On Problems in the Representation Theory of Symmetric Groups

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution except as declared in the preface and specified in the text.

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Abstract

In this thesis, we study the representation theory of the symmetric groups $\mathfrak{S}_n$, their Sylow p-subgroups P_n and related algebras.

For all primes p and natural numbers n, we determine the maximum number of distinct irreducible constituents of degree coprime to p of restrictions of irreducible characters of $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$, and show that every value between 1 and this maximum is attained. These results can be stated graph-theoretically in terms of the Young lattice, which describes branching for symmetric groups. We present new graph isomorphisms between certain subgraphs of the Young lattice and find self-similar structures. This generalises from $p = 2$ to all p work of Ayyer, Prasad and Spallone which was central in the construction of character correspondences for symmetric groups in the context of the McKay Conjecture, a fundamental open problem in the representation theory of finite groups.

Linear characters of Sylow subgroups have also played a central role in character correspondences verifying the McKay Conjecture, becoming the focus of much current interest. For instance, a consequence of recent work of Giannelli and Navarro shows the existence of linear constituents in the restriction of every irreducible character of a symmetric group to its Sylow p-subgroups. We now identify these linear constituents, using a mixture of algebraic and combinatorial techniques including Mackey theory and an analysis of Littlewood–Richardson coefficients.

We determine precisely when the trivial character $\mathbb{1}_{P_n}$ appears as a constituent of the restriction of an irreducible character of $\mathfrak{S}_n$, for all n and odd p. As a consequence, we determine the irreducible characters of the Hecke algebra corresponding to the permutation character $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$. Analogous results are obtained for the alternating groups $\mathfrak{A}_n$. We then extend our scope to arbitrary linear characters of P_n, proving in particular that for all p, given linear characters ϕ and ϕ' of P_n, the induced characters $\phi\big\uparrow^{\mathfrak{S}_n}$ and $\phi'\big\uparrow^{\mathfrak{S}_n}$ are equal if and only if ϕ and ϕ' are $N_{\mathfrak{S}_n}(P_n)$–conjugate.

Finally, we consider the representation theory of Schur algebras in all characteristics. We classify the classical Schur algebras $S(n,r)$ which are Ringel self-dual, using decomposition numbers for symmetric groups, tilting module multiplicities and combinatorial methods.

To Mum, Dad and Adrian

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Contents

Chapter 1

Introduction

Symmetries exist in the world all around us, from vast planetary orbits to microscopic molecules. They play an active role in the way we process information: the existence of symmetry allows us to filter data efficiently in order to simplify and solve many complex problems. These natural phenomena can be studied in the abstract mathematical framework of group theory and representation theory, in particular that of the symmetric groups, whose importance extends beyond algebra and mathematics to all areas of science. From Cayley's theorem, stating that any finite group can be embedded into a symmetric group, to the role of symmetric groups as pioneering examples for which theories and conjectures concerning groups are first investigated, symmetric groups have always been central in and continue to lie at the forefront of research in representation theory.

Local–Global Conjectures. The Local–Global Conjectures form one of the most significant families of conjectures in representation theory. Throughout, let G denote a finite group and p a prime number. Though the ordinary representation theory of finite groups was first developed over a century ago by mathematicians such as Frobenius, Burnside and Schur, and modular representation theory by Brauer some decades after, even today there are still many fundamental open problems in this vast and active area of research.

Lying at the heart of modern representation theory, the Local–Global Conjectures posit that certain information about the representation theory of a group G — the 'global' level — can be described using corresponding information about the p-local structure of G, such as Sylow p-subgroups and Sylow normalisers, and in particular the *local* subgroups of G, namely $N_G(P)$ for non-trivial p-subgroups P of G. We refer the reader to [45] and [51] for detailed surveys on the topic.

Obtaining information at a global level by investigating local behaviours is both a natural and fruitful process, most notably evidenced by the classification of finite simple groups. Indeed, a promising strategy towards proving these conjectures is to reduce to (smaller) groups which are well-understood, or to situations where specific machinery may be developed, such as partitioning problems about a group into problems about certain subgroups, or problems about group algebras into problems concerning their blocks; verifying and understanding the conjectured statements in the case of well-known or large families of groups; and reductions to simple groups.

Following this last approach, significant progress has already been made towards Brauer's Height Zero Conjecture by Kessar and Malle, who proved completely one direction of the equivalence in its statement in 2013 [42], and also the McKay Conjecture by Malle and Späth in 2016, who resolved it completely for $p = 2$ [46]. Moreover, these ideas have led to the development of novel tools, techniques and results with applications extending beyond the immediate origins of group theory and representation theory.

All of these new developments point to the existence of some rich, underlying theory which would explain the various local–global phenomena, but this theory is as yet elusive. These topics have therefore generated much interest and led to a plethora of international research activities both past and upcoming on representation theory and finite group theory, and in particular on the Local–Global Conjectures.

The McKay Conjecture. We now focus on one key conjecture in particular. A central member of the Local–Global Conjectures is the *McKay Conjecture*. As explained by Navarro in [51], it is no exaggeration to say that this open problem is the crux of modern representation theory. Beyond its own importance, the McKay Conjecture has a fundamental place in this family of conjectures as an origin from which several others were conceived, including the Alperin–McKay Conjecture, the Dade Conjecture and Broué's Abelian Defect Group Conjecture [45]. For a finite group G, let $\mathrm{Irr}(G)$ denote the set of ordinary irreducible characters of G. For p a prime, let $\mathrm{Irr}_{p'}(G)$ denote the subset of those irreducible characters of degree coprime to p.

Conjecture (McKay, 1972 [49])**.** *Let G be a finite group and p be a prime. Let P be a Sylow p-subgroup of G. Then*

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N_G(P))|$$

where $N_G(P)$ denotes the normaliser of P.

There have been many results over the last few decades on the McKay Conjecture, including Isaacs [35], who verified the conjecture for all finite groups of odd order at all primes p, and Olsson [53], for symmetric groups and general linear groups. While deceptively simple to state, the general McKay Conjecture is still open, though remarkable progress has been made in recent years. A landmark paper of Isaacs, Malle and Navarro [37] reduced this to a problem concerning simple groups in 2007 using certain equivariant maps between sets of characters, leading to a major breakthrough by Malle and Späth who verified the conjecture completely for $p = 2$ in 2016 [46].

The process of restriction of characters has played a key role in the construction of bijections between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N_G(P))$ in the case that a Sylow p-subgroup P is self-normalizing, see for example [50] and [52] for odd p. We remark that under this assumption, $\mathrm{Irr}_{p'}(N_G(P)) = \mathrm{Irr}_{p'}(P)$ coincides exactly with the set of linear characters of P. It was later shown in [26] that symmetric groups $\mathfrak{S}_n$ where n is a power of 2 exhibit the same phenomenon when $p = 2$: one finds that the restriction of every $\chi \in \mathrm{Irr}_{p'}(G)$ to P contains a unique irreducible constituent χ^* of degree coprime to p, and $\chi \mapsto \chi^*$ is a correspondence of characters witnessing the truth of the McKay Conjecture. A remarkable feature of these correspondences is that restriction is a choice-free process, and indeed a natural operation to consider in the context of characters of a group and its subgroups. The existence of 'natural' bijections respecting certain algebraic structures or properties is a strong indication towards a theory that would explain the deeper algebraic connections between G and its local subgroups.

Following this, the bijection in [26] became fundamental in the construction of a canonical bijection between $\mathrm{Irr}_{2'}(\mathfrak{S}_n)$ and $\mathrm{Irr}_{2'}(P_n)$ for all natural numbers n in [28], where P_n denotes a Sylow 2-subgroup of $\mathfrak{S}_n$. Here canonical refers to the property that the bijection commutes with the action of Galois and group automorphisms, and a purely algebraic description of this canonical McKay bijection was later given in [38]. Another key ingredient used in [38] is [1, Theorem 1], showing that every odd-degree irreducible character of a symmetric group $\mathfrak{S}_n$ contains a unique odd-degree irreducible constituent upon restriction to $\mathfrak{S}_{n-1}$. The first main result of this thesis can be summarised as follows:

> *For all primes p, we describe the number of irreducible constituents of degree coprime to p of restrictions $\chi\big\downarrow_{\mathfrak{S}_{n-1}}$, for $\chi \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$.*

This extends [1] from $p = 2$ to all p, and is described in more detail in Chapter 3.

A central theme in the character theory of finite groups is the relationship between $\mathrm{Irr}(G)$ and $\mathrm{Irr}(P)$. Though the aforementioned results illustrate the importance of considering the restrictions of characters of finite groups to their Sylow subgroups, surprisingly little is known in general when, for instance, we do not impose the condition $P = N_G(P)$. This is the case even for symmetric groups, that is, when p is odd. A very recent step towards providing a fuller picture of such restrictions is the following consequence of work by Giannelli and Navarro [31]: for any prime p and any $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$, the restriction of χ to a Sylow p-subgroup always contains a linear constituent. In fact, they show that if p divides the degree of χ, then the restriction contains at least p different linear constituents.

Despite this, it is not known a priori *which* linear constituents appear in such restrictions. Investigating such linear constituents is the primary focus of this thesis (see Chapters 4, 5 and 6). The second main result of this thesis shows that it suffices to consider only a subfamily of those linear characters of the Sylow subgroup.

> *Let p be any prime. Given linear characters ϕ and ψ of a Sylow p-subgroup P_n of $\mathfrak{S}_n$, we show that the induced characters $\phi\big\uparrow^{\mathfrak{S}_n}$ and $\psi\big\uparrow^{\mathfrak{S}_n}$ are equal if and only if ϕ and ψ are $N_{\mathfrak{S}_n}(P_n)$–conjugate.*

This is an analogue for symmetric groups of a theorem of Navarro for p-solvable groups [50], and is described in Chapter 4. Following this, we wish to *identify* those linear constituents appearing in the restrictions of irreducible characters of symmetric groups to their Sylow subgroups. The third main result of this thesis can be summarised as follows:

> *Given a linear character ϕ of a Sylow p-subgroup P_n of $\mathfrak{S}_n$, we describe the set of irreducible constituents of $\phi\big\uparrow^{\mathfrak{S}_n}$.*

The case of the trivial character ϕ is studied in Chapter 5, and arbitrary ϕ in Chapter 6.

Schur algebras. In a related but distinct line of research, we investigate certain properties of Schur algebras. Around the turn of the twentieth century, Frobenius and Schur discovered a fundamental link between the complex representation theories of the finite symmetric groups $\mathfrak{S}_r$ and the general linear groups $\mathrm{GL}_n(\mathbb{C})$, for natural numbers n and r, via what are now known as the Schur algebras $S(n,r)$. These algebras lie at the intersection of significant areas in representation theory, capturing algebraic group theoretic properties from GL_n, but also being readily analysed using combinatorial techniques analogous to those well-known in the study of $\mathfrak{S}_r$.

To understand the rational representation theory of GL_n, it is enough to understand their polynomial representations, and these in turn are equivalent to representations of $S(n,r)$. On the other hand, the application of so-called Schur functors passes structural information from module categories of the Schur algebras to those of certain symmetric groups. This is described in Green's prominent monograph [33], which underpinned much of the work in this area following its publication in 1980.

While we do not make explicit use of the following in our work, we must mention the importance of Schur–Weyl duality, which relates the representations of GL_n and $\mathfrak{S}_r$ through the tensor power $E^{\otimes r}$ of the natural n-dimensional module E. This, amongst other results, has motivated much work in recent decades to understand the close relationship between general linear groups and symmetric groups through the use of Schur algebras.

The natural actions of GL_n and $\mathfrak{S}_r$ on $E^{\otimes r}$ motivate an equivalent definition of the Schur algebra $S(n,r)$ as $\mathrm{End}_{\mathfrak{S}_r}(E^{\otimes r})$, showing that it is an endomorphism ring of certain permutation modules for symmetric groups. This form allows us to make a generalisation parallel to the one from symmetric group algebras to Hecke algebras in type A, from (classical) Schur algebras $S(n,r)$ to the quantized q-Schur algebras $S_q(n,r)$ as introduced by Dipper and James in [10]. The role of the general linear group is taken over by a certain Hopf algebra which is a quantized version of GL_n. We will not comment further here on these quantum general linear groups, except to say that [13] provides a comprehensive introduction, in particular to their standard homological properties.

In some sense, Schur algebras are also more well-behaved than their motivating counterparts. They are finite-dimensional, unlike the group algebras of GL_n, and they are quasi-hereditary algebras, unlike the group algebras of symmetric groups. The latter point is of particular importance. Quasi-hereditary algebras were first introduced by Cline, Parshall and Scott in [4], connecting the rational representation theory of reductive algebraic groups in positive characteristic with the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ for semisimple complex Lie algebras. Quasi-hereditary algebras come in pairs called *Ringel duals*, and a Ringel dual of S can be defined in terms of tilting modules for S. This duality can be used to phrase the Kazhdan–Lusztig conjectures, another significant family of conjectures in representation theory, in terms of maps between tilting modules and also composition factors in good or cogood filtrations [43]. Chapter 7 contains the final main result of this thesis, which is as follows:

We classify those (classical) Schur algebras which are Ringel self-dual.

Structure. We now describe the content of each chapter in turn.

In Chapter 3, we give best-possible bounds on the maximum number of distinct irreducible constituents of degree coprime to p of the restriction $\chi\big\downarrow_{S_{n-1}}$, as χ runs over $\mathrm{Irr}(S_n)$, for any prime p and natural number n. We further determine all of the attainable values for the numbers of such constituents. These results can also be stated combinatorially in terms of the Young graph, a well-studied object at the interface of representation theory and algebraic combinatorics describing the branching behaviour of the symmetric groups. This work generalises from $p = 2$ to all primes p Theorem 1 of [1], which was central in the construction of character correspondences in [28] for symmetric groups in the context of the McKay Conjecture. We then give analogous results for character inductions, observing that more complex behaviours are exhibited in this case, and describe graph isomorphisms between certain subgraphs of the Young graph. This generalises from $p = 2$ to all primes p Theorems 2 and 3 of [1].

Fix a prime p. For each natural number n, let P_n denote a Sylow p-subgroup of the symmetric group $\mathfrak{S}_n$. In Chapter 4, for all natural numbers n and all primes p we show that if ϕ and ψ are linear characters of P_n, then the inductions $\phi\big\uparrow^{\mathfrak{S}_n}$ and $\psi\big\uparrow^{\mathfrak{S}_n}$ are equal if and only if ϕ and ψ are conjugate via an element of the normaliser $N_{\mathfrak{S}_n}(P_n)$. This is an analogue for symmetric groups of a result of Navarro for p-solvable groups [50].

In Chapter 5, we determine the set of $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ such that the trivial character $\mathbb{1}_{P_n}$ of P_n appears as a constituent of $\chi\big\downarrow_{P_n}$, for all natural numbers n and odd primes p. We prove analogous results for the alternating groups $\mathfrak{A}_n$, and consequently determine the irreducible characters of the Hecke algebras corresponding to the permutation characters $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$ and $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{A}_n}$.

Extending our investigations from the trivial character to arbitrary linear characters ϕ of P_n, we describe the set $\Omega(\phi)$, the subset of $\mathrm{Irr}(\mathfrak{S}_n)$ consisting of the irreducible constituents of $\phi\big\uparrow^{\mathfrak{S}_n}$. This is done in Chapter 6 using new results on Littlewood–Richardson coefficients.

In Chapter 7, we determine when the Schur algebra $S(n, r)$ is Ringel self-dual for all natural numbers $3 \leq n < r$. In particular, we complete the classification of classical Schur algebras which are Ringel self-dual, following work of Donkin [12] and Erdmann and Henke [21].

Chapter 2

Preliminaries

2.1 Notation

We first record some notational conventions that will be used throughout this thesis.

Sets of numbers

We denote the set $\{1, 2, 3, \dots\}$ of natural numbers by $\mathbb{N}$, and the set of non-negative integers by $\mathbb{N}_0$.

For $m \in \mathbb{N}$, we let $\mathbb{N}_{\geq m} = \{m, m+1, m+2, \dots\}$ and $\mathbb{N}_{>m} = \{m+1, m+2, m+3, \dots\}$.

Furthermore, $[m] := \{1, 2, \dots, m\}$ and $[\overline{m}] := \{0, 1, 2, \dots, m-1\}$.

Groups and characters

For $n \in \mathbb{N}$, let $\mathfrak{S}_n$ denote the symmetric group on n points, and $\mathfrak{A}_n$ the alternating subgroup of $\mathfrak{S}_n$. Also, let C_n denote a cyclic group of order n. For a set X, let $\mathrm{Sym}X$ be the group of permutations of X.

Let G be a finite group. Then $\mathrm{Irr}(G)$ denotes a complete set of ordinary irreducible characters of G, and $\mathrm{Lin}(G)$ denotes the subset of those characters which are linear (i.e. of degree 1). The trivial character of G is denoted by $\mathbb{1}_G$. We also let $\mathrm{Char}(G)$ denote the set of all (ordinary) characters of G.

Let p be a prime number. Define $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid p \nmid \chi(1)\}$. Also, $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of G.

Let $H \leq G$. We denote conjugation of group elements by $h^g := g^{-1}hg$, and extend this to subgroups so $H^g := \{h^g \mid h \in H\}$. For $\chi \in \mathrm{Char}(H)$ and $g \in G$, we define $\chi^g \in \mathrm{Char}(H^g)$ by setting $\chi^g(x) := \chi(gxg^{-1})$ for all $x \in H^g$.

The restriction of the character χ from G to H is denoted by $\chi\big\downarrow^G_H$, or simply $\chi\big\downarrow_H$ when the original group is understood. Similarly, if ϕ is a character of H then $\phi\big\uparrow^G_H$ (or simply $\phi\big\uparrow^G$) denotes the induction of ϕ from H to G. (If the meaning is clear from context, we may also denote induction and restriction without the arrows.)

Symbols

Let M and N be finite-dimensional, not necessarily irreducible modules for some (finite-dimensional) algebra A, usually a group algebra. We write $M \mid N$ to mean that M is a direct summand of N, and use the same terminology and notation for their corresponding characters. That is, if M (resp. N) affords the character χ_M (resp. χ_N), then we also write $\chi_M \mid \chi_N$ and say that χ_M is a direct summand of χ_N. We use $\nmid$ to indicate 'is not a direct summand of'.

As usual, δ_{ij} or $\delta_{i,j}$ denotes the Kronecker delta for variables i and j, taking value 1 if $i = j$ and 0 otherwise. When the meaning is clear, we use the Kronecker delta for more general objects i and j than just numbers, such as (ordered) sequences of numbers, or characters of a group.

For emphasis, disjoint unions may sometimes be written using $\sqcup$. This does not preclude $A \cap B = \emptyset$ when we simply write $A \cup B$.

To ease notation, we omit extra sets of parentheses when the meaning is clear from context. For instance, if $s = (s_1, \ldots, s_n)$ is a sequence and f is a function taking such a sequence as its input, we will sometimes write $f(s_1, \ldots, s_n)$ for $f((s_1, \ldots, s_n))$. Similarly, when we concatenate sequences, say $t = (t_1, \ldots, t_m)$ and $u = (u_1, \ldots, u_n)$, we may write (t, u) for the sequence $(t_1, \ldots, t_m, u_1, \ldots, u_n)$.

p-adic expansions

Let $n \in \mathbb{N}$ and p be a prime number. We notate the p-adic expansion or base p expansion of n in two ways (depending on convenience for the context at hand): either

(1) $n = \sum_{j=1}^{t} a_i p^{n_i}$, in which case $t \in \mathbb{N}$, $a_i \in [p-1]$ for all $i \in [t]$ and $0 \leq n_1 < n_2 < \cdots < n_t$ are integers (we will always specify the order of the indices n_i); or

(2) $n = \sum_{i=0}^{t} a_i p^i$ for some $t \in \mathbb{N}$, so $a_i \in \{0, 1, \ldots, p-1\}$ for $i \in \{0, 1, \ldots, t\}$ and $a_t \neq 0$. (Alternatively we may also write $n = \sum_{i \geq 0} a_i p^i$ to mean there exists some $t \in \mathbb{N}$ such that $a_i = 0$ for all $i > t$.)

We denote by $\nu_p(n)$ the p-adic valuation of n. That is, $p^{\nu_p(n)}$ is the highest power of p dividing n.

2.1.1 Partitions

By a *partition*, we mean a finite non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of natural numbers. We say that λ is a partition of a natural number n, written $\lambda \vdash n$, if $\lambda_1 + \cdots + \lambda_k = n$. We also say that n is the *size* of the partition λ, and write $n = |\lambda|$. We denote by $\mathcal{P}(n)$ the set of partitions of n and we let

$$\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}(n).$$

Hence we sometimes also write $\lambda \in \mathcal{P}(n)$ in place of $\lambda \vdash n$.

The λ_i are known as the *parts* of the partition. The *length* of λ, often written $l(\lambda)$, is the number of parts of λ, i.e. $l(\lambda) = k$. Repeated parts are often denoted using index notation for convenience; the meaning should always be clear from context. For instance, $(2, 1, 1, 1) = (2, 1^3) \neq (2, 1)$, while (p^k) could denote a single part of size p^k or $(p, \ldots, p)$ where the part p appears k times, and we interpret this based on context by specifying $(p^k) \vdash p^k$ or $(p^k) \vdash kp$ respectively.

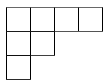
Given a partition λ, its *conjugate partition* is the partition $\lambda' = (\mu_1, \mu_2, \ldots, \mu_t)$ where $t = \lambda_1$ and $\mu_i := |\{j \in [l(\lambda)] \mid \lambda_j \geq i\}|$.

Young diagrams

The *Young diagram* $[\lambda]$ corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is the subset of the Cartesian plane defined by:

$$[\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq k,\ 1 \leq j \leq \lambda_i\},$$

where we view the diagram in matrix orientation, with the node $(1, 1)$ in the upper left corner. Pictorially, $[\lambda]$ is often drawn using left-aligned boxes (nodes) such that there are λ_i boxes in row i, with the rows numbered downward (so that the top row is numbered 1). For example,



is the Young diagram of the partition $(4, 2, 1)$. In particular, the Young diagram of the conjugate partition λ' can be obtained by reflecting $[\lambda]$ about the main diagonal $y = -x$.

Call a box in the Young diagram $[\lambda]$ *removable* if there are no boxes to the right or below it. In other words, if $\lambda = (\lambda_1, \ldots, \lambda_k)$ then a box in $[\lambda]$ is removable if and only if it is the rightmost box of a row i where $\lambda_i > \lambda_{i+1}$ or $i = k$. *Addable* positions are defined similarly: they are (empty) positions to which a box may be added such that the resulting shape is the Young diagram of a partition. It is easy to see that the number of addable positions for any $[\lambda]$ is one more than the number of removable boxes of $[\lambda]$.

We use λ and $[\lambda]$ interchangeably when the meaning is clear from context; for instance, for partitions μ and λ we say $[\mu] \subseteq [\lambda]$ if $l(\mu) \leq l(\lambda)$ and $\mu_i \leq \lambda_i$ for all $i \leq l(\mu)$, or equally write $\mu \subseteq \lambda$. In this case we say that μ is a subpartition of λ (or λ contains μ, or μ is contained in λ).

If $\mu \subseteq \lambda$ (possibly $\mu = \emptyset$), then we may define a skew diagram (or skew shape) $[\lambda \setminus \mu] := [\lambda] \setminus [\mu]$, and call $\lambda \setminus \mu$ a skew partition. We refer the reader to [41, §1.4] for more detail.

Combinatorics of partitions

The *dominance ordering* $\trianglelefteq$ on the set $\mathcal{P}$ of all partitions is a partial order defined by

$$\mu = (\mu_1, \ldots, \mu_k) \ \trianglelefteq \ \lambda = (\lambda_1, \ldots, \lambda_k) \quad \Longleftrightarrow \quad \sum_{i=1}^{m} \mu_i \leq \sum_{i=1}^{m} \lambda_i \quad \forall\, m \in [k],$$

where $k = \max\{l(\mu), l(\lambda)\}$ (and we append trailing zeros to μ or λ as necessary). We sometimes also use the *lexicographical ordering* on $\mathcal{P}(n)$ for $n \in \mathbb{N}$, which is the total order given by $\mu < \lambda$ if $\mu_i < \lambda_i$, where $i := \min\{j \mid \lambda_j \neq \mu_j\}$.

A partition λ is called a *hook* or *hook partition* if $\lambda_2 \leq 1$. Equivalently, its Young diagram does not contain the box in position $(2, 2)$.

For $e \in \mathbb{N}$, we also use extensively the notions of e-hooks, e-rim hooks, leg lengths of hooks, e-cores and e-quotients of partitions in Chapter 3. We give a brief summary below, and refer the reader to [54, Chapter I] or [41, §2.3] for detailed definitions.

If (i, j) is a box in the Young diagram of a partition $[\lambda]$, then the (i, j)*-hook of* λ (often denoted $H_\lambda(i, j)$) is the set of boxes

$$\{(i', j') \in [\lambda] \mid i = i' \text{ and } j' \geq j, \text{ or } j = j' \text{ and } i' > i\}.$$

A *hook of* λ is $H_\lambda(i, j)$ for some $(i, j) \in [\lambda]$.

The *length* or *size* e of a hook is the number of boxes in it, in which case we call it an e-hook of λ. We denote by $\mathcal{H}(\lambda)$ the set of hooks of λ and by $\mathcal{H}_e(\lambda)$ the subset of $\mathcal{H}(\lambda)$ consisting of those hooks of λ having length divisible by e.

We also sometimes denote the size of $H_\lambda(i, j)$ by $|H_\lambda(i, j)|$ or $h_\lambda(i, j)$; in particular, $h_\lambda(i, j) = (\lambda_i - j) + (\lambda'_j - i) + 1$. The *rim* of λ is $\mathcal{R}(\lambda) = \{(i', j') \in [\lambda] \mid (i'+1, j'+1) \notin [\lambda]\}$, and the (i, j)*-rim hook of* λ is $R_\lambda(i, j) = \{(i', j') \in \mathcal{R}(\lambda) \mid i' \geq i \text{ and } j' \geq j\}$. The *leg length* of a hook or rim hook is one less than the number of rows it occupies, so for instance, the leg length of $H_\lambda(i, j)$ is $\lambda'_j - i = |\lambda| - 1 - \lambda_i + j$.

Informally, the e-core of a partition λ is the partition obtained by successively removing e-(rim) hooks from λ until no more can be removed, and is denoted by $C_e(\lambda)$; this process turns out to be well-defined, see for instance [54, Lemma 3.1]. The e-quotient $Q_e(\lambda)$ of a partition λ is more readily described on a construction known as *James' abacus*; we describe these objects in more detail in Chapter 3 for ease of reference.

Specific conventions

We conclude this section with some notation which is not necessarily standard in the literature.

If $n \in \mathbb{N}$ and α is a partition with $\alpha_1 \leq n$, then we let (n, α) denote the concatenation of the partitions (n) and α. More generally, if β is another partition such that $\alpha_{l(\alpha)} \geq \beta_1$, then we simply denote the concatenation of α and β by (α, β). The meaning should always be clear from context.

For two partitions λ and μ, let the sum of λ and μ be the partition $\lambda + \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$ (where we introduce trailing zeros to λ or μ as necessary). Clearly this definition extends to sums of multiple partitions.

Finally, we introduce some useful definitions concerning sets of partitions.

Definition 2.1. *Suppose $A \subseteq \mathcal{P}$. We define $A' := \{\lambda' \mid \lambda \in A\}$ and $A^\circ := A \cup A'$.*

For $n \in \mathbb{N}$ and m a positive real number, define

$$\mathcal{B}_n(m) = \{\lambda \vdash n \mid \lambda_1 \leq m \text{ and } l(\lambda) \leq m\}.$$

Thus $\mathcal{B}_n(m)$ is the set of those partitions of n whose Young diagrams fit inside an $m \times m$ square grid. We will usually take m to be an integer. In particular, $\mathcal{B}_n(m)$ is closed under taking conjugates of partitions, i.e. $\mathcal{B}_n(m)^\circ = \mathcal{B}_n(m)$.

2.2 The representation theory of symmetric groups

For each $n \in \mathbb{N}$, the complete set $\mathrm{Irr}(\mathfrak{S}_n)$ of irreducible characters of $\mathfrak{S}_n$ is naturally in bijection with $\mathcal{P}(n)$, the set of all partitions of n (see [41, Theorem 2.1.11] or [40, §11], for example). For $\lambda \vdash n$, we denote the corresponding irreducible character by χ^λ. We sometimes identify the labelling partition with the corresponding irreducible character, and hence write $\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ to denote at once the partition λ of n and the irreducible character χ^λ; the meaning of this notation will always be clear from context. We refer the reader to [40], [41] and [54] for detailed accounts of the representation theory of symmetric groups and related algebraic combinatorics.

Conjugate partitions

Under the natural bijection between $\mathrm{Irr}(\mathfrak{S}_n)$ and $\mathcal{P}(n)$, the trivial character of $\mathfrak{S}_n$ corresponds to (n), and the sign or alternating character to (1^n) [41, 2.1.7]. We record another easy and useful fact.

Lemma 2.2. *Let p be an odd prime, $n \in \mathbb{N}$ and $P_n \in \mathrm{Syl}_p(\mathfrak{S}_n)$. Let $\lambda \vdash n$. Then $\chi^\lambda\big\downarrow_{P_n} = \chi^{\lambda'}\big\downarrow_{P_n}$, where $P_n \in \mathrm{Syl}_p(\mathfrak{S}_n)$.*

Proof. It is well-known that $\chi^{\lambda'} = \chi^\lambda \cdot \chi^{(1^n)}$ (see [41, 2.1.8], for example). Since p is odd, P_n is contained in the alternating subgroup of $\mathfrak{S}_n$, and the assertion follows. □

The Murnaghan–Nakayama Rule

The Murnaghan–Nakayama rule (see [41, 2.4.7] or [40, 21.1], for example) provides a combinatorial formula for computing the values of the ordinary irreducible characters of symmetric groups. This is described using skew shapes. A border strip is defined to be a skew shape which is a rim hook in the Young diagram of some partition. If γ is a border strip, then $h(\gamma)$ is the leg length of such a corresponding rim hook.

Theorem 2.3 (Murnaghan–Nakayama rule). *Let $r, n \in \mathbb{N}$ with $r < n$. Suppose that $\pi\rho \in \mathfrak{S}_n$ where ρ is an r-cycle and π is a permutation of the remaining $n - r$ numbers. Then*

$$\chi^\lambda(\pi\rho) = \sum (-1)^{h(\lambda\setminus\mu)} \chi^\mu(\pi),$$

where the sum runs over all $\mu \subseteq \lambda$ such that $[\lambda \setminus \mu]$ is a border strip of size r. In particular, if $\lambda \vdash n$ and $\sigma \in \mathfrak{S}_n$ is an n-cycle, then

$$\chi^\lambda(\sigma) = \begin{cases} 0 & \text{if } \lambda \text{ is not a hook,} \\ (-1)^l & \text{if } \lambda \text{ is a hook of leg length } l. \end{cases}$$

Corollary 2.4. *Let p be a prime and $\lambda \vdash p$. Let $\sigma \in \mathfrak{S}_p$ be a p-cycle, $P = \langle \sigma \rangle$ and ψ be the regular character of P. Then*

$$\chi^\lambda\big\downarrow^{\mathfrak{S}_p}_P = \begin{cases} m \cdot \psi & \text{if } \lambda \text{ is not a hook,} \\ m' \cdot \psi + (-1)^l \cdot \mathbb{1}_P & \text{if } \lambda \text{ is a hook of leg length } l, \end{cases}$$

for some integers m and m'.

Proof. It follows from Theorem 2.5 (below; see also Theorem 3.12) that $p \nmid \chi^\lambda(1)$ if and only if λ is a hook, since $\lambda \vdash p$.

If λ is not a hook, then $\chi^\lambda(\sigma^i) = 0$ for all $i \in [p-1]$ by Theorem 2.3. Hence χ^λ is a multiple of ψ, since $\psi(1) = p$ and $\psi(\sigma^i) = 0$ for all $i \in [p-1]$.

If λ is a hook of leg length l, then $\chi^\lambda(\sigma^i) = (-1)^l$ for all $i \in [p-1]$ by Theorem 2.3, while $\chi^\lambda(1) = \binom{p-1}{l}$ (see the hook length formula below, for instance). But $\binom{p-1}{l} \equiv (-1)^l \pmod p$, so the result follows. □

Degrees of irreducible characters of symmetric groups

Let p be a prime. For a partition $\lambda \vdash n$, we write $\lambda \vdash_{p'} n$ if the corresponding character χ^λ labelled by λ has p'-degree, that is, degree coprime to p. Thus $\lambda \vdash_{p'} n$ is equivalent to $\chi^\lambda \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$, and in this case we also say that λ is a p'-partition of n (and sometimes simply write $\lambda \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$).

While the remarkable *hook length formula* (see e.g. [41, Theorem 2.3.21])

$$\chi^\lambda(1) = \frac{n!}{\prod_{(i,j)\in[\lambda]} h_\lambda(i,j)} \qquad \forall\ \lambda \vdash n$$

provides a purely combinatorial method for calculating the degrees of irreducible characters of symmetric groups, it turns out that a recursive description of p'-partitions given by Macdonald [44] (and later developed by Olsson [54] using the theory of p-core towers) is more convenient for our purposes. We introduce these towers briefly here, and refer the reader to [54] for a detailed description.

We can more generally define e-quotient towers and e-core towers for all $e \in \mathbb{N}$. Let λ be a partition, and let its e-quotient $Q_e(\lambda)$ be denoted by $(\lambda^{(0)}, \ldots, \lambda^{(e-1)})$.

(The e-quotient of a partition is more easily described using James' abacus, so we postpone its definition to Section 3.1.1 below; we refer the reader to [54] for further detail.) This is a sequence of e partitions. We then recursively set $Q_e(\lambda^{(i_1,\dots,i_j)}) = (\lambda^{(i_1,\dots,i_j,0)},\dots,\lambda^{(i_1,\dots,i_j,e-1)})$ for all $j \in \mathbb{N}$ for all $(i_1,\dots,i_j) \in [\overline{e}]^j$.

Now, let $T^Q(\lambda)_0 = (\lambda)$ and for all $j \in \mathbb{N}$, let

$$T^Q(\lambda)_j = (\lambda^{(0,\dots,0)},\dots,\lambda^{(e-1,\dots,e-1)}) = (\lambda^{(i_1,\dots,i_j)})_{(i_1,\dots,i_j)\in[\overline{e}]^j}$$

where the indexing sequences $(i_1,\dots,i_j)$ are taken in lexicographical order. Each $T^Q(\lambda)_j$ is a sequence of e^j partitions, and the collection of all of the sequences $T^Q(\lambda)_j$ for $j \in \mathbb{N}_0$ is known as the *e-quotient tower of* λ, denoted by $T^Q(\lambda)$ (or $T^Q_e(\lambda)$). The *e-core tower of* λ, denoted by $T^C(\lambda)$ (or $T^C_e(\lambda)$) is obtained from $T^Q(\lambda)$ by replacing each partition by its e-core. That is, $T^C(\lambda) = (T^C(\lambda)_0, T^C(\lambda)_1,\dots)$ where $T^C(\lambda)_0 = \big(C_e(\lambda)\big)$ and

$$T^C(\lambda)_j = \big(C_e(\lambda^{(0,\dots,0)}),\dots,C_e(\lambda^{(e-1,\dots,e-1)})\big) = \big(C_e(\lambda^{(i_1,\dots,i_j)})\big)_{(i_1,\dots,i_j)\in[\overline{e}]^j}$$

with indexing sequences taken in lexicographical order, for each $j \in \mathbb{N}$.

Given a tower sequence $T = (T_0, T_1,\dots)$ where each $T_j = (\mu^{(1)},\dots,\mu^{(e^j)})$ is a sequence of e^j partitions, we define $|T_j| := |\mu^{(1)}| + \cdots + |\mu^{(e^j)}|$. The following result was first proven by MacDonald in [44] and is fundamental to our work in Chapter 3.

Theorem 2.5. *Let p be a prime. Let $n \in \mathbb{N}$ with p-adic expansion $n = \sum_{j\geq 0} a_j p^j$. Let $\lambda \vdash n$. Then*

$$\nu_p\big(\chi^\lambda(1)\big) = \frac{\sum_{j\geq 0}|T^C(\lambda)_j| - \sum_{j\geq 0} a_j}{p-1}.$$

In particular, $\nu_p(\chi^\lambda(1)) = 0$ if and only if $|T^C(\lambda)_j| = a_j$ for all $j \in \mathbb{N}_0$.

We reformulate MacDonald's result in language that will be convenient for our purposes in Theorems 3.12 and 3.13.

The Branching Theorem and the Young graph

Recall that $\mathcal{P}$ denotes the set of partitions of natural numbers. For $\lambda \vdash n$, we let $(\lambda,\mu) \in \mathcal{E}$ if and only if χ^μ is an irreducible constituent of the restriction $\chi^\lambda\big\downarrow_{\mathfrak{S}_{n-1}}$. The *Young graph* $\mathbb{Y}$ has $\mathcal{P}$ as its set of vertices and $\mathcal{E}$ as its set of edges.

We recall the *Branching Theorem* (or *branching rule for symmetric groups*) (see [40, Chapter 9] or [41, Theorem 2.4.3], for instance) which tells us that

$$\chi^\lambda\big\downarrow_{\mathfrak{S}_{n-1}} = \sum_{\mu\in\lambda^-}\chi^\mu$$

for any $\chi^\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$, where λ^- denotes the set of all partitions $\mu \vdash n-1$ such that $[\mu]$ is obtained from $[\lambda]$ by removing a single box. (In particular, such a box must be a removable box in $[\lambda]$.) By Frobenius Reciprocity, we have that

$$\chi^\lambda\big\uparrow^{\mathfrak{S}_{n+1}} = \sum_{\mu\in\lambda^+} \chi^\mu,$$

where λ^+ denotes the set of all partitions $\mu \vdash n+1$ such that $[\mu]$ is obtained from $[\lambda]$ by adding a single box. Since the Young graph describes branching for symmetric groups, it is sometimes also called the branching graph in this context. Notice, in particular, that branching for symmetric groups is multiplicity-free.

It is useful to let $\mathfrak{S}_0$ be the trivial 1-element group, with $\mathcal{P}(0) = \{\emptyset\}$ where $\chi^\emptyset$ denotes the irreducible character of $\mathfrak{S}_0$. In this case we may add a root vertex labelled $\emptyset$ to the Young graph $\mathbb{Y}$, and an edge connecting it to the vertex (1).

For p a prime, let $\mathbb{Y}_{p'}$ be the subgraph of $\mathbb{Y}$ induced by the subset of vertices (partitions) labelling irreducible characters of p'-degree. By analogy, we let

$$\lambda^-_{p'} = \{\mu \in \lambda^- \mid p \nmid \chi^\mu(1)\} \quad \text{and} \quad \lambda^+_{p'} = \{\mu \in \lambda^+ \mid p \nmid \chi^\mu(1)\}.$$

These sets describe the neighbourhood of λ in $\mathbb{Y}_{p'}$ whenever λ is itself a p'-partition.

The branching theorem turns out to be a special case of the *Littlewood–Richardson rule*, as we will see now.

2.2.1 Littlewood–Richardson coefficients

Littlewood–Richardson coefficients arise in many contexts, appearing in the decomposition of tensor products of irreducible representations of symmetric groups (and of course, the closely related general linear groups), as coefficients when a product of two Schur polynomials is expressed as a linear combination of Schur polynomials in the ring of symmetric polynomials, and also in geometry and topology (see [25] and [33], for example).

Let $m, n \in \mathbb{N}$ with $m < n$. For $\mu \vdash m$ and $\nu \vdash n-m$, the Littlewood–Richardson rule (see [40, Chapter 16]) describes the decomposition into irreducible constituents of the induced character

$$(\chi^\mu \times \chi^\nu)\big\uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_m\times\mathfrak{S}_{n-m}},$$

with Littlewood–Richardson coefficients arising as the multiplicities.

Before we recall the Littlewood–Richardson rule, we introduce some notation and technical definitions. By convention, the highest row of $[\lambda]$ for a partition λ is numbered 1, but the highest row of a skew shape $\gamma = [\lambda \setminus \mu] := [\lambda] \setminus [\mu]$ need not be 1.

Definition 2.6. *Let $n \in \mathbb{N}$. Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ and let $\mathcal{C} = (c_1, \dots, c_n)$ be a sequence of positive integers. We say that $\mathcal{C}$ is of* weight λ *if*

$$|\{i \in \{1, \dots, n\} \ : \ c_i = j\}| = \lambda_j$$

for all $j \in \{1, \dots, k\}$. We say that an element c_j of $\mathcal{C}$ is good *if $c_j = 1$ or if*

$$|\{i \in \{1, 2, \dots, j-1\} \ : \ c_i = c_j - 1\}| > |\{i \in \{1, 2, \dots, j-1\} \ : \ c_i = c_j\}|.$$

Finally, we say that the sequence $\mathcal{C}$ is good *if c_j is good for every $j \in \{1, \dots, n\}$.*

We can now describe the Littlewood–Richardson coefficients $c^{\lambda}_{\mu\nu}$, which we also sometimes denote by $c^{\lambda}_{\mu,\nu}$ for clarity.

Theorem 2.7 (Littlewood–Richardson rule)**.** *Let $m, n \in \mathbb{N}$ with $m < n$. Let $\mu \vdash m$ and $\nu \vdash n - m$. Then*

$$(\chi^{\mu} \times \chi^{\nu})\big\uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}} = \sum_{\lambda \vdash n} c^{\lambda}_{\mu\nu} \ \chi^{\lambda}$$

where $c^{\lambda}_{\mu\nu}$ equals the number of ways to replace the nodes of $[\lambda \setminus \mu]$ by natural numbers such that

(i) the sequence obtained by reading the numbers from right to left, top to bottom is a good sequence of weight ν;

(ii) the numbers are non-decreasing (weakly increasing) left to right along rows; and

(iii) the numbers are strictly increasing down columns.

Let ν be a partition. We call a way of replacing the nodes of a skew shape γ with $|\nu|$ boxes by numbers satisfying conditions (i)–(iii) of Theorem 2.7 a *Littlewood–Richardson filling of γ of weight ν*. It is easy to see that every skew shape has at least one Littlewood–Richardson filling. For convenience, let $\mathcal{LR}(\gamma)$ denote the set of all possible weights of Littlewood–Richardson fillings of a skew shape γ. For example, $\mathcal{LR}([(4,1) \setminus (2)]) = \{(3), (2,1)\}$.

Moreover, the Littlewood–Richardson coefficients described in Theorem 2.7 are symmetric: $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$ for all partitions μ, ν and all partitions $\lambda \vdash |\mu| + |\nu|$. We write $\gamma \cong [\lambda]$ if γ is an orientation-preserving translation of the Young diagram of the partition λ in the plane. In other words, $\gamma = \{(i + a, j + b) \mid (i, j) \in [\lambda]\}$ for some fixed $a, b \in \mathbb{Z}$. We denote by γ° the 180°-rotation of γ (up to translation). For A a set of partitions and/or skew shapes, we also write $\gamma \in A$ if $\gamma \cong \alpha$ for some $\alpha \in A$.

We record below some useful lemmas.

Lemma 2.8 ([3, Lemma 4.4])**.** *Let μ and γ be partitions such that $[\gamma] \subsetneq [\mu]$. The following are equivalent:*

(i) $|\mathcal{LR}([\mu \setminus \gamma])| = 1$;

(ii) there is a unique Littlewood–Richardson filling of $[\mu \setminus \gamma]$;

(iii) $[\mu \setminus \gamma] \cong [\nu]$ or $[\mu \setminus \gamma]^{\circ} \cong [\nu]$, for some partition $\nu \vdash |\mu| - |\gamma|$.

Lemma 2.9. *Let γ be a skew shape. Suppose the non-empty rows of γ are numbered $1 \leq r_1 < r_2 < \ldots < r_t$. Then in any Littlewood–Richardson filling of γ, the boxes in row r_i can only be filled with numbers from $\{1, 2, \dots, i\}$, for each $i \in \{1, \dots, t\}$.*

Proof. This is immediate from conditions (i)–(iii) of Theorem 2.7. ☐

Lemma 2.10. *Let μ and ν be partitions. Let λ be a partition of $|\mu| + |\nu|$ and suppose that $c^{\lambda}_{\mu\nu} > 0$. Then $\lambda_1 \leq \mu_1 + \nu_1$ and $l(\lambda) \leq l(\mu) + l(\nu)$.*

Proof. Fix a Littlewood–Richardson filling of $[\lambda \setminus \mu]$ of weight ν: all of the boxes in the first row of this skew shape (which has length $\lambda_1 - \mu_1$) must be filled with the number 1. Hence $\lambda_1 - \mu_1 \leq \nu_1$, since there may be other 1s elsewhere in this filling of $[\lambda \setminus \mu]$.

Similarly, consider the numbers that have been filled into the first column of $[\lambda \setminus \mu]$ in this filling of weight ν. These numbers must be distinct, and hence correspond to different parts of the partition ν. Hence $l(\lambda) - l(\mu) \leq l(\nu)$. ☐

We can similarly define *iterated Littlewood–Richardson coefficients* $c^{\lambda}_{\mu^1,\dots,\mu^r}$ as follows. Let $r \in \mathbb{N}$ and $\mu^1, \dots, \mu^r$ be partitions, and let $\lambda \vdash n := |\mu^1| + \cdots + |\mu^r|$. Then $c^{\lambda}_{\mu^1,\dots,\mu^r}$ is the multiplicity of χ^{λ} as a constituent of $(\chi^{\mu^1} \times \cdots \times \chi^{\mu^r})\big\uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_{|\mu^1|} \times \cdots \times \mathfrak{S}_{|\mu^r|}}$. When $r = 2$, these are the usual Littlewood–Richardson coefficients as defined above, and letting $m = |\mu^1| + \cdots + |\mu^{r-1}|$ when $r \geq 2$, it is easy to see that

$$c^{\lambda}_{\mu^1,\dots,\mu^r} = \sum_{\gamma \vdash m} c^{\gamma}_{\mu^1,\dots,\mu^{r-1}} \cdot c^{\lambda}_{\gamma,\mu^r}. \tag{2.1}$$

From (2.1) and Theorem 2.7 we observe that if $c^{\lambda}_{\mu^1,\dots,\mu^r} > 0$ then $\lambda \leq \mu^1 + \cdots + \mu^r$ in the lexicographical ordering on partitions, and that $c^{\mu^1+\cdots+\mu^r}_{\mu^1,\dots,\mu^r} = 1$. The iterated Littlewood–Richardson coefficients are also symmetric under any permutation of the partitions $\mu^1, \dots, \mu^r$. An iterated Littlewood–Richardson (LR) filling of $[\lambda]$ by $\mu^1, \dots, \mu^r$ is a way of replacing the nodes of $[\lambda]$ by numbers defined recursively as follows: if $r = 1$ then $[\lambda] = [\mu^1]$ has a unique LR filling, of weight μ^1; if $r \geq 2$ then we mean an iterated LR filling of $[\gamma]$ by $\mu^1, \dots, \mu^{r-1}$ together with an LR filling of $[\lambda \setminus \gamma]$ of weight μ^r (for some $\gamma \subseteq \lambda$ such that this is possible).

Lemma 2.11. *Let $a, b_1, \dots, b_a \in \mathbb{N}$. Let $\nu^1, \dots, \nu^a$ be partitions such that $b_i \geq |\nu^i|$ for all i and let $c = |\nu^1| + \cdots + |\nu^a|$. Let $\mu \vdash c$ and let $\lambda = (b_1 + b_2 + \cdots + b_a, \mu)$. Then the iterated Littlewood–Richardson coefficients $c^{\lambda}_{(b_1,\nu^1),\dots,(b_1,\nu^a)}$ and $c^{\mu}_{\nu^1,\dots,\nu^a}$ are equal.*

Proof. Clearly $c^{\mu}_{\nu^1,\dots,\nu^a} \leq c^{\lambda}_{(b_1,\nu^1),\dots,(b_1,\nu^a)}$, since we may take any Littlewood–Richardson filling of $[\mu]$ by $\nu^1, \dots, \nu^a$ and replace each number i by $i + 1$, then combine with the first row of $[\lambda]$ filled with all 1s to produce a Littlewood–Richardson filling of $[\lambda]$ by $(b_1, \nu^1), \dots, (b_a, \nu^a)$. Conversely, any such filling of $[\lambda]$ contains 1s in exactly the first row of $[\lambda]$ since $\lambda_1 = b_1 + \cdots + b_a$, so this process is bijective. Thus $c^{\mu}_{\nu^1,\dots,\nu^a} = c^{\lambda}_{(b_1,\nu^1),\dots,(b_1,\nu^a)}$. ☐

We conclude this section by introducing an operator that will be useful later.

Definition 2.12. *For $n, m \in \mathbb{N}$ and $A \subseteq \mathcal{P}(n)$, $B \subseteq \mathcal{P}(m)$, let*

$$A \star B := \{\lambda \vdash n + m \mid \exists\ \mu \in A,\ \nu \in B \text{ such that } c^{\lambda}_{\mu\nu} > 0\}.$$

Clearly $\star$ is commutative, which follows from the symmetry of the Littlewood–Richardson coefficients, and associative.

2.3 Characters of wreath products

Let G be a finite group and let H be a subgroup of $\mathfrak{S}_n$ for some $n \in \mathbb{N}$. We denote by $G^{\times n}$ the direct product of n copies of G. The natural action of $\mathfrak{S}_n$ on the direct factors of $G^{\times n}$ induces an action of $\mathfrak{S}_n$ (and therefore of $H \leq \mathfrak{S}_n$) via automorphisms of $G^{\times n}$, giving the wreath product $G \wr H := G^{\times n} \rtimes H$. (In this thesis, we consider only finite wreath products, in which case the restricted wreath product and unrestricted wreath product agree and we make no distinction.) We sometimes refer to $G^{\times n}$ as the base group of the wreath product $G \wr H$.

As in [41, Chapter 4], we denote the elements of $G \wr H$ by $(g_1, \ldots, g_n; h)$ for $g_i \in G$ and $h \in H$. Let V be a $\mathbb{C}G$–module and suppose it affords the character ϕ. We let $V^{\otimes n} := V \otimes \cdots \otimes V$ (n copies) be the corresponding $\mathbb{C}G^{\times n}$–module. The left action of $G \wr H$ on $V^{\otimes n}$ defined by linearly extending

$$(g_1, \ldots, g_n; h) \;:\; \quad v_1 \otimes \cdots \otimes v_n \longmapsto g_1 v_{h^{-1}(1)} \otimes \cdots \otimes g_n v_{h^{-1}(n)} \tag{2.2}$$

turns $V^{\otimes n}$ into a $\mathbb{C}(G \wr H)$–module, which we denote by $\widetilde{V^{\otimes n}}$ (see [41, (4.3.7)]). We denote by $\tilde{\phi}$ the character afforded by the $\mathbb{C}(G \wr H)$–module $\widetilde{V^{\otimes n}}$. For any $\psi \in \mathrm{Char}(H)$, we let ψ also denote its inflation to $G \wr H$ and let

$$\mathcal{X}(\phi; \psi) := \tilde{\phi} \cdot \psi$$

be the character of $G \wr H$ obtained as the product of $\tilde{\phi}$ and ψ. If $K \leq G$ and $L \leq H$ are finite groups, then we have by the definition of $\mathcal{X}$ that

$$\mathcal{X}(\phi; \psi)\big\downarrow^{G \wr H}_{K \wr L} = \mathcal{X}(\phi\big\downarrow^{G}_{K}; \psi\big\downarrow^{H}_{L}).$$

Let $\phi \in \mathrm{Irr}(G)$ and let $\phi^{\times n} := \phi \times \cdots \times \phi$ denote the corresponding irreducible character of $G^{\times n}$. Observe that $\tilde{\phi} \in \mathrm{Irr}(G \wr H)$ is an extension of $\phi^{\times n}$. For $\psi \in \mathrm{Irr}(H)$ we have that $\mathcal{X}(\phi; \psi) \in \mathrm{Irr}(G \wr H \mid \phi^{\times n})$, the set of irreducible characters χ of $G \wr H$ whose restriction $\chi\big\downarrow_{G^{\times n}}$ contains $\phi^{\times n}$ as an irreducible constituent. Indeed, Gallagher's Theorem [36, Corollary 6.17] gives

$$\mathrm{Irr}(G \wr H \mid \phi^{\times n}) = \{\mathcal{X}(\phi; \psi) \mid \psi \in \mathrm{Irr}(H)\}.$$

More generally, if $K \leq G$ and $\psi \in \mathrm{Irr}(K)$ then we denote by $\mathrm{Irr}(G \mid \psi)$ the set of characters $\chi \in \mathrm{Irr}(G)$ such that ψ is an irreducible constituent of the restriction $\chi\big\downarrow_K$. When clear from context, we also abbreviate $\mathcal{X}(\chi^\gamma; \chi^\nu)$ involving characters of symmetric groups (so γ and ν are partitions) to $\mathcal{X}(\gamma; \nu)$.

Wreath product characters of the form $\mathcal{X}(\phi; \psi)$ will play an important role in Chapters 4 to 6. The general form of an irreducible character of a wreath product group will also be important; for a precise description of the full set of pairwise non-isomorphic irreducible characters of an arbitrary wreath product $G \wr H$, we refer the reader to [41, §4.3]. Here we simply record the following: let $\xi_1, \ldots, \xi_k$ be representatives for the orbits of the conjugation action of $G \wr H$ on $\mathrm{Irr}(G^{\times n})$. By Clifford theory (see [36, Theorem 6.2], for instance), we have that

$$\mathrm{Irr}(G \wr H) = \bigsqcup_{i=1}^{k} \mathrm{Irr}(G \wr H \mid \xi_i).$$

To describe each $\mathrm{Irr}(G \wr H \mid \xi_i)$, fix i and suppose $\xi_i \in \mathrm{Irr}(G^{\times n})$ is given by $\xi_i = \phi_{i_1} \times \cdots \times \phi_{i_n}$ for some i_j, where $\mathrm{Irr}(G) = \{\phi_1, \ldots, \phi_t\}$. We partition $[n]$ into subsets $\alpha_1, \ldots, \alpha_k$ according to the relation $i_x = i_y$; that is, $x, y \in [n]$ belong to the same subset if and only if $i_x = i_y$. Define I to be the Young subgroup $\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k}$ of $\mathfrak{S}_n$. By Gallagher's Theorem [36, Corollary 6.17], for all $\chi \in \mathrm{Irr}(G \wr H \mid \xi_i)$ there exists a unique $\theta \in \mathrm{Irr}(H \cap I)$ such that

$$\chi = \widetilde{\xi_i} \cdot \theta\big\uparrow_{G\wr(H\cap I)}^{G\wr H},$$

where $\widetilde{\xi_i}$ is an extension of ξ_i from $\mathrm{Irr}(G^{\times n})$ to $\mathrm{Irr}(G \wr (H \cap I))$ with action as defined in (2.2), and where θ also denotes its inflation from $\mathrm{Irr}(H \cap I)$ to $\mathrm{Irr}(G \wr (H \cap I))$. In fact, the following map is a bijection (see [36, Theorem 6.11]):

$$\mathrm{Irr}(G \wr (H \cap I) \mid \xi_i) \longrightarrow \mathrm{Irr}(G \wr H \mid \xi_i)\ , \qquad \eta \longmapsto \eta\big\uparrow^{G\wr H}.$$

In the special case where $n = p$ is a prime number and $H = C_p$, we see that $H \cap I$ can only be the trivial group or all of H itself (according to whether $\phi_{i_1}, \ldots, \phi_{i_n}$ are all not equal or are all equal, respectively), and so necessarily the inertia group $G \wr (H \cap I)$ is either simply $G^{\times p} = G \wr 1$ or $G \wr H$. Thus every $\psi \in \mathrm{Irr}(G \wr C_p)$ is either of the form

(a) $\psi = \phi_{i_1} \times \cdots \times \phi_{i_p}\big\uparrow_{G^{\times p}}^{G\wr C_p}$, where $\phi_{i_1}, \ldots, \phi_{i_p} \in \mathrm{Irr}(G)$ are not all equal; or

(b) $\psi = \mathcal{X}(\phi; \theta)$ for some $\phi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(C_p)$.

When (a) holds, $\psi\big\downarrow_{G^{\times p}}$ is the sum of the p irreducible characters of $G^{\times p}$ whose p factors are a cyclic permutation of $\phi_{i_1}, \ldots, \phi_{i_p}$. When (b) holds, $\psi\big\downarrow_{G^{\times p}} = \phi^{\times p} \cdot \theta(1) = \phi^{\times p}$.

2.3.1 Irreducible constituents of characters of wreath products

We record some results concerning characters of wreath products that will be useful later in this thesis.

Lemma 2.13 ([41, Lemma 4.3.9])**.** *Let $n \in \mathbb{N}$. Let $H \leq \mathfrak{S}_n$ and G be finite groups. Let*

$\phi \in \mathrm{Irr}(G)$ *and* $\psi \in \mathrm{Irr}(H)$. *Then for all* $f_1, \ldots, f_n \in G$ *and* $\pi \in H$,

$$\mathcal{X}(\phi;\psi)(f_1,\ldots,f_n;\pi) = \prod_{v=1}^{c(\pi)} \phi(f_{j_v} \cdot f_{\pi^{-1}(j_v)} \cdot f_{\pi^{-2}(j_v)} \cdots f_{\pi^{-l_v+1}(j_v)}) \cdot \psi(\pi),$$

where $c(\pi)$ *is the number of disjoint cycles in* π, l_v *is the length of the* v^{th} *cycle, and for each* v, j_v *is some fixed element in the* v^{th} *cycle.*

The element $f_{j_v} \cdot f_{\pi^{-1}(j_v)} \cdot f_{\pi^{-2}(j_v)} \cdots f_{\pi^{-l_v+1}(j_v)} \in G$ is called the v^{th} cycle product of $(f_1, \ldots, f_n; \pi)$, and is determined up to conjugation (a different choice of j_v yields a G-conjugate of the given element). The character formula in Lemma 2.13 is well-defined since ϕ is a character of G. For example, if $n = 8$ and $\pi = (1,3,7,2)(5,8,6)(4)$, then $\mathcal{X}(\phi;\psi)(f_1,\ldots,f_8;\pi) = \phi(f_2f_7f_3f_1) \cdot \phi(f_6f_8f_5) \cdot \phi(f_4) \cdot \psi(\pi)$.

Lemma 2.14 (Associativity of wreath products)**.** *Let* $l, m, n \in \mathbb{N}$ *and let* $G \leq \mathfrak{S}_l$, $H \leq \mathfrak{S}_m$ *and* $I \leq \mathfrak{S}_n$. *Then the following map* $\theta : (G \wr H) \wr I \longrightarrow G \wr (H \wr I)$ *is an isomorphism of groups:*

$$\begin{aligned}\big((g_{11},\ldots,g_{1m};h_1),\ldots,(g_{n1},\ldots,g_{nm};h_n)\ ;\ \pi\big)&\\ \longmapsto \big(g_{11},\ldots,g_{1m},g_{21},\ldots,g_{2m},\ldots,g_{n1},\ldots,g_{nm}\ ;\ (h_1,\ldots,h_n;\pi)\big),&\end{aligned}$$

where $g_{ji} \in G$, $h_i \in H$ *and* $\pi \in I$. *Moreover, for* $\alpha \in \mathrm{Char}(G)$, $\beta \in \mathrm{Char}(H)$ *and* $\gamma \in \mathrm{Char}(I)$, *we have that*

$$\mathcal{X}\big(\mathcal{X}(\alpha;\beta);\gamma\big)(x) = \mathcal{X}\big(\alpha;\mathcal{X}(\beta;\gamma)\big)(\theta(x))$$

for all $x \in (G \wr H) \wr I$.

Proof. The first statement is a routine check, following the notational convention in [41, §4.1]. The second statement follows from Lemma 2.13. □

In particular, associativity for three terms as in Lemma 2.14 then gives associativity for k-term wreath products for all $k \geq 3$, and so from now on we simply write $G_1 \wr G_2 \wr \cdots \wr G_k$ without internal parentheses when referring to such groups, and identify corresponding elements under such isomorphisms.

We remark that the map θ above is 'natural', in the sense that θ behaves well with respect to the canonical permutation representations of wreath products described in [41, 4.1.18]. Specifically, for $G \leq \mathfrak{S}_l$ and $H \leq \mathfrak{S}_m$ we have a permutation representation $\psi : G \wr H \to \mathfrak{S}_{lm} = \mathrm{Sym}\{1,2,\ldots,lm\}$ given by the map

$$(g_1,\ldots,g_m;h) \quad\longmapsto\quad \big((j-1)l+i \ \longmapsto\ (h(j)-1)l+g_{h(j)}(i)\big) \tag{2.3}$$

for all $j \in [m]$ and $i \in [l]$. In the same vein, we may define permutation representations $\psi' : (G \wr H) \wr I \to \mathfrak{S}_{lmn}$ and $\psi'' : G \wr (H \wr I) \to \mathfrak{S}_{lmn}$. Then $\psi'(x) = \psi''(\theta(x))$, for all $x \in (G \wr H) \wr I$.

Next, we record some results describing the irreducible constituents of restrictions and inductions of characters of wreath products.

Lemma 2.15. *Let G be a finite group and $H \leq \mathfrak{S}_n$ for some $n \in \mathbb{N}$. Let $\chi \in \mathrm{Irr}(G)$. Then*

$$\chi^{\times n}\big\uparrow_{G^{\times n}}^{G \wr H} = \sum_{\theta \in \mathrm{Irr}(H)} \theta(1) \cdot \mathcal{X}(\chi; \theta).$$

Proof. Observe since $\sum_{\theta \in \mathrm{Irr}(H)} \theta(1)^2 = |H|$, we have that

$$\deg(\chi^{\times n}\big\uparrow_{G^{\times n}}^{G \wr H}) = |G \wr H : G^{\times n}| \cdot (\deg \chi)^n = |H|(\deg \chi)^n = \deg\left(\sum_{\theta \in \mathrm{Irr}(H)} \theta(1) \cdot \mathcal{X}(\chi; \theta)\right).$$

Moreover, for any $\theta \in \mathrm{Irr}(H)$,

$$\langle \chi^{\times n}\big\uparrow_{G^{\times n}}^{G \wr H}, \mathcal{X}(\chi; \theta)\rangle = \langle \chi^{\times n}, \mathcal{X}(\chi; \theta)\big\downarrow_{G^{\times n}}\rangle = \langle \chi^{\times n}, \theta(1) \cdot \chi^{\times n}\rangle = \theta(1),$$

and the claim follows. □

Lemma 2.16 ([36, Problem 5.2])**.** *Let G be a finite group. Suppose $H, K \leq G$ with $KH = G$. Let ϕ be a character of H. Then $\phi\big\uparrow_H^G\big\downarrow_K = \phi\big\downarrow_{H \cap K}^H\big\uparrow^K$.*

Proof. By Mackey's Theorem (see [36, §5], for example),

$$\phi\big\uparrow_H^G\big\downarrow_K = \sum_{g \in H \backslash G / K} \phi^g\big\downarrow_{H^g \cap K}^{H^g}\big\uparrow^K$$

where the sum runs over a set of (H, K)–double coset representatives g. However, $KH = G$ implies (that $HK = G$ also, and hence) we can simply take the single representative $g = 1$, from which the claim follows immediately. □

Remark 2.17. We often wish to apply Lemma 2.16 to G of the form $\mathfrak{S}_m \wr L$ for some finite group $L \leq \mathfrak{S}_l$ where $l, m \in \mathbb{N}$, with $H = (\mathfrak{S}_m)^{\times l}$ and $K = P \wr L$ for some subgroup P of $\mathfrak{S}_m$ (usually a Sylow subgroup). Indeed, since

$$K = \{(f_1, \dots, f_l; \pi) \mid f_i \in P,\ \pi \in L\} \quad \text{and} \quad H = \{(g_1, \dots, g_l; 1) \mid g_i \in \mathfrak{S}_m\},$$

then

$$KH = \{(f_1 g_{\pi^{-1}(1)}, \dots, f_l g_{\pi^{-1}(l)}; \pi)\}$$

which ranges over all of G as f_i, g_j and π vary accordingly. Hence $KH = G$. ◊

Lemma 2.18. *Let p be an odd prime and G be a finite group. Let $\eta \in \mathrm{Char}(G)$ and $\varphi \in \mathrm{Irr}(G)$. If $\langle \eta, \varphi \rangle \geq 2$, then*

$$\langle \mathcal{X}(\eta; \tau), \mathcal{X}(\varphi; \theta)\rangle \geq 2$$

for all $\tau, \theta \in \mathrm{Irr}(C_p)$.

Proof. Let $\eta = \varphi + \Delta$, so $\langle \Delta, \varphi \rangle \geq 1$. Fix some $\tau \in \mathrm{Irr}(C_p)$. We first decompose $\mathcal{X}(\varphi + \Delta; \tau)$ into various summands by considering the corresponding $G \wr C_p$–module. Let φ be afforded by G–module V, Δ by G–module W, and τ by C_p–module A. Then $\mathcal{X}(\varphi+\Delta;\tau)$ is afforded by $\widetilde{(V \oplus W)^{\otimes p}} \otimes A$, which has a decomposition into $M_0 \oplus M_1 \oplus \cdots \oplus M_p$ as $G \wr C_p$–modules where

$$M_p = \widetilde{V^{\otimes p}} \otimes A, \qquad M_0 = \widetilde{W^{\otimes p}} \otimes A$$

and for $i \in [p-1]$, N_i is the vector space direct sum of the (external) tensor products of all ordered sequences of Vs and Ws of length p with exactly i Vs, and $M_i = N_i \otimes A$. For example,

$$M_1 = (VW \cdots W + WVW \cdots W + \cdots + W \cdots WV) \otimes A, \text{ and}$$
$$M_2 = (VVW \cdots W + VWVW \cdots W + \cdots + W \cdots WVV) \otimes A,$$

where XY denotes $X \otimes Y$ for $X, Y \in \{V, W\}$ and $+$ denotes a direct sum of vector spaces. Clearly M_p affords the character $\mathcal{X}(\varphi;\tau)$ and M_0 the character $\mathcal{X}(\Delta;\tau)$. Letting ψ_i be the character of M_i for $i \in [p-1]$, we now wish to determine ψ_i.

