

Then, proceeding in the same way as with the calculation of interest rates and using similar expansions, we obtain equation (45) for the risk premium.

Finally, we obtain volatility  $\sigma_t$  and jump size  $J_t$  in normal times. In the unconstrained region  $\underline{v} < v_t < \bar{v}$ , stock price  $S_t$ , dividend  $D_t$ , and state variable  $v_t$  follow processes:  $dS_t = S_t[\mu_t dt + \sigma_t dw_t + J_t dj_t]$ ,  $dD_t = D_t[\mu_D dt + \sigma_D dw_t + J_D dj_t]$ , and  $dv_t = \hat{\mu}_v dt + \hat{\sigma}_v dw_t + (\max\{\underline{v}; v_t + \hat{J}_v\} - v_t) dj_t$ . Applying Ito's lemma to  $S_t = (1 - l_A - l_B)\hat{\Psi}(v_t; -\gamma_A)s(v_t)^{\gamma_A}D_t$ , and matching  $dw_t$  and  $dj_t$  terms, after some algebra, we obtain

$$\sigma_t = \sigma_D + \left( \frac{\hat{\Psi}'(v_t; -\gamma_A)}{\hat{\Psi}(v_t; -\gamma_A)} - \frac{\gamma_A(1 - s(v_t))}{\gamma_A(1 - s(v_t)) + \gamma_B s(v_t)} \right) \hat{\sigma}_v, \quad (\text{A72})$$

$$J_t = \frac{(1 + J_D)\hat{\Psi}(\max\{\underline{v}; v_t + \hat{J}_v\}; -\gamma_A)s(\max\{\underline{v}; v_t + \hat{J}_v\})^{\gamma_A}}{\hat{\Psi}(v_t; -\gamma_A)s(v_t)^{\gamma_A}} - 1. \quad \blacksquare \quad (\text{A73})$$

#### Lemma A.4 (Useful expansions).

1) For small increment  $\Delta v_t = v_{t+\Delta t} - v_t$  the ratio  $(s(v_{t+\Delta t})/s(v_t))^{-\gamma_A}$  has expansion:

$$\left( \frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} = 1 + a_t \Delta v_t + b_t (\Delta v_t)^2 + o(\Delta t), \quad (\text{A74})$$

where coefficients  $a_t$  and  $b_t$  are given by:

$$a_t = \frac{(1 - s_t)\Gamma_t}{\gamma_B}, \quad b_t = \frac{1}{2\gamma_B^2}(1 - s_t)^2\Gamma_t^2 + \frac{1}{2\gamma_A^2\gamma_B^2}s_t(1 - s_t)\Gamma_t^3, \quad (\text{A75})$$

$\Gamma_t = \gamma_A \gamma_B / (\gamma_A(1-s) + \gamma_B s)$  is the risk aversion of the representative investor and  $s_t$  is consumption share of investor A that solves equation (14).

2) For the case  $J_D = 0$ , the SPD in a one-investor economy can be expanded as follows:

$$e^{-\rho \Delta t} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} = 1 - r_A \Delta t - \kappa_A \Delta w_t + o(\Delta t), \quad (\text{A76})$$

where  $r_A$  and  $\kappa_A$  are the riskless rate and the Sharpe ratio in an economy populated only by investor A, given by:

$$r_A = \rho + \gamma_A(\mu_D + \delta_A \sigma_D) - \frac{\gamma_A(1 + \gamma_A)}{2} \sigma_D^2, \quad \kappa_A = \gamma_A \sigma_D. \quad (\text{A77})$$

**Proof of Lemma A.4.** 1) We expand the ratio on the left-hand side of (A74) using Taylor's formula, and observe that  $a_t = (s(v_t)^{-\gamma_A})' / s(v_t)^{-\gamma_A}$  and  $b_t = 0.5(s(v_t)^{-\gamma_A})'' / s(v_t)^{-\gamma_A}$ . Differentiating, we obtain the following expressions for  $a_t$  and  $b_t$ :

$$a_t = -\gamma_A \frac{s'(v_t)}{s(v_t)}, \quad b_t = \frac{\gamma_A(1 + \gamma_A)}{2} \left( \frac{s'(v_t)}{s(v_t)} \right)^2 - \frac{\gamma_A}{2} \frac{s''(v)}{s(v)}. \quad (\text{A78})$$

To find derivatives  $s'(v)$  and  $s''(v)$ , we differentiate equation (14) twice with respect to  $v$ , and obtain two equations for the derivatives:

$$1 = - \left( \frac{\gamma_A}{s_t} + \frac{\gamma_B}{1-s_t} \right) s'(v_t), \quad (\text{A79})$$

$$0 = \left( \frac{\gamma_A}{s_t^2} - \frac{\gamma_B}{(1-s_t)^2} \right) (s'(v_t))^2 - \left( \frac{\gamma_A}{s_t} + \frac{\gamma_B}{1-s_t} \right) s''(v_t). \quad (\text{A80})$$

Finding  $s'(v)$  and  $s''(v)$  from the system (A79)–(A80) and substituting them into expressions (A78) for coefficients  $a_t$  and  $b_t$ , after some algebra, we obtain expressions (A75).

2) Substituting  $D_{t+\Delta t}/D_t$  from (1) into equation (A76), after some algebra, we obtain:

$$\begin{aligned} e^{-\rho \Delta t} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} &= e^{-\rho \Delta t} (1 + \mu_D \Delta t + \sigma_D \Delta w_t)^{-\gamma_A} \\ &= (1 - \rho \Delta t) \left( 1 - \left( \gamma_A \mu_D - \frac{\gamma_A(1 + \gamma_A)}{2} \sigma_D^2 \right) \Delta t - \gamma_A \sigma_D \Delta w_t \right) + o(\Delta t) \\ &= 1 - r_A \Delta t - \kappa_A \Delta w_t + o(\Delta t). \quad \blacksquare \end{aligned} \quad (\text{A81})$$

**Proof of Proposition 4.** Consider a reflected arithmetic Brownian motion with boundaries  $\underline{v}$  and  $\bar{v}$  and dynamics  $dv_t = \hat{\mu}_v dt + \hat{\sigma}_v dw_t$  when  $\underline{v} < v_t < \bar{v}$ , where  $w_t$  is a Brownian

motion. The transition density for this process is given by (see Veestraeten, 2004):

$$\begin{aligned}
f_v(v, \tau; v_t, t) &= \frac{1}{\sqrt{2\pi\hat{\sigma}_v^2(\tau-t)}} \sum_{n=-\infty}^{+\infty} \left[ \exp\left(-\frac{2\hat{\mu}_v}{\hat{\sigma}_v^2}n(\bar{v}-\underline{v}) - \frac{(v-v_t-\hat{\mu}_v(\tau-t)+2n(\bar{v}-\underline{v}))^2}{2\hat{\sigma}_v^2(\tau-t)}\right) \right. \\
&+ \exp\left(-\frac{2\hat{\mu}_v}{\hat{\sigma}_v^2}(v_t-\underline{v}+n(\bar{v}-\underline{v})) - \frac{(v-v_t-\hat{\mu}_v(\tau-t)+2(v_t-\underline{v}+n[\bar{v}-\underline{v}]))^2}{2\hat{\sigma}_v^2(\tau-t)}\right) \left. \right] \\
&+ \frac{2\hat{\mu}_v}{\hat{\sigma}_v^2} \sum_{n=0}^{+\infty} \left[ \exp\left(-\frac{2\hat{\mu}_v}{\hat{\sigma}_v^2}(\bar{v}-v+n[\bar{v}-\underline{v}])\right) \mathcal{N}\left(\frac{v_t+\hat{\mu}_v(\tau-t)-v-2(\bar{v}-v+n[\bar{v}-\underline{v}])}{\hat{\sigma}_v\sqrt{\tau-t}}\right) \right. \\
&- \exp\left(\frac{2\hat{\mu}_v}{\hat{\sigma}_v^2}(v-\underline{v}+n[\bar{v}-\underline{v}])\right) \left(1-\mathcal{N}\left(\frac{v_t+\hat{\mu}_v(\tau-t)-v+2(v-\underline{v}+n[\bar{v}-\underline{v}])}{\hat{\sigma}_v\sqrt{\tau-t}}\right)\right) \left. \right], \tag{A82}
\end{aligned}$$

where  $\mathcal{N}(\cdot)$  is the cumulative distribution of a standard normal distribution. By  $F_v(v, \tau; v_t, t) = \text{Prob}\{v_\tau \leq v|v_t\}$  we denote the corresponding cumulative distribution function of  $v$  conditional on observing  $v_t$  at time  $t$ . We observe that  $s_t = s(v_t)$  is a decreasing function of  $v_t$  implicitly defined by equation (14). From the latter equation we also find that  $s^{-1}(x) = \gamma_B \ln(1-s) - \gamma_A \ln(s)$ . The cumulative distribution function of consumption share  $s_\tau$  at time  $\tau$  conditional on observing  $s_t$  at time  $t$  can then be found as follows:

$$\begin{aligned}
F(x, \tau; s_t, t) &= \text{Prob}\{s_\tau \leq x|s_t\} \equiv \text{Prob}\{s(v_\tau) \leq x|s_t\} \\
&= 1 - \text{Prob}\{v_\tau \leq s^{-1}(x)|v_t\} \\
&= 1 - \text{Prob}\{v_\tau \leq \gamma_B \ln(1-x) - \gamma_A \ln(x)|v_t\} \\
&= 1 - F_v(\gamma_B \ln(1-x) - \gamma_A \ln(x), \tau; v_t, t). \tag{A83}
\end{aligned}$$

Substituting  $v_t = \gamma_B \ln(1-s_t) - \gamma_A \ln(s_t)$  into (A83), differentiating CDF  $F(x, \tau; s_t, t)$  with respect to  $x$  and setting  $x = s$ , we find that the transition PDF for  $s$  is given by:

$$f(s, \tau; s_t, t) = \left(\frac{\gamma_A}{s} + \frac{\gamma_B}{1-s}\right) f_v(\gamma_B \ln(1-s) - \gamma_A \ln(s), \tau; \gamma_B \ln(1-s_t) - \gamma_A \ln(s_t), t), \tag{A84}$$

where transition density  $f_v(v, \tau; v_t, t)$  is given by equation (A82).