Since $\dim A = 1$, the restriction $\psi_i \big\downarrow^{G \wr C_p}_{G^{\times p}}$ is the sum of the $\binom{p}{i}$ characters of $G^{\times p}$ whose summands are permutations of $\varphi^{\times i} \times \Delta^{\times (p-i)}$. Let $s(\varphi, \Delta)$ denote an ordered sequence of length p with entries taken from $\{\varphi, \Delta\}$ (and suppose that both φ and Δ appear in the sequence), and let $\overline{s}(\varphi, \Delta)$ denote the corresponding character of $G^{\times p}$. We denote by $s(V, W)$ the sequence obtained from $s(\varphi, \Delta)$ by replacing φ, Δ with V, W respectively, and let $\overline{s}(V, W)$ denote the vector subspace of N_i given by the tensor product corresponding to $s(V, W)$ where $s(\varphi, \Delta)$ has i terms equal to φ. Given $s(V, W)$, let $\hat{s}(V, W)$ denote the vector space direct sum of $\overline{t}(V, W)$ over all p cyclic permutations $t(V, W)$ of $s(V, W)$. Then by inspection of the action of $G^{\times p}$ on $\widetilde{(V \oplus W)^{\otimes p}}$ and observing that $\{(1, \ldots, 1; \sigma) \mid \sigma \in C_p\}$ is a set of coset representatives for $G^{\times p}$ in $G \wr C_p$, we find that $\hat{s}(V, W)$ is a $G \wr C_p$–module affording the character $\overline{s}(\varphi, \Delta) \big\uparrow^{G \wr C_p}_{G^{\times p}}$. Hence ψ_i is the sum of $\binom{p}{i}/p$ characters of the form $\overline{s}(\varphi, \Delta) \big\uparrow^{G \wr C_p}_{G^{\times p}} \cdot \tau$, where $s(\varphi, \Delta)$ runs over a set of orbit representatives for the permutation action of C_p on the set of such length p sequences with i terms equal to φ. For example, we may take the single representative $(\varphi, \Delta, \cdots, \Delta)$ for $i = 1$, while for $i = 2$, the $\frac{p-1}{2}$ sequences

$$(\varphi, \varphi, \Delta, \ldots, \Delta), \quad (\varphi, \Delta, \varphi, \Delta, \ldots, \Delta), \quad \cdots, \quad (\varphi, \underbrace{\Delta, \ldots, \Delta}_{(p-3)/2}, \varphi, \Delta, \ldots, \Delta)$$

form a set of representatives. Moreover, the induced character $\overline{s}(\varphi, \Delta) \big\uparrow^{G \wr C_p}_{G^{\times p}}$ is zero on elements $(g_1, \ldots, g_p; \sigma) \in G \wr C_p \setminus G^{\times p}$, that is, whenever $\sigma \in C_p \setminus \{1\}$. On the other hand, $\tau((g_1, \ldots, g_p; \sigma)) = \tau(\sigma)$ by the definition of inflation. Hence

$$\overline{s}(\varphi, \Delta) \big\uparrow^{G \wr C_p}_{G^{\times p}} \cdot \tau = \overline{s}(\varphi, \Delta) \big\uparrow^{G \wr C_p}_{G^{\times p}},$$

since τ is linear. In particular, we conclude that

$$\mathcal{X}(\varphi+\Delta;\tau)=\mathcal{X}(\varphi;\tau)+\mathcal{X}(\Delta;\tau)+\sum_{i=1}^{p-1}\sum\overline{s}(\varphi,\Delta)\big\uparrow_{G^{\times p}}^{G\wr C_p}$$

where the last summation runs over the appropriate representatives, i.e. there are $\binom{p}{i}/p$ terms in the i^{th} sum. Since $\langle\Delta,\varphi\rangle\geq 1$, $\varphi^{\times p}\big\uparrow_{G^{\times p}}^{G\wr C_p}$ is a direct summand of every $\overline{s}(\varphi,\Delta)\big\uparrow_{G^{\times p}}^{G\wr C_p}$, whence $\varphi^{\times p}\big\uparrow_{G^{\times p}}^{G\wr C_p}$ is a direct summand of $\mathcal{X}(\varphi+\Delta;\tau)$ of multiplicity m where

$$m\geq\sum_{i=1}^{p-1}\frac{\binom{p}{i}}{p}\geq 2.$$

Finally, $\varphi^{\times p}\big\uparrow_{G^{\times p}}^{G\wr C_p}=\sum_{\theta\in\mathrm{Irr}(C_p)}\mathcal{X}(\varphi;\theta)$ by Lemma 2.15, so the claim follows. □

Lemma 2.19. *Let G, H be finite groups with $H\leq\mathfrak{S}_m$ for some $m\in\mathbb{N}$, and let $\theta\in\mathrm{Irr}(H)$. Let $\alpha\in\mathrm{Irr}(G)$ and $\Delta\in\mathrm{Char}(G)$ be such that $\langle\Delta,\alpha\rangle=1$. Then for any $\beta\in\mathrm{Irr}(H)$,*

$$\langle\mathcal{X}(\Delta;\theta),\mathcal{X}(\alpha;\beta)\rangle=\langle\theta,\beta\rangle=\delta_{\theta,\beta}.$$

Proof. Let $\zeta=\mathcal{X}(\Delta;\theta)$ and $c=\langle\zeta,\mathcal{X}(\alpha;\theta)\rangle$. Clearly $c\geq 1$, since α is a constituent of Δ. (More generally, if $\Delta=\psi_1+\cdots+\psi_r$ is a decomposition into irreducible characters ψ_i which are not necessarily distinct, then $\sum_i\mathcal{X}(\psi_i;\theta)$ is a direct summand of $\mathcal{X}(\Delta;\theta)$.) Now $\zeta\big\downarrow_{G^{\times m}}=\theta(1)\cdot\Delta^{\times m}$, so

$$\begin{aligned}\theta(1)=\theta(1)\cdot(\langle\Delta,\alpha\rangle)^m&=\langle\zeta\big\downarrow_{G^{\times m}},\alpha^{\times m}\rangle=\sum_{\gamma\in\mathrm{Irr}(G\wr H)}\langle\zeta,\gamma\rangle\cdot\langle\gamma\big\downarrow_{G^{\times m}},\alpha^{\times m}\rangle\\&\geq\sum_{\beta\in\mathrm{Irr}(H)}\langle\zeta,\mathcal{X}(\alpha;\beta)\rangle\cdot\langle\mathcal{X}(\alpha;\beta)\big\downarrow_{G^{\times m}},\alpha^{\times m}\rangle=\sum_{\beta\in\mathrm{Irr}(H)}\beta(1)\cdot\langle\zeta,\mathcal{X}(\alpha;\beta)\rangle\\&\geq\theta(1)\cdot c\geq\theta(1).\end{aligned}$$

Thus the above inequalities in fact hold with equality and the claim follows, since $\beta(1)\in\mathbb{N}$ and $\langle\zeta,\mathcal{X}(\alpha;\beta)\rangle\in\mathbb{N}_{\geq 0}$. □

We conclude by mentioning two useful results concerning wreath products of symmetric groups.

Theorem 2.20 ([32, Theorem 3.5])**.** *Let p be an odd prime and let $k\in\mathbb{N}$. Let $K:=\mathfrak{S}_{p^{k-1}}\wr\mathfrak{S}_p\leq\mathfrak{S}_{p^k}$ and let $\chi\in\mathrm{Irr}_{p'}(\mathfrak{S}_{p^k})$. The restriction $\chi\big\downarrow_K$ has a unique irreducible constituent χ^* lying in $\mathrm{Irr}_{p'}(K)$, appearing with multiplicity 1. Moreover, the map $\chi\mapsto\chi^*$ is a bijection between $\mathrm{Irr}_{p'}(\mathfrak{S}_{p^k})$ and $\mathrm{Irr}_{p'}(K)$.*

More precisely, such a character χ is equal to χ^λ for a hook partition $\lambda\vdash p^k$. If $\lambda=(p^k-(mp+x),1^{mp+x})$ for some $x\in\{0,1,\dots,p-1\}$, then $\chi^*\in\{\mathcal{X}(\mu;\nu_1),\mathcal{X}(\mu;\nu_2)\}$, where

$$\mu=(p^{k-1}-m,1^m),\ \nu_1=(p-x,1^x)\text{ and }\nu_2=(x+1,1^{p-1-x}).$$

Theorem 2.21 ([55, Corollary 9.1]). *Let $m, n \in \mathbb{N}$. Let $\mu = (\mu_1, \dots, \mu_k) \vdash m$ and $\nu = (\nu_1, \dots, \nu_l) \vdash n$. The lexicographically greatest partition λ of mn such that χ^λ is an irreducible constituent of $\mathcal{X}(\mu;\nu)\big\uparrow^{\mathfrak{S}_{mn}}_{\mathfrak{S}_m \wr \mathfrak{S}_n}$ is*

$$\lambda = (n\mu_1, \dots, n\mu_{k-1}, n(\mu_k - 1) + \nu_1, \nu_2, \dots, \nu_l).$$

Moreover, χ^λ occurs as a constituent with multiplicity 1.

2.3.2 Sylow subgroups of symmetric groups

We recall some facts about Sylow subgroups of symmetric groups, and refer the reader to [41, Chapter 4] for a more detailed discussion. Fix a prime p and let $n \in \mathbb{N}$. Let P_n denote a Sylow p-subgroup of $\mathfrak{S}_n$. Clearly P_1 is the trivial group while P_p is cyclic of order p. More generally, $P_{p^k} = (P_{p^{k-1}})^{\times p} \rtimes P_p = P_{p^{k-1}} \wr P_p \cong P_p \wr \cdots \wr P_p$ (k-fold wreath product) for all $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ and let $n = \sum_{i=1}^t a_i p^{n_i}$ be its p-adic expansion, where $0 \leq n_1 < \cdots < n_t$. Then $P_n \cong (P_{p^{n_1}})^{\times a_1} \times \cdots \times (P_{p^{n_t}})^{\times a_t}$.

Conjugating by an appropriate element of $\mathfrak{S}_n$, we may assume the following:

- For each $k \in \mathbb{N}$, P_{p^k} is generated by $\sigma_1, \dots, \sigma_k \in \mathfrak{S}_{p^k} = \mathrm{Sym}\{1, 2, \dots, p^k\}$ where

$$\sigma_i = \prod_{j=1}^{p^{i-1}} (j\ ,\ p^{i-1} + j\ ,\ 2p^{i-1} + j\ ,\ \cdots\ ,\ (p-1)p^{i-1} + j)$$

  for each i. For example, when $p = 3$ we have

$$\sigma_1 = (1, 2, 3), \quad \sigma_2 = (1, 4, 7)(2, 5, 8)(3, 6, 9), \quad \sigma_3 = (1, 10, 19) \cdots (9, 18, 27),$$

  and so on. This choice of P_{p^k} for all $k \in \mathbb{N}$ is compatible with the identification of k-fold wreath products in Lemma 2.14 and realises the permutation representation (2.3).

- For $n = \sum_{i=1}^t a_i p^{n_i}$, P_n is a direct product of factors $P_{p^{n_i}} \leq \mathfrak{S}_{p^{n_i}}$ permuting disjoint subsets of $\{1, 2, \dots, n\}$. For instance, if $i \in [t]$ and $j \in [a_i]$, then the j^{th} factor $P_{p^{n_i}} \leq \mathfrak{S}_{p^{n_i}}$ permutes the numbers $\{r + 1, r + 2, \dots, r + p^{n_i}\}$, where $r = a_1 p^{n_1} + \cdots + a_{i-1} p^{n_{i-1}} + (j-1) p^{n_i}$.

We record a characterisation of p^k-cycles in P_{p^k}.

Lemma 2.22. *Let $k \in \mathbb{N}$ and p be a prime. Let $x \in P_{p^k} = P_{p^{k-1}} \wr P_p \leq \mathfrak{S}_{p^k}$, so $x = (f_1, \dots, f_p; \sigma)$ for some $f_i \in P_{p^{k-1}}$ and $\sigma \in P_p$. If x has a fixed point, then $\sigma = 1$. Moreover, x is a p^k-cycle if and only if $\sigma \neq 1$ and $f_{\sigma^{p-1}(1)} \cdots f_{\sigma(1)} \cdot f_1$ is a p^{k-1}-cycle.*

Proof. We embed $P_{p^k} \leq \mathfrak{S}_{p^k}$ via the permutation representation (2.3). For $i \in [p]$, let $J_i = \{(i-1)p^{k-1} + t \mid t \in [p^{k-1}]\}$. If $\sigma = 1$, then x permutes J_i for each i, while if $\sigma \neq 1$, then x sends elements of J_i to $J_{\sigma(i)} \neq J_i$. Thus if x has a fixed point then $\sigma = 1$, while if x is a p^k-cycle then $\sigma \neq 1$.

So now suppose $\sigma \neq 1$. We may represent $x = (f_1, \dots, f_p; \sigma)$ as a permutation of $\{1, \dots, p^k\}$ pictorially as follows (see Figure 2.1):

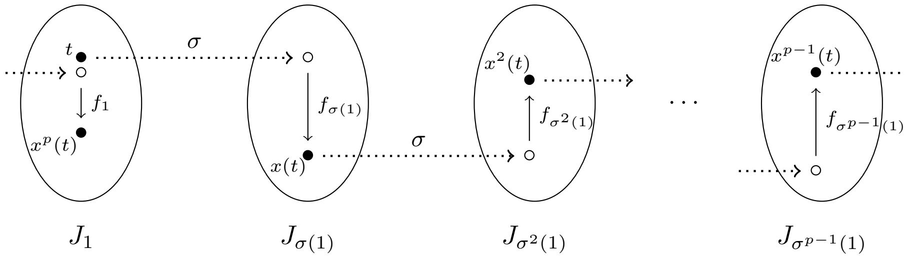


Figure 2.1: The permutation $x = (f_1, \dots, f_p; \sigma)$ on $\{1, 2, \dots, p^k\}$.

That is, for each $i \in [p]$ and $t \in [p^{k-1}]$, σ sends $ip^{k-1} + t$ to $\sigma(i)p^{k-1} + t$, while the f_i component of x sends $(i-1)p^{k-1} + t$ to $(i-1)p^{k-1} + f_i(t)$. Given $t \in J_1$, $x^j(t) \in J_1$ if and only if $j \mid p$. Moreover, $x^p(t) = f_1 \cdot f_{\sigma^{p-1}(1)} \cdots f_{\sigma(1)}(t)$, so x is a p^k-cycle if and only if $\min\{j \in \mathbb{N} \mid x^j(t) = t\} = p^{k-1}$, in other words, if and only if $g := f_1 \cdot f_{\sigma^{p-1}(1)} \cdots f_{\sigma(1)}$ is a p^{k-1}-cycle. Finally, observe that $f_{\sigma^{p-1}(1)} \cdots f_{\sigma(1)} \cdot f_1 = f_1^{-1} g f_1$ has the same cycle type as g. $\square$

Let $\mathrm{Irr}(P_p) = \{\phi_0, \phi_1, \dots, \phi_{p-1}\} = \mathrm{Lin}(P_p)$, where $\phi_0 = \mathbb{1}_{P_p}$ is the trivial character of the cyclic group P_p. (This labelling follows from the fact that we may write $P_p = \langle g \rangle$ and $\phi_j(g) = \omega^j$ for each $j \in \{0, 1, \dots, p-1\} = [\overline{p}]$, where $\omega = e^{2\pi i/p}$.) Note that the regular character of P_p equals $\sum_{i=0}^{p-1} \phi_i$. When $m \geq 2$, an easy application of [36, Corollary 6.17] shows that

$$\mathrm{Lin}(P_{p^m}) = \bigsqcup_{\phi \in \mathrm{Lin}(P_{p^{m-1}})} \mathrm{Irr}(P_{p^m} \mid \phi^{\times p}).$$

In particular, $\mathrm{Irr}(P_{p^m} \mid \phi^{\times p}) = \{\mathcal{X}(\phi; \psi) \mid \psi \in \mathrm{Lin}(P_p)\}$.

Using the above observations, we may naturally define a bijection $s \longleftrightarrow \phi(s)$ between the set $[\overline{p}]^m$ of sequences of length m with elements from $[\overline{p}]$ and the set $\mathrm{Lin}(P_{p^m})$. More precisely, if $m = 0$ we let the empty sequence of length 0 correspond to the trivial character of P_1, and if $m = 1$ we let $s = (x)$ correspond to ϕ_x, for each $x \in [\overline{p}]$. If $m \geq 2$ then for any $s = (s_1, \dots, s_m) \in [\overline{p}]^m$, we recursively define

$$\phi(s) := \mathcal{X}\big(\phi(s^-); \phi(s_m)\big),$$

where $s^- = (s_1, \dots, s_{m-1}) \in [\overline{p}]^{m-1}$. By Lemma 2.14,

$$\phi(s) = \mathcal{X}(\phi(s_1, \dots, s_i); \phi(s_{i+1}, \dots, s_m))$$

for any $i \in [m-1]$. We remark that the abelianisation P_{p^k}/P'_{p^k} is isomorphic to $(C_p)^k$, by [53, Lemma 1.4]. Once we fix a natural isomorphism $P_{p^k}/P'_{p^k} \to (C_p)^k$ (see Lemma 4.3

below), then our indexing of $\mathrm{Lin}(P_{p^k})$ can in fact be obtained equivalently from the canonical bijection $\mathrm{Lin}(P_{p^k}) \longleftrightarrow \mathrm{Irr}(P_{p^k}/P'_{p^k})$.

Now let $n \in \mathbb{N}$ and let $n = \sum_{i=1}^t a_i p^{n_i}$ be its p-adic expansion, where $0 \leq n_1 < \cdots < n_t$. Since $P_n \cong (P_{p^{n_1}})^{\times a_1} \times \cdots \times (P_{p^{n_t}})^{\times a_t}$,

$$\mathrm{Lin}(P_n) = \{\phi(\underline{\mathbf{s}}) \mid \underline{\mathbf{s}} = \big(\mathbf{s}(1,1), \dots, \mathbf{s}(1,a_1), \mathbf{s}(2,1), \dots, \mathbf{s}(2,a_2), \dots, \mathbf{s}(t,a_t)\big)\}, \tag{2.4}$$

where for all $i \in [t]$ and $j \in [a_i]$ we have that $\mathbf{s}(i,j) \in [\overline{p}]^{n_i}$, and

$$\phi(\underline{\mathbf{s}}) := \phi(\mathbf{s}(1,1)) \times \cdots \times \phi(\mathbf{s}(1,a_1)) \times \phi(\mathbf{s}(2,1)) \times \cdots \times \phi(\mathbf{s}(2,a_2)) \times \cdots \times \phi(\mathbf{s}(t,a_t)).$$

When we suppose that $\phi(\underline{\mathbf{s}})$ is a linear character of P_n, we mean that $\underline{\mathbf{s}}$ is a sequence of sequences, of the form described in (2.4) above.

Sylow normalisers

Next, we describe the structure of the normaliser $N_{\mathfrak{S}_n}(P_n)$. Following the notation $n = \sum_{i=1}^t a_i p^{n_i}$, by [53, Lemma 4.1] we have that

$$N_{\mathfrak{S}_n}(P_n) \cong N_1 \wr \mathfrak{S}_{a_1} \times \cdots \times N_t \wr \mathfrak{S}_{a_t},$$

where $N_i = N_{\mathfrak{S}_{p^{n_i}}}(P_{p^{n_i}})$. Moreover, for $k \in \mathbb{N}$,

$$N_{\mathfrak{S}_{p^k}}(P_{p^k}) \cong P_{p^k} \rtimes (C_{p-1})^{\times k}, \tag{2.5}$$

from which it follows that $N_{\mathfrak{S}_n}(P_n) = P_n$ for all n when $p = 2$. The structure of Sylow normalisers for symmetric groups is well-known; for our purposes, we record the following presentation:

$$N_{\mathfrak{S}_{p^k}}(P_{p^k}) = \langle P_{p^k}, \rho_1^{(k)}, \rho_2^{(k)}, \dots, \rho_k^{(k)} \rangle$$

where $P_{p^k} = \langle \sigma_1, \dots, \sigma_k \rangle$ as above, and $\rho_i^{(j)}$ are defined recursively as follows. Let c be a primitive root modulo p, and set $\rho_1^{(1)} = (c_1, c_2, \dots, c_{p-1}) \in \mathrm{Sym}\{1, \dots, p\}$ where $c_i \in [p]$ is such that $c_i \equiv c^i \pmod{p}$. For an integer m, let $\tau_m \in \mathfrak{S}_{p^k}$ be the permutation $i \mapsto i+m$ with numbers modulo p^k (taken in the range $\{1, \dots, p^k\}$). For $1 \leq j < k$, set

$$\rho_j^{(k)} = \prod_{i=0}^{p-1} \tau_{ip^{k-1}} \cdot \rho_j^{(k-1)} \cdot \tau_{-ip^{k-1}}$$

and

$$\rho_k^{(k)} = \prod_{i=0}^{p^{k-1}-1} \tau_{-i} \cdot (c_1 p^{k-1}, c_2 p^{k-1}, \dots, c_{p-1} p^{k-1}) \cdot \tau_i$$

with numbers modulo p^k (taken in the range $\{1, \dots, p^k\}$). Notice that for each k, the permutation σ_k is a product of p-cycles; $\rho_k^{(k)}$ is simply the product over all p-cycles

$(a_1, a_2, \ldots, a_p)$ in σ_k of $(a_{c_1}, a_{c_2}, \ldots, a_{c_{p-1}})$. By construction, each $\rho_j^{(k)}$ is a product of $(p-1)$-cycles, and the $\rho_j^{(k)}$ commute for all j for each fixed k. We deduce that $N_{\mathfrak{S}_{p^k}}(P_{p^k}) = P_{p^k} \rtimes \langle \rho_j^{(k)} \mid j \in [k] \rangle \cong P_{p^k} \rtimes (C_{p-1})^{\times k}$.

For example, when $p = 5$ we may choose $c = 2$ so that $\rho_1^{(1)} = (2, 4, 3, 1)$,

$$\rho_1^{(2)} = (2,4,3,1)(7,9,8,6)(12,14,13,11)(17,19,18,16)(22,24,23,21)$$

and

$$\rho_2^{(2)} = (6,16,11,1)(7,17,12,2)(8,18,13,3)(9,19,14,4)(10,20,15,5).$$

Relation to the alternating group $\mathfrak{A}_n$

Finally, we relate the normalisers of P_n to the alternating subgroup $\mathfrak{A}_n$ of $\mathfrak{S}_n$. Note when p is odd then $P_n \leq \mathfrak{A}_n$ for all n, and so $\mathrm{Syl}_p(\mathfrak{A}_n) = \mathrm{Syl}_p(\mathfrak{S}_n)$. It is clear to see that $\chi^\lambda \big\downarrow_{\mathfrak{A}_n} = \chi^{\lambda'} \big\downarrow_{\mathfrak{A}_n}$. The ordinary irreducible characters of $\mathfrak{A}_n$ can be indexed as follows:

$$\mathrm{Irr}(\mathfrak{A}_n) = \{\chi^\lambda \big\downarrow_{\mathfrak{A}_n} \mid \lambda \neq \lambda' \in \mathcal{P}(n)\} \ \cup \ \{\psi_+^\lambda, \psi_-^\lambda \mid \lambda = \lambda' \in \mathcal{P}(n)\}.$$

We refer the reader to [41, Chapter 2.5] for a detailed discussion of the representation theory of $\mathfrak{A}_n$.

Lemma 2.23. *Let p be a prime and let $n \in \mathbb{N}_{\geq p}$. Then there exists $g \in N_{\mathfrak{S}_n}(P_n) \setminus \mathfrak{A}_n$. In particular, if $\lambda \vdash n$ is self-conjugate, then $(\psi_+^\lambda)^g = \psi_-^\lambda$.*

Proof. If $p = 2$, then P_{2^k} contains a transposition for all $k \in \mathbb{N}$, since $P_2 = \langle (12) \rangle$. If p is odd, then $N_{\mathfrak{S}_{p^k}}(P_{p^k})$ contains an element of cycle type $(p-1)\cdots(p-1)$ (p^{k-1} times). The case of n not a power of p then follows. The final assertion follows from the definition of $\psi_\pm^\lambda$ (see [41], for example). □

Corollary 2.24. *Let p be a prime and let $n \in \mathbb{N}$. Suppose $\lambda \vdash n$ is self-conjugate. Then $\langle \psi_+^\lambda \big\downarrow_{P_n}, \mathbb{1}_{P_n} \rangle = \langle \psi_-^\lambda \big\downarrow_{P_n}, \mathbb{1}_{P_n} \rangle$.*

Proof. If $n < p$ then the assertion is clear since P_n is trivial and $\deg \psi_+^\lambda = \deg \psi_-^\lambda$. Otherwise, let $g \in N_{\mathfrak{S}_n}(P_n) \setminus \mathfrak{A}_n$. Then since $(\mathbb{1}_{P_n})^g = \mathbb{1}_{P_n}$, we have that

$$\langle \psi_-^\lambda \big\downarrow_{P_n}, \mathbb{1}_{P_n} \rangle = \langle (\psi_+^\lambda)^g \big\downarrow_{P_n}, (\mathbb{1}_{P_n})^g \rangle = \langle (\psi_+^\lambda \big\downarrow_{P_n})^g, (\mathbb{1}_{P_n})^g \rangle = \langle \psi_+^\lambda \big\downarrow_{P_n}, \mathbb{1}_{P_n} \rangle.$$

□

Chapter 3

On the p'-subgraph of the Young graph

The first part of this chapter which is centred around character restrictions is based on the paper [30], joint with Dr Eugenio Giannelli and Dr Stuart Martin. The results in Section 3.1 were obtained in collaboration, with Dr Giannelli and I contributing equally to the proofs and Dr Martin providing guidance throughout. In the latter parts of this chapter, we extend our investigations to character inductions and further properties of the Young graph.

As described in the introduction, a key ingredient in the character bijection in [28] between $\mathrm{Irr}_{2'}(\mathfrak{S}_n)$ and $\mathrm{Irr}_{2'}(P_n)$ where $P_n \in \mathrm{Syl}_2(\mathfrak{S}_n)$ is Theorem 1 of [1], which states the following: for any natural number n and irreducible character χ of $\mathfrak{S}_n$ of odd degree, the restriction $\chi\big\downarrow_{\mathfrak{S}_{n-1}}$ contains a unique irreducible constituent of odd degree. In their same paper [28], Giannelli, Kleshchev, Navarro and Tiep give a generalisation of this result by changing the ambient group from the symmetric groups to general linear groups and special linear groups. Isaacs, Navarro, Olsson and Tiep extend [1, Theorem 1] for symmetric groups in a different direction, changing the depth of restriction: they show for any natural numbers $2^k \leq n$ and any $\chi \in \mathrm{Irr}_{2'}(\mathfrak{S}_n)$ that the restriction $\chi\big\downarrow_{\mathfrak{S}_{n-2^k}}$ contains a unique irreducible constituent of odd degree appearing with odd multiplicity [38]. We now generalise the third main ingredient of [1, Theorem 1]: the prime p itself.

3.1 Restriction

Let p be a prime number. We study the restriction to $\mathfrak{S}_{n-1}$ of irreducible characters of $\mathfrak{S}_n$ of degree coprime to p. In particular, we study the combinatorial properties of the subgraph $\mathbb{Y}_{p'}$ of the Young graph $\mathbb{Y}$. This is an extension to odd primes of the work done in [1] for $p = 2$.

The Young graph, as described in Section 2.2, is a well-understood and extensively studied combinatorial object, deeply connected to the representation theory of symmetric groups. It is thus somewhat surprising that only recently in [1], the following remarkable fact was shown to hold.

Theorem 3.1 ([1, Theorem 1])**.** *Let $n \in \mathbb{N}$ and let $\chi \in \mathrm{Irr}_{2'}(\mathfrak{S}_n)$. Then the restriction $\chi\big\downarrow_{\mathfrak{S}_{n-1}}$ has a unique irreducible constituent of odd degree.*

Theorem 3.1 shows that the *odd subgraph* $\mathbb{Y}_{2'}$ of the Young graph $\mathbb{Y}$ is a rooted tree. Starting from this observation, the rest of [1] is devoted to describing the combinatorial structure of $\mathbb{Y}_{2'}$. We remark that the relevance of [1] transcends the study of the Young graph: in fact, Theorem 3.1 was recently used in the construction of several types of character correspondences (see [26], [28] and [38]).

In this chapter, we study the combinatorial structure of $\mathbb{Y}_{p'}$ for any odd prime p. As remarked in [1, Section 7], $\mathbb{Y}_{3'}$ is not a tree. Indeed, for every odd prime p, there exists an irreducible character χ of p'-degree of some $\mathfrak{S}_n$ whose restriction $\chi\big\downarrow_{\mathfrak{S}_{n-1}}$ has more than one irreducible constituent of p'-degree (namely $\chi^{(2,1)}\big\downarrow_{\mathfrak{S}_2} = \chi^{(2)} + \chi^{(1^2)}$, to give the smallest example). Yet notably, given *any* prime p and any irreducible character χ of p'-degree of $\mathfrak{S}_n$, Theorems 3.2 and 3.3 below give sharp bounds on the number of irreducible constituents of p'-degree of $\chi\big\downarrow_{\mathfrak{S}_{n-1}}$. In particular, this is a generalisation of Theorem 3.1 to all primes.

Let p be any prime. Given a partition $\lambda \vdash n$, recall from Section 2.2 that $\lambda^-_{p'}$ denotes the set consisting of all partitions $\mu \vdash_{p'} n-1$ such that χ^μ is an irreducible constituent of $\chi^\lambda\big\downarrow_{\mathfrak{S}_{n-1}}$. Next, we define $\mathcal{E}_p(n)$ to be the set

$$\mathcal{E}_p(n) = \left\{ |\lambda^-_{p'}| \ : \ \lambda \vdash_{p'} n \right\},$$

and we let $br_p(n)$ be the maximal value in $\mathcal{E}_p(n)$. Note that $br_p(n)$ is well-defined: clearly $\mathcal{E}_p(n)$ is non-empty since the trivial character of $\mathfrak{S}_n$ has degree 1. When p is fixed and understood, we will also write $\mathcal{E}(n)$ and $br(n)$, without the subscript p. Our first result describes $\mathcal{E}_p(n)$ and gives a recursive formula for the exact value of $br_p(n)$.

Theorem 3.2. *Let $n \in \mathbb{N}$ and let p be a prime. Let $n = \sum_{j=1}^t a_j p^{n_j}$ be its p-adic expansion, where $0 \leq n_1 < n_2 < \cdots < n_t$. Then $\mathcal{E}_p(n) = \{1, 2, \ldots, br_p(n)-1, br_p(n)\}$ and*

$$br_p(n) = br_p(a_1 p^{n_1}) + \sum_{j=2}^{t} \Phi(a_j, br_p(m_j))$$

where $m_j = \sum_{i=1}^{j-1} a_i p^{n_i}$, and where Φ is the function described explicitly in Definition 3.6 below.

Theorem 3.2 is proven in Section 3.1.2. In Section 3.1.4 we determine $br_p(ap^k)$ for any prime p, any $k \in \mathbb{N}_0$ and any $a \in \{1, \ldots, p-1\}$. The following result serves as the base case for computing $br_p(n)$ for any natural number n, using the recursive expression given in Theorem 3.2.

Theorem 3.3. *Let p be an odd prime, $k \in \mathbb{N}_0$ and $a \in \{1, \ldots, p-1\}$. Then*

$$br_p(ap^k) = \begin{cases} f(2a) & \text{if } k = 0, \\ p - 1 + 2\lfloor \frac{2a-(p-1)}{6} \rfloor & \text{if } k = 1 \text{ and } \frac{p}{2} < a < p, \\ 2a & \text{otherwise.} \end{cases}$$

Here $f(x) = \max\{y \in \mathbb{N}_0 \mid y(y+1) \leq x\}$.

Theorems 3.2 and 3.3 provide us with a recursive formula for $br_p(n)$, the maximal number of downward edges from a vertex on level n of $\mathbb{Y}_{p'}$ to level $n-1$. Later in this chapter we show that the slightly involved expression for the value of $br_p(n)$ described in Theorem 3.2 can be bounded from above by a simpler function of the p-adic digits of n.

Corollary 3.4. *Let $n \in \mathbb{N}$ and let p be a prime. Let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be its p-adic expansion, where $0 \leq n_1 < n_2 < \cdots < n_t$. Then $1 \leq br_p(n) \leq \mathbb{B}_p(n)$, where*

$$\mathbb{B}_p(n) := br_p(a_1 p^{n_1}) + \sum_{j=2}^{t} \left\lfloor \frac{a_j}{2} \right\rfloor \leq 2a_1 + \sum_{j=2}^{t} \left\lfloor \frac{a_j}{2} \right\rfloor.$$

Theorem 3.3 and Corollary 3.4 are proven in Section 3.1.4. Corollary 3.4 has some interesting consequences (see Section 3.1.3). For instance, in Remark 3.29 below, we observe that when $p \in \{2, 3\}$ then $\mathbb{B}_p(n) = br_p(n)$. In particular, our result is a generalisation of Theorem 3.1. Moreover, for any prime p we observe that the upper bound $\mathbb{B}_p(n)$ is attained for every n having all of its p-adic digits lying in $\{0, 1, 2, 3\}$.

We further show that the upper bound $\mathbb{B}_p(n)$ given in Corollary 3.4 is indeed a good approximation of $br_p(n)$. In fact, the following result shows that the difference $\varepsilon_p(n) := \mathbb{B}_p(n) - br_p(n)$ can be bounded by a function depending only on the prime p, and not on $n \in \mathbb{N}$.

Proposition 3.5. *For any $n \in \mathbb{N}$, we have $\varepsilon_p(n) < \frac{p}{2}\log_2(p)$.*

Proposition 3.5 is proven in Section 3.1.3. A consequence is that for any odd prime p we have $\sup\{br_p(n) \mid n \in \mathbb{N}\} = \infty$. This is false when $p = 2$, since by Theorem 3.1 we have that $br_2(n) = 1$ for all $n \in \mathbb{N}$.

3.1.1 James' abacus

We fix some notation that will be used throughout this chapter. We begin by introducing a technical definition necessary for stating and proving Theorem 3.2.

Definition 3.6. *For $a \in \mathbb{N}_0$ and $L \in \mathbb{N}$, define*

$$\Phi(a, L) := \max\left\{ \sum_{i=1}^{L} f(a_i) \;\middle|\; a_1 + \cdots + a_L \leq a \text{ and } a_i \in \mathbb{N}_0 \;\; \forall\, i \in [L] \right\},$$

where $f(x) = \max\{y \in \mathbb{N}_0 \mid y(y+1) \leq x\}$.

We now record some properties of this function Φ which will be useful for later proofs.

Lemma 3.7. *Let $a \in \mathbb{N}_0$ and $L \in \mathbb{N}$. Then $\Phi(a, L) \leq \lfloor \frac{a}{2} \rfloor$. Moreover, if $L \geq \lfloor \frac{a}{2} \rfloor$ then $\Phi(a, L) = \lfloor \frac{a}{2} \rfloor$.*

Proof. Suppose $\Phi(a, L) = f(a_1) + \cdots + f(a_L)$ such that $a_i \in \mathbb{N}_0$ and $a_1 + \cdots + a_L \leq a$. Observe that for all integers $x \geq 2$, we have $f(x) \leq f(2) + f(x-2)$. Hence

$$f(a_i) \leq \left\lfloor \frac{a_i}{2} \right\rfloor \cdot f(2) + f(\delta_i)$$

for all $i \in [L]$, where $\delta_i = a_i - 2\lfloor \frac{a_i}{2} \rfloor \in \{0, 1\}$. Thus

$$\Phi(a, L) \leq \sum_{i=1}^{L} \Big(\left\lfloor \frac{a_i}{2} \right\rfloor \cdot f(2) + f(\delta_i) \Big) = \sum_{i=1}^{L} \left\lfloor \frac{a_i}{2} \right\rfloor \leq \left\lfloor \frac{a}{2} \right\rfloor,$$

where the middle equality follows from the fact that $f(2) = 1$ and $f(1) = f(0) = 0$.

Finally, if $L \geq \lfloor \frac{a}{2} \rfloor$ then we see that $\Phi(a, L) = \lfloor \frac{a}{2} \rfloor$ by considering

$$a_1 = a_2 = \cdots = a_{\lfloor \frac{a}{2} \rfloor} = 2 \text{ and } a_{\lfloor \frac{a}{2} \rfloor + 1} = \cdots = a_L = 0,$$

which satisfy $\sum_{i=1}^{L} a_i = 2 \cdot \lfloor \frac{a}{2} \rfloor \leq a$ and $\sum_{i=1}^{L} f(a_i) = \lfloor \frac{a}{2} \rfloor$. □

Lemma 3.8. *Let $k \in \mathbb{N}$. Then $2^{k-1} \leq \Phi(2^k + 2, 2^{k-1}) \leq 2^{k-1} + 1$.*

Proof. When $k = 1$, we note that $\Phi(4, 1) = 1$. Now assume $k \geq 2$. The upper bound follows from Lemma 3.7. The lower bound follows from the fact that $2^k + 2 = 6 + 2 \cdot (2^{k-1} - 2) + 0$, and $f(6) + f(2) \cdot (2^{k-1} - 2) + f(0) = 2^{k-1}$. □

Let λ be a partition. For any natural number e, we denote by $C_e(\lambda)$ and $Q_e(\lambda) = (\lambda^0, \lambda^1, \ldots, \lambda^{e-1})$ the *e-core* and *e-quotient* of λ respectively (see [54, Chapter I] for precise definitions). The *e-weight* of λ is the natural number $w_e(\lambda)$ defined by $w_e(\lambda) = |\lambda^0| + |\lambda^1| + \cdots + |\lambda^{e-1}|$. We remark that given a partition λ of n, the e-quotient $Q_e(\lambda)$ is uniquely determined up to a cyclic permutation of its components. Moreover, it is well-known that, up to such cyclic permutations, any partition is uniquely determined by its e-core and e-quotient; we refer the reader to [54] for a detailed discussion.

Recall that $\mathcal{H}(\lambda)$ denotes the set of hooks of λ and $\mathcal{H}_e(\lambda)$ the subset of $\mathcal{H}(\lambda)$ consisting of those hooks of λ having length divisible by e. We let $\mathcal{H}(Q_e(\lambda)) = \cup_{i=0}^{e-1} \mathcal{H}(\lambda^i)$. As explained in [54, Theorem 3.3], there is a bijection between $\mathcal{H}_e(\lambda)$ and $\mathcal{H}(Q_e(\lambda))$ mapping hooks in λ of length ex to hooks of length x in the quotient of λ. Moreover, the bijection respects the process of hook removal. Namely, any partition μ obtained by removing an ex-hook from λ is such that $C_e(\mu) = C_e(\lambda)$ and the e-quotient of μ is obtained by removing a x-hook from one of the e partitions involved in the e-quotient of λ. A fundamental result is the following.

Proposition 3.9 ([54, Proposition 3.6]). *Let $\lambda \in \mathcal{P}(n)$. The number of e-hooks that must be removed from λ to obtain $C_e(\lambda)$ is $w_e(\lambda)$. Moreover, $w_e(\lambda) = |\mathcal{H}_e(\lambda)| = (|\lambda| - |C_e(\lambda)|)/e$.*

All of the operations on partitions concerning addition and removal of e-hooks described above are best illustrated on James' abacus. We give here a brief description of this important object (in particular, fixing our convention for the orientation and labelling of the abaci that we will use), and refer the reader to [41, Chapter 2] for a complete account of the combinatorial properties of James' abacus.

An e-abacus configuration A consists of e vertical runners, labelled $A_0, A_1, \ldots, A_{e-1}$ from left to right, and the rows are labelled by integers such that row numbers increase downwards. Each position (i, j), i.e. the position in row i on runner A_j, in the abacus configuration either contains a bead or not; we also call an empty position a *gap*. As is customary, all abaci contain finitely many rows and hence finitely many beads, but in all instances enough to perform all of the necessary operations. We say that position (i, j) is the *first* gap in A if there are beads in positions (x, y) for all $x < i$ and all y, and in positions (i, y) for all $y < j$.

The partition λ corresponding to an abacus configuration A is given as follows: if a bead b on the abacus lies in position (i, j), let λ_b be the number of gaps (x, y) such that either $x < i$, or $x = i$ and $y < j$. Then $\{\lambda_b \mid b \text{ is a bead on } A\}$ gives the multiset of parts of the partition λ, from which we remove zeros and sort its elements into non-increasing order to produce λ. We also sometimes simply say that A is an e-abacus for λ, or that A represents the partition λ.

For $j \in \{0, \ldots, e-1\}$, denote by $|A_j|$ the number of beads on runner j. Moreover, we denote by $A^{\uparrow}$ the e-abacus obtained from A by sliding all of the beads on each runner upwards as much as possible. Extending the notation just introduced, we denote by $A^{\uparrow}_0, \ldots, A^{\uparrow}_{e-1}$ the runners of $A^{\uparrow}$. As explained in [41, Chapter 2], $A^{\uparrow}$ is an e-abacus for the e-core $C_e(\lambda)$ of λ, and (up to a cyclic permutation of the runners) the individual runners $A_0, \ldots A_{e-1}$ are 1-abacus configurations for the partitions $\lambda^0, \ldots, \lambda^{e-1}$ in the e-quotient $Q_e(\lambda)$ of λ.

Let the operation of sliding any single bead down (resp. up) one row on its runner be called a *down-move* (resp. *up-move*). Of course, such a move is only possible for a bead in position (i, j) if the respective position $(i \pm 1, j)$ was empty initially. On the level of partitions, performing a down- or up-move corresponds to adding or removing an e-hook, respectively. In analogy with the notation used for partitions, we denote by $w(A)$ the total number of up-moves needed to obtain $A^{\uparrow}$ from A. Similarly, for $i \in \{0, \ldots, e-1\}$ we let $w(A_i)$ be the number of those up-moves that were performed on runner i in the transition from A to $A^{\uparrow}$. It is easy to see that $w_e(\lambda) = w(A) = w(A_0) + \cdots + w(A_{e-1})$.

Suppose that c is a bead in position (i, j) of A. We say that c is a *removable bead* if $j \neq 0$ and there is no bead in $(i, j-1)$, or if $j = 0$ and there is no bead in $(i-1, e-1)$. Denote by $A^{\leftarrow c}$ the abacus obtained by sliding c into position $(i, j-1)$ (respectively $(i-1, e-1)$). Clearly removable beads in an abacus A for λ correspond to removable nodes in $[\lambda]$, so the set of such $A^{\leftarrow c}$ is in natural bijection with λ^-. *Addable beads* are defined analogously and correspond to elements of λ^+.

Finally, for $j \in \{0, \ldots, e-1\}$ we denote by $\mathrm{Rem}(A_j)$ the number of removable beads in A lying on runner A_j. In particular, we have that $|\lambda^-| = \mathrm{Rem}(A_0)+\cdots+\mathrm{Rem}(A_{e-1})$. Similarly, we let $\mathrm{Add}(A_j)$ denote the number of addable beads in A lying on A_j.

When we depict partitions on James' abacus, we adopt the convention of denoting beads on the abacus by X, and empty positions by O (or no symbol at all when the meaning is clear).

Lemma 3.10. *Let $e \in \mathbb{N}$. Let λ be a partition and let A be an e-abacus for λ. Suppose c is a removable bead on runner A_j and let $\mu \vdash n-1$ be the partition represented by $A^{\leftarrow c}$. Then*

$$w_e(\mu) - w_e(\lambda) = \begin{cases} |A_j| - |A_{j-1}| - 1 & \text{if } j \neq 0 \\ |A_0| - |A_{e-1}| - 2 & \text{if } j = 0. \end{cases}$$

Proof. First suppose $j \neq 0$. Without loss of generality we can relabel the rows of the e-abacus A such that there is no empty position in any row labelled by a negative integer. Let $B := A^{\leftarrow c}$. Clearly $w(A_i) = w(B_i)$ for all $i \in \{0, \ldots, e-1\} \setminus \{j-1, j\}$. Hence

$$w_e(\mu) - w_e(\lambda) = w(B_{j-1}) + w(B_j) - w(A_{j-1}) - w(A_j).$$

Let s and t be the numbers of beads lying in rows labelled by non-negative integers in runners A_{j-1} and A_j respectively. Suppose that the s beads on A_{j-1} lie in rows $0 \leq x_1 < \cdots < x_s$ and that the t beads on A_j lie in rows $0 \leq y_1 < \cdots < y_t$. Then

$$w(A_{j-1})+w(A_j) = \sum_{i=1}^{s}(x_i-(i-1))+\sum_{i=1}^{t}(y_i-(i-1)) = \sum_{i=1}^{s} x_i+\sum_{i=1}^{t} y_i - \tfrac{s(s-1)}{2} - \tfrac{t(t-1)}{2}.$$

Suppose that the bead c lies in row y_l for some $l \in [t]$. Since c is removable, $y_l \neq x_i$ for all $i \in [s]$. Thus the beads on B_{j-1} lie in rows $0 \leq x'_1 < \cdots < x'_{s+1}$ with $\{x'_1, \ldots, x'_{s+1}\} = \{x_1, \ldots, x_s, y_l\}$ and the beads on B_j lie in rows $0 \leq y'_1 < \cdots < y'_{t-1}$ with $\{y'_1, \ldots, y'_{t-1}\} = \{y_1, \ldots, y_{l-1}, y_{l+1}, \ldots, y_t\}$. Hence

$$w(B_{j-1})+w(B_j) = \sum_{i=1}^{s+1}(x'_i-(i-1))+\sum_{i=1}^{t-1}(y'_1-(i-1)) = \sum_{i=1}^{s} x_i+\sum_{i=1}^{t} y_i- \tfrac{s(s+1)}{2} - \tfrac{(t-1)(t-2)}{2}$$

and we conclude that $w_e(\mu) - w_e(\lambda) = t - s - 1 = |A_j| - |A_{j-1}| - 1$.

The case when $j = 0$ is similar. □

Remark 3.11. Given a partition λ and a fixed e-abacus A for λ we let λ^i be the partition corresponding to the runner A_i, considered as a 1-abacus. The resulting e-quotient $(\lambda^0, \lambda^1, \ldots, \lambda^{e-1})$ depends on the choice of the abacus A (a different choice of e-abacus, e.g. having first gap in a different position, may induce a cyclic permutation of the components of the e-quotient). Nevertheless, all of the results presented in this chapter hold independently of this observation. For instance, the e-weight $w_e(\lambda)$ does not depend on the choice of e-abacus; the same discussion holds for Theorem 3.13 below. $\diamondsuit$

Let p be a prime. As outlined in Section 2.2, the irreducible characters of $\mathfrak{S}_n$ of p'-degree were characterised in [44]. We restate this result in language convenient for our purposes.

Theorem 3.12. *Let $n \in \mathbb{N}$ and let $\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$. Let $a \in \{1, \dots, p-1\}$ and $k \in \mathbb{N}_0$ be such that $ap^k \leq n < (a+1)p^k$. Then $\lambda \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$ if and only if $C_{p^k}(\lambda) \in \mathrm{Irr}_{p'}(\mathfrak{S}_{n-ap^k})$.*

Theorem 3.12 says that λ is a p'-partition if and only if $w_{p^k}(\lambda) = a$ and the partition $C_{p^k}(\lambda)$ obtained from λ by successively removing all possible p^k-hooks is a p'-partition of $n - ap^k$. It will sometimes be useful to use the following equivalent version of Theorem 3.12.

Theorem 3.13. *Let $n \in \mathbb{N}$ and let $n = \sum_{j=0}^{k} a_j p^j$ be its p-adic expansion. Let $\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ and let $Q_p(\lambda) = (\lambda^0, \lambda^1, \dots, \lambda^{p-1})$. Then $\lambda \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$ if and only if*

(i) $C_p(\lambda) \vdash a_0$, and

(ii) for all $t \in \{0, 1, \dots, p-1\}$ there exists $b_{1t}, b_{2t}, \dots, b_{kt} \in \mathbb{N}_0$ such that

$$\sum_{t=0}^{p-1} b_{jt} = a_j \text{ for all } j \in \{1, \dots k\}, \text{ and } \lambda_t \vdash_{p'} \sum_{j=1}^{k} b_{jt} p^{j-1}.$$

Proof. This characterisation of p'-partitions of $n \in \mathbb{N}$ follows from considering the p-core tower associated to any partition of n (see Section 2.2 and [54, Chapters I and II]). $\square$

3.1.2 The core map

Fix an arbitrary prime p. In this section we state some combinatorial results crucial to the proofs of the main theorems of this chapter. As a consequence of these observations, we are able to give a proof of Theorem 3.2. As appropriately remarked later in this section, the proofs of Theorem 3.3 and Corollary 3.4 are postponed to Section 3.1.4 to improve readability.

Notation 3.14. *Unless otherwise stated, in this section we fix $n \in \mathbb{N}$ such that $n = ap^k + m$ for some $k \in \mathbb{N}$, $a \in [p-1]$ and $0 < m < p^k$. To be precise, this will be the standing assumption from Theorem 3.15 to Proposition 3.23.*

The following result, which we believe is of independent interest, is one of the key steps in proving Theorem 3.2.

Theorem 3.15. *Let $\lambda \vdash_{p'} n$ and let $\alpha \in \lambda^-_{p'}$. Then $C_{p^k}(\alpha) \in \mu^-_{p'}$, where $\mu := C_{p^k}(\lambda)$. In particular, we deduce that the map*

$$C_{p^k} : \lambda^-_{p'} \longrightarrow \mu^-_{p'},$$

is well-defined. Moreover, it is surjective.

Proof. Let A be the p^k-abacus for μ having first gap in position $(0,0)$. It is easy to see that rows $i \geq 1$ must be empty, since $|\mu| = m < p^k$. (We will not need rows i with $|i| > a$, so we may assume row $-a$ is the top row of the abacus and $+a$ the bottom row.) So $|A_0| = a$ and $|A_j| \in \{a, a+1\}$, for all $j \in \{0, 1, \ldots, p^k - 1\}$. Let B be the p^k-abacus for λ such that $B^\uparrow = A$. By Proposition 3.9, we have that $w_{p^k}(\lambda) = a$ and B is obtained from A after performing exactly a down-moves.

Let c be the bead in B such that $B^{\leftarrow c}$ represents α, and suppose c lies on runner B_j. Since α is a p'-partition of $n - 1 = ap^k + (m-1) \geq ap^k$ we deduce from Theorem 3.12 that $w_{p^k}(\alpha) = a$. By Lemma 3.10 we have that $|B_j| = 1 + |B_{j-1}|$ (j cannot be 0 because $|B_l| = |A_l| \in \{a, a+1\}$ for all $l \in \{0, \ldots, p^k - 1\}$). It follows that there exists a bead d in position $(0, j)$ of A and that position $(0, j-1)$ of A is empty. Hence $A^{\leftarrow d}$ is a p^k-abacus for $C_{p^k}(\alpha)$, which by Theorem 3.12 must be a p'-partition. Thus $C_{p^k}(\alpha) \in \mu^-_{p'}$ and the map $C_{p^k} : \lambda^-_{p'} \longrightarrow \mu^-_{p'}$ is well-defined.

To show that the map is surjective we proceed as follows. Let A be the p^k-abacus for μ as described above. For any $\beta \in \mu^-_{p'}$ there exists a bead d in A such that $A^{\leftarrow d}$ is a p^k-abacus for β. Let $j \in \{1, \ldots, p^k - 1\}$ be such that d is in position $(0, j)$ in A and such that position $(0, j-1)$ is empty. Let B be the p^k-abacus for λ described above. Clearly we have that $|B_j| = |A_j| = 1 + |A_{j-1}| = 1 + |B_{j-1}|$. Hence there exists a row $y \in \{-a, \ldots, a\}$ such that position $(y, j-1)$ of B is empty and such that there is a bead (say e) in position (y, j). Let α be the partition corresponding to the p^k-abacus $B^{\leftarrow e}$. By Lemma 3.10 we deduce that $w_{p^k}(\alpha) = a$. Moreover it is clear that $C_{p^k}(\alpha) = \beta \in \mathrm{Irr}_{p'}(\mathfrak{S}_{n-ap^k})$. By Theorem 3.12 we deduce that $\alpha \in \lambda^-_{p'}$ and therefore C_{p^k} is surjective. □

Corollary 3.16. *Let $\lambda \vdash_{p'} n$. Then $|C_{p^k}(\lambda)^-_{p'}| \leq |\lambda^-_{p'}|$.*

Keeping $n = ap^k + m$ as in Notation 3.14, we now introduce the following definition. Given $\gamma \vdash_{p'} m$, define

$$br_p(n, \gamma) := \max\{|\lambda^-_{p'}| \ \mid \ \lambda \vdash_{p'} n \ \text{ and } \ C_{p^k}(\lambda) = \gamma\}.$$

(As usual, we omit the subscript p when it is understood.) Clearly $br(n)$, the main object of our study, is equal to the maximal $br(n, \gamma)$ over all p'-partitions γ of m. Corollary 3.16 allows us to give the following definition.

Definition 3.17. *Let $n = ap^k + m$ be as in Notation 3.14, and let $\gamma \vdash_{p'} m$. We define $N(a, p^k, \gamma) \in \mathbb{N}_0$ to be such that $|\gamma^-_{p'}| + N(a, p^k, \gamma) = br(n, \gamma)$.*

Proposition 3.18. *Let $\gamma \vdash_{p'} m$ and let $L = |\gamma^-_{p'}|$. Then $N(a, p^k, \gamma) = \Phi(a, L)$, where Φ is as described in Definition 3.6.*

In order to prove Proposition 3.18, we introduce the following combinatorial concepts.

Definition 3.19. *Let $n = ap^k + m$ be as in Notation 3.14, and let $\gamma \vdash_{p'} m$. Denote by A_γ the p^k-abacus for γ having first gap in position $(0,0)$. Define $\mathcal{R}_{A_\gamma}$ to be the subset of*

$\{0, 1, \ldots, p^k - 1\}$ *such that* $j \in \mathcal{R}_{A_\gamma}$ *if and only if there is a removable bead* c *on runner* j *of* A_γ *and the partition corresponding to the* p^k*-abacus* $A_\gamma^{\leftarrow c}$ *is a* p'*-partition of* $m-1$.

Since A_γ has first gap in position $(0,0)$ and since $|\gamma| = m < p^k$ we deduce that all removable beads in A_γ lie in row 0. Hence $|\mathcal{R}_{A_\gamma}| = |\gamma_{p'}^-|$. By the definition of removable beads, we have in particular that $0 \notin \mathcal{R}_{A_\gamma}$, and for $j \in [p^k - 2]$ we have that if $j \in \mathcal{R}_{A_\gamma}$ then $j + 1 \notin \mathcal{R}_{A_\gamma}$.

Lemma 3.20. *Let* $\gamma \vdash_{p'} m$*. Let* $\lambda \vdash_{p'} n$ *be such that* $C_{p^k}(\lambda) = \gamma$ *and let* B *be the* p^k*-abacus for* λ *such that* $B^\uparrow = A_\gamma$*. Let* c *be a removable bead on runner* j *of* B *and let* μ *be the partition of* $n-1$ *corresponding to* $B^{\leftarrow c}$*. Then* μ *is a* p'*-partition if and only if* $j \in \mathcal{R}_{A_\gamma}$.

Proof. Let $A := A_\gamma$. First suppose $j \in \mathcal{R}_A$. In particular, $j \neq 0$. Then

$$|B_j| = |A_j| = |A_{j-1}| + 1 = |B_{j-1}| + 1,$$

so $w_{p^k}(\mu) = a$ by Lemma 3.10. We also have that $(B^{\leftarrow c})^\uparrow$ is an abacus configuration for $C_{p^k}(\mu)$. Moreover if d is the bead in position $(0, j)$ of A then $(B^{\leftarrow c})^\uparrow = A^{\leftarrow d}$. Therefore we deduce that $C_{p^k}(\mu) \in \gamma_{p'}^-$ and hence $\mu \vdash_{p'} n-1$ by Theorem 3.12.

Now suppose that $j \notin \mathcal{R}_A$. If $j = 0$ then $|B_0| = |A_0| \neq |A_{p^k-1}| + 2 = |B_{p^k-1}| + 2$. Hence $w_{p^k}(\mu) \neq a$ by Lemma 3.10, so μ is not a p'-partition by Theorem 3.12. Otherwise, suppose that $j \neq 0$. Then $C_{p^k}(\mu)$ is represented by the p^k-abacus $(B^{\leftarrow c})^\uparrow = A^{\leftarrow d}$, where d is a bead lying in position $(0, j)$ of A. Since $j \notin \mathcal{R}_A$ we deduce that $C_{p^k}(\mu)$ is not a p'-partition, and so μ is not a p'-partition by Theorem 3.12. □

Corollary 3.21. *Let* $\gamma \vdash_{p'} m$ *and let* $\lambda \vdash_{p'} n$ *be such that* $C_{p^k}(\lambda) = \gamma$*. Let* B *be the* p^k*-abacus for* λ *such that* $B^\uparrow = A_\gamma$*. Then*

$$|\lambda_{p'}^-| = \sum_{j \in \mathcal{R}_{A_\gamma}} \mathrm{Rem}(B_j).$$

Recall from Definition 3.6 that $f(x) = \max\{y \in \mathbb{N}_0 \mid y(y+1) \leq x\}$. The following lemma describes the key relationship between this function f and certain removable beads, which will be necessary for the proof of Proposition 3.18 (below).

Lemma 3.22. *Let* $\lambda \in \{\emptyset, (1)\}$ *and let* T_λ *denote the* 2*-abacus for* λ *having first gap in position* $(0,0)$*. Let* $x \in \mathbb{N}_0$ *and let* $\mathcal{T}_\lambda(x)$ *be the set of all* 2*-abaci* U *such that* $w(U) = x$ *and* $U^\uparrow = T_\lambda$*. Then*

$$\max\{\mathrm{Rem}(U_1) \mid U \in \mathcal{T}_\lambda(x)\} = \begin{cases} f(x) + 1 & \text{if } \lambda = (1), \\ \lfloor \sqrt{x} \rfloor & \text{if } \lambda = \emptyset. \end{cases}$$

Proof. This is clear if $x = 0$ or $x = 1$, so we may assume now that $x \geq 2$ (and hence $f(x) > 0$). We first fix $\lambda = (1)$; this is the case that we use in the proof of Proposition 3.18 below. Since λ is now fixed, we ease the notation by letting $T = T_{(1)}$

and $\mathcal{T}(x) = \mathcal{T}_{(1)}(x)$ for all $x \in \mathbb{N}_0$. Moreover, let $F(x) := \max\{\mathrm{Rem}(U_1) \mid U \in \mathcal{T}(x)\}$. We first show that there exists $A \in \mathcal{T}(x)$ satisfying $w(A_0) = 0$ (equivalently $w(A_1) = x$) and $\mathrm{Rem}(A_1) = F(x)$.

Let $U \in \mathcal{T}(x)$ be such that $w(U_0) = \ell$ and $\mathrm{Rem}(U_1) = r$ for some $l \in \{1, 2, \ldots, x\}$ and some $r \in \{0, 1, \ldots, F(x)\}$. Then there exists a 2-abacus $V \in \mathcal{T}(y)$ for some $y \leq x$ such that $w(V_0) < \ell$ and $\mathrm{Rem}(V_1) \geq r$. This follows from the following observation. Since $\ell \geq 1$ there exists $i \in \mathbb{Z}$ such that there is a bead in position $(i, 0)$ of U but not in $(i-1, 0)$. Recalling that beads are denoted by X and gaps by O, consider the four possibilities for rows $i-1$ and i of U (with the left- and right-hand runners labelled by 0,1 respectively):

$i-1$	OO	OO	OX	OX
i	XX	XO	XX	XO

In the first three instances, we can move the bead in $(i, 0)$ to $(i-1, 0)$ to obtain the desired V. In the fourth (i.e. rightmost) case, we need to additionally move the bead in $(i-1, 1)$ to $(i, 1)$. Hence, if $B \in \mathcal{T}(x)$ satisfies $\mathrm{Rem}(B_1) = F(x)$ then there exists $y \leq x$ and $A' \in \mathcal{T}(y)$ such that $\mathrm{Rem}(A'_1) = F(x)$, $w(A'_0) = 0$ and $w(A'_1) = y$. Let $(i, 1)$ be the lowest position occupied by a bead (say d) in A'. Moving d to position $(i + (x - y), 1)$ we obtain a 2-abacus configuration $A \in \mathcal{T}(x)$ such that $\mathrm{Rem}(A_1) = \mathrm{Rem}(A'_1) = F(x)$, $w(A_0) = 0$ and $w(A_1) = x$, as desired.

It remains to show that $F(x) = f(x) + 1$. First suppose for a contradiction that $F(x) \geq f(x) + 2$, and let $A \in \mathcal{T}(x)$ be such that $\mathrm{Rem}(A_1) = F(x)$ and $w(A_0) = 0$. By construction there exist integers $0 \leq j_1 < j_2 < \cdots < j_{f(x)+2}$ such that there is a bead in position $(j_k, 1)$ of A for all $k \in [f(x) + 2]$. This implies that $w(A) = w(A_1) \geq (f(x) + 1)(f(x) + 2) > x$, a contradiction. Hence $F(x) \leq f(x) + 1$.

Now let $y := f(x) \cdot (f(x) + 1) \leq x$. Let B be the 2-abacus obtained from T by first sliding down the bead in position $(0, 1)$ to position $(f(x) + x - y, 1)$ and then sliding down the bead in position $(i, 1)$ to position $(i + f(x), 1)$ for each $i \in \{-1, -2, \ldots, -f(x)\}$. Clearly $B \in \mathcal{T}(x)$ and $\mathrm{Rem}(B_1) = f(x) + 1$. Thus $F(x) = f(x) + 1$, as desired.

The case $\lambda = \emptyset$ is similar. □

Proof of Proposition 3.18. Let $\lambda \vdash_{p'} n$ be such that $C_{p^k}(\lambda) = \gamma$ and $|\lambda^-_{p'}| = br(n, \gamma)$. Let B be the p^k-abacus for λ such that $B^\uparrow = A_\gamma$. In particular, B is obtained from A_γ by performing a down-moves. Let $\mathcal{R}_{A_\gamma} = \{j_1, \ldots, j_L\}$. Then by Corollary 3.21, we have

$$L + N(a, p^k, \gamma) = br(n, \gamma) = |\lambda^-_{p'}| = \sum_{i=1}^{L} \mathrm{Rem}(B_{j_i}).$$

Let $a_i = w(B_{j_i - 1}) + w(B_{j_i})$ for $i \in [L]$, so $a_1 + \cdots + a_L \leq a$. Since no two numbers in $\mathcal{R}_{A_\gamma}$ are consecutive (as remarked after Definition 3.19), we can regard the pairs of runners of (B_{j_1-1}, B_{j_1}), $(B_{j_2-1}, B_{j_2}), \ldots, (B_{j_L-1}, B_{j_L})$ as L disjoint 2-abaci, whose 2-cores are all equal to the 2-abacus $T_{(1)}$ considered in Lemma 3.22. It is easy to see that the 2-abacus identified with the pair (B_{j_i-1}, B_{j_i}) lies in $\mathcal{T}_{(1)}(a_i)$ for all $i \in [L]$. Lemma 3.22, together with the maximality of $|\lambda^-_{p'}|$ among all the p'-partitions of n with

p^k-core equal to γ, allows us to deduce that $\mathrm{Rem}(B_{j_i}) = f(a_i)+1$, for all $i \in [L]$. Hence we obtain

$$N(a, p^k, \gamma) = \sum_{i=1}^{L} \mathrm{Rem}(B_{j_i}) - L = \sum_{i=1}^{L} f(a_i).$$

We conclude the proof by showing that

$$N(a, p^k, \gamma) = \max\left\{ \sum_{i=1}^{L} f(a_i') \mid a_1' + \cdots + a_L' \leq a,\ a_i' \in \mathbb{N}_0\ \forall\ i \right\} = \Phi(a, L).$$

Suppose for a contradiction that there exists a natural number $y \leq a$ and $(a_1', \dots, a_L')$ a composition of y such that $\sum_{i=1}^{L} f(a_i') > N(a, p^k, \gamma)$. Since f is a non-decreasing function, without loss of generality we can assume that $y = a$. Then by using constructions analogous to those in the proof of Lemma 3.22, we can construct a partition $\tilde{\lambda} \vdash_{p'} n$ with $C_{p^k}(\tilde{\lambda}) = \gamma$, $w_{p^k}(\tilde{\lambda}) = a$ and p^k-abacus $\tilde{B}$ satisfying $\tilde{B}^{\uparrow} = A_\gamma$ such that $w(\tilde{B}_{j_i}) = a_i'$ and $\mathrm{Rem}(\tilde{B}_{j_i}) = f(a_i') + 1$ for all $i \in [L]$. This implies that

$$br(n, \gamma) \geq |\tilde{\lambda}_{p'}^-| = L + \sum_{i=1}^{L} f(a_i') > L + N(a, p^k, \gamma) = |\lambda_{p'}^-| = br(n, \gamma),$$

which is a contradiction. Hence $N(a, p^k, \gamma) = \Phi(a, L)$. ☐

Proposition 3.23. *Let $\gamma \vdash_{p'} m$. Then $br(n) = br(n, \gamma)$ if and only if $|\gamma_{p'}^-| = br(m)$. In particular, $br(n) = br(m) + \Phi(a, br(m))$.*

Proof. Suppose that $br(n) = br(n, \gamma)$. Let $\lambda \vdash_{p'} n$ be such that $C_{p^k}(\lambda) = \gamma$ and $|\lambda_{p'}^-| = br(n)$, so that $br(n) = |\gamma_{p'}^-| + \Phi(a, |\gamma_{p'}^-|)$ by Proposition 3.18. Let $\delta \vdash_{p'} m$ be such that $|\delta_{p'}^-| = br(m)$. Then, since Φ is non-decreasing in each argument (when the other argument is fixed), we have

$$br(n) \geq br(n, \delta) = |\delta_{p'}^-| + \Phi(a, |\delta_{p'}^-|) = br(m) + \Phi(a, br(m)) \geq |\gamma_{p'}^-| + \Phi(a, |\gamma_{p'}^-|) = br(n),$$

whence we in fact have equalities everywhere. This proves all three statements: $br(m) = |\gamma_{p'}^-|$ gives the only if direction; $br(n) = br(n, \delta)$ gives the if direction (with δ in place of γ); and the final assertion is clear. ☐

Corollary 3.24. *Let $n \in \mathbb{N}$. Let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be its p-adic expansion, where $0 \leq n_1 < \cdots < n_t$. Let $m_j = \sum_{i=1}^{j-1} a_i p^{n_i}$. Then*

$$br(n) = br(a_1 p^{n_1}) + \sum_{j=2}^{t} \Phi(a_j, br(m_j)).$$

Thus we have shown that the second statement of Theorem 3.2 holds. In the last part of this section we complete the proof of Theorem 3.2 by studying the set $\mathcal{E}(n) = \{|\lambda_{p'}^-| \ :\ \lambda \vdash n \text{ and } p \nmid \chi^\lambda(1)\}$.

Theorem 3.25. *Let p be a prime, $k \in \mathbb{N}_0$ and $a \in \{1, 2, \ldots, p-1\}$. Then $\mathcal{E}(ap^k) = \{1, 2, \ldots, br(ap^k)\}$.*

The proof of Theorem 3.25 is rather more technical and so has been postponed to Section 3.1.4. More precisely, Theorem 3.25 follows from Propositions 3.30, 3.33 and 3.43, which are proved in Section 3.1.4 below.

The next statement extends the observations already made in Lemma 3.22, and is crucial to completing the description of the set $\mathcal{E}(n)$.

Lemma 3.26. *Let $B = T_{(1)}$ denote the 2-abacus for the partition (1) having first gap in position $(0, 0)$. Let $x \in \mathbb{N}_0$ and let $\mathcal{T}(x)$ be the set consisting of all 2-abaci U such that $w(U) = x$ and $U^{\uparrow} = B$. Then $\{\mathrm{Rem}(U_1) \mid U \in \mathcal{T}(x)\} = \{1, 2, \ldots, f(x) + 1\}$.*

Proof. From Lemma 3.22 we know that the maximal value of $\{\mathrm{Rem}(U_1) \mid U \in \mathcal{T}(x)\}$ is $f(x)+1$. For any $r \in \{0, 1, \ldots, f(x)\}$, let $U(r)$ be the 2-abacus obtained from B by first sliding down the bead in position $(0, 1)$ to position $(x - r(r+1), 1)$ and then (if $r > 0$) sliding down the bead in position $(i, 1)$ to position $(i+r, 1)$ for each $i \in \{-1, -2, \ldots, -r\}$. Clearly $U(r) \in \mathcal{T}(x)$ and $\mathrm{Rem}(U(r)_1) = r + 1$. □

Theorem 3.27. *Let p be a prime and let $n \in \mathbb{N}$. Let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be its p-adic expansion, where $0 \leq n_1 < n_2 < \cdots < n_t$. Then $\mathcal{E}(n) = \{1, 2, \ldots, br(n)\}$.*

Proof. We proceed by induction on t, the p-adic length of n. If $t = 1$ then the statement follows from Theorem 3.25.

Now assume that $t \geq 2$. Let $m = \sum_{j=1}^{t-1} a_j p^{n_j}$ and let γ be a p'-partition of m such that $|\gamma_{p'}^{-}| = br(m)$. For convenience, let $L = br(m)$ and $k = n_t$. As in Definition 3.19 let $A := A_\gamma$ be the p^k-abacus for γ having first gap in position $(0, 0)$. Moreover, let $\mathcal{R}_A = \{j_1, \ldots, j_L\}$.

Applying Lemma 3.26 to the L pairs of runners $(A_{j_i - 1}, A_{j_i})$ of A, we see that for each $r \in \{0, 1, \ldots, \Phi(a_t, L)\}$, there exists a sequence of a_t down-moves that can be performed on A to produce a p^k-abacus B^r such that

$$\sum_{j \in \mathcal{R}_A} \mathrm{Rem}(B_j^r) = L + r.$$

Let $\lambda(r)$ be the partition of n corresponding to B^r. Clearly $C_{p^k}(\lambda(r)) = \gamma$ and by Theorem 3.12 we deduce that $\lambda(r) \vdash_{p'} n$. Moreover, $|\lambda(r)_{p'}^{-}| = L + r$ by Corollary 3.21. Hence $L + r \in \mathcal{E}(n)$, and thus $\{L, L+1, \ldots, br(n)\} \subseteq \mathcal{E}(n)$, noting that $L + \Phi(a_t, L) = br(n, \gamma) = br(n)$ by Proposition 3.23.