The stationary distribution of variable  $v$ , calculated in Veestraeten (2004), is given by:

$$f_v(v) = \frac{2\hat{\mu}_v}{\hat{\sigma}_v^2} \frac{\exp\left(\frac{2\hat{\mu}_v}{\hat{\sigma}_v^2}v\right)}{\exp\left(\frac{2\hat{\mu}_v}{\hat{\sigma}_v^2}\bar{v}\right) - \exp\left(\frac{2\hat{\mu}_v}{\hat{\sigma}_v^2}\underline{v}\right)}. \tag{A85}$$

Proceeding in the same way as for the derivation of transition PDF (A84), we obtain stationary PDF (47) for consumption share  $s$ . The PDF for the case  $\hat{\mu}_v = 0$  is obtained from the case  $\hat{\mu}_v \neq 0$  by taking the limit  $\hat{\mu}_v \rightarrow 0$ . ■

**Proof of Corollary 2.** The proof easily follows by substituting boundary conditions (37) into the equation (A72) for volatility  $\sigma_t$  at the boundary values  $\underline{v}$  and  $\bar{v}$ . ■

**Proof of Proposition 5.** Consider Lagrangian (A9) for the dynamic optimization of investor  $i$ . Differentiating this Lagrangian with respect to  $l_i$  and  $c_{it}$ , we obtain:

$$\frac{\partial V_i(W_{it}^*, v_t; l_i)}{\partial l_i} = \eta_{it} D_t \Delta t + e^{-\rho \Delta t} \mathbb{E}_t^i \left[ \frac{\partial V_i(W_{i,t+\Delta t}^*, v_{t+\Delta t}; l_i)}{\partial l_i} \right], \quad (\text{A86})$$

$$u'(c_{it}^*) = \eta_{it}. \quad (\text{A87})$$

By the envelope theorem (e.g., Back (2010, p.162)):

$$\frac{\partial V_i(W_{it}, v_t; l_i)}{\partial W} = u'_i(c_{it}^*), \quad (\text{A88})$$

$$\frac{\partial V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)}{\partial W} = u'_i(c_{i,t+\Delta t}^*). \quad (\text{A89})$$

Substituting (50), (A87), (A88), and (A89) into equation (A86), and simplifying, we find:

$$\widehat{S}_{it} = D_t \Delta t + \mathbb{E}_t^i \left[ e^{-\rho \Delta t} \frac{u'_i(c_{i,t+\Delta t}^*)}{u'_i(c_{it}^*)} \widehat{S}_{i,t+\Delta t} \right]. \quad (\text{A90})$$

From equation (32), we recall that the SPD of investor  $A$  is given by

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t + \Delta U_t} \frac{(c_{A,t+\Delta t}^*)^{-\gamma_A}}{(c_{At}^*)^{-\gamma_A}} \frac{D_{t+\Delta t}}{D_t}, \quad (\text{A91})$$

where  $\Delta U_t = \max\{0; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \bar{v}\}$ . Rewriting equation (A90) for investor  $A$  in terms of SPD (A91), we obtain:

$$\widehat{S}_{At} = D_t \Delta t + \mathbb{E}_t^A \left[ e^{-\Delta U_t} \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \widehat{S}_{A,t+\Delta t} \right]. \quad (\text{A92})$$

Following the same steps as in the proof of Lemma 2, we find that  $\widehat{S}_{At} = \widehat{\Psi}_i(v_t; -\gamma_A) s(v_t)^{\gamma_A} D_t$ , where  $\widehat{\Psi}_i(v; \theta)$  satisfies differential-difference equation (36) with boundary conditions (53).

Iterating equation (26) for stock and equation (A92) for shadow prices, we obtain:

$$S_t + (1 - l_A - l_B) D_t \Delta t = \frac{1}{\xi_t} \mathbb{E}_t^A \left[ \sum_{\tau=t}^{\infty} \xi_{\tau} (1 - l_A - l_B) D_{\tau} \Delta t \right], \quad (\text{A93})$$

$$\widehat{S}_{At} = \frac{1}{\xi_t} \mathbb{E}_t^A \left[ \sum_{\tau=t}^{\infty} e^{-(U_{\tau} - U_t)} \xi_{\tau} D_{\tau} \Delta t \right]. \quad (\text{A94})$$

Inequality  $(S_t + (1 - l_A - l_B) D_t \Delta t) / (1 - l_A - l_B) > \widehat{S}_{At}$  follows from the fact that  $U_t = \sum_{\tau=0}^t \Delta U_{\tau}$  is a nondecreasing process. In the continuous-time limit, we obtain that  $S_t / (1 - l_A - l_B) > \widehat{S}_{At}$ . Hence, the liquidity premium  $\Lambda_{At}$  is positive. The derivation of the shadow price of investor  $B$  is analogous. ■

# Internet Appendix for “Collateral Constraints and Asset Prices”

Georgy Chabakauri and Brandon Yueyang Han

## IA.1 Additional results

**Proposition IA.1 (Existence of boundaries  $\underline{v}$  and  $\bar{v}$ ).** *There exist constant boundaries  $\underline{v}$  and  $\bar{v}$  for the state variable  $v_t$  process (31) that solve equations  $\Phi_A(\bar{v}) = 0$  and  $\Phi_B(\underline{v}) = 0$ .*

**Proof of Proposition IA.1.** We here show the existence of  $\bar{v}$  that solves  $\Phi_A(\bar{v}) = 0$ , where  $\Phi_A(v)$  is given by equation (A26) in the Appendix of the paper. The proof for  $\underline{v}$  is analogous.

We note that  $\Phi_A(v_t) \geq 0$  because of the constraint  $W_{At} \geq 0$ . Suppose,  $\bar{v}$  does not exist, and hence  $\Phi_A(v_t) > 0$  for all  $v_t$ . From equation (14) for consumption share  $s$  we observe that  $s(v_t) \rightarrow 0$  when  $v_t \rightarrow +\infty$ . For arbitrary  $\varepsilon \in (0, l_A)$  choose  $v_t$  sufficiently large, so that  $s(v_t) - l_A < -\varepsilon$ . Let  $T(v_t)$  be the stopping time, defined as

$$T(v_t) = \inf\{\tau : s(v_\tau) - l_A \geq -\varepsilon\}. \quad (\text{IA.1})$$

From equation (A26) for  $\Phi_A(v_t)$  we obtain the following inequality:

$$\begin{aligned} \Phi_A(v_t)s(v_t)^{1-\gamma_A} &\leq -\varepsilon \mathbb{E}_t^A \left[ \sum_{\tau=t}^{T(v_t)} e^{-\rho(\tau-t)} \left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A} s(v_\tau)^{-\gamma_A} \Delta t \right] \\ &+ \mathbb{E}_t^A \left[ \sum_{\tau=T(v_t)+\Delta t}^{+\infty} e^{-\rho(\tau-t)} \left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A} s(v_\tau)^{-\gamma_A} (s(v_\tau) - \varepsilon) 1_{\{s(v_\tau) \geq \varepsilon\}} \Delta t \right] \\ &\leq -\varepsilon (l_A - \varepsilon)^{-\gamma_A} \mathbb{E}_t^A \left[ \sum_{\tau=t}^{T(v_t)} e^{-\rho(\tau-t)} \left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A} \Delta t \right] \\ &+ \max(1; \varepsilon^{1-\gamma_A}) \mathbb{E}_t^A \left[ \sum_{\tau=T(v_t)+\Delta t}^{+\infty} e^{-\rho(\tau-t)} \left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A} \Delta t \right]. \end{aligned} \quad (\text{IA.2})$$

Next, we show that  $T(v_t) \rightarrow +\infty$  as  $v_t \rightarrow +\infty$ . Let  $\hat{v}$  be such that  $s(\hat{v}) = l_A - \varepsilon$ . Then, because  $s(v_t)$  is a decreasing function, the stopping time (IA.1) can be rewritten as  $T(v_t) = \inf\{\tau : v_\tau \leq \hat{v}\}$ . We note that  $T(v_t) \geq \hat{T}$ , where  $\hat{T}$  is the minimal time required to get from  $v_t$  to  $\hat{v}$ , which is the time when  $\Delta w_t = -\sqrt{\Delta t}$  and  $\Delta j_t = 1$  along the path. Time

$\hat{T}$  is found from the condition  $v_t + (\hat{T}/\Delta t)(\mu_v \Delta t - \sigma_v \sqrt{\Delta t} + J_v) = \hat{v}$ , where  $J_v < 0$  and  $\mu_v \Delta t - \sigma_v \sqrt{\Delta t} < 0$  for small  $\Delta t$ . Hence,  $\hat{T} \rightarrow +\infty$  as  $v_t \rightarrow +\infty$ , and hence  $T(v_t) \rightarrow +\infty$ . We also note that  $\mathbb{E}_t[\sum_{\tau=t}^{\infty} e^{-r(\tau-t)} D_{\tau}^{1-\gamma_A} \Delta t] < +\infty$  by condition (15). Therefore, for a sufficiently large  $v_t$  we obtain from inequality (IA.2) that  $\Phi_A(v_t) < 0$ , which contradicts initial assumption that  $\Phi_A(v_t) \geq 0$  for all  $v_t$ . Hence, there exists  $\bar{v}$  such that  $\Phi_A(\bar{v}) = 0$ . ■

**Lemma IA.1 (Unconstrained optimization).** *Consider an infinitesimal unconstrained investor with risk aversion  $\gamma_i$  and labor income  $l_i D_t$ ,  $i = A, B$ , who lives in the economy in which the state price density is given by (32). The investor's value function is given by*

$$V_i^{unc}(W_t, v_t) = \frac{(W_t + l_i/(1 - l_A - l_B)S_t)^{1-\gamma_i}}{1 - \gamma_i} h_i(v_t)^{\gamma_i}, \quad (\text{IA.3})$$

where  $h(v_t)$  is a uniformly bounded wealth-consumption ratio, given by:

$$h_i(v_t) = \mathbb{E}_t^i \left[ \sum_{\tau=t}^{+\infty} \left( \frac{\xi_{i\tau}}{\xi_{it}} \right)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i} \Delta t \right]. \quad (\text{IA.4})$$

The investor's optimal consumption is given by  $c_{i\tau}^* = \ell(\xi_{i\tau} e^{\rho(\tau-t)})^{-1/\gamma_i}$ , where  $\ell$  is a constant. Moreover, for all feasible consumptions  $c_t$  the following inequalities are satisfied:

$$\mathbb{E}_t^i \left[ \sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_i(c_{\tau}) \Delta t \right] \leq \mathbb{E}_t^i \left[ \sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_i(c_{\tau}^*) \Delta t \right] = V_i^{unc}(W_t, v_t), \quad (\text{IA.5})$$

$$\limsup_{T \rightarrow \infty} e^{-\rho T} \mathbb{E}_t^i \left[ V_i^{unc}(W_T, v_T) \right] \leq 0. \quad (\text{IA.6})$$

**Proof of Lemma IA.1.** We solve the problem using the martingale method. The static budget constraint is given by:

$$\mathbb{E}_t^i \left[ \sum_{\tau=t}^{+\infty} \frac{\xi_{i\tau}}{\xi_{it}} c_{\tau}^* \right] = W_t + \frac{l_i S_t}{1 - l_A - l_B}, \quad (\text{IA.7})$$

where the last term is the value of the labor income. Because the dividends and labor incomes are collinear, the value of the labor income is given by:

$$\mathbb{E}_t^i \left[ \sum_{\tau=t}^{+\infty} \frac{\xi_{i\tau}}{\xi_{it}} (l_i D_{\tau}) \right] = \frac{l_i S_t}{1 - l_A - l_B}.$$

The first order condition gives the optimal consumption  $c_{\tau}^* = \ell(\xi_{i\tau}/\xi_{it} e^{\rho(\tau-t)})^{-1/\gamma_i}$ , where  $\ell$  is the Lagrange multiplier that can be found by substituting  $c_{\tau}^*$  into (IA.7). Finding

the multiplier  $\ell$  and substituting  $c_\tau^*$  into the objective function, we obtain the value function (IA.3), where  $h(v_t)$  is given by (IA.4).