If $L = 1$ then the proof is complete; otherwise, using the inductive hypothesis we have that for any $i \in \{1, 2, \ldots, L-1\}$, there exists $\gamma(i) \vdash_{p'} m$ such that $|\gamma(i)_{p'}^{-}| = i$. Taking $r = 0$ and replacing γ by $\gamma(i)$ in the above construction, we construct $\beta(i) \vdash_{p'} n$ such that $C_{p^k}(\beta(i)) = \gamma(i)$ and $|\beta(i)_{p'}^{-}| = i + 0$. Hence $\{1, 2, \ldots, L-1\} \subseteq \mathcal{E}(n)$, and we conclude that $\mathcal{E}(n) = \{1, 2, \ldots, br(n)\}$. □

Proof of Theorem 3.2. This follows directly from Corollary 3.24 and Theorem 3.27. □

3.1.3 The upper bound $\mathbb{B}(n)$

In this section we prove Proposition 3.5. Fix a prime p and let $n \in \mathbb{N}$. Let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be its p-adic expansion, where $0 \leq n_1 < \cdots < n_t$. Recall that $\mathbb{B}_p(n)$ is defined as follows:

$$\mathbb{B}_p(n) = br_p(a_1 p^{n_1}) + \sum_{j=2}^{t} \left\lfloor \frac{a_j}{2} \right\rfloor .$$

From Lemma 3.7 and Corollary 3.24, we see that $br_p(n) \leq \mathbb{B}_p(n)$, and the difference $\varepsilon_p(n) = \mathbb{B}_p(n) - br_p(n)$ can be written as

$$\varepsilon_p(n) = \sum_{j=2}^{t} \left(\left\lfloor \frac{a_j}{2} \right\rfloor - \Phi(a_j, br_p(m_j)) \right)$$

where $m_j = \sum_{i=1}^{j-1} a_i p^{n_i}$. The following statement will be useful in the proof of Proposition 3.5, below.

Lemma 3.28. *Let $s, t \in \mathbb{N}_0$ with $s \leq t$. Let $b_0, b_1, \ldots, b_t \in \{0, 1, \ldots, p-1\}$ with $b_0, b_1, \ldots, b_s$ not all zero. Then $br\left(\sum_{j=0}^{s} b_j p^j\right) \leq br\left(\sum_{j=0}^{t} b_j p^j\right)$.*

Proof. This follows directly from Proposition 3.23. □

Proof of Proposition 3.5. Fix $n \in \mathbb{N}$ and its p-adic expansion as above. Let $\varepsilon(j) = \lfloor \frac{a_j}{2} \rfloor - \Phi(a_j, br(m_j))$. If $a_j \leq 3$ then $\varepsilon(j) = 0$ by Lemma 3.7, since $br(m_j) \geq 1$. Hence if $a_j \leq 3$ for all $j \geq 2$, then in fact $\varepsilon_p(n) = 0$. Thus if $p \leq 3$ then $\varepsilon_p(n) = 0$, so from now on we may assume $p \geq 5$ and that there exists $i \in \{2, \ldots, t\}$ such that $a_i \geq 4$. In particular, there exists a unique $k \in \mathbb{N}$ and integers $1 = i_0 < i_1 < i_2 < \cdots < i_k \leq t$ such that for all $j \in [k]$,

$$i_j = \min \left\{ x \in \{i_{j-1}+1, \ldots, t-1, t\} \mid a_x \geq 2^j + 2 \right\},$$

and $\{x \in \{i_k + 1, \ldots, t-1, t\} \mid a_x \geq 2^{k+1} + 2\} = \emptyset$. Note that k must satisfy $2^k < p$, because if $2^k \geq p$ then $a_{i_k} \geq 2^k + 2 > p - 1$, contradicting the fact that a_{i_k} is a p-adic digit.

We first show that $br(m_{i_j}) \geq 2^{j-1}$ for all $j \in [k]$ by induction. This is clear for $j = 1$. For $j \in \{2, \ldots, k\}$, we have

$$\begin{aligned} br(m_{i_j}) &\geq br(m_{i_{j-1}+1}) = br(m_{i_{j-1}}) + \Phi(a_{i_{j-1}}, br(m_{i_{j-1}})) \\ &\geq 2^{j-2} + \Phi(2^{j-1} + 2, 2^{j-2}) \geq 2^{j-1}. \end{aligned}$$

The inequalities above hold by Lemma 3.28, the fact that Φ is non-decreasing in each argument, the inductive hypothesis, and Lemma 3.8, while the equality follows from Proposition 3.23. Thus for all $x \geq i_j + 1$ we have

$$br(m_x) \geq br(m_{i_j+1}) = br(m_{i_j}) + \Phi(a_{i_j}, br(m_{i_j})) \geq 2^{j-1} + \Phi(2^j + 2, 2^{j-1}) \geq 2^j.$$

Now let $x \in \{2, \dots, t\}$ be such that $i_j < x < i_{j+1}$ for some $j \in [k-1]$. Since $x < i_{j+1}$ *and* $x > i_j$, we have $a_x \leq 2^{j+1}+1$, and since $x > i_j$, we have by the above discussion that $br(m_x) \geq 2^j$. Therefore $br(m_x) \geq \lfloor \frac{a_x}{2} \rfloor$ and hence $\varepsilon(x) = 0$ by Lemma 3.7. Similarly if $x < i_1$ then $a_x \leq 3$ and so $\varepsilon(x) = 0$, while if $x > i_k$ then $br(m_x) \geq 2^k \geq \lfloor \frac{a_x}{2} \rfloor$ and thus $\varepsilon(x) = 0$ also. Hence

$$\varepsilon_p(n) = \sum_{j=1}^{k} \varepsilon(i_j).$$

Finally, for each $j \in [k]$, we have by Lemma 3.8 that

$$\varepsilon(i_j) = \left\lfloor \frac{a_{i_j}}{2} \right\rfloor - \Phi(a_{i_j}, br(m_{i_j})) \leq \frac{p-1}{2} - \Phi(2^j + 2, 2^{j-1}) \leq \frac{p-1}{2} - 2^{j-1}.$$

Hence

$$\varepsilon_p(n) \leq \sum_{i=0}^{k-1} \left(\frac{p-1}{2} - 2^i \right) = k \cdot \frac{p-1}{2} - (2^k - 1) < k \cdot \frac{p}{2} < \frac{p}{2} \log_2 p.$$

□

Remark 3.29. Proposition 3.5 shows that the difference between the upper bound $B(n)$ and the actual value of $br(n)$ is small, and is bounded independently of n. If $p \in \{2, 3\}$ then $\varepsilon_n = 0$, as observed in the first part of the proof of Proposition 3.5 above. In particular, fixing $p = 2$ we recover [1, Theorem 1]. As already mentioned in the introduction, the proof of Proposition 3.5 also shows that for any prime p, we have $\mathbb{B}(n) = br(n)$ whenever all of the p-adic digits of n are at most 3. ◊

3.1.4 p'-constituents when $n = ap^k$

The main goals in this section are to prove Theorem 3.3 (determining the value of $br_p(ap^k)$), Corollary 3.4 and Theorem 3.25 (showing that $\mathcal{E}_p(ap^k)$ is the set of consecutive integers $\{1, 2, \dots, br_p(ap^k)\}$). These two results play the role of base cases for Theorem 3.2.

For the rest of this section, let p be an odd prime. The case when $k = 0$ is straightforward and is described in the following proposition.

Proposition 3.30. *Let* $a \in \{1, 2, \dots, p-1\}$. *Then* $\mathcal{E}(a) = \{1, 2, \dots, br(a)\}$ *and* $br(a) = f(2a)$.

Proof. Every partition of $a-1$ is a p'-partition, and we can always construct a partition λ of a such that $|\lambda^-| = m$ for any $m \in \{1, 2, \dots, f(2a)\}$, since $f(2a)$ is the maximum number of parts of distinct size achieved by a partition of a. □

In the following proposition we provide a naive upper bound for $br(ap^k)$, for all $k \in \mathbb{N}$ and $a \in \{1, \dots, p-1\}$. As we will show in the rest of this section, this bound turns out to be tight for almost all values of a and k.

Proposition 3.31. *Let* $a \in \{1, 2, \dots, p-1\}$ *and let* $k \in \mathbb{N}$. *Then* $br(ap^k) \leq 2a$.

Proof. Let C and D be p^k-abacus configurations such that D is obtained from C by performing a single down-move. It is easy to see that the number of removable beads in D is at most the number of removable beads in C plus two. Hence if λ is a partition such that $C_{p^k}(\lambda) = \emptyset$ then $|\lambda^-| \leq 2w_{p^k}(\lambda)$. Now let $n = ap^k$ and let $\lambda \vdash_{p'} n$ satisfy $|\lambda^-_{p'}| = br(n)$. From Theorem 3.12 we know that $C_{p^k}(\lambda) = \emptyset$ and $w_{p^k}(\lambda) = a$. The result follows. □

Proof of Corollary 3.4. This is a straightforward consequence of Lemma 3.7, Corollary 3.24 and Proposition 3.31. □

To complete the proof of Theorem 3.3, it will be convenient to split the remainder of this section into two parts. In each part we will appropriately fix the natural numbers a and k according to the statement of Theorem 3.3.

Part I

In this first part, we consider the case $k = 1$ and $a < \frac{p}{2}$, and the case $k \geq 2$.

Proposition 3.32. *Let* $a \in \{1, 2, \dots, p-1\}$ *and let* $k \in \mathbb{N}$. *If* $k = 1$ *and* $a < \frac{p}{2}$, *or if* $k \geq 2$, *then* $br(ap^k) = 2a$.

Proof. It is enough to construct $\lambda \vdash_{p'} ap^k$ such that $|\lambda^-_{p'}| = 2a$, by Proposition 3.31.
(i) First suppose $k = 1$ and $a < \frac{p}{2}$. Let $\lambda = (p-1, p-2, \dots, p-a, a, a-1, \dots, 2, 1) \vdash ap$. Figure 3.1 depicts the p-abacus configuration for λ having first gap in position $(0, 0)$, where we have indicated the row numbers on the left and the runner numbers above each column.

	0	1	2	3	$\cdots$	$2a-2$	$2a-1$	$2a$	$\cdots$	$p-1$
-1	×	×	×	×	$\cdots$	×	×	×	$\cdots$	×
0	○	×	○	×	$\cdots$	○	×	○	$\cdots$	○
1	×	○	×	○	$\cdots$	×	○	○	$\cdots$	○

Figure 3.1: The partition $\lambda = (p-1, p-2, \dots, p-a, a, a-1, \dots, 2, 1) \vdash ap$.

Since $C_p(\lambda) = \emptyset$ we have that $\lambda \vdash_{p'} ap$ by Theorem 3.12. Moreover, we observe that $w_p(\mu) = w_p(\lambda) - 1 = a - 1$ for each $\mu \in \lambda^-$, by Lemma 3.10, and so $C_p(\mu) \vdash p - 1$ by Proposition 3.9. But every partition of $p-1$ is of p'-degree, so by Theorem 3.12 we have that $\mu \vdash_{p'} ap - 1$ for every $\mu \in \lambda^-$, whence $\lambda^-_{p'} = \lambda^-$ and so $|\lambda^-_{p'}| = 2a$.
(ii) Suppose now that $k \geq 2$. Let $r = p^{k-1} - a > 0$ and let

$$\lambda^j = a + p - 2 + rp + (a - j)(p - 1) = p^k - (j - 1)(p - 1) - 1$$

for each $j \in \{1, 2, \dots, a\}$. Let

$$\lambda = \left(\lambda^1, \lambda^2, \dots, \lambda^a, a, (a-1)^{p-1}, (a-2)^{p-1}, \dots, 2^{p-1}, 1^{p-1}\right) \vdash ap^k.$$

The best way to verify that λ has the required properties is to look at it on James' abacus. We describe below and depict in Figure 3.2 a p-abacus configuration A corresponding to λ such that:

- the first gap is in position $(1, 0)$;
- rows $1 \leq i \leq a-1$ have a gap only in position $(i, 0)$;
- row a has a bead only in position $(a, 1)$;
- rows $a+1$ to $a+r$ are all empty;
- rows $a+1+r \leq i \leq 2a+r$ have a bead only in position $(i, 0)$;
- there is a gap in position (x, y) for all $x > 2a+r$.

	0	1	2	$\cdots$	$p-1$
1	$\circ$	$\times$	$\times$	$\cdots$	$\times$
$\vdots$	$\vdots$				$\vdots$
$a-1$	$\circ$	$\times$	$\times$	$\cdots$	$\times$
a	$\circ$	$\times$	$\circ$	$\cdots$	$\circ$
$a+1$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$
$\vdots$	$\vdots$				$\vdots$
$a+r$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$
$a+1+r$	$\times$	$\circ$	$\circ$	$\cdots$	$\circ$
$\vdots$	$\vdots$				$\vdots$
$2a+r$	$\times$	$\circ$	$\circ$	$\cdots$	$\circ$

Figure 3.2: The partition $\lambda = \big(\lambda^1, \lambda^2, \ldots, \lambda^a, a, (a-1)^{p-1}, (a-2)^{p-1}, \ldots, 2^{p-1}, 1^{p-1}\big) \vdash ap^k$.

We observe that $Q_p(\lambda) = (\lambda^0, \emptyset, \ldots, \emptyset)$, where $\lambda^0 = (p^{k-1}, \ldots, p^{k-1}) \vdash ap^{k-1}$. From [54, Theorem 3.3], we deduce that $w_{p^k}(\lambda) = w_{p^{k-1}}(\lambda^0) = a$ and $C_{p^k}(\lambda) = \emptyset$. Thus $\lambda \vdash_{p'} ap^k$, by Theorem 3.12.

Notice that λ has exactly $2a$ removable nodes, corresponding to the $2a$ removable beads in A lying in positions $(i, 1)$ and $(a+r+i, 0)$ for $i \in [a]$. Let c be a removable bead in position $(i, 1)$ of A, for some $i \in [a]$. Then $A^{\leftarrow c}$ corresponds to the partition $\mu \vdash ap^k - 1$ such that $C_p(\mu) = (p-1) \vdash p-1$ and $Q_p(\mu) = (\mu^0, \mu^1, \emptyset, \ldots, \emptyset)$, where

$$\mu^0 = (p^{k-1}-1, \ldots, p^{k-1}-1, i-1) \vdash a(p^{k-1}-1) + i - 1 \quad \text{and} \quad \mu^1 = (1^{a-i}) \vdash a - i.$$

We observe that $\mu^0 \vdash_{p'} (a-1)p^{k-1} + m$, where $m := p^{k-1} - a + (i-1)$. This follows from Theorem 3.12 since $w_{p^{k-1}}(\mu^0) = a-1$ and $C_{p^{k-1}}(\mu^0) = (m) \vdash_{p'} m$. Moreover, $\mu^1 \vdash_{p'} a - i$. We can now use Theorem 3.13 to deduce that $\mu \vdash_{p'} ap^k - 1$ and therefore $\mu \in \lambda^-_{p'}$.

A similar argument shows that for every $j \in [a]$ the p-abacus $A^{\leftarrow d}$ obtained from A by sliding the bead d in position $(a+r+j, 0)$ to position $(a+r+j-1, p-1)$, corresponds to a p'-partition μ of $ap^k - 1$, that is, $\mu \in \lambda^-_{p'}$. Thus $|\lambda^-_{p'}| = 2a$. $\square$

Proposition 3.33. *Let $a \in \{1, 2, \ldots, p-1\}$ and let $k \in \mathbb{N}$. If $k = 1$ and $a < \frac{p}{2}$, or if $k \geq 2$, then $\mathcal{E}(ap^k) = \{1, 2, \ldots br(ap^k)\}$.*

Proof. It is enough to construct $\lambda \vdash_{p'} ap^k$ such that $|\lambda_{p'}^-| = m$ for each $m \in [2a-1]$, by Proposition 3.32.

(i) First suppose that $k = 1$ and $a < \frac{p}{2}$. We first exhibit $\lambda(j) \vdash_{p'} ap$ such that $|\lambda(j)_{p'}^-| = 2j$ for each $j \in [a-1]$:

· let $\lambda(1) = (ap-1, 1)$;

· for each fixed $j \in \{2, \ldots, a-1\}$, let $\lambda(j) = (\lambda_1, \lambda_2, \ldots, \lambda_{2j})$ where

- $\lambda_1 = (a-j+1)p - 2j + 1$,
- $\lambda_x = p + 2 - x$ for $x \in \{2, \ldots, j\}$, and
- $\lambda_y = 2j + 1 - y$ for $y \in \{j+1, \ldots, 2j\}$.

The p-abacus for $\lambda(j)$ having first gap in position $(0,0)$ is depicted in Figure 3.3.

	0	1	2	3	$\cdots$	$2j-2$	$2j-1$	$2j$	$\cdots$	$p-1$
-1	$\times$	$\times$	$\times$	$\times$	$\cdots$	$\times$	$\times$	$\times$	$\cdots$	$\times$
0	$\circ$	$\times$	$\circ$	$\times$	$\cdots$	$\circ$	$\times$	$\circ$	$\cdots$	$\circ$
1	$\circ$	$\circ$	$\times$	$\circ$	$\cdots$	$\times$	$\circ$	$\circ$	$\cdots$	$\circ$
2	$\circ$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$
$\vdots$	$\vdots$									$\vdots$
$a-j+1$	$\times$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$

Figure 3.3: The partition $\lambda(j)$.

That $\lambda(j) \vdash_{p'} ap$ follows from Theorem 3.12. Moreover, $w_p(\mu) = w_p(\lambda(j)) - 1 = a-1$ for each $\mu \in \lambda(j)^-$ by Lemma 3.10, and so $|C_p(\mu)| = p-1$ by Proposition 3.9. But then $C_p(\mu) \vdash_{p'} p-1$ and so by Theorem 3.12 we have that $\mu \vdash_{p'} ap-1$ for each $\mu \in \lambda(j)^-$, whence $\lambda(j)_{p'}^- = \lambda(j)^-$ and so $|\lambda(j)_{p'}^-| = 2j$. Hence $\{2, 4, \ldots, 2a-2\} \subseteq \mathcal{E}(ap)$.

Next we exhibit $\beta(j) \vdash_{p'} ap$ such that $|\beta(j)_{p'}^-| = 2j-1$ for each $j \in [a]$:

· let $\beta(1) = \big((a-1)p+1, 1^{p-1}\big)$;

· let $\beta(a) = (2a-1, 2a-2, \ldots, a+1, a^{p-2a+2}, a-1, \ldots, 2, 1)$;

· for each fixed $j \in \{2, \ldots, a-1\}$, let $\beta(j) = (\beta_1, \ldots, \beta_p)$ where

- $\beta_1 = (a-j)p + 1$,
- $\beta_x = 2j + 2 - x$ for $x \in \{2, \ldots, j\}$,
- $\beta_y = j$ for $y \in \{j+1, \ldots, p-j+1\}$, and
- $\beta_z = p + 1 - z$ for $z \in \{p-j+2, \ldots, p\}$.

The p-abacus for $\beta(j)$ having first gap in position $(0,0)$ is depicted in Figure 3.4.

	0	1	2	3	$\cdots$	$2j-2$	$2j-1$	$2j$	$\cdots$	$p-1$
-1	$\times$	$\times$	$\times$	$\times$	$\cdots$	$\times$	$\times$	$\times$	$\cdots$	$\times$
0	$\circ$	$\times$	$\circ$	$\times$	$\cdots$	$\circ$	$\times$	$\times$	$\cdots$	$\times$
1	$\circ$	$\circ$	$\times$	$\circ$	$\cdots$	$\times$	$\circ$	$\circ$	$\cdots$	$\circ$
2	$\circ$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$
$\vdots$	$\vdots$									$\vdots$
$a-j+1$	$\times$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$

Figure 3.4: The partition $\beta(j)$.

That $\beta(j) \vdash_{p'} ap$ follows from Theorem 3.12. By Lemma 3.10, if $j \neq a$ then $|\beta(j)^-| = 2j$ and $|\beta(j)^- \setminus \beta(j)^-_{p'}| = 1$, while if $j = a$ then $|\beta(j)^-_{p'}| = |\beta(j)^-| = 2a-1$. In both cases we have $|\beta(j)^-_{p'}| = 2j-1$, giving $\{1, 3, \ldots, 2a-1\} \subseteq \mathcal{E}(ap)$. Thus $\mathcal{E}(ap) = \{1, 2, \ldots, 2a\}$ as claimed.

(ii) Suppose now that $k \geq 2$. We first construct a partition $\lambda(j) \vdash_{p'} ap^k$ such that $|\lambda(j)^-_{p'}| = 2a - j$, for all $j \in [a-1]$. Let $r = p^{k-1} - a > 0$ and let

$$\lambda(j) := \big(\eta_{a-1}, \ldots, \eta_j, \theta_j, \ldots, \theta_1, a, (a-1)^{p-1}, \ldots, (j+1)^{p-1}, j^{p-2}, (j-1)^{p-1}, \ldots, 1^{p-1}\big),$$

where $\theta_t = a + pr + t(p-1)$ and $\eta_t = \theta_t + (p-2)$ for $t \in [a-1]$. We describe below and depict a p-abacus A^j for $\lambda(j)$ in Figure 3.5:

· the first gap is in position $(1, 1)$;
· rows $1 \leq x \leq j$ have a gap only in position $(x, 1)$;
· rows $j + 1 \leq x \leq a - 1$ have a gap only in position $(x, 0)$;
· row a has a bead only in position $(a, 1)$;
· rows $a + 1$ to $a + r$ are all empty;
· rows $a + r + 1 \leq x \leq a + r + j$ have a bead only in position $(x, 1)$;
· rows $a + r + j + 1 \leq x \leq 2a + r$ have a bead only in position $(x, 0)$;
· there is a gap in position (x, y) for all $x > 2a + r$.

	0	1	2	$\cdots$	$p-1$
1	×	○	×	$\cdots$	×
$\vdots$	$\vdots$				$\vdots$
j	×	○	×	$\cdots$	×
$j+1$	○	×	×	$\cdots$	×
$\vdots$	$\vdots$				$\vdots$
$a-1$	○	×	×	$\cdots$	×
a	○	×	○	$\cdots$	○
$a+1$	○	○	○	$\cdots$	○
$\vdots$	$\vdots$				$\vdots$
$a+r$	○	○	○	$\cdots$	○
$a+r+1$	○	×	○	$\cdots$	○
$\vdots$	$\vdots$				$\vdots$
$a+r+j$	○	×	○	$\cdots$	○
$a+r+j+1$	×	○	○	$\cdots$	○
$\vdots$	$\vdots$				$\vdots$
$2a+r$	×	○	○	$\cdots$	○

Figure 3.5: A p-abacus A^j for the partition $\lambda(j)$.

Since j is fixed, we denote $\lambda(j)$ by λ and A^j by A from now on. Arguing as in the proof of Proposition 3.32, we deduce that $\lambda \vdash_{p'} ap^k$. Moreover, it is clear that $|\lambda^-| = 2a$. Let $x \in [j]$ and let c be the bead lying in position $(x, 2)$ of A. Let $\mu^{(x)}$ be the partition of $ap^k - 1$ corresponding to the p-abacus $A^{\leftarrow c}$. Then $C_p(\mu^{(x)}) = (p, 1^{p-1})$. Therefore $\mu^{(x)}$ is not a p'-partition, by Theorem 3.13. It follows that $|\lambda^-_{p'}| \leq 2a - j$.

We now show that all of the other $2a - j$ removable beads in A correspond to p'-partitions of $ap^k - 1$. Let $x \in \{j+1, j+2, \ldots, a\}$ and let c be the bead in position $(x, 1)$ of A. Let $\mu^{(x)}$ be the partition of $ap^k - 1$ corresponding to the p-abacus $A^{\leftarrow c}$. Then $C_p(\mu^{(x)}) = (p-1) \vdash_{p'} p - 1$ and $Q_p(\mu^{(x)}) = (\mu^0, \mu^1, \emptyset, \ldots, \emptyset)$, where

$$\mu^0 = \big((p^{k-1} - 1)^{a-j}, x - j - 1\big) \quad \text{and} \quad \mu^1 = \big((r + j + 1)^j, (j+1)^{a-x}, j^{x-j-1}\big).$$

By Theorem 3.12, both μ^0 and μ^1 are p'-partitions, since

$$|\mu^0| = (a - j - 1)p^{k-1} + (p-1)\sum_{i=1}^{k-2} p^i + [(p-1) - (a-x)]$$

and

$$|\mu^1| = jp^{k-1} + (a - x).$$

This implies $\mu^{(x)} \vdash_{p'} ap^k - 1$, by Theorem 3.13.

Now let c be the bead in position $(a + r + x, 1)$ for some $x \in [j]$, and let $\mu^{(x)}$ be the partition corresponding to the p-abacus $A^{\leftarrow c}$. Arguing as before, we deduce from Theorem 3.13 that $\mu^{(x)} \vdash_{p'} ap^k - 1$.

Finally, let c be the bead in position $(a + r + x, 0)$ for some $x \in \{j+1, \ldots, a\}$ and let $\mu^{(x)}$ be the partition corresponding to $A^{\leftarrow c}$. First, we observe that $C_p(\mu^{(x)}) = (p-2, 1) \vdash_{p'} p - 1$. Moreover, $Q_p(\mu^{(x)}) = (\mu^0, \mu^1, \emptyset, \ldots, \emptyset, \mu^{p-1})$, where

$$\mu^0 = \big((p^{k-1} + 1)^{a-x}, (p^{k-1})^{x-j-1}\big), \quad \mu^1 = \big((r+j)^j, j^{a-j}\big), \quad \text{and} \quad \mu^{p-1} = (r + x - 1).$$

Again, $\mu^{(x)} \vdash_{p'} ap^k - 1$ by Theorem 3.13, and so $|\lambda^-_{p'}| = 2a - j$. Thus $\{a+1, a+2, \ldots, 2a-1\} \subseteq \mathcal{E}_{ap^k}$.

Finally, we construct a partition $\beta(j) \vdash_{p'} ap^k$ such that $|\beta(j)^-_{p'}| = a - j$, for all $j \in \{0, 1, \ldots, a-1\}$. Let B^j be the p-abacus obtained from the p-abacus A^j described above by replacing the bead in position $(a, 1)$ with a gap so that row a is now empty. Let $\beta(j)$ be the partition of ap^k corresponding to the p-abacus B^j. Since j is fixed we denote B^j by B and $\beta(j)$ by β.

It is clear that $\beta \vdash_{p'} ap^k$ and $|\beta^-| = 2a - j - 1$. Moreover, if c is one of the $a - 1$ removable beads lying on runner 1 of B and μ is the partition of $ap^k - 1$ corresponding to the p-abacus $B^{\leftarrow c}$, then $C_p(\mu) = (p, 1^{p-1})$ and therefore μ is not a p'-partition by Theorem 3.13. Hence $|\beta^-_{p'}| \leq a - j$. Arguing as before, the partition corresponding to the p-abacus $B^{\leftarrow c}$ for any removable bead c lying on runner 0 of B is a p'-partition of $ap^k - 1$. Hence $|\beta^-_{p'}| = a - j$, and so $\{1, 2, \ldots, a\} \subseteq \mathcal{E}(ap^k)$.

Thus $\mathcal{E}(ap^k) = \{1, 2, \ldots, 2a\}$ as claimed. □

Part II

In this second part of Section 3.1.4, we fix $k = 1$ and $a \in \mathbb{N}$ such that $\frac{p}{2} < a < p$. The main aim in Part II is to prove the following fact.

Proposition 3.34. *Let $a \in \mathbb{N}$ and suppose that $\frac{p}{2} < a < p$. Then $br(ap) = p - 1 + 2\lfloor \frac{2a-(p-1)}{6} \rfloor$.*

The proof of Proposition 3.34 is split into a series of technical lemmas. We start by fixing some notation which will be kept throughout Part II.

Notation 3.35. *Let $a \in \mathbb{N}$ satisfy $\frac{p}{2} < a < p$. Let $x := a - \frac{p-1}{2}$, and write $x = 3q + \delta$ for some $q \in \mathbb{N}_0$ and $\delta \in \{0, 1, 2\}$. In particular, $q = \lfloor \frac{x}{3} \rfloor = \lfloor \frac{2a-(p-1)}{6} \rfloor$.*

Definition 3.36. *Denote by $A_\emptyset$ the p-abacus for the empty partition $\emptyset$ such that $A_\emptyset$ has first gap in position $(0, 0)$. We then define $\mathcal{Z}(a)$ to be the set of p-abaci B such that $w(B) = a$ and $B^\uparrow = A_\emptyset$.*

It is clear by Theorem 3.12 that $\mathcal{Z}(a)$ is naturally in bijection with $\mathrm{Irr}_{p'}(\mathfrak{S}_{ap})$.

Lemma 3.37. *Let $\lambda \vdash_{p'} ap$ and let $B \in \mathcal{Z}(a)$ be the p-abacus corresponding to λ. Then*

$$|\lambda^-_{p'}| = \sum_{i=1}^{p-1} \mathrm{Rem}(B_i) \quad \text{and} \quad br(ap) = \max_{B \in \mathcal{Z}(a)} \sum_{i=1}^{p-1} \mathrm{Rem}(B_i).$$

Proof. The statement follows directly from Lemma 3.10 and Theorem 3.12. □

Lemma 3.38. *For $a \in \mathbb{N}$ such that $\frac{p}{2} < a < p$, we have $br(ap) \geq p - 1 + 2q$.*

Proof. We exhibit a partition $\beta \vdash_{p'} ap$ such that $|\beta^-_{p'}| = p - 1 + 2q$. If $\delta = 0$ then let

$$\beta = (p + 2q, p + 2q - 1, \ldots, p + q + 1, p + q - 1, \ldots, q + 1, q^{p-2q+1}, q - 1, \ldots, 2, 1),$$

while if $\delta \neq 0$ then let

$$\beta = \big(p(\delta+1)+2, p+2q+1, p+2q, \ldots, p+q+3, p+q-1, \ldots, q+1, q^{p-2q+1}, q-1, \ldots, 1\big).$$

We describe below and depict a p-abacus $B_\beta \in \mathcal{Z}(a)$ for β in Figure 3.6:

	0	1	2	3	4	$\cdots$	$2q-2$	$2q-1$	$2q$	$2q+1$	$2q+2$	$\cdots$	$p-3$	$p-2$	$p-1$
-3	×	×	×	×	×	$\cdots$	×	×	×	×	×	$\cdots$	×	×	×
-2	×	∘	×	∘	×	$\cdots$	×	∘	×	×	×	$\cdots$	×	×	×
-1	×	∘	×	∘	×	$\cdots$	×	∘	×	∘	×	$\cdots$	×	∘	×
0	∘	×	∘	×	∘	$\cdots$	∘	×	∘	×	∘	$\cdots$	∘	×	∘
1	∘	∘	∘	×	∘	$\cdots$	∘	×	∘	∘	∘	$\cdots$	∘	∘	∘
2	∘	∘	∘	∘	∘	$\cdots$	∘	∘	∘	∘	∘	$\cdots$	∘	∘	∘
$\vdots$	$\vdots$														$\vdots$
$1+\delta$	∘	×	∘	∘	∘	$\cdots$	∘	∘	∘	∘	∘	$\cdots$	∘	∘	∘

Figure 3.6: A p-abacus B_β for the partition β.

· for $j \in \{0, 2, \ldots, p-3, p-1\}$, runner j has beads in positions (x, j) for all $x \leq -1$;
· runner 1 has beads in positions $(0, 1)$, $(1 + \delta, 1)$ and $(y, 1)$ for all $y \leq -3$;
· for $j \in \{3, 5, \ldots, 2q - 1\}$, runner j has beads in positions $(0, j)$, $(1, j)$ and (y, j) for all $y \leq -3$;

· for $j \in \{2q+1, 2q+3, \dots, p-2\}$, runner j has beads in positions $(0, j)$ and (y, j) for all ≤ -2.

Observe that $C_p(\beta) = \emptyset$ and $w_p(\beta) = a$, whence $\beta \vdash_{p'} ap$ by Theorem 3.12. Moreover, by Lemma 3.37 we have $\beta^- = \beta^-_{p'}$. Hence $br(ap) \geq |\beta^-_{p'}| = p - 1 + 2q$. □

Thus it remains to show that $|\lambda^-_{p'}| \leq p - 1 + 2q$ for all $\lambda \vdash_{p'} ap$. In order to do this we introduce a new combinatorial object.

Definition 3.39. *Let $T_\emptyset$ be the 2-abacus for the empty partition $\emptyset$ having first gap in position $(0,0)$. Let $U^{(0)}$, $U^{(1)} \dots, U^{(p-1)}$ be 2-abaci such that $(U^{(i)})^\uparrow = T_\emptyset$ for all $i \in \{0, 1, \dots, p-1\}$. If $w(U^{(0)}) + w(U^{(1)}) + \cdots + w(U^{(p-1)}) = w \in \mathbb{N}_0$ then we call the sequence $\underline{U} = (U^{(0)}, U^{(1)}, \dots, U^{(p-1)})$ a* doubled p-abacus of weight w *and write $w(\underline{U}) = w$ in this case. Moreover, we denote by $\mathcal{D}(w)$ the set of doubled p-abaci of weight w.*

Finally, given any $w \in \mathbb{N}_0$ we let $M(w) = \max\{\rho(\underline{U}) \mid \underline{U} \in \mathcal{D}(w)\}$, where for any $\underline{U} \in \mathcal{D}(w)$ we define $\rho(\underline{U})$ as

$$\rho(\underline{U}) = \sum_{i=1}^{p-1} \mathrm{Rem}(U_1^{(i)}).$$

We denote by $U_0^{(i)}$ (resp. $U_1^{(i)}$) the left- (resp. right-) hand runner of the 2-abacus $U^{(i)}$.

Remark 3.40. Let $\lambda \vdash_{p'} ap$ and let $B \in \mathcal{Z}(a)$ correspond to λ. For $i \in [p-1]$, let $U^{(i)} = (B_{i-1}, B_i)$ and let $U^{(0)} = (B_{p-1}, B_0)$. Then $\underline{U} := (U^{(0)}, U^{(1)}, \dots, U^{(p-1)}) \in \mathcal{D}(2a)$ and $\rho(\underline{U}) = |\lambda^-_{p'}|$, by Lemma 3.37. With this in mind, we define $\mathcal{D}(\mathcal{Z}(a))$ to be the subset of $\mathcal{D}(2a)$ of sequences $\underline{U} := (U^{(0)}, U^{(1)}, \dots, U^{(p-1)})$ such that $U_0^{(i)} = U_1^{(i-1)}$ for all $i \in \{0, 1, \dots, p-1\}$ (here two runners are equal if they coincide as 1-abaci; that is, they have beads in exactly the same rows). Clearly the set $\mathcal{D}(\mathcal{Z}(a))$ is naturally in bijection with $\mathcal{Z}(a)$ via the construction described above. ♢

Lemma 3.41. *Let a and x be as in Notation 3.35. Then $br(ap) \leq M(2a) = p-1+\lfloor \frac{2x}{3} \rfloor$.*

Proof. It follows from Remark 3.40 that $br(ap) \leq M(2a)$, so it remains to prove $M(2a) = p - 1 + \lfloor \frac{2x}{3} \rfloor$.

Let $\underline{U} = (U^{(0)}, U^{(1)}, \dots, U^{(p-1)}) \in \mathcal{D}(2a)$ be such that $\rho(\underline{U}) = M(2a)$. Let $w_i = w(U^{(i)})$. Clearly $w_1 + w_2 + \cdots w_{p-1} \leq 2a$. Moreover, arguing as in the proof of Lemma 3.22 we can assume that $w(U_1^{(i)}) = w_i$ and $w(U_0^{(i)}) = 0$ for all $i \in [p-1]$. From the maximality of $\rho(\underline{U})$ we deduce using Lemma 3.22 (in the case $\lambda = \emptyset$) that $\mathrm{Rem}(U_1^{(i)}) = \lfloor \sqrt{w_i} \rfloor$ and hence

$$M(2a) = \max \left\{ \sum_{i=1}^{p-1} \lfloor \sqrt{b_i} \rfloor \;\middle|\; b_1 + \cdots + b_{p-1} \leq 2a \text{ and } b_i \in \mathbb{N}_0 \;\; \forall\, i \in [p-1] \right\}.$$

Let $b = (b_1, \dots, b_{p-1})$ be such that $b_i \in \mathbb{N}_0$ for all i, $\sum_i b_i \leq 2a$ and $\sum_i \lfloor \sqrt{b_i} \rfloor = M(2a)$; we will call any $(p-1)$-tuple satisfying these conditions *maximal*. If there exists i

such that $b_i \geq 9$, then there exists j such that $b_j \leq 1$. This follows since $\sum_i b_i \leq 2a < 2p$. Replacing b_i by $b'_i = b_i - 4$ and b_j by $b'_j = b_j + 4$ in b we obtain a new maximal sequence b'. Hence we may assume without loss of generality that our maximal sequence b has $b_i \leq 8$ for all $i \in [p-1]$.

Now if there exists i such that $b_i = 0$ then there exists j such that $b_j \geq 2$, because $2a > p$. In this case, replacing b_i by $b'_i = 1$ and b_j by $b'_j = b_j - 1$ in b we obtain a new maximal sequence b'. Hence we may further assume that b has $b_i \geq 1$ for all $i \in [p-1]$. The observations above show that without loss of generality we may assume

$$\lfloor\sqrt{b_1}\rfloor = \cdots = \lfloor\sqrt{b_t}\rfloor = 2, \ \lfloor\sqrt{b_{t+1}}\rfloor = \cdots = \lfloor\sqrt{b_{p-1}}\rfloor = 1,$$

for some $t \in \{0, \ldots, p-1\}$.

In particular, $b_i \in \{4, \ldots, 8\}$ for $i \in [t]$ and $b_j \in \{1, 2, 3\}$ for $j \in \{t+1, \ldots, p-1\}$. Thus $4t + (p-1-t) \leq \sum_i b_i \leq 2a$, which gives $t \leq \lfloor\frac{2x}{3}\rfloor$ since t is an integer. This in turn implies that $M(2a) = 2t + (p-1-t) \leq p - 1 + \lfloor\frac{2x}{3}\rfloor$.

Finally, equality holds because we can construct $\underline{U} \in \mathcal{D}(2a)$ such that $w(U^{(1)}) = \cdots = w(U^{(t)}) = 4$, $w(U^{(t+1)}) = \cdots = w(U^{(p-1)}) = 1$ and $w(U^{(0)}) = 2a - 3t - (p-1)$, where $t = \lfloor\frac{2x}{3}\rfloor$, with $\mathrm{Rem}(U_1^{(j)}) = 2$ for $j \in [t]$ and $\mathrm{Rem}(U_1^{(j)}) = 1$ for $j \in \{t+1, \ldots, p-1\}$. ☐

Lemmas 3.38 and 3.41 show that $p-1+2\lfloor\frac{x}{3}\rfloor \leq br(ap) \leq p-1+\lfloor\frac{2x}{3}\rfloor$. In particular, if $\delta \neq 2$ then we have that $\lfloor\frac{2x}{3}\rfloor = 2q + \lfloor\frac{2\delta}{3}\rfloor = 2q = 2\lfloor\frac{x}{3}\rfloor$. In this case we have $br(ap) = M(2a) = p - 1 + 2q$. To deal with the remaining case of $\delta = 2$ where $p - 1 + 2q \leq br(ap) \leq M(2a) = p - 1 + 2q + 1$, we have the following lemma.

Lemma 3.42. *Let $a \in \mathbb{N}$ be as in Notation 3.35 and suppose that $\delta = 2$. Then $br(ap) \leq M(2a) - 1$.*

Proof. From Remark 3.40 it is enough to show that if $\underline{U} \in \mathcal{D}(2a)$ and $\rho(\underline{U}) = M(2a)$, then $\underline{U} \notin \mathcal{D}(\mathcal{Z}(a))$. To do this we will show that if $\rho(\underline{U}) = M(2a)$ then there exists $i \in \{0, 1, \ldots, p-1\}$ such that $U_0^{(i)} \neq U_1^{(i-1)}$.

For $i \in \{0, 1, \ldots, p-1\}$, let $b_i = w(U^{(i)})$. Arguing as in the proof of Lemma 3.41 we see that $\rho(\underline{U}) = \sum_{i=1}^{p-1}\lfloor\sqrt{b_i}\rfloor$. Moreover, given any composition $\underline{w} = (w_1, \ldots, w_{p-1})$ such that $w_1 + \cdots + w_{p-1} \leq 2a$ there exists $\underline{V} \in \mathcal{D}(2a)$ such that $w(V^i) = w_i$ for all $i \in [p-1]$, $w(V^0) = 2a - (w_1 + \cdots + w_{p-1})$ and $\rho(\underline{V}) = \sum_{i=1}^{p-1}\lfloor\sqrt{w_i}\rfloor$.

Let $\underline{b} = (b_1, \ldots, b_{p-1})$ and suppose that $b_i \geq 9$ for some $i \in [p-1]$.

· If there exists j such that $b_j = 0$, then replacing (b_i, b_j) by $(b'_i, b'_j) := (b_i - 4, 4)$ in $\underline{b}$ we obtain a new composition $\underline{b'}$ such that $\sum_{i=1}^{p-1}\lfloor\sqrt{w_i}\rfloor > \rho(\underline{U})$, contradicting the maximality of $\rho(\underline{U})$.

· If $b_i \geq 10$, then there exists $j \neq l$ such that $b_j = b_l = 1$ since $a < p$. But then we may replace (b_i, b_j, b_l) by $(b_i - 6, 4, 4)$ in $\underline{b}$ to obtain a contradiction as before.

· If there exists $i' \neq i$ such that $b_{i'} \geq 9$, then since we cannot have $b_{i'} \geq 10$ we deduce that $b_{i'} = 9$. In particular, $2a \geq 18$ so $p > 3$. Since $a < p$, there exist distinct j, j', j''

such that $b_j = b_{j'} = b_{j''} = 1$. But then we may replace $(9, 9, 1, 1, 1)$ by $(5, 4, 4, 4, 4)$ in $\underline{b}$ to obtain a contradiction.

The above observations show that if $b_i \geq 9$ for some $i \in [p-1]$ then in fact $b_i = 9$ and $1 \leq b_i \leq 8$ for all $j \neq i$. In particular, there exists $t \in \{0, 1, \ldots, p-2\}$ such that $\underline{b}$ has t parts satisfying $\lfloor\sqrt{b_j}\rfloor = 2$ and $p-2-t$ parts satisfying $\lfloor\sqrt{b_j}\rfloor = 1$. Hence $M(2a) = 3 + 2t + (p-2-t) = p - 1 + \lfloor\frac{2x}{3}\rfloor = p - 1 + 2q + 1$, so $t = 2q - 1$. But this implies that

$$2a \geq \sum_{m=1}^{p-1} b_m \geq 9 + 4t + (p-2-t) = p - 1 + 6q + 5.$$

Therefore $6q + 5 \leq 2a - (p-1) = 2x = 6q + 4$, a contradiction. Thus $b_i \leq 8$ for all $i \in [p-1]$.

So suppose there are t values of i for which $\lfloor\sqrt{b_i}\rfloor = 2$, s values for which it is 1, and $p - 1 - s - t$ values for which it is 0. Then

$$p + 2q = M(2a) = 2t + s \leq p - 1 + t,$$

so $t \geq 2q + 1$. In particular $t \geq 1$, so there exists i with $\lfloor\sqrt{b_i}\rfloor = 2$. If there exists $j \neq l$ such that $b_j = b_l = 0$, then we may replace (b_i, b_j, b_l) by $(b_i - 2, 1, 1)$ in $\underline{b}$ to obtain a contradiction to the maximality of $\rho(\underline{U})$. So there is at most one $b_j = 0$ and thus $s + t \in \{p-2, p-1\}$.

If $s + t = p - 2$, then $p + 2q = M(2a) = 2t + s$ implies $t = 2q + 2$, and so

$$6q + 4 - b_0 = 2x - b_0 = \sum_{m=1}^{p-1} b_m - (p-1) \geq 4t + s - (p-1) = 6q + 5,$$

which is a contradiction. Thus $s + t = p - 1$ and $t = 2q + 1$. Since

$$6q + 4 - b_0 = \sum_{m=1}^{p-1} b_m - (p-1) \geq 4t + s - (p-1) = 6q + 3,$$

one of the following must hold:

(i) $|\{i : b_i = 4\}| = t$, $|\{i : b_i = 1\}| = s$ and $b_0 = 1$; or

(ii) $|\{i : b_i = 4\}| = t - 1$, $|\{i : b_i = 5\}| = 1$, $|\{i : b_i = 1\}| = s$ and $b_0 = 0$; or

(iii) $|\{i : b_i = 4\}| = t$, $|\{i : b_i = 2\}| = 1$, $|\{i : b_i = 1\}| = s - 1$ and $b_0 = 0$.

Now, suppose for a contradiction that $\underline{U} \in \mathcal{D}(\mathcal{Z}(a))$. Then we have that *the bead configurations on* $U_1^{(i-1)}$ *and* $U_0^{(i)}$ *are equal for all* i: call this property $(\star)$. The key in the following is to notice that $t = |\{i : b_i \geq 4\}| = 2q + 1$ is odd.

In case (i), let $i \in [p-1]$ be such that $b_i = 4$. Then $(w(U_0^{(i)}), w(U_1^{(i)})) = (j, 4 - j)$ for some $j \in \{0, 1, \ldots, 4\}$. If $j = 2$ then $(\star)$ would imply $b_{i+1} \geq 2$, and hence $b_{i+1} = 4$, since $b_l \in \{1, 4\}$ for all l. This then gives $w(U_0^{(i+1)}) = w(U_1^{(i+1)}) = 2$. We can iterate this argument to deduce that $w(U_0^{(y)}) = w(U_1^{(y)}) = 2$ for all $y \in \{0, 1, \dots, p-1\}$, which is a contradiction. Thus $j \in \{0, 1, 3, 4\}$.

If $j = 0$, then $w(U_1^{(i)}) = 4$, so $(\star)$ implies that $w(U_0^{(i+1)}) = 4$ and hence $b_{i+1} = 4$ also. Similarly if $j = 1$, then $w(U_0^{(i+1)}) = 3$ and hence $b_{i+1} = 4$. On the other hand, if $j = 3$ or $j = 4$ then similarly we deduce that $b_{i-1} = 4$. These observations imply that t is an even natural number (because if $j \in \{0, 1\}$ then we may pair off i and $i+1$ where $b_i = b_{i+1} = 4$, and if $j \in \{3, 4\}$ then we may pair off i and $i-1$ where $b_i = b_{i-1} = 4$). This gives a contradiction, and so $\underline{U} \notin \mathcal{D}(\mathcal{Z}(a))$, as desired. The analyses of cases (ii) and (iii) are similar. $\square$

Thus when $\delta = 2$ we also have that $br(ap) = p - 1 + 2\lfloor \frac{x}{3} \rfloor$, by Lemmas 3.38, 3.41 and 3.42. This proves Proposition 3.34.

Proof of Theorem 3.3. This follows directly from Propositions 3.30, 3.32 and 3.34. $\square$

We devote the final part of this section to the description of $\mathcal{E}(ap)$ for any $\frac{p}{2} < a < p$.

Proposition 3.43. *Let $a \in \mathbb{N}$ be such that $\frac{p}{2} < a < p$. Then $\mathcal{E}(ap) = \{1, 2, \dots, br(ap)\}$.*

Proof. Let $\beta \vdash_{p'} ap$ with p-abacus $B := B_\beta$ as defined in Lemma 3.38. In particular, we proved that $|\beta_{p'}^-| = br(ap) = p - 1 + 2q$, with q defined as in Notation 3.35.

Denote by b the bead in position $(1+\delta, 1)$ of B. For $i \in \{1, 2, \dots, \frac{p-1}{2}\}$ let c_i be the bead in position $(0, p-2i)$ of B and let $B(i)$ be the p-abacus obtained from B by sliding b down to position $(1+\delta+i, 1)$ and by sliding c_j up to position $(-1, p-2j)$ for all $j \in \{1, \dots, i\}$. Let $\mu(i) \vdash ap$ be the partition corresponding to the p-abacus $B(i)$. From Theorem 3.12 we have that $\mu(i) \vdash_{p'} ap$ and $|\mu(i)_{p'}^-| = |\beta_{p'}^-| - 2i$. It follows that

$$\{2q, 2q+2, \cdots, br(ap) - 2, br(ap)\} \subseteq \mathcal{E}(ap).$$

Now let $A := B(\frac{p-1}{2})$. For $i \in \{1, 2, \dots, q-1\}$ let $A(i)$ be the p-abacus obtained from A by sliding down bead b from position $(1+\delta+\frac{p-1}{2}, 1)$ to position $(1+\delta+\frac{p-1}{2}+3i, 1)$ and by replacing runner A_{2j+1} with $A_{2j+1}^{\uparrow}$ for all $j \in \{1, \dots, i\}$. This step is depicted in Figure 3.7:

	$2j$	$2j+1$	$2j+2$			$2j$	$2j+1$	$2j+2$
-2	$\times$	$\circ$	$\times$		-2	$\times$	$\times$	$\times$
-1	$\times$	$\times$	$\times$	$\longrightarrow$	-1	$\times$	$\times$	$\times$
0	$\circ$	$\circ$	$\circ$		0	$\circ$	$\circ$	$\circ$
1	$\circ$	$\times$	$\circ$		1	$\circ$	$\circ$	$\circ$

Figure 3.7: Obtaining $A(i)$ from A.

Let $\nu(i) \vdash ap$ be the partition corresponding to the p-abacus $A(i)$. Since $w(A_{2i+1}) = 3$ for all $i \in \{1, 2, \dots, q-1\}$, it follows from Theorem 3.12 that $\nu(i) \vdash_{p'} ap$ and $|\nu(i)_{p'}^-| = |\mu(\frac{p-1}{2})_{p'}^-| - 2i$. Thus $\{2, 4, 6, \cdots, 2q-2\} \subseteq \mathcal{E}(ap)$, and so it remains to show $\{1, 3, \dots, br(ap) - 1\} \subseteq \mathcal{E}(ap)$.

First suppose $q \geq 1$. Consider the p-abacus C obtained from B by sliding down the bead in position $(-1,0)$ to position $(0,0)$ and by sliding up the bead in position $(0,1)$ to position $(-1,1)$.

Let γ be the partition corresponding to C. It is easy to see that $\gamma \vdash_{p'} ap$ and that $|\gamma_{p'}^-| = br(ap) - 1$. We can now repeat the strategy used above to see that $\{3,5,\ldots,br(ap)-1\} \subseteq \mathcal{E}(ap)$. Of course, $1 \in \mathcal{E}(ap)$ by considering the trivial partition $(ap) \vdash_{p'} ap$.

If $q = 0$ we begin with the p-abacus C' obtained from B by swapping runners 0 and 1, instead of C. The same argument then shows $\{1,3,\ldots,br(ap)-1\} \subseteq \mathcal{E}(ap)$. ☐

Proof of Theorem 3.25. This follows from Propositions 3.30, 3.33 and 3.43. ☐

3.2 Induction

Let p be a prime number. In the first part of this chapter, we studied the restrictions of irreducible characters of the symmetric group $\mathfrak{S}_n$ of degree coprime to p to $\mathfrak{S}_{n-1}$, giving a generalisation from $p = 2$ to all primes p of [1, Theorem 1]. In this section, we now investigate the more complex behaviours exhibited by character inductions to $\mathfrak{S}_{n+1}$, generalising [1, Theorem 2] from $p = 2$ to all p as a consequence.

3.2.1 Main results

Let p be any prime. For $n \in \mathbb{N}$ and $\lambda \vdash n$, define

$$\lambda_{p'}^+ = \{\mu \vdash_{p'} n+1 \ : \ \chi^\mu \mid \chi^\lambda\big\uparrow^{\mathfrak{S}_{n+1}}\},$$

$$\mathcal{E}_p^+(n) = \{|\lambda_{p'}^+| \ : \ \lambda \vdash_{p'} n\}, \quad \text{and} \quad br_p^+(n) = \max \mathcal{E}_p^+(n).$$

(As usual, we omit the subscript p when it is understood.) Our main results in this section are the following:

Theorem 3.44. *Let $n \in \mathbb{N}_0$ and let p be any prime. Let $n+1 = \sum_{j=1}^t a_j p^{n_j}$ be the p-adic expansion of $n+1$, where $0 \leq n_1 < n_2 < \ldots < n_t$. Then*

$$br^+(n) = br^+(a_1 p^{n_1} - 1) + \sum_{j=2}^{t} \Phi(a_j, br^+(m_j - 1)) + \Delta(n,p),$$

where $m_j = \sum_{i=1}^{j-1} a_i p^{n_i}$ and Φ and Δ are defined in Definition 3.49 below. In particular, $br^+(n) = 1$ if and only if $n = 0$ or $n = \sum_{j=1}^u p^{k_j}$ for some $u \in \mathbb{N}$ and $1 \leq k_1 < \cdots < k_u$.

Theorem 3.45. *Let $n \in \mathbb{N}_0$ and let p be any prime. Then $1 \in \mathcal{E}^+(n)$ if and only if $p \mid n$, and*

$$0 \in \mathcal{E}^+(n) \quad \text{if and only if} \quad \begin{cases} p \mid n+1 & \text{if } p \geq 5, \\ 9 \mid n+1 & \text{if } p = 3, \\ 8 \mid n+1 & \text{if } p = 2. \end{cases}$$

Moreover, when $br^+(n) > 1$*, then*

$$\{2, 3, \dots, br^+(n)\} \subseteq \mathcal{E}^+(n) \subseteq \{0, 1, 2, \dots, br^+(n)\}.$$

Theorem 3.46. *Let* p *be a prime,* $a \in [p-1]$ *and* $k \in \mathbb{N}_0$*. Then*

$$br^+(ap^k - 1) = \begin{cases} \lfloor \sqrt{a} \rfloor + 1 & \text{if } k > 0, \\ f(2a-2) + 1 & \text{if } k = 0 \end{cases}$$

where f *is as defined in Definition 3.49 below.*

Theorems 3.44 and 3.46 give an exact recursive formula for the exact value of $br^+(n)$, while Theorem 3.45 determines the set of achievable values for the quantity $|\lambda^+_{p'}|$ as λ runs over the p'-partitions of n. In other words, we determine all possible numbers of upward edges that a vertex in the Young subgraph $\mathbb{Y}_{p'}$ can have.

When $p = 2$, the expressions in Theorems 3.44 and 3.45 afford a simpler form: we obtain Corollary 3.66 below, which records the values of $br^+_2(n)$ and $\mathcal{E}^+_2(n)$ for all $n \in \mathbb{N}_0$, thus recovering [1, Theorem 2]. When p is odd, we can further give a bound to easily estimate the size of $br^+(n)$ in terms of the p-adic digits of n.

Corollary 3.47. *Let the notation be as in Theorem 3.44, and suppose further that p is odd. Then*

$$br^+(n) \leq \mathbb{B}^+(n) := 2 + \sqrt{2a_1} + \sum_{j=2}^{t} \left\lfloor \frac{a_j}{2} \right\rfloor.$$

Moreover, $\mathbb{B}^+(n) - br^+(n) < p\log_2(p)$ *for all* n*. Thus*

$$\sup\{br^+(n) \mid n \in \mathbb{N}\} = \infty.$$

Notice that this is in contrast to $\sup\{br^+_2(n) \mid n \in \mathbb{N}\} = 2$ when $p = 2$, which follows from [1, Theorem 2] (or Corollary 3.66). The proofs of our main results appear in Section 3.2.3, below.

3.2.2 Differences between restriction and induction

Our main results extend from $p = 2$ to all primes p Theorem 2 of [1], which we restate in our present notation below.

Theorem 3.48 ([1, Theorem 2]). *Let* $n \in \mathbb{N}$*. Then* $br^+_2(n) \leq 2$*, so* $\mathbb{Y}_{2'}$ *is an incomplete binary tree. Moreover,* $\mathcal{E}^+_2(n) = \{1\}$ *if* n *is even, and* $\mathcal{E}^+_2(n) \subseteq \{0, 2\}$ *if* n *is odd. In particular, for* $\lambda \vdash_{2'} n$ *when* n *is odd,* $|\lambda^+_{2'}| = 0$ *if and only if* $C_{2^{\nu_2(n+1)}}(\lambda)$ *is not a hook.*

Before we prove our main results in Section 3.2.3 below, we discuss in the present section some behaviours exhibited by character inductions and their p'-constituents which differ from those exhibited by the character restrictions investigated in the previous chapter.

Indeed, we can already see from Theorem 3.48 that 1 is not always an element of $\mathcal{E}_2^+(n)$. Moreover, by Theorem 3.44, for every prime p there exists $n \in \mathbb{N}$ such that $0 \in \mathcal{E}_p^+(n)$, in contrast to the situation of restriction where $\mathcal{E}_p(n) = \{1, 2, \dots, br(n)\} \not\ni 0$ for all $n \in \mathbb{N}$.

Notice that starting at *any* vertex $\lambda \vdash_{p'} n$, there exists a sequence $\lambda = \lambda^{(n)}$, $\lambda^{(n-1)}, \dots, \lambda^{(1)}, \lambda^{(0)}$ in $\mathbb{Y}_{p'}$ such that $\lambda^{(i)} \vdash_{p'} i$ and $\lambda^{(i)} \in \lambda_{p'}^{(i+1)-}$ for all $i \in \{0, 1, \dots, n-1\}$; that is, we may always trace a downward path to the root vertex $\emptyset$. However, not every $\lambda \in \mathbb{Y}_{p'}$ lies on an infinite ray $\{\lambda^{(i)}\}_{i=0}^\infty$ such that $\lambda^{(i)} \vdash_{p'} i$ and $\lambda^{(i)} \in \lambda_{p'}^{(i+1)-}$ for all i. By Theorem 3.12 it is easy to see that $\mathrm{Irr}_{p'}(\mathfrak{S}_{p^k})$ consists exactly of the hook partitions of p^k, and thus the only infinite rays in $\mathbb{Y}_{p'}$ are $\{(i)\}_{i=0}^\infty$ and $\{(1^i)\}_{i=0}^\infty$, corresponding respectively to the trivial and sign representations of the symmetric groups.

We determine the values of n for which $0 \in \mathcal{E}_p^+(n)$. Before we do this, we set up some preliminaries. (The definitions of Φ and f were given in Definition 3.6; for convenience we restate them here.)

Definition 3.49. *For $a \in \mathbb{N}_0$ and $L \in \mathbb{N}$, define*

$$\Phi(a, L) := \max\left\{ \sum_{i=1}^{L} f(a_i) \;\middle|\; a_1 + \cdots + a_L \leq a \quad \text{and} \quad a_i \in \mathbb{N}_0 \quad \forall\, i \in [L] \right\},$$

where $f(x) = \max\{y \in \mathbb{N}_0 \mid y(y+1) \leq x\}$. Let $\zeta(x) = \max\{y \in \mathbb{N}_0 \mid y(y+2) \leq x\}$, and define

$$g(a) = \begin{cases} 2f(a) - 1 & \text{if } f(a) > \zeta(a), \\ 2f(a) & \text{otherwise.} \end{cases}$$

Also define $M(a) = \max\{f(a-b) + g(b) \mid b \in \{0, 1, \dots, a\}\}$.

Now let $n \in \mathbb{N}_0$ and p be a prime. Let $n = \sum_{i \geq 0} d_i p^i$ be the p-adic expansion of n, and let $d(n) := (d_1, d_0)$. Define

$$\Delta(n, p) = \begin{cases} 1 & \text{if } p = 5 \text{ and } d(n) = (3,3), \text{ or } p = 7 \text{ and } d(n) \in \{(3,5), (5,5)\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\zeta(a) \leq f(a) \leq g(a) \leq 2f(a)$ for all $a \in \mathbb{N}_0$. We remark that $f(a) > \zeta(a)$ occurs precisely when $y(y+1) \leq a < y(y+2)$ for some $y \in \mathbb{N}_0$, in which case $y = f(a)$ by the definition of f.

Next, we record the addable bead analogue of Lemma 3.10.

Lemma 3.50. *Let $e \in \mathbb{N}$. Let λ be a partition and let A be an e-abacus for λ. Suppose c is an addable bead on runner A_j and let $\mu \vdash n+1$ be the partition represented by $A^{c\to}$. Then*

$$w_e(\mu) - w_e(\lambda) = \begin{cases} |A_j| - |A_{j+1}| - 1 & \text{if } j \neq e-1, \\ |A_{e-1}| - |A_0| & \text{if } j = e-1. \end{cases}$$

Proof. This follows from Lemma 3.10. □

Let p be a prime. Let $k \in \mathbb{N}$ and let $\gamma \vdash_{p'} m$ where $0 < m < p^k$. By Proposition 3.18, $|\lambda^-_{p'}| = |\gamma^-_{p'}|$ for all $\lambda \vdash_{p'} p^k + m$ such that $C_{p^k}(\lambda) = \gamma$, since $\Phi(1, L) = 0$ for all L. Thus when $a = 1$ the core map in Theorem 3.15 is in fact bijective. We record its useful induction analogue; the ideas used its proof are completely analogous to those in Theorem 3.15.

Corollary 3.51. *Let p be a prime and $k \in \mathbb{N}$. Let $n = ap^k + m$ where $a \in [p-1]$ and $0 \leq m \leq p^k - 2$. Let $\lambda \vdash_{p'} n$. Then the map*

$$C_{p^k} : \lambda^+_{p'} \longrightarrow \big(C_{p^k}(\lambda)\big)^+_{p'}$$

is well-defined and surjective.

Proof. Well-definition follows directly from Theorem 3.15: let $\delta \in \lambda^+_{p'}$, so $\lambda \in \delta^-_{p'}$. Then $C_{p^k}(\lambda) \in C_{p^k}(\delta)^-_{p'}$, so $C_{p^k}(\delta) \in C_{p^k}(\lambda)^+_{p'}$.

For surjectivity, let A be the p^k-abacus for $\mu := C_{p^k}(\lambda)$ with first gap in position $(0,0)$, so $|A_j| \in \{|A_0|, |A_0|+1\}$ for each $j \in \{0, 1, \ldots, p^k - 1\}$. In particular, $|\mu| = m \leq p^k - 2$ so position $(0, p^k - 1)$ is also empty. Let B be the p^k-abacus for λ such that $B^\uparrow = A$. Let $\beta \in \mu^+_{p'}$ and let d be the addable bead in A such that $A^{d\rightarrow}$ (defined in the obvious way) is a p^k-abacus for β. Suppose d lies in position $(0, j)$ for some $j \in [p^k - 2]$ (and so position $(0, j+1)$ is empty). Surjectivity then follows from Lemma 3.50 and an analogous argument (using $A^{d\rightarrow}$ instead of $A^{\leftarrow d}$) to that in the proof of Theorem 3.15. □

We remark that the above corollary does not hold if $m = p^k - 1$: in this case $C_{p^k}(\lambda)^+_{p'} \subseteq \mathcal{P}(p^k)$, while $C_{p^k}(\mu) = \emptyset$ for $\mu \in \lambda^+_{p'}$. (This is analogous to the case of $m = 0$ in Theorem 3.15, since $\emptyset^-$ is undefined.)

Before we deduce that the core map in Corollary 3.51 is in fact also bijective when $a = 1$, we remark that we can now characterise when $0 \in \mathcal{E}^+_p(n)$.

Proposition 3.52. *Let $n \in \mathbb{N}_0$ and let p be a prime. Then*

$$0 \in \mathcal{E}^+_p(n) \quad \textit{if and only if} \quad \begin{cases} p \mid n+1 & \text{if } p \geq 5, \\ 9 \mid n+1 & \text{if } p = 3, \\ 8 \mid n+1 & \text{if } p = 2. \end{cases}$$

Proof. We prove this proposition in steps.

(1) If $p \nmid n+1$, then $|\lambda^+_{p'}| > 0$ for every $\lambda \vdash_{p'} n$: this is immediate since $\chi^\lambda \big\uparrow^{S_{n+1}}_{S_n}(1) = |S_{n+1} : S_n| \cdot \chi^\lambda(1) = (n+1) \cdot \chi^\lambda(1)$.

(2) If $p \geq 5$ and $p \mid n+1$, then $\lambda = (n - \frac{p-1}{2}, \frac{p-1}{2})$ satisfies $\lambda \vdash_{p'} n$ and $|\lambda^+_{p'}| = 0$: this clearly holds for $n = p-1$, and holds by inspection for $n > p-1$ by observing that $\lambda^+ = \{(n - \frac{p-1}{2} + 1, \frac{p-1}{2}), (n - \frac{p-1}{2}, \frac{p-1}{2} + 1), (n - \frac{p-1}{2}, \frac{p-1}{2}, 1)\}$ but $\lambda^+_{p'} = \emptyset$, by Theorem 3.12 (or the hook length formula).

(3) If $9 \nmid n+1$, then $|\lambda^+_{3'}| > 0$ for every $\lambda \vdash_{3'} n$: by (1), it remains to consider $n \in \mathbb{N}$ such that $3 \mid n+1$ but $9 \nmid n+1$. Let the 3-adic expansion of n be $a_t 3^{n_t} + \cdots + a_1 3^{n_1} + 3\delta + 2$

where $\delta \in \{0,1\}$, $t \in \mathbb{N}_0$, $2 \le n_1 < \cdots < n_t$ and $a_i \in \{1,2\}$ for all i. From Corollary 3.51, we have that

$$|\lambda^+_{3'}| \ge |C_{3^{n_t}}(\lambda)^+_{3'}| \ge \cdots \ge |C_{3^{n_1}}(\cdots(C_{3^{n_t}}(\lambda))\cdots)^+_{3'}| = |C_{3^{n_1}}(\lambda)^+_{3'}|$$

and $C_{3^{n_1}}(\lambda) \vdash_{3'} 3\delta+2 \in \{2,5\}$ by Theorem 3.12. By inspection, $|\mu^+_{3'}| > 0$ for all $\mu \vdash_{3'} 2$ and all $\mu \vdash_{3'} 5$. Thus $|\lambda^+_{3'}| > 0$.

(4) If $9 \mid n+1$, *then* $\lambda = (n-4,4)$ *satisfies* $\lambda \vdash_{3'} n$ *and* $|\lambda^+_{3'}| = 0$: this clearly holds for $n = 8$, and holds by inspection for $n > 8$ by observing that $\lambda^+ = \{(n-3,4),(n-4,5),(n-4,4,1)\}$ but $\lambda^+_{3'} = \emptyset$.

(5) If $8 \nmid n+1$, *then* $|\lambda^+_{2'}| > 0$ *for every* $\lambda \vdash_{2'} n$: let the binary expansion of n be $2^{n_t} + \cdots + 2^{n_1} + c$ where $t \in \mathbb{N}_0$, $3 \le n_1 < \cdots < n_t$ and $c \in \{1,3,5\}$ (by (1), we may now assume n is odd). Then by Corollary 3.51,

$$|\lambda^+_{2'}| \ge |C_{2^{n_t}}(\lambda)^+_{2'}| \ge \cdots \ge |C_{2^{n_1}}(\cdots(C_{2^{n_t}}(\lambda))\cdots)^+_{2'}| = |C_{2^{n_1}}(\lambda)^+_{2'}|$$

and $C_{2^{n_1}}(\lambda) \vdash_{2'} c$ by Theorem 3.12. By inspection, all odd partitions μ of $c \in \{1,3,5\}$ satisfy $|\mu^+_{2'}| = 2$. Hence $|\lambda^+_{2'}| > 0$.

(6) If $8 \mid n+1$, *then* $\lambda = (n-3,2,1)$ *satisfies* $\lambda \vdash_{2'} n$ *and* $|\lambda^+_{2'}| = 0$: this holds by inspection by observing that $\lambda^+ = \{(n-2,2,1),(n-3,3,1),(n-3,2,2),(n-3,2,1,1)\}$ but $\lambda^+_{2'} = \emptyset$. ☐

In fact, we can further characterise when $|\lambda^+_{p'}| = 0$ in terms of cores.

Lemma 3.53. *Let* p *be a prime. Suppose* $a \in [p-1]$, $k \in \mathbb{N}$ *and* $\lambda \vdash_{p'} ap^k - 1$. *Then*

- *(i)* $|\lambda^+_{p'}| \ge |C_{p^k}(\lambda)^+_{p'}|$, *and*
- *(ii)* $|\lambda^+_{p'}| = 0$ *if and only if* $|C_{p^k}(\lambda)^+_{p'}| = 0$.

Proof. By Theorem 3.12, $C_{p^k} \vdash_{p'} p^k - 1$, so the assertions are trivially true if $a = 1$. From now on we may assume $a \ge 2$.

(i) That $|\lambda^+_{p'}| \ge |C_{p^k}(\lambda)^+_{p'}|$ is clear if $|C_{p^k}(\lambda)^+_{p'}| = 0$, so suppose $\xi \in C_{p^k}(\lambda)^+_{p'}$. Then $\xi \vdash_{p'} p^k$, so ξ is a hook. In particular, $C_{p^k}(\lambda) \subset \xi$ must then also be a hook, so $C_{p^k}(\lambda) = (p^k-1-t, 1^t)$ of degree $\binom{p^k-2}{t}$ for some $t \in \{0,1,\dots,p^k-2\}$. Then $C_{p^k}(\lambda)^+ = \{(p^k-t,1^t),(p^k-1-t,1^{t+1}),(p^k-1-t,2,1^{t-1})\}$, where the first two elements are hooks of degree $\binom{p^k-1}{t}$ and $\binom{p^k-1}{t+1}$ respectively. Since

$$\tbinom{p^k-1}{t}(p^k-1-t) = \tbinom{p^k-2}{t}(p^k-1) = \tbinom{p^k-1}{t+1}(t+1)$$

and $p \nmid \binom{p^k-2}{t}$ as $C_{p^k}(\lambda)$ is a p'-partition, and also $(p^k-1-t,2,1^{t-1})$ is not a hook, we have that $C_{p^k}(\lambda)^+_{p'} = \{(p^k-t,1^t),(p^k-1-t,1^{t+1})\}$.

Letting A be the p^k-abacus for $C_{p^k}(\lambda)$ with first gap in $(0,0)$, A has beads precisely in positions (i,j) for all $i \le -1$ and all j, $(0,1),(0,2),\dots,(0,t)$ (if $t \ne 0$) and $(0,p^k-1)$. (That $|C_{p^k}(\lambda)^+_{p'}| = 2$ may also be verified on the abacus by using Lemma 3.50 and

Theorem 3.12.) Let B be the p^k-abacus for λ obtained from A by performing $w_{p^k}(\lambda) = a-1$ down-moves, so $B^\uparrow = A$. If d is an addable bead on runner B_{p^k-1} and ζ is represented by $B^{d\rightarrow}$, then $w_{p^k}(\zeta) - w_{p^k}(\lambda) = |B_{p^k-1}| - |B_0| = |A_{p^k-1}| - |A_0| = 1$ by Lemma 3.50, whence $w_{p^k}(\zeta) = a$. Moreover, $C_{p^k}(\zeta)$ is represented by the p^k-abacus A' obtained from A by deleting the bead in position $(0, p^k-1)$ and creating a bead in position $(0,0)$, showing that $C_{p^k}(\zeta) = \emptyset$. Thus $\zeta \vdash_{p'} ap^k$ by Theorem 3.12, and in particular $\zeta \in \lambda^+_{p'}$. Finally, since $|B_{p^k-1}| = |B_0| + 1$, there must be at least 2 addable beads on B_{p^k-1}. Thus $|\lambda^+_{p'}| \geq 2 = |C_{p^k}(\lambda)^+_{p'}|$.

(ii) To show that $|\lambda^+_{p'}| = 0$ if and only if $|C_{p^k}(\lambda)^+_{p'}| = 0$, we have already seen from (i) that if $|C_{p^k}(\lambda)^+_{p'}| > 0$, then $|\lambda^+_{p'}| > 0$. Conversely, if $|\lambda^+_{p'}| > 0$ then let $\beta \in \lambda^+_{p'}$. Thus $\beta \vdash_{p'} ap^k$, $C_{p^k} = \emptyset$ and $w_{p^k}(\beta) = a$. Let B be the p^k-abacus for β such that $B^\uparrow$ (representing the empty partition) has beads on all runners in rows $i < 0$, and is empty in rows $i \geq 0$. Then λ is represented by $B^{\leftarrow c}$ for some bead c on B_j, and moreover $j \neq 0$ by Lemma 3.10 since $w_{p^k}(\lambda) = a-1$. Hence $C_{p^k}(\lambda)$ is represented by $(B^{\leftarrow c})^\uparrow$, which is obtained from $B^\uparrow$ by deleting the bead in $(-1, j)$ and creating a bead in $(0, j-1)$, and we read off from the abacus that $C_{p^k}(\lambda) = (j, 1^{p^k-1-j})$. Setting $t = p^k - 1 - j$, we see from (i) that $C_{p^k}(\lambda)^+_{p'} = \{(p^k - t, 1^t), (p^k - 1 - t, 1^{t+1})\}$, which completes the proof. □

Proposition 3.54. *Let p be a prime. Suppose $n+1 \in \mathbb{N}$ has p-adic expansion $\sum_{i=1}^t a_i p^{n_i}$ where $t \in \mathbb{N}$ and $0 \leq n_1 < \cdots < n_t$. Let $\lambda \vdash_{p'} n$. Then $|\lambda^+_{p'}| = 0$ if and only if $|C_{p^{n_1}}(\lambda)^+_{p'}| = 0$.*

Proof. If $n_1 = 0$ then $p \nmid n+1$ and also $p \nmid |C_{p^{n_1}}(\lambda)| + 1$. Thus by Proposition 3.52, $|\lambda^+_{p'}| > 0$ and $|C_{p^{n_1}}(\lambda)^+_{p'}| > 0$. From now on, we may assume $n_1 \geq 1$.

When $t = 1$, the assertion follows from Lemma 3.53.

Now suppose $t \geq 2$. Since $C_e = C_e \circ C_{ef}$ for all $e, f \in \mathbb{N}$, by Theorem 3.12,

$$\alpha := C_{p^{n_2}}(\lambda) = C_{p^{n_2}}(C_{p^{n_3}}(\cdots C_{p^{n_t}}(\lambda)\cdots)) \vdash_{p'} a_1 p^{n_1} - 1.$$

Let A be the p^{n_2}-abacus for α with first gap in position $(0,0)$. Since $|\alpha| \leq p^{n_2} - 2$, there are no beads in rows $i \geq 1$ or in position $(0, p^{n_2}-1)$ of A.

First suppose $\mu \in \alpha^+_{p'}$. Then μ is represented by $A^{c\rightarrow}$ for some bead c in position $(0,j)$ of A where $j \in \{0, 1, \ldots, p^{n_2} - 2\}$. Let $\beta = C_{p^{n_3}}(\lambda)$, a p'-partition of $a_2 p^{n_2} + a_1 p^{n_1} - 1$ by Theorem 3.12. Then $\alpha = C_{p^{n_2}}(\beta)$ and $w_{p^{n_2}}(\beta) = a_2$, and there is a p^{n_2}-abacus B representing β obtained from A by performing a_2 down-moves. Since $|B_j| = |A_j| = |A_{j+1}| + 1 = |B_{j+1}| + 1$, there exists an addable bead d on runner B_j. The partition ν represented by $B^{d\rightarrow}$ satisfies $w_{p^{n_2}}(\nu) = a_2$ by Lemma 3.50, and $C_{p^{n_2}}(\nu) = \mu$ since $B^\uparrow = A$. Thus ν is a p'-partition by Theorem 3.12, so $\nu \in \beta^+_{p'}$. Next we consider β on a p^{n_3}-abacus A' with first gap in position $(0,0)$, so ν is represented by $(A')^{c'\rightarrow}$ for some bead c' in position $(0,j')$ of A' where $j' \in \{0, 1, \ldots, p^{n_3} - 2\}$ since $|\beta| \leq p^{n_3} - 2$. Letting $\gamma = C_{p^{n_4}}(\lambda)$, we deduce as above that there exists some $\omega \in \gamma^+_{p'}$. Iterating this procedure, we produce a partition in $C_{p^{n_4}}(\lambda)^+_{p'}, \ldots, C_{p^{n_t}}(\lambda)^+_{p'}$, and finally $\lambda^+_{p'}$.

Conversely, suppose $\xi \in \lambda^+_{p'}$. By Corollary 3.51,

$$C_{p^{n_2}}(\xi) = C_{p^{n_2}}(C_{p^{n_3}}(\cdots C_{p^{n_t}}(\xi)\cdots)) \in \big(C_{p^{n_2}}(C_{p^{n_3}}(\cdots C_{p^{n_t}}(\lambda)\cdots))\big)^+_{p'} = \big(C_{p^{n_2}}(\lambda)\big)^+_{p'}.$$

Thus we have shown that $|\lambda^+_{p'}| = 0$ if and only if $|\alpha^+_{p'}| = 0$. Finally, by Lemma 3.53 we have that $|\alpha^+_{p'}| = 0$ if and only if $|C_{p^{n_1}}(\lambda)^+_{p'}| = 0$, since $C_{p^{n_1}}(\alpha) = C_{p^{n_1}}(\lambda)$. □

Remark 3.55. Proposition 3.54 allows us to we recover the characterisation of when $|\lambda^+_{2'}| = 0$ in Theorem 3.48 by observing that $|C_{p^{n_1}}(\lambda)^+_{p'}| = 0$ if and only if $C_{p^{n_1}}(\lambda)$ is not a hook. This is because $C_{p^{n_1}}(\lambda) \vdash_{p'} p^{n_1} - 1$, and the p'-partitions of p^{n_1} are precisely the hook partitions. ♢

Finally, we characterise when $1 \in \mathcal{E}^+(n)$, noting for contrast that $1 \in \mathcal{E}(n)$ for all $n \in \mathbb{N}$ in the case of restriction.

Lemma 3.56. *Let $n \in \mathbb{N}_0$ and let p be a prime. Then $1 \in \mathcal{E}^+(n)$ if and only if $p \mid n$.*

Proof. If $p \mid n$, then $\lambda = (n) \vdash_{p'} n$ and $\lambda^+_{p'} = \{(n+1)\}$. Thus $1 \in \mathcal{E}^+(n)$.

Conversely, suppose $\lambda \vdash_{p'} n$ with $|\lambda^+_{p'}| = 1$. Write $n + 1 = \sum_{i=1}^t a_i p^{n_i}$ where $t \in \mathbb{N}$ and $0 \leq n_1 < \cdots < n_t$. If $t \geq 2$, let $\alpha := C_{p^{n_2}}(\lambda)$ and observe by Corollary 3.51 that

$$|\lambda^+_{p'}| \geq |C_{p^{n_t}}(\lambda)^+_{p'}| \geq |C_{p^{n_2}}(\cdots(C_{p^{n_t}}(\lambda))\cdots)^+_{p'}| = |C_{p^{n_2}}(\lambda)^+_{p'}| = |\alpha^+_{p'}|,$$

while if $t = 1$ then set $\alpha := \lambda$. In all cases, $\alpha \vdash_{p'} a_1 p^{n_1} - 1$ by Theorem 3.12. Let $\beta = C_{p^{n_1}}(\lambda)$. Then $|\beta^+_{p'}| > 0$ by Proposition 3.54 since $|\lambda^+_{p'}| > 0$.

If $n_1 \geq 1$, then $|\alpha^+_{p'}| \geq |\beta^+_{p'}|$ by Lemma 3.53. But from part (i) of the proof of Lemma 3.53 we find that $|\beta^+_{p'}| > 0$ implies $|\beta^+_{p'}| = 2$, contradicting $1 = |\lambda^+_{p'}| \geq |\alpha^+_{p'}|$. Hence $n_1 = 0$.

Thus $\alpha \vdash a_1 - 1$. But then $1 = |\lambda^+_{p'}| \geq |\alpha^+_{p'}| = |\alpha^+| = |\alpha^-| + 1$, from which we deduce $|\alpha^-| = 0$. Thus implies $\alpha = \emptyset$, the unique partition of zero; in particular, $a_1 = 1$. Hence $n + 1 = \sum_{i=2}^t a_i p^{n_i} + 1$ (where $n_2 \geq 1$ if $t \geq 2$), and so $p \mid n$. □

Returning to the discussion of the crucial core map of Corollary 3.51, we now set up some constructions analogous to those regarding character restrictions in Section 3.1, and analyse the differences that arise.

Notation 3.57. *Unless otherwise stated, we fix a prime p and $n \in \mathbb{N}$ such that $n = ap^k + m$ for some $k \in \mathbb{N}$, $a \in [p-1]$ and $0 \leq m \leq p^k - 2$. To be precise, this will be the standing assumption from here until the end of Section 3.2.2.*

Given $\gamma \vdash_{p'} m$, we may now define

$$br^+_p(n, \gamma) = br^+(n, \gamma) : \max\{|\lambda^+_{p'}| \mid \lambda \vdash_{p'} n \text{ and } C_{p^k}(\lambda) = \gamma\},$$

and

$$N^+(a, p^k, \gamma) := br^+(n, \gamma) - |\gamma^+_{p'}| \in \mathbb{N}_0.$$

In order to determine $N^+(a, p^k, \gamma)$, we need to analyse certain properties of the functions f, g, Φ and M.

Definition 3.58. *Let $\gamma \vdash_{p'} m$. Let A_γ be the p^k-abacus for γ with first gap $(0, p^k - 1)$. Define $\mathcal{R}_{A_\gamma}$ to be the subset of $\{0, 1, \ldots, p^k - 1\}$ such that $j \in \mathcal{R}_{A_\gamma}$ if and only if there is an addable bead c on runner j of A_γ and the partition corresponding to $A_\gamma^{c\to}$ has p'-degree.*

Since $m \leq p^k - 2$, we deduce that there are no beads in A_γ in rows $i \geq 2$ or in positions $(1, p^k - 2)$ and $(1, p^k - 1)$. Notice that there is an addable bead in position $(0, p^k - 2)$, and all other addable beads lie in $(1, j)$ for some $0 \leq j \leq p^k - 3$. Moreover, observe that $|\mathcal{R}_{A_\gamma}| = |\gamma_{p'}^+|$, $p^k - 1 \notin \mathcal{R}_{A_\gamma}$, and $\{j, j+1\} \subseteq \mathcal{R}_{A_\gamma}$ implies $j = p^k - 3$. In particular, if there *is* a bead in A_γ in position $(1, p^k - 3)$, then γ is necessarily a hook partition and $m = p^k - 2$. This is because if the number of beads occupying positions $(1, j)$ where $0 \leq j < p^k - 3$ in A_γ is t, then each of those beads corresponds to a part of γ, and the bead in $(1, p^k - 3)$ corresponds to $\gamma_1 = p^k - 2 - t$. Hence $m = |\gamma| \geq (p^k - 2 - t) + t \cdot 1 = p^k - 2$, whence equality holds. Thus if $\{p^k - 3, p^k - 2\} \subseteq \mathcal{R}_{A_\gamma}$, then in fact $|\mathcal{R}_{A_\gamma}| \in \{2, 3\}$ since γ is a hook.

Lemma 3.59. *Let $\gamma \vdash_{p'} m$. Let $\lambda \vdash_{p'} n$ satisfy $C_{p^k}(\lambda) = \gamma$ and let B be the p^k-abacus for γ such that $B^\uparrow = A_\gamma$. Let c be an addable bead on B_j and suppose $B^{c\to}$ represents $\mu \vdash n + 1$. Then $p \nmid \chi^\mu(1)$ if and only if $j \in \mathcal{R}_{A_\gamma}$. In particular,*

$$|\lambda_{p'}^+| = \sum_{j \in \mathcal{R}_{A_\gamma}} \mathrm{Add}(B_j).$$

Proof. The proof is entirely analogous to that of Lemma 3.20. □

Proposition 3.60. *Let $\gamma \vdash_{p'} m$. Then*

$$N^+(a, p^k, \gamma) = \begin{cases} g(a) & \text{if } \mathcal{R}_{A_\gamma} = \{p^k - 3, p^k - 2\}, \\ M(a) & \text{if } \{p^k - 3, p^k - 2\} \subsetneqq \mathcal{R}_{A_\gamma}, \\ \Phi(a, |\gamma_{p'}^+|) & \text{otherwise.} \end{cases}$$

The fact that $\mathcal{R}_{A_\gamma}$ may now contain consecutive integers, in contrast to Definition 3.19, gives rise to the more complex behaviour of the quantity $N^+(a, p^k, \gamma)$. We begin by dealing with the familiar case.

Lemma 3.61. *Let $\gamma \vdash_{p'} m$. If $\{p^k - 3, p^k - 2\} \not\subseteq \mathcal{R}_{A_\gamma}$, then $N^+(a, p^k, \gamma) = \Phi(a, |\gamma_{p'}^+|)$.*

Proof. Let $\mathcal{R}_{A_\gamma} = \{j_1, \ldots, j_L\}$, where $L = |\gamma_{p'}^+|$. Under the given assumption, $\mathcal{R}_{A_\gamma}$ contains no consecutive integers. Thus we may regard $(B_{j_i}, B_{j_i+1}), \ldots, (B_{j_L}, B_{j_L+1})$ as L disjoint 2-abaci whose 2-cores are equal to the 2-abacus representing $\emptyset$ with first gap $(0, 1)$ (resp. $(1, 1)$) if $j_i = p^k - 2$ (resp. $j_i \neq p^k - 2$). Let V be the 2-abacus for $\emptyset$ with first gap $(0, 1)$.

Observe that for $x \in \mathbb{N}_0$ and W a 2-abacus such that $W^{\uparrow} = V$ and $w(W) = x$, $\mathrm{Add}(W_0)$ is equal to $\mathrm{Rem}(U_1)$ for a 2-abacus $U \in \mathcal{T}_{(1)}(x)$ as defined in Lemma 3.22. (Consider flipping or mirroring the 2-abaci about a vertical axis.) Hence

$$\max\{\mathrm{Add}(W_0) \mid W^{\uparrow} = V,\ w(W) = x\} = f(x) + 1.$$

That $N^+(a, p^k, \gamma) = \Phi(a, L)$ then follows by the same argument as in the proof of Proposition 3.18. ☐

In order to analyse the case when $\{p^k - 3, p^k - 2\} \subseteq \mathcal{R}_{A_\gamma}$, we have the following lemma.

Lemma 3.62. *Let X be the 3-abacus for (1) with first gap $(0, 2)$. For $b \in \mathbb{N}_0$, let $\mathcal{Y}(b)$ be the set of all 3-abaci Y such that $Y^{\uparrow} = X$ and $w(Y) = b$. Then*

$$\max\{\mathrm{Add}(Y_0) + \mathrm{Add}(Y_1) \mid Y \in \mathcal{Y}(b)\} = 2 + g(b).$$

Proof. The claim is clear if $b \in \{0, 1\}$, so from now on we may assume $b \geq 2$ (in particular $f(b) \geq 1$). First, we exhibit Y such that $\mathrm{Add}(Y_0) + \mathrm{Add}(Y_1) = 2 + g(b)$.

Notice that $\mathrm{Add}(X_0) = \mathrm{Add}(X_1) = 1$. For $u \in \mathbb{N}$, let $Z(u)$ be the 3-abacus with beads in precisely $\{(i, 0) \mid i \leq -1\} \cup \{(i, 1) \mid i \leq -u - 1,\ i = 0,\ 2 \leq i \leq u + 1\} \cup \{(i, 2) \mid i \leq -1\}$. Observe that

$$Z(u)^{\uparrow} = X, \quad w(Z(u)) = u(u + 2) \quad \text{and} \quad \mathrm{Add}(Z(u)_0) + \mathrm{Add}(Z(u)_1) = 2u + 2.$$

Let $Z'(u)$ be the 3-abacus with beads in precisely $\{(i, 0) \mid i \leq 1\} \cup \{(i, 1) \mid i \leq -u,\ 2 \leq i \leq u + 1\} \cup \{(i, 2) \mid i \leq -1\}$. Then

$$Z'(u)^{\uparrow} = X, \quad w(Z'(u)) = u(u + 1) \quad \text{and} \quad \mathrm{Add}(Z'(u)_0) + \mathrm{Add}(Z'(u)_1) = 2u + 1.$$

The abaci X, $Z(u)$ and $Z'(u)$ are depicted in Figure 3.8.

If $g(b) = 2f(b)$, then the abacus $Y \in \mathcal{Y}(b)$ obtained from $Z(f(b))$ by performing $b - f(b) \cdot (f(b) + 2)$ down-moves on the bead in position $(f(b) + 1, 1)$ satisfies $\mathrm{Add}(Y_0) + \mathrm{Add}(Y_1) = 2f(b) + 2 = g(b) + 2$. On the other hand, if $g(b) = 2f(b) - 1$ then the abacus $Y' \in \mathcal{Y}(b)$ obtained from $Z'(f(b))$ by performing $b - f(b) \cdot (f(b) + 1)$ down-moves on the bead in position $(f(b) + 1, 1)$ satisfies $\mathrm{Add}(Y'_0) + \mathrm{Add}(Y'_1) = 2f(b) + 1 = g(b) + 2$.

Next, suppose $A \in \mathcal{Y}(b)$ is such that $a := \mathrm{Add}(A_0) + \mathrm{Add}(A_1)$ is maximal. From above, we already know that $a \geq 2 + g(b)$. By a similar argument to the proof of Lemma 3.22, we can construct from A a 3-abacus $B \in \mathcal{Y}(b')$ such that $w(B) = w(B_1) = b'$ (i.e. $w(B_0) = w(B_2) = 0$) and $\mathrm{Add}(B_0) + \mathrm{Add}(B_1) = a$, for some $b' \leq b$. Notice that there must be a bead in position $(i, 1)$ of B for all $i \leq -b$, since $w(B) \leq b$. Thus there are exactly b beads in $(i, 1)$ of B where $i > -b$, since $B^{\uparrow} = X$. Moreover, $\mathrm{Add}(B_0) = |\{j \in \{-b + 1, \ldots, -1, 0, 1\} \mid B_1 \text{ has a gap in row } j\}|$ and $\mathrm{Add}(B_1) = |\{j \geq$

	0	1	2
		⋮	
-1	×	×	×
0	×	×	○
1	×	○	○
2	○	○	○
		⋮	

	0	1	2
		⋮	
$-u-1$	×	×	×
$-u$	×	○	×
⋮		⋮	
-1	×	○	×
0	×	×	○
1	×	○	○
2	○	×	○
⋮		⋮	
$u+1$	○	×	○
$u+2$	○	○	○
		⋮	

	0	1	2
		⋮	
$-u$	×	×	×
$-u+1$	×	○	×
⋮		⋮	
-1	×	○	×
0	×	○	○
1	×	○	○
2	○	×	○
⋮		⋮	
$u+1$	○	×	○
$u+2$	○	○	○
		⋮	

Figure 3.8: The 3-abaci X, $Z(u)$ and $Z'(u)$.

$0 \mid B_1$ has a bead in row $j\}|$. Hence

$$a = \mathrm{Add}(B_0) + \mathrm{Add}(B_1) = 1 + \delta_0 - \delta_1 + 2u$$

where $u := |\{j > 0 \mid B_1 \text{ has a bead in row } j\}|$, and $\delta_i = 1$ if B_1 has a bead in row i and $\delta = 0$ otherwise. In particular, $u \geq 1$ since $b \geq 2$. We split into four cases depending on the values of $\delta_0, \delta_1 \in \{0, 1\}$.

If $\delta_1 = 1$ and $\delta_0 = 0$, then $a = 2u$. Moreover, the $u - 1$ beads in rows $i > 1$ of B_1 must occupy rows $i_2 < \ldots < i_u$ where $i_j \geq j$ for all $j \in \{2, \ldots, u\}$. Since $B_1^{\uparrow} = X_1$ (noting that the lowest u beads on X_1 lie in rows $0, -1, \ldots, -u+1$), we must have

$$b \geq b' = w(B) \geq u(u-1) + u = u^2.$$

Thus $u - 1 \leq f(b)$ and so $a = 2u \leq 2f(b) + 2$. We know that $g(b) \in \{2f(b) - 1, 2f(b)\}$; suppose $g(b) = 2f(b) - 1$. By definition of g, we must have $f(b) > \zeta(b)$ and hence $f(b) \cdot (f(b) + 1) \leq b < f(b) \cdot (f(b) + 2)$. But $a \geq 2 + g(b) = 2f(b) + 1$ implies $u = f(b) + 1$, giving $b \geq u^2 > f(b) \cdot (f(b) + 2)$, a contradiction. Thus $g(b) = 2f(b)$ and so $a \leq 2f(b) + 2 = g(b) + 2$. The other three cases are similar. □

Lemma 3.63. *Let $\gamma \vdash_{p'} m$. If $\{p^k - 3, p^k - 2\} \subseteq \mathcal{R}_{A_\gamma}$, then $\mathcal{R}_{A_\gamma} \in \{2, 3\}$ and*

$$N^+(a, p^k, \gamma) = \begin{cases} g(a) & \text{if } |\mathcal{R}_{A_\gamma}| = 2, \\ M(a) & \text{if } |\mathcal{R}_{A_\gamma}| = 3. \end{cases}$$

Proof. We have already shown following Definition 3.58 that if $\{p^k - 3, p^k - 2\} \subseteq \mathcal{R}_{A_\gamma}$, then γ is a hook and hence $\mathcal{R}_{A_\gamma} \in \{2, 3\}$. Let $\lambda \vdash_{p'} n$ satisfy $C_{p^k}(\lambda) = \gamma$ and $|\lambda_{p'}^+| = br^+(n, \gamma)$, and let B be the p^k-abacus for λ such that $B^{\uparrow} = A_\gamma$.

If $\mathcal{R}_{A_\gamma} = \{p^k - 3, p^k - 2\}$, then $|\lambda_{p'}^+| = \mathrm{Add}(B_{p^k-3}) + \mathrm{Add}(B_{p^k-2})$ by Lemma 3.59.

Therefore $|\lambda^+_{p'}| = 2 + g(a)$ by Lemma 3.62 and the maximality of $|\lambda^+_{p'}|$.

Otherwise, suppose $|\mathcal{R}_{A_\gamma}| = 3$. If γ is the hook $(p^k - 2 - t, 1^t)$, then $|\gamma^+| \geq |\gamma^+_{p'}| = 3$ implies $0 < t < p^k - 3$. In fact, the beads in row 1 of A_γ lie precisely in runners $0, 1, \dots, t-1$, and we deduce that $\mathcal{R}_{A_\gamma} = \{t-1, p^k - 3, p^k - 2\}$. Since the sets of runners (B_{t-1}, B_t) and $(B_{p^k-3}, B_{p^k-2}, B_{p^k-1})$ are disjoint, the claim $N^+(a, p^k, \gamma) = M(a)$ follows from the maximality of $|\lambda^+_{p'}|$, the definition of $M(a)$ (see Definition 3.49), and Lemmas 3.59 and 3.62. □

Proof of Proposition 3.60. This follows from Lemmas 3.61 and 3.63. □

Corollary 3.64. *The core map in Corollary 3.51 is bijective when $a = 1$.*

Proof. This follows from Proposition 3.60 and the fact that $g(1) = M(1) = \Phi(1, L) = 0$ for all $L \in \mathbb{N}$. □

3.2.3 Proofs of main results

Proposition 3.65. *Let p be a prime, $a \in [p-1]$ and $k \in \mathbb{N}_0$. Then*

$$br^+(ap^k - 1) = \begin{cases} \lfloor \sqrt{a} \rfloor + 1 & \text{if } k > 0, \\ f(2a-2) + 1 & \text{if } k = 0 \end{cases}$$

and $\{2, 3, \dots, br^+(ap^k - 1)\} \subseteq \mathcal{E}^+(ap^k - 1)$ whenever $ap^k - 1 > 0$.

Proof. We begin with the case when $k = 0$, which is in analogy with Proposition 3.30. When $a = 1$ then clearly $br^+(ap^k - 1) = br^+(0) = 1$, and $f(0) = 0$. Next, observe that $|\lambda^+| = |\lambda^-| + 1$ for any partition λ, and $\lambda^+ = \lambda^+_{p'}$ if $|\lambda| \leq p - 2$. Thus for $a > 1$ we have that $br^+(a-1) = f(2a-2) + 1$, since $f(2a-2)$ is the maximal number of parts of distinct size in any partition of $a - 1$, and hence the maximal number of removable boxes in any partition of $a - 1$. Moreover, it is easy to see that there exists $\lambda \vdash a - 1$ such that $|\lambda^-| = m$ for any $m \in \{1, 2, \dots, f(2a-2)\}$, and hence $|\lambda^+| = m + 1 \in \{2, \dots, f(2a-2), f(2a-2) + 1\}$.

From now on, we may assume that $k > 0$. We proceed in steps, showing that:

(i) $2 \in \mathcal{E}^+(ap^k - 1)$;

(ii) $\{3, 4, \dots, \lfloor \sqrt{a} \rfloor + 1\} \subseteq \mathcal{E}^+(ap^k - 1)$ (if $a \geq 4$); and

(iii) $br^+(ap^k - 1) \leq \lfloor \sqrt{a} \rfloor + 1$.

(i) Writing $n = ap^k - 1$, we have $\lambda = (n) \vdash_{p'} n$ and $\lambda^+ = \{(n+1), (n, 1)\} = \lambda^+_{p'}$. Hence $2 \in \mathcal{E}^+(ap^k - 1)$.

(ii) We show that whenever $a \geq (b+2)^2$ for some $b \in \mathbb{N}_0$, then $b + 3 \in \mathcal{E}^+(ap^k - 1)$, from which the claim that $\{3, 4, \dots, \lfloor \sqrt{a} \rfloor + 1\} \subseteq \mathcal{E}^+(ap^k - 1)$ follows by setting $b + 2 = 2, 3, \dots, \lfloor \sqrt{a} \rfloor$. We exhibit a partition $\lambda \vdash_{p'} ap^k - 1$ such that $|\lambda^+_{p'}| = b + 3$, describing below and depicting a p^k-abacus for λ with first gap in $(0, q-1)$ in Figure 3.9 (where $q = p^k$ for convenience):

· rows $0 \le x \le b+1$ have a gap only in position $(x, q-1)$;
· rows $b+2 \le x \le 2b+1$ have a bead only in position $(x, q-1)$;
· row $2b+2$ has a bead only in position $(2b+2, q-2)$;
· rows $2b+3 \le x \le 2b+t+2$ are empty;
· row $2b+t+3$ has a bead only in position $(2b+t+3, q-1)$.

Explicitly, λ is the following partition:

$$\lambda = \big((b+t+2)q, (b+1)q, bq+2, \dots, 2q+b, q+b+1, (b+1)^{q-1} \dots, 2^{q-1}, 1^{q-1}\big).$$

	0	$\cdots$	$q-3$	$q-2$	$q-1$
0	$\times$	$\cdots$	$\times$	$\times$	$\circ$
$\vdots$	$\vdots$				$\vdots$
$b+1$	$\times$	$\cdots$	$\times$	$\times$	$\circ$
$b+2$	$\circ$	$\cdots$	$\circ$	$\circ$	$\times$
$\vdots$	$\vdots$				$\vdots$
$2b+1$	$\circ$	$\cdots$	$\circ$	$\circ$	$\times$
$2b+2$	$\circ$	$\cdots$	$\circ$	$\times$	$\circ$
$2b+3$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$
$\vdots$	$\vdots$				$\vdots$
$2b+t+2$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$
$2b+t+3$	$\circ$	$\cdots$	$\circ$	$\circ$	$\times$

Figure 3.9: The p^k-abacus for $\lambda \vdash_{p'} ap^k - 1$ with first gap $(0, q-1)$.

From the abacus it is clear that $w_q(\lambda) = a-1$ and $C_q(\lambda) = (q-1) \vdash_{p'} q-1$, whence $\lambda \vdash_{p'} ap^k - 1$ by Theorem 3.12. By Lemma 3.50, the $b+3$ addable beads on runner $q-2$ (in rows $0, 1, \dots, b+1, 2b+2$) correspond to elements of $\lambda^+_{p'}$, while the $b+1$ addable beads on runner $q-1$ (in rows $b+2, \dots, 2b+1, 2b+t+3$) correspond to elements of $\lambda^+ \setminus \lambda^+_{p'}$, and there are no other addable beads. Hence $|\lambda^+_{p'}| = b+3$ as claimed.

(iii) Let $\lambda \vdash_{p'} ap^k - 1$ and suppose $|\lambda^+_{p'}| > 0$. If $a = 1$, then $\lambda^+_{p'}$ is the set of hook partitions in λ^+. Thus $|\lambda^+_{p'}| > 0$ implies that λ itself is a hook. Therefore $|\lambda^+_{p'}| = 2$. Since $\lambda \vdash_{p'} p^k - 1$ was arbitrary, $br^+(p^k - 1) \le 2$.

Now suppose $a \in \{2, \dots, p-1\}$ and fix some $\mu \in \lambda^+_{p'}$. In particular, $\mu \vdash_{p'} ap^k$ so $C_{p^k} = \emptyset$ and $w_{p^k}(\mu) = a$. Let A be the p^k-abacus for $C_{p^k}(\mu)$ with first gap in $(0,0)$ (so all rows $x < 0$ are filled, and all rows $x \ge 0$ are empty). Let B be the p^k-abacus for μ such that $B^\uparrow = A$, so B is obtained from A by performing a down-moves. Since $\lambda \in \mu^-_{p'}$, λ is represented by the abacus $X := B^{\leftarrow c}$ for some removable bead c in B. But $w_{p^k}(\lambda) = a-1$ since $\lambda \vdash_{p'} ap^k - 1 = (a-1)p^k + (p-1)\sum_{i=0}^{k-1} p^i$, by Theorem 3.12, so c lies on runner B_j for some $j \ne 0$ by Lemma 3.10. Thus the abacus $X^\uparrow$ which represents $C_{p^k}(\lambda)$ has form as depicted in Figure 3.10 (where if $j = 1$ then we omit the column labelled $j-2$, and if $j = p^k - 1$ then we omit column $j+1$):

	0	⋯	$j-2$	$j-1$	j	$j+1$	⋯	p^k-1
-2	×	⋯	×	×	×	×	⋯	×
-1	×	⋯	×	×	∘	×	⋯	×
0	∘	⋯	∘	×	∘	∘	⋯	∘
1	∘	⋯	∘	∘	∘	∘	⋯	∘

Figure 3.10: The p^k-abacus $X^{\uparrow}$ for $C_{p^k}(\lambda)$.

For any $\nu \in \lambda^+_{p'}$, ν is represented by $X^{d\to}$ for some addable bead d in X. Since $w_{p^k}(\nu) = a$ and $w_{p^k}(\lambda) = a-1$, by Lemma 3.50 the bead d must lie on runner X_{j-1}, as $|X_i| - |X_{i+1}| - 1 = 1$ if and only if $i = j-1$ (when $i \neq p^k - 1$) and $|X_{p^k-1}| - |X_0| \neq 1$. Therefore $\lambda^+_{p'}$ is in bijection with the set of addable beads on X_{j-1}. It remains to show that the maximal number m of addable beads on runner $j-1$ attained after performing $w_{p^k}(\lambda) = a-1$ down-moves on the p^k-abacus $X^{\uparrow}$ is (at most) $\lfloor\sqrt{a}\rfloor + 1$.

Observe that m is also the maximal number of addable beads on runner 0 attained after performing *at most* $a-1$ down-moves on T, the 2-abacus for the partition (1) with first gap $(0,-1)$. (As runners, T_0 coincides with $X^{\uparrow}_{j-1}$ and T_1 with $X^{\uparrow}_j$.) By a similar argument to that in the proof of Lemma 3.22, this optimum m is achieved by a 2-abacus U such that $U^{\uparrow} = T$, $w(U_1) = 0$ and $w(U_0) \leq a-1$. Thus there exist integers $-1 \leq j_1 < j_2 < \cdots < j_m$ such that there is a bead in position $(j_k, 0)$ of U for all $k \in [m]$. Hence $w(U) = w(U_0) \geq m(m-2)$, since the beads on T_0 occupied precisely those rows $x \leq 0$. But then if $m \geq \lfloor\sqrt{a}\rfloor + 2$, then $m(m-2) \geq (\lfloor\sqrt{a}\rfloor+1)^2 - 1 > a-1$, a contradiction since $w(U_0) \leq a-1$. Thus $m \leq \lfloor\sqrt{a}\rfloor + 1$. □

Proof of Theorem 3.46. This follows directly from Proposition 3.65. □

We can now fully recover Theorem 3.48 (Theorem 2 of [1]) by setting $p = 2$ in our results thus far.

Corollary 3.66. *Let $n \in \mathbb{N}_0$. Then $br^+_2(n) = 1$ if n is even, while $br^+_2(n) = 2$ if n is odd. Moreover,*

$$\mathcal{E}^+_2(n) = \begin{cases} \{1\} & \text{if } 2 \mid n, \\ \{0,2\} & \text{if } 8 \mid n+1, \\ \{2\} & \text{otherwise.} \end{cases}$$

Proof. Suppose $n+1$ has binary expansion $\sum_{j=1}^t 2^{n_j}$ where $t \in \mathbb{N}$ and $0 \leq n_1 < \cdots < n_t$. For all $\lambda \vdash_{2'} n = \sum_{j=2}^t 2^{n_j} + (2^{n_1} - 1)$, we have $|\lambda^+_{2'}| = |C_{2^{n_2}}(\lambda)^+_{2'}|$ by Corollary 3.64, where we note that $C_{2^{n_2}}(\lambda) \vdash_{2'} 2^{n_1} - 1$ by Theorem 3.12. Also by Theorem 3.12, given any $\mu \vdash_{2'} 2^{n_1} - 1$ there exists some $\lambda \vdash_{2'} n$ such that $C_{2^{n_2}}(\lambda) = \mu$. Hence $br^+_2(n) = br^+_2(2^{n_1} - 1)$. If n is even (i.e. $n_1 = 0$), then $br^+_2(n) = br^+_2(0) = 1$. If n is odd, then $br^+_2(n) = 2$ by Proposition 3.65. The final assertion then follows from Proposition 3.52 and Lemma 3.56. □

We remark that Corollary 3.66 is exactly the statement of Theorem 3.44 and Theorem 3.45 when $p = 2$, since the only non-zero binary digit is 1 and $\Phi(1, L) = 0$ for all

$L \in \mathbb{N}$. Therefore, for the remainder of this section we may assume p is an odd prime.

In order to prove a recursive formula for the value of $br^+(n)$ (namely Theorem 3.44), we first investigate a single step of the recursion. That is, we relate the quantities $br^+(n)$ and $br^+(m)$, where $n = ap^k + m$ with $a \in [p-1]$ and $m < p^k - 1$.

Proposition 3.67. *Let p be an odd prime and $k \in \mathbb{N}$. Let $n = ap^k + m$ where $a \in [p-1]$ and $0 \leq m \leq p^k - 2$. Then*

$$br^+(n) = br^+(m) + \Phi(a, br^+(m)) + \delta$$

where $\delta = 1$ if $k = 1$, $m = p - 2$ and $(p, a) \in \{(5, 3), (7, 3), (7, 5)\}$, and $\delta = 0$ otherwise.

Proof. If $m \neq p^k - 2$, then $\{p^k - 3, p^k - 2\} \not\subseteq \mathcal{R}_{A_\gamma}$ for all $\gamma \vdash_{p'} m$ (recall the set $\mathcal{R}_{A_\gamma}$ from Definition 3.58) and thus $br^+(n) = br^+(m) + \Phi(a, br^+(m))$ by exactly the same argument as in Proposition 3.23. Notice that $\delta = 0$ in this case.

From now on, we may suppose $m = p^k - 2$. Let $\gamma \vdash_{p'} m$ be such that $|\gamma_{p'}^+| = br^+(m)$. Let $\lambda \vdash_{p'} n$ be such that $C_{p^k}(\lambda) = \gamma$ and $|\lambda_{p'}^+| = br^+(n, \gamma)$. Since $g(a), M(a), \Phi(a, L) \geq 1$ whenever $a \geq 2$ (for all $L \in \mathbb{N}$), then $br^+(n) \geq |\lambda_{p'}^+| \geq br^+(m) + 1$ by Proposition 3.60. Hence we have the following inequality:

$$br^+(n) \geq br^+(m) + 1 \quad \text{if} \quad a \geq 2. \tag{3.1}$$

First, suppose $br^+(m) \leq 3$. Since $m = p^k - 2 = (p-1)\sum_{i=1}^{k-1} p^i + (p-2)$, then $br^+(m) \geq br^+(p-2) + k - 1$ by (3.1). By Proposition 3.65, this implies $k + f(2p-4) \leq 3$ since $p - 1 \geq 2$. Hence $(p, k) \in \{(3, 1), (3, 2), (5, 1), (7, 1)\}$. We find by direct computation that $br^+(n) = br^+(m) + \Phi(a, br^+(m))$ in all of these cases *except* if $(p, k, a) \in \{(5, 1, 3), (7, 1, 3), (7, 1, 5)\}$, in which case $br^+(n) = br^+(m) + \Phi(a, br^+(m)) + 1$.

We may now assume that $br^+(m) \geq 4$; in particular, $\delta = 0$. Since $|\gamma_{p'}^+| = br^+(m)$, the partition γ cannot be a hook. In particular, $\{p^k - 3, p^k - 2\} \not\subseteq \mathcal{R}_{A_\gamma}$ and so $N^+(a, p^k, \gamma) = \Phi(a, |\gamma_{p'}^+|)$, by Proposition 3.60. Thus $br^+(n) \geq br^+(n, \gamma) = br^+(m) + \Phi(a, br^+(m))$.

Let $\alpha \vdash_{p'} n$ be such that $|\alpha_{p'}^+| = br^+(n)$. Let $\beta = C_{p^k}(\alpha)$, so $\beta \vdash_{p'} m$ and $|\beta_{p'}^+| \leq br^+(m)$. If $\{p^k - 3, p^k - 2\} \not\subseteq \mathcal{R}_{A_\beta}$, then $N^+(a, p^k, \beta) = \Phi(a, |\beta_{p'}^+|)$ and hence

$$br^+(n) = |\alpha_{p'}^+| = |\beta_{p'}^+| + \Phi(a, |\beta_{p'}^+|) \leq br^+(m) + \Phi(a, br^+(m)) \leq br^+(n).$$

Thus $br^+(n) = br^+(m) + \Phi(a, br^+(m))$.

On the other hand, if $\{p^k - 3, p^k - 2\} \subseteq \mathcal{R}_{A_\beta}$, then $N^+(a, p^k, \beta) = g(a)$ or $M(a)$, in which case $|\beta_{p'}^+| = 2$ or 3 respectively. We claim that $br^+(m) + \Phi(a, br^+(m)) \geq 3 + M(a) > 2 + g(a)$ for any $a \in [p-1]$: for clarity, this is proven separately in Lemma 3.68 below. But this lemma then gives us

$$br^+(n) \geq br^+(m) + \Phi(a, br^+(m)) \geq |\beta_{p'}^+| + N^+(a, p^k, \beta) = br^+(n, \beta) = br^+(n),$$

whence $br^+(n) = br^+(m) + \Phi(a, br^+(m))$ as desired. □

Lemma 3.68. *Let the notation be as in Proposition 3.67, and further suppose that* $m = p^k - 2$ *and* $br^+(m) \geq 4$*. Then* $br^+(m) + \Phi(a, br^+(m)) \geq 3 + M(a) > 2 + g(a)$*.*

Proof. It is clear from Definition 3.49 that $3 + M(a) > 2 + g(a)$. To show the first inequality, first suppose $br^+(m) \leq 5$. Since $m = p^k - 2$, then $br^+(m) \geq k + f(2p - 4)$ by (3.1) and Proposition 3.65. Hence $p \geq 13$, and we verify directly that $\Phi(a, 5) + 5 > 3 + M(a)$ for all $a \in [12]$. If in fact $br^+(m) = 4$, then necessarily $p \leq 11$, and we verify directly that $\Phi(a, 4) + 4 \geq 3 + M(a)$ for all $a \in [10]$.

Now suppose that $br^+(m) \geq 6$, so $br^+(m) + \Phi(a, br^+(m)) \geq 6 + \Phi(a, 6)$. Since $f(x) \in \{\lfloor\sqrt{x}\rfloor - 1, \lfloor\sqrt{x}\rfloor\}$ for all $x \in \mathbb{N}_0$, we have that

$$\begin{aligned}\Phi(a, 6) &\geq 6 \cdot f\left(\lfloor \tfrac{a}{6} \rfloor\right) \geq 6\left(\lfloor(\sqrt{\lfloor \tfrac{a}{6} \rfloor})\rfloor - 1\right) = 6\left(\lfloor\sqrt{\tfrac{a}{6}}\rfloor - 1\right)\\ &\geq 6\left(\sqrt{\tfrac{a}{6}} - 2\right) = \sqrt{6a} - 12.\end{aligned}$$

On the other hand, since $f(b) \leq 2g(b)$ we have that

$$M(a) \leq \max\{f(a-b) + 2f(b) \mid b \in \{0, 1, \ldots, a\}\} \leq \max_{b \in [0,a]}(\sqrt{a-b} + 2\sqrt{b}) = \sqrt{5a}.$$

Thus $6 + \Phi(a, 6) \geq \sqrt{6a} - 6 \geq 3 + \sqrt{5a} \geq 3 + M(a)$ for all $a \geq 1778$, and for $a \leq 1777$ we verify that $6 + \Phi(a, 6) \geq 3 + M(a)$ computationally. □

Proof of Theorem 3.44 for odd p. This follows from Proposition 3.67 by induction on the p-adic length t of $n + 1$. □

Proof of Theorem 3.45 for odd p. Proposition 3.52 and Lemma 3.56 characterise when 0 and 1 belong to $\mathcal{E}^+(n)$ respectively. Now suppose $br^+(n) \geq 2$. We wish to show that $\{2, 3, \ldots, br^+(n)\} \subseteq \mathcal{E}^+(n)$. We proceed by induction on t, the p-adic length of $n + 1$, and observe that the case $t = 1$ has been shown in Proposition 3.65.

Now suppose $t \geq 2$. Let $m = \sum_{j=1}^{t-1} a_j p^{n_j} - 1 = n - a_t p^{n_t}$, and write $L = br^+(m)$, $k = n_t$ and $a = a_t$ for convenience. Since $br^+(n) \geq 2$, then n is not of the form $\sum_j p^{n_j}$ with $p \mid n$, by Theorem 3.44. In particular, m is also not of this form, so $br^+(m) \geq 2$.

We first show that $\{2, 3, \ldots, L\} \subseteq \mathcal{E}^+(n)$: by the inductive hypothesis, for each $i \in \{2, \ldots, L\}$ there exists $\delta \vdash_{p'} m$ such that $|\delta^+_{p'}| = i$. Recall the p^k-abacus A_δ from Definition 3.58, and fix some $j \in \mathcal{R}_{A_\delta}$. Let $\lambda \vdash_{p'} n$ be the partition represented by the p^k-abacus B obtained from A_δ by performing a down-moves on the unique addable bead on runner j of A_δ. By Lemma 3.59, $|\lambda^+_{p'}| = |\mathcal{R}_{A_\delta}| = |\delta^+_{p'}|$, and hence $\{2, 3, \ldots, L\} \subseteq \mathcal{E}^+(n)$.

To show that $\{L + 1, \ldots, br^+(n)\} \subseteq \mathcal{E}^+(n)$: let $\gamma \vdash_{p'} m$ be such that $|\gamma^+_{p'}| = L$. If $\{p^k - 3, p^k - 2\} \not\subseteq \mathcal{R}_{A_\gamma}$, then $N^+(a, p^k, \gamma) = \Phi(a, L)$, and $\{L, L + 1, \ldots, L + \Phi(a, L)\} \subseteq \mathcal{E}^+(n)$ follows from exactly the same argument as in Theorem 3.27. Otherwise, (i) $\mathcal{R}_{A_\gamma} = \{p^k - 3, p^k - 2\}$ or (ii) $\mathcal{R}_{A_\gamma} = \{p^k - 3, p^k - 2, t\}$ for some $0 \leq t < p^k - 4$. Let $A = A_\gamma$.

(i) In this case, $L = 2$ and $br^+(n) = 2 + g(a)$, by Proposition 3.60. Observe that the runners $(A_{p^k-3}, A_{p^k-2}, A_{p^k-1})$ when viewed as a 3-abacus represents the partition (1) with first gap (0,2); that is, it coincides with the 3-abacus X defined in Lemma 3.62. Recall also the 3-abaci $Z(u)$ and $Z'(u)$ from Lemma 3.62.

If $g(a) = 2f(a) - 1$, then for each $u \in \{1, \dots, f(a)\}$, the abacus Y' obtained from $Z'(u)$ by performing $a - u(u+1)$ down-moves on the bead in position $(1, u+1)$ satisfies $w(Y') = a$ and $\mathrm{Add}(Y'_0) + \mathrm{Add}(Y'_1) = 2u + 1$. The partition λ represented by Y therefore satisfies $\lambda \vdash_{p'} n$ and $|\lambda^+_{p'}| = 2u + 1$. Hence $\{3, 5, \dots, 2 + g(a)\} \subseteq \mathcal{E}^+(n)$. Moreover, for each $u \in \{1, \dots, f(a) - 1\}$, the abacus Y obtained from $Z(u)$ by performing $a - u(u+2)$ down-moves on the bead in position $(1, u+1)$ satisfies $w(Y) = a$ and $\mathrm{Add}(Y_0) + \mathrm{Add}(Y_1) = 2u + 2$, from which we deduce $\{2, 4, \dots, 1 + g(a)\} \subseteq \mathcal{E}^+(n)$. Thus $\{L + 1, \dots, br^+(n)\} \subseteq \mathcal{E}^+(n)$ as required.

The case $g(a) = 2f(a)$ is similar.

(ii) In this case, $L = 3$ and $br^+(n) = 3 + M(a)$. Suppose $b \in \{0, 1, \dots, a\}$ satisfies $M(a) = f(a - b) + g(b)$. Then for each $i \in \{2, 3, \dots, g(b) + 2\}$, there exists some 3-abacus Y such that $Y^\uparrow = X$, $w(Y) = b$ and $\mathrm{Add}(Y_0) + \mathrm{Add}(Y_1) = i$, by case (i) above. Also, for each $j \in \{1, 2, \dots, f(a - b) + 1\}$, there exists some 2-abacus U such that $U^\uparrow = V$, the 2-abacus for $\emptyset$ with first gap (0,1), $w(U) = a - b$ and $\mathrm{Add}(U_0) = j$, by the same ideas as in the proof of Lemma 3.26. Let B be the p^k-abacus obtained from A by replacing the runners (A_t, A_{t+1}) with the 2-abacus U, and $(A_{p^k-3}, A_{p^k-2}, A_{p^k-1})$ by Y. Then $w(B) = a$ and $\sum_{x \in \mathcal{R}_A} \mathrm{Add}(B_x) = i + j$. Thus the partition λ represented by B satisfies $\lambda \vdash_{p'} n$ and $|\lambda^+_{p'}| = i + j$, by Lemma 3.59. Therefore $\{3, 4, \dots, 3 + M(a)\} = \{2, \dots, 2 + g(b)\} + \{1, \dots, f(a - b) - 1\} \subseteq \mathcal{E}^+(n)$, as required. □

Proof of Corollary 3.47. By Theorem 3.44, we have that

$$br^+(n) = br^+(a_1 p^{n_1} - 1) + \sum_{j=2}^{t} \Phi(a_j, br^+(m_j - 1)) + \Delta(n, p).$$

By Proposition 3.65, $br^+(a_1 p^{n_1} - 1) \le \max\{\lfloor \sqrt{a_1} \rfloor, f(2a_1 - 2)\} + 1 < \sqrt{2a_1} + 1$. Combining this with Lemma 3.7, we get

$$br^+(n) < (\sqrt{2a_1} + 1) + \sum_{j=2}^{t} \left\lfloor \frac{a_j}{2} \right\rfloor + 1 = \mathbb{B}^+(n)$$

as desired. To bound the difference $\mathbb{B}^+(n) - br^+(n)$, observe that

$$\mathbb{B}^+(n) - br^+(n) < 2\sqrt{p} + 1 + \sum_{j=2}^{t} \varepsilon(j) \quad \text{where} \quad \varepsilon(j) := \left\lfloor \frac{a_j}{2} \right\rfloor - \Phi(a_j, br^+(m_j - 1)).$$

We show that $\varepsilon := \sum_{j=2}^{t} \varepsilon(j) < \frac{p}{2} \log_2(p) - 1$.

If $a_j \le 3$ then $\varepsilon(j) = 0$, by Lemma 3.7 and the fact that $br^+(m) \ge 1$ for all $m \in \mathbb{N}_0$. Hence if $a_j \le 3$ for all $j \in \{2, \dots, t\}$, then $\varepsilon = 0$. In particular, $\varepsilon = 0$ if $p = 3$, and so $\mathbb{B}^+(n) - br^+(n) < 3 \log_2(3)$.

Otherwise, there exists $j \in \{2, \dots, t\}$ such that $a_j \ge 4$ (and so $p \ge 5$). Then there exists a unique $k \in \mathbb{N}$ and integers $1 = i_0 < i_1 < \cdots < i_k \leq t$ such that $i_j = \min\{x \in \{i_{j-1}+1, \ldots, t\} \mid a_x \geq 2^j+2\}$ for all $j \in [k]$ and $\{x \in \{i_k+1, \ldots, t\} \mid a_x \geq 2^{k+1}+2\} = \emptyset$. In particular, $2^k < p$. By a similar argument to that in the proof of Proposition 3.5 (replacing every instance of $br(m_j)$ by $br^+(m_j - 1)$), we find that

$$\varepsilon \leq \sum_{i=0}^{k} \left(\frac{p-1}{2} - 2^i \right) < \frac{kp}{2} - 1.$$

Hence $\mathbb{B}^+(n) - br^+(n) < 2\sqrt{p} + 1 + \varepsilon < 2\sqrt{p} + \frac{p}{2}\log_2(p) < p\log_2(p)$ as desired. □

3.3 Self-similarities in the Young graph

3.3.1 Graph isomorphisms

In [1, Theorem 3] it was shown that the tree $\mathbb{Y}_{2'}$ exhibited 'self-similarities at all scales'. To state this more precisely: given a partition $\lambda \in \mathbb{Y}_{2'}$ and $k \in \mathbb{N}_0$, let $\lambda^{+[0,k]}$ denote the induced subtree rooted at λ consisting of those vertices μ of $\mathbb{Y}_{2'}$ such that $\mu \geq \lambda$ in the dominance partial ordering on partitions, and $|\lambda| \leq |\mu| \leq |\lambda| + k$.

Theorem 3.69 ([1, Theorem 3]). *Let $n, \nu \in \mathbb{N}$ and suppose $\nu_2(n) \geq \nu$. Let $\lambda \vdash_{2'} n$. Then*

$$C_{2^\nu} : \lambda^{+[0,2^\nu-1]} \to \emptyset^{+[0,2^\nu-1]}$$

is an isomorphism of trees.

In other words, for each $k \in \mathbb{N}$, the subtree of $\mathbb{Y}_{2'}$ consisting of partitions λ such that $|\lambda| \leq 2^k - 1$ is 'repeated infinitely often' inside the full tree $\mathbb{Y}_{2'}$.

On the other hand, it is clear that the subgraph $\mathbb{Y}_{p'}$ is never a tree when p is odd. Nevertheless, there are still certain isomorphic structures in the induced subgraphs of $\mathbb{Y}_{p'}$ for all p. Our main result is the following:

Theorem 3.70. *Let p be any prime. Suppose $n = \sum_{i=1}^{k} p^{n_i}$ for some $k \in \mathbb{N}$ and integers $0 \leq n_1 < n_2 < \ldots < n_k$. Let $\lambda \vdash_{p'} n$ and $\nu \leq n_1$. Then*

$$C_{p^\nu} : \lambda^{\uparrow[0,p^\nu-1]} \longrightarrow \emptyset^{\uparrow[0,p^\nu-1]}$$

is a graph isomorphism.

The induced subgraphs $\lambda^{\uparrow[p^\nu-1]}$ and $\emptyset^{\uparrow[0,p^\nu-1]}$ of $\mathbb{Y}_{p'}$ are defined explicitly in Definition 3.72. Examples of these graph isomorphisms are presented in Figures 3.11 and 3.12, where we say a partition λ lies on level n if $\lambda \vdash n$. Theorem 3.70 is proved in Section 3.3.2 below, and we comment on the general case of $n = \sum_{i=1}^{k} a_i p^{n_i}$ at the end of the section in Remark 3.78.

In particular, observe that we recover all of the isomorphisms of subtrees in [1] for the tree $\mathbb{Y}_{2'}$ when we set $p = 2$ in Theorem 3.70. These isomorphisms can be also

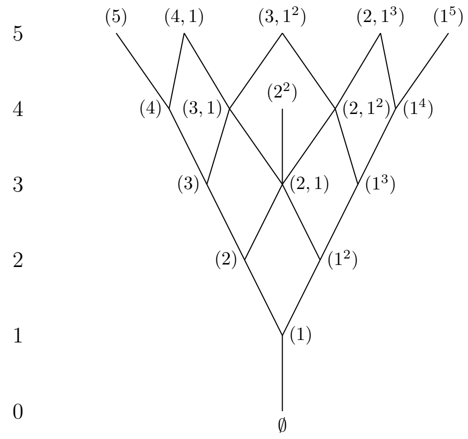


Figure 3.11: $\mathbb{Y}_{5'}$ from levels 0 to 5.

thought of as refinements of the following observation, which can be seen by considering partitions on James' abacus.

Observation 3.71. *Let $n \in \mathbb{N}$ and p be a prime. Let $n = ap^k + m$ where $k \in \mathbb{N}$, $a \in \{1, 2, \ldots, p-1\}$ and $0 \leq m < p^k$. Then*

$$C_{p^k} : \mathrm{Irr}_{p'}(\mathfrak{S}_n) \longrightarrow \mathrm{Irr}_{p'}(\mathfrak{S}_{n-ap^k})$$

is surjective and exactly $|\mathrm{Irr}_{p'}(\mathfrak{S}_{ap^k})|$-to-1.

(Here we have identified partitions λ with their corresponding characters χ^λ.)

3.3.2 On hooks and their leg lengths

Definition 3.72. *Let p be any prime. Let $s \leq t \in \mathbb{N}_0$ and let $\lambda^{\uparrow[s,t]}$ denote the induced subgraph of $\mathbb{Y}_{p'}$ on the set of vertices $\mu \in \mathbb{Y}_{p'}$ such that $|\mu| = |\lambda| + m$ for some $s \leq m \leq t$, and such that there exist partitions $\mu^{(0)} = \lambda$, $\mu^{(1)}$, $\mu^{(2)}, \ldots, \mu^{(m)} = \mu$ satisfying $\mu^{(i)} \in \mathbb{Y}_{p'}$ and $\mu^{(i-1)} \in \mu^{(i)-}_{p'}$ for all $i \in [m]$. When $s = t$, we simply denote $\lambda^{\uparrow[s,t]}$ by $\lambda^{\uparrow[s]}$.*

The partitions $\mu^{(m)}, \ldots, \mu^{(0)}$ form a path of minimal length from μ to λ inside $\mathbb{Y}_{p'}$. Let such a path be called a p-downpath from μ to λ*, or simply a* downpath from μ to λ *whenever the value of p is clear from context. Thus $\lambda^{\uparrow[0,t]}$ contains exactly those partitions μ with a downpath to λ such that $|\lambda| \leq |\mu| \leq |\lambda| + t$.*

When we write $\mu \in \mathbb{G}$ for some subgraph $\mathbb{G}$ of $\mathbb{Y}$, we will always mean that μ is a vertex of $\mathbb{G}$, and hence μ is a partition. Clearly $\mathbb{Y}$ and hence $\mathbb{Y}_{p'}$ is a graded poset with rank function r given by $r(\lambda) = |\lambda|$. Thus informally, $\lambda^{\uparrow[0,t]}$ is the 'cone-like' subgraph of $\mathbb{Y}_{p'}$ with apex λ, between ranks $|\lambda|$ and $|\lambda| + t$.

A beautiful result of Bessenrodt [2] on leg lengths of addable and removable hooks which we will refer to later is the following.

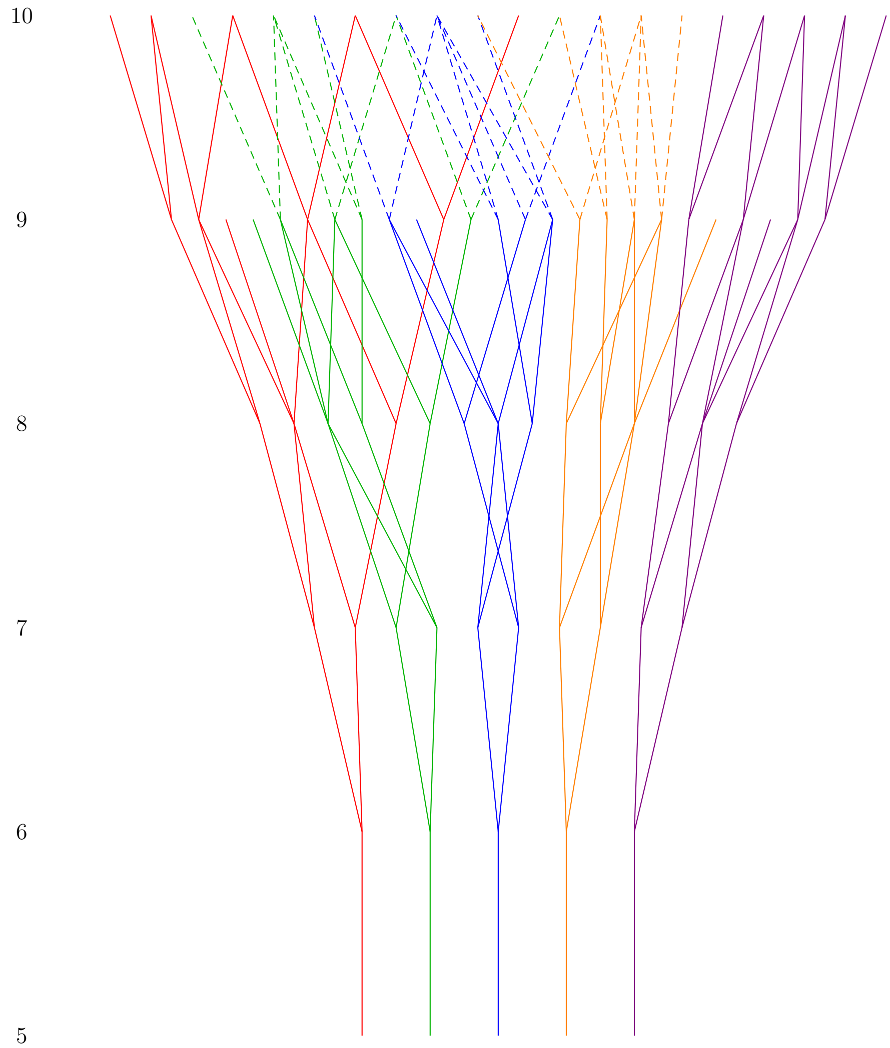


Figure 3.12: $\mathbb{Y}_{5'}$ from levels 5 to 10. The five coloured subgraphs between levels 5 to 9 are each isomorphic to levels 0 to 4. The red (leftmost) and violet (rightmost) subgraphs between levels 9 and 10 are isomorphic to levels 4 to 5, while the remaining three subgraphs are not (indicated by dashed lines).

Theorem 3.73 ([2, Theorem 1.1])**.** *Let λ be a partition of $n \in \mathbb{N}$. For $k \in \mathbb{N}$, $k \leq n$ and $0 \leq l \leq k-1$, let $A_{k,l}(\lambda)$ be the number of k-hooks of leg length l that can be added to $[\lambda]$ to give the Young diagram of a partition of $n+k$, and let $R_{k,l}(\lambda)$ be the number of k-hooks of leg length l that can be removed from $[\lambda]$ to give the Young diagram of a partition of $n-k$. Then $A_{k,l}(\lambda) = 1 + R_{k,l}(\lambda)$.*

The key step in proving Theorem 3.70 is the following.

Proposition 3.74. *Let p be a prime. Suppose $n = \sum_{i=1}^{k} p^{n_i}$ for some $k \in \mathbb{N}$ and integers $0 \leq n_1 < n_2 < \ldots < n_k$. Let $\lambda \vdash_{p'} n$. Then*

$$C_{p^{n_k}} : \lambda^{\uparrow[0,p^{n_1}-1]} \longrightarrow \left(C_{p^{n_k}}(\lambda)\right)^{\uparrow[0,p^{n_1}-1]}$$

is a graph isomorphism.

Before we prove this proposition, we present some useful lemmas. Let p be any prime. Let $\lambda \vdash_{p'} n$ where $p^k < n < 2p^k$ for some $k \in \mathbb{N}$. Then $|\mathcal{H}_{p^k}(\lambda)| = 1$ by Theorem 3.12 (and in fact, the hook in $\mathcal{H}_{p^k}(\lambda)$ has length precisely p^k since $n < 2p^k$). Similarly, μ has a unique p^k-hook for all $\mu \in \lambda^{-}_{p'}$. Thus it makes sense to refer to *the* p^k-hook of λ (and similarly for μ); this notation will be kept in Lemma 3.75 below.

Lemma 3.75. *Let $\mu \in \lambda^{-}_{p'}$. Then the p^k-hooks of λ and μ have the same leg length.*

Proof. Let A be the p^k-abacus for λ with first gap $(0,0)$. Positions (i,j) in A are empty whenever $i \geq 2$, since $|\lambda| < 2p^k$. Since λ has a unique p^k-hook, there is a unique $t \in \{0,1,\ldots,p^k-1\}$ such that $(0,t)$ is empty but $(1,t)$ contains a bead. Moreover, its leg length is given by $|\{(i,j) \text{ contains a bead in } A : i=0,\ j>t \text{ or } i=1,\ j<t\}| =: b_A(t)$.

Let c be the bead in A such that $B := A^{\leftarrow c}$ is a p^k-abacus for μ. Similarly, there is a unique $s \in \{0,1,\ldots,p^k-1\}$ such that $(0,s)$ is empty but $(1,s)$ contains a bead, and the leg length of this hook is $b_B(s)$. It remains to observe that $b_A(t) = b_B(s)$.

If c is not in position $(0,t+1)$ or $(1,t)$ in A, then $s=t$ and so $b_A(t) = b_B(s)$ by inspection. (If $t = p^k-1$ then set $(0,t+1) := (1,0)$.) If c is in position $(0,t+1)$, then $s \neq t$. Clearly $b_A(t) = b_B(s)$ if $s \neq t+1$ also, while if $s = t+1$ then $b_A(t) = b_B(s)$ since $(0,t+1)$ contains a bead in A but not in B and $(1,t)$ contains a bead in both A and B. The case if c lies in $(1,t)$ in A is similar. □

In Corollary 3.76 and Lemma 3.77 below, let $n = \sum_{i=1}^{k} p^{n_i}$ for some $k \in \mathbb{N}$ and integers $0 \leq n_1 < n_2 < \cdots < n_k$, and let $m \in \{0,1,\ldots,p^{n_1}-1\}$. Fix $\lambda \vdash_{p'} n$. By Theorem 3.12, every partition α of p'-degree such that $n \leq |\alpha| \leq n+m$ has a unique p^{n_k}-hook, since $p^{n_k} \leq n$ and $n + p^{n_1} - 1 < 2p^{n_k}$.

Corollary 3.76. *Let $\lambda \vdash_{p'} n$ and suppose $\mu \in \lambda^{\uparrow[m]}$. Then the p^{n_k}-hook of μ and he p^{n_k}-hook of λ have the same leg length.*

Proof. We proceed by induction on m; the assertion is clear for $m=0$ so now assume $m \geq 1$. Since $\mu \in \lambda^{\uparrow[m]}$, there exists $\delta \in \mu^{-}_{p'} \cap \lambda^{\uparrow[m-1]}$. By the inductive hypothesis, the leg length of the p^{n_k}-hook of δ equals that of λ. The assertion then follows from Lemma 3.75. □

In fact, the converse is also true.

Lemma 3.77. *Let $\mu \vdash_{p'} n+m$. Suppose that $C_{p^{n_k}}(\mu) \in C_{p^{n_k}}(\lambda)^{\uparrow[m]}$. Further suppose that the leg lengths of the p^{n_k}-hooks of μ and λ are equal. Then $\mu \in \lambda^{\uparrow[m]}$.*

Proof. Let l be the leg length of the p^{n_k}-hook of λ. We proceed by induction on m. When $m = 0$, $C_{p^{n_k}}(\mu) = C_{p^{n_k}}(\lambda)$. By Theorem 3.73, λ is the unique partition of n obtained from $C_{p^{n_k}}(\lambda)$ by adding a p^{n_k}-hook of leg length l, since $C_{p^{n_k}}(\lambda)$ has no removable p^{n_k}-hooks. Hence $\mu = \lambda$.