Next, we show that  $h(v_t)$  is uniformly bounded. First, we consider the case  $\gamma_i \geq 1$ . Using equation (IA.4) and Hölder's inequality, we obtain:

$$h_i(v_t) = \mathbb{E}_t^i \left[ \sum_{\tau=t}^{\infty} \left( \frac{\xi_{i\tau}}{\xi_{it}} \right)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i} \right] \leq \left( \mathbb{E}_t^i \left[ \sum_{\tau=t}^{\infty} \frac{\xi_{i\tau} D_\tau}{\xi_{it} D_t} \right] \right)^{1-1/\gamma_i} \left( \mathbb{E}_t^i \left[ \sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)} \left( \frac{D_\tau}{D_t} \right)^{1-\gamma_i} \right] \right)^{1/\gamma_i}.$$

We note that both multipliers on the right-hand side of the latter inequality are bounded. The first multiplier equals the price-dividend ratio and is bounded by Proposition 2. The second multiplier is bounded due to condition (15) on the model parameters. Consider now the case  $\gamma_i \leq 1$ . From the FOCs (29) and the fact that  $\underline{s} \leq s \leq \bar{s}$ , we obtain:

$$\frac{\xi_{iT}}{\xi_{it}} \geq e^{-\rho(T-t)} \left( \frac{c_T^*}{c_t^*} \right)^{-\gamma_i} \geq e^{-\rho(T-t)} \left( \frac{D_T}{D_t} \right)^{-\gamma_i} \left( \frac{\bar{s}}{\underline{s}} \right)^{-\gamma_i}.$$

From the latter inequality it follows that

$$\mathbb{E}_t^i \left[ \left( \frac{\xi_{i\tau}}{\xi_{it}} \right)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i} \right] \leq \left( \frac{\bar{s}}{\underline{s}} \right)^{1-\gamma_i} \mathbb{E}_t^i \left[ e^{-\rho(\tau-t)} \left( \frac{D_\tau}{D_t} \right)^{1-\gamma_i} \right]. \quad (\text{IA.8})$$

The inequality (IA.8) and condition (15) imply that the infinite series in (IA.4) converges and function  $h_i(v)$  is uniformly bounded. We also observe that  $h_i(v) \geq \Delta t > 0$ .

Now, we prove inequality (IA.5). We consider feasible consumption streams satisfying condition  $W_t + l_i/(1-l_A-l_B)S_t \geq 0$  for all  $t$ , which means that investor's aggregate wealth is nonnegative at all times so that investor does not go bankrupt. From the investor's budget constraint and the latter inequality, for all feasible consumptions we obtain:

$$W_t + \frac{l_i S_t}{1-l_A-l_B} \geq \mathbb{E}_t^i \left[ \sum_{\tau=t}^T \frac{\xi_{i\tau}}{\xi_{it}} c_\tau \Delta t \right] + \mathbb{E}_t^i \left[ \frac{\xi_{iT}}{\xi_{it}} \left( W_T + \frac{l_i S_T}{1-l_A-l_B} \right) \right] \geq \mathbb{E}_t^i \left[ \sum_{\tau=t}^T \frac{\xi_{i\tau}}{\xi_{it}} c_\tau \Delta t \right]. \quad (\text{IA.9})$$

Consider the weighting function  $w_t$  given by

$$w_\tau = \frac{\left( \frac{\xi_{i\tau}}{\xi_{it}} \right)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i}}{\hat{h}_{iT}(v_t)}, \quad \text{where } \hat{h}_{iT}(v_t) = \mathbb{E}_t^i \left[ \sum_{\tau=t}^T \left( \frac{\xi_{i\tau}}{\xi_{it}} \right)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i} \Delta t \right]. \quad (\text{IA.10})$$

We note that  $\mathbb{E}_t^i \left[ \sum_{\tau=t}^T w_\tau \Delta t \right] = 1$ . Using Jensen's inequality and inequality (IA.9), we

obtain:

$$\begin{aligned}
\mathbb{E}_t^i \left[ \sum_{\tau=t}^T \frac{e^{-\rho(\tau-t)} c_\tau^{1-\gamma_i}}{1-\gamma_i} \Delta t \right] &= \mathbb{E}_t^i \left[ \sum_{\tau=t}^T \frac{\left( (\xi_{i\tau}/\xi_{it})^{1/\gamma_i} e^{\rho(\tau-t)/\gamma_i} c_\tau \right)^{1-\gamma_i} w_\tau \Delta t}{1-\gamma_i} \right] \hat{h}_{iT}(v_t) \\
&\leq \frac{\left( \mathbb{E}_t^i \left[ \sum_{\tau=t}^T (\xi_{i\tau}/\xi_{it})^{1/\gamma_i} e^{\rho(\tau-t)/\gamma_i} c_\tau w_\tau \Delta t \right] \right)^{1-\gamma_i}}{1-\gamma_i} \hat{h}_{iT}(v_t) \\
&= \frac{\left( \mathbb{E}_t^i \left[ \sum_{\tau=t}^T (\xi_{i\tau}/\xi_{it}) c_\tau \Delta t \right] \right)^{1-\gamma_i}}{1-\gamma_i} \hat{h}_{iT}(v_t)^{\gamma_i} \leq \frac{\left( W_t + \frac{l_i S_t}{1-l_A-l_B} \right)^{1-\gamma_i}}{1-\gamma_i} \hat{h}_{iT}(v_t)^{\gamma_i}.
\end{aligned} \tag{IA.11}$$

Taking limit  $T \rightarrow \infty$  in (IA.11), and noting that  $\hat{h}_{iT}(v_t) \rightarrow h_i(v_t)$ , we obtain (IA.5).

Finally, we prove inequality (IA.6). Because  $c_\tau \geq 0$ , from inequality (IA.9), we obtain:

$$\mathbb{E}_t^i \left[ \frac{\xi_{iT}}{\xi_{it}} \left( W_t + \frac{l_i S_t}{1-l_A-l_B} \right) \right] \leq W_t + \frac{l_i S_t}{1-l_A-l_B}. \tag{IA.12}$$

Using Jensen's inequality following the same steps as in inequality (IA.11), we obtain:

$$\begin{aligned}
\frac{\mathbb{E}_t^i \left[ \left( W_t + \frac{l_i S_t}{1-l_A-l_B} \right)^{1-\gamma_i} \right]}{1-\gamma_i} &\leq \frac{\left( \mathbb{E}_t^i \left[ \frac{\xi_{iT}}{\xi_{it}} \left( W_t + \frac{l_i S_t}{1-l_A-l_B} \right) \right] \right)^{1-\gamma_i}}{1-\gamma_i} \left( \mathbb{E}_t^i \left[ \left( \frac{\xi_{iT}}{\xi_{it}} \right)^{-\frac{1-\gamma_i}{\gamma_i}} \right] \right)^{\gamma_i} \\
&\leq \frac{\left( W_t + \frac{l_i S_t}{1-l_A-l_B} \right)^{1-\gamma_i}}{1-\gamma_i} \left( \mathbb{E}_t^i \left[ \left( \frac{\xi_{iT}}{\xi_{it}} \right)^{-\frac{1-\gamma_i}{\gamma_i}} \right] \right)^{\gamma_i}.
\end{aligned}$$

The above inequality and the boundedness of  $h_i(v_t)$  then imply the following inequality:

$$e^{-\rho(\tau-t)} \mathbb{E}_t^i [V_{iT}^{unc}] \leq Const \times V_{it}^{unc} \left( \mathbb{E}_t^i \left[ \left( \frac{\xi_{iT}}{\xi_{it}} \right)^{-\frac{1-\gamma_i}{\gamma_i}} e^{-\rho(\tau-t)/\gamma_i} \right] \right)^{\gamma_i}. \tag{IA.13}$$

Inequality (IA.13) also holds for  $\gamma_i = 1$  if CRRA preferences are replaced with logarithmic preferences. Suppose,  $\gamma_i > 1$ . Then, inequality (IA.6) is satisfied because  $V_i^{unc} < 0$ . Suppose,  $\gamma_i \leq 1$ . Then, using inequalities (IA.8), (IA.13), and condition (15), we obtain:

$$e^{-\rho(\tau-t)} \mathbb{E}_t^i [V_{iT}^{unc}] \leq Const \times \left( \mathbb{E}_t^i \left[ e^{-\rho(\tau-t)} \left( \frac{D_\tau}{D_t} \right)^{1-\gamma_i} \right] \right)^{\gamma_i} \rightarrow 0, \text{ as } T \rightarrow \infty. \blacksquare$$

**Lemma IA.2.** *Let  $\mathcal{P}(V)$  be a point-wise monotone operator such that for all point-wise bounded functions  $V_1$  and  $V_2$  such that  $V_1 \leq V_2 \Rightarrow \mathcal{P}(V_1) \leq \mathcal{P}(V_2)$ . Suppose further there exist point-wise bounded functions  $\underline{V}$  and  $\bar{V}$  such that  $\underline{V} \leq \bar{V}$ ,  $\mathcal{P}(\underline{V}) \geq \underline{V}$ , and  $\mathcal{P}(\bar{V}) \leq \bar{V}$ . Then, there exists a point-wise bounded function  $V^*$  such that: 1)  $\underline{V} \leq V^* \leq \bar{V}$ ; 2)  $V^* \leq \mathcal{P}(V^*)$ ; 3)  $\mathcal{P}^n(\underline{V}) \rightarrow V^*$  point-wise as  $n \rightarrow \infty$ .*



**Proof of Lemma IA.2.** From the monotonicity of the operator  $\mathcal{P}(V)$  and the definitions of  $\underline{V}$  and  $\bar{V}$ , we obtain:

$$\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}(\bar{V}) \leq \bar{V}. \quad (\text{IA.14})$$

Applying the operator  $\mathcal{P}$  to inequalities (IA.14), and then using the definitions of  $\underline{V}$  and  $\bar{V}$ , we obtain:  $\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}^2(\underline{V}) \leq \bar{V}$ . Proceeding in the same way  $n$  times we obtain  $\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}^2(\underline{V}) \leq \dots \leq \mathcal{P}^n(\underline{V}) \leq \bar{V}$ . Consequently,  $\mathcal{P}^n(\underline{V})$  is point-wise increasing and bounded, and hence, converges to some function  $V^*$  such that  $\underline{V} \leq V^* \leq \bar{V}$  and  $\mathcal{P}^n(\underline{V}) \leq V^*$ . Applying operator to both sides of the latter inequality, we find that  $\mathcal{P}^{n+1}(\underline{V}) \leq \mathcal{P}(V^*)$ . Taking limit, we find that  $V^* \leq \mathcal{P}(V^*)$ . ■

**Proposition IA.2 (Verification of optimality).** *Consider an infinitesimal investor  $i$  who lives in an economy in which the state price density is given by equation (32). Suppose, this investor maximizes expected discounted utility (6) subject to a self-financing budget constraint and the collateral constraint (9). Then, there exists unique bounded value function  $V_i^*$  satisfying the dynamic programming equation (A4) and the transversality condition, such that for all feasible consumptions*

$$V_{it}^* \geq \mathbb{E}_t^i \left[ \sum_{\tau=t}^{+\infty} u(c_{i\tau}) \Delta t \right], \quad (\text{IA.15})$$

and, moreover,

$$V_{it}^* = \mathbb{E}_t^i \left[ \sum_{\tau=t}^{+\infty} u(c_{i\tau}^*) \Delta t \right], \quad (\text{IA.16})$$

for the optimal consumptions given by FOCs (29).