Now assume $m \geq 1$. Since $C_{p^{n_k}}(\mu) \in C_{p^{n_k}}(\lambda)^{\uparrow[m]}$, there exists $\delta \in C_{p^{n_k}}(\mu)^-_{p'} \cap C_{p^{n_k}}(\lambda)^{\uparrow[m-1]}$. By Theorem 3.15, there exists $D \in \mu^-_{p'}$ such that $C_{p^{n_k}}(D) = \delta$. By Lemma 3.75, the p^{n_k}-hooks of D and μ have equal leg length, and hence by the inductive hypothesis, $D \in \lambda^{\uparrow[m-1]}$. Thus $\mu \in D^+_{p'} \subseteq \lambda^{\uparrow[m]}$. □

Proof of Proposition 3.74. The assertion is trivially true if $n_1 = 0$ (since both graphs consist of a single point), so we may assume $n_1 > 0$ from now on. We show that (i) the stated map is well-defined; (ii) it is a bijection on the sets of vertices; and (iii) (μ, δ) is an edge in $\lambda^{\uparrow[0,p^{n_1}-1]}$ if and only if $\big(C_{p^{n_k}}(\mu), C_{p^{n_k}}(\delta)\big)$ is an edge in $\big(C_{p^{n_k}}(\lambda)\big)^{\uparrow[0,p^{n_1}-1]}$.

(i) Well-definition: we first show that $\lambda^{\uparrow[m]}$ maps into $C_{p^{n_k}}(\lambda)^{\uparrow[m]}$ for each $m \in \{0,1,\ldots,p^{n_1}-1\}$, which implies that $C_{p^{n_k}} : \lambda^{\uparrow[0,p^{n_1}-1]} \longrightarrow \big(C_{p^{n_k}}(\lambda)\big)^{\uparrow[0,p^{n_1}-1]}$ is well-defined. We proceed by induction on m. When $m = 0$, we have $\lambda^{\uparrow[0]} = \{\lambda\}$, $C_{p^{n_k}}(\lambda)^{\uparrow[0]} = \{C_{p^{n_k}}(\lambda)\}$ and $\lambda \mapsto C_{p^{n_k}}(\lambda)$.

Now let $m \geq 1$ and let $\mu \in \lambda^{\uparrow[m]}$. Since $p^{n_k} \leq n < |\mu| \leq n + p^{n_1} - 1 < 2p^{n_k}$, we have $C_{p^{n_k}}(\mu) \vdash_{p'} |\mu| - p^{n_k} = |C_{p^{n_k}}(\lambda)| + m$ by Theorem 3.12. Thus it suffices to find a downpath from $C_{p^{n_k}}(\mu)$ to $C_{p^{n_k}}(\lambda)$.

Since $\mu \in \lambda^{\uparrow[m]}$, there exists some downpath from μ to λ. In particular, there exists some $\delta \in \mu^-_{p'}$ on this downpath, so $\delta \in \lambda^{\uparrow[m-1]}$. By the inductive hypothesis, $C_{p^{n_k}}(\delta) \in C_{p^{n_k}}(\lambda)^{\uparrow[m-1]}$ so there exists some downpath

$$\gamma^{(m-1)} = C_{p^{n_k}}(\delta), \gamma^{(m-2)}, \ldots, \gamma^{(1)}, \gamma^{(0)} = C_{p^{n_k}}(\lambda)$$

in $\mathbb{Y}_{p'}$. By Theorem 3.15, $C_{p^{n_k}}(\delta) \in C_{p^{n_k}}(\mu)^-_{p'}$, so we may set $\gamma^{(m)} = C_{p^{n_k}}(\mu)$ and see that $\gamma^{(i-1)} \in \gamma^{(i)-}_{p'}$ for all $i \in [m]$ as required.

(ii) Bijection on vertices: By (i), the map

$$C_{p^{n_k}} : \lambda^{\uparrow[m]} \longrightarrow C_{p^{n_k}}(\lambda)^{\uparrow[m]}$$

is well-defined for each $m \in \{0,1,\ldots,p^{n_1}-1\}$. We now show that this is a bijection on the vertices for each m.

Surjectivity: we proceed by induction on m. This is clear for $m = 0$, so assume $m \geq 1$. Let $\delta \in C_{p^{n_k}}(\lambda)^{\uparrow[m]}$. Then there exists some $\varepsilon \in \delta^-_{p'} \cap C_{p^{n_k}}(\lambda)^{\uparrow[m-1]}$. By the inductive hypothesis, there exists a partition $E \in \lambda^{\uparrow[m-1]}$ such that $C_{p^{n_k}}(E) = \varepsilon$. Also, $p^{n_k} \leq n \leq |E| = n + m - 1 \leq n + p^{n_1} - 2 < 2p^{n_k} - 1$, so by Corollary 3.51 there exists a partition $D \in E^+_{p'}$ such that $C_{p^{n_k}}(D) = \delta$. Since $E \in \lambda^{\uparrow[m-1]}$, we have $D \in \lambda^{\uparrow[m]}$.

Injectivity: suppose

$$C_{p^{n_k}}(A) = C_{p^{n_k}}(B) =: \alpha \in C_{p^{n_k}}(\lambda)^{\uparrow[m]}$$

for some m and some $A, B \in \lambda^{\uparrow[m]}$. Since α has no removable p^{n_k}-hooks, α has a unique addable p^{n_k}-hook of each leg length $l \in \{0, 1, \ldots, p^{n_k} - 1\}$. By Corollary 3.76, the p^{n_k}-hooks of A and B have the same leg length, so $A = B$.

(iii) Edges: suppose (μ, δ) is an edge in $\lambda^{\uparrow[0,p^{n_1}-1]}$. So $\delta \in \mu^-_{p'}$, $\mu \in \lambda^{\uparrow[m]}$ and $\delta \in \lambda^{\uparrow[m-1]}$ for some $m \in \{1, 2, \ldots, p^{n_1}-1\}$. By (i), we know that $C_{p^{n_k}}(\mu) \in C_{p^{n_k}}(\lambda)^{\uparrow[m]}$ and $C_{p^{n_k}}(\delta) \in C_{p^{n_k}}(\lambda)^{\uparrow[m-1]}$. But by Theorem 3.15, $\delta \in \mu^-_{p'}$ implies $C_{p^{n_k}}(\delta) \in C_{p^{n_k}}(\mu)^-_{p'}$, and so $\big(C_{p^{n_k}}(\mu), C_{p^{n_k}}(\delta)\big)$ is an edge in $\big(C_{p^{n_k}}(\lambda)\big)^{\uparrow[0,p^{n_1}-1]}$.

Conversely, suppose that $\alpha \in C_{p^{n_k}}(\lambda)^{\uparrow[m]}$ is joined to $\beta \in C_{p^{n_k}}(\lambda)^{\uparrow[m-1]}$ for some $m \in \{1, 2, \ldots, p^{n_1} - 1\}$. By (ii), there exists a unique partition $A \in \lambda^{\uparrow[m]}$ such that $C_{p^{n_k}}(A) = \alpha$, and a unique $B \in \lambda^{\uparrow[m-1]}$ such that $C_{p^{n_k}}(B) = \beta$. Consider the map $C_{p^{n_k}} : B^+_{p'} \longrightarrow \beta^+_{p'}$. This map is surjective by Corollary 3.51, and $\alpha \in \beta^+_{p'}$, so suppose A' lies in the preimage of α. Then $A' \in B^+_{p'}$ and $B \in \lambda^{\uparrow[m-1]}$, so $A' \in \lambda^{\uparrow[m]}$ and $C_{p^{n_k}}(A') = \alpha$. By uniqueness of A, we have $A = A'$ and thus $A \in B^+_{p'}$. Therefore (A, B) is an edge in $\lambda^{\uparrow[0,p^{n_1}-1]}$. □

Let us now highlight a link between the graph isomorphisms of Proposition 3.74 and Bessenrodt's theorem [2] on hooks. Let $n = \sum_{i=1}^{k} p^{n_i}$ be as above and let $\mu \vdash_{p'} n - p^{n_k}$. Since $|\mu| < p^{n_k}$, μ has a unique addable p^{n_k}-hook H_l of each leg length $l \in \{0, 1, \ldots, p^{n_k} - 1\}$. Let $\lambda^{(l)} \vdash n$ be the result of μ with H_l added. Then $\lambda^{(l)} \vdash_{p'} n$ for all l by Theorem 3.12. If $\lambda \vdash_{p'} n$ is such that $C_{p^{n_k}}(\lambda) = \mu$, then $\lambda = \lambda^{(l)}$ for some l.

Moreover, the map

$$C_{p^{n_k}} : {\lambda^{(l)}}^{\uparrow[0,p^{n_1}-1]} \longrightarrow \mu^{\uparrow[0,p^{n_1}-1]}$$

is a graph isomorphism, and ${\lambda^{(l)}}^{\uparrow[0,p^{n_1}-1]}$ and ${\lambda^{(l')}}^{\uparrow[0,p^{n_1}-1]}$ are edge and vertex disjoint whenever $l \neq l'$. (This is because if $\gamma \in \lambda^{(l)\uparrow[m]} \cap \lambda^{(l')\uparrow[m]}$, then the unique p^{n_k}-hook of γ has leg length $l = l'$.)

Therefore the disjoint copies of $\mu^{\uparrow[0,p^{n_1}-1]}$ obtained by adding p^{n_k}-hooks are indexed precisely by the leg length of the added hook. It is easy to see that the cones $\lambda^{\uparrow[0,p^{n_1}-1]}$ and $\tilde{\lambda}^{\uparrow[0,p^{n-1}-1]}$ are also disjoint if $C_{p^{n_k}}(\lambda) \neq C_{p^{n_k}}(\tilde{\lambda})$, for $\lambda, \tilde{\lambda} \vdash_{p'} n$.

Proof of Theorem 3.70. We proceed by induction on k. When $k = 1$ and $n = p^{n_1}$, by Theorem 3.12 we have that $C_{p^{n_1}}(\lambda) = \emptyset$. The assertion then follows from Proposition 3.74 since $C_{p^\nu} = C_{p^\nu} \circ C_{p^{n_1}}$ whenever $\nu \leq n_1$.

For the inductive step, suppose $k \geq 2$. Composing the successive graph isomorphisms

$$\lambda^{\uparrow[0,p^{n_1}-1]} \xrightarrow{C_{p^{n_k}}} C_{p^{n_k}}(\lambda)^{\uparrow[0,p^{n_1}-1]} \xrightarrow{C_{p^{n_{k-1}}}} C_{p^{n_{k-1}}}(\lambda)^{\uparrow[0,p^{n_1}-1]} \xrightarrow{C_{p^{n_{k-2}}}} \cdots$$
$$\cdots \xrightarrow{C_{p^{n_1}}} C_{p^{n_2}}(\lambda)^{\uparrow[0,p^{n_1}-1]} \xrightarrow{C_{p^{n_2}}} C_{p^{n_1}}(\lambda)^{\uparrow[0,p^{n_1}-1]} = \emptyset^{\uparrow[0,p^{n_1}-1]}$$

given by Proposition 3.74, we find that $C_{p^{n_1}} = C_{p^{n_1}} \circ C_{p^{n_2}} \circ \cdots \circ C_{p^{n_k}}$ gives an isomorphism of graphs $\lambda^{\uparrow[p^\nu-1]} \longrightarrow \emptyset^{\uparrow[0,p^\nu-1]}$. The result for $\nu \leq n_1$ then follows. □

Remark 3.78. (1) Since n_1 is the p-adic valuation of n, setting $p = 2$ exactly recovers [1, Theorem 3].

(2) Let $\lambda \vdash_{p'} n = \sum_{i=1}^k p^{n_i}$, and let $\mu \vdash_{p'} |\lambda| + m$ where $m \in \{0, 1, \ldots, p^{n_1} - 1\}$. Then a corollary of these graph isomorphisms is that

$$\mu \in \lambda^{\uparrow[0,p^{n_1}-1]} \quad \Longleftrightarrow \quad \mu \geq \lambda$$

(where the latter condition denotes the dominance partial ordering), since this assertion clearly holds when $\lambda = \emptyset$. Note $p^{n_1} - 1$ cannot be exceeded in general: for example, $\mu = (4, 1) \vdash_{3'} 5$ dominates $\lambda = (2, 1) \vdash_{3'} 3$ but there is no downpath from μ to λ since $\mu_{3'}^- = \{(4)\}$.

(3) Theorem 3.70 is stated for n such that all of the p-adic digits of n belong to $\{0, 1\}$. The result does not hold in general when n has p-adic digits greater than 1. For example, let $p = 3$, $n = 6$, $\lambda = (4, 1^2)$ and $\nu = n_1 = 1$. Then $\lambda_{3'}^+ = \{(4, 2, 1), (4, 1^3)\}$, but $C_3(\lambda) = \emptyset$ and $|\emptyset_{3'}^+| = 1$.

Also, the graph isomorphism does not in general extend beyond $\nu \leq n_1$. Consider the following example where $\nu > n_1$: let $p = 2$, $n = 7$, $\nu = 1 > 0 = n_1$ and $\lambda = (4, 2, 1)$. Then $|\lambda_{2'}^+| = 0$, but $C_2(\lambda) = (1)$ and $|(1)_{2'}^+| = 2$.

(4) Finally, recall from Definition 3.72 that the induced subgraph $\lambda^{\uparrow[s,t]}$ (for some $0 \leq s \leq t$) is said to contain the vertex μ if *there exists* a downpath from μ to λ in $\mathbb{Y}_{p'}$. With the notation as in Theorem 3.70, letting $\mu \in \lambda^{\uparrow[1,p^{n_1}-1]}$, in fact *every* element of $\mu_{p'}^-$ lies in $\lambda^{\uparrow[0,p^{n_1}-1]}$. Indeed, Theorem 3.70 implies

$$|\mu_{p'}^- \cap \lambda^{\uparrow[0,p^{n_1}-1]}| = |C_{p^{n_1}}(\mu)_{p'}^- \cap \emptyset^{\uparrow[0,p^{n_1}-1]}| = |C_{p^{n_1}}(\mu)_{p'}^-|.$$

But the maps $C_{p^{n_i}} : C_{p^{n_{i+1}}}(\mu)_{p'}^- \longrightarrow C_{p^{n_i}}(\mu)_{p'}^-$ are bijections for all $1 \leq i < k$ (where we let $C_{p^{n_{k+1}}}(\mu) := \mu$), so $|\mu_{p'}^-| = |C_{p^{n_1}}(\mu)_{p'}^-|$. Hence $\mu_{p'}^- \subseteq \lambda^{\uparrow[0,p^{n_1}-1]}$. ◊

Chapter 4

Linear characters of Sylow subgroups of symmetric groups

This chapter is based on joint work with Dr Eugenio Giannelli and Jason Long. The results in Section 4.3 were obtained in collaboration with J. Long, and the results in Sections 4.2 and 4.4 with Dr Giannelli.

We investigate the linear constituents of restrictions of irreducible characters of symmetric groups to their Sylow subgroups. Specifically, let p be any prime and fix a Sylow p-subgroup P_n of the symmetric group $\mathfrak{S}_n$. Let ϕ and ψ be linear characters of P_n and let $N = N_{\mathfrak{S}_n}(P_n)$. We show that if the inductions of ϕ and ψ to $\mathfrak{S}_n$ are equal, then ϕ and ψ are N-conjugate. This is an analogue for symmetric groups of a result of Navarro for p-solvable groups [50]. We further show that the set of irreducible constituents of the induced character determines the N–orbit of ϕ when n is a power of p.

4.1 Outline

In recent years, the restriction of characters from a finite group G to a Sylow subgroup P of G has played a major role in character correspondences in the context of the McKay Conjecture (see [28], [38] and [52], for example). Little is known about such restrictions in general, however, even in the case of symmetric groups.

A consequence of a recent result [31] of Giannelli and Navarro is the existence of a linear constituent in any restriction of an irreducible character of $\mathfrak{S}_n$ to P_n, for all n and p. A natural question is to *identify* which linear characters appear in such restrictions, or equivalently, to describe the irreducible constituents of the induction $\phi\big\uparrow^{\mathfrak{S}_n}$ for every linear character ϕ of P_n, for all n and p.

For any finite group G and P a Sylow subgroup of G, the normaliser $N = N_G(P)$ acts on the set of linear characters of P by conjugation. It is easy to see that if two linear characters are N–conjugate then their inductions to N, and hence G, are equal. Thus when considering induced characters $\phi\big\uparrow^{G}$, it is sufficient to consider a set of orbit representatives ϕ under this action of N. However, is the converse true? That is, if ϕ and ψ are two linear characters of P such that $\phi\big\uparrow^G = \psi\big\uparrow^G$, must ϕ and ψ be N–conjugate?

This was answered in the affirmative for all p-solvable groups by Navarro in [50], though there exist finite groups (such as $\mathrm{PSL}(3,3)$ with $p=3$) for which the answer is negative. In the course of investigating character restrictions and inductions for the symmetric groups and their Sylow subgroups, we prove that the answer is also affirmative for all $\mathfrak{S}_n$ and all primes p.

Theorem 4.1. *Let p be any prime and let $n \in \mathbb{N}$. Let $P_n \in \mathrm{Syl}_p(\mathfrak{S}_n)$ and let $N = N_{\mathfrak{S}_n}(P_n)$. Let ϕ and ψ be linear characters of P_n. Then $\phi\big\uparrow_{P_n}^{\mathfrak{S}_n} = \psi\big\uparrow_{P_n}^{\mathfrak{S}_n}$ if and only if ϕ and ψ are N–conjugate.*

As part of the proof of Theorem 4.1, we provide an entirely combinatorial condition equivalent to the algebraic statement that two linear characters are N–conjugate, and an explicit description of certain character values, both of which we believe to be of independent interest. The details are given in Sections 4.2 and 4.3 respectively.

Finally, the action of N just described is also related to the action given by Galois conjugation on the set of linear characters. In Section 4.4 we show that the partition by N–orbits is a strict coarsening of the partition by Galois orbits whenever $n \geq 2p$, and also compare these to the equivalence classes given by the relation on linear characters ϕ, ψ of P_n defined via $\Omega(\phi) = \Omega(\psi)$, where $\Omega(\phi)$ is the set of irreducible constituents of the induced character $\phi\big\uparrow^{\mathfrak{S}_n}$.

We record a proof of the easy direction of Theorem 4.1.

Lemma 4.2. *Let G be a finite group and p be a prime. Let $P \in \mathrm{Syl}_p(G)$ and $N = N_G(P)$. Suppose $\phi, \psi \in \mathrm{Char}(P)$ and $\psi = \phi^n$ for some $n \in N$. Then $\phi\big\uparrow_P^N = \psi\big\uparrow_P^N$, and hence $\phi\big\uparrow_P^G = \psi\big\uparrow_P^G$.*

Proof. Let $\alpha \in \mathrm{Irr}(N)$. Note that $\alpha^n = \alpha$. Then by Frobenius reciprocity,

$$\langle \phi\big\uparrow^N, \alpha\rangle = \langle \phi, \alpha\big\downarrow_P\rangle = \langle \phi^n, (\alpha\big\downarrow_P)^n\rangle = \langle \psi, (\alpha^n)\big\downarrow_P\rangle = \langle \psi\big\uparrow^N, \alpha\rangle.$$

Thus $\phi\big\uparrow^N = \psi\big\uparrow^N$, as α is arbitrary. □

4.2 On a conjugacy action of Sylow normalisers

Throughout this chapter, let p denote an arbitrary, fixed prime, and let $n \in \mathbb{N}$. The main aim of this section is to prove Theorem 4.1 for all primes p. Recall the notation from (2.4) in Section 2.3.2 for linear characters $\phi(\underline{\mathbf{s}})$ of P_n. We begin by proving an equivalent condition on the indexing sequences $\underline{\mathbf{s}}$ and $\underline{\mathbf{t}}$ for the corresponding linear characters $\phi(\underline{\mathbf{s}})$ and $\phi(\underline{\mathbf{t}})$ of P_n to be $N_{\mathfrak{S}_n}(P_n)$–conjugate, in Lemma 4.3 (the case $n = p^k$) and Lemma 4.5 (for arbitrary $n \in \mathbb{N}$) below.

Given $k \in \mathbb{N}$ and $s \in [\overline{p}]^k$ we let

$$\Sigma(s) := \{t \in [\overline{p}]^k \mid t_j = 0 \text{ if and only if } s_j = 0, \ \forall\, j \in [k]\}.$$

If $t \in \Sigma(s)$, then we say also that $\phi(t) \in \Sigma(s)$, where $\phi(t)$ is the linear character of P_{p^k} corresponding to $t \in [\overline{p}]^k$. (We refer the reader to Section 2.3.2 for a description of $\mathrm{Lin}(P_{p^k})$.) When we say $\phi \in \Sigma(s)$, we mean that $\phi = \phi(t)$ for some $t \in [\overline{p}]^k$ such that $t \in \Sigma(s)$. It will be clear from context whether we refer to $\Sigma(s)$ as a subset of $\mathrm{Lin}(P_{p^k})$ or of $[\overline{p}]^k$.

Lemma 4.3. *Let $k \in \mathbb{N}$ and let $\phi, \psi \in \mathrm{Lin}(P_{p^k})$. Then ϕ and ψ are $N_{\mathfrak{S}_{p^k}}(P_{p^k})$–conjugate if and only if there exists $s \in [\overline{p}]^k$ such that $\phi, \psi \in \Sigma(s)$.*

Proof. Let $G = \mathfrak{S}_{p^k}$, $P = P_{p^k}$ and $N = N_{\mathfrak{S}_{p^k}}(P_{p^k})$. Since P' is characteristic in P we have that $P' \lhd N$. Moreover, the standard map

$$\mathrm{Lin}(P) \longrightarrow \mathrm{Irr}(P/P') = \mathrm{Lin}(P/P'), \qquad \phi \longmapsto \widetilde{\phi}, \quad \widetilde{\phi}(gP') := \phi(g) \ \forall\, g \in P$$

is well-defined and a bijection. Let $g \in N$, $x \in P$ and $\phi \in \mathrm{Lin}(P)$. Then

$$(\widetilde{\phi})^{gP'}(xP') = \widetilde{\phi}(gxg^{-1}P') = \phi(gxg^{-1}) = \phi^g(x) = \widetilde{\phi^g}(xP').$$

Hence $(\widetilde{\phi})^{gP'} = \widetilde{\phi^g}$. From [53, Lemma 1.4] we have that $P/P' \cong (P_{p^{k-1}}/{P_{p^{k-1}}}') \times C_p \cong (C_p)^{\times k}$. Specifically, define

$$\theta_k : P \longrightarrow (P_{p^{k-1}}/{P_{p^{k-1}}}') \times C_p, \qquad (x_1, \dots, x_p; \sigma) \longmapsto (x_1 \cdots x_p {P_{p^{k-1}}}', \sigma),$$

where $x_i \in P_{p^{k-1}}$ and $\sigma \in P_p$. This is a surjective homomorphism with kernel

$$\{(x_1, \dots, x_p; \sigma) \mid x_1 \cdots x_p \in {P_{p^{k-1}}}', \ \sigma = 1\},$$

which by [53, Lemma 1.4] is exactly P'. Thus $P/P' \cong (P_{p^{k-1}}/{P_{p^{k-1}}}') \times C_p$, so by iterating we find that $P/P' \cong (C_p)^{\times k}$. We also have from a direct application of [53, Proposition 1.5] that

$$N/P' \cong (N_{\mathfrak{S}_{p^{k-1}}}(P_{p^{k-1}})/{P_{p^{k-1}}}') \times N_{\mathfrak{S}_p}(C_p) \cong (N_{\mathfrak{S}_p}(C_p))^{\times k} \cong (C_p \rtimes C_{p-1})^{\times k}.$$

In particular, if $(x_1, \dots, x_k) \in [\overline{p}]^k$, $\chi = \phi_{x_1} \times \cdots \times \phi_{x_k} \in \mathrm{Lin}(P/P') = \mathrm{Irr}((C_p)^{\times k})$ and $h = (h_1, \dots, h_k) \in N/P' \cong (N_{\mathfrak{S}_p}(C_p))^{\times k}$ then we have that

$$\chi^h = (\phi_{x_1})^{h_1} \times \cdots \times (\phi_{x_k})^{h_k}.$$

Since $N_{S_p}(C_p)$ acts on $\mathrm{Lin}(C_p)$ by fixing the trivial character ϕ_0 and transitively permuting $\phi_1, \phi_2, \dots, \phi_{p-1}$, it follows that $\phi_{x_1} \times \cdots \times \phi_{x_k}$ and $\phi_{y_1} \times \cdots \times \phi_{y_k}$ are N/P'–conjugates if and only if there exists $s \in [\overline{p}]^k$ such that $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \Sigma(s)$. This shows that in order to conclude the proof, it remains to show that if $\phi = \phi(x) \in \mathrm{Lin}(P)$ for some $x = (x_1, \ldots, x_k) \in [\overline{p}]^k$, then $\widetilde{\phi} = \phi_{x_1} \times \cdots \times \phi_{x_k}$. This can be seen inductively as follows. The case $k = 1$ is clear. Let $k \geq 2$ and let γ be the p-cycle $(1, 2 \ldots, p) \in P_p$. We now let $\gamma_1 = \gamma \in P_p$ and for $j \in \{2, 3, \ldots, k-1\}$ we denote by γ_j the element of $P_{p^j} = P_{p^{j-1}} \wr P_p$ defined by $\gamma_j = (1, 1, \ldots, 1; \gamma)$. Using the description of P' given in [53, Lemma 1.4] we deduce that

$$P/P' = \left\langle \omega_1^{(k)} P' \right\rangle \times \left\langle \omega_2^{(k)} P' \right\rangle \times \ldots \times \left\langle \omega_k^{(k)} P' \right\rangle,$$

where the elements $\omega_j^{(k)} \in P = P_{p^k}$ are defined as follows, recalling that P_{p^k} is the k-fold wreath product $P_p \wr \cdots \wr P_p$: let $\omega_k^{(k)} = \gamma_k$. Then let $\omega_{k-1}^{(k)} = (\gamma_{k-1}, 1, \ldots, 1; 1)$. Then let $\omega_{k-2}^{(k)} = \big((\gamma_{k-2}, 1, \ldots, 1; 1), 1, \ldots, 1; 1\big)$, and similarly define $\omega_j^{(k)}$ to be the 'nested' element $\big(\big(\ldots\big((\gamma_j, 1, \ldots, 1; 1), 1, \ldots, 1; 1\big)\ldots\big)\big)$ for all $j \in [k-1]$. (See Example 4.4 below.)

Finally, given any $j \in [k]$ we have by Lemma 2.13 that

$$\begin{aligned}\widetilde{\phi}(\omega_j^{(k)} P') &= \phi(\omega_j^{(k)}) = \phi(x_1, \ldots, x_k)(\omega_j^{(k)}) \\ &= \mathcal{X}\big(\phi(x_1, \ldots, x_{j-1}); \phi_{x_j}\big)(\gamma_j) = \phi_{x_j}(\gamma) = \big(\phi_{x_1} \times \cdots \times \phi_{x_k}\big)(\omega_j^{(k)} P'),\end{aligned}$$

This shows that $\widetilde{\phi} = \phi_{x_1} \times \cdots \times \phi_{x_k}$, as desired. □

Example 4.4. Let $k = 2$ and let $Q = P_{p^2}$. Then $Q/Q' \cong (P_p/P_p') \times C_p$. The element $(1, \ldots, 1; \gamma) \in P_p \wr P_p = Q$ maps into the second direct factor C_p under θ_2. The factor P_p/P_p' is isomorphic to $C_p = \langle \gamma \rangle$, which is mapped onto by $(\gamma, 1, \ldots, 1; 1) \in Q$. Thus $Q/Q' \cong \langle \omega_1^{(2)} Q' \rangle \times \langle \omega_2^{(2)} Q' \rangle$ where $\omega_1^{(2)} = (\gamma, 1, \ldots, 1; 1) \in Q$ and $\omega_2^{(2)} = \gamma_2 = (1, \ldots, 1; \gamma) \in Q$.

Now let $k = 3$. The above two generators $\omega_1^{(2)}$, $\omega_2^{(2)}$ can be 'lifted' via $P_{p^3}/{P_{p^3}}' \cong (Q/Q') \times C_p$ to give

$$\omega_1^{(3)} = \big((\gamma, 1, \ldots, 1; 1), 1, \ldots, 1; 1\big) \in P_{p^3}, \qquad \omega_2^{(3)} = \big((1, \ldots, 1; \gamma), 1, \ldots, 1; 1\big) \in P_{p^3}.$$

The final C_p factor is generated by $\omega_3^{(3)} {P_{p^3}}'$ where $\omega_3^{(3)} = (1, \ldots, 1; \gamma) \in P_{p^3}$, by the definition of θ_3.

More generally, $\gamma_k \in P_{p^k}$ viewed as an element of $\mathfrak{S}_{p^k} = \mathrm{Sym}\{1, \ldots, p^k\}$ via the permutation representation (2.3) coincides with σ_k as defined in Section 2.3.2 for all $k \in \mathbb{N}$, and $\omega_j^{(k)} = \gamma_j \in \mathrm{Sym}\{1, \ldots, p^j\} \leq \mathrm{Sym}\{1, \ldots, p^k\}$ for all $j \leq k$. ◊

Lemma 4.5. *Let $n \in \mathbb{N}$ and let $n = \sum_{i=1}^{k} a_i p^{n_i}$ be its p-adic expansion. Let $\phi(\underline{\mathbf{t}}), \phi(\underline{\mathbf{u}}) \in$* $\mathrm{Lin}(P_n)$. *Then $\phi(\underline{\mathbf{t}})$ and $\phi(\underline{\mathbf{u}})$ are $N_{\mathfrak{S}_n}(P_n)$–conjugate if and only if there exists $\sigma \in$* $\mathrm{Sym}[a_1] \times \cdots \times \mathrm{Sym}[a_k]$ *such that for each $i \in [k]$, there exists $\mathbf{s}(i) \in [\overline{p}]^{n_i}$ satisfying*

$$\mathbf{t}(i, \sigma(j)),\ \mathbf{u}(i, j) \in \Sigma\big(\mathbf{s}(i)\big)$$

for all $j \in [a_i]$.

Proof. For each $i \in [k]$, let $N_i = N_{\mathfrak{S}_{p^{n_i}}}(P_{p^{n_i}})$. Since $N_{\mathfrak{S}_n}(P_n) = N_1 \wr \mathfrak{S}_{a_1} \times \cdots \times N_k \wr \mathfrak{S}_{a_k}$, the statement follows from Lemma 4.3. □

Next, we have two technical lemmas.

Lemma 4.6. *Let $n, m \in \mathbb{N}$. Let A and B be characters of $\mathfrak{S}_n$ and let Z be a non-zero character of $\mathfrak{S}_m$. Then*

$$(A \times Z)\big\uparrow_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} = (B \times Z)\big\uparrow_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} \quad \text{if and only if} \quad A = B.$$

Proof. The 'if' direction is clear, so now suppose that $(A \times Z)\big\uparrow^{\mathfrak{S}_{n+m}} = (B \times Z)\big\uparrow^{\mathfrak{S}_{n+m}}$ and assume for a contradiction that $A \neq B$. For $X \in \{A, B, Z\}$, let $c_\lambda^X := \langle X, \chi^\lambda \rangle$, where λ is a partition of n (resp. m) if $X \in \{A, B\}$ (resp. $X = Z$). We define the following sets:

$$\mathcal{M} = \{\lambda \vdash n \mid c_\lambda^A \neq c_\lambda^B\} \text{ and } \mathcal{N} = \{\mu \vdash m \mid c_\mu^Z \neq 0\},$$

which by assumption are non-empty. Let $\overline{\lambda}$ and $\overline{\mu}$ be the lexicographically greatest partitions in $\mathcal{M}$ and $\mathcal{N}$ respectively, and let α be the partition of $n+m$ defined by $\alpha = \overline{\lambda} + \overline{\mu} := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$. By the Littlewood–Richardson rule, we have that

$$\begin{aligned}\Big\langle (A \times Z)\big\uparrow^{\mathfrak{S}_{n+m}}, \chi^\alpha \Big\rangle &= c_{\overline{\lambda}}^A c_{\overline{\mu}}^Z c_{\overline{\lambda}\overline{\mu}}^\alpha + \sum_{\lambda > \overline{\lambda}} \sum_{\mu \in \mathcal{N}} c_\lambda^A c_\mu^Z c_{\lambda\mu}^\alpha = c_{\overline{\lambda}}^A c_{\overline{\mu}}^Z c_{\overline{\lambda}\overline{\mu}}^\alpha + \sum_{\lambda > \overline{\lambda}} \sum_{\mu \in \mathcal{N}} c_\lambda^B c_\mu^Z c_{\lambda\mu}^\alpha \\ &\neq c_{\overline{\lambda}}^B c_{\overline{\mu}}^Z c_{\overline{\lambda}\overline{\mu}}^\alpha + \sum_{\lambda > \overline{\lambda}} \sum_{\mu \in \mathcal{N}} c_\lambda^B c_\mu^Z c_{\lambda\mu}^\alpha = \Big\langle (B \times Z)\big\uparrow^{\mathfrak{S}_{n+m}}, \chi^\alpha \Big\rangle.\end{aligned}$$

since $c_{\overline{\mu}}^Z c_{\overline{\lambda}\overline{\mu}}^\alpha \neq 0$. This contradicts $(A \times Z)\big\uparrow = (B \times Z)\big\uparrow$. □

Lemma 4.7. *Let $a, n, m \in \mathbb{N}$ and suppose $n > m$. Let $P \times Q \leq \mathfrak{S}_{an} \times \mathfrak{S}_m \leq \mathfrak{S}_{an+m}$ be such that P contains an element σ which is a product of a disjoint n-cycles. Let $g \in Q$. Let χ be a character of P and η be a character of Q. Then*

$$(\chi \times \eta)\big\uparrow_{P \times Q}^{\mathfrak{S}_{an+m}}(\sigma g) = \chi\big\uparrow_P^{\mathfrak{S}_{an}}(\sigma) \cdot \eta\big\uparrow_Q^{\mathfrak{S}_m}(g).$$

Proof. This follows from the definition of induced characters, after observing that $\sigma^x \in P \times Q$ if and only if $\sigma^x \in P$, for all $x \in \mathfrak{S}_{an+m}$. □

It turns out that the difficult part of Theorem 4.1 is the case when $n = ap^k$, which we have stated as Theorem 4.8 below and whose proof has been postponed to Section 4.3.

Theorem 4.8. *Let $a \in [p-1]$ and $k \in \mathbb{N}$. Let $\phi, \psi \in \mathrm{Lin}(P_{ap^k})$ be such that $\phi\big\uparrow^{\mathfrak{S}_{ap^k}} = \psi\big\uparrow^{\mathfrak{S}_{ap^k}}$. Then ϕ and ψ are $N_{\mathfrak{S}_{ap^k}}(P_{ap^k})$–conjugate.*

Assuming Theorem 4.8, we are able to prove Theorem 4.1.

Proof of Theorem 4.1. Let $n = a_1 p^{n_1} + \cdots + a_k p^{n_k}$ be the p-adic expansion of n, with $k \in \mathbb{N}$, $a_i \in [p-1]$ for all i and $0 \leq n_1 < \cdots < n_k$. We proceed by induction on k. If

$k=1$ then the statement holds by Theorem 4.8. Suppose that $k\geq 2$ and assume for a contradiction that ϕ and ψ are not $N_{\mathfrak{S}_n}(P_n)$–conjugate. Let $m=a_kp^{n_k}$ and write $\phi=\phi_1\times\phi_2$ and $\psi=\psi_1\times\psi_2$ where $\phi_1,\psi_1\in\mathrm{Lin}(P_m)$ and $\phi_2,\psi_2\in\mathrm{Lin}(P_{n-m})$. Since ϕ and ψ are not $N_{\mathfrak{S}_n}(P_n)$–conjugate and $N_{\mathfrak{S}_n}(P_n)\cong N_{\mathfrak{S}_m}(P_m)\times N_{\mathfrak{S}_{n-m}}(P_{n-m})$, at least one of the following two statements must hold:

(i) ϕ_1 and ψ_1 are not $N_{\mathfrak{S}_m}(P_m)$–conjugate;

(ii) ϕ_2 and ψ_2 are not $N_{\mathfrak{S}_{n-m}}(P_{n-m})$–conjugate.

Since $P_n\cong P_m\times P_{n-m}$, we have that

$$(\phi_1\big\uparrow_{P_m}^{\mathfrak{S}_m}\times\phi_2\big\uparrow_{P_{n-m}}^{\mathfrak{S}_{n-m}})\big\uparrow^{\mathfrak{S}_n}=\phi\big\uparrow_{P_n}^{\mathfrak{S}_n}=\psi\big\uparrow_{P_n}^{\mathfrak{S}_n}=(\psi_1\big\uparrow_{P_m}^{\mathfrak{S}_m}\times\psi_2\big\uparrow_{P_{n-m}}^{\mathfrak{S}_{n-m}})\big\uparrow^{\mathfrak{S}_n}$$

and so using Lemmas 4.2, 4.6 and the inductive hypothesis, we deduce that both conditions (i) and (ii) must hold. Let $g\in\mathfrak{S}_{n-m}$ be such that $\phi_2\big\uparrow^{\mathfrak{S}_{n-m}}(g)\neq\psi_2\big\uparrow^{\mathfrak{S}_{n-m}}(g)$; such an element exists by the inductive hypothesis. Let $\sigma\in P_m\leq\mathfrak{S}_m$ be a product of a_k disjoint p^{n_k}-cycles. We now denote by h the element of $\mathfrak{S}_m\times\mathfrak{S}_{n-m}\leq\mathfrak{S}_n$ defined as follows:

$$h=\begin{cases}\sigma & \text{if } \phi_1\big\uparrow^{\mathfrak{S}_m}(\sigma)\neq\psi_1\big\uparrow^{\mathfrak{S}_m}(\sigma),\\ \sigma g & \text{otherwise.}\end{cases}$$

Then $\phi\big\uparrow^{\mathfrak{S}_n}(h)\neq\psi\big\uparrow^{\mathfrak{S}_n}(h)$ by Lemma 4.7, a contradiction. □

4.3 Induced character values

Throughout this section, let $n=ap^k$ where $k\in\mathbb{N}$ and $a\in[p-1]$. Recall from (2.4) that the linear characters of P_n are parametrised as $\phi(\underline{\mathbf{u}})$ where $\underline{\mathbf{u}}=\big(\mathbf{u}(1),\mathbf{u}(2),\dots,\mathbf{u}(a)\big)$ with $\mathbf{u}(i)\in[\overline{p}]^k$ for each i. By Lemmas 4.2 and 4.5, we need only distinguish when the elements of the sequences $\mathbf{u}(i)$ are equal to 0 (corresponding to $\mathbb{1}_{P_p}$) or not (corresponding to some non-trivial linear character of P_p). Thus we may identify all of the values which are not equal to 0. Our proofs involve computing certain character values, for which the following notational convention will be useful, in particular for Lemma 4.13.

Notation 4.9. *In this section (Section 4.3) only, we sometimes rewrite* $\mathbf{u}(i)\in[\overline{p}]^k$ *as* $\mathsf{u}^i=(\mathsf{u}^i_1,\dots,\mathsf{u}^i_k)\in\{0,1\}^k$ *where*

$$\mathsf{u}^i_j=\begin{cases}1 & \text{if } \mathbf{u}(i)_j=0\\ 0 & \text{if } \mathbf{u}(i)_j\in\{1,2,\dots,p-1\},\end{cases}$$

and let $\underline{\mathsf{u}}=(\mathsf{u}^1,\dots,\mathsf{u}^a)$.

So for instance, in this section we do not distinguish between characters $\phi(s)$, $\phi(t)\in\mathrm{Lin}(P_{p^k})$ for $s,t\in[\overline{p}]^k$ if $\phi(t)\in\Sigma(s)$. If $\mathsf{u}\in\{0,1\}^k$ then by $\phi(\mathsf{u})$ we mean a (any) linear character $\phi(s) \in \mathrm{Lin}(P_{p^k})$ where $s \in [\overline{p}]^k$ such that u corresponds to $u \in [\overline{p}]^k$ in the sense of Notation 4.9, and s and u have 0s in the same positions. We shall adhere to using italics or bold (e.g. s, $\mathbf{s}$) to denote elements of $[\overline{p}]^k$, and sans serif (e.g. u) for elements in $\{0,1\}^k$. We then extend these notational conventions from $\mathrm{Lin}(P_{p^k})$ to $\mathrm{Lin}(P_m)$ for all $m \in \mathbb{N}$.

The aim of this section is to prove the following statement, which, in light of Lemma 4.5, is equivalent to Theorem 4.8.

Theorem 4.10. *Let $k \in \mathbb{N}$ and let $a \in [p-1]$. Let $\phi(\underline{\mathsf{s}}), \phi(\underline{\mathsf{t}}) \in \mathrm{Lin}(P_{ap^k})$. Suppose $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_{ap^k}} = \phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_{ap^k}}$. Then there exists a permutation $\sigma \in \mathrm{Sym}\{1,\dots,a\}$ such that $\mathsf{s}^i = \mathsf{t}^{\sigma(i)}$ for all i.*

The key idea is to consider the values $\phi\big\uparrow^{\mathfrak{S}_n}(g)$ where $\phi \in \mathrm{Lin}(P_n)$ and g is a product of disjoint cycles whose lengths are distinct powers of p. By the definition of induced characters,

$$\phi\big\uparrow_{P_n}^{\mathfrak{S}_n}(g) = \frac{|C_{\mathfrak{S}_n}(g)|}{|P_n|} \sum_{x \in \mathrm{ccl}_{\mathfrak{S}_n}(g) \cap P_n} \phi(x)$$

for any $g \in \mathfrak{S}_n$, where $\mathrm{ccl}_{\mathfrak{S}_n}(g)$ denotes the conjugacy class of g in $\mathfrak{S}_n$.

Remark 4.11. Our proof involves computing the values of induced characters at elements of $\mathfrak{S}_n$ of several distinct cycle types. In general, a single cycle type may or may not be enough to distinguish the inductions $\phi\big\uparrow^{\mathfrak{S}_n}$ as ϕ runs over a set of orbit representatives in $\mathrm{Lin}(P_n)$ under the action of $N_{\mathfrak{S}_n}(P_n)$. That is, we do not know whether there exists $g \in \mathfrak{S}_n$ with the property that $\phi\big\uparrow^{\mathfrak{S}_n}(g) = \psi\big\uparrow^{\mathfrak{S}_n}(g)$ for $\phi, \psi \in \mathrm{Lin}(P_n)$ if and only if ϕ and ψ are $N_{\mathfrak{S}_n}(P_n)$–conjugate.

Nevertheless, since there is no loss in considering elements of several different cycle types compared with limiting ourselves to just one type, in our arguments below we consider all g which are products of distinct p-power length cycles. $\lozenge$

Definition 4.12. *Let $b \in [p-1]$ and suppose that $l_1, l_2, \dots, l_b$ are distinct elements of $[k]$. Let $g \in \mathfrak{S}_{p^k}$ have disjoint cycles of length $p^{l_1}, p^{l_2}, \cdots, p^{l_b}, 1, \dots, 1$ (we also say g* has cycle type $p^{l_1}p^{l_2}\cdots p^{l_b}$*) and let $\mathsf{u} \in \{0,1\}^k$. Define*

$$\Gamma_{l_1 l_2 \dots l_b;k}(\mathsf{u}) = \sum_{x \in \mathrm{ccl}_{\mathfrak{S}_{p^k}}(g) \cap P_{p^k}} \phi(\mathsf{u})(x).$$

If $b \geq 2$ and $l_i = k$ for some i, then we set $\Gamma_{l_1 l_2 \dots l_b;k}(\mathsf{u})$ to be 0. More generally, we define $\Gamma_{l_1 l_2 \dots l_b;k}(\mathsf{u})$ for any distinct natural numbers $l_1, \dots, l_b$ by setting the value to be 0 if $l_i > k$ for any i.

In particular,

$$\phi(\mathsf{u})\big\uparrow_{P_{p^k}}^{\mathfrak{S}_{p^k}}(g) = \frac{|C_{\mathfrak{S}_{p^k}}(g)|}{|P_{p^k}|} \cdot \Gamma_{l_1 \dots l_b;k}(\mathsf{u}).$$

Thus for such a fixed element g, when we compare the values of $\phi(\mathsf{u})\big\uparrow_{P_{p^k}}^{\mathfrak{S}_{p^k}}(g)$ and

$\phi(\mathsf{u}')\big\uparrow_{P_{p^k}}^{\mathfrak{S}_{p^k}}(g)$ for some u and u', it is enough to compare the values of $\Gamma_{l_1\ldots l_b;k}(\mathsf{u})$ and $\Gamma_{l_1\ldots l_b;k}(\mathsf{u}')$.

Lemma 4.13. *Let $l \in [k]$ and let $\mathsf{u} = (\mathsf{u}_1, \ldots, \mathsf{u}_k) \in \{0,1\}^k$. Then*

$$\Gamma_{l;k}(\mathsf{u}) = p^k \cdot C_l(\mathsf{u}), \quad \text{where} \quad C_l(\mathsf{u}) = p^{\frac{p^l-1}{p-1}-2l} \prod_{m=1}^{l} (p\mathsf{u}_m - 1).$$

Remark 4.14. It is useful to observe that $C_l(\mathsf{u})$ does not depend on $\mathsf{u}_{l+1}, \ldots, \mathsf{u}_k$; that is, $C_l(\mathsf{u}) = C_l\big((\mathsf{u}_1, \ldots, \mathsf{u}_l)\big)$. ♢

Proof of Lemma 4.13. Observe that if $l = k = 1$, then

$$\Gamma_{1;1}(\mathsf{u}) = \sum_{x \in P_p \setminus \{1\}} \phi(\mathsf{u})(x) = \begin{cases} p-1 & \text{if } \phi(\mathsf{u}) = \mathbb{1}_{P_p} \quad (\text{i.e. if } \mathsf{u} = (1)), \\ -1 & \text{otherwise.} \end{cases}$$

(In other words, $\Gamma_{1;1}(\mathsf{u}) = p\mathsf{u}_1 - 1$.) Now let $k \geq 2$ and first suppose $l = k$. Let $x = (f_1, \ldots, f_p; \sigma) \in P_{p^k} = P_{p^{k-1}} \wr P_p$ where $f_i \in P_{p^{k-1}}$ and $\sigma \in P_p$. If x is a p^k-cycle, then $\sigma \neq 1$ and $f_{\sigma^{p-1}(1)} \cdots f_{\sigma(1)} \cdot f_1$ must be a p^{k-1}-cycle in $P_{p^{k-1}}$ by Lemma 2.22. Letting $\mathsf{u}^- = (\mathsf{u}_1, \ldots, \mathsf{u}_{k-1})$ and Y be the set of elements in $P_{p^{k-1}}$ of cycle type p^{k-1}, we find by Lemma 2.13 that

$$\begin{aligned}
\Gamma_{k;k}(\mathsf{u}) &= \sum_{\substack{x=(f_1,\ldots,f_p;\sigma)\in P_{p^k} \\ \text{of cycle type } p^k}} \underbrace{\mathcal{X}\big(\phi(\mathsf{u}^-); \phi(\mathsf{u}_k)\big)}_{\phi(u)} \big((f_1, \ldots, f_p; \sigma)\big) \\
&= \sum_{\substack{x=(f_1,\ldots,f_p;\sigma), \\ \sigma \in P_p\setminus\{1\},\ f_1,\ldots,f_p \in P_{p^{k-1}}, \\ f_{\sigma^{p-1}(1)}\cdots f_{\sigma(1)}\cdot f_1 \in Y}} \phi(\mathsf{u}^-)(f_{\sigma^{p-1}(1)} \cdots f_{\sigma(1)} \cdot f_1) \cdot \phi(\mathsf{u}_k)(\sigma) \\
&= \sum_{\sigma \in P_p \setminus \{1\}} \phi(\mathsf{u}_k)(\sigma) \cdot |P_{p^{k-1}}|^{p-1} \cdot \sum_{y \in Y} \phi(\mathsf{u}^-)(y) \\
&= (pu_k - 1) \cdot p^{p^{k-1}-1} \cdot \Gamma_{k-1;k-1}(\mathsf{u}^-),
\end{aligned}$$

where the third equality holds since for any fixed $y \in Y$, we may choose the elements $f_1, \ldots, f_{\sigma^{p-2}(1)}$ in $P_{p^{k-1}}$ freely, after which $f_{\sigma^{p-1}(1)} \cdots f_{\sigma(1)} \cdot f_1 = y$ uniquely determines $f_{\sigma^{p-1}(1)}$. Inductively, we have

$$\Gamma_{k;k}(\mathsf{u}) = p^{(p^{k-1}+p^{k-2}+\cdots+1)-k} \cdot \prod_{m=1}^{k} (p\mathsf{u}_m - 1) = p^{\frac{p^k-1}{p-1}-k} \cdot \prod_{m=1}^{k} (p\mathsf{u}_m - 1).$$

Next, let $1 \leq l < k$. If $x = (f_1, \ldots, f_p; \sigma) \in P_{p^k}$ has cycle type p^l then it must have a fixed point as $l < k$. Thus $\sigma = 1$ and f_i has cycle type p^l for a unique $1 \leq i \leq p$ and

$f_j = 1$ for all $j \neq i$. Letting Z be the set of elements in $P_{p^{k-1}}$ of cycle type p^l,

$$\begin{aligned}\Gamma_{l;k}(\mathsf{u}) &= \sum_{\substack{x=(f_1,\dots,f_p;1),\\ \exists! i,\ f_i\in Z,\ f_j=1\ \forall\ j\neq i}} \phi(\mathsf{u}^-)(f_1)\cdots\phi(\mathsf{u}^-)(f_p)\cdot\phi(\mathsf{u}_k)(1)\\ &= p\sum_{z\in Z}\phi(\mathsf{u}^-)(z) = p\cdot\Gamma_{l;k-1}(\mathsf{u}^-) = p^{k-l}\cdot\Gamma_{l;l}\big((\mathsf{u}_1,\dots,\mathsf{u}_l)\big).\end{aligned}$$

Hence for all $1\leq l\leq k$,

$$\Gamma_{l;k}(\mathsf{u}) = p^{k-l}\cdot p^{\frac{p^l-1}{p-1}-l}\cdot\prod_{m=1}^{l}(p\mathsf{u}_m-1).$$

□

Remark 4.15. Lemma 4.13 is already enough to prove Theorem 4.10 when $a=1$. Indeed, let $\phi(\mathsf{s}),\phi(\mathsf{t})\in\mathrm{Lin}(P_{p^k})$ for some $\mathsf{s},\mathsf{t}\in\{0,1\}^k$ and suppose $\phi(\mathsf{s})\big\uparrow^{\mathfrak{S}_{p^k}} = \phi(\mathsf{t})\big\uparrow^{\mathfrak{S}_{p^k}}$. Then $\phi(\mathsf{s})\big\uparrow^{\mathfrak{S}_{p^k}}(g) = \phi(\mathsf{t})\big\uparrow^{\mathfrak{S}_{p^k}}(g)$ for each $g\in\mathfrak{S}_{p^k}$, in particular g of cycle type p^l for every $l\in[k]$. By Lemma 4.13, this implies

$$\prod_{m=1}^{l}(p\mathsf{s}_m-1) = \prod_{m=1}^{l}(p\mathsf{t}_m-1)$$

for all $l\in[k]$. Therefore $\mathsf{s}_m=\mathsf{t}_m$ for all $m\in[k]$, and thus $\mathsf{s}=\mathsf{t}$.

However, Lemma 4.13 is not enough when $a>1$. For example, let $a=2$, $k=3$ and consider $\underline{\mathsf{s}} = \big((1,0,0),(0,1,1)\big)$ and $\underline{\mathsf{t}} = \big((1,0,1),(0,1,0)\big)$. The induced characters $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_{2p^3}}$ and $\phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_{2p^3}}$ agree on p, p^2 and p^3-cycles, though are not equal. This motivates considering more complicated cycle types, see Proposition 4.17 below. ◊

From now on, we may assume $2\leq a<p$, and hence also that p is odd.

Definition 4.16. *For a set A, let* $\mathrm{Part}\,A = \{X \mid \bigsqcup_{Y\in X} Y = A\}$ *be the set of partitions of A. (Our convention is that $Y\neq\emptyset$, i.e. $\emptyset\notin X$.) Suppose $X=\{Y_1,\dots,Y_m\}$ is a partition of the set A, with $y_i=|Y_i|$ for each i and $y_1\geq\cdots\geq y_m\geq 1$. We say that $(y_1,y_2,\dots,y_m)$, a partition of the number $\sum_i y_i = |A|$, is the* type *of X.*

Proposition 4.17. *Let $b\in\{2,3,\dots,p-1\}$. Let $1\leq l_1<l_2<\cdots<l_b\leq k$ be integers and let $\mathsf{u}\in\{0,1\}^k$. Then*

$$\Gamma_{l_1l_2\dots l_b;k}(\mathsf{u}) = p^k\cdot C_{l_1}(\mathsf{u})\cdots C_{l_b}(\mathsf{u})\cdot(p^k-p^{l_b})(p^k-p^{l_b}-p^{l_{b-1}})\cdots(p^k-p^{l_b}-\cdots-p^{l_2}).$$

Proof. Both sides of the equation equal 0 if $l_b=k$, so from now on assume $l_b<k$. We proceed by induction on b, beginning with the base case $b=2$. Let $\mathsf{u}^-=(\mathsf{u}_1,\dots,\mathsf{u}_{k-1})$. Let $x=(f_1,\dots,f_p;\sigma)\in P_{p^k}=P_{p^{k-1}}\wr P_p$ be of cycle type $p^{l_1}p^{l_2}$. Then it must have a fixed point as $l_1<l_2$, and so $\sigma=1$. Let $g_z\in P_{p^{k-1}}$ be of cycle type p^{l_z} for $z\in\{1,2\}$, let $g_3\in P_{p^{k-1}}$ be of cycle type $p^{l_1}p^{l_2}$ and let $G_z=\mathrm{ccl}_{\mathfrak{S}_{p^{k-1}}}(g_z)\cap P_{p^{k-1}}$ for $z\in\{1,2,3\}$.

Then by Lemma 2.13,

$$\begin{aligned}
\Gamma_{l_1 l_2;k}(\mathsf{u}) &= \sum_{\substack{x=(f_1,\ldots,f_p;1)\in P_{p^k} \\ \text{of cycle type } p^{l_1}p^{l_2}}} \phi(\mathsf{u})(x) \\
&= \sum_{i=1}^{p} \sum_{\substack{j=1 \\ j\neq i}}^{p} \sum_{\substack{f_i\in G_1,\ f_j\in G_2, \\ f_h=1\ \forall\ h\neq i,j}} \phi(\mathsf{u}^-)(f_i)\cdot\phi(\mathsf{u}^-)(f_j)\cdot\big(\phi(\mathsf{u}^-)(1)\big)^{p-2}\cdot\phi(\mathsf{u}_k)(1) \\
&\qquad + \sum_{i=1}^{p} \sum_{\substack{f_i\in G_3, \\ f_h=1\ \forall\ h\neq i}} \phi(\mathsf{u}^-)(f_i)\cdot\big(\phi(\mathsf{u}^-)(1)\big)^{p-1}\cdot\phi(\mathsf{u}_k)(1) \\
&= p(p-1)\cdot\Gamma_{l_1;k-1}(\mathsf{u}^-)\cdot\Gamma_{l_2;k-1}(\mathsf{u}^-) + p\cdot\Gamma_{l_1 l_2;k-1}(\mathsf{u}^-).
\end{aligned}$$

Recalling that $\Gamma_{l_1 l_2;k'}(\mathsf{u}) = 0$ if $l_2 \geq k'$, we therefore have by Lemma 4.13 that

$$\begin{aligned}
\Gamma_{l_1 l_2;k}(\mathsf{u}) &= (p-1)\sum_{i=l_2}^{k-1} p^{k-i}\cdot\Gamma_{l_1;i}\big((\mathsf{u}_1,\ldots,\mathsf{u}_i)\big)\cdot\Gamma_{l_2;i}\big((\mathsf{u}_1,\ldots,\mathsf{u}_i)\big) \\
&= (p-1)\sum_{i=l_2}^{k-1} p^{k-i}\cdot p^i C_{l_1}(\mathsf{u})\cdot p^i C_{l_2}(\mathsf{u}) \\
&= p^k\cdot C_{l_1}(\mathsf{u})\cdot C_{l_2}(\mathsf{u})\cdot(p-1)\sum_{i=l_2}^{k-1} p^i = p^k\cdot C_{l_1}(\mathsf{u})\cdot C_{l_2}(\mathsf{u})\cdot(p^k - p^{l_2}).
\end{aligned}$$

This concludes the base case $b = 2$.

For the inductive step: if $x = (f_1,\ldots,f_p;\sigma) \in P_{p^k}$ has cycle type $p^{l_1}\cdots p^{l_b}$ then it must have a fixed point as the l_i are distinct. Hence $\sigma = 1$, and thus the cycle type of $x = (f_1,\ldots,f_p;1)$ is the product of the cycle types of $f_1,\ldots,f_p$. By Lemma 2.13,

$$\begin{aligned}
\Gamma_{l_1\ldots l_b;k}(\mathsf{u}) &= \sum_{\substack{x=(f_1,\ldots,f_p;1)\in P_{p^k} \\ \text{of cycle type } p^{l_1}\cdots p^{l_b}}} \phi(\mathsf{u})(x) \\
&= \sum_{x} \phi(\mathsf{u}^-)(f_1)\cdot\phi(\mathsf{u}^-)(f_2)\cdots\phi(\mathsf{u}^-)(f_p)\cdot\phi(\mathsf{u}_k)(1).
\end{aligned}$$

(Notice that $\phi(\mathsf{u}_k)(1) = 1$ since $\phi(\mathsf{u}_k)$ is linear.)

By considering the cycle type of each f_i, we can rewrite this sum as follows. Let $I = \{i \mid f_i = 1\}$ and suppose $|I| = p - L$ for some $L \in \{0,1,\ldots,p-1\}$. Say $[p]\setminus I = \{i_1,\ldots,i_L\}$, and suppose for each $j \in [L]$ that the cycle type of f_{i_j} is $\prod_{m\in\nu_j} p^{l_m}$. Then $\{\nu_j\}_{j=1}^{L}$ is a partition of the set $[b]$, since x has cycle type $p^{l_1}\cdots p^{l_b}$. (Note that for each j, $\nu_j \neq \emptyset$ since $i_j \notin I$, and ν_j is a genuine set rather than a multiset since $l_1,\ldots,l_b$ are distinct.)

We sum over such partitions $\{\nu_j\}_{j=1}^{L}$ of $[b]$, grouping by type (see Definition 4.16). In particular, if $\{\nu_j\}_{j=1}^{L}$ has type $\lambda \vdash b$, then $L = l(\lambda)$ and so $p - l(\lambda)$ many of the elements $f_1,\ldots,f_p$ are equal to 1. Conversely, given $\lambda \vdash b$ and some $\nu = \{\nu_1,\ldots,\nu_{l(\lambda)}\} \in \mathrm{Part}[b]$

of type λ, there are $\frac{p!}{(p-l(\lambda))!}$ many injective mappings F from ν to $[p]$. Each such F represents a different assignment of cycle types to the elements $f_1,\ldots,f_p$: for $i\in[p]$, if $i\notin \mathrm{Im}(F)$ then $f_i=1$, and if $i=F(\nu_j)$ then the cycle type of f_i is $\prod_{m\in\nu_j}p^{l_m}$.

If $\omega=\{w_1,\ldots,w_t\}\subset[b]$, let $\Gamma_{\omega;k-1}(\mathsf{u}^-)$ denote $\Gamma_{l_{w_1}\cdots l_{w_t};k-1}(\mathsf{u}^-)$. Since and $\Gamma_{\nu_j;k-1}(\mathsf{u}^-)=\sum_y \phi(\mathsf{u}^-)(y)$ as y runs over the elements of $P_{p^{k-1}}$ of cycle type $\prod_{m\in\nu_j}p^{l_m}$, and $\phi(\mathsf{u}^-)$ is linear, then

$$\begin{aligned}\Gamma_{l_1\ldots l_b;k}(\mathsf{u}) &= \sum_{\substack{x=(f_1,\ldots,f_p;1)\in P_{p^k}\\ \text{of cycle type } p^{l_1}\cdots p^{l_b}}} \phi(\mathsf{u}^-)(f_1)\cdot\phi(\mathsf{u}^-)(f_2)\cdots\phi(\mathsf{u}^-)(f_p)\\ &=\sum_{\lambda\vdash b}\sum_{\substack{\nu\in\mathrm{Part}[b]\\ \text{of type }\lambda}}\frac{p!}{\big(p-l(\lambda)\big)!}\cdot\prod_{\omega\in\nu}\Gamma_{\omega;k-1}(\mathsf{u}^-)\cdot\big(\phi(\mathsf{u}^-)(1)\big)^{p-l(\lambda)}\\ &= p\cdot\sum_{\substack{\lambda\vdash b\\ \lambda\neq(b)}}\sum_{\substack{\nu\in\mathrm{Part}[b]\\ \text{of type }\lambda}}\frac{(p-1)!}{\big(p-l(\lambda)\big)!}\cdot\prod_{\omega\in\nu}\Gamma_{\omega;k-1}(\mathsf{u}^-)+p\cdot\Gamma_{l_1\ldots l_b;k-1}(\mathsf{u}^-).\end{aligned}$$

Inductively, we therefore have

$$\Gamma_{l_1\ldots l_b;k}(\mathsf{u})=\sum_{i=l_b}^{k-1}p^{k-i}\sum_{\substack{\lambda\vdash b\\ \lambda\neq(b)}}\sum_{\substack{\nu\in\mathrm{Part}[b]\\ \text{of type }\lambda}}\frac{(p-1)!}{\big(p-l(\lambda)\big)!}\cdot\prod_{\omega\in\nu}\Gamma_{\omega;i}\big((\mathsf{u}_1,\ldots,\mathsf{u}_i)\big)\tag{4.1}$$

since $l_1<\cdots<l_b$ and $\Gamma_{l_1\ldots l_b;j}(\mathsf{u})=0$ if $l_i>j$ for any i.

Since $\lambda\neq(b)$, every ω appearing in (4.1) satisfies $|\omega|<b$. Therefore, if $\omega=\{w_1>w_2>\cdots>w_t\}$ then by the inductive hypothesis and Remark 4.14,

$$\Gamma_{\omega;i}\big((\mathsf{u}_1,\ldots,\mathsf{u}_i)\big)=p^i\cdot C_{l_{w_1}}(\mathsf{u})\cdots C_{l_{w_t}}(\mathsf{u})\cdot(-1)^{t-1}\cdot P_i^\omega$$

where

$$P_i^\omega:=(-p^i+p^{l_{w_1}})(-p^i+p^{l_{w_1}}+p^{l_{w_2}})\cdots(-p^i+p^{l_{w_1}}+p^{l_{w_2}}+\cdots+p^{l_{w_{t-1}}})$$

if $t>1$, or $P_i^\omega:=1$ if $t=1$. Substituting this into (4.1),

$$\begin{aligned}\Gamma_{l_1\ldots l_b;k}(\mathsf{u})&=\sum_{i=l_b}^{k-1}p^{k-i}\sum_{\substack{\lambda\vdash b\\ \lambda\neq(b)}}\sum_{\substack{\nu\in\mathrm{Part}[b]\\ \text{of type }\lambda}}\frac{(p-1)!}{\big(p-l(\lambda)\big)!}\cdot\prod_{\omega\in\nu}\left[p^i\left(\prod_{w\in\omega}C_{l_w}(\mathsf{u})\right)\cdot(-1)^{|\omega|-1}\cdot P_i^\omega\right]\\ &=\sum_{i=l_b}^{k-1}p^{k-i}\sum_{\substack{\lambda\vdash b\\ \lambda\neq(b)}}\sum_{\substack{\nu\in\mathrm{Part}[b]\\ \text{of type }\lambda}}\frac{(p-1)!\cdot p^{il(\lambda)}\cdot(-1)^{b-l(\lambda)}}{\big(p-l(\lambda)\big)!}\cdot C_{l_1}(\mathsf{u})\cdots C_{l_b}(\mathsf{u})\cdot\prod_{\omega\in\nu}P_i^\omega\\ &=(-1)^{b-1}p^k\prod_{i=1}^b C_{l_i}(\mathsf{u})\sum_{i=l_b}^{k-1}\sum_{\substack{\lambda\vdash b\\ \lambda\neq(b)}}\sum_{\substack{\nu\in\mathrm{Part}[b]\\ \text{of type }\lambda}}\frac{(p-1)!\cdot p^{i(l(\lambda)-1)}\cdot(-1)^{l(\lambda)-1}}{\big(p-l(\lambda)\big)!}\cdot\prod_{\omega\in\nu}P_i^\omega.\end{aligned}$$

Thus, to conclude the proof of the proposition, it suffices to show that the following equality holds:

$$
\begin{aligned}
&(-p^k+p^{l_b})(-p^k+p^{l_b}+p^{l_{b-1}})\cdots(-p^k+p^{l_b}+\cdots+p^{l_2})\\
&\qquad=\sum_{i=l_b}^{k-1}\sum_{\substack{\lambda\vdash b\\ \lambda\neq(b)}}\sum_{\substack{\nu\in\mathrm{Part}[b]\\ \text{of type }\lambda}}\frac{(p-1)!\cdot p^{i(l(\lambda)-1)}\cdot(-1)^{l(\lambda)-1}}{\big(p-l(\lambda)\big)!}\cdot\prod_{\omega\in\nu}P_i^{\omega}.
\end{aligned}
\tag{4.2}
$$

Observe by the inductive hypothesis that (4.2) holds (replacing b by b') for all $b' < b$, and indeed for all sets X of size b' (by replacing $[b]$ by X). We rewrite the right-hand side of (4.2) as a sum over $\mu \vdash b-1$ and $\gamma \in \mathrm{Part}\{2,3,\ldots,b\}$ where ν is either γ with one of its elements X replaced by $X \cup \{1\}$, or $\nu = \gamma \cup \{\{1\}\}$. Thus the right-hand side of (4.2) equals

$$
\begin{aligned}
&\sum_{i=l_b}^{k-1}\Bigg[\sum_{\substack{\mu\vdash b-1\\ \mu\neq(b-1)}}\sum_{\substack{\gamma\in\mathrm{Part}\{2,\ldots,b\}\\ \text{of type }\mu}}\Bigg(\underbrace{\frac{(p-1)!\cdot p^{i(l(\mu)-1)}\cdot(-1)^{l(\mu)-1}}{\big(p-l(\mu)\big)!}\prod_{\omega\in\gamma}P_i^{\omega}\cdot\sum_{\omega\in\gamma}(-p^i+\textstyle\sum_{h=1}^{|\omega|}p^{l_{w_h}})}_{\nu\ =\ \gamma\text{ with 1 added to an existing member of }\gamma}\\
&\quad+\underbrace{\frac{(p-1)!\cdot p^{il(\mu)}\cdot(-1)^{l(\mu)}}{\big(p-l(\mu)-1\big)!}\cdot\prod_{\omega\in\gamma}P_i^{\omega}}_{\nu\ =\ \gamma\ \cup\ \{\{1\}\}}\Bigg)+\underbrace{\frac{(p-1)!\cdot p^i\cdot(-1)}{(p-2)!}\cdot P_i^{\{2,3,\ldots,b\}}\cdot P_i^{\{1\}}}_{\substack{\mu=(b-1)\text{ term: }\gamma=\{\{2,3,\ldots,b\}\},\\ \lambda\neq(b)\text{ so }\nu=\{\{1\},\{2,\ldots,b\}\}\text{ only}}}\Bigg]\\
&=\sum_{i=l_b}^{k-1}\Bigg[\sum_{\substack{\mu\vdash b-1\\ \mu\neq(b-1)}}\sum_{\substack{\gamma\in\mathrm{Part}\{2,\ldots,b\}\\ \text{of type }\mu}}\frac{(p-1)!\cdot p^{i(l(\mu)-1)}\cdot(-1)^{l(\mu)-1}}{\big(p-l(\mu)\big)!}\prod_{\omega\in\gamma}P_i^{\omega}\cdot(p^{l_2}+\cdots+p^{l_b}-p^{i+1})\\
&\quad-(p-1)p^i\cdot P_i^{\{2,\ldots,b\}}\Bigg].
\end{aligned}
$$

Let

$$
Q_i=\sum_{\substack{\mu\vdash b-1\\ \mu\neq(b-1)}}\sum_{\substack{\gamma\in\mathrm{Part}\{2,\ldots,b\}\\ \text{of type }\mu}}\frac{(p-1)!\cdot p^{i(l(\mu)-1)}\cdot(-1)^{l(\mu)-1}}{\big(p-l(\mu)\big)!}\prod_{\omega\in\gamma}P_i^{\omega}.
$$

Since (4.2) holds for $b' = b-1$ and the set $X = \{l_2, \ldots, l_b\}$, we have that

$$
(-p^k+p^{l_b})(-p^k+p^{l_b}+p^{l_{b-1}})\cdots(-p^k+p^{l_b}+\cdots+p^{l_3})=\sum_{i=l_b}^{k-1}Q_i.
$$

Since the only condition on k for this immediately preceding equation to hold is that $k > l_b$, we also know for all $i > l_b$ (by replacing k by i) that

$$
(-p^i+p^{l_b})(-p^i+p^{l_b}+p^{l_{b-1}})\cdots(-p^i+p^{l_b}+\cdots+p^{l_3})=\sum_{j=l_b}^{i-1}Q_j.
$$

Thus the right-hand side of (4.2) is equal to

$$\sum_{i=l_b}^{k-1} \left[Q_i \cdot (p^{l_2} + \cdots + p^{l_b} - p^{i+1}) - (p-1)p^i(-p^i + p^{l_b}) \cdots (-p^i + p^{l_b} + \cdots p^{l_3})\right]$$
$$= (-p^k + p^{l_b}) \cdots (-p^k + p^{l_b} + \cdots + p^{l_3})(p^{l_2} + \cdots + p^{l_b})$$
$$- \sum_{i=l_b}^{k-1} \left[p^{i+1}Q_i + (p-1)p^i(-p^i + p^{l_b}) \cdots (-p^i + p^{l_b} + \cdots + p^{l_3})\right].$$

To show that (4.2) holds for b and the set $[b]$ as we originally required, and hence to conclude the inductive step and the proof of this proposition, it therefore remains to show that $p^k(-p^k + p^{l_b}) \cdots (-p^k + p^{l_b} + \cdots + p^{l_3})$ equals

$$\sum_{i=l_b}^{k-1} \left[p^{i+1}Q_i + (p-1)p^i(-p^i + p^{l_b}) \cdots (-p^i + p^{l_b} + \cdots + p^{l_3})\right].$$

This is clear if $l_b = k-1$, so now assuming that $l_b < k-1$, we have

$$\sum_{i=l_b}^{k-1} \left[p^{i+1}Q_i + (p-1)p^i(-p^i + p^{l_b}) \cdots (-p^i + p^{l_b} + \cdots + p^{l_3})\right]$$
$$= \sum_{i=l_b}^{k-1} p^{i+1}Q_i + (p-1) \sum_{i=l_b+1}^{k-1} p^i \sum_{j=l_b}^{i-1} Q_j$$
$$= \sum_{h=l_b}^{k-2} Q_h \cdot \left(p^{h+1} + (p-1) \sum_{z=h+1}^{k-1} p^z\right) + p^k \cdot Q_{k-1}$$
$$= p^k \cdot \sum_{h=l_b}^{k-1} Q_h = p^k(-p^k + p^{l_b}) \cdots (-p^k + p^{l_b} + \cdots + p^{l_3})$$

as required. □

Proposition 4.18. *Let $a \in \{2, 3, \dots, p-1\}$. Let $\phi(\underline{\mathsf{s}})$, $\phi(\underline{\mathsf{t}}) \in \mathrm{Lin}(P_{ap^k})$. Suppose that $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_{ap^k}} = \phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_{ap^k}}$. Let $b \in [a]$ and let $l_1, \dots, l_b$ be distinct integers in $[k]$. Then*

$$\sum_{j=1}^{a} C_{l_1}(\mathsf{s}^j) \cdot C_{l_2}(\mathsf{s}^j) \cdots C_{l_b}(\mathsf{s}^j) = \sum_{j=1}^{a} C_{l_1}(\mathsf{t}^j) \cdot C_{l_2}(\mathsf{t}^j) \cdots C_{l_b}(\mathsf{t}^j).$$

For clarity, we postpone the proof of Proposition 4.18 to the end of this section. We continue with a series of lemmas, culminating in the proof of Theorem 4.10.