**Proof of Proposition IA.2.** Consider the following operator:

$$\mathcal{P}_i(V) = \max_{c_t} \left\{ u_i(c_t) \Delta t + e^{-\rho \Delta t} \mathbb{E}_t^i [V_{i,t+\Delta t}] \right\}, \quad i = A, B \quad (\text{IA.17})$$

where maximization is subject to budget constraint (A5) and collateral constraint (A6). Consider the following functions:

$$\underline{V}_{it} = \begin{cases} 0, & \gamma_i < 1, \\ \mathbb{E}_t^i \left[ \sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_i(l_i D_\tau) \Delta t \right], & \gamma_i \geq 1, \end{cases} \quad \bar{V}_{it} = \begin{cases} V_{it}^{unc}, & \gamma_i \leq 1, \\ 0, & \gamma_i > 1, \end{cases} \quad (\text{IA.18})$$

where  $V_t^{unc}$  is given by (IA.3).

We observe that for  $\gamma_i \geq 1$  function  $\underline{V}_i$  is bounded due to condition (15) imposed on model parameters. Because  $c_t = l_i D_t$  is feasible, we obtain that

$$\mathcal{P}(\underline{V}_i) \geq u_i(l_i D_t) + e^{-\rho \Delta t} \mathbb{E}_t^i \left[ \sum_{\tau=t+\Delta t}^{+\infty} e^{-\rho(\tau-t)} u_i(l_i D_\tau) \Delta t \right] = \underline{V}_i.$$

For  $\gamma_i < 1$  it is easy to see that  $\mathcal{P}(\underline{V}_i) \geq \underline{V}_i$  because  $u_i(c) > 0$ . Next, we prove that  $\mathcal{P}_i(\bar{V}_i) \leq \bar{V}_i$ . The latter inequality is straightforward for  $\gamma_i > 1$  because  $\mathcal{P}_i(0) \leq 0$ . Suppose now,  $\gamma_i \leq 1$ . Consider operator  $\tilde{\mathcal{P}}_i(V_i)$  given by equation (IA.17), where the maximization is subject to the budget constraint (A5), but without the collateral constraint (A6). Hence,  $\mathcal{P}_i(V_i) \leq \tilde{\mathcal{P}}_i(V_i)$ . By Lemma IA.1,  $\bar{V}_i^{unc}$  is the solution of the unconstrained optimization, and hence  $\bar{V}_i = \tilde{\mathcal{P}}_i(\bar{V}_i)$ . Therefore,  $\mathcal{P}_i(\bar{V}_i) \leq \tilde{\mathcal{P}}_i(\bar{V}_i) = \bar{V}_i$ .

We drop subscript and superscript  $i$  for convenience. Consider the sequence  $V_{n+1} = \mathcal{P}(V_n)$ , with  $V_0 = \underline{V}$ , where  $\underline{V}$  is given in (IA.18). Then, by Lemma IA.2,  $V_n \rightarrow V^*$  point-wise as  $n \rightarrow \infty$ . Next, we show that  $V^*$  is the value function and  $\mathcal{P}(V^*) = V^*$ . By the definition operator  $\mathcal{P}(V)$  in (IA.17), for all feasible consumption streams

$$\begin{aligned} V_{n+1} &\geq u(c_t)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t[V_n(W_{t+\Delta t}; v_{t+\Delta t})] \\ &\geq \mathbb{E}_t\left[\sum_{\tau=t}^{n\Delta t} e^{-\rho(\tau-t)}u(c_\tau)\Delta t\right] + e^{-\rho n\Delta t}\mathbb{E}_t[\underline{V}]. \end{aligned} \tag{IA.19}$$

Taking point-wise limit  $n \rightarrow \infty$  in (IA.19) and taking into account that  $\mathbb{E}_t[\underline{V}]$  is point-wise bounded, we obtain inequality (IA.15).

By Lemma IA.2,  $V^* \leq \mathcal{P}(V^*)$  and  $V^* \leq \bar{V}$ , where  $\bar{V}$  is given in (IA.18), and hence

$$\begin{aligned} V^*(W_t, v_t) &\leq u(c_t^*)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t[V^*(W_{t+\Delta t}; v_{t+\Delta t})] \\ &\leq \mathbb{E}_t\left[\sum_{\tau=t}^T u(c_\tau^*)\Delta t\right] + e^{-\rho T}\mathbb{E}_t[V^*(W_T, v_T)] \\ &\leq \mathbb{E}_t\left[\sum_{\tau=t}^T u(c_\tau^*)\Delta t\right] + e^{-\rho T}\mathbb{E}_t[\bar{V}(W_T, v_T)], \end{aligned} \tag{IA.20}$$

where  $c^*$  is the optimal consumption that solves optimization in equation (IA.17).

We note that  $\bar{V} = 0$  for  $\gamma > 1$  and  $\limsup e^{-\rho T}\mathbb{E}_t[\bar{V}(W_T, v_T)] \leq 0$  as  $T \rightarrow \infty$  for  $\gamma \leq 1$ , by Lemma IA.1. Taking the limit  $T \rightarrow \infty$  in (IA.20) we find that  $V^* \leq \mathbb{E}_t\left[\sum_{\tau=t}^{+\infty} u(c_\tau^*)\Delta t\right]$ , which along with inequality (IA.15) yields (IA.16). Equation (IA.16) along with inequality (IA.20) also imply that  $V^* = \mathcal{P}(V^*)$ . Moreover,  $V^*$  is point-wise bounded because  $\underline{V} \leq V^* \leq \bar{V}$ . Then, given the existence of the value function, the optimal consumptions are given by (29). Finally, we show that  $V^*$  satisfies the transversality condition. We note that  $e^{-\rho(T-t)}\mathbb{E}_t[\underline{V}_T] \leq e^{-\rho(T-t)}\mathbb{E}_t[V_T^*] \leq e^{-\rho(T-t)}\mathbb{E}_t[\bar{V}_T]$ . Taking limit  $T \rightarrow \infty$  we find that the upper and lower bound in the latter equation converge to 0, and hence the transversality condition is satisfied for  $V^*$ . ■

### Proposition IA.3 (Closed-form solutions).

1) In the limit  $\Delta t \rightarrow 0$  the price-dividend ratio  $\Psi$  and wealth-consumption ratios  $\Phi_i$  are given by equations (34) and (35), where function  $\widehat{\Psi}(v; \theta)$  is given by:

$$\widehat{\Psi}(v; \theta) = \int_{\underline{v}}^v s(y)^\theta \widehat{\psi}(v-y) dy + \frac{\int_{\underline{v}}^{\bar{v}} s(y)^\theta [\widehat{\psi}'(\bar{v}-y) - \widehat{\psi}(\bar{v}-y)] dy}{1 + H \left( \widehat{\psi}(\bar{v}-\underline{v}) - \int_0^{\bar{v}-\underline{v}} \widehat{\psi}(y) dy \right)} \left( 1 - H \int_0^{v-\underline{v}} \widehat{\psi}(y) dy \right), \quad (\text{IA.21})$$

where  $s(y)$  solves equation (14),<sup>1</sup> and  $\widehat{\psi}(x)$ ,  $H$  and some auxiliary variables are given by:

$$\widehat{\psi}(x) = \frac{2}{\widehat{\sigma}_v^2} \sum_{n=0}^{\infty} \left[ \left( \frac{2\lambda_A(1+J_D)^{1-\gamma_A}}{\widehat{\sigma}_v^2} \right)^n \frac{\exp\left(\frac{(\zeta_+ + \zeta_-)(x + n\widehat{J}_v)}{2}\right)}{(\zeta_+ - \zeta_-)^{2n+1} n!} \right. \quad (\text{IA.22})$$

$$\left. \times Q_n \left( \frac{(\zeta_+ - \zeta_-)(x + n\widehat{J}_v)}{2} \right) \mathbf{1}_{\{x+n\widehat{J}_v \geq 0\}} \right], \quad (\text{IA.23})$$

$$Q_n(x) = \exp(-x) \sum_{m=0}^n (2x)^{n-m} \frac{(n+m)!}{m!(n-m)!} - \exp(x) \sum_{m=0}^n (-2x)^{n-m} \frac{(n+m)!}{m!(n-m)!}, \quad (\text{IA.24})$$

$$H = \lambda_A + \rho - (1 - \gamma_A)\mu_D + \frac{(1 - \gamma_A)\gamma_A}{2} \sigma_D^2 - \lambda_A(1 + J_D)^{1-\gamma_A}, \quad (\text{IA.25})$$

$$\zeta_{\pm} = -\frac{\widehat{\mu}_v^A + (1 - \gamma_A)\widehat{\sigma}_v\sigma_D \mp \sqrt{(\widehat{\mu}_v^A + (1 - \gamma_A)\widehat{\sigma}_v\sigma_D)^2 + 2\widehat{\sigma}_v^2 \left( \lambda_A + \rho - (1 - \gamma_A)\mu_D^A + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2 \right)}}{\widehat{\sigma}_v^2}, \quad (\text{IA.26})$$

$$\widehat{\mu}_v^A = \widehat{\mu}_v + \delta_A \widehat{\sigma}_v, \quad \mu_D^A = \mu_D + \delta_A \sigma_D. \quad (\text{IA.27})$$

2) Stock return volatility in normal times and the jump size  $J_t$  are given by:

$$\sigma_t = \sigma_D + \left( \frac{\widehat{\Psi}'(v_t; -\gamma_A)}{\widehat{\Psi}(v_t; -\gamma_A)} - \frac{\gamma_A(1-s(v_t))}{\gamma_A(1-s(v_t)) + \gamma_B s(v_t)} \right) \widehat{\sigma}_v, \quad (\text{IA.28})$$

$$J_t = \frac{(1 + J_D)\widehat{\Psi}(\max\{v; v_t + \widehat{J}_v\}; -\gamma_A) s(\max\{v; v_t + \widehat{J}_v\})^{\gamma_A}}{\widehat{\Psi}(v_t; -\gamma_A) s(v_t)^{\gamma_A}} - 1. \quad (\text{IA.29})$$

Numbers of shares  $n_{i,st}^*$  and leverage  $L_{it} = -b_{it}B_{it}$  to market price  $S_t$  ratio are given by:

$$n_{i,st}^* = \frac{\widehat{\Phi}_i(v_t)\sigma_D + \widehat{\Phi}_i'(v_t)\widehat{\sigma}_v}{\Psi(v_t)\sigma_t}, \quad \frac{L_{it}}{S_t} = n_{i,st} - \frac{\widehat{\Phi}_i(v_t)}{\Psi(v_t)(1-l_A-l_B)}, \quad (\text{IA.30})$$

where  $\widehat{\Phi}_A(v_t) = \Phi_A(v_t)s(v_t)$  and  $\widehat{\Phi}_B(v_t) = \Phi_B(v_t)(1-s(v_t))$ .