For notational convenience, we denote multisets by asterisks. For example, the multiset equality of (i) in Lemma 4.19 below may be rewritten as $\{s_1, \dots, s_a\}^* = \{t_1, \dots, t_a\}^*$.

Lemma 4.19. *Let $q \in \mathbb{N}_{\geq 2}$. Let $a \in [q]$ and $s_j, t_j \in \mathbb{N}_0$ for $j = 1, 2, \dots, a$. If*

$$\sum_{j=1}^{a} (-q)^{s_j} = \sum_{j=1}^{a} (-q)^{t_j},$$

then either

(i) $\{s_1,\ldots,s_a\}=\{t_1,\ldots,t_a\}$ *is an equality of multisets; or*

(ii) $a=q$ *and the multisets* $\{s_1,\ldots,s_a\}$ *and* $\{t_1,\ldots,t_a\}$ *are* $\{w,w-1,\ldots,w-1\}$ *and* $\{w-2,\ldots,w-2\}$ *for some* $w\in\mathbb{N}_{\geq 2}$.

Proof. We proceed by induction on a. The assertion is clear if $a=1$, so now assume $2\leq a\leq q$, and suppose $\{s_1,\ldots,s_a\}^*\neq\{t_1,\ldots,t_a\}^*$. If $s_i=t_j$ for some $i,j\in[a]$, then by the inductive hypothesis for $a-1\neq q$, we have $\{s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_a\}^*=\{t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_a\}^*$. But then $\{s_1,\ldots,s_a\}^*=\{t_1,\ldots,t_a\}^*$, a contradiction. Thus $s_i\neq t_j$ for all $i,j\in[a]$. Without loss of generality suppose $s_1=\max\{s_i,t_j\}_{i,j\in[a]}$, so in particular $t_j<s_1$ for all j. By multiplying both sides of the equality of sums in the statement of the lemma by $-q$ if necessary, we may further assume that s_1 is even. Then

$$q^{s_1-1}\leq q^{s_1}-(a-1)q^{s_1-1}\leq\sum_{j=1}^{a}q^{s_j}(-1)^{s_j}=\sum_{j=1}^{a}q^{t_j}(-1)^{t_j}\leq aq^{s_1-2}\leq q^{s_1-1}.$$

Hence all inequalities in the above must hold with equalities, implying that $a=q$, $s_j=s_1-1$ for all $j\neq 1$, and $t_j=s_1-2$ for all $j\in[a]$. This is exactly case (ii). $\square$

Recall $n=ap^k$ where $k\in\mathbb{N}$ and now $a\in\{2,3,\ldots,p-1\}$, following Remark 4.15.

Lemma 4.20. *Let* $a\in\{2,3,\ldots,p-1\}$. *Let* $\phi(\underline{\mathsf{s}}),\ \phi(\underline{\mathsf{t}})\in\mathrm{Lin}(P_n)$ *and suppose* $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_n}=\phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_n}$. *Let* $l\in[k]$. *Then*

$$\left\{\sum_{m=1}^{l}\mathsf{s}^1_m,\ldots,\sum_{m=1}^{l}\mathsf{s}^a_m\right\}^*=\left\{\sum_{m=1}^{l}\mathsf{t}^1_m,\ldots,\sum_{m=1}^{l}\mathsf{t}^a_m\right\}^*.$$

Proof. First suppose $l<k$. Let $\mathsf{u}\in\{0,1\}^k$ and let $\mathsf{u}^-=(\mathsf{u}_1,\ldots,\mathsf{u}_{k-1})$. Define

$$\Delta_{l;k}(\mathsf{u})=\sum_{\substack{x\in P_{p^k}\text{ of}\\ \text{cycle type }p^lp^l}}\phi(\mathsf{u})(x),\ \text{ for } l\in[k-1]\quad\text{and}\quad\Delta_{k;k}(\mathsf{u})=0.$$

Then

$$\begin{aligned}\Delta_{l;k}(\mathsf{u})&=\sum_{\substack{x=(f_1,\ldots,f_p;1)\in P_{p^k}\\ \text{of cycle type }p^lp^l}}\phi(\mathsf{u})(x)\\ &=\binom{p}{2}\cdot\sum_{\substack{f_1,f_2\in P_{p^{k-1}}\\ \text{both of cycle type }p^l}}\phi(\mathsf{u}^-)(f_1)\cdot\phi(\mathsf{u}^-)(f_2)+p\cdot\sum_{\substack{f_1\in P_{p^{k-1}}\text{ of}\\ \text{cycle type }p^lp^l}}\phi(\mathsf{u}^-)(f_1)\\ &=\binom{p}{2}\cdot\Gamma_{l;k-1}(\mathsf{u}^-)\cdot\Gamma_{l;k-1}(\mathsf{u}^-)+p\cdot\Delta_{l;k-1}(\mathsf{u}^-)\\ &=\binom{p}{2}\sum_{i=l}^{k-1}p^{k-1-i}\cdot\Gamma_{l;i}\big((\mathsf{u}_1,\ldots,\mathsf{u}_i)\big)^2=\binom{p}{2}\sum_{i=l}^{k-1}p^{k-1+i}\cdot C_l(\mathsf{u})^2\end{aligned}$$

by Lemma 4.13. Now let $\phi(\underline{\mathsf{u}}) \in \mathrm{Lin}(P_n)$ where $\underline{\mathsf{u}} = (\mathsf{u}^1, \dots, \mathsf{u}^a)$. Recalling that $P_n = P_{p^k} \times \cdots \times P_{p^k}$ (a times), we have that

$$\begin{aligned}
\sum_{\substack{x \in P_n \text{ of} \\ \text{cycle type } p^l p^l}} \phi(\underline{\mathsf{u}})(x) &= \sum_{\substack{x = x_1 \cdots x_a,\ x_i \in P_{p^k} \\ x \text{ of cycle type } p^l p^l}} \phi(\mathsf{u}^1)(x_1) \cdots \phi(\mathsf{u}^a)(x_a) \\
&= \sum_{\{i,j\} \subseteq [a]} \sum_{\substack{x_i, x_j \in P_{p^k} \text{ each} \\ \text{of cycle type } p^l}} \phi(\mathsf{u}^i)(x_i) \cdot \phi(\mathsf{u}^j)(x_j) + \sum_{i=1}^{a} \sum_{\substack{x_i \in P_{p^k} \text{ of} \\ \text{cycle type } p^l p^l}} \phi(\mathsf{u}^i)(x_i) \\
&= \sum_{i=1}^{a} \sum_{\substack{j=1 \\ j \neq i}}^{a} \Gamma_{l;k}(\mathsf{u}^i) \cdot \Gamma_{l;k}(\mathsf{u}^j) + \sum_{i=1}^{a} \Delta_{l;k}(\mathsf{u}^i) \\
&= \sum_{i=1}^{a} \sum_{\substack{j=1 \\ j \neq i}}^{a} p^k C_l(\mathsf{u}^i) \cdot p^k C_l(\mathsf{u}^j) + \sum_{i=1}^{a} \binom{p}{2} \sum_{h=l}^{k-1} p^{k-1+h} \cdot C_l(\mathsf{u}^i)^2 \\
&= p^{2k} \left(\sum_{i=1}^{a} C_l(\mathsf{u}^i) \right) \left(\sum_{j=1}^{a} C_l(\mathsf{u}^j) \right) + \sum_{i=1}^{a} C_l(\mathsf{u}^i)^2 \left[-p^{2k} + \binom{p}{2} \sum_{h=l}^{k-1} p^{k-1-h} \right] \\
&= p^{2k} \left(\sum_{i=1}^{a} C_l(\mathsf{u}^i) \right) \left(\sum_{j=1}^{a} C_l(\mathsf{u}^j) \right) + \left[-\tfrac{p^k}{2}(p^k + p^l) \right] \cdot \sum_{i=1}^{a} C_l(\mathsf{u}^i)^2. \qquad (4.3)
\end{aligned}$$

Note $\underline{\mathsf{s}} = (\mathsf{s}^1, \dots, \mathsf{s}^a)$ and $\underline{\mathsf{t}} = (\mathsf{t}^1, \dots, \mathsf{t}^a)$. Since $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_n}(g) = \phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_n}(g)$ for $g \in \mathfrak{S}_n$ of cycle type p^l, we have that

$$\sum_{i=1}^{a} \Gamma_{l;k}(\mathsf{s}^i) = \sum_{i=1}^{a} \Gamma_{l;k}(\mathsf{t}^i)$$

and thus

$$\sum_{i=1}^{a} C_l(\mathsf{s}^i) = \sum_{i=1}^{a} C_l(\mathsf{t}^i)$$

by Lemma 4.13. Using expression (4.3) and the fact that $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_n}(g) = \phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_n}(g)$ for $g \in \mathfrak{S}_n$ of cycle type $p^l p^l$ then gives

$$\sum_{i=1}^{a} C_l(\mathsf{s}^i)^2 = \sum_{i=1}^{a} C_l(\mathsf{t}^i)^2,$$

and therefore

$$\sum_{i=1}^{a} \prod_{m=1}^{l} (p\mathsf{s}^i - 1)^2 = \sum_{i=1}^{a} \prod_{m=1}^{l} (p\mathsf{t}^i - 1)^2$$

by Lemma 4.13. Thus,

$$\sum_{i=1}^{a} (-q)^{\sigma_j} = \sum_{i=1}^{a} (-q)^{\tau_j} \quad \text{where} \quad q = p-1,\ \sigma_j = 2\sum_{m=1}^{l} \mathsf{s}^j_m \text{ and } \tau_j = 2\sum_{m=1}^{l} \mathsf{t}^j_m,$$

so by Lemma 4.19 we must have $\{\sigma_1, \ldots, \sigma_a\}^* = \{\tau_1, \ldots, \tau_a\}^*$ (case (ii) is not possible as all σ_j and τ_j are even). The assertion of the present lemma for $l < k$ then follows directly.

Finally, for $l = k$: by a similar argument we obtain an expression similar to (4.3) where instead of the term $-\frac{p^k}{2}(p^k + p^l)$ we have $-p^{2k}$. The rest then follows as in the case $l < k$. $\square$

Proposition 4.21. *Let $a \in \{2, 3, \ldots, p-1\}$. Let $\phi(\underline{\mathsf{s}}),\ \phi(\underline{\mathsf{t}}) \in \mathrm{Lin}(P_n)$ and suppose $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_n} = \phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_n}$. Let $b \in [a]$ and let $l_1, \ldots, l_b$ be distinct integers in $[k]$. Then $\{\sigma_1, \ldots, \sigma_a\}^* = \{\tau_1, \ldots, \tau_a\}^*$, where*

$$\sigma_j = \sum_{i=1}^{b} \sum_{m=1}^{l_i} \mathsf{s}_m^j \quad \text{and} \quad \tau_j = \sum_{i=1}^{b} \sum_{m=1}^{l_i} \mathsf{t}_m^j$$

for each $j \in [a]$.

Proof. By Proposition 4.18,

$$\sum_{j=1}^{a} \prod_{i=1}^{b} \prod_{m=1}^{l_i} (p\mathsf{s}_m^j - 1) = \sum_{j=1}^{a} \prod_{i=1}^{b} \prod_{m=1}^{l_i} (p\mathsf{t}_m^j - 1),$$

and hence $\sum_{j=1}^{a}(-p+1)^{\sigma_j} = \sum_{j=1}^{a}(-p+1)^{\tau_j}$. The assertion follows by Lemma 4.19: case (ii) cannot occur because $\sum_{j=1}^{a} \sigma_j = \sum_{j=1}^{a} \tau_j$ by Lemma 4.20. $\square$

Definition 4.22. *Let $b, k \in \mathbb{N}$. Given natural numbers $l_1, \ldots, l_b \leq k$ and a sequence $\mathsf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_k) \in \{0,1\}^k$, define*

$$f(l_1, \ldots, l_b; \mathsf{s}) = \sum_{i=1}^{b} \sum_{m=1}^{l_i} \mathsf{s}_m.$$

Let $a \in \mathbb{N}$. Given an a-tuple $\underline{\mathsf{s}} = (\mathsf{s}^1, \ldots, \mathsf{s}^a)$ where $\mathsf{s}^i \in \{0,1\}^k$ for all i, define

$$f(l_1, \ldots, l_b; \underline{\mathsf{s}}) = \{f(l_1, \ldots, l_b; \mathsf{s}^1), \ldots, f(l_1, \ldots, l_b; \mathsf{s}^a)\}^*.$$

Thus the result of Proposition 4.21 may be restated as

$$f(l_1, \ldots, l_b; \underline{\mathsf{s}}) = f(l_1, \ldots, l_b; \underline{\mathsf{t}})$$

for all distinct integers $l_1, \ldots, l_b$ in $[k]$, where $b \leq a < p$.

Lemma 4.23. *Let $\{l_1, \ldots, l_b\}^* = \{m_1, \ldots, m_b\}^*$. Suppose that in addition we have*

$$\{l_1 + 1, \ldots, l_c + 1, l_{c+1}, \ldots, l_b\}^* = \{m_1 + 1, \ldots, m_c + 1, m_{c+1}, \ldots, m_b\}^*$$

for some $c \in [b-1]$. Then $\{l_1, \ldots, l_c\}^* = \{m_1, \ldots, m_c\}^*$.

Proof. Suppose for the sake of contradiction that $\{l_1,\dots,l_c\}^* \neq \{m_1,\dots,m_c\}^*$. Let us suppose without loss of generality that $l_1 \leq \cdots \leq l_c$ and $m_1 \leq \cdots \leq m_c$. Let $j \leq c$ be maximal such that $l_j \neq m_j$, and without loss of generality we assume that $l_j < m_j$.

Given a multiset S and $v \in \mathbb{N}$ we let $S_{\geq v} = \{x : x \in S,\ x \geq v\}^*$. Observe that since $\{l_1,\dots,l_b\}^* = \{m_1,\dots,m_b\}^*$ we have that $\{l_1,\dots,l_b\}^*_{\geq v} = \{m_1,\dots,m_b\}^*_{\geq v}$ for any v. Similarly

$$\{l_1+1,\dots,l_c+1,l_{c+1},\dots,l_b\}^*_{\geq v} = \{m_1+1,\dots,m_c+1,m_{c+1},\dots,m_b\}^*_{\geq v}$$

for any v. We now consider $v = m_j+1$. Note that

$$\begin{aligned}&|\{l_1+1,\dots,l_c+1,l_{c+1},\dots,l_b\}^*_{\geq m_j+1}| - |\{l_1,\dots,l_b\}^*_{\geq m_j+1}|\\ &= |\{l_{j+1}+1,\dots,l_c+1\}^*_{\geq m_j+1}| - |\{l_{j+1},\dots,l_c\}^*_{\geq m_j+1}|\end{aligned}$$

by cancelling off equal elements in the two multisets and noting that $l_i < m_j$ for all $i \leq j$. Moreover,

$$\begin{aligned}&|\{m_1+1,\dots,m_c+1,m_{c+1},\dots,m_b\}^*_{\geq m_j+1}| - |\{m_1,\dots,m_b\}^*_{\geq m_j+1}|\\ &\geq |\{m_j+1,\dots,m_c+1\}^*_{\geq m_j+1}| - |\{m_j,\dots,m_c\}^*_{\geq m_j+1}|\\ &= 1 + |\{m_{j+1}+1,\dots,m_c+1\}^*_{\geq m_j+1}| - |\{m_{j+1},\dots,m_c\}^*_{\geq m_j+1}|\\ &= 1 + |\{l_{j+1}+1,\dots,l_c+1\}^*_{\geq m_j+1}| - |\{l_{j+1},\dots,l_c\}^*_{\geq m_j+1}|\end{aligned}$$

by maximality of j. In particular, we have

$$\begin{aligned}&|\{m_1+1,\dots,m_c+1,m_{c+1},\dots,m_b\}^*_{\geq m_j+1}| - |\{m_1,\dots,m_b\}^*_{\geq m_j+1}|\\ &\geq 1 + |\{l_1+1,\dots,l_c+1,l_{c+1},\dots,l_b\}^*_{\geq m_j+1}| - |\{l_1,\dots,l_b\}^*_{\geq m_j+1}|\end{aligned}$$

which is a contradiction. □

Theorem 4.24. *Let $a,k \in \mathbb{N}$. Let $\underline{\mathsf{s}} = (\mathsf{s}^1,\dots,\mathsf{s}^a)$ and $\underline{\mathsf{t}} = (\mathsf{t}^1,\dots,\mathsf{t}^a)$ where $\mathsf{s}^i,\mathsf{t}^i \in \{0,1\}^k$ for all i. Suppose that for any distinct integers $l_1,l_2,\dots,l_b \in [k]$ such that $b \in [a]$ we have*

$$f(l_1,\dots,l_b;\underline{\mathsf{s}}) = f(l_1,\dots,l_b;\underline{\mathsf{t}}). \tag{4.4}$$

Then there exists a permutation $\sigma \in \mathrm{Sym}[a]$ such that $\mathsf{s}^i = \mathsf{t}^{\sigma(i)}$ for all i.

Proof. We prove the assertion for (a,k) by induction on $a+k$. When $k=1$ and a is arbitrary, the assertion is clear since the single term of each sequence s^i is simply $f(1;\mathsf{s}^i)$. When $a=1$ and k is arbitrary, note that $\mathsf{s}^1_1 = f(1;\mathsf{s}^1) = f(1;\mathsf{t}^1) = \mathsf{t}^1_1$, and $\mathsf{s}^1_r = f(r;\mathsf{s}^1) - f(r-1;\mathsf{s}^1) = f(r;\mathsf{t}^1) - f(r-1;\mathsf{t}^1) = \mathsf{t}^1_r$ for all $r \in \{2,3,\dots,k\}$, so $\mathsf{s}^1 = \mathsf{t}^1$ as required.

Now suppose $a,k \geq 2$. Write $\hat{\mathsf{s}}^i$ for the sequence $(\mathsf{s}^i_j)^k_{j=2} \in \{0,1\}^{k-1}$ and let $\hat{\underline{\mathsf{s}}} = (\hat{\mathsf{s}}^1,\dots,\hat{\mathsf{s}}^a)$. Define $\hat{\mathsf{t}}^i$ and $\hat{\underline{\mathsf{t}}}$ similarly.

First suppose that $\mathsf{s}_1^i = \mathsf{t}_1^i = z$ for all $i \in [a]$, for some $z \in \{0,1\}$. Observe that for any distinct integers $l_1, \ldots, l_b \in [k-1]$ such that $b \in [a]$ we have

$$f(l_1+1, \ldots, l_b+1; \mathsf{s}^i) = f(l_1, \ldots, l_b; \hat{\mathsf{s}}^i) + bz,$$

and similarly

$$f(l_1+1, \ldots, l_b+1; \mathsf{t}^i) = f(l_1, \ldots, l_b; \hat{\mathsf{t}}^i) + bz.$$

Thus by (4.4), we have that

$$f(l_1, \ldots, l_b; \underline{\hat{\mathsf{s}}}) = f(l_1, \ldots, l_b; \underline{\hat{\mathsf{t}}}).$$

By the inductive hypothesis for $(a, k-1)$, there exists a permutation $\sigma \in \mathrm{Sym}[a]$ such that $\hat{\mathsf{s}}^i = \hat{\mathsf{t}}^{\sigma(i)}$ for all $i \in [a]$. Therefore $\mathsf{s}^i = \mathsf{t}^{\sigma(i)}$ as required.

Otherwise, we may now suppose that not all s_1^i and t_1^i are equal. Let $I_s = \{i \in [a] : \mathsf{s}_1^i = 1\}$ and define I_t similarly. Since $f(1; \underline{\mathsf{s}}) = f(1; \underline{\mathsf{t}})$, then $|I_s| = |I_t|$, so we may without loss of generality reorder $\underline{\mathsf{t}}$ to assume that $\mathsf{s}_1^i = \mathsf{t}_1^i$ for all $i \in [a]$. Let $I = I_s = I_t$ and note that $I \in [a-1]$.

For any distinct integers $l_1, \ldots, l_b \in [k-1]$ such that $b \in [a-1]$ we have

$$f(l_1+1, \ldots, l_b+1; \mathsf{s}^i) = b\mathsf{s}_1^i + f(l_1, \ldots, l_b; \hat{\mathsf{s}}^i)$$

and

$$f(l_1+1, \ldots, l_b+1; \mathsf{t}^i) = b\mathsf{s}_1^i + f(l_1, \ldots, l_b; \hat{\mathsf{t}}^i)$$

since $\mathsf{s}_1^i = \mathsf{t}_1^i$. Thus

$$\begin{aligned} f(l_1+1, \ldots, l_b+1; \underline{\mathsf{s}}) &= \{b\mathsf{s}_1^i + f(l_1, \ldots, l_b; \hat{\mathsf{s}}^i) \ : \ i \in [a]\}^* \\ = f(l_1+1, \ldots, l_b+1; \underline{\mathsf{t}}) &= \{b\mathsf{s}_1^i + f(l_1, \ldots, l_b; \hat{\mathsf{t}}^i) \ : \ i \in [a]\}^*. \end{aligned} \tag{4.5}$$

In addition,

$$\begin{aligned} f(1, l_1+1, \ldots, l_b+1; \underline{\mathsf{s}}) &= \{(b+1)\mathsf{s}_1^i + f(l_1, \ldots, l_b; \hat{\mathsf{s}}^i) \ : \ i \in [a]\}^* \\ = f(1, l_1+1, \ldots, l_b+1; \underline{\mathsf{t}}) &= \{(b+1)\mathsf{s}_1^i + f(l_1, \ldots, l_b; \hat{\mathsf{t}}^i) \ : \ i \in [a]\}^*. \end{aligned}$$

Therefore, by Lemma 4.23, we have that

$$\{b + f(l_1, \ldots, l_b; \hat{\mathsf{s}}^i) \ : \ i \in I\}^* = \{b + f(l_1, \ldots, l_b; \hat{\mathsf{t}}^i) \ : \ i \in I\}^*,$$

which implies

$$\{f(l_1, \ldots, l_b; \hat{\mathsf{s}}^i) \ : \ i \in I\}^* = \{f(l_1, \ldots, l_b; \hat{\mathsf{t}}^i) \ : \ i \in I\}^* \tag{4.6}$$

and therefore by (4.5) also

$$\{f(l_1,\ldots,l_b;\hat{\mathsf{s}}^i)\ :\ i\in[a]\setminus I\}^* = \{f(l_1,\ldots,l_b;\hat{\mathsf{t}}^i)\ :\ i\in[a]\setminus I\}^*. \tag{4.7}$$

Let $\hat{\underline{\mathsf{s}}}^{(1)}$ and $\hat{\underline{\mathsf{s}}}^{(0)}$ be the sequences $(\hat{\mathsf{s}}^i)_{i\in I}$ and $(\hat{\mathsf{s}}^i)_{i\notin I}$ respectively, and define $\hat{\underline{\mathsf{t}}}^{(1)}$ and $\hat{\underline{\mathsf{t}}}^{(0)}$ similarly. Since (4.6) and (4.7) hold for any valid choice of $\{l_i\}$, this tells us that for any distinct integers $l_1,\ldots,l_b\in[k-1]$ such that $b\in[a-1]$ we have

$$f(l_1,\ldots,l_b;\hat{\underline{\mathsf{s}}}^{(1)}) = f(l_1,\ldots,l_b;\hat{\underline{\mathsf{t}}}^{(1)}) \quad\text{and}\quad f(l_1,\ldots,l_b;\hat{\underline{\mathsf{s}}}^{(0)}) = f(l_1,\ldots,l_b;\hat{\underline{\mathsf{t}}}^{(0)}).$$

Since $|I|$ and $a-|I|$ are both at most $a-1$, we may apply the inductive hypotheses for $(|I|,k-1)$ and $(a-|I|,k-1)$ to obtain permutations $\sigma_1\in\operatorname{Sym}I$ such that $\hat{\mathsf{s}}^i=\hat{\mathsf{t}}^{\sigma_1(i)}$ for all $i\in I$ (and hence $\mathsf{s}^i=\mathsf{t}^{\sigma_1(i)}$) and $\sigma_0\in\operatorname{Sym}([a]\setminus I)$ such that $\hat{\mathsf{s}}^i=\hat{\mathsf{t}}^{\sigma_0(i)}$ for all $i\notin I$ (and hence $\mathsf{s}^i=\mathsf{t}^{\sigma_0(i)}$). Finally, let $\sigma=\sigma_0\cdot\sigma_1\in\operatorname{Sym}[a]$, so $\mathsf{s}^i=\mathsf{t}^{\sigma(i)}$ for all i as desired. □

Proof of Theorem 4.10. This follows from Proposition 4.21 and Theorem 4.24. □

4.3.1 Proof of Proposition 4.18

Let $k\in\mathbb{N}$ and $n=ap^k$ where $a\in\{2,3,\ldots,p-1\}$. Let $b\in[a]$ and let $l_1,\ldots,l_b$ be distinct integers in $[k]$. Let $\phi(\underline{\mathsf{s}})$, $\phi(\underline{\mathsf{t}})\in\operatorname{Lin}(P_{ap^k})$, and suppose that $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_{ap^k}}=\phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_{ap^k}}$. The statement we wish to prove is

$$\sum_{j=1}^{a} C_{l_1}(\mathsf{s}^j)\cdot C_{l_2}(\mathsf{s}^j)\cdots C_{l_b}(\mathsf{s}^j) = \sum_{j=1}^{a} C_{l_1}(\mathsf{t}^j)\cdot C_{l_2}(\mathsf{t}^j)\cdots C_{l_b}(\mathsf{t}^j).$$

We proceed by induction on by b. The case $b=1$ follows from evaluating $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_n}=\phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_n}$ on an element of $\mathfrak{S}_n$ of cycle type p^{l_1}. We present our argument for $b=2$ explicitly as an illustrative template for the general inductive argument.

Let $1\le l_1<l_2\le k$ and let $\phi(\underline{\mathsf{u}})\in\operatorname{Lin}(P_n)$. We have that

$$\begin{aligned}\sum_{\substack{x\in P_n\text{ of}\\ \text{cycle type}\\ p^{l_1}p^{l_2}}}\phi(\underline{\mathsf{u}})(x) &= \sum_{i=1}^{a}\sum_{\substack{j=1\\ j\neq i}}^{a}\Gamma_{l_1;k}(\mathsf{u}^i)\cdot\Gamma_{l_2;k}(\mathsf{u}^j)+\sum_{i=1}^{a}\Gamma_{l_1l_2;k}(\mathsf{u}^i)\\ &= \left(\sum_{i=1}^{a}\Gamma_{l_1;k}(\mathsf{u}^i)\right)\left(\sum_{j=1}^{a}\Gamma_{l_2;k}(\mathsf{u}^j)\right)-\sum_{j=1}^{a}\Gamma_{l_1;k}(\mathsf{u}^j)\cdot\Gamma_{l_2;k}(\mathsf{u}^j)+\sum_{j=1}^{a}\Gamma_{l_1l_2;k}(\mathsf{u}^j).\end{aligned} \tag{4.8}$$

Observe by Lemma 4.13 and Proposition 4.17 that

$$-\sum_{j=1}^{a}\Gamma_{l_1;k}(\mathsf{u}^j)\cdot\Gamma_{l_2;k}(\mathsf{u}^j)+\sum_{j=1}^{a}\Gamma_{l_1l_2;k}(\mathsf{u}^j) = -p^k\cdot p^{l_2}\cdot\sum_{j=1}^{a}C_{l_1}(\mathsf{u}^j)\cdot C_{l_2}(\mathsf{u}^j). \tag{4.9}$$

By the inductive hypothesis (that is, the case $b=1$), $\sum_{i=1}^{a}\Gamma_{l;k}(\mathsf{s}^i)=\sum_{i=1}^{a}\Gamma_{l;k}(\mathsf{t}^i)$ for any $l\in[k]$. Since the coefficient $-p^k\cdot p^{l_2}$ of $\sum_{j=1}^{a}C_{l_1}(\mathsf{u}^j)\cdot C_{l_2}(\mathsf{u}^j)$ in (4.9) is non-zero, by evaluating $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_n}=\phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_n}$ on an element of $\mathfrak{S}_n$ of cycle type $p^{l_1}p^{l_2}$, we find using (4.8) that

$$\sum_{j=1}^{a}C_{l_1}(\mathsf{s}^j)\cdot C_{l_2}(\mathsf{s}^j)=\sum_{j=1}^{a}C_{l_1}(\mathsf{t}^j)\cdot C_{l_2}(\mathsf{t}^j)$$

as claimed.

In the general inductive argument, the main steps are as follows:

(i) write $\sum_{x\in P_n,\text{ of cycle type }p^{l_1}\cdots p^{l_b}}\phi(\underline{\mathsf{u}})(x)$ as a sum of terms $\Gamma_{l_h\cdots l_i;k}(\mathsf{u}^j)$ for subsets $\{h,\dots,i\}$ of $[b]$: in the $b=2$ example, this is the first line of (4.8);

(ii) replace sums with 'restricted' indices by sums with 'unrestricted' indices: when $b=2$, we replaced the sum over the restricted index $j\neq i$ by sums involving only unrestricted indices i and j which were free to run over $1,2,\dots,a$, in the second line of (4.8);

(iii) consider those sums of products of Γ terms involving all of $l_1,\dots,l_b$, rearrange to obtain a product of some coefficient with $\sum_{j=1}^{a}C_{l_1}(\mathsf{u}^j)\cdots C_{l_b}(\mathsf{u}^j)$ and show that this coefficient is non-zero: when $b=2$ this coefficient (up to sign) is $p^k\cdot p^{l_2}\neq 0$.

When evaluating $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_n}=\phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_n}$ on an element of $\mathfrak{S}_n$ of cycle type $p^{l_1}\cdots p^{l_b}$, by the inductive hypothesis those sums of products of Γ terms involving only a strict subset of $l_1,\dots,l_b$ in Step (ii) will be equal for $\underline{\mathsf{u}}=\underline{\mathsf{s}}$ and $\underline{\mathsf{u}}=\underline{\mathsf{t}}$. Combined with the fact that the coefficient in Step (iii) is non-zero, we find that $\sum_{j=1}^{a}C_{l_1}(\mathsf{u}^j)\cdots C_{l_b}(\mathsf{u}^j)$ is equal for $\underline{\mathsf{u}}=\underline{\mathsf{s}}$ and $\underline{\mathsf{u}}=\underline{\mathsf{t}}$, as required.

Before proceeding with the inductive argument for general b, we describe the process of replacing 'restricted' sums in Step (ii) more formally. Let $N\in\mathbb{N}$ and let $F_1,\dots,F_N$ be functions from domain $[a]$ to some codomain, usually $\mathbb{Z}$. Consider the expression

$$\mathcal{F}(F_1,\dots,F_N;N):=\sum_{i_1=1}^{a}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{a}\cdots\sum_{\substack{i_N=1\\ i_N\neq i_1,i_2,\dots,i_{N-1}}}^{a}F_1(i_1)\cdot F_2(i_2)\cdots F_N(i_N).$$

We express $\mathcal{F}(F_1,\dots,F_N;N)$ in terms of $\mathcal{F}(G;1)$ for various functions G to obtain an 'unrestricted' sum *expression* $\mathcal{U}(F_1,\dots,F_N;N)$ (whose *value* equals that of $\mathcal{F}$) as follows. Note that when $N=1$, $\mathcal{F}(\mathbf{F};\mathbf{1})=\sum_{i=1}^{a}F(i)$ is already an 'unrestricted' sum (meaning that the summation index i is free to range over $[a]$), so define $\mathcal{U}(F;1)=\mathcal{F}(F;1)$. When $N=2$,

$$\mathcal{F}(F_1,F_2;2)=\sum_{i=1}^{a}\sum_{\substack{j=1,\\ j\neq i}}^{a}F_1(i)F_2(j)=\underbrace{\mathcal{F}(F_1;1)\cdot\mathcal{F}(F_2;1)-\mathcal{F}(\mathbf{F_1F_2};\mathbf{1})}_{=:\,\mathcal{U}(F_1,F_2;2)},$$

so define $\mathcal{U}(F_1,F_2;2)$ to be the expression following the last equals sign in the above.

Note that F_1F_2 denotes pointwise multiplication, $F_1F_2(i) = F_1(i)F_2(i)$. When $N = 3$:

$$\begin{aligned}\mathcal{F}(F_1, F_2, F_3; 3) &= \mathcal{F}(F_1; 1) \cdot \mathcal{F}(F_2; 1) \cdot \mathcal{F}(F_3; 1) - \mathcal{F}(F_1, F_2F_3; 2) - \mathcal{F}(F_2, F_1F_3; 2) \\ &\quad - \mathcal{F}(F_3, F_1F_2; 2) - \mathcal{F}(F_1F_2F_3; 1) \\ &= \prod_{i=1}^{3} \mathcal{F}(F_i; 1) - \mathcal{F}(F_1; 1) \cdot \mathcal{F}(F_2F_3; 1) - \mathcal{F}(F_2; 1) \cdot \mathcal{F}(F_1F_3; 1) \\ &\quad - \mathcal{F}(F_3; 1) \cdot \mathcal{F}(F_1F_2; 1) + \mathbf{2}\mathcal{F}(\mathbf{F_1F_2F_3; 1})\end{aligned}$$

so define $\mathcal{U}(F_1, F_2, F_3; 3)$ to be the expression following the last equals sign in the above. Observe that

$$\begin{aligned}\prod_{i=1}^{N} \mathcal{F}(F_i; 1) &= \sum_{\lambda \vdash N} \sum_{\substack{\nu \in \mathrm{Part}[N] \\ \text{of type } \lambda}} \sum_{i_1=1}^{a} F_{\nu_{11}} F_{\nu_{12}} \cdots F_{\nu_{1\lambda_1}}(i_1) \cdots \sum_{\substack{i_y=1 \\ i_y \neq i_1, \dots \\ \dots, i_{y-1}}}^{a} F_{\nu_{y1}} F_{\nu_{y2}} \cdots F_{\nu_{y\lambda_y}}(i_y) \\ &= \sum_{\lambda \vdash N} \sum_{\substack{\nu \in \mathrm{Part}[N] \\ \text{of type } \lambda}} \mathcal{F}(F_{\nu_{11}} \cdots F_{\nu_{1\lambda_1}}, \dots, F_{\nu_{y1}} \cdots F_{\nu_{y\lambda_y}}; y)\end{aligned}$$

where $y = l(\lambda)$, and for each ν we fix some ordering $\nu = \{\nu_1, \dots, \nu_{l(\lambda)}\}$ such that $|\nu_j| = \lambda_j$ and let $\nu_j = \{\nu_{j1}, \nu_{j2}, \dots, \nu_{j\lambda_j}\}$. Thus we give the following recursive definition for $\mathcal{U}$ for general $N \in \mathbb{N}$:

$$\mathcal{U}(F_1, \dots, F_N; N) = \prod_{i=1}^{N} \mathcal{F}(F_i; 1) - \sum_{\substack{\lambda \vdash N \\ \lambda \neq (1^N)}} \sum_{\substack{\nu \in \mathrm{Part}[N] \\ \text{of type } \lambda}} \mathcal{U}(F_{\nu_{11}} \cdots F_{\nu_{1\lambda_1}}, \dots, F_{\nu_{y1}} \cdots F_{\nu_{y\lambda_y}}; y)$$

and note this is well-defined because $\lambda \neq (1^N)$ implies that $y = l(\lambda) < N$ for all such λ.

Lemma 4.25. *Let $a, N \in \mathbb{N}$ and let $F_1, \dots, F_N$ be functions defined on $[a]$. Let D_N denote the coefficient of $\mathcal{F}(F_1 \cdots F_N; 1)$ in the expression $\mathcal{U}(F_1, \dots, F_N; N)$. Then*

$$D_N = (-1)^{N-1}(N-1)!.$$

Proof. We proceed by induction on N. By the examples calculated above, we can see that the assertion holds for $N = 1, 2, 3$ ($D_N = +\mathbf{1}, -\mathbf{1}, +\mathbf{2}$ respectively, from $+\mathcal{F}(F; 1)$, $-\mathcal{F}(F_1F_2; 1)$ and $+2\mathcal{F}(F_1F_2F_3; 1)$ which we highlighted in bold above). From the recursive definition of $\mathcal{U}$, we have that

$$\begin{aligned}D_N &= -\sum_{\substack{\lambda \vdash N \\ \lambda \neq (1^N)}} \sum_{\substack{\nu \in \mathrm{Part}[N] \\ \text{of type } \lambda}} D_{l(\lambda)} \\ &= -\Bigg[\sum_{\substack{\mu \vdash N-1 \\ \mu \neq (1^{N-1})}} \sum_{\substack{\gamma \in \mathrm{Part}[N-1] \\ \text{of type } \mu}} \Big(\underbrace{l(\mu) \cdot D_{l(\mu)}}_{\substack{\nu = \gamma \text{ with } N \text{ added} \\ \text{to an existing} \\ \text{member of } \gamma}} + \underbrace{D_{l(\mu)+1}}_{\substack{\nu = \gamma \text{ with} \\ \{1\} \text{ added}}} \Big) \Bigg] - \underbrace{(N-1)D_{N-1}}_{\mu = (1^{N-1}) \text{ term}}\end{aligned}$$

$$= -(N-1)D_{N-1}$$

since $D_l = (-1)^{l-1}(l-1)! = -(l-1)D_{l-1}$ for all $l = l(\mu) \leq N-1$ by the inductive hypothesis. Thus $D_N = (-1)^{N-1}(N-1)!$ as claimed. □

Proof of Proposition 4.18. As stated already, we proceed by induction on b and it remains to show the inductive argument. Now suppose $b \geq 3$ and that the assertion of the proposition holds for all $b' < b$. Let $\phi(\underline{\mathsf{u}}) \in \mathrm{Lin}(P_n)$ and let $g \in \mathfrak{S}_n$ be an element of cycle type $p^{l_1} \cdots p^{l_b}$. Then

$$\sum_{x \in \mathrm{ccl}_{\mathfrak{S}_n}(g) \cap P_n} \phi(\underline{\mathsf{u}})(x) = \sum_{\lambda \vdash b} \sum_{\substack{\nu \in \mathrm{Part}[b] \\ \text{of type } \lambda}} \sum_{i_1=1}^{a} \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^{a} \cdots \sum_{\substack{i_{l(\lambda)}=1 \\ i_{l(\lambda)} \neq i_1, \dots \\ \dots, i_{l(\lambda)-1}}}^{a} \Gamma_{\nu_1;k}(\mathsf{u}^{i_1}) \cdots \Gamma_{\nu_{l(\lambda)};k}(\mathsf{u}^{i_{l(\lambda)}}) \tag{4.10}$$

where for each ν we fix an ordering $\nu = \{\nu_1, \dots, \nu_{l(\lambda)}\}$ such that $|\nu_j| = \lambda_j$, and if $\nu_j = \{w_1, w_2, \dots, w_{\lambda_j}\}$ then $\Gamma_{\nu_j;k}(\mathsf{u})$ denotes $\Gamma_{l_{w_1} l_{w_2} \dots l_{w_{\lambda_j}}}(\mathsf{u})$. Next we fix some ν and consider the expression

$$\mathcal{U}\Big(\Gamma_{\nu_1;k}(\mathsf{u}^{(-)}),\ \dots,\ \Gamma_{\nu_{l(\lambda)};k}(\mathsf{u}^{(-)});\ l(\lambda)\Big). \tag{4.11}$$

where $(-)$ denotes the argument of the function $\Gamma_{\nu_j;k}(\mathsf{u}^{(-)})$, for each j. This is a sum of products of terms of the form $\sum_{i=1}^{a} \Gamma_{\nu_{j_1};k}(\mathsf{u}^i) \cdots \Gamma_{\nu_{j_m};k}(\mathsf{u}^i)$ for subsets $\{j_1, \dots, j_m\}$ of $\{1, \dots, l(\lambda)\}$. Let $V = \nu_{j_1} \cup \cdots \cup \nu_{j_m}$. By Proposition 4.17 and its proof (including the definition of P_i^{ω}), we have

$$\sum_{i=1}^{a} \Gamma_{\nu_{j_1};k}(\mathsf{u}^i) \cdots \Gamma_{\nu_{j_m};k}(\mathsf{u}^i) = \sum_{i=1}^{a} p^{mk} \cdot \prod_{w \in V} C_{l_w}(\mathsf{u}^i) \cdot (-1)^{m-1} \prod_{h=1}^{m} P_k^{\nu_{j_h}}. \tag{4.12}$$

If $|V| < b$ then by the inductive hypothesis, since $\phi(\underline{\mathsf{s}})\big\uparrow^{\mathfrak{S}_n} = \phi(\underline{\mathsf{t}})\big\uparrow^{\mathfrak{S}_n}$, the right-hand side of (4.12) is equal for $\underline{\mathsf{u}} = \underline{\mathsf{s}}$ and $\underline{\mathsf{u}} = \underline{\mathsf{t}}$. Subtracting this from (4.10), we find that the expression given by the sum of only the $|V| = b$ terms from the $\mathcal{U}$ expression in (4.11) is equal for $\underline{\mathsf{u}} = \underline{\mathsf{s}}$ and $\underline{\mathsf{u}} = \underline{\mathsf{t}}$; by Lemma 4.25, this is the following:

$$\sum_{\lambda \vdash b} \sum_{\substack{\nu \in \mathrm{Part}[b] \\ \text{of type } \lambda}} (-1)^{l(\lambda)-1} \cdot \big(l(\lambda)-1\big)! \cdot \sum_{i=1}^{a} \Gamma_{\nu_1;k}(\mathsf{u}^i) \cdots \Gamma_{\nu_{l(\lambda)};k}(\mathsf{u}^i)$$

$$= \sum_{\lambda \vdash b} \sum_{\substack{\nu \in \mathrm{Part}[b] \\ \text{of type } \lambda}} (-1)^{l(\lambda)-1} \cdot \big(l(\lambda)-1\big)! \cdot \sum_{i=1}^{a} p^{kl(\lambda)} \cdot \prod_{h=1}^{b} C_{l_b}(\mathsf{u}^i) \cdot (-1)^{b-l(\lambda)} \cdot \prod_{\omega \in \nu} P_k^{\omega}$$

$$= (-1)^{b-1} \sum_{i=1}^{a} p^{kl(\lambda)} \cdot C_{l_1}(\mathsf{u}^i) \cdots C_{l_b}(\mathsf{u}^i) \cdot \sum_{\lambda \vdash b} \sum_{\substack{\nu \in \mathrm{Part}[b] \\ \text{of type } \lambda}} \big(l(\lambda)-1\big)! \cdot p^{kl(\lambda)} \cdot \prod_{\omega \in \nu} P_k^{\omega}.$$

Thus it remains to show that the coefficient of $\sum_{i=1}^{a} C_{l_1}(\mathsf{u}^i) \cdots C_{l_b}(\mathsf{u}^i)$ is non-zero in order to see that $\sum_{i=1}^{a} C_{l_1}(\mathsf{s}^i)\cdot C_{l_2}(\mathsf{s}^i)\cdots C_{l_b}(\mathsf{s}^i) = \sum_{i=1}^{a} C_{l_1}(\mathsf{t}^i)\cdot C_{l_2}(\mathsf{t}^i)\cdots C_{l_b}(\mathsf{t}^i)$ and conclude the proof. We do this by proving the following for all integers $1 \leq l_1 < l_2 < \cdots < l_b \leq k$ and $2 \leq b \leq a < p$:

$$\sum_{\lambda\vdash b}\ \sum_{\substack{\nu\in\mathrm{Part}[b]\\ \text{of type }\lambda}} p^{kl(\lambda)}\cdot\big(l(\lambda)-1\big)!\cdot p^{kl(\lambda)}\cdot\prod_{\omega\in\nu}P_k^{\omega} = p^k\cdot p^{l_b}(p^{l_b}+p^{l_{b-1}})\cdots(p^{l_b}+p^{l_{b-1}}+\cdots+p^{l_2}) \tag{4.13}$$

and noting that the right-hand side of (4.13) is non-zero, while the left-hand side is $(-1)^{b-1}$ times the coefficient we are interested in. To prove that (4.13) holds, we proceed by induction on b; the cases $b=2$ and $b=3$ are straightforward to verify, so we now show the inductive step. Notice by the inductive hypothesis that (4.13) holds with b replaced by b', for any $b' < b$, and with $\{l_1,\ldots,l_b\}$ replaced by any subset of $[k]$ of size b'. Observe that

$$\begin{aligned}
&\sum_{\lambda\vdash b}\ \sum_{\substack{\nu\in\mathrm{Part}[b]\\ \text{of type }\lambda}} p^{kl(\lambda)}\cdot\big(l(\lambda)-1\big)!\cdot p^{kl(\lambda)}\cdot\prod_{\omega\in\nu}P_k^{\omega}\\
&=\sum_{\mu\vdash b-1}\ \sum_{\substack{\gamma\in\mathrm{Part}\{2,\ldots,b\}\\ \text{of type }\mu}}\Big[\underbrace{p^{kl(\mu)}\cdot\big(l(\mu)-1\big)!\prod_{\omega\in\gamma}P_k^{\omega}\cdot\sum_{\omega\in\gamma}(-p^k+\sum_{w\in\omega}p^w)}_{\nu=\gamma\text{ with 1 added to an existing member of }\gamma}\\
&\qquad\qquad+\underbrace{p^{k(l(\mu)+1)}\cdot l(\mu)!\prod_{\omega\in\gamma}P_k^{\omega}}_{\nu=\gamma\text{ with }\{1\}\text{ added}}\Big]\\
&=\sum_{\mu\vdash b-1}\ \sum_{\substack{\gamma\in\mathrm{Part}\{2,\ldots,b\}\\ \text{of type }\mu}} p^{kl(\mu)}\cdot\big(l(\mu)-1\big)!\prod_{\omega\in\gamma}P_k^{\omega}\cdot(p^{l_b}+p^{l_{b-1}}\cdots p^{l_2})\\
&=p^k\cdot p^{l_b}(p^{l_b}+p^{l_{b-1}})\cdots(p^{l_b}+p^{l_{b-1}}+\cdots p^{l_3})\cdot(p^{l_b}+p^{l_{b-1}}\cdots p^{l_2})
\end{aligned}$$

and thus the proof is complete. □

4.4 Equivalence relations on $\mathrm{Lin}(P_n)$

In this section, we compare the orbits of $\mathrm{Lin}(P_n)$ under the conjugation action of $N_{\mathfrak{S}_n}(P_n)$ to those under the action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\phi)/\mathbb{Q})$ for $\phi\in\mathrm{Lin}(P_n)$, and to the equivalence classes given by the relation $\Omega(\phi)=\Omega(\psi)$ for $\phi,\psi\in\mathrm{Lin}(P_n)$ (we recall the definition of $\Omega(-)$ below).

Let p be a prime and let ω denote a primitive p^{th} root of unity in $\mathbb{C}$. It is clear that

$$\{\phi(g)\mid\phi\in\mathrm{Lin}(P_p),\ g\in P_p\}=\{1,\omega,\omega^2,\ldots,\omega^{p-1}\}=:\mu_p.$$

By Lemma 2.13, we also have

$$\{\phi(g) \mid \phi \in \mathrm{Lin}(P_n),\ g \in P_n\} = \mu_p$$

for any $n \in \mathbb{N}$ such that $n \geq p$. (When $n < p$, P_n is the trivial group and so the only character value is 1.) Indeed, the field of character values $\mathbb{Q}(\phi)$, obtained by adjoining all values of ϕ to $\mathbb{Q}$, is equal to $\mathbb{Q}(\omega)$ for every $\phi \in \mathrm{Lin}(P_n) \setminus \{\mathbb{1}_{P_n}\}$. Thus we may consider the action of the Galois group $\mathbb{G} := \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ on the characters of P_n. Note that this is trivial when $p = 2$, so from now on we fix p to be an odd prime.

Let $n \in \mathbb{N}$ and let its p-adic expansion be $n = \sum_{i=1}^{k} a_i p^{n_i}$ where $0 \leq n_1 < \cdots < n_k$. Let $\phi(\underline{\mathbf{s}}), \phi(\underline{\mathbf{t}}) \in \mathrm{Lin}(P_n)$. We say $\phi(\underline{\mathbf{s}})$ and $\phi(\underline{\mathbf{t}})$ are *Galois conjugates*, which we denote by $\phi(\underline{\mathbf{s}}) \sim \phi(\underline{\mathbf{t}})$, if there exists $\sigma \in \mathbb{G}$ such that $\phi(\underline{\mathbf{s}})^\sigma = \phi(\underline{\mathbf{t}})$. That is, $\phi^\sigma(g) := (\phi(g))^\sigma = \sigma(\phi(g))$ for $g \in P_n$ and ϕ a character of P_n.

For $\phi \in \mathrm{Lin}(P_n)$, let $\Omega(\phi)$ denote the set of irreducible characters of $\mathfrak{S}_n$ containing ϕ in its restriction. Equivalently,

$$\Omega(\phi) := \{\chi \in \mathrm{Irr}(\mathfrak{S}_n) \ : \ \chi \mid \phi\big\uparrow^{\mathfrak{S}_n}\}.$$

If $\phi = \phi(\underline{\mathbf{s}})$, then we also denote $\Omega(\phi)$ by $\Omega(\underline{\mathbf{s}})$. Note that since p is odd, the set $\Omega(\phi)$ is closed under conjugation of partitions, by Lemma 2.2.

We thus have three equivalence relations on the set $\mathrm{Lin}(P_n)$, given by the following conditions for $\phi, \psi \in \mathrm{Lin}(P_n)$:

(i) $\Omega(\phi) = \Omega(\psi)$, i.e. the inductions have the same set of irreducible constituents;

(ii) $N_{\mathfrak{S}_n}(P_n)$–conjugacy, i.e. $\phi\big\uparrow^{\mathfrak{S}_n} = \psi\big\uparrow^{\mathfrak{S}_n}$ by Theorem 4.1; and

(iii) Galois conjugacy, $\phi \sim \psi$.

Clearly if $\phi\big\uparrow^{\mathfrak{S}_n} = \psi\big\uparrow^{\mathfrak{S}_n}$ then $\Omega(\phi) = \Omega(\psi)$. It is also easy to see that if $\phi \sim \psi$ then $\phi\big\uparrow^{\mathfrak{S}_n} = \psi\big\uparrow^{\mathfrak{S}_n}$, since all characters of symmetric groups are integer-valued. Indeed, since $\Omega(\mathbb{1}_{P_p}) = \mathrm{Irr}(\mathfrak{S}_p) \setminus \{\chi^{(p-1,1)}, \chi^{(2,1^{p-2})}\}$ and $\Omega(\phi) = \mathrm{Irr}(\mathfrak{S}_p) \setminus \{\chi^{(p)}, \chi^{(1^p)}\}$ for all $\phi \in \mathrm{Lin}(P_p) \setminus \{\mathbb{1}_{P_p}\}$, it follows that all three conditions (i) – (iii) are equivalent whenever $p \leq n < 2p$, as $P_n \cong P_p \times P_{n-p} \cong P_p$ in this case (and vacuous when $n < p$ as $|\mathrm{Lin}(P_n)| = 1$).

However, the reverse implications do not hold in general, and we give explicit counterexamples below.

Lemma 4.26. *Let $n \in \mathbb{N}$ be such that $n \geq 2p$. Then there exist $\phi(\underline{\mathbf{s}})$, $\phi(\underline{\mathbf{t}}) \in \mathrm{Lin}(P_n)$ such that $\phi(\underline{\mathbf{s}})\big\uparrow^{\mathfrak{S}_n} = \phi(\underline{\mathbf{t}})\big\uparrow^{\mathfrak{S}_n}$ but $\phi(\underline{\mathbf{s}}) \not\sim \phi(\underline{\mathbf{t}})$.*

Proof. First let $n = p^k$ with $k \geq 2$. By Theorem 4.1 and Lemma 4.3, it suffices to exhibit two sequences $s, t \in [\overline{p}]^k$ such that $t \in \Sigma(s)$ but $\phi(s) \not\sim \phi(t)$. Recall $P_p = \langle g \rangle$ with $\phi_i(g) = \omega^i$ for $i \in [\overline{p}]$; we may without loss of generality take $g = (p, p-1, \ldots, 2, 1)$. We show that $\phi(s) \not\sim \phi(t)$ where $s = (1, \ldots, 1) \in [\overline{p}]^k$ and $t = (1, \ldots, 1, 2) \in [\overline{p}]^k$. Let

$u = (1,\dots,1) \in [\overline{p}]^{k-1}$. Suppose for the sake of contradiction that $\phi(s)^\sigma = \phi(t)$ for some $\sigma \in \mathbb{G}$. Then for any $g_1,\dots,g_p \in P_{p^{k-1}}$, setting $\gamma = (g_1,\dots,g_p;g) \in P_{p^k}$ and $\gamma' = (g_1,\dots,g_p;1) \in P_{p^k}$ gives

$$\phi(s)^\sigma(\gamma) = \phi(t)(\gamma) \quad \text{and} \quad \phi(s)^\sigma(\gamma') = \phi(t)(\gamma'),$$

which by Lemma 2.13 implies

$$\big(\phi(u)(g_1\cdots g_p)\cdot\phi_1(g)\big)^\sigma = \phi(u)(g_1\cdots g_p)\cdot\phi_2(g)$$

and

$$\big(\phi(u)(g_1)\cdots\phi(u)(g_p)\big)^\sigma = \phi(u)(g_1)\cdots\phi(u)(g_p)$$

respectively. Setting $g_1 = \ldots = g_p = 1$, we have $(\phi_1(g))^\sigma = \phi_2(g)$ and thus σ is determined as the unique element of $\mathbb{G}$ satisfying $\sigma(\omega) = \omega^2$. But setting $g_2 = \ldots = g_p = 1$, we find that $(\phi(u)(g_1))^\sigma = \phi(u)(g_1)$ for all $g_1 \in P_{p^{k-1}}$. Since $\phi(u) \neq \mathbb{1}_{P_{p^{k-1}}}$, there exists $g_1 \in P_{p^{k-1}}$ such that $\phi(u)(g_1) = \omega^j$ for some $j \in [p-1]$. But then $\omega^{2j} = \sigma(\omega^j) = \omega^j$, a contradiction.

Next let $n > p^2$. Letting $n = \sum_{i=1}^t a_i p^{n_i}$ be its p-adic expansion where $0 \leq n_1 < \cdots < n_t$, then $k := n_t \geq 2$. Let $m = n - p^k$. Then $\phi(\underline{\mathbf{s}}) := \mathbb{1}_{P_m} \times \phi(s)$ and $\phi(\underline{\mathbf{t}}) := \mathbb{1}_{P_m} \times \phi(t)$ are not Galois conjugates, but $\phi(\underline{\mathbf{s}})\big\uparrow^{\mathfrak{S}_n} = \phi(\underline{\mathbf{t}})\big\uparrow^{\mathfrak{S}_n}$ by Lemmas 4.5 and 4.2.

For $n = 2p$, we may take $\phi(\underline{\mathbf{s}}) = \phi_1 \times \phi_1$ and $\phi(\underline{\mathbf{t}}) = \phi_1 \times \phi_2$, while for $2p < n < p^2$ we may take $\phi(\underline{\mathbf{s}}) = \mathbb{1}_{P_{n-2p}} \times \phi_1 \times \phi_1$ and $\phi(\underline{\mathbf{t}}) = \mathbb{1}_{P_{n-2p}} \times \phi_1 \times \phi_2$. □

Similarly, $\Omega(\phi) = \Omega(\psi)$ clearly does not imply $\phi\big\uparrow^{\mathfrak{S}_n} = \psi\big\uparrow^{\mathfrak{S}_n}$ in general. Indeed, an equality of induced characters implies that every $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ appears with the *same* multiplicity in $\phi\big\uparrow^{\mathfrak{S}_n}$ as in $\psi\big\uparrow^{\mathfrak{S}_n}$, while $\Omega(\phi) = \Omega(\psi)$ simply says that one multiplicity is non-zero if and only if the other is non-zero.

Example 4.27. Let $p \geq 5$ be a prime. We present infinitely many natural numbers n and pairs of linear characters $\phi, \psi \in \mathrm{Lin}(P_n)$ such that $\Omega(\phi) = \Omega(\psi)$ but $\phi\big\uparrow^{\mathfrak{S}_n} \neq \psi\big\uparrow^{\mathfrak{S}_n}$.

Let $k \neq l \in \mathbb{N}_{\geq 2}$ and set $n = p^k + p^l$. Let $\phi(\underline{\mathbf{s}}),\ \phi(\underline{\mathbf{t}}) \in \mathrm{Lin}(P_n)$ where $\underline{\mathbf{s}} = \big((0^{k-1},1),(0^l)\big)$ and $\underline{\mathbf{t}} = \big((0^k),(0^{l-1},1)\big) \in [\overline{p}]^k \times [\overline{p}]^l$, with 0^m denoting the all 0s sequence of length m. By Theorem 5.1,

$$\Omega(0^k) = \mathrm{Irr}(\mathfrak{S}_{p^k}) \setminus \{\chi^{(p^k-1,1)}, \chi^{(2,1^{p^k-2})}\},$$

and by direct verification (or by Lemma 6.4 in Chapter 6 later),

$$\Omega(0^{k-1},1) = \mathrm{Irr}(\mathfrak{S}_{p^k}) \setminus \{\chi^{(p^k)}, \chi^{(1^{p^k})}\}$$

for all $k \geq 2$. By the Littlewood–Richardson rule, we find that

$$\Omega(\underline{\mathbf{s}}) = \Omega(\underline{\mathbf{t}}) = \mathrm{Irr}(\mathfrak{S}_n) \setminus \{\chi^{(n)}, \chi^{(1^n)}\},$$

but since $k \neq l$ we also have $\phi(\underline{\mathbf{s}})\big\uparrow^{\mathfrak{S}_n} \neq \phi(\underline{\mathbf{t}})\big\uparrow^{\mathfrak{S}_n}$ by Lemma 4.5 and Theorem 4.1. $\Diamond$

Surprisingly, when n is a power of p then knowing just the set $\Omega(\phi)$ of irreducible constituents *without* the multiplicities with which these constituents appear *is* enough to determine the $N_{\mathfrak{S}_n}(P_n)$–orbit of the linear character ϕ.

Lemma 4.28. *Let $k \in \mathbb{N}$ and $\phi, \psi \in \mathrm{Lin}(P_{p^k})$. If $\Omega(\phi) = \Omega(\psi)$, then $\phi\big\uparrow^{\mathfrak{S}_{p^k}} = \psi\big\uparrow^{\mathfrak{S}_{p^k}}$.*

Lemma 4.28 is immediate from the following lemma. Since there is a natural bijection between $\mathrm{Irr}(\mathfrak{S}_n)$ and $\mathcal{P}(n)$, for $\phi \in \mathrm{Lin}(P_n)$ we may equally view $\Omega(\phi)$ as a subset of $\mathcal{P}(n)$. Below, $\leq$ denotes the lexicographical ordering on partitions.

Lemma 4.29. *Let $k \in \mathbb{N}$.*

(a) Let $s \in [\overline{p}]^k$ and let λ be the lexicographically greatest partition in $\Omega(s)$. If $s_k \neq 0$, then λ contains a part of size 1.

(b) Let $s, t \in [\overline{p}]^k$ be such that $t \notin \Sigma(s)$. Let $x \in [k]$ be minimal such that $\{s_x, t_x\}$ contains exactly one 0, and suppose that $s_x = 0$ and $t_x \neq 0$. Let α be the lexicographically greatest partition in $\Omega(s)$. Then $\langle \chi^\alpha\big\downarrow_{P_{p^k}}, \phi(s)\rangle = 1$, and also $\alpha > \nu$ for all $\nu \in \Omega(t)$.

Before we prove Lemma 4.29, we show how to deduce Lemma 4.28 from it.

Proof of Lemma 4.28. Suppose $\phi\big\uparrow^{\mathfrak{S}_{p^k}} \neq \psi\big\uparrow^{\mathfrak{S}_{p^k}}$. Then ϕ and ψ are not $N_{\mathfrak{S}_{p^k}}(P_{p^k})$–conjugate, by Lemma 4.2. Thus $\phi = \phi(s)$ and $\psi = \phi(t)$ for some $s, t \in [\overline{p}]^k$ such that $t \notin \Sigma(s)$ by Lemma 4.3. Then Lemma 4.29 (b) shows that $\Omega(\phi) \neq \Omega(\psi)$. $\square$

The proof of Lemma 4.29 uses two results from Chapter 6 later, where we further investigate the sets $\Omega(\phi)$ for $\phi \in \mathrm{Lin}(P_n)$.

Proof of Lemma 4.29. **(a)** If $s_i = 0$ for all $i < k$, then $\lambda = (p^k - 1, 1)$ by Lemma 6.4 (notice that Lemma 6.4 and its proof hold as stated also when $p = 3$ with the single exception $\Omega(0,1) = \mathcal{B}_9(8) \setminus \{(3^3)\}$). If $s_i \neq 0$ for some $i < k$, then the assertion follows by induction on the number of non-zero entries of s, using Lemma 6.11 (which holds in its entirety also when $p = 3$) combined with Lemma 6.4.

(b) We proceed by induction on k. The base case $k = 1$ is clear since $s = (0)$, $\Omega(s) = \mathcal{P}(p) \setminus \{(p-1,1), (2, 1^{p-2})\}$, $\lambda = (p)$, $t = (1)$ and $\Omega(t) = \mathcal{P}(p) \setminus \{(p), (1^p)\}$.

Now assume $k \geq 2$, and consider the following subgroups of $\mathfrak{S}_{p^k}$: let $P = P_{p^k} = P_{p^{k-1}} \wr P_p$ and let B be its base group, namely $P = B \rtimes P_p$ and $B \cong (P_{p^{k-1}})^{\times p}$. Let $Y = (\mathfrak{S}_{p^{k-1}})^{\times p}$ be the Young subgroup of $\mathfrak{S}_{p^k}$ naturally containing B. We define two further subgroups of $\mathfrak{S}_{p^k}$ as follows: $H := Y \rtimes \mathfrak{S}_p \cong \mathfrak{S}_{p^{k-1}} \wr \mathfrak{S}_p$ and $W := Y \rtimes P_p \cong \mathfrak{S}_{p^{k-1}} \wr P_p$. Clearly $P \leq W \leq H$. Let $s^- = (s_1, \ldots, s_{k-1})$ and $t^- = (t_1, \ldots, t_{k-1})$. Let $\mu = (\mu_1, \ldots, \mu_r)$ be the lexicographically greatest partition in $\Omega(s^-)$. We split into two cases according to $x = k$ or $x < k$.

Case 1: $x = k$. In this case, $t^- \in \Sigma(s^-)$ so $\Omega(s^-) = \Omega(t^-)$. Let $\lambda = (p\mu_1, \ldots, p\mu_r)$.

1. $\lambda \in \Omega(s)$: observe that $\langle \mathcal{X}(\mu;(p)), \chi^\lambda\big\downarrow_H\rangle = 1$ by Theorem 2.21. Since $\phi(s) = \mathcal{X}(\phi(s^-);\phi_0)$, then $\phi(s) \mid \mathcal{X}(\mu;(p))\big\downarrow_P^H \mid \chi^\lambda\big\downarrow_P$.

2. λ is lexicographically greatest in $\Omega(s)$: suppose $\nu \in \Omega(s)$. Then $\phi(s^-)^{\times p} = \phi(s)\big\downarrow_B \mid \chi^\nu\big\downarrow_B = (\chi^\nu\big\downarrow_Y)\big\downarrow_B$. So there exists an irreducible constituent of $\chi^\nu\big\downarrow_Y$, say $\chi^{\eta_1} \times \cdots \times \chi^{\eta_p} \in \mathrm{Irr}(Y)$, such that $\eta_i \in \Omega(s^-)$ for all i. But $c^\nu_{\eta_1,\dots,\eta_p} > 0$ implies $\nu \leq \eta_1 + \cdots + \eta_p$, and $\eta_i \leq \mu$ by definition of μ. Hence $\nu \leq \eta_1 + \cdots + \eta_p \leq \mu + \cdots + \mu = \lambda$.

3. $\langle \chi^\lambda\big\downarrow_{P_{p^k}}, \phi(s)\rangle = 1$: applying the argument in Step 2 to $\nu = \lambda$, we see that the only irreducible constituent $\chi^{\eta_1} \times \cdots \times \chi^{\eta_p}$ of $\chi^\lambda\big\downarrow_Y$ such that $\eta_i \in \Omega(s^-)$ for all i is $(\chi^\mu)^{\times p}$, and it occurs with multiplicity 1. Since $(\chi^\mu)^{\times p} \mid \mathcal{X}(\mu;(p))\big\downarrow_Y$ and $\mathcal{X}(\mu;(p)) \mid \chi^\lambda\big\downarrow_H$, it follows that

$$\begin{aligned}\langle \chi^\lambda\big\downarrow_{P_{p^k}}, \phi(s)\rangle &= \langle \mathcal{X}(\mu;(p))\big\downarrow_P^H, \phi(s)\rangle = \langle \mathcal{X}(\mu;\phi_0)\big\downarrow_P^W, \phi(s)\rangle \\ &= \langle \mathcal{X}(\mu\big\downarrow_{P_{p^{k-1}}};\phi_0), \mathcal{X}(\phi(s^-);\phi_{s_k})\rangle = 1\end{aligned}$$

where the final equality follows from Lemma 2.19 since $\langle \chi^\mu\big\downarrow_{P_{p^{k-1}}}, \phi(s^-)\rangle = \delta_{0,s_k} = 1$ by the inductive hypothesis.

4. $\lambda > \nu$ for all $\nu \in \Omega(t)$: suppose $\nu \in \Omega(t)$. By the same argument as in Step 2, there exists $\eta_1,\dots,\eta_p \in \Omega(t^-)$ such that $\chi^{\eta_1} \times \cdots \times \chi^{\eta_p} \mid \chi^\nu\big\downarrow_Y$. But $\Omega(t^-) = \Omega(s^-)$, so $\eta_i \leq \mu$ for all i and we similarly obtain $\nu \leq \eta_1 + \cdots + \eta_p \leq \mu + \cdots + \mu = \lambda$. Thus it remains to show $\lambda \notin \Omega(t)$. As in Step 3, $\langle \chi^\lambda\big\downarrow_{P_{p^k}}, \phi(t)\rangle = \langle \mathcal{X}(\mu;(p))\big\downarrow_P^H, \phi(t)\rangle$ since the only irreducible constituent $\chi^{\eta_1} \times \cdots \times \chi^{\eta_p} \mid \chi^\lambda\big\downarrow_Y$ with $\eta_i \in \Omega(t^-)$ for all i is $(\chi^\mu)^{\times p}$ with multiplicity 1. Finally, observe that $\mathcal{X}(\mu;(p))\big\downarrow_P^H, \phi(t)\rangle = \delta_{0,t_k} = 0$ by Lemma 2.19.

Case 2: $x < k$. In this case, $\mu > \gamma$ for all $\gamma \in \Omega(t^-)$, by the inductive hypothesis. Let

$$\lambda = \begin{cases} (p\mu_1,\dots,p\mu_r) & \text{if } s_k = 0, \\ (p\mu_1,\dots,p\mu_{r-1},p-1,1) & \text{if } s_k \neq 0. \end{cases}$$

Notice that if $s_k \neq 0$ then $\lambda = (p\mu_1,\dots,p\mu_{r-1},p\mu_r - 1,1)$ by (a), and λ immediately precedes $(p\mu_1,\dots,p\mu_r)$ in lexicographical order.

1. $\lambda \in \Omega(s)$: by Theorem 2.21, $\langle \mathcal{X}(\mu;\theta), \chi^\lambda\big\downarrow_H\rangle = 1$, so $\phi(s) \mid \chi^\lambda$.

2. λ is lexicographically greatest in $\Omega(s)$: suppose $\nu \in \Omega(s)$. By the same argument as in Step 2 of Case 1, we find that $\nu \leq (p\mu_1,\dots,p\mu_r)$. Thus $\nu \leq \lambda$ if $s_k = 0$. On the other hand, if $s_k \neq 0$ then it remains to prove that if $\nu = (p\mu_1,\dots,p\mu_r)$ then $\nu \notin \Omega(s)$.

   Suppose $\nu \in \Omega(s)$. Since the only irreducible constituent $\chi^{\eta_1} \times \cdots \times \chi^{\eta_p} \mid \chi^\nu\big\downarrow_Y$ with $\eta_i \in \Omega(s^-)$ for all i is $(\chi^\mu)^{\times p}$ with multiplicity 1, there is a unique $\psi \in \mathrm{Irr}(W)$

such that $\psi \mid \chi^\nu\big\downarrow_W$ and $\phi(s) \mid \psi\big\downarrow_P$. Moreover, $\psi = \mathcal{X}(\mu;\phi_j)$ for some $j \in [\overline{p}]$ since necessarily $\psi \in \mathrm{Irr}(W \mid (\chi^\mu)^{\times p})$. From Step 1, we know that $\mathcal{X}(\mu;\theta) = \mathcal{X}(\mu;(p-1,1)) \mid \chi^\nu\big\downarrow_H$. But $\chi^{(p-1,1)}\big\downarrow_{P_p} = \sum_{i=1}^{p-1}\phi_i$, so $\mathcal{X}(\mu;\phi_i) \mid \chi^\nu\big\downarrow_W$ for all $i \in [p-1]$, a contradiction.

3. $\langle \chi^\lambda\big\downarrow_{P_{p^k}}, \phi(s)\rangle = 1$: consider all irreducible constituents $\chi^{\eta_1} \times \cdots \times \chi^{\eta_p} \mid \chi^\lambda\big\downarrow_Y$ such that $\eta_i \in \Omega(s^-)$ for all i.

   If $s_k = 0$, then $c^\lambda_{\eta_1,\ldots,\eta_p} > 0$ implies $\lambda \leq \eta_1 + \cdots + \eta_p \leq \mu + \cdots + \mu = \lambda$, so the only such constituent is $(\chi^\mu)^{\times p}$, and it occurs with multiplicity 1. Then $\langle \chi^\lambda\big\downarrow_{P_{p^k}}, \phi(s)\rangle = 1$ follows by the same argument as in Step 3 of Case 1.

   If $s_k \neq 0$, then $\lambda = (p\mu_1,\ldots,p\mu_{r-1},p\mu_r - 1, 1) = (p\mu_1,\ldots,p\mu_{r-1},p-1,1)$. Hence the only such constituent is $(\chi^\mu)^{\times p}$, and it occurs with multiplicity $p-1$. Thus

$$\begin{aligned}\langle \chi^\lambda\big\downarrow_{P_{p^k}}, \phi(s)\rangle &= \langle \mathcal{X}(\mu;(p-1,1))\big\downarrow^H_P, \phi(s)\rangle \\ &= \sum_{i=1}^{p-1}\langle \mathcal{X}(\mu\big\downarrow_{P_{p^{k-1}}};\phi_i), \phi(s)\rangle = \sum_{i=1}^{p-1}\delta_{i,s_k} = 1\end{aligned}$$

   by Lemma 2.19.

4. $\lambda > \nu$ for all $\nu \in \Omega(t)$: suppose $\nu \in \Omega(t)$. Then there exists $\eta_1,\ldots,\eta_p \in \Omega(t^-)$ such that $\chi^{\eta_1} \times \cdots \times \chi^{\eta_p} \mid \chi^\nu\big\downarrow_Y$. Since $\eta_i \lneqq \mu$ for all i and $p \geq 3$, then $\nu \leq \eta_1 + \cdots + \eta_p \lneqq (p\mu_1,\ldots,p\mu_{r-1},p\mu_r - 1, 1) \leq \lambda$.

Thus in all cases, we have shown that the lexicographically greatest partition in $\Omega(s)$ has the claimed properties. □

Therefore, in terms of the three equivalence relations (i), (ii) and (iii), we have that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) and (iii) $\not\Longleftarrow$ (ii) $\not\Longleftarrow$ (i) in general, but (i) $\Longleftrightarrow$ (ii) when $n = p^k$.

The results of Chapter 4 are first steps towards the goal of describing the sets $\Omega(\phi)$ explicitly. We say more on this in Chapters 5 and 6.

Chapter 5

On permutation characters and Sylow p-subgroups of $\mathfrak{S}_n$

This chapter is based on the paper [29], joint with Dr Eugenio Giannelli. Here, we are able to present a shorter proof of Theorem 5.14 ([29, Theorem 3.2]) using new results on Littlewood–Richardson coefficients (Section 5.2), proved in collaboration with J. Long.

In this chapter, we identify all of the irreducible characters of the symmetric group $\mathfrak{S}_n$ containing the trivial character as a constituent upon restriction to a Sylow p-subgroup, for all n and odd primes p.

We would like to mention that following the publication of our article [29], our main result Theorem 5.1 was applied to the representation theory of simple groups by Malle and Zalesski in [47] as part of a study of so-called Syl_p-regular characters and Steinberg-like characters, culminating in their classification of projective indecomposable modules of certain dimensions for simple groups G.

5.1 Outline

We investigate the decomposition into irreducible constituents of the permutation character $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$, where n is any natural number, p is an odd prime and P_n is a Sylow p-subgroup of $\mathfrak{S}_n$. More precisely, our main result determines all of the irreducible constituents of the permutation module induced by the action of $\mathfrak{S}_n$ on the cosets of a Sylow p-subgroup P_n, whose character is $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$.

Theorem 5.1. *Let p be an odd prime, let n be a natural number and let $\lambda \in \mathcal{P}(n)$. Then χ^λ is not an irreducible constituent of $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$ if and only if $n = p^k$ for some $k \in \mathbb{N}$ and $\lambda \in \{(p^k-1,1),(2,1^{p^k-2})\}$, or $p=3$ and λ is one of the following partitions:*

$$(2,2);\quad (3,2,1);\quad (5,4),(2^4,1),(4,3,2),(3^2,2,1);\quad (5,5),(2^5).$$

Excluding the few exceptions arising for small symmetric groups at the prime 3, Theorem 5.1 shows that given any natural number n which is not a power of p, the restriction to P_n of any irreducible character of $\mathfrak{S}_n$ has the trivial character $\mathbb{1}_{P_n}$ as a constituent. We remark that this clearly does not hold for $p = 2$. For instance, the sign representation of $\mathfrak{S}_n$ restricts irreducibly and non-trivially to a Sylow 2-subgroup of $\mathfrak{S}_n$. More generally, when n is a power of 2, [26, Theorem 1.1] shows that no non-trivial irreducible character of odd degree of $\mathfrak{S}_n$ appears as an irreducible constituent of $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$, where $P_n \in \mathrm{Syl}_2(\mathfrak{S}_n)$. The above observations underline that for the prime 2 the situation is notably less regular than for odd primes, and at the time of writing we do not have a conjecture for a characterisation of the subset of $\mathcal{P}(n)$ labelling irreducible characters appearing as constituents of $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$ when $p = 2$.

Let $\mathcal{H} := \mathcal{H}(\mathfrak{S}_n, P_n, \mathbb{1}_{P_n})$ be the Hecke algebra naturally corresponding to the permutation character $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$. We refer the reader to [8, Chapter 11D] for the complete definition and properties of this correspondence. It is well-known that the number of irreducible representations of $\mathcal{H}$ equals the number of distinct irreducible constituents of the corresponding permutation character (see for example [8, Theorem (11.25)(ii)]). Therefore our Theorem 5.1 has the following consequence.

Corollary 5.2. *Let p be an odd prime and let $n > 10$ be a natural number. If $n \neq p^k$ (respectively $n = p^k$) then the Hecke algebra $\mathcal{H}$ has exactly $|\mathcal{P}(n)|$ (respectively $|\mathcal{P}(n)|-2$) irreducible representations.*

As explained in [8, Theorem 11.25(iii)], understanding the dimensions of the irreducible representations of $\mathcal{H}$ is equivalent to determining the multiplicities of the irreducible constituents of $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$. For this reason we believe that it would be interesting to find a solution to the following problem.

Question 5.3. *Is there a combinatorial description of the map $f : \mathcal{P}(n) \longrightarrow \mathbb{N}_0$, where $f(\lambda)$ equals the multiplicity of χ^λ as an irreducible constituent of $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$?*

A second consequence of Theorem 5.1 is a precise description of the constituents of the permutation character $\mathbb{1}_{Q_n}\big\uparrow^{\mathfrak{A}_n}$, where $\mathfrak{A}_n$ is the alternating group of degree n and Q_n is a Sylow p-subgroup of $\mathfrak{A}_n$. Recall that $\chi^\lambda\big\downarrow_{\mathfrak{A}_n} = \chi^{\lambda'}\big\downarrow_{\mathfrak{A}_n}$, and that the ordinary irreducible characters of $\mathfrak{A}_n$ can be labelled as

$$\mathrm{Irr}(\mathfrak{A}_n) = \{\chi^\lambda\big\downarrow_{\mathfrak{A}_n} \mid \lambda \neq \lambda' \in \mathcal{P}(n)\} \ \cup \ \{\psi_+^\lambda, \psi_-^\lambda \mid \lambda = \lambda' \in \mathcal{P}(n)\}.$$

Theorem 5.4. *Let $p \geq 5$ be a prime, let n be a natural number and let $\psi \in \mathrm{Irr}(\mathfrak{A}_n)$. Then ψ is not an irreducible constituent of $\mathbb{1}_{Q_n}\big\uparrow^{\mathfrak{A}_n}$ if and only if $n = p^k$ for some $k \in \mathbb{N}$ and $\psi = \chi^\lambda\big\downarrow_{\mathfrak{A}_n}$ with $\lambda \in \{(p^k-1, 1), (2, 1^{p^k-2})\}$.*

If $p = 3$, then $\psi \in \mathrm{Irr}(\mathfrak{A}_n)$ is not an irreducible constituent of $\mathbb{1}_{Q_n}\big\uparrow^{\mathfrak{A}_n}$ if and only if one of the following holds:

- *$n = 3^k$ for some $k \geq 2$ and $\psi = \chi^\lambda\big\downarrow_{\mathfrak{A}_n}$ with $\lambda \in \{(3^k-1, 1), (2, 1^{3^k-2})\}$; or*

∘ $n \leq 10$ *and* $\psi \in \{\psi_{\pm}^{(2,1)}, \psi_{\pm}^{(2,2)}, \psi_{\pm}^{(3,2,1)}, \chi^\lambda\big\downarrow_{\mathfrak{A}_n}\}$ *where*

$$\lambda \in \{(5,4), (2^4,1), (4,3,2), (3^2,2,1), (5^2), (2^5)\}.$$

Theorem 5.4 follows from Theorem 5.1 and Corollary 2.24 by observing that when p is odd, Q_n is a Sylow p-subgroup of $\mathfrak{S}_n$.

We conclude by mentioning that Theorem 5.1 gives information on the eigenvalues of the irreducible representations of $\mathfrak{S}_n$, at elements of odd prime power order. This may already be known to experts, but we were not able to find a reference in the literature.

Corollary 5.5. *Let $p \geq 5$ be a prime and let n be a natural number. Let $\lambda \in \mathcal{P}(n)$ and let ρ^λ be a representation of $\mathfrak{S}_n$ affording χ^λ. If n is not a power of p, or if $n = p^k$ but $\lambda \notin \{(p^k-1,1), (2,1^{p^k-2})\}$, then $\rho^\lambda(g)$ has an eigenvalue equal to 1 for any $g \in \mathfrak{S}_n$ of order a power of p. In particular, if P is a fixed Sylow p-subgroup of $\mathfrak{S}_n$ then for all $g \in P$ the matrices $\rho^\lambda(g)$ have a common eigenvector for the eigenvalue 1.*

An analogous study was done extensively in [58] in the case of Chevalley groups. The case of elements of prime order was discussed in [59] for quasi-simple groups.

5.2 Littlewood–Richardson combinatorics

In this section we prove some results concerning Littlewood–Richardson coefficients, which we believe are of independent interest, that will be useful in proving Theorem 5.1 as well as several key results later in Chapter 6. Recall the notation $\mathcal{B}_n(m)$ from Definition 2.1, and the operator $\star$ from Definition 2.12.

Lemma 5.6. *Let $t, t' \in \mathbb{N}$. Then $\mathcal{B}_{2t-1}(t) \star \mathcal{B}_{2t'-1}(t') = \mathcal{B}_{2t+2t'-2}(t+t')$.*

Proof. That $\mathcal{B}_{2t-1}(t) \star \mathcal{B}_{2t'-1}(t') \subseteq \mathcal{B}_{2t+2t'-2}(t+t')$ follows directly from Lemma 2.10. To prove the converse, we proceed by induction on $t+t'$. The base case follows from the observation that for any natural numbers N and M such that $N < 2M$, we have $\mathcal{B}_N(M) \star \mathcal{B}_1(1) \supseteq \mathcal{B}_{N+1}(M+1)$: given any partition $\lambda \in \mathcal{B}_{N+1}(M+1)$, either $\lambda \in \mathcal{B}_{N+1}(M)$ in which case considering any removable box of λ shows that $\lambda \in \mathcal{B}_N(M) \star \mathcal{B}_1(1)$; or $\lambda_1 = M+1$, in which case $\lambda_2 < M+1$ since $N < 2M$, and so considering $\mu = (\lambda_1 - 1, \lambda_2, \dots) \in \lambda^-$ shows that $\lambda \in \mathcal{B}_N(M) \star \mathcal{B}_1(1)$ (and the case if $l(\lambda) = M+1$ is dealt with similarly).

We may now assume that $t, t' \geq 2$. For the inductive step, we take as inductive hypothesis $\mathcal{B}_{2t-3}(t-1) \star \mathcal{B}_{2t'-1}(t') = \mathcal{B}_{2t+2t'-4}(t+t'-1)$. By applying $- \star \mathcal{B}_1(1)$ to both sides, we find

$$\mathcal{B}_{2t-3}(t-1) \star \mathcal{B}_{2t'}(t'+1) = \mathcal{B}_{2t+2t'-3}(t+t'),$$

and then applying $\mathcal{B}_1(1) \star -$ to both sides, we find

$$\mathcal{B}_{2t-2}(t) \star \mathcal{B}_{2t'}(t'+1) = \mathcal{B}_{2t+2t'-2}(t+t'+1).$$

Hence

$$\mathcal{B}_{2t+2t'-2}(t+t') \subseteq \mathcal{B}_{2t+2t'-2}(t+t'+1) = \mathcal{B}_{2t-2}(t) \star \mathcal{B}_{2t'}(t'+1).$$

Thus, letting $\lambda \in \mathcal{B}_{2t+2t'-2}(t+t')$, there exist partitions $\mu \in \mathcal{B}_{2t-2}(t)$ and $\nu \in \mathcal{B}_{2t'}(t'+1)$ such that $c^{\lambda}_{\mu\nu} > 0$. In particular, fix a Littlewood–Richardson filling F of weight ν of the skew shape $[\lambda \setminus \mu]$.

To complete the inductive step, we construct $\hat{\mu} \in \mathcal{B}_{2t-1}(t)$ and $\hat{\nu} \in \mathcal{B}_{2t'-1}(t')$ such that $c^{\lambda}_{\hat{\mu}\hat{\nu}} > 0$, from which we conclude therefore that $\mathcal{B}_{2t+2t'-2}(t+t') \subseteq \mathcal{B}_{2t-1}(t) \star \mathcal{B}_{2t'-1}(t')$. The main idea is to remove an appropriate box b from the skew shape $[\lambda \setminus \mu]$, set $[\hat{\mu}] = [\mu] \cup \mathsf{b}$ and exhibit an appropriate filling F' of $[\lambda \setminus \hat{\mu}]$ of weight $\hat{\nu}$, whence $c^{\lambda}_{\hat{\mu}\hat{\nu}} > 0$.

Since all sets considered are closed under conjugation of partitions, we may without loss of generality assume $\nu_1 \geq l(\nu)$ (by taking λ', μ' and ν' instead of λ, μ and ν if necessary). Let $k \geq 1$ be such that $\nu_1 = \nu_2 = \ldots = \nu_k > \nu_{k+1}$, and let x denote the box containing the last 1 in the Littlewood–Richardson reading order of the filling F (namely right to left, top to bottom). Clearly this must lie at the top of its column and leftmost in its row in $[\lambda \setminus \mu]$, and so must be an addable box for μ. We split into three cases according to the position of x.

Case (i): if the position of x *is neither* $(1, t+1)$ *nor* $(t+1, 1)$. Since x is an addable box for $\mu \in \mathcal{B}_{2t-2}(t)$, setting $\hat{\mu}$ to be the partition whose Young diagram is $[\mu] \cup \mathsf{x}$ we find that $\hat{\mu} \in \mathcal{B}_{2t-1}(t)$.

If $k = 1$ then the filling F' defined as F restricted to the boxes of $[\lambda \setminus \hat{\mu}]$ is a Littlewood–Richardson filling of weight $\hat{\nu} := (\nu_1 - 1, \nu_2, \ldots, \nu_{l(\nu)}) \in \mathcal{B}_{2t'-1}(t'+1)$. Moreover, $\hat{\nu}_1 = \nu_1 - 1 \leq t' + 1 - 1 = t'$, and $l(\hat{\nu}) = l(\nu) \leq t'$ since $l(\nu) \leq \nu_1$ and $|\nu| = 2t'$. Thus $\hat{\nu} \in \mathcal{B}_{2t'-1}(t')$.

If $k > 1$, then $(\nu_1 - 1, \nu_2, \ldots, \nu_{l(\nu)})$ is not a partition: in this case we define F' and $\hat{\nu}$ as follows. Let $i \in \{2, \ldots, k\}$ and consider the position of the last i in the reading order of the filling F. By the definition of Littlewood–Richardson fillings, the last i must appear later in the reading order than the last $i-1$ since $\nu_{i-1} = \nu_i$. Since this holds for all $i \in \{2, \ldots, k\}$, the box containing the last i must be the leftmost i in its row in $[\lambda \setminus \mu]$ (and hence leftmost in its row), and either at the top of its column or immediately below the box containing the last $i-1$ in the reading order of F. Thus we may define a Littlewood–Richardson filling F' of $[\lambda \setminus \hat{\mu}]$ to be obtained from F by removing the 1 corresponding to the box x, then relabelling the last i in F by the number $i-1$, for each $2 \leq i \leq k$. In particular, the weight of F' is the partition $\hat{\nu} := (\nu_1, \ldots, \nu_{k-1}, \nu_k - 1, \nu_{k+1}, \ldots, \nu_{l(\nu)})$. Moreover, $k > 1$ and $l(\nu) \leq \nu_1$ imply that $\nu \in \mathcal{B}_{2t'}(t')$, and hence $\hat{\nu} \in \mathcal{B}_{2t'-1}(t')$. An example is shown in Figure 5.1.

Thus for all values of k, setting $\mathsf{b} = \mathsf{x}$ and taking $\hat{\mu}$, $\hat{\nu}$ as described above we find that $\lambda \in \mathcal{B}_{2t-1}(t) \star \mathcal{B}_{2t'-1}(t')$ as claimed.

Case (ii): if x *lies in position* $(1, t+1)$. Then $\mu_1 = t$ and $\lambda_1 = \mu_1 + \nu_1$ since x contains the last 1 of F. Let y denote the box containing the last 2 in the reading order of F; this exists as $\nu \neq (2t')$. The box y must be leftmost in its row, as all of the 1s in F

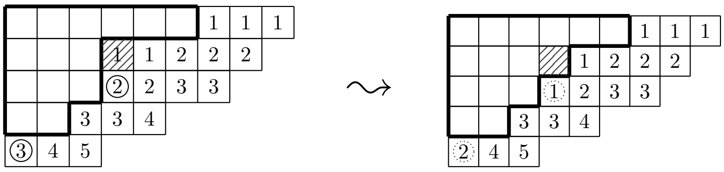


Figure 5.1: Example of case (i): $t = 8$, $t' = 9$, $\lambda = (9, 8, 7, 5, 3) \vdash 32$, $\mu = (6, 3, 3, 2)$, $\nu = (5^3, 2, 1)$, $k = 3$ and F as shown. On the left, the box x is shaded, and the last i of F is circled for $2 \leq i \leq k$. On the right, F' is shown with $\hat{\mu} = (6, 4, 3, 2)$ and $\hat{\nu} = (5^2, 4, 2, 1)$. The boxes containing the circled numbers have been relabelled to produce F'.

lie precisely in the first row of $[\lambda \setminus \mu]$. If it does not lie at the top of its column, it must lie immediately under a 1 in F, from which we deduce that y occupies position $(2, j)$ for some $j \geq t + 1$. But then $\mu_2 \geq t$, contradicting $|\mu| = 2t - 2$. Thus y lies at the top of its column and is an addable box for μ. Moreover, y cannot lie in position $(t + 1, 1)$ or else $|\mu| \geq \mu_1 + l(\mu) - 1 = t + t - 1$. Thus, if y occupies position $(r, \mu_r + 1)$ then $\hat{\mu} := (\mu_1, \ldots, \mu_{r-1}, \mu_r + 1, \mu_{r+1}, \ldots, \mu_{l(\mu)}) \in \mathcal{B}_{2t-1}(t)$ (note $\hat{\mu}$ is well-defined since $\mu_r < \mu_{r-1}$).

Let $j \geq 2$ be such that $\nu_2 = \nu_3 = \ldots = \nu_j > \nu_{j+1}$. Similarly to case (i), we define a Littlewood–Richardson filling F' of $[\lambda \setminus \hat{\mu}]$ to be obtained from F by removing the 2 corresponding to the box y, then relabelling the last i in F by the number $i - 1$ for each $3 \leq i \leq j$ (or no relabelling required if $j = 2$). The resulting weight is $\hat{\nu} := (\nu_1, \nu_2, \ldots, \nu_{j-1}, \nu_j - 1, \nu_{j+1}, \ldots, \nu_{l(\nu)}) \in \mathcal{B}_{2t'-1}(t' + 1)$. Since $\lambda_1 = \mu_1 + \nu_1 \leq t + t'$, we must have $l(\nu) \leq \nu_1 \leq t'$, and so in fact $\hat{\nu} \in \mathcal{B}_{2t'-1}(t')$.

Thus setting $\mathsf{b} = \mathsf{y}$ and taking $\hat{\mu}$, $\hat{\nu}$ as described we find that $\lambda \in \mathcal{B}_{2t-1}(t) \star \mathcal{B}_{2t'-1}(t')$ as claimed.

Case (iii): if x *lies in position* $(t + 1, 1)$. Let z denote the box containing the second-to-last 1 in the reading order of F; this exists as $\nu \neq (1^{2t'})$. It cannot be in position $(t+1, 2)$, or else $|\mu| \geq \mu'_1 + \mu'_2 = 2t$. Thus z must be leftmost in its row (in some row $r < t$) and lie at the top of its column, so it must be an addable box for μ. Moreover, z cannot be in position $(1, t + 1)$ as $|\mu| = 2t - 2$ and so $\hat{\mu} := (\mu_1, \ldots, \mu_{r-1}, \mu_r + 1, \mu_{r+1}, \ldots, \mu_{l(\mu)}) \in \mathcal{B}_{2t-1}(t)$.

Recall $\nu_1 = \ldots = \nu_k > \nu_{k+1}$. If $k = 1$, then the filling F' defined as F restricted to the boxes of $[\lambda \setminus \hat{\mu}]$ is a Littlewood–Richardson filling of weight $\hat{\nu} := (\nu_1 - 1, \nu_2, \ldots, \nu_{l(\nu)}) \in \mathcal{B}_{2t'-1}(t')$. If $k > 1$, then since the last 1 lies in the box x at position $(t + 1, 1)$, the last i lies in position $(t + i, 1)$ for each $2 \leq i \leq k$, and notice that $\mu'_2 \leq t - 2$ since $l(\mu) = t$. Similarly to case (i), we define a Littlewood–Richardson filling F' of $[\lambda \setminus \hat{\mu}]$ to be obtained from F by removing the 1 corresponding to the box z, then relabelling the second-to-last i in F by the number $i - 1$, for each $2 \leq i \leq k$. The resulting weight is $\hat{\nu} := (\nu_1, \ldots, \nu_{k-1}, \nu_k - 1, \nu_{k+1}, \ldots, \nu_{l(\nu)}) \in \mathcal{B}_{2t'-1}(t')$.

Thus setting $\mathsf{b} = \mathsf{z}$ and taking $\hat{\mu}$, $\hat{\nu}$ as described we find that $\lambda \in \mathcal{B}_{2t-1}(t) \star \mathcal{B}_{2t'-1}(t')$ as claimed. $\square$

Proposition 5.7. *Let $n, n', t, t' \in \mathbb{N}$ be such that $\frac{n}{2} < t \leq n$ and $\frac{n'}{2} < t' \leq n'$. Then*

$$\mathcal{B}_n(t) \star \mathcal{B}_{n'}(t') = \mathcal{B}_{n+n'}(t+t').$$

Proof. That $\mathcal{B}_n(t) \star \mathcal{B}_{n'}(t') \subseteq \mathcal{B}_{n+n'}(t+t')$ follows from Definition 2.12. For the reverse inclusion, and hence equality of sets, we proceed by induction on the quantity $2t - n + 2t' - n' \geq 2$, with the base case given by Lemma 5.6. Now suppose $2t - n + 2t' - n' > 2$, so without loss of generality assume $t' - 1 > \frac{n'-1}{2}$. Then $\mathcal{B}_{n'-1}(t'-1) \star \mathcal{B}_1(1) = \mathcal{B}_{n'}(t')$ and $\mathcal{B}_n(t) \star \mathcal{B}_{n'-1}(t'-1) = \mathcal{B}_{n+n'-1}(t+t'-1)$ by the inductive hypothesis. Thus

$$\begin{aligned} \mathcal{B}_n(t) \star \mathcal{B}_{n'}(t') &= \mathcal{B}_n(t) \star \big(\mathcal{B}_{n'-1}(t'-1) \star \mathcal{B}_1(1)\big) \\ &= \big(\mathcal{B}_n(t) \star \mathcal{B}_{n'-1}(t'-1)\big) \star \mathcal{B}_1(1) \\ &= \mathcal{B}_{n+n'-1}(t+t'-1) \star \mathcal{B}_1(1) \\ &= \mathcal{B}_{n+n'}(t+t') \end{aligned}$$

as claimed. □

Lemma 5.8. *Let $n, m, t \in \mathbb{N}$ and suppose that $\frac{m}{2} < t \leq m$. If $n \geq 5$, then*

$$\mathcal{B}_m(t) \star (\mathcal{B}_n(n-2) \cup \{(n)\}^\circ) = \mathcal{B}_{m+n}(t+n).$$

In particular, $\mathcal{P}(m+n) = \mathcal{P}(m) \star (\mathcal{P}(n) \setminus \{(n-1,1)\}^\circ)$.

Proof. If $t = 1$ then $m = 1$ and the result follows from the branching rule (see Section 2.2), so from now on we may assume $t \geq 2$.

Let $X := \mathcal{B}_m(t) \star (\mathcal{B}_n(n-2) \cup \{(n)\}^\circ)$. Since $n \geq 5$, we have that $n - 2 > \frac{n}{2}$, and so $\mathcal{B}_{m+n}(t+n-2) \subseteq X$ by Proposition 5.7. Moreover, $X \subseteq \mathcal{B}_m(t) \star \mathcal{P}(n) = \mathcal{B}_{m+n}(t+n)$, by Proposition 5.7. Since $X^\circ = X$, it remains to show that if $\lambda \vdash m+n$ with $\lambda_1 \in \{t+n-1, t+n\}$, then $\lambda \in X$.

First suppose $\lambda = t+n$, so $\lambda = (t+n, \mu)$ for some $\mu \vdash m-t < t$. Observe that $\chi^{(t,\mu)} \times \chi^{(n)} \mid \chi^\lambda\big\downarrow_{\mathfrak{S}_m \times \mathfrak{S}_n}$ and $\mu \in \mathcal{B}_m(t)$, so $\lambda \in X$.

Otherwise we have $\lambda_1 = t+n-1$, so $\lambda = (t+n-1, \mu)$ for some $\mu \vdash m-t+1$. If $\mu_1 \geq t$, then $m = 2t-1$ and thus $\lambda = (t+n-1, t)$. Since $\chi^{(t,t-1)} \times \chi^{(n)} \mid \chi^\lambda\big\downarrow_{\mathfrak{S}_m \times \mathfrak{S}_n}$ and $(t, t-1) \in \mathcal{B}_m(t)$, then $\lambda \in X$. If $l(\mu) \geq t$ then similarly $m = 2t-1$ and $\lambda = (t+n-1, 1^t)$, and we similarly observe that $\lambda \in X$ since $(t, 1^{t-1}) \in \mathcal{B}_m(t)$. Otherwise, $\mu \in \mathcal{B}_{m-t+1}(t-1)$, so $(t-1, \mu) \in \mathcal{B}_m(t)$. But clearly $\chi^{(t-1,\mu)} \times \chi^{(n)} \mid \chi^\lambda\big\downarrow_{\mathfrak{S}_m \times \mathfrak{S}_n}$, so $\lambda \in X$. □

Recall that $\mathcal{LR}(\gamma)$ denotes the set of weights of Littlewood–Richardson fillings of a skew shape γ, and ν^+ denotes the set of partitions indexing the irreducible constituents in the induced character $\chi^\nu\big\uparrow^{\mathfrak{S}_{|\nu|+1}}$, for any partition ν.

Lemma 5.9. *Let $X = [\lambda \setminus \mu]$ be a skew shape, and suppose $\nu \in \mathcal{LR}(X)$. Let Y be a skew shape obtained from X by adding a single box. Then $\mathcal{LR}(Y) \cap \nu^+ \neq \emptyset$.*

Proof. Let $|\lambda| = n$ and $|\mu| = m$. First suppose Y is obtained from X by adding a box externally, that is, $Y = [\tilde{\lambda} \setminus \mu]$ for some $\tilde{\lambda} \in \lambda^+$. Since $\nu \in \mathcal{LR}(X)$, the iterated Littlewood–Richardson coefficient $c^{\tilde{\lambda}}_{\mu,\nu,(1)} = \langle \chi^{\tilde{\lambda}}, \chi^\mu \times \chi^\nu \times \chi^{(1)} \big\uparrow^{\mathfrak{S}_{n+m+1}}_{\mathfrak{S}_n \times \mathfrak{S}_m \times \mathfrak{S}_1} \rangle$ is positive. But by the branching rule,

$$\begin{aligned} 0 < c^{\tilde{\lambda}}_{\mu,\nu,(1)} &= \left\langle \chi^{\tilde{\lambda}}, \chi^\mu \times (\chi^\nu \times \chi^{(1)}) \big\uparrow^{\mathfrak{S}_n \times \mathfrak{S}_{m+1}}_{\mathfrak{S}_n \times \mathfrak{S}_m \times \mathfrak{S}_1} \big\uparrow^{\mathfrak{S}_{n+m+1}}_{\mathfrak{S}_n \times \mathfrak{S}_{m+1}} \right\rangle \\ &= \left\langle \chi^{\tilde{\lambda}}, \sum_{\tilde{\nu} \in \nu^+} \chi^\mu \times \chi^{\tilde{\nu}} \big\uparrow^{\mathfrak{S}_{n+m+1}}_{\mathfrak{S}_n \times \mathfrak{S}_{m+1}} \right\rangle \\ &= \sum_{\tilde{\nu} \in \nu^+} \left\langle \chi^{\tilde{\lambda}}, \chi^\mu \times \chi^{\tilde{\nu}} \big\uparrow^{\mathfrak{S}_{n+m+1}}_{\mathfrak{S}_n \times \mathfrak{S}_{m+1}} \right\rangle \end{aligned}$$

so there exists $\tilde{\nu} \in \nu^+$ such that $\left\langle \chi^{\tilde{\lambda}}, \chi^\mu \times \chi^{\tilde{\nu}} \big\uparrow^{\mathfrak{S}_{n+m+1}}_{\mathfrak{S}_n \times \mathfrak{S}_{m+1}} \right\rangle > 0$. Hence $\tilde{\nu} \in \mathcal{LR}(Y)$.

Otherwise, if Y is obtained from X by adding a box internally, that is, $Y = [\lambda \setminus \tilde{\mu}]$ for some $\tilde{\mu} \in \mu^-$, a similar argument considering the subgroup $\mathfrak{S}_{n-1} \times \mathfrak{S}_1 \times \mathfrak{S}_m \leq \mathfrak{S}_{n+m}$ and observing that $\chi^{(1)} \times \chi^\nu \big\uparrow^{\mathfrak{S}_{m+1}}_{\mathfrak{S}_1 \times \mathfrak{S}_m} = \chi^\nu \times \chi^{(1)} \big\uparrow^{\mathfrak{S}_{m+1}}_{\mathfrak{S}_m \times \mathfrak{S}_1} = \sum_{\tilde{\nu} \in \nu^+} \chi^{\tilde{\nu}}$ shows that $\mathcal{LR}(Y) \cap \nu^+ \neq \emptyset$ in this case also. □

The following definition will be useful for the next section.

Definition 5.10. *Let $q, y \in \mathbb{N}$ be such that $q \geq 2$ and let $\mathcal{B} \subseteq \mathcal{P}(y)$. Let $H = (\mathfrak{S}_y)^{\times q} \leq \mathfrak{S}_{qy}$. We let $\mathcal{D}(q, y, \mathcal{B})$ be the subset of $\mathcal{P}(qy)$ consisting of all those partitions $\lambda \in \mathcal{P}(qy)$ for which there exists $\mu_1, \mu_2, \dots, \mu_q \in \mathcal{B}$, not all equal, such that*

$$\chi^{\mu_1} \times \chi^{\mu_2} \times \cdots \times \chi^{\mu_q} \;\Big|\; \chi^\lambda \big\downarrow_H.$$

We remark that if $\mathcal{B}^\circ = \mathcal{B}$, then $\mathcal{D}(q, y, \mathcal{B})^\circ = \mathcal{D}(q, y, \mathcal{B})$: this follows from the fact that $\chi^{\lambda'} = \chi^\lambda \cdot \chi^{(1^{qy})}$ for all $\lambda \vdash qy$, and $\chi^{(1^{qy})} \big\downarrow_{(\mathfrak{S}_y)^{\times q}} = \chi^{(1^y)} \times \cdots \times \chi^{(1^y)}$ (see Section 2.2).

Proposition 5.11. *Let $m, t \in \mathbb{N}$ and suppose $\frac{m}{2} + 1 < t \leq m$. Let $\lambda \in \mathcal{B}_{2m}(2t-1)$. Then either $\lambda \in \mathcal{D}\big(2, m, \mathcal{B}_m(t)\big)$, or $\chi^\lambda \big\downarrow_{\mathfrak{S}_m \times \mathfrak{S}_m}$ has two irreducible constituents $\chi^\alpha \times \chi^\alpha$ and $\chi^\beta \times \chi^\beta$ where $\alpha \neq \beta \in \mathcal{B}_m(t)$.*

Proof. First suppose $\lambda = (m, m)$. Notice that $\chi^\alpha \times \chi^\alpha$ is a constituent of $\chi^\lambda \big\downarrow_{\mathfrak{S}_m \times \mathfrak{S}_m}$ where $\alpha \in \mathcal{B}_m(t)$ if and only if $\alpha = (\alpha_1, m - \alpha_1)$ with $\frac{m}{2} \leq \alpha_1 \leq t$ (the conditions of the proposition imply that $t \geq 2$). But $t > \frac{m}{2} + 1$, so there are at least two possible integer values that $\alpha_1 \in [\frac{m}{2}, t]$ can take. Thus we find two irreducible constituents $\chi^\alpha \times \chi^\alpha$ and $\chi^\beta \times \chi^\beta$ where $\alpha \neq \beta \in \mathcal{B}_m(t)$ as required.

Moreover, $\chi^{\lambda'} = \chi^\lambda \times \chi^{(1^n)}$ where $\chi^{(1^n)}$ is the sign character, so $\chi^{\alpha'} \times \chi^{\alpha'}$ and $\chi^{\beta'} \times \chi^{\beta'}$ are two different irreducible constituents of $\chi^{\lambda'} \big\downarrow_{\mathfrak{S}_m \times \mathfrak{S}_m}$, and $\alpha' \neq \beta' \in \mathcal{B}_m(t)$ since $\mathcal{B}_m(t)$ is closed under conjugation.

Now let $\lambda \in \mathcal{B}_{2m}(2t-1) \setminus \{(m,m)\}^\circ$. By Proposition 5.7, there exist partitions $\mu \in \mathcal{B}_m(t)$ and $\nu \in \mathcal{B}_m(t-1)$ such that $c^\lambda_{\mu\nu} > 0$. If $\mu \neq \nu$ then $\lambda \in \mathcal{D}\big(2, 2m, \mathcal{B}_m(t)\big)$ and we are done, so assume that $\mu = \nu \in \mathcal{B}_m(t-1)$. By 'passing a box' between $[\mu]$ and $[\lambda \setminus \mu]$, we construct partitions

(i) $\hat{\mu} \in \mathcal{B}_{m+1}(t)$ and $\hat{\nu} \in \mathcal{B}_{m-1}(t-1)$ such that $c^{\lambda}_{\hat{\mu}\hat{\nu}} > 0$; then

(ii) $\tilde{\mu} \in \mathcal{B}_m(t)$ and $\tilde{\nu} \in \mathcal{B}_m(t)$ such that $c^{\lambda}_{\tilde{\mu}\tilde{\nu}} > 0$, *and* $\tilde{\mu} \neq \mu$,

whence the assertion of the proposition follows. We now explain in detail the constructions (i) and (ii).

Step (i): Fix a Littlewood–Richardson filling F of $[\lambda \setminus \mu]$ of weight ν. Let b denote the box of $[\lambda \setminus \mu]$ containing the last 1 in the reading order of F; clearly this is an addable box for μ. We split into three cases depending on the shape of $[\mu] + \mathsf{b}$.

Case (a): If $[\mu] + \mathsf{b}$ is not a rectangle, then define $\hat{\mu}$ via $[\hat{\mu}] := [\mu] + \mathsf{b}$. Let $k \in \mathbb{N}$ be such that $\nu_1 = \nu_2 = \cdots = \nu_k > \nu_{k+1}$. Define F' to be obtained from F by removing the 1 corresponding to the box b, and then if $k > 1$ additionally relabelling the last i in F by the number $i-1$, for each $2 \leq i \leq k$. Thus F' is a Littlewood–Richardson filling of $[\lambda \setminus \hat{\mu}]$ of weight $\hat{\nu} := (\nu_1, \ldots, \nu_{k-1}, \nu_k - 1, \nu_{k+1}, \ldots, \nu_{l(\nu)})$, by the same argument as in the proof of Lemma 5.6.

Now we may assume $[\mu] + \mathsf{b}$ is a rectangle. Notice $m \geq 3$, so either $(2,1) \subseteq \mu$ or $\mu \in \{(m), (1^m)\}$. If $\mu = (m)$, then F being a filling of weight $\nu = \mu$ and the definition of b together imply that $\lambda = (2m)$, a contradiction. Similarly if $\mu = (1^m)$ then $\lambda = (1^{2m}) \notin \mathcal{B}_{2m}(2t-1)$. Thus when $[\mu] + \mathsf{b}$ is a rectangle then $(2,1) \subseteq \mu$.

Case (b): If $[\mu] + \mathsf{b}$ is a rectangle and $l(\lambda) > l(\mu)$, let c be the box in row $l(\mu)+1$, column 1, and define $[\hat{\mu}] := [\mu] + \mathsf{c}$. Suppose in F the box c is filled with the number j. Since the rows of $[\lambda \setminus \mu]$ are filled weakly increasingly, and the columns strictly increasingly, the j in c must be the last j that appears in the reading order of F. Suppose $\nu_j = \nu_{j+1} = \ldots = \nu_l > \nu_{l+1}$. Define F' to be obtained from F by removing the j corresponding to the box c, and then if $l > j$ additionally relabelling the last i in F by the number $i-1$, for each $j+1 \leq i \leq l$. Thus F' is a Littlewood–Richardson filling of $[\lambda \setminus \hat{\mu}]$ of weight $\hat{\nu} := (\nu_1, \ldots, \nu_{l-1}, \nu_l - 1, \nu_{l+1}, \ldots, \nu_{l(\nu)})$, by the same argument as in the proof of Lemma 5.6.

Case (c): Otherwise $[\mu] + \mathsf{b}$ is a rectangle but $l(\lambda) = l(\mu)$. If $l(\mu) > 2$, then the number 2 appears in F precisely as the entries in the second row of $[\lambda \setminus \mu]$, and thus $\nu_2 = \lambda_2 - \mu_2$. But the number ν_1 of 1s in F is equal to $\lambda_1 - \mu_1 + 1$ (they appear in the first row of $[\lambda \setminus \mu]$ and b). Thus $\mu = \nu$ and $\mu_1 = \mu_2$ give $\lambda_2 = \lambda_1 + 1$, a contradiction. Thus $l(\mu) = 2$, in which case μ is of the form $(a, a-1) \vdash m$, but since $l(\lambda) = l(\mu) = 2$ then in fact $\lambda = (m, m)$, a contradiction. Thus case (c) in fact cannot occur.

Observe that in cases (a) and (b), $[\hat{\mu}]$ is obtained from $[\mu]$ by adding a single addable box, so $\mu \in \mathcal{B}_m(t-1)$ implies $\hat{\mu} \in \mathcal{B}_{m+1}(t)$. Also since $\nu \in \mathcal{B}_m(t-1)$, clearly $\hat{\nu} \in \mathcal{B}_{m-1}(t-1)$.

Step (ii): Let $\mathsf{x} = [\hat{\mu}] \setminus [\mu]$. By construction, x is not the only removable box of $[\hat{\mu}]$. Choose a removable box of $\hat{\mu}$ different from x, say y. Let $\tilde{\mu}$ be defined via $[\tilde{\mu}] := [\hat{\mu}] - \mathsf{y}$, so $\tilde{\mu} \neq \mu$. Also $\hat{\mu} \in \mathcal{B}_{m+1}(t)$, so $\tilde{\mu} \in \mathcal{B}_m(t)$. By Lemma 5.9, there exists a Littlewood–Richardson filling of $[\lambda \setminus \hat{\nu}] \cup \mathsf{y}$ of weight $\tilde{\nu}$, for some $\tilde{\nu} \in \hat{\nu}^{+}$. But $\hat{\nu} \in \mathcal{B}_{m-1}(t-1)$, so $\tilde{\nu} \in \mathcal{B}_m(t)$. $\square$

Proposition 5.12. *Let $m, t \in \mathbb{N}$ be such that $\frac{m}{2} + 1 < t \leq m$. Let $q \in \mathbb{N}_{\geq 3}$. Then*

$$\mathcal{B}_{qm}(qt-1) \subseteq \mathcal{D}\big(q, m, \mathcal{B}_m(t)\big).$$

Proof. We proceed by induction on q, beginning with the base case $q = 3$. Let $\lambda \in \mathcal{B}_{3m}(3t-1)$. Then $\mathcal{B}_{2m}(2t-1) \star \mathcal{B}_m(t) = \mathcal{B}_{3m}(3t-1)$ by Proposition 5.7, and so there exists $\mu \in \mathcal{B}_{2m}(2t-1)$ and $\nu \in \mathcal{B}_m(t)$ such that $c^\lambda_{\mu\nu} > 0$. By Proposition 5.11, one of the following holds:

(i) $\mu \in \mathcal{D}\big(2, m, \mathcal{B}_m(t)\big)$, in which case $c^\mu_{\sigma\tau} > 0$ for some $\sigma \neq \tau \in \mathcal{B}_m(t)$. Then $c^\lambda_{\sigma\tau\nu} > 0$ and hence $\lambda \in \mathcal{D}\big(3, m, \mathcal{B}_m(t)\big)$; or

(ii) $\chi^\alpha \times \chi^\alpha$, $\chi^\beta \times \chi^\beta$ are both constituents of $\chi^\mu\big\downarrow_{\mathfrak{S}_m \times \mathfrak{S}_m}$ where $\alpha \neq \beta \in \mathcal{B}_m(t)$. Then $c^\lambda_{\alpha\alpha\nu}$, $c^\lambda_{\beta\beta\nu} > 0$, but either $\nu \neq \alpha$ or $\nu \neq \beta$ and so we have $\lambda \in \mathcal{D}\big(3, m, \mathcal{B}_m(t)\big)$ in this case also.

Now suppose $q \geq 4$ and assume the statement of the proposition holds for $q-1$. Let $\lambda \in \mathcal{B}_{qm}(qt-1)$. Then there exists $\mu \in \mathcal{B}_{(q-1)m}\big((q-1)t-1\big)$ and $\nu \in \mathcal{B}_m(t)$ such that $c^\lambda_{\mu\nu} > 0$, by Proposition 5.7. By the inductive hypothesis, $\mu \in \mathcal{D}\big(q-1, m, \mathcal{B}_m(t)\big)$, so there exists $\mu_1, \ldots, \mu_{q-1} \in \mathcal{B}_m(t)$ which are not all equal such that $c^\mu_{\mu_1 \cdots \mu_{q-1}} > 0$. Hence $c^\lambda_{\mu_1 \cdots \mu_{q-1}\nu} > 0$, which gives $\lambda \in \mathcal{D}\big(q, m, \mathcal{B}_m(t)\big)$. □

5.3 The prime power case

Fix an odd prime p. The aim of this section is to prove Theorem 5.1 for $n = p^k$. As we will see, this is the crucial part of Theorem 5.1. In fact, the complete statement for all natural numbers follows relatively easily from the prime power case.

Definition 5.13. *For $n \in \mathbb{N}_{\geq 3}$, let $\Delta(n) = \mathcal{P}(n) \setminus \{(n-1, 1), (2, 1^{n-2})\}$. For $q \in \mathbb{N}_{\geq 2}$, let $\mathcal{D}(q, n) := \mathcal{D}(q, n, \Delta(n))$.*

The main objective of this section is to establish the following:

Theorem 5.14. *Let $k \in \mathbb{N}$ and $\lambda \vdash p^k \neq 9$. Then $\langle \chi^\lambda\big\downarrow_{P_{p^k}}, \mathbb{1}_{P_{p^k}} \rangle = 0$ if and only if $\lambda \notin \Delta(p^k)$. If $p^k = 9$ then $\langle \chi^\lambda\big\downarrow_{P_9}, \mathbb{1}_{P_9} \rangle = 0$ if and only if*

$$\lambda \in \{(8,1), (5,4), (4,3,2), (3^2,2,1), (2^4,1), (2,1^7)\}.$$

Our proof is by induction on $k \in \mathbb{N}$. We start with the base case $k = 1$.

Lemma 5.15. *Let $n \in \mathbb{N}$ and suppose $n \leq p$. Let $\lambda \vdash n$. Then $\langle \chi^\lambda\big\downarrow_{P_n}, \mathbb{1}_{P_n} \rangle = 0$ if and only if $n = p$ and $\lambda \in \{(p-1, 1), (2, 1^{p-2})\}$.*

Proof. This follows from Corollary 2.4. □

The following proposition is one of the key steps in our proof of Theorem 5.14.

Proposition 5.16. *Let $k \in \mathbb{N}$. Let $\mu_1, \ldots, \mu_p$ be partitions of p^k, not all the same, such that for all $i \in [p]$, we have $\langle \chi^{\mu_i}\big\downarrow_{P_{p^k}}, \mathbb{1}_{P_{p^k}} \rangle \neq 0$. Let $\lambda \in \mathcal{P}(p^{k+1})$ be such that $\chi^{\mu_1} \times \cdots \times \chi^{\mu_p}$ is an irreducible constituent of $\chi^\lambda\big\downarrow_{\mathfrak{S}_{p^k}^{\times p}}$. Then $\langle \chi^\lambda\big\downarrow_{P_{p^{k+1}}}, \mathbb{1}_{P_{p^{k+1}}} \rangle \neq 0$.*

Proof. Let $G = \mathfrak{S}_{p^{k+1}}$, $H = (\mathfrak{S}_{p^k})^{\times p} \leq G$ and set $\psi = \chi^{\mu_1} \times \cdots \times \chi^{\mu_p} \in \mathrm{Irr}(H)$. Let $P = P_{p^{k+1}}$ be such that $P = B \rtimes D$ where $(P_{p^k})^{\times p} \cong B \leq H$ and $P_p \cong D \leq G$, naturally acting on H by permuting (as blocks for its action) the p direct factors of H. Hence $W := H \rtimes D$ satisfies $H \leq W \leq G$ and $W \cong \mathfrak{S}_{p^k} \wr P_p$, and D is chosen such that $P \leq W$.

Since $\chi^\lambda \in \mathrm{Irr}(G|\psi)$, there exists $\chi \in \mathrm{Irr}(W|\psi)$ such that χ is a constituent of $\chi^\lambda\big\downarrow_W$. Since $\mu_1, \ldots, \mu_p$ are not all equal, then $\chi = \psi\big\uparrow_H^W$ by the description of $\mathrm{Irr}(\mathfrak{S}_{p^k} \wr P_p)$ in Section 2.3. It is clear that $PH = W$, so $\chi\big\downarrow_P = \psi\big\downarrow_B\big\uparrow^P$ by Lemma 2.16.

Moreover, $\psi\big\downarrow_B^H = \chi^{\mu_1}\big\downarrow_{P_{p^k}} \times \cdots \times \chi^{\mu_p}\big\downarrow_{P_{p^k}}$, so since $\langle \chi^{\mu_i}\big\downarrow_{P_{p^k}}, \mathbb{1}_{P_{p^k}} \rangle \neq 0$ for all i, we have that $\mathbb{1}_B$ is a constituent of $\psi\big\downarrow_B$. Thus $\mathbb{1}_B\big\uparrow^P$ is a direct summand of $\psi\big\downarrow_B\big\uparrow^P$. But $\mathbb{1}_P = \mathcal{X}(\mathbb{1}_{P_{p^k}} \wr P_p)$ is a constituent of $\mathbb{1}_B\big\uparrow^P$ by Lemma 2.15, so $\langle \chi^\lambda\big\downarrow_P, \mathbb{1}_P \rangle > 0$ as claimed. $\square$

In light of Proposition 5.16, we now focus on the study of the restriction of irreducible characters of $\mathfrak{S}_{p^{k+1}}$ to the Young subgroup $\mathfrak{S}_{p^k}^{\times p}$.

Our next goal is to show that $\mathcal{D}(p, p^k)$ is a very large subset of $\Delta(p^{k+1})$, where we recall the notation $\mathcal{D}(p, p^k)$ from Definitions 5.10 and 5.13. This is done in Corollary 5.18 below. Recall the definition of the set $\mathcal{B}_n(m)$ for $m, n \in \mathbb{N}$ from Definition 2.1.

Proposition 5.17. *Let $k \in \mathbb{N}$ be such that $p^k \notin \{3, 5, 9\}$. Then for all $q \in \{3, \ldots, p\}$,*

$$\mathcal{D}(q, p^k) = \mathcal{B}_{qp^k}(qp^k - 2).$$

Proof. Clearly $\mathcal{D}(q, p^k) \subseteq \mathcal{B}_{qp^k}(qp^k - 2)$. Since $p^k > 6$, Proposition 5.12 shows that $\mathcal{B}_{qp^k}(qp^k - 2q - 1) \subseteq \mathcal{D}(q, p^k)$. Since both $\mathcal{D}(q, p^k)$ and $\mathcal{B}_{qp^k}(qp^k - 2)$ are closed under conjugation, it remains to prove that if $\lambda = (qp^k - r, \mu)$ where $r \in \{2, 3, \ldots, 2q\}$ and $\mu \vdash r$, then $\lambda \in \mathcal{D}(q, p^k)$.

If $k \geq 2$, then $r \leq (q-1)p^k$ and $p^k - r \geq r = |\mu|$, so $(p^k - r, \mu) \in \mathcal{LR}([\lambda \setminus ((q-1)p^k)])$. Thus $\chi^{(p^k)} \times \cdots \times \chi^{(p^k)} \times \chi^{(p^k - r, \mu)} \mid \chi^\lambda\big\downarrow_{(\mathfrak{S}_{p^k})^{\times q}}$, so $\lambda \in \mathcal{D}(q, p^k)$ since $(p^k - r, \mu) \in \Delta(p^k)$.

Otherwise, $k = 1$. Since $r \in \{2, 3, \ldots, 2q\}$, we can write $r = m_1 + \ldots + m_q$ where $m_i \in \{0, 2, 3\}$ and m_i are not all equal (for $r = 2q$, we take $m_1 = \ldots = m_{q-3} = 2$, $m_{q-2} = m_{q-1} = 3$ and $m_q = 0$). We may reorder the m_i such that $m_i \neq 0$ for all $i \in [j]$ and $m_i = 0$ for all $i > j$, for some $j \in [q]$. Then there exist $\nu_i \vdash m_i$ for each $i \in [j]$ such that $c^\mu_{\nu_1, \ldots, \nu_j} > 0$. Since $p \geq 7$ (as $k = 1$), we have by Lemma 2.11 that

$$c^{(jp-r, \mu)}_{(p-m_1, \nu_1), \ldots, (p-m_j, \nu_j)} = c^\mu_{\nu_1, \ldots, \nu_j} > 0,$$

noting $\sum_{i=1}^j m_i = r$. Hence $c^\lambda_{(p-m_1, \nu_1), \ldots, (p-m_j, \nu_j), (p), \ldots, (p)} > 0$, from which we deduce that $\lambda \in \mathcal{D}(q, p)$ since (p) and $(p - m_i, \nu_i) \in \Delta(p)$ for all i. $\square$

Corollary 5.18. *Let $k \in \mathbb{N}$, and if $p = 3$ then further assume $k \geq 3$. Then*

$$\mathcal{D}(p, p^k) = \mathcal{B}_{p^{k+1}}(p^{k+1} - 2).$$

Proof. When $p^k \neq 5$, the statement follows from Proposition 5.17 by setting $q = p$. If $p^k = 5$, then direct verification shows that $\mathcal{D}(5, 5) = \mathcal{B}_{25}(23)$. □

We are now ready to prove Theorem 5.14.

Proof of Theorem 5.14. We proceed by induction on $k \in \mathbb{N}$ for $p \geq 5$ and on $k \in \mathbb{N}_{\geq 3}$ for $p = 3$. The base case for $p \geq 5$ follows from Lemma 5.15, while the assertion may be verified computationally for $k \leq 3$ if $p = 3$. Now assume the statement holds for some $k \in \mathbb{N}$ (where $k \geq 3$ if $p = 3$). To ease the notation, let $n = p^{k+1}$, $P = P_n$ and let A be the set defined by

$$A = \{\lambda \vdash n \mid \langle \chi^\lambda\big\downarrow_P, \mathbb{1}_P\rangle \neq 0\}.$$

From Proposition 5.16 together with the inductive hypothesis, we deduce that $\mathcal{D}(p, p^k) \subseteq A$. Moreover, $(n), (1^n) \in A$ since $\chi^{(n)}\big\downarrow_P = \mathbb{1}_P = \chi^{(1^n)}\big\downarrow_P$. Hence we have that $\Delta(n) \subseteq A$, by Corollary 5.18. By Lemma 2.2 it remains to show that $(n-1, 1) \notin A$.

Let $B = \mathfrak{S}_{p^k}^{\times p}$, $B \leq \mathfrak{S}_{p^k} \wr \mathfrak{S}_p \leq \mathfrak{S}_{p^{k+1}}$ and let $C \leq B$ where $C \cong P_{p^k}^{\times p}$. From [32, Lemma 3.2] and the Littlewood–Richardson rule we see that $\chi^{(n-1,1)}\big\downarrow_B = (p-1)\mathbb{1}_B + \Theta$, where

$$\Theta = (\chi^\mu \times \mathbb{1} \times \cdots \times \mathbb{1}) + (\mathbb{1} \times \chi^\mu \times \cdots \times \mathbb{1}) + \cdots + (\mathbb{1} \times \cdots \times \mathbb{1} \times \chi^\mu) \text{ and } \mu = (p^k - 1, 1).$$

(Here $\mathbb{1}$ denotes $\mathbb{1}_{\mathfrak{S}_{p^k}}$.) From [27, Theorem 4.2, Proposition 4.3] there exists $\nu \in \{(p-1, 1), (2, 1^{p-2})\}$ such that

$$\chi^{(n-1,1)}\big\downarrow_{\mathfrak{S}_{p^k} \wr \mathfrak{S}_p} = \mathcal{X}((p^k); \nu) + \Delta,$$

where Δ is a sum of irreducible characters of $\mathfrak{S}_{p^k} \wr \mathfrak{S}_p$ each of which has degree divisible by p. Since $\mathcal{X}((p^k); \nu)\big\downarrow_B = (p-1)\mathbb{1}_B$, we have that $\Delta\big\downarrow_B = \Theta$. Using the inductive hypothesis, we see that $\mathbb{1}_{P_{p^k}}$ is not a constituent of $\chi^\mu\big\downarrow_{P_{p^k}}$. Hence $\langle \Theta\big\downarrow_C, \mathbb{1}_C\rangle = 0$. Together these show that $\langle \Delta\big\downarrow_C, \mathbb{1}_C\rangle = \langle \Theta\big\downarrow_C, \mathbb{1}_C\rangle = 0$ and hence $\langle (\chi^{(n-1,1)})\big\downarrow_P, \mathbb{1}_P\rangle = \langle \mathcal{X}((p^k); \nu)\big\downarrow_P, \mathbb{1}_P\rangle$.

Finally, by Lemma 5.15 we know that $\langle \chi^\nu\big\downarrow_{P_p}, \mathbb{1}_{P_p}\rangle = 0$. Since $\mathbb{1}_P = \mathcal{X}(\mathbb{1}_{P_{p^k}}; \mathbb{1}_{P_p})$ we deduce that $\langle \mathcal{X}((p^k); \nu)\big\downarrow_P, \mathbb{1}_P\rangle = 0$, whence $\langle (\chi^{(n-1,1)})\big\downarrow_P, \mathbb{1}_P\rangle = 0$ as required. Thus the statement of the theorem holds for $k + 1$. This concludes the proof. □

5.4 Proof of Theorem 5.1

In the final section of this chapter, we prove Theorem 5.1 for all odd primes p and all natural numbers n. We begin with a short technical lemma.

Lemma 5.19. *Let γ be a skew shape and let $m = |\gamma| \geq 4$. Suppose $(m-1,1) \in \mathcal{LR}(\gamma)$. Then one of the following holds:*

(i) $\gamma \cong [(m-1,1)]$ *or* $\gamma \cong [(m-1,1)]^\circ$*;*

(ii) $\mathcal{LR}(\gamma) \cap \{(m),(m-2,2),(m-2,1,1)\} \neq \emptyset$.

Proof. Since $(m-1,1) \in \mathcal{LR}(\gamma)$, no three boxes of γ lie in the same column, and γ has at most one column containing two boxes. Suppose (i) does not hold. Then $(m) \in \mathcal{LR}(\gamma)$ if (a) no two boxes of γ lie in the same column; or $(m-2,1,1) \in \mathcal{LR}(\gamma)$ if (b) γ has precisely two connected components, one of which is a row of $m-2$ boxes and the other of which is a column of two boxes.

Now assume γ satisfies neither (a) nor (b). Then γ has a unique connected component δ whose boxes lie in exactly two rows, say rows j and $j+1$, and each of the other components lies entirely within one row. Moreover, if $\delta = \gamma$ is the unique connected component then δ has at least two boxes in each of rows j and $j+1$, while if δ contains only two boxes then by assumption γ has at least three connected components. In all instances, $(m-2,2) \in \mathcal{LR}(\gamma)$. □

Now, we prove Theorem 5.1, beginning with the case when $p \geq 5$.

Proposition 5.20. *Let $p \geq 5$ be a prime and $n \in \mathbb{N}$. Let $\lambda \vdash n$. Then $\langle \chi^\lambda\big\downarrow_{P_n}, \mathbb{1}_{P_n}\rangle = 0$ if and only if $n = p^k$ for some $k \in \mathbb{N}$ and $\lambda \in \{(p^k-1,1),(2,1^{p^k-2})\}$.*

Proof. Let $\Sigma(n)$ denote the sum of the p-adic digits of n, that is, the sum of the digits when n is expressed in base p. We prove the assertion by induction on $\Sigma(n)$. Theorem 5.14 and Lemma 5.15 show that the statement holds when $\Sigma(n) = 1$, and when $n \leq p$.

Now assume that $n > p$ and $\Sigma(n) \geq 2$. Let k be such that $p^k < n < p^{k+1}$ and set $m = n - p^k$. Clearly $k > 0$ and $\Sigma(m) = \Sigma(n) - 1$. Call $(\mu,\nu) \in \mathcal{P}(m) \times \mathcal{P}(p^k)$ a *suitable pair* for $\lambda \in \mathcal{P}(n)$ if $c^\lambda_{\mu\nu} \neq 0$ and $\langle \chi^\mu\big\downarrow_{P_m}, \mathbb{1}_{P_m}\rangle \cdot \langle \chi^\nu\big\downarrow_{P_{p^k}}, \mathbb{1}_{P_{p^k}}\rangle > 0$. We denote by $\mathcal{S}(\lambda)$ the set of suitable pairs for λ. It is clear that if $\mathcal{S}(\lambda) \neq \emptyset$ then $\langle \chi^\lambda\big\downarrow_{P_n}, \mathbb{1}_{P_n}\rangle > 0$, since $P_n \cong P_m \times P_{p^k}$. We now show that $\mathcal{S}(\lambda) \neq \emptyset$ for all $\lambda \in \mathcal{P}(n)$.

First suppose that $\Sigma(m) > 1$ and let $\lambda \in \mathcal{P}(n)$. Theorem 5.14 together with the inductive hypothesis shows that $\mathcal{S}(\lambda) = \{(\mu,\nu) \in \mathcal{P}(m) \times \Delta(p^k) \mid c^\lambda_{\mu\nu} \neq 0\}$. If $\lambda_2 \geq 2$ then there exists $\nu \in \Delta(p^k)$ such that $[\nu] \subseteq [\lambda]$. Hence $\mathcal{LR}([\lambda \setminus \nu]) \times \{\nu\} \subseteq \mathcal{S}(\lambda) \neq \emptyset$. Otherwise λ is a hook partition. Since $|\lambda| > p^k$, there exists some hook partition $\nu \notin \{(p^k-1,1),(2,1^{p^k-2})\}$ such that $[\nu] \subset [\lambda]$. Therefore again we have $\mathcal{LR}([\lambda \setminus \nu]) \times \{\nu\} \subseteq \mathcal{S}(\lambda) \neq \emptyset$.

Now we may assume that $\Sigma(m) = 1$, that is, $m = p^l \leq p^k$ for some integer l. First suppose $l = k$. Since $\mathcal{S}(\lambda) \neq \emptyset$ for all $\lambda \vdash 10$, we may assume $p^k \geq 7$. By Proposition 5.17, if $\lambda \in \mathcal{B}_{2p^k}(2p^k-5)$ then $\mathcal{S}(\lambda) \neq 0$ since $\mathcal{B}_{p^k}(p^k-2) \subset \Delta(p^k)$. Since $\mathcal{S}(\lambda) \neq \emptyset$ if and only if $\mathcal{S}(\lambda') \neq \emptyset$, it remains to consider $\lambda \vdash 2p^k$ such that $\lambda_1 \geq 2p^k - 4$. If $\lambda_1 = 2p^k-4$ then $(\alpha,\alpha) \in \mathcal{S}(\lambda)$ for some $\alpha \in \{(p^k-2,2),(p^k-2,1^2)\}$;

if $\lambda_1 \in \{2p^k-3, 2p^k-2\}$ then $((\lambda_1-p^k,\mu),(p^k)) \in \mathcal{S}(\lambda)$ where $\lambda=(\lambda_1,\mu)$; and if $\lambda_1 \geq 2p^k-1$ then $((p^k),(p^k)) \in \mathcal{S}(\lambda)$.

Suppose finally that $k>l$. As above, $|\lambda|>p^l$ implies that there exists some $\mu \in \Delta(p^l)$ such that $[\mu] \subset [\lambda]$. Let $\nu \in \mathcal{LR}([\lambda\setminus\mu])$. If $\nu \in \Delta(p^k)$ then $(\mu,\nu) \in \mathcal{S}(\lambda) \neq \emptyset$. Otherwise, $\mathcal{LR}([\lambda\setminus\mu]) \subseteq \{(p^k-1,1),(2,1^{p^k-2})\}$. By Lemmas 5.19 and 2.8, we must have

$$[\lambda\setminus\mu] \in \{[(p^k-1,1)],[(p^k-1,1)]^\circ,[(2,1^{p^k-2})],[(2,1^{p^k-2})]^\circ\}.$$

Since $k>l$, we observe that $[\lambda\setminus\mu] \not\cong [(p^k-1,1)]^\circ$ and $[\lambda\setminus\mu] \not\cong [(2,1^{p^k-2})]^\circ$. Hence if $\mu=(\mu_1,\dots,\mu_s)$, we must have either

(a) $\lambda=(\mu_1+p^k-1,\mu_2+1,\mu_3,\dots,\mu_s)$ and $\mu_1=\mu_2$, or

(b) $\lambda=(\mu_1,\dots,\mu_s,2,1^{p^k-2})$ and $\mu_s \geq 2$.