<sup>1</sup>Although  $s(y)$  is not in closed form, we observe from equation (14) that its inverse is given by  $s^{-1}(x) = \gamma_B \ln(x) - \gamma_A \ln(1-x)$ . The change of variable  $x = s(y)$  eliminates implicit functions, similar to Chabakauri (2015). We keep all integrals in terms of  $s(y)$  because  $s(y)$  is intuitive and easily computable from (14).

**Proof of Proposition IA.3.** 1) First, we solve the differential-difference equation in Lemma 2. We denote  $g(x) = \widehat{\Psi}(x + \underline{v}; \theta)$  and apply the following changes of variables:

$$\begin{aligned} x &= v - \underline{v}, & \tilde{\sigma} &= \widehat{\sigma}_v, & \tilde{\mu} &= \widehat{\mu}_v + \delta_A \widehat{\sigma}_v + (1 - \gamma_A) \sigma_D \widehat{\sigma}_v, & \tilde{J} &= -\widehat{J}_v, \\ \tilde{\lambda} &= \lambda_A (1 + J_D)^{1-\gamma_A}, & \tilde{\rho} &= \lambda_A + \rho - (1 - \gamma_A)(\mu_D + \delta_A \sigma_D) + \frac{(1 - \gamma_A)\gamma_A}{2} \sigma_D^2. \end{aligned} \quad (\text{IA.31})$$

Equations (36) and (37) with new variables now become:

$$\frac{\tilde{\sigma}^2}{2} g''(x) + \tilde{\mu} g'(x) - \tilde{\rho} g(x) + \tilde{\lambda} g(\max\{x - \tilde{J}, 0\}) + s(x + \underline{v})^\theta = 0, \quad (\text{IA.32})$$

$$g'(0) = 0, \quad g(\bar{v} - \underline{v}) - g'(\bar{v} - \underline{v}) = 0. \quad (\text{IA.33})$$

Let  $\mathcal{L}[g(x)] = \int_0^\infty e^{-zx} g(x) dx$  be the Laplace transform of  $g(x)$ , and similarly for other functions. The Laplace transforms of  $g'(x)$ ,  $g''(x)$  and  $g(\max\{x - \tilde{J}, 0\})$  are given by:

$$\begin{aligned} \mathcal{L}[g'(x)] &= z\mathcal{L}[g(x)] - g(0), \\ \mathcal{L}[g''(x)] &= z^2\mathcal{L}[g(x)] - zg(0) - g'(0), \\ \mathcal{L}[g(\max\{x - \tilde{J}, 0\})] &= \int_0^\infty e^{-zx} g(\max\{x - \tilde{J}, 0\}) dx \\ &= \int_0^{\tilde{J}} e^{-zx} g(0) dx + \int_{\tilde{J}}^\infty e^{-zx} g(x - \tilde{J}) dx \\ &= \frac{1}{z} (1 - e^{-\tilde{J}z}) g(0) + e^{-\tilde{J}z} \mathcal{L}[g(x)]. \end{aligned} \quad (\text{IA.34})$$

Applying the transform to equation (IA.32), we arrive at the following equation:

$$\begin{aligned} \frac{\tilde{\sigma}^2}{2} \left( z^2 \mathcal{L}[g(x)] - zg(0) - g'(0) \right) + \tilde{\mu} (z\mathcal{L}[g(x)] - g(0)) - \tilde{\rho} \mathcal{L}[g(x)] \\ + \tilde{\lambda} \left( e^{-\tilde{J}z} \mathcal{L}[g(x)] + \frac{1}{z} (1 - e^{-\tilde{J}z}) g(0) \right) + \mathcal{L}[s(x + \underline{v})^\theta] = 0. \end{aligned} \quad (\text{IA.35})$$

Applying boundary condition  $g'(0) = 0$  and solving for  $\mathcal{L}[g(x)]$ , we obtain:

$$\mathcal{L}[g(x)] = \frac{\mathcal{L}[s(x + \underline{v})^\theta]}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} + g(0) \left( \frac{1}{z} - \frac{\tilde{\rho} - \tilde{\lambda}}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} \cdot \frac{1}{z} \right). \quad (\text{IA.36})$$

Now define a new function  $\widehat{\psi}(x)$  through inverse Laplace transform

$$\widehat{\psi}(x) = \mathcal{L}^{-1} \left[ \frac{1}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} \right]. \quad (\text{IA.37})$$

Next, we apply inverse transform to each term in (IA.36). Noting that  $\mathcal{L}^{-1}[1/z] = 1$  and using the theorem which states that Laplace transform of a convolution is the product of Laplace transforms, we derive the following inverse transforms:

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[s(x+\underline{v})^\theta\right]}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}}\right]&= \int_0^x s(y+\underline{v})^\theta \cdot \hat{\psi}(x-y)dy, \\ \mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}} \cdot \frac{1}{z}\right]&= \int_0^x \mathbf{1}_{\{y \geq 0\}} \cdot \hat{\psi}(x-y)dy = \int_0^x \hat{\psi}(y)dy.\end{aligned}\tag{IA.38}$$

The linearity of the Laplace transform gives the following equation:

$$g(x) = \mathcal{L}^{-1}[\mathcal{L}[g(x)]] = \int_0^x s(y+\underline{v})^\theta \cdot \hat{\psi}(x-y)dy + g(0) \left[1 - (\tilde{\rho} - \tilde{\lambda}) \int_0^x \hat{\psi}(y)dy\right]. \tag{IA.39}$$

We calculate  $g(0)$  below, and then after changing the variable back from  $x$  to  $v = x + \underline{v}$ , substituting in expressions for  $\tilde{\rho}$  and  $\tilde{\lambda}$  from (IA.31), we obtain (IA.21).

Next, we solve for  $\hat{\psi}(x)$  in closed form. We expand  $\mathcal{L}[\hat{\psi}(x)]$  as series, and sum up the inverse transforms of each term in the summation to get  $\hat{\psi}(x)$ .

$$\begin{aligned}\mathcal{L}[\hat{\psi}(x)] &= \frac{1}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}} \\ &= (\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2)^{-1} \cdot \left(1 - \frac{\tilde{\lambda}e^{-\tilde{J}z}}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2}\right)^{-1} \\ &= \sum_{n=0}^{\infty} \frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2)^{n+1}}.\end{aligned}\tag{IA.40}$$

The above series converges for  $z$  such that  $|\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2| > |\tilde{\lambda}\exp(-\tilde{J}z)|$ . This holds if the real part of  $z$  is sufficiently large, e.g.,  $\Re(z) > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho}+\tilde{\lambda}}$ . The inverse Laplace transform can then be calculated along the line  $(\bar{z}-i\infty, \bar{z}+i\infty)$  in the complex domain where  $\bar{z} > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho}+\tilde{\lambda}}$ , and hence, the inequality for  $\Re(z)$  is satisfied.

Let  $\zeta_- < \zeta_+$  be roots of  $\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2=0$ , given by (IA.26). We use the following inversion formula for  $1/[(z-\zeta_+)(z-\zeta_-)]^{n+1}$  from Gradshteyn and Ryzhik (2007, p. 1117):

$$\mathcal{L}^{-1}\left[\frac{1}{[(z-\zeta_+)(z-\zeta_-)]^{n+1}}\right] = \frac{\sqrt{\pi}}{\Gamma(n+1)} \frac{x^{n+\frac{1}{2}}}{(\zeta_+-\zeta_-)^{n+\frac{1}{2}}} e^{\frac{\zeta_++\zeta_-}{2}x} I_{n+\frac{1}{2}}\left(\frac{\zeta_+-\zeta_-}{2}x\right). \tag{IA.41}$$

Function  $e^{-n\tilde{J}z}$  in the complex domain corresponds to a shift from  $x$  to  $x - n\tilde{J}$ . Therefore,

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2)^{n+1}} \right] &= \tilde{\lambda}^n \left( -\frac{\tilde{\sigma}^2}{2} \right)^{-n-1} \mathbf{1}_{x \geq n\tilde{J}} \\ &\times \frac{\sqrt{\pi}}{\Gamma(n+1)} \frac{(x - n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+ - \zeta_-)^{n+\frac{1}{2}}} e^{\frac{\zeta_+ + \zeta_-}{2}(x - n\tilde{J})} I_{n+\frac{1}{2}} \left( \frac{(\zeta_+ - \zeta_-)(x - n\tilde{J})}{2} \right). \end{aligned} \quad (\text{IA.42})$$

Consequently, the explicit expression for  $\hat{\psi}(x)$  is given by:

$$\hat{\psi}(x) = \sum_{n=0}^{\infty} \tilde{\lambda}^n \left( -\frac{\tilde{\sigma}^2}{2} \right)^{-n-1} \mathbf{1}_{\{x \geq n\tilde{J}\}} \frac{\sqrt{\pi}}{\Gamma(n+1)} \frac{(x - n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+ - \zeta_-)^{n+\frac{1}{2}}} e^{\frac{\zeta_+ + \zeta_-}{2}(x - n\tilde{J})} I_{n+\frac{1}{2}} \left( \frac{(\zeta_+ - \zeta_-)(x - n\tilde{J})}{2} \right), \quad (\text{IA.43})$$

where function  $I_{n+\frac{1}{2}}(\cdot)$  is a modified Bessel function of the first kind,  $\zeta_- < \zeta_+$  are given by (IA.26) and  $\tilde{\rho}$ ,  $\tilde{\mu}$ ,  $\tilde{\sigma}$ ,  $\tilde{\lambda}$ , and  $\tilde{J}$  are defined in (IA.31). Bessel function  $I_{n+\frac{1}{2}}(\cdot)$  is given by (see equation 8.467 in Gradshteyn and Ryzhik (2007)):

$$I_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left[ e^z \sum_{m=0}^n \frac{(-1)^m (n+m)!}{m!(n-m)!(2z)^m} + (-1)^{n+1} e^{-z} \sum_{m=0}^n \frac{(n+m)!}{m!(n-m)!(2z)^m} \right]. \quad (\text{IA.44})$$

Substituting (IA.44) into (IA.43), after minor algebra, we obtain expression (IA.23) for  $\hat{\psi}(x)$ . The infinite series (IA.43) has only a finite number of nonzero terms because for a fixed  $x$  indicators  $\mathbf{1}_{\{x \geq n\tilde{J}\}}$  vanish for sufficiently large  $n$ , and hence, (IA.43) is well-defined.

To find  $g(0)$  in equation (IA.39), we first evaluate  $\hat{\psi}(0)$ . From the above formula (IA.43), because  $\mathbf{1}_{\{0 \geq n\tilde{J}\}} = 0$  for all  $n > 0$ , we obtain

$$\hat{\psi}(0) = -\frac{2}{\tilde{\sigma}^2} \cdot \frac{e^{\zeta_+ \cdot 0} - e^{\zeta_- \cdot 0}}{\zeta_+ - \zeta_-} = 0. \quad (\text{IA.45})$$

Differentiating (IA.39) and using  $\hat{\psi}(0) = 0$ , we find:

$$g'(x) = \int_0^x s(y + \underline{v})^\theta \cdot \hat{\psi}'(x - y) dy - g(0) \cdot (\tilde{\rho} - \tilde{\lambda}) \hat{\psi}(x), \quad (\text{IA.46})$$

We solve for  $g(0)$  from the boundary condition  $g(\bar{v} - \underline{v}) - g'(\bar{v} - \underline{v}) = 0$  and obtain:

$$g(0) = \frac{\int_0^{\bar{v} - \underline{v}} s(y + \underline{v})^\theta \cdot [\hat{\psi}'(\bar{v} - \underline{v} - y) - \hat{\psi}(\bar{v} - \underline{v} - y)] dy}{1 - (\tilde{\rho} - \tilde{\lambda}) \int_0^{\bar{v} - \underline{v}} \hat{\psi}(y) dy + (\tilde{\rho} - \tilde{\lambda}) \hat{\psi}(\bar{v} - \underline{v})}. \quad (\text{IA.47})$$

Substituting (IA.47) into (IA.39), we derive equation (IA.21) for  $\hat{\Psi}(v; \theta)$ .

2) Equations (IA.28) for  $\sigma_t$  and (IA.29) for  $J_t$  are the same as equations (A72) for  $\sigma_t$  and (A73) in the Appendix of the paper. Next, we find the trading strategies. Equation (8) for  $W_{i,t+\Delta t}$ , implies the following expressions for  $n_{i,st}^*$  and  $b_{it}^*$ :

$$n_{i,st}^* = \sqrt{\frac{\text{var}_t[W_{i,t+\Delta t} - W_{it}|\text{normal}]}{\text{var}_t[\Delta S_t + (1 - l_A - l_B)D_t\Delta t|\text{normal}]}}$$

$$b_{it}^* = \mathbb{E}_t[W_{i,t+\Delta t}|\text{normal}] - n_{it}\mathbb{E}_t[S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t|\text{normal}].$$

Taking limit  $\Delta t \rightarrow 0$  in the above expressions and using expansions similar to those in the proof of Lemma 2, we obtain the number of stocks and the leverage per the market value of stocks in equation (IA.30). ■

**Proposition IA.4.** *Consider two economies with the following restrictions on model parameters: 1)  $\lambda_A = \lambda_B = 0$  and 2)  $\lambda_A = \lambda_B \equiv \lambda > 0$ ,  $\gamma_A = \gamma_B \equiv \gamma > 0$ . Then, the differential-difference equation (36) is an ordinary differential equation. Moreover, there exist constants  $C_+ > 0$  and  $C_- > 0$  such that the solution of equation (36) satisfying boundary conditions (37) is given by*

$$\widehat{\Psi}(v) = C_- e^{\varphi_- v} + C_+ e^{\varphi_+ v} + \widehat{\Psi}^{unc}(v), \quad (\text{IA.48})$$

where function  $\widehat{\Psi}^{unc}(v)$  corresponds to an unconstrained model, constants  $C_{\pm}$  are given by

$$C_+ = \frac{(1 - \varphi_-)e^{\varphi_- \bar{v}}(\widehat{\Psi}^{unc})'(\underline{v}) - \varphi_- e^{\varphi_- \bar{v}}((\widehat{\Psi}^{unc})(\bar{v}) - (\widehat{\Psi}^{unc})'(\bar{v}))}{\varphi_+(\varphi_- - 1)e^{\varphi_- \bar{v} + \varphi_+ \underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+ \bar{v} + \varphi_- \underline{v}}},$$

$$C_- = \frac{(\varphi_+ - 1)e^{\varphi_+ \bar{v}}(\widehat{\Psi}^{unc})'(\underline{v}) + \varphi_+ e^{\varphi_+ \bar{v}}((\widehat{\Psi}^{unc})(\bar{v}) - (\widehat{\Psi}^{unc})'(\bar{v}))}{\varphi_+(\varphi_- - 1)e^{\varphi_- \bar{v} + \varphi_+ \underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+ \bar{v} + \varphi_- \underline{v}}}, \quad (\text{IA.49})$$

and  $\varphi_{\pm}$  are a positive and a negative solutions, respectively, of equation  $h(\varphi) = 0$ , where  $h(\varphi)$  is a quadratic characteristic polynomial given by:

$$h(\varphi) = \begin{cases} \rho - (1 - \gamma_A)(\mu_D + \delta_A \sigma_D) + \frac{(1 - \gamma_A)\gamma_A}{2}\sigma_D^2 \\ \quad - (\widehat{\mu}_v + \delta_A \widehat{\sigma}_v + (1 - \gamma_A)\sigma_D \widehat{\sigma}_v)\varphi - \frac{\widehat{\sigma}_v^2 \varphi^2}{2}, & \text{if } \lambda_A = \lambda_B = 0, \\ \rho - (1 - \gamma)(\mu_D + \delta_A \sigma_D) + \frac{(1 - \gamma)\gamma}{2}\sigma_D^2 + \lambda(1 - (1 + J_D)^{1-\gamma}) \\ \quad - (\widehat{\mu}_v + \delta_A \widehat{\sigma}_v + (1 - \gamma)\sigma_D \widehat{\sigma}_v)\varphi - \frac{\widehat{\sigma}_v^2 \varphi^2}{2}, & \text{if } \lambda_i = \lambda, \gamma_i = \gamma. \end{cases} \quad (\text{IA.50})$$

**Proof of Proposition IA.4.** From equation (40), we observe that the term involving jump  $\widehat{J}_v$  vanishes when  $\lambda_A = \lambda_B = 0$ , and the jump  $\widehat{J}_v$  is zero when  $\lambda_A = \lambda_B > 0$  and  $\gamma_A = \gamma_B$ . Hence, the differential-difference equation (36) becomes a linear ODE. Solution  $\widehat{\Psi}^{unc}(v)$  corresponding to the unconstrained model is a particular solution of (36). The general solution is then given by (IA.48). Constants  $C_{\pm}$  can be easily found from the boundary conditions (37) by solving the following system of linear equations:

$$\begin{aligned} C_+ \varphi_+ e^{\varphi_+ v} + C_- \varphi_- e^{\varphi_- v} &= -(\Psi^{unc})'(v), \\ C_+(\varphi_+ - 1)e^{\varphi_+ \bar{v}} + C_-(\varphi_- - 1)e^{\varphi_- \bar{v}} &= \Psi^{unc}(\bar{v}) - (\Psi^{unc})'(\bar{v}). \end{aligned} \quad (\text{IA.51})$$

It remains to prove that constants  $C_{\pm}$  are positive. First, we show that they have the same sign. Because the state price density in the unconstrained economy is given by  $\xi_{\tau} = \ell(s(v_{\tau})D_{\tau})^{-\gamma_A}$ , where  $\ell$  is a constant, the P/D ratio in this economy is given by:

$$\Psi^{unc}(v) = \mathbb{E}_t^A \left[ \int_t^{\infty} e^{-\rho(\tau-t)} \left( \frac{s(v_{\tau})D_{\tau}}{s(v)D_t} \right)^{-\gamma_A} \frac{D_{\tau}}{D_t} d\tau \right], \quad (\text{IA.52})$$

where  $s(v)$  is the consumption share satisfying equation (14). Define function  $\widehat{\Psi}^{unc}(v) = \Psi^{unc}(v)s(v)^{-\gamma_A}$ , which is given by:

$$\widehat{\Psi}^{unc}(v) = \mathbb{E}_t^A \left[ \int_t^{\infty} e^{-\rho(\tau-t)} \left( \frac{D_{\tau}}{D_t} \right)^{1-\gamma_A} s(v_{\tau})^{-\gamma_A} d\tau \right]. \quad (\text{IA.53})$$

Using Itô's Lemma it can be easily shown that (IA.53) is a particular solution of the differential equation (36). Proposition 1 implies that in the unconstrained economy  $v_{\tau} = v + \widehat{\mu}_v(\tau - t) + \widehat{\sigma}_v(w_{\tau} - w_t)$ , and hence  $v_{\tau}$  is a linear function of  $v$ . Differentiating the equation (14) for the consumption share we find that  $s'(v) = -1/(\gamma_A/s(v) + \gamma_B/(1 - s(v)))$ . Using  $s'(v)$  and differentiating (IA.53) w.r.t.  $v$ , we obtain:

$$\begin{aligned} (\widehat{\Psi}^{unc})'(v) &= -\gamma_A \mathbb{E}_t^A \left[ \int_t^{\infty} e^{-\rho(\tau-t)} \left( \frac{D_{\tau}}{D_t} \right)^{1-\gamma_A} s(v_{\tau})^{-\gamma_A-1} s'(v_{\tau}) d\tau \right] \\ &= \mathbb{E}_t^A \left[ \int_t^{\infty} e^{-\rho(\tau-t)} \left( \frac{D_{\tau}}{D_t} \right)^{1-\gamma_A} \frac{\gamma_A/s(v_{\tau})}{\gamma_A/s(v_{\tau}) + \gamma_B/(1 - s(v_{\tau}))} s(v_{\tau})^{-\gamma_A} d\tau \right]. \end{aligned} \quad (\text{IA.54})$$

From (IA.54) it can be easily observed that

$$(\widehat{\Psi}^{unc})'(v) > 0, \quad \widehat{\Psi}^{unc}(v) - (\widehat{\Psi}^{unc})'(v) > 0. \quad (\text{IA.55})$$

From the latter two inequalities it follows that constants  $C_{\pm}$  given by (IA.49) have the same sign if  $\varphi_+ > 0 > \varphi_-$  and  $\varphi_+ > 1$ . The latter inequalities are proven in Lemma IA.3



below, and hence,  $C_+$  and  $C_-$  have the same sign. The fact that  $C_{\pm} > 0$  follows from Proposition 2 which shows that the P/D ratio in the unconstrained economy is lower than in the constrained economy, which implies  $\widehat{\Psi}(v) = C_-e^{\varphi-v} + C_+e^{\varphi+v} + \widehat{\Psi}^{unc}(v) \geq \widehat{\Psi}^{unc}(v)$ , and hence  $C_{\pm} > 0$ . ■

**Lemma IA.3.** *Consider characteristic polynomials  $h(\varphi)$  given by (IA.50). Denote by  $\varphi_-$  and  $\varphi_+$  the smallest and the largest solutions of equation  $h(\varphi) = 0$ . Then,  $\varphi_+ > 0 > \varphi_-$  and  $\varphi_+ > 1$ .*

**Proof Lemma IA.3.** We note that  $h(0) > 0$ , which is a continuous-time analogue of condition (15) for investor  $A$  imposed for the existence of equilibrium. Next, we show that  $h(1) > 0$ . For simplicity, we only look at the case when  $\lambda_A = \lambda_B = 0$  and risk aversions  $\gamma_A$  and  $\gamma_B$  are arbitrary. The other case of Proposition IA.4 in which  $\lambda_A = \lambda_B = \lambda$  and  $\gamma_A = \gamma_B$  can be proven analogously. Substituting  $\widehat{\mu}_v$  and  $\widehat{\sigma}_v$  from equations (38) and (39) into equation (IA.50), after some algebra, we obtain:

$$h(1) = \rho - (1 - \gamma_B)(\mu_D + \delta_B \sigma_D) + \frac{(1 - \gamma_B)\gamma_B}{2} \sigma_D^2.$$

The inequality  $h(1) > 0$  is a continuous-time analogue of inequality (15) for investor  $B$ . Then, because  $h(-\infty) = h(+\infty) = -\infty$ ,  $h(0) > 0$  and  $h(1) > 0$  it follows that there exist roots  $\varphi_{\pm}$  such that  $\varphi_+ > 1$  and  $0 > \varphi_-$ . ■

**Proposition IA.5. (U-shaped P/D ratios)** *Let  $\Psi$  and  $\Psi^{unc}$  be the price-dividend ratios in the constrained and unconstrained economies, respectively.*

- 1)  $\Psi$  is a decreasing (increasing) function of consumption share  $s$  near the boundary  $\underline{s}$  ( $\bar{s}$ ).
- 2) Consider two economies with the following parameters: a)  $\lambda_A = \lambda_B = 0$ ,  $\gamma_A \geq \gamma_B \geq 1$  and b)  $\lambda_A = \lambda_B \equiv \lambda > 0$ ,  $\gamma_A = \gamma_B \equiv \gamma \geq 1$ . Then,  $\Psi - \Psi^{unc}$  is a positive and convex function of  $s$ , and is a decreasing (increasing) function of  $s$  near the boundary  $\underline{s}$  ( $\bar{s}$ ).
- 3) Suppose  $\gamma_A = \gamma_B > 1$ . Then,  $\Psi^{unc}$  is a convex function of  $s$ .

**Proof Proposition IA.5.**

- 1) Lemma 2 shows that  $\Psi = \widehat{\Psi}s^{\gamma_A}$ , where  $\widehat{\Psi}(v)$  satisfies boundary conditions (37). Now, consider  $\widehat{\Psi}(v)$  as a function of  $s$  and differentiate w.r.t. to  $s$ :

$$\frac{\partial \Psi}{\partial s} = \widehat{\Psi}'(v) \frac{\partial v}{\partial s} s^{\gamma_A} + \gamma_A \widehat{\Psi}(v) s^{\gamma_A - 1} = \left( -\widehat{\Psi}'(v) \left( \frac{\gamma_A}{s} + \frac{\gamma_B}{1-s} \right) + \widehat{\Psi}(v) \frac{\gamma_A}{s} \right) s^{\gamma_A}, \quad (\text{IA.56})$$

where  $\partial v/\partial s$  is obtained by differentiating equation (14) for the consumption share. Using boundary conditions (37) and the fact that  $\underline{s} = s(\bar{v})$  and  $\bar{s} = s(\underline{v})$  (because  $s(v)$  is a decreasing function of  $v$ ), we obtain:

$$\frac{\partial \Psi}{\partial s}\Big|_{s=\underline{s}} = -\widehat{\Psi}(\bar{v})\frac{\gamma_B \underline{s}^{\gamma_A}}{1-\underline{s}} < 0, \quad \frac{\partial \Psi}{\partial s}\Big|_{s=\bar{s}} = \widehat{\Psi}(\underline{v})\gamma_A \bar{s}^{\gamma_A-1} > 0.$$

2)  $\Psi = \widehat{\Psi}s^{\gamma_A}$ ,  $\Psi^{unc} = \widehat{\Psi}^{unc}s^{\gamma_A}$ . Using equation (IA.48) for  $\widehat{\Psi}$ , the fact that in the latter equation  $C_{\pm} > 0$  (Proposition IA.4), and the equation (14) for  $v$  in terms of  $s$ , we obtain:

$$\Psi - \Psi^{unc} = C_-(1-s)^{\varphi-\gamma_B}s^{(1-\varphi_-)\gamma_A} + C_+(1-s)^{\varphi+\gamma_B}s^{(1-\varphi_+)\gamma_A}.$$

Differentiating the latter equation, we obtain:

$$\begin{aligned} (\Psi - \Psi^{unc})' &= C_-\left(\frac{(1-\varphi_-)\gamma_A}{s} - \frac{\varphi-\gamma_B}{1-s}\right)(1-s)^{\varphi-\gamma_B}s^{(1-\varphi_-)\gamma_A} \\ &\quad + C_+\left(\frac{(1-\varphi_+)\gamma_A}{s} - \frac{\varphi+\gamma_B}{1-s}\right)(1-s)^{\varphi+\gamma_B}s^{(1-\varphi_+)\gamma_A}. \end{aligned} \tag{IA.57}$$

Differentiating one more time, we obtain:

$$\begin{aligned} (\Psi - \Psi^{unc})'' &= C_-\left(\frac{(1-\varphi_-)^2\gamma_A^2 - (1-\varphi_-)\gamma_A}{s^2} + \frac{\varphi_-^2\gamma_B^2 - \varphi-\gamma_B}{(1-s)^2} - \frac{2\gamma_A\gamma_B\varphi_-(1-\varphi_-)}{s(1-s)}\right)\frac{(1-s)^{\varphi-\gamma_B}}{s^{(\varphi-1)\gamma_A}} \\ &\quad + C_+\left(\frac{(1-\varphi_+)^2\gamma_A^2 - (1-\varphi_+)\gamma_A}{s^2} + \frac{\varphi_+^2\gamma_B^2 - \varphi+\gamma_B}{(1-s)^2} - \frac{2\gamma_A\gamma_B\varphi_+(1-\varphi_+)}{s(1-s)}\right)\frac{(1-s)^{\varphi+\gamma_B}}{s^{(\varphi+1)\gamma_A}}. \end{aligned}$$

Using the facts that  $\varphi_+ > 0 > \varphi_-$ ,  $\varphi_+ > 1$ , and  $\gamma_i \geq 1$  from the above equation we find that  $(\Psi - \Psi^{unc})'' > 0$ , and hence,  $\Psi - \Psi^{unc}$  is convex.

Next, we verify that  $\Psi - \Psi^{unc}$  is decreasing (increasing) around  $\underline{s}$  ( $\bar{s}$ ). Taking into account equation (14) for  $v$  as a function of  $s$ , we rewrite  $(\Psi - \Psi^{unc})'$  in (IA.57) as follows:

$$\begin{aligned} (\Psi - \Psi^{unc})' &= \left(C_-\left(\frac{(1-\varphi_-)\gamma_A}{s} - \frac{\varphi-\gamma_B}{1-s}\right)e^{\varphi-v} + C_+\left(\frac{(1-\varphi_+)\gamma_A}{s} - \frac{\varphi+\gamma_B}{1-s}\right)e^{\varphi+v}\right)s^{\gamma_A} \\ &= \left(\frac{\gamma_A}{s}\left(C_-(1-\varphi_-)e^{\varphi-v} + C_+(1-\varphi_+)e^{\varphi+v}\right) - \frac{\gamma_B}{1-s}\left(C_-e^{\varphi-v} + C_+e^{\varphi+v}\right)\right)s^{\gamma_A}. \end{aligned} \tag{IA.58}$$

Using equations (IA.51) for constants  $C_{\pm}$ , inequalities (IA.55) for the derivatives of  $\widehat{\Psi}^{unc}$  and  $C_{\pm} > 0$  (Proposition IA.4), and the fact that  $\underline{s} = s(\bar{v})$  and  $\bar{s} = s(\underline{v})$ , from (IA.58) we obtain:

$$\begin{aligned} (\Psi - \Psi^{unc})'(\underline{s}) &= \frac{\gamma_A}{\underline{s}}\left(\widehat{\Psi}^{unc}(\bar{v}) - \widehat{\Psi}^{unc}(\bar{v})\right) + \frac{\gamma_B}{1-\underline{s}}\left(\widehat{\Psi}^{unc}(\bar{v}) - \widehat{\Psi}^{unc}(\bar{v}) - C_+e^{\varphi+\bar{v}} - C_-e^{\varphi-\bar{v}}\right) < 0, \\ (\Psi - \Psi^{unc})'(\bar{s}) &= \frac{\gamma_A}{\bar{s}}\left(\widehat{\Psi}^{unc}(\underline{v}) + C_+e^{\varphi+\underline{v}} + C_-e^{\varphi-\underline{v}}\right) + \frac{\gamma_B}{1-\bar{s}}\left(\widehat{\Psi}^{unc}(\underline{v})\right) > 0. \end{aligned}$$

3) Suppose  $\gamma_A = \gamma_B = \gamma$ . Then, from equation (14) for variable  $v$  in terms of consumption share  $s$  we find that  $s_t = 1/(1 + \exp(v_t/\gamma))$ . Using the latter equation, we obtain:

$$s_\tau = \frac{s_t}{s_t + (1 - s_t)e^{(v_\tau - v_t)/\gamma}}. \quad (\text{IA.59})$$

Substituting  $s_\tau$  into equation (IA.52) for the unconstrained P/D ratio, we obtain:

$$\Psi_t^{unc} = \mathbb{E}_t^A \left[ \int_t^\infty e^{-\rho(\tau-t)} \left( \frac{D_\tau}{D_t} \right)^{1-\gamma} \left( s_t + (1 - s_t)e^{(v_\tau - v_t)/\gamma} \right)^\gamma d\tau \right]. \quad (\text{IA.60})$$

We note that because state variable  $v_t$  in the unconstrained case follows an arithmetic Brownian motion (see Proposition 2), then  $v_\tau - v_t$  does not depend on  $v_t$  or  $s_t$ . Then, twice differentiating (IA.60) w.r.t.  $s_t$ , we obtain:

$$(\Psi_t^{unc})''_{ss} = \gamma(\gamma-1)\mathbb{E}_t^A \left[ \int_t^\infty e^{-\rho(\tau-t)} \left( \frac{D_\tau}{D_t} \right)^{1-\gamma} \left( s_t + (1 - s_t)e^{(v_\tau - v_t)/\gamma} \right)^{\gamma-2} \left( 1 - e^{(v_\tau - v_t)/\gamma} \right)^2 d\tau \right].$$

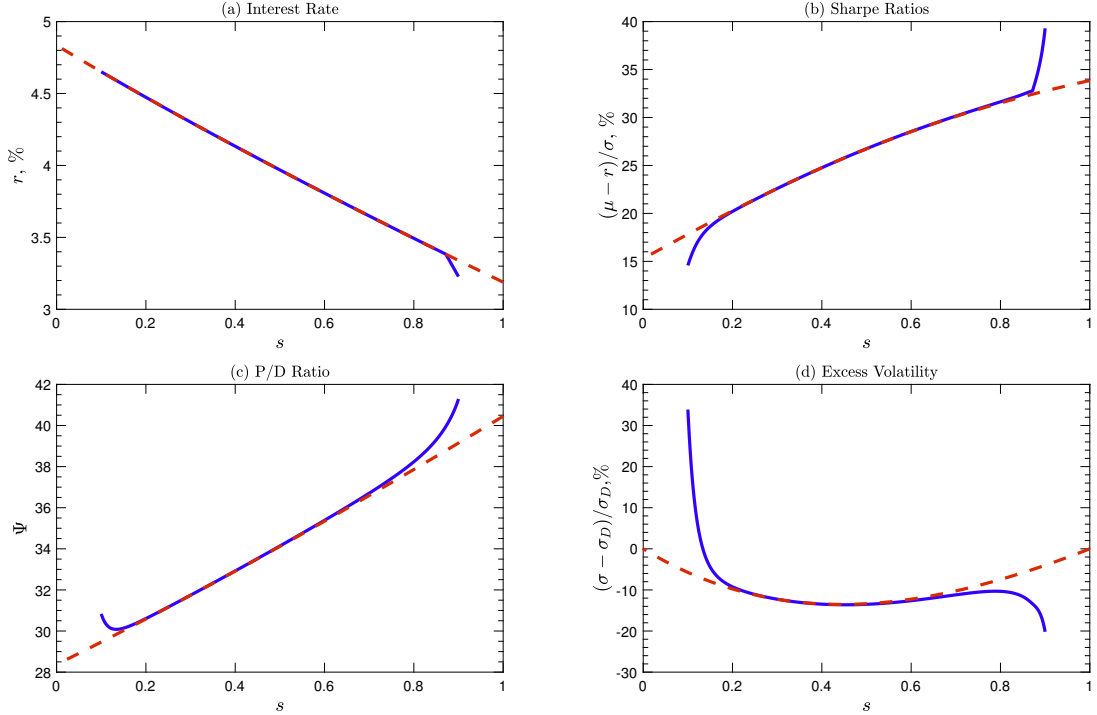
We observe that  $(\Psi_t^{unc})'' \geq 0$  when  $\gamma \geq 1$ , and hence,  $\Psi_t^{unc}$  is a convex function of  $s$ . ■

## IA.2 Alternative parameter specifications

The results in our baseline analysis are derived assuming that the more risk averse investor is also more pessimistic about the output growth rate, and the main calibration in Section 4 shows equilibrium processes when investors have the same risk aversions but different beliefs. In this section, we demonstrate the robustness of our results on the effects of constraints on interest rates, Sharpe ratios, P/D ratios, and volatilities by considering two economies with alternative specifications of exogenous model parameters.

Figure IA.1 shows the equilibrium processes when investor  $A$  is more risk averse but also more optimistic than investor  $B$ . Figure IA.2 shows the equilibrium processes when the investors have different risk aversions and identical beliefs about the probabilities of states, that is,  $\lambda_A = \lambda_B$  and  $\delta_A = \delta_B = 0$ . The exact values of model parameters are given in the legends of the figures. From both figures, we observe that interest rates, Sharpe ratios, P/D ratios, and volatilities are affected by the collateral constraints in the same way as in our baseline analysis in Section 4. In particular, interest rates go down, Sharpe ratios spike in bad times, P/D ratios are U-shaped and sensitive to small shocks, and volatilities are higher (lower) in good (bad) times. However, we note that the magnitudes of these effects are smaller than in our baseline calibration.

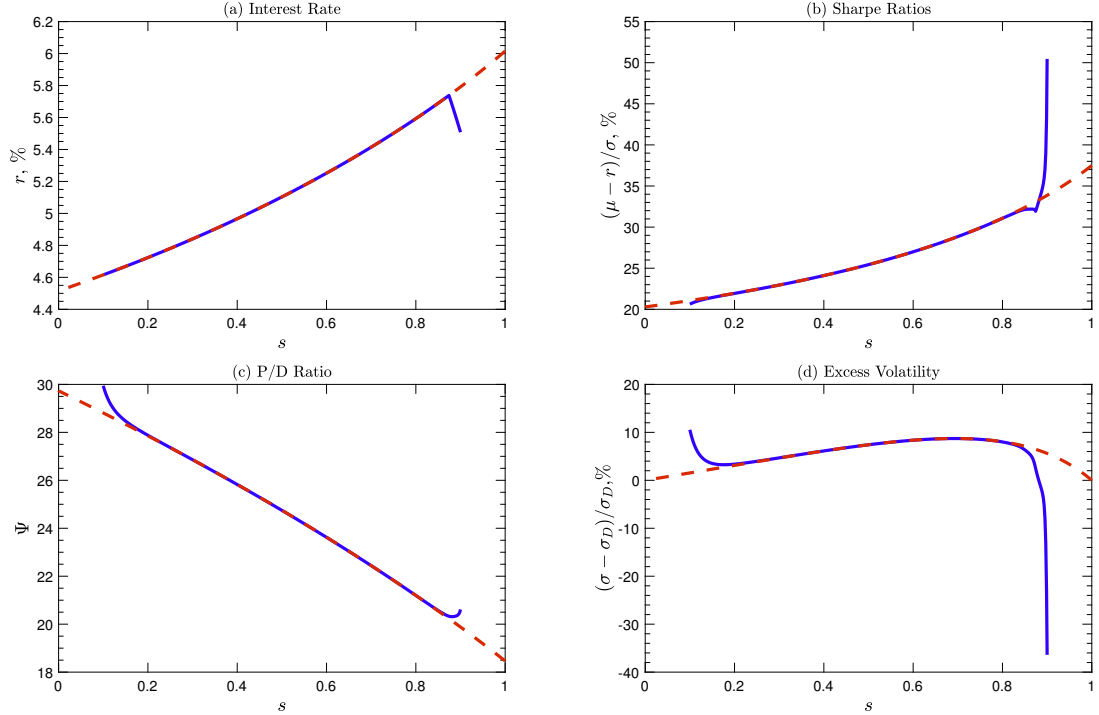
Finally, we discuss general conditions for the counter- and procyclicality of consumption share  $s$ . It is intuitive that share  $s$  is countercyclical in our baseline analysis in Section 4, in



**Figure IA.1**  
**Equilibrium processes**

Panels (a)–(d) show interest rate  $r_t$ , Sharpe ratio  $(\mu_t - r_t)/\sigma_t$ , price-dividend ratio  $\Psi_t$ , and excess volatility  $(\sigma_t - \sigma_D)/\sigma_D$  as functions of  $s_t = c_{A,t}^*/D_t$  for the constrained (solid lines) and unconstrained (dashed lines) economies. The model parameters are:  $\gamma_A = 1.5$ ,  $\gamma_B = 2$ ,  $\delta_B = -\delta_A = 0.05$ ,  $\lambda_A = 0.02$ ,  $\lambda_B = 0.01$ ,  $\rho = 0.02$ ,  $\mu_D = 1.8\%$ ,  $\sigma_D = 3.2\%$ ,  $J_D = -0.25$ .

which we assume that the less risk averse investor is also more optimistic than the more risk averse investor because in good (bad) time the distribution of wealth and consumption shift toward the less (more) risk averse investor. Without restricting risk aversions and beliefs, consumption share  $s$  is countercyclical if the volatility of state variable  $v$  given by (39) is positive,  $\hat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta_B - \delta_A > 0$ , and is procyclical otherwise. The variable  $v$  follows a reflected Brownian motion, so that  $v_{t+dt} = \max\{\underline{v}; \min\{\bar{v}; v_t + \hat{\mu}_v dt + \hat{\sigma}_v dw_t + \hat{J}_v dj_t\}\}$ . Hence, if  $\hat{\sigma}_v > 0$  then positive shocks  $dw_t > 0$  increase the state variable by  $\hat{\sigma}_v dw_t$  and decrease consumption share  $s$  because  $s(v)$  is a decreasing function of  $v$ , as can be seen from equation (14). As a result, consumption share  $s$  is higher in bad (good) times, following periods of consecutive negative (positive) shocks  $dw_t$ . We note that condition  $\hat{\sigma}_v > 0$  is satisfied for the parameters used for the equilibrium processes on Figure IA.1, and hence



**Figure IA.2**

### Equilibrium processes

Panels (a)–(d) show interest rate  $r_t$ , Sharpe ratio  $(\mu_t - r_t)/\sigma_t$ , price-dividend ratio  $\Psi_t$ , and excess volatility  $(\sigma_t - \sigma_D)/\sigma_D$  as functions of  $s_t = c_{At}^*/D_t$  for the constrained (solid lines) and unconstrained (dashed lines) economies. The model parameters are:  $\gamma_A = 4$ ,  $\gamma_B = 2$ ,  $\delta_B = \delta_A = 0$ ,  $\lambda_A = \lambda_B = 0.01$ ,  $\rho = 0.02$ ,  $\mu_D = 1.8\%$ ,  $\sigma_D = 3.2\%$ ,  $J_D = -0.25$ .

share  $s$  is countercyclical. All qualitative results remain the same when  $\hat{\sigma}_v < 0$ .<sup>2</sup>

## References

Gradshteyn, I.S., and I.M. Ryzhik, 2007, *Table of Integrals, Series, and Products* (6th ed.), New York: Academic Press.

<sup>2</sup>We also remark that in the derivation of the differential-difference equation (36) we use the fact that  $\hat{J}_v < 0$ , where  $\hat{J}_v$  is given by equation (40). The condition  $\hat{J}_v < 0$  is satisfied in our baseline analysis where  $\gamma_A > \gamma_B$  and  $\lambda_A > \lambda_B$ . We note that this condition is without loss of generality because if it is violated it can be restored by relabeling investors  $A$  and  $B$  as investors  $B$  and  $A$ , respectively.