However, any partition satisfying (b) is conjugate to a partition satisfying (a), so by Lemma 2.2 it remains to consider only one of the two cases. Suppose we are in the situation of case (a). Letting $\tilde{\mu}=(\mu_1+1,\mu_2,\dots,\mu_{s-1},\mu_s-1)$, we have that $(p^k) \in \mathcal{LR}([\lambda\setminus\tilde{\mu}])$. Moreover, $\mu_1=\mu_2$ implies that $\tilde{\mu} \neq (p^l-1,1)$. Hence $(\tilde{\mu},(p^k)) \in \mathcal{S}(\lambda)$ unless $\tilde{\mu}=(2,1^{p^l-2})$. But in this case we would have $\mu=(1^{p^l})$, $\lambda=(p^k,2,1^{p^l-2})$ and therefore $((2^2,1^{p^l-4}),(p^k-2,1^2)) \in \mathcal{S}(\lambda) \neq \emptyset$. Thus in all instances we have found a suitable pair for $\lambda \vdash n$, and hence $\langle \chi^\lambda\big\downarrow_{P_n}, \mathbb{1}_{P_n}\rangle > 0$. ☐

Finally, we conclude by verifying Theorem 5.1 for $p=3$.

Proposition 5.21. *Let $p=3$. Then $\langle \chi^\lambda\big\downarrow_{P_n}, \mathbb{1}_{P_n}\rangle = 0$ if and only if $n=3^k$ for some $k\in\mathbb{N}$ and $\lambda \in \{(3^k-1,1),(2,1^{3^k-2})\}$, or $n \leq 10$ and λ is one of the following partitions:*

$$(2,2);\quad (3,2,1);\quad (5,4),(2^4,1),(4,3,2),(3^2,2,1);\quad (5,5),(2^5).$$

Proof. The same argument as in the proof of Proposition 5.20 shows that the assertion holds for all $n\in\mathbb{N}$ divisible by 27. Since the assertion may be verified computationally for $n \leq 27$, it remains to consider n of the form $27t+u$ where $t,u\in\mathbb{N}$ and $u<27$.

Given $\lambda \vdash n$, it is clear that there exists $\mu \vdash 27t$ (and if $27t$ is a power of 3, say 3^k, we can further choose μ such that $\mu \in \Delta(3^k)$) such that $[\mu] \subset [\lambda]$. Let $\nu \in \mathcal{LR}([\lambda\setminus\mu])$. If $u \notin \{3,4,6,9,10\}$, then

$$\langle \chi^\lambda\big\downarrow_{P_n}, \mathbb{1}_{P_n}\rangle \geq \langle \chi^\mu\times\chi^\nu\big\downarrow_{P_{27t}\times P_u}, \mathbb{1}_{P_{27t}\times P_u}\rangle = \langle \chi^\mu\big\downarrow_{P_{27t}}, \mathbb{1}_{P_{27t}}\rangle \cdot \langle \chi^\nu\big\downarrow_{P_u}, \mathbb{1}_{P_u}\rangle > 0.$$

From now on we may assume $u \in \{3,4,6,9,10\}$. Let $\lambda=(\lambda_1,\dots,\lambda_t)$ and let

$$\begin{aligned} &T(3)=\{(2,1)\},\quad T(4)=\{(2,2)\},\quad T(6)=\{(3,2,1)\},\\ &T(9)=\{(8,1),(5,4),(4,3,2),(3^2,2,1),(2^4,1),(2,1^7)\},\quad T(10)=\{(5,5),(2^5)\}. \end{aligned} \tag{5.1}$$

By Lemma 5.22 (below), there exists $\mu \in \mathcal{P}(u)\setminus T(u)$ satisfying $[\mu]\subset[\lambda]$. Thus, if $27t$ is not a power of 3 then we may take any $\nu \in \mathcal{LR}([\lambda\setminus\mu])$ to see that $\langle \chi^\lambda\big\downarrow_{P_n}, \mathbb{1}_{P_n}\rangle \geq$

$\langle \chi^\mu\big\downarrow_{P_u}, \mathbb{1}_{P_u}\rangle \cdot \langle \chi^\nu\big\downarrow_{P_{27t}}, \mathbb{1}_{P_{27t}}\rangle > 0$.

So now we may assume $27t$ is a power of 3. Let $27t = 3^k$ for some $k \geq 3$. Taking $\mu \in \mathcal{P}(u) \setminus T(u)$ such that $[\mu] \subset [\lambda]$, we may take any $\nu \in \mathcal{LR}([\lambda \setminus \mu])$ to see that $\langle \chi^\lambda\big\downarrow_{P_n}, \mathbb{1}_{P_n}\rangle \geq \langle \chi^\mu\big\downarrow_{P_u}, \mathbb{1}_{P_u}\rangle \cdot \langle \chi^\nu\big\downarrow_{P_{27t}}, \mathbb{1}_{P_{27t}}\rangle > 0$, *unless* $\mathcal{LR}([\lambda \setminus \mu]) \subseteq \{(3^k-1,1),(2,1^{3^k-2})\}$.

Thus it remains to deal with the case $\mathcal{LR}([\lambda \setminus \mu]) \subseteq \{(3^k-1,1),(2,1^{3^k-2})\}$. By Lemmas 5.19 and 2.8, $[\lambda \setminus \mu] \in \{[(3^k-1,1)],[(3^k-1,1)]^\circ,[(2,1^{3^k-2})],[(2,1^{3^k-2})]^\circ\}$. Recall that we have reduced to $u \in \{3,4,6,9,10\}$, so in particular $u \leq 10$. Letting $\mu = (\mu_1,\dots,\mu_s)$, we must have either

(a) $\lambda = (\mu_1 + 3^k - 1, \mu_2 + 1, \mu_3, \dots, \mu_s)$ and $\mu_1 = \mu_2$, or

(b) $\lambda = (\mu_1, \dots, \mu_s, 2, 1^{p^3-2})$ and $\mu_s \geq 2$.

However, any partition satisfying (b) is conjugate to a partition satisfying (a), so by Lemma 2.2 it remains to consider only one of the two cases. Suppose we are in the situation of case (a). If $\mu = (1^u)$, then $\lambda = (3^k, 2, 1^{u-2})$ and $\chi^\lambda\big\downarrow_{P_{3^k}\times P_u}$ has $\chi^{(3^k-2,2)} \times \chi^{(3,1^{u-3})}$ as a constituent. Otherwise, $\chi^\lambda\big\downarrow_{P_{3^k}\times P_u}$ has $\chi^\gamma \times \chi^{(u)}$ as a constituent where $\gamma = (u_1 + 3^k - u, \mu_2, \dots, \mu_s) \in \Delta(3^k)$. □

Lemma 5.22. *Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition such that $|\lambda| \geq 28$. Let $u \in \{3,4,6,9,10\}$ and let $T(u)$ be as defined in (5.1). Then there exists $\mu \in \mathcal{P}(u) \setminus T(u)$ satisfying $[\mu] \subset [\lambda]$.*

Proof. If $u \in \{3,4,6\}$ then the assertion is clear since either $\lambda_1 \geq u$ or $t \geq u$, whence $[(u)] \subseteq [\lambda]$ or $[(1^u)] \subseteq [\lambda]$. For $u \in \{9,10\}$, we proceed with proof by contradiction: suppose that $[\mu] \subseteq [\lambda]$ implies $\mu \in T(u)$ whenever μ is a partition of u.

If $u = 9$, then $[(9)],[(1^9)],[(3^3)] \not\subseteq [\lambda]$ so $\lambda_1 \leq 8$, $\lambda_3 \leq 2$ and $t \leq 8$. Thus $|\lambda| \leq 8\cdot 2 + 2 \cdot 6 = 28$ with equality if and only if $\lambda = (8^2, 2^6)$, but $(7,2) \subset (8^2,2^6)$.

Finally if $u = 10$, then $[(10)],[(1^{10})] \not\subseteq [\lambda]$ so $\lambda_1 \leq 9$ and $t \leq 9$. Since $|\lambda| \geq 28$, we must have $\lambda_4 \geq 1$. Since $[(7,1^3)] \not\subseteq [\lambda]$, we have $\lambda_1 \leq 6$. Also $\lambda_3 \leq 2$ since $[(3^3,1)] \not\subseteq [\lambda]$. But then $|\lambda| \leq 6 \cdot 2 + 2 \cdot 7 < 28$, a contradiction. □

To conclude, we remark that we can in fact say more about the multiplicity with which $\mathbb{1}_{P_n}$ appears in the restriction of irreducible characters of $\mathfrak{S}_n$, and hence about the degrees of the irreducible representations of $\mathcal{H}(\mathfrak{S}_n, P_n, \mathbb{1}_{P_n})$. If $k \in \mathbb{N}$ and $\lambda \vdash p^{k+1}$, and $\mu_1, \dots, \mu_p \vdash p^k$ are not all equal and satisfy $\chi^{\mu_1} \times \cdots \times \chi^{\mu_p} \mid \chi^\lambda\big\downarrow^{\mathfrak{S}_{p^{k+1}}}_{(\mathfrak{S}_{p^k})^{\times p}}$, then

$$\langle \chi^\lambda\big\downarrow_{P_{p^{k+1}}}, \mathbb{1}_{P_{p^{k+1}}}\rangle \geq \prod_{i=1}^{p} \langle \chi^{\mu_i}\big\downarrow_{P_{p^k}}, \mathbb{1}_{P_{p^k}}\rangle.$$

This follows immediately from the proof of Proposition 5.16. Using this, one can for instance compute when $f(\lambda) = 1$, where f is the function described in Question 5.3; indeed, Theorem 5.1 determines precisely when $f(\lambda) = 0$.

Chapter 6

Identifying linear constituents in character restrictions

This chapter is based on joint work with Dr Eugenio Giannelli, and uses our new results on Littlewood–Richardson coefficients in Section 5.2.

We describe the sets $\Omega(\phi)$ introduced in Section 4.4 consisting of the irreducible constituents of the induced character $\phi\big\uparrow^{\mathfrak{S}_n}$, for all natural numbers n, primes $p \geq 5$, and linear characters ϕ of a Sylow p-subgroup P_n of the symmetric group $\mathfrak{S}_n$. We give sharp bounds for $\Omega(\phi)$ which afford an explicit, combinatorial description in terms of the indexing set for $\phi \in \mathrm{Lin}(P_n)$. This extends the work in Chapter 5, where we considered only the trivial character $\phi = \mathbb{1}_{P_n}$.

6.1 Outline

Let $p \geq 5$ be a prime and let P_n be a Sylow p-subgroup of $\mathfrak{S}_n$, for $n \in \mathbb{N}$. For $\phi \in \mathrm{Lin}(P_n)$, recall from Section 4.4 that $\Omega(\phi) := \{\chi \in \mathrm{Irr}(\mathfrak{S}_n) : \chi \mid \phi\big\uparrow^{\mathfrak{S}_n}\}$. By identifying irreducible characters of symmetric groups with their indexing partitions, throughout this chapter we view $\Omega(\phi)$ as a subset of $\mathcal{P}(n)$. In other words, we set

$$\Omega(\phi) = \{\lambda \vdash n \ : \ \chi^\lambda \mid \phi\big\uparrow^{\mathfrak{S}_n}\}.$$

We remark that $\Omega(\phi)$ is closed under conjugation of partitions, by Lemma 2.2.

To describe $\Omega(\phi)$, we first recall the notion $\mathcal{B}_n(m)$: this is the set of partitions $\lambda \vdash n$ whose Young diagrams fit inside an $m \times m$ square grid. As we will see below, it turns out that $\Omega(\phi)$ is always of the form $\mathcal{B}_n(m) \sqcup B$ where B is small (or empty), and m is a natural number depending on ϕ. In fact, every partition $\lambda \in B$ satisfies $m < \lambda_1 \leq M$ or $m < l(\lambda) \leq M$, where $M \in \mathbb{N}$ and $M - m$ is small.

In order to formalise this description, we introduce some technical definitions. For a linear character ϕ of P_n, define

$$m(\phi) = \max\{x \in \mathbb{N} \mid \mathcal{B}_n(x) \subseteq \Omega(\phi)\} \quad \text{and} \quad M(\phi) = \min\{x \in \mathbb{N} \mid \Omega(\phi) \subseteq \mathcal{B}_n(x)\}.$$

When $n = p^k$ for some $k \in \mathbb{N}$, recall that $\phi \in \mathrm{Lin}(P_n)$ may be indexed as $\phi = \phi(s)$ where s runs over $[\overline{p}]^k$, with $\mathbb{1}_{P_n}$ corresponding to $s = (0, \dots, 0)$. (Recall that if $\phi = \mathbb{1}_{P_{p^k}}$, then $M(\phi)$ and $m(\phi)$ are already known, by Theorem 5.1.) For $s \neq (0, \dots, 0)$, define

$$f(s) = \min\{i \in [k] \mid s_i \neq 0\}.$$

Furthermore, if $|\{i \in [k] : s_i \neq 0\}| \geq 2$, then define

$$g(s) = \min\{i > f(s) \mid s_i \neq 0\}.$$

For arbitrary $n \in \mathbb{N}$, $\mathrm{Lin}(P_n) = \{\phi(\underline{\mathbf{s}}) \mid \underline{\mathbf{s}}\}$ as in (2.4). Throughout, if $\phi = \phi(s)$ (for some indexing label s), then we also refer to $m(\phi)$, $M(\phi)$ and $\Omega(\phi)$ as $m(s)$, $M(s)$ and $\Omega(s)$ respectively. The main result of this chapter is the following:

Theorem 6.1. *Let $p \geq 5$ be a prime. Let $k \in \mathbb{N}$, and suppose $\phi = \phi(s) \in \mathrm{Lin}(P_{p^k}) \setminus \{\mathbb{1}_{P_{p^k}}\}$. Then $M(\phi) = p^k - p^{k-f(s)}$, and*

$$m(\phi) = \begin{cases} p^k - p^{k-f(s)} - 1 + \delta_{f(s),k} & \text{if } |\{i \in [k] : s_i \neq 0\}| = 1, \\ p^k - p^{k-f(s)} - p^{k-g(s)} & \text{if } |\{i \in [k] : s_i \neq 0\}| \geq 2. \end{cases}$$

Let $n \in \mathbb{N}$ and suppose it has p-adic expansion $n = \sum_{i=1}^t a_i p^{n_i}$, with $0 \leq n_1 < \cdots < n_t$. For $\phi = \phi(\underline{\mathbf{s}}) = \phi(\mathbf{s}(1,1)) \times \cdots \times \phi(\mathbf{s}(t, a_t)) \in \mathrm{Lin}(P_n)$, we have

$$M(\phi) = \sum_{(i,j)} M(\mathbf{s}(i,j)) \qquad \text{and} \qquad m(\phi) = \sum_{(i,j)} N(\mathbf{s}(i,j)),$$

where the sums run over all $i \in [t]$ and $j \in [a_i]$, and $N(\mathbf{s}(i,j))$ is as in Definition 6.18 below.

Theorem 6.1 (combined with Theorem 5.1 for $\phi = \mathbb{1}_{P_n}$) shows that a large proportion of $\mathcal{P}(n)$ is contained inside $\Omega(\phi)$ for each $\phi \in \mathrm{Lin}(P_n)$. Let Ω_n be the intersection of all the sets $\Omega(\phi)$ where ϕ is free to run among the elements of $\mathrm{Lin}(P_n)$. A corollary of Theorems 6.1 and 5.1 is the following:

Corollary 6.2.

$$\lim_{n \to \infty} \frac{|\Omega_n|}{|\mathcal{P}(n)|} = 1.$$

The structure of this chapter is as follows: in Section 6.2, we consider the case where n is a power of the prime p, and in Section 6.3 we extend our scope to arbitrary natural numbers n. The precise statements and proofs of the various parts of Theorem 6.1 can be found as follows:

· for $n = p^k$, $M(\phi)$ is determined in Theorem 6.8;

· for $n = p^k$, $m(\phi)$ is determined in Lemma 6.4, Theorem 6.10 and Lemma 6.12;

· for arbitrary n, $M(\phi)$ is determined in Theorem 6.17; and

· for arbitrary n, $m(\phi)$ is determined in Theorems 6.19 and 6.20.

A proof of Corollary 6.2 appears in Section 6.3.

6.2 Types of sequences

Throughout this section, fix a prime $p \geq 5$. Let $k \in \mathbb{N}$ and let $\phi = \phi(s) \in \mathrm{Lin}(P_{p^k})$ for some $s \in [\overline{p}]^k$. The aim of this section is to determine the following numbers:

$$m(s) = \max\{x \in \mathbb{N} \mid \mathcal{B}_{p^k}(x) \subseteq \Omega(s)\} \quad \text{and} \quad M(s) = \min\{x \in \mathbb{N} \mid \Omega(s) \subseteq \mathcal{B}_{p^k}(x)\}.$$

We have already determined these values when $\phi = \mathbb{1}_{P_{p^k}}$ (corresponding to $s = (0, \dots, 0) \in [\overline{p}]^k$) in Theorem 5.1, so we now treat the non-trivial linear characters of P_{p^k}. It will be useful to recall that whenever $n \in \mathbb{N}$ is not a power of p then $\Omega(\mathbb{1}_{P_n}) = \mathcal{P}(n)$, while if $n = p^k$ then $\Omega(\mathbb{1}_{P_{p^k}}) = \mathcal{P}(p^k) \setminus \{(p^k - 1, 1)\}^\circ$. In this section (Section 6.2), when $k \in \mathbb{N}$ and $s \in [\overline{p}]^k$, we will assume that $s = (s_1, s_2, \dots, s_k)$ and denote by s^- the sequence $(s_1, s_2, \dots, s_{k-1}) \in [\overline{p}]^{k-1}$.

After beginning with some useful lemmas, we determine $M(s)$ and $m(s)$ for sequences $s \in [\overline{p}]^k$ corresponding to non-trivial $\phi(s)$. From the results in this section, we will see that the form of $\Omega(s)$ falls into four types (see Definition 6.14), and we summarise our findings in Remark 6.15.

Lemma 6.3. *Let $x \in [\overline{p}]$. Then*

$$\Omega(x) = \begin{cases} \mathcal{P}(p) \setminus \{(p-1, 1), (2, 1^{p-2})\} & \text{if } x = 0, \\ \mathcal{P}(p) \setminus \{(p), (1^p)\} = \mathcal{B}_p(p-1) & \text{if } x \in [p-1]. \end{cases}$$

Proof. By Corollary 2.4, if λ is not a hook then $\chi^\lambda\big\downarrow_{P_p}$ is a multiple of the regular character $\sum_{i=0}^{p-1} \phi_i$ of P_p; otherwise if λ is a hook of leg length l then $\chi^\lambda(1) = \binom{p-1}{l} \equiv (-1)^l \pmod{p}$. In particular, $\chi^\lambda(1) = 1$ if and only if $\lambda \in \{(p), (1^p)\}$, and $\chi^\lambda(1) = p-1$ if and only if $\lambda \in \{(p-1, 1), (2, 1^{p-2})\}$, so the assertion follows. □

Lemma 6.4. *Let $k \in \mathbb{N}_0$ and let $s = (0, \dots, 0, x) \in [\overline{p}]^{k+1}$ where $x \neq 0$. Then $\Omega(s) = \mathcal{B}_{p^{k+1}}(p^{k+1} - 1)$. Moreover, $\langle \chi^{(p^{k+1}-1,1)}\big\downarrow_{P_{p^{k+1}}}, \phi(s) \rangle = 1$.*

Proof. The assertion follows directly from Lemma 6.3 when $k = 0$. Now assume $k \geq 1$, so $\mathcal{D}(p, p^k) = \mathcal{B}_{p^{k+1}}(p^{k+1} - 2)$ by Corollary 5.18. Let $\lambda \in \mathcal{D}(p, p^k)$. Arguing exactly as in the proof of Proposition 5.16, we see that $\mathbb{1}_B\big\uparrow^P$ is a direct summand of $\chi^\lambda\big\downarrow_P$ where $P = P_{p^{k+1}}$ and $B \cong (P_{p^k})^{\times p} \leq P$. But $\mathbb{1}_B\big\uparrow^P = \sum_{\theta \in \mathrm{Irr}(P_p)} \mathcal{X}(\mathbb{1}_{P_{p^k}}; \theta)$ by

Lemma 2.15, so setting $\theta = \phi_x$ shows that $\phi(s)$ is an irreducible constituent of $\chi^\lambda\big\downarrow_P$. Hence $\mathcal{D}(p, p^k) \subseteq \Omega(s)$.

Since $\Omega(s)$ is closed under conjugation, in order to conclude that $\Omega(s) = \mathcal{B}_{p^{k+1}}(p^{k+1}-1)$ it remains to show $(p^{k+1}) \notin \Omega(s)$ and $(p^{k+1}-1,1) \in \Omega(s)$. Clearly if $\lambda = (p^{k+1})$ then $\chi^\lambda\big\downarrow_P = \mathbb{1}_P \neq \phi(s)$, so $(p^{k+1}) \notin \Omega(s)$. On the other hand, if $\lambda = (p^{k+1}-1,1)$, then by Theorem 2.20, $\chi^\lambda\big\downarrow^{\mathfrak{S}_{p^{k+1}}}_{\mathfrak{S}_{p^k}\wr\mathfrak{S}_p}$ contains $\mathcal{X}((p^k);\nu)$ as a constituent for some $\nu \in \{(p-1,1)\}^\circ$. But then

$$\mathcal{X}((p^k);\nu)\big\downarrow^{\mathfrak{S}_{p^k}\wr\mathfrak{S}_p}_{P} = \mathcal{X}\left(\chi^{(p^k)}\big\downarrow^{\mathfrak{S}_{p^k}}_{P_{p^k}};\chi^\nu\big\downarrow^{\mathfrak{S}_p}_{P_p}\right) = \sum_{z=1}^{p-1}\mathcal{X}(\mathbb{1}_{P_{p^k}};\phi_z).$$

This shows that $\phi(s)$ is an irreducible constituent of $\chi^\lambda\big\downarrow_P$. Hence $(p^{k+1}-1,1) \in \Omega(s)$.

Keeping $\lambda = (p^{k+1}-1,1)$, we now wish to show that $\langle\chi^\lambda\big\downarrow_P, \phi(s)\rangle = 1$. Let $H \cong (\mathfrak{S}_{p^k})^{\times p}$ with $B \leq H$. From [32, Lemma 3.2] and the Littlewood–Richardson rule we see that $\chi^\lambda\big\downarrow_H = (p-1)\mathbb{1}_H + \Theta$, where

$$\Theta = (\chi^\mu \times \mathbb{1} \times \cdots \times \mathbb{1}) + (\mathbb{1} \times \chi^\mu \times \cdots \times \mathbb{1}) + \cdots + (\mathbb{1} \times \cdots \times \mathbb{1} \times \chi^\mu) \text{ and } \mu = (p^k-1,1).$$

(Here $\mathbb{1}$ denotes $\mathbb{1}_{\mathfrak{S}_{p^k}}$.) Since we already know that $\mathcal{X}((p^k);\nu)$ is a constituent of $\chi^\lambda\big\downarrow_{\mathfrak{S}_{p^k}\wr\mathfrak{S}_p}$, then by the description of $\mathrm{Irr}(\mathfrak{S}_{p^k}\wr P_p)$ from Section 2.3,

$$\chi^\lambda\big\downarrow_{\mathfrak{S}_{p^k}\wr P_p} = \sum_{i=1}^{p-1}\mathcal{X}(\mathbb{1}_{\mathfrak{S}_{p^k}};\phi_i) + (\underbrace{\chi^\mu \times \mathbb{1} \times \cdots \times \mathbb{1}}_{=:\alpha})\big\uparrow_H^{\mathfrak{S}_{p^k}\wr P_p}.$$

Finally, $\langle\mathcal{X}(\mathbb{1}_{\mathfrak{S}_{p^k}};\phi_i),\phi(s)\rangle = \langle\mathcal{X}(\phi(s^-),\phi_i),\mathcal{X}(\phi(s^-),\phi_x)\rangle = \delta_{ix}$, and

$$\langle\alpha\big\uparrow_H^{\mathfrak{S}_{p^k}\wr P_p}\big\downarrow_P,\phi(s)\rangle = \langle\alpha\big\downarrow_B\big\uparrow^P,\phi(s)\rangle = \langle\alpha\big\downarrow_B,\phi(s)\big\downarrow_B\rangle = \langle\chi^\mu\big\downarrow_{P_{p^k}},\mathbb{1}_{P_{p^k}}\rangle = 0,$$

by Lemma 2.16 and Theorem 5.1. Since $x \in [p-1]$, we deduce that $\langle\chi^\lambda\big\downarrow_P,\phi(s)\rangle = 1$. □

Next, recall the notation $\mathcal{D}(q,y,\mathcal{B})$ from Definition 5.10. This allows us to relate the sets $\Omega(s^-)$ and $\Omega(s)$, for $\phi(s) \in \mathrm{Lin}(P_{p^k})$ and $k \in \mathbb{N}_{\geq 2}$.

Lemma 6.5. *Let $k \in \mathbb{N}_{\geq 2}$, $s = (s_1,\dots,s_k) \in [\overline{p}]^k$ and let $s^- = (s_1,\dots,s_{k-1})$. Then*

$$\mathcal{D}(p,p^{k-1},\Omega(s^-)) \subseteq \Omega(s).$$

Proof. We consider the following subgroups of $\mathfrak{S}_{p^k}$: let $P = P_{p^k} = P_{p^{k-1}} \wr P_p$. Let $B = (P_{p^{k-1}})^{\times p}$ be the base group of the wreath product P, and let $H = (\mathfrak{S}_{p^{k-1}})^{\times p} \leq \mathfrak{S}_{p^k}$ naturally contain B. Let $W = BH = H \rtimes P_p \leq \mathfrak{S}_{p^k}$, so $W \cong \mathfrak{S}_{p^{k-1}} \wr P_p$.

Let $\lambda \in \mathcal{D}(p,p^{k-1},\Omega(s^-))$, so $\chi^\lambda\big\downarrow_H$ has a constituent $\psi := \chi^{\mu_1} \times \cdots \times \chi^{\mu_p} \in \mathrm{Irr}(H)$ such that the partitions $\mu_i \vdash p^{k-1}$ are not all equal, and $\mu_i \in \Omega(s^-)$ for all $i \in [p]$. Since $\chi^\lambda \in \mathrm{Irr}(\mathfrak{S}_{p^k} \mid \psi)$, there exists $\chi \in \mathrm{Irr}(W \mid \psi)$ such that χ is a constituent of $\chi^\lambda\big\downarrow_W$. Since $\mu_1,\dots,\mu_p$ are not all equal, then $\chi = \psi\big\uparrow_H^W$ by the description of $\mathrm{Irr}(\mathfrak{S}_{p^k} \wr P_p)$ in

Section 2.3. It is clear that $PH = W$, so $\chi\big\downarrow_P = \psi\big\downarrow_B\big\uparrow^P$ by Lemma 2.16. Moreover, $\psi\big\downarrow_B = \chi^{\mu_1}\big\downarrow_{P_{p^{k-1}}} \times\cdots\times \chi^{\mu_p}\big\downarrow_{P_{p^{k-1}}}$, so $\phi(s^-)^{\times p}$ is a constituent of $\psi\big\downarrow_B$ since $\mu_i \in \Omega(s^-)$ for all $i \in [p]$. Thus $\phi(s^-)^{\times p}\big\uparrow^P$ is a direct summand of $\chi^\lambda\big\downarrow_P$.

But by Lemma 2.15, $\phi(s^-)^{\times p}\big\uparrow^P = \sum_{\theta\in\mathrm{Irr}(P_p)} \theta(1)\cdot \mathcal{X}(\phi(s^-);\theta)$, so taking $\theta = \phi_{s_k}$ shows that $\phi(s) = \mathcal{X}(\phi(s^-);\phi_{s_k})$ is a constituent of $\chi^\lambda\big\downarrow_P$. Thus $\lambda \in \Omega(s)$, which concludes the proof. $\square$

It turns out that the leading non-zeros in the sequence s govern the form of $\Omega(s)$, the set of irreducible constituents of $\phi(s)\big\uparrow^{\mathfrak{S}_{p^k}}$. In order to describe $m(s)$ and $M(s)$, we give the following definition, recalling $f(s)$ and $g(s)$ from Section 6.1.

Definition 6.6. *Let $k \in \mathbb{N}$ and $s \in [\overline{p}]^k$.*

- *For $z \in \{0,1,\dots,k\}$, let $U_k(z) = \{s \in [\overline{p}]^k : |\{i \in [k] : s_i \neq 0\}| = z\}$. Note $U_0(z)$ is empty for $z \in \mathbb{N}$, and $U_0(0) = \{\emptyset\}$.*
- *If $s \in U_k(z)$ where $z \geq 1$, then define $f(s) = \min\{i \in [k] \mid s_i \neq 0\}$.*
- *If $s \in U_k(z)$ where $z \geq 2$, then define $g(s) = \min\{i > f(s) \mid s_i \neq 0\}$ and set $\eta(s) = p^k - p^{k-f(s)} - p^{k-g(s)}$.*

First, we determine the value of $M(s)$.

Proposition 6.7. *Let $k \in \mathbb{N}$, and let $s \in [\overline{p}]^k \setminus U_k(0)$. Then $\Omega(s) \subseteq \mathcal{B}_{p^k}(p^k - p^{k-f(s)})$, that is, $M(s) \leq p^k - p^{k-f(s)}$.*

Proof. We proceed by induction on $k - f(s)$. The base case $f(s) = k$ follows from Lemma 6.4. Now suppose $f(s) < k$ and consider s^-. In particular, $f(s) = f(s^-)$. Let $\lambda \notin \mathcal{B}_{p^k}(p^k - p^{k-f(s)})$, so we may without loss of generality assume

$$\lambda_1 > p^k - p^{k-f(s)} = p(p^{k-1} - p^{k-1-f(s^-)}).$$

Then for each irreducible constituent $\chi^{\mu_1}\times\cdots\times\chi^{\mu_p}$ of $\chi^\lambda\big\downarrow_{(\mathfrak{S}_{p^{k-1}})^{\times p}}$ (so each μ_i is a partition of p^{k-1}), by the Littlewood–Richardson rule there exists some $1 \leq i \leq p$ such that $(\mu_i)_1 > p^{k-1} - p^{k-1-f(s^-)}$. Thus by the inductive hypothesis $\mu_i \notin \Omega(s^-)$, since $\mu_i \notin \mathcal{B}_{p^{k-1}}(p^{k-1} - p^{k-1-f(s^-)})$.

Suppose that $\lambda \in \Omega(s)$, so then $\phi(s)\big\downarrow_{(P_{p^{k-1}})^{\times p}} = \phi(s^-)^{\times p}$ is a constituent of $\chi^\lambda\big\downarrow_{(P_{p^{k-1}})^{\times p}}$. Since $\phi(s^-)^{\times p}$ is irreducible, it must therefore be a constituent of

$$\chi^{\mu_1}\big\downarrow_{P_{p^{k-1}}} \times\cdots\times \chi^{\mu_p}\big\downarrow_{P_{p^{k-1}}}$$

for some $\chi^{\mu_1}\times\cdots\times\chi^{\mu_p}$ as described above. In particular, this implies that $\mu_1,\dots,\mu_p \in \Omega(s^-)$, a contradiction. Hence $\lambda \notin \Omega(s)$, and so $\mathcal{P}(p^k) \setminus \mathcal{B}_{p^k}(p^k - p^{k-f(s)}) \subseteq \mathcal{P}(p^k) \setminus \Omega(s)$. $\square$

Theorem 6.8. *Let $k \in \mathbb{N}$, and let $s \in [\overline{p}]^k \setminus U_k(0)$. Then $M(s) = p^k - p^{k-f(s)}$.*

Proof. It remains to exhibit a partition $\lambda \in \Omega(s)$ such that $\lambda_1 = p^k - p^{k-f(s)}$, since we already know by Proposition 6.7 that $\Omega(s) \subseteq \mathcal{B}_{p^k}(p^k - p^{k-f(s)})$. We proceed by induction on $k - f(s)$. For the base case $f(s) = k$, we have $\Omega(s) = \mathcal{B}_{p^k}(p^k - 1)$ from Lemma 6.4, which implies that $\lambda = (p^k - 1, 1) \in \Omega(s)$.

Now suppose $f(s) < k$ and consider s^-. In particular, $f(s) = f(s^-)$. There exists some partition $\mu = (\mu_1, \ldots, \mu_m) \in \Omega(s^-)$ such that $\mu_1 = p^{k-1} - p^{k-1-f(s^-)}$, by the inductive hypothesis. First suppose $s_k \neq 0$. Let

$$\lambda = (p\mu_1, p\mu_2, \ldots, p\mu_{m-1}, p(\mu_m - 1) + p - 1, 1).$$

Then by Theorem 2.21, $\mathcal{X}\big(\mu; (p-1,1)\big)$ is a constituent of $\chi^\lambda\big\downarrow^{\mathfrak{S}_{p^k}}_{\mathfrak{S}_{p^{k-1}} \wr \mathfrak{S}_p}$, whence

$$\mathcal{X}\big(\mu; (p-1,1)\big)\big\downarrow_{P_{p^k}} \;\Big|\; \chi^\lambda\big\downarrow_{P_{p^k}}.$$

Observe that

$$\mathcal{X}\big(\mu; (p-1,1)\big)\big\downarrow_{P_{p^k}} = \mathcal{X}\big(\chi^\mu\big\downarrow^{\mathfrak{S}_{p^{k-1}}}_{P_{p^{k-1}}}; \chi^{(p-1,1)}\big\downarrow^{\mathfrak{S}_p}_{P_p}\big) = \sum_{i=1}^{p-1} \mathcal{X}\big(\chi^\mu\big\downarrow^{\mathfrak{S}_{p^{k-1}}}_{P_{p^{k-1}}}; \phi_i\big),$$

and $\mathcal{X}(\phi(s^-); \phi_i)$ is a summand of $\mathcal{X}(\chi^\mu\big\downarrow^{\mathfrak{S}_{p^{k-1}}}_{P_{p^{k-1}}}; \phi_i)$ since $\mu \in \Omega(s^-)$. Since $s_k \neq 0$, we see therefore that

$$\phi(s) = \mathcal{X}(\phi(s^-); \phi_{s_k}) \;\Big|\; \mathcal{X}\big(\mu; (p-1,1)\big)\big\downarrow_{P_{p^k}} \;\Big|\; \chi^\lambda\big\downarrow_{P_{p^k}}.$$

Thus $\lambda \in \Omega(s)$, and $\lambda_1 = p\mu_1 = p^k - p^{k-f(s)}$.

Otherwise, suppose $s_k = 0$. Let

$$\lambda = (p\mu_1, p\mu_2, \ldots, p\mu_{m-1}, p(\mu_m - 1) + p).$$

Then by Theorem 2.21, $\mathcal{X}\big(\mu; (p)\big)$ is a constituent of $\chi^\lambda\big\downarrow^{\mathfrak{S}_{p^k}}_{\mathfrak{S}_{p^{k-1}} \wr \mathfrak{S}_p}$, whence

$$\mathcal{X}\big(\mu; (p)\big)\big\downarrow_{P_{p^k}} \mid \chi^\lambda\big\downarrow_{P_{p^k}}.$$

But $\mathcal{X}\big(\mu; (p)\big)\big\downarrow_{P_{p^k}} = \mathcal{X}\big(\mu\big\downarrow^{\mathfrak{S}_{p^{k-1}}}_{P_{p^{k-1}}}; \mathbb{1}_{P_p}\big)$ contains $\mathcal{X}\big(\phi(s^-); \phi_0\big)$ as a summand, since $\mu \in \Omega(s^-)$. Hence

$$\phi(s) = \mathcal{X}\big(\phi(s^-); \phi_{s_k}\big) \;\Big|\; \mathcal{X}\big(\mu; (p)\big)\big\downarrow_{P_{p^k}} \;\Big|\; \chi^\lambda\big\downarrow_{P_{p^k}}.$$

Thus $\lambda \in \Omega(s)$, and $\lambda_1 = p\mu_1 = p^k - p^{k-f(s)}$. □

Next, we determine the value of $m(s)$. We prove more, in fact, about the structure of $\Omega(s)$.

Definition 6.9. *We say that a partition λ is* thin *if λ is a hook, $l(\lambda) \leq 2$, or $\lambda_1 \leq 2$.*

Theorem 6.10. *Let $m, k \in \mathbb{N}$ with $2 \leq m \leq k$. Let $s \in U_k(m)$. Then*

(i) $m(s) = \eta(s)$*;*

(ii) $\Omega(s) \setminus \mathcal{B}_{p^k}(\eta(s))$ *contains no thin partitions;*

(iii) $\Omega(s) \cap \{\lambda \in \mathcal{P}(p^k) \mid \lambda_1 = M(s)\}^\circ = \{(M(s), \mu) \mid \mu \in \Omega(s_{f(s)+1}, \ldots, s_k)\}^\circ$*; and*

(iv) $\langle \chi^\lambda \big\downarrow_{P_{p^k}}, \phi(s) \rangle \geq 2$ *for all* $\lambda \in \{(\eta(s), p^k - \eta(s)), (\eta(s), 1^{p^k - \eta(s)})\}^\circ$.

We remark that the value of $m(s)$ for $s \in U_k(1)$ is determined in Lemmas 6.4 and 6.12. We first show that statement (iii) of Theorem 6.10 holds.

Lemma 6.11. *Let $k, m \in \mathbb{N}$, and let $s \in U_k(m)$ be such that $f(s) < k$. Then*

$$\Omega(s) \cap \{\lambda \vdash p^k \mid \lambda_1 = M(s)\}^\circ = \{(M(s), \mu) \mid \mu \in \Omega(s_{f(s)+1}, \ldots, s_k)\}^\circ.$$

In particular, if $m \geq 2$ then $\Omega(s) \cap \{\lambda \vdash p^k | \lambda_1 = M(s)\}^\circ$ contains no thin partitions.

Proof. Let $f = f(s)$, $t = (s_1, \ldots, s_f)$ and $u = (s_{f+1}, \ldots, s_k)$. Let $W = \mathfrak{S}_{p^f} \wr \mathfrak{S}_{p^{k-f}} \leq \mathfrak{S}_{p^k}$ and let Y be the base group of the wreath product W, namely $Y = (\mathfrak{S}_{p^f})^{\times p^{k-f}} \leq W$. Let $P = P_{p^k}$, and note that since $P = P_{p^f} \wr P_{p^{k-f}}$ we have that $P \leq W$. Finally we denote by B the base group of P, that is, $B = (P_{p^f})^{\times p^{k-f}} \leq Y$.

Let $\lambda = (M(s), \mu) \in \mathcal{P}(p^k)$, for some $\mu \in \mathcal{P}(p^{k-f})$. It suffices to prove the following two statements:

(i) $\langle \chi^\lambda \big\downarrow_P, \phi(s) \rangle = \langle \mathcal{X}((p^f - 1, 1); \mu)\big\downarrow_P, \phi(s) \rangle$; and

(ii) $\langle \mathcal{X}((p^f - 1, 1); \mu)\big\downarrow_P, \phi(s) \rangle > 0$ if and only if $\mu \in \Omega(u)$.

The first assertion of the lemma then follows, since $\Omega(s)$ is closed under conjugation. The second statement follows simply from the observation that if $m \geq 2$ then $u \neq (0, \ldots, 0) \in [\overline{p}]^{k-f}$, and hence $\{(p^{k-f}), (1^{p^{k-f}})\} \cap \Omega(u) = \emptyset$.

We now prove (i) and (ii). For convenience, let $\alpha = (p^f - 1, 1)$ and $q = p^{k-f}$.

(i) By Theorem 6.8, $M(s) = p^k - p^{k-f} = q(p^f - 1)$. Hence for $\mu_1, \ldots, \mu_q \vdash p^f$, if $c^\lambda_{\mu_1, \ldots, \mu_q} > 0$ then either $\mu_1 = \cdots = \mu_q = \alpha$ or there exists $j \in [q]$ such that $\mu_j = (p^f)$. Since $(p^f) \notin \Omega(t)$ by Lemma 6.4, it follows that

$$\langle \chi^\lambda \big\downarrow_B, \phi(t)^{\times q} \rangle = \langle \chi^\lambda \big\downarrow_Y, (\chi^\alpha)^{\times q} \rangle \cdot \langle (\chi^\alpha)^{\times q} \big\downarrow_B, \phi(t)^{\times q} \rangle.$$

Moreover, by Lemma 2.11 we have that

$$\langle \chi^\lambda \big\downarrow_Y, (\chi^\alpha)^{\times q} \rangle = c^\lambda_{\alpha, \ldots, \alpha} = c^\mu_{(1), \ldots, (1)} = \chi^\mu(1),$$

and thus $\langle \chi^\lambda \big\downarrow_B, \phi(t)^{\times q} \rangle = \chi^\mu(1) \cdot \left(\langle \chi^\alpha \big\downarrow_{P_{p^f}}, \phi(t) \rangle\right)^q$. By Theorem 2.21 we know that $\mathcal{X}(\alpha; \mu)$ is an irreducible constituent of $\chi^\lambda \big\downarrow_W$. Moreover,

$$\langle \mathcal{X}(\alpha; \mu)\big\downarrow_Y, (\chi^\alpha)^{\times q} \rangle = \chi^\mu(1).$$

Writing $\chi^\lambda\big\downarrow_W = \mathcal{X}(\alpha;\mu) + \Delta$ for some character Δ of W, and $\mathcal{X}(\alpha;\mu)\big\downarrow_Y = \chi^\mu(1) \cdot (\chi^\alpha)^{\times q} + \theta$ for some character θ of Y, we have that

$$\begin{aligned}\langle \chi^\lambda\big\downarrow_B, \phi(t)^{\times q}\rangle &= \langle \mathcal{X}(\alpha;\mu)\big\downarrow^W_B, \phi(t)^{\times q}\rangle + \langle \Delta\big\downarrow^W_B, \phi(t)^{\times q}\rangle \\ &= \chi^\mu(1)\cdot\big(\langle \chi^\alpha\big\downarrow_{P_{p^f}}, \phi(t)\rangle\big)^q + \langle \theta\big\downarrow^Y_B, \phi(t)^{\times q}\rangle + \langle \Delta\big\downarrow^W_B, \phi(t)^{\times q}\rangle,\end{aligned}$$

and therefore

$$\langle \theta\big\downarrow^Y_B, \phi(t)^{\times q}\rangle = \langle \Delta\big\downarrow^W_B, \phi(t)^{\times q}\rangle = 0.$$

Letting $c = \langle \Delta\big\downarrow^W_P, \phi(s)\rangle$, then since $\phi(s)\big\downarrow^P_B = \phi(t)^{\times q}$, we have that

$$0 = \langle \Delta\big\downarrow^W_B, \phi(t)^{\times q}\rangle \geq c\langle \phi(s)\big\downarrow^P_B, \phi(t)^{\times q}\rangle = c,$$

from which we conclude $c = 0$. Thus $\big\langle \chi^\lambda\big\downarrow_P, \phi(s)\big\rangle = \big\langle \mathcal{X}(\alpha;\mu)\big\downarrow_P, \phi(s)\big\rangle$.

(ii) Now let $\gamma = \chi^\alpha\big\downarrow^{\mathfrak{S}_{p^f}}_{P_{p^f}}$. By Lemma 6.4, $\langle \gamma, \phi(t)\rangle = 1$. Moreover, we observe that

$$\mathcal{X}(\alpha;\mu)\big\downarrow_P = \mathcal{X}(\alpha;\mu)\big\downarrow^{\mathfrak{S}_{p^f}\wr\mathfrak{S}_q}_{P_{p^f}\wr P_q} = \mathcal{X}(\gamma;\chi^\mu\big\downarrow^{\mathfrak{S}_q}_{P_q}) = \sum_{\tau\in\mathrm{Irr}(P_q)} \langle \chi^\mu\big\downarrow_{P_q}, \tau\rangle\cdot\mathcal{X}(\gamma;\tau).$$

Since $\phi(s) = \mathcal{X}(\phi(t);\phi(u))$, we have that

$$\begin{aligned}\langle \mathcal{X}(\alpha;\mu)\big\downarrow_P, \phi(s)\rangle &= \sum_{\tau\in\mathrm{Irr}(P_q)} \langle \chi^\mu\big\downarrow_{P_q}, \tau\rangle\cdot\langle \mathcal{X}(\gamma;\tau), \phi(s)\rangle \\ &= \sum_{\tau\in\mathrm{Irr}(P_q)} \langle \chi^\mu\big\downarrow_{P_q}, \tau\rangle\cdot\delta_{\phi(u),\tau} = \langle \chi^\mu\big\downarrow_{P_q}, \phi(u)\rangle,\end{aligned}$$

by Lemma 2.19. By definition of $\Omega(u)$, $\langle \chi^\mu\big\downarrow_{P_q}, \phi(u)\rangle > 0$ if and only if $\mu \in \Omega(u)$. $\square$

Recall that $\eta(s)$ was defined in Definition 6.6 for sequences $s \in [\overline{p}]^k$ containing at least two non-zero entries.

Lemma 6.12. *Let $k \in \mathbb{N}$, $s \in U_k(1)$ and $x \in [\overline{p}]$. Let $f = f(s)$. Then*

(a) $\Omega(s,x) = \mathcal{B}_{p^{k+1}}(p^{k+1} - p^{k+1-f} - 1) \sqcup \{(p^{k+1} - p^{k+1-f}, \mu) : \mu \in \Omega(s_{f+1}, \ldots, s_k, x)\}^\circ$;

(b) moreover, if $x \neq 0$ and $\lambda \in \{(\eta(s,x), p^{k+1-f}+1), (\eta(s,x), 1^{p^{k+1-f}+1})\}^\circ$, then $\langle \chi^\lambda\big\downarrow_{P_{p^{k+1}}}, \phi(s,x)\rangle \geq 2$.

Proof. **(a)** Let $f = f(s)$, $t = (s_1, \ldots, s_f)$ and $u = (s_{f+1}, \ldots, s_k)$. We proceed by induction on k and distinguish between two cases depending on the value of s_k.

Case 1. First suppose that $s_k \neq 0$. In particular, $f = k$. Then $\Omega(s) = \mathcal{B}_{p^k}(p^k - 1)$, by Lemma 6.4. By Proposition 5.12, we deduce that

$$\mathcal{B}_{p^{k+1}}(p^{k+1} - p - 1) \subseteq \mathcal{D}\big(p, p^k, \mathcal{B}_{p^k}(p^k - 1)\big),$$

so by Lemma 6.5 we find that $\mathcal{B}_{p^{k+1}}(p^{k+1} - p - 1) \subseteq \Omega(s,x)$ for all $x \in [\overline{p}]$. On the other hand, by Theorem 6.8 we know that $\Omega(s,x) \subseteq \mathcal{B}_{p^{k+1}}(p^{k+1}-p)$. Hence $p^{k+1}-p-1 \leq m(s,x) \leq p^{k+1}-p$. The statement (a) now follows directly from Lemma 6.11.

In particular, we observe that we did not need to use an inductive hypothesis in Case 1. Moreover, we showed that the base case $k=1$ of our induction holds.

Case 2. Now suppose that $s_k = 0$ (so necessarily $k \geq 2$). Then $f(s^-) = f \in [k-1]$, and by the inductive hypothesis applied to $s = (s^-, s_k)$ we have that

$$\Omega(s) = \mathcal{B}_{p^k}(p^k - p^{k-f} - 1) \sqcup \{(p^k - p^{k-f}, \mu) \mid \mu \in \Omega(u)\}^\circ. \tag{6.1}$$

By Proposition 5.12 and Lemma 6.5, we deduce that

$$\mathcal{B}_{p^{k+1}}(p^{k+1} - p^{k+1-f} - p - 1) \subseteq \mathcal{D}\big(p, p^k, \mathcal{B}_{p^k}(p^k - p^{k-f} - 1)\big) \subseteq \mathcal{D}(p, p^k, \Omega(s)) \subseteq \Omega(s,x).$$

We now want to show that for all $r \in \{0,1,\dots,p-1\}$ and all $\mu \vdash p^{k+1-f} + p - r$, the partition $\lambda := (p^{k+1} - p^{k+1-f} - p + r, \mu)$ belongs to $\Omega(s,x)$. This would allow us to conclude that $\mathcal{B}_{p^{k+1}}(p^{k+1} - p^{k+1-f} - 1) \subseteq \Omega(s,x)$, since $\Omega(s,x)$ is closed under conjugation.

If $r=0$ then $\mu \vdash p^{k+1-f}+p$. Since $\Omega(u) = \mathcal{P}(p^{k-f}) \setminus \{(p^{k-f}-1,1)\}^\circ$ by Theorem 5.1, there certainly exists a partition $\nu_1 \in \Omega(u)$ such that $\nu_1 \subseteq \mu$. Hence there exist partitions $\nu_2 \vdash p^{k-f}+2$ and $\nu_3, \dots, \nu_p \vdash p^{k-f}+1$ such that $c^\mu_{\nu_1,\dots,\nu_p} > 0$. By Lemma 2.11 we deduce that

$$c^\lambda_{(p^k-p^{k-f},\nu_1),(p^k-p^{k-f}-2,\nu_2),(p^k-p^{k-f}-1,\nu_3),\dots,(p^k-p^{k-f}-1,\nu_p)} = c^\mu_{\nu_1,\dots,\nu_p} > 0.$$

Since $p \geq 5$ and $f \in [k-1]$, we have that $p^{k-f}+2 \leq p^k - p^{k-f} - 2$, whence $(p^k - p^{k-f}, \nu_1)$, $(p^k - p^{k-f}-2, \nu_2)$ and $(p^k - p^{k-f} - 1, \nu_i)$ for all $i \in \{3,\dots,p\}$ are all partitions. Moreover, they belong to $\Omega(s)$ by (6.1), so by Lemma 6.5 we conclude that $\lambda \in \mathcal{D}(p, p^k, \Omega(s)) \subseteq \Omega(s,x)$, for all $x \in [\overline{p}]$.

If $r \in [p-1]$ then $\mu \vdash p^{k+1-f} + p - r$ and there exists a partition $\nu \vdash rp^{k-f}$ such that $\nu \subseteq \mu$. (If $r=1$ then we choose $\nu \in \Omega(u)$; this is possible by Theorem 5.1.) By Theorem 5.1 we know that the trivial character of $P_{rp^{k-f}}$ is an irreducible constituent of $\chi^\nu\big\downarrow_{P_{rp^{k-f}}}$. Thus there exist $\nu_1, \dots, \nu_r \in \Omega(u)$ such that $c^\nu_{\nu_1,\dots,\nu_r} > 0$, since $\phi(u) = \mathbb{1}_{P_{p^{k-f}}}$. Moreover, there exist partitions $\nu_{r+1}, \dots, \nu_p \vdash p^{k-f}+1$ such that $c^\mu_{\nu_1,\dots,\nu_r,\nu_{r+1},\dots,\nu_p} > 0$. Using Lemma 2.11 we deduce that

$$c^\lambda_{(p^k-p^{k-f},\nu_1),\dots,(p^k-p^{k-f},\nu_r),(p^k-p^{k-f}-1,\nu_{r+1}),\dots,(p^k-p^{k-f}-1,\nu_p)} = c^\mu_{\nu_1,\dots,\nu_p} > 0.$$

Note that $(p^k - p^{k-f}, \nu_i)$ for $i \in [r]$ and $(p^k - p^{k-f} - 1, \nu_j)$ for $j \in \{r+1,\dots,p\}$ are indeed partitions as $p^{k-f}+1 \leq p^k - p^{k-f} - 1$. Moreover, they belong to $\Omega(s)$ by (6.1), so $\lambda \in \mathcal{D}(p, p^k, \Omega(s)) \subseteq \Omega(s,x)$ for all $x \in [\overline{p}]$, by Lemma 6.5.

Thus we have shown that $\mathcal{B}_{p^{k+1}}(p^{k+1} - p^{k+1-f} - 1) \subseteq \Omega(s,x)$ for all x. The statement (a) now follows from Lemma 6.11, since $M(s,x) = p^{k+1} - p^{k+1-f}$ by Theorem 6.8.

(b) We turn to the proof of statement (b). Let $t=(s,x)$ and observe that $f(t)=f(s)=f$. Let $P=P_{p^{k+1}}=P_{p^k}\wr P_p$ and let B be its base group, namely $P=B\rtimes P_p$ and $B\cong (P_{p^k})^{\times p}$. Let $Y=(\mathfrak{S}_{p^k})^{\times p}$ be the Young subgroup of $\mathfrak{S}_{p^{k+1}}$ naturally containing B. We define two further subgroups of $\mathfrak{S}_{p^{k+1}}$ as follows: $H:=Y\rtimes\mathfrak{S}_p\cong\mathfrak{S}_{p^k}\wr\mathfrak{S}_p$ and $W:=Y\rtimes P_p\cong\mathfrak{S}_{p^k}\wr P_p$. Clearly $P\le W\le H$.

First, we let $\lambda=(\eta(t),p^{k+1}-\eta(t))$ and define $\mu,\nu\vdash p^k$ as follows:

$$\mu=(p^k-p^{k-f},p^{k-f})\quad\text{and}\quad\nu=(p^k-p^{k-f}-1,p^{k-f}+1).$$

Note $\mu,\nu\in\Omega(s)$ (by part (a) of the present lemma if $f<k$, and by Lemma 6.4 if $f=k$). Moreover, it is easy to see that letting $\mu_1=\cdots=\mu_{p-1}=\mu$, we have that $c^{\lambda}_{\mu_1,\dots,\mu_{p-1},\nu}=1$. Since $\theta:=(\chi^{\mu})^{\times(p-1)}\times\chi^{\nu}$ is an irreducible constituent of $\chi^{\lambda}\big\downarrow_Y$, there exists $\rho\in\operatorname{Irr}(W|\theta)$ such that $\rho\mid\chi^{\lambda}\big\downarrow_W$. But $\mu\ne\nu$, so by the description of $\operatorname{Irr}(\mathfrak{S}_{p^k}\wr P_p)$ in Section 2.3, we have that $\rho=\theta\big\uparrow_Y^W$. From Lemma 2.16, we see that $\rho\big\downarrow_P=\theta\big\downarrow_B\big\uparrow^P$, which has $\phi(s)^{\times p}\big\uparrow^P$ as a direct summand, and hence $\langle\rho\big\downarrow_P,\phi(t)\rangle\ge 1$ by Lemma 2.15 since $\phi(t)=\mathcal{X}(\phi(s);\phi_x)$. On the other hand, $\mathcal{X}(\mu;(p-1,1))\mid\chi^{\lambda}\big\downarrow_H$ by [9, Theorem 1.5]. Thus $\beta:=\mathcal{X}(\mu;\phi_x)$ is an irreducible constituent of $\chi^{\lambda}\big\downarrow_W$, since $\chi^{(p-1,1)}\big\downarrow_{P_p}^{\mathfrak{S}_p}=\sum_{i=1}^{p-1}\phi_i$, and clearly $\langle\beta\big\downarrow_P,\phi(t)\rangle\ge 1$. Since $\rho\ne\beta$ are both irreducible, we find that

$$\langle\chi^{\lambda}\big\downarrow_P,\phi(t)\rangle\ge\langle\rho\big\downarrow_P,\phi(t)\rangle+\langle\beta\big\downarrow_P,\phi(t)\rangle\ge 2.$$

For $\lambda=(\eta(t),1^{p^{k+1}-\eta(t)})$, a similar argument using $\mu=(p^k-p^{k-f},1^{p^{k-f}})$ and $\nu=(p^k-p^{k-f}-1,1^{p^{k-f}+1})$, and using Theorem 2.20 to show that $\mathcal{X}(\mu;\tau)\mid\chi^{\lambda}\big\downarrow_H$ for some $\tau\in\{(p-1,1)\}^{\circ}$ shows that $\langle\chi^{\lambda}\big\downarrow_P,\phi(t)\rangle\ge 2$.

Finally, since $\chi^{\lambda}\big\downarrow_P=\chi^{\lambda'}\big\downarrow_P$, statement (b) follows. □

Remark 6.13. If $s\in U_k(2)$ and $s_k\ne 0$, then Lemma 6.12 shows that $\Omega(s)\setminus\mathcal{B}_{p^k}(p^k-p^{k-f(s)}-1)$ contains no thin partitions. This follows from the observation that

$$(p^{k-f(s)}),(1^{p^{k-f(s)}})\notin\Omega(s_{f(s)+1},\dots,s_k)=\mathcal{B}_{p^{k-f}}(p^{k-f}-1),$$

by Lemma 6.4. ◊

We are now ready to prove Theorem 6.10.

Proof of Theorem 6.10. We proceed by induction on k, where the base case is $k=2$.

If either $k=2$, or $k\ge 3$, $m=2$ and $s_k\ne 0$ (so $s^-\in U_{k-1}(1)$), then statements (i) and (iii) follow from Lemma 6.12 (a), statement (ii) from Remark 6.13, and statement (iv) from Lemma 6.12 (b).

Suppose now that $m\ge 3$, or that $m=2$ and $s_k=0$ (so necessarily $k\ge 3$). Then $s^-\in U_{k-1}(m')$ where $m'\ge 2$. Also $f(s^-)=f(s)=:f\in[k-1]$. By the inductive hypothesis, we have that

$$\Omega(s^-)=\mathcal{B}_{p^{k-1}}(\eta(s^-))\sqcup A(s^-)\sqcup\{(p^{k-1}-p^{k-1-f},\mu):\mu\in\Omega(s_{f+1},\dots,s_{k-1})\}^{\circ},$$

where $A(s^-) := \Omega(s^-) \cap \{\lambda \vdash p^{k-1} : \eta(s^-) < \lambda_1 < M(s^-)\}^\circ$ contains no thin partitions. By Proposition 5.12 and Lemma 6.5, we deduce that

$$\mathcal{B}_{p^k}(p\eta(s^-)-1) \subseteq \mathcal{D}\big(p, p^{k-1}, \mathcal{B}_{p^{k-1}}(\eta(s^-))\big) \subseteq \mathcal{D}\big(p, p^{k-1}, \Omega(s^-)\big) \subseteq \Omega(s).$$

Note that $\eta(s) = p\eta(s^-)$, so letting $\rho(s) = p^k - \eta(s)$ and $\rho(s^-) = p^{k-1} - \eta(s^-)$, we also have $\rho(s) = p\rho(s^-)$.

Let $\lambda = (\eta(s), \mu)$ for some $\mu \vdash \rho(s)$. If $\mu \notin \{(\rho(s))\}^\circ$, then by Proposition 5.12 there exist $\nu_1, \ldots, \nu_p \vdash \rho(s^-)$, not all equal, such that $c^{\mu}_{\nu_1,\ldots,\nu_p} > 0$. Indeed, $\mu \in \mathcal{B}_{\rho(s)}(\rho(s)-1) \subseteq \mathcal{D}\big(p, \rho(s^-), \mathcal{P}(\rho(s^-))\big)$. Hence by Lemma 2.11 we have that

$$c^{\lambda}_{(\eta(s^-),\nu_1),\ldots,(\eta(s^-),\nu_p)} = c^{\mu}_{\nu_1,\ldots,\nu_p} > 0.$$

Notice that $(\eta(s^-), \nu_i)$ is indeed a partition for all i, and in fact belongs to $\Omega(s^-)$, since $\eta(s^-) \geq 1 + \rho(s^-)$ follows from $f \geq 1$, $g \geq 2$ and $k \geq 3$. Hence $\lambda \in \mathcal{D}(p, p^{k-1}, \Omega(s^-)) \subseteq \Omega(s)$, by Lemma 6.5.

Otherwise if $\mu \in \{(\rho(s)), (1^{\rho(s)})\}$, then $\lambda \in \{\lambda^0, \lambda^1\}$ where $\lambda^0 = (\eta(s), \rho(s))$ and $\lambda^1 = (\eta(s), 1^{\rho(s)})$. Let $\nu \vdash p^{k-1}$ be the partition defined as follows:

$$\nu = \nu(\lambda^i) = \begin{cases} (\eta(s^-), \rho(s^-)) & \text{if } i = 0, \\ (\eta(s^-), 1^{\rho(s^-)}) & \text{if } i = 1. \end{cases}$$

With ν thus defined, let $H = \mathfrak{S}_{p^{k-1}} \wr \mathfrak{S}_p$ and let $\mathcal{X} \in \mathrm{Irr}(H)$ be defined as follows:

$$\mathcal{X} = \mathcal{X}(\lambda^i) = \begin{cases} \mathcal{X}\big(\nu(\lambda^i); (p)\big) & \text{if } i = 0, \\ \mathcal{X}\big(\nu(\lambda^i); \alpha\big) & \text{if } i = 1, \end{cases}$$

where $\alpha \in \{(p), (1^p)\}$ is chosen such that $\mathcal{X}(\lambda^1) \mid \chi^{\lambda^1}\big\downarrow_H$, according to Theorem 2.20. Moreover, $\mathcal{X}(\lambda^0) \mid \chi^{\lambda^0}\big\downarrow_H$, by Theorem 2.21. By the inductive hypothesis, we know that $\langle \chi^{\nu(\lambda^i)}\big\downarrow_{P_{p^{k-1}}}, \phi(s^-) \rangle \geq 2$ for $i \in \{0, 1\}$. Hence from Lemma 2.18 we deduce that

$$\langle \chi^{\lambda^i}\big\downarrow_{P_{p^k}}, \phi(s) \rangle \geq \langle \mathcal{X}(\lambda^i)\big\downarrow_{P_{p^k}}, \phi(s) \rangle \geq 2.$$

This shows at once that $\eta(s) \leq m(s)$, since $\Omega(s)^\circ = \Omega(s)$ so $\mathcal{B}_{p^k}(\eta(s)) \subseteq \Omega(s)$, and also that statement (iv) holds, since $\chi^{\lambda}\big\downarrow_{P_{p^k}} = \chi^{\lambda'}\big\downarrow_{P_{p^k}}$.

Next, we turn to the proof of statement (ii). In particular, statement (i) that $m(s) = \eta(s)$ then follows immediately, since (ii) implies $(\eta(s)+1, p^k - \eta(s) - 1) \notin \Omega(s)$, for instance. In order to prove (ii), it suffices to consider partitions $\lambda \vdash p^k$ such that $\lambda_1 > \eta(s)$, since $\Omega(s)^\circ = \Omega(s)$.

Let $x \in \{1, 2, \ldots, p^{k-g(s)}\}$ and first let $\lambda = (\eta(s)+x, \rho(s)-x)$. (We remark that x is chosen so that λ_1 varies between $\eta(s)+1$ and $M(s)$.) Since $\lambda_1 > \eta(s) = p\eta(s^-)$, we have that for any sequence of partitions $(\mu^1, \ldots, \mu^p) \in \mathcal{P}(p^{k-1})^{\times p}$ such that $c^{\lambda}_{\mu^1,\ldots,\mu^p} > 0$, there exists $j \in [p]$ such that $(\mu^j)_1 > \eta(s^-)$. Moreover, $\mu^j \subseteq \lambda$ so $l(\mu^j) \leq 2$, and thus $\mu^j \notin \Omega(s^-)$ by the inductive hypothesis. Letting $B = (P_{p^{k-1}})^{\times p}$, we deduce that $\langle \chi^\lambda \big\downarrow_B, \phi(s^-)^{\times p} \rangle = 0$, and therefore that $\lambda \notin \Omega(s)$.

To conclude the proof of (ii), it remains to consider $\lambda = (\eta(s) + x, 1^{\rho(s)-x})$. In this case $\lambda \notin \Omega(s)$ follows from a similar argument, noticing instead that since λ is a hook then $\mu^1, \ldots, \mu^p \subseteq \lambda$ must also be hooks. Thus we have proven statement (ii), and as described previously also statement (i).

Finally, statement (iii) follows from Lemma 6.11, since $f(s) < k$. □

We conclude this section by introducing the following definition, which allows us to summarise the values of $m(s)$ obtained thus far, and will be useful for our discussion in the next section. Recall the notation $U_k(z)$, $f(s)$ and $\eta(s)$ from Definition 6.6.

Definition 6.14. *Let $k \in \mathbb{N}$ and $s \in [\overline{p}]^k$. The* type *of the sequence s is the number $\tau(s) \in \{1, 2, 3, 4\}$ defined as follows:*

$$\tau(s) = \begin{cases} 1 & \text{if } s \in U_k(0), \\ 2 & \text{if } s \in U_k(1) \text{ and } s_k = 0, \\ 3 & \text{if } s \in U_k(1) \text{ and } s_k \neq 0, \\ 4 & \text{if } s \in U_k(z) \text{ for some } z \geq 2. \end{cases}$$

Remark 6.15. We collect here a description of $m(s)$ for $s \in [\overline{p}]^k$, $k \in \mathbb{N}$, depending on the type of s:

$$m(s) = \begin{cases} p^k - 2 & \text{if } \tau(s) = 1, \\ p^k - p^{k-f(s)} - 1 & \text{if } \tau(s) = 2, \\ p^k - 1 & \text{if } \tau(s) = 3, \\ \eta(s) & \text{if } \tau(s) = 4. \end{cases}$$

If $\tau(s) \in \{1, 2, 3\}$ then we have a complete description of $\Omega(s)$, namely

$$\Omega(s) = \begin{cases} \mathcal{B}_{p^k}(m(s)) \cup \{(p^k)\}^\circ & \text{if } \tau(s) = 1, \\ \mathcal{B}_{p^k}(m(s)) \cup \{(m(s)+1, \mu) : \mu \in \Omega(\mathbb{1}_{P_{p^{k-f(s)}}})\}^\circ & \text{if } \tau(s) = 2, \\ \mathcal{B}_{p^k}(m(s)) & \text{if } \tau(s) = 3, \end{cases}$$

If $\tau(s) = 4$, then $\Omega(s) \setminus \mathcal{B}_{p^k}(m(s))$ contains no thin partitions by Theorem 6.10 .

For convenience, when $k = 0$ and s is the empty sequence we set $\tau(s) = 1$, though notice $m(s) = p^k = 1$ as $\Omega(s) = \mathcal{P}(1)$.

Finally, we remark that for all $k \in \mathbb{N}_0$, we have that $m(s) > \frac{p^k}{2}$ for $s \in [\overline{p}]^k$ of all types. ◊

6.3 Bounding $\Omega(\phi)$

Let $p \geq 5$ be a prime. Following on from the previous section, the aim of the present section is to determine the numbers $m(\phi)$ and $M(\phi)$ for all $\phi \in \mathrm{Lin}(P_n)$ where n is now an arbitrary natural number.

Let $n \in \mathbb{N}$ and let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be its p-adic expansion, where $0 \leq n_1 < \cdots < n_t$. Recall that we may write $\phi = \phi(\underline{\mathbf{s}}) = \phi(\mathbf{s}(1,1)) \times \cdots \times \phi(\mathbf{s}(t,a_t))$ as in (2.4), and recall the operator $\star$ from Section 2.2.1.

Lemma 6.16. *For all $n \in \mathbb{N}$ and $\phi(\underline{\mathbf{s}}) \in \mathrm{Lin}(P_n)$,*

$$\Omega(\underline{\mathbf{s}}) = \Omega(\mathbf{s}(1,1)) \star \cdots \star \Omega(\mathbf{s}(i,j)) \star \cdots \star \Omega(\mathbf{s}(t,a_t)).$$

Proof. Since $\phi(\underline{\mathbf{s}}) = \phi(\mathbf{s}(1,1)) \times \cdots \times \phi(\mathbf{s}(t,a_t))$, the statement follows from the definitions of Ω and $\star$ by considering the chain of subgroups

$$\mathfrak{S}_n \geq (\mathfrak{S}_{p^{n_1}})^{\times a_1} \times \cdots \times (\mathfrak{S}_{p^{n_t}})^{\times a_t} \geq (P_{p^{n_1}})^{\times a_1} \times \cdots \times (P_{p^{n_t}})^{\times a_t} = P_n.$$

□

Theorem 6.17. *For all $n \in \mathbb{N}$ and $\phi(\underline{\mathbf{s}}) \in \mathrm{Lin}(P_n)$, $M(\mathbf{s}) = \sum_{(i,j)} M(\mathbf{s}(i,j))$.*

Proof. Let $M := \sum_{(i,j)} M(\mathbf{s}(i,j))$. For $k \in \mathbb{N}_0$ and $s \in [\overline{p}]^k$, we have that $M(s) = p^k - p^{k-f(s)}$ by Theorem 6.8 if $s \neq (0,\ldots,0)$, and $M(0,\ldots,0) = p^k$ since $\chi^{(p^k)} = \mathbb{1}_{\mathfrak{S}_{p^k}} \in \Omega(0,\ldots,0) = \Omega(\mathbb{1}_{P_{p^k}})$. Hence $M(\mathbf{s}(i,j)) > p^{n_i}/2$ for all (i,j), so by Lemma 6.16 and Proposition 5.7 we have that

$$\begin{aligned}\Omega(\underline{\mathbf{s}}) &= \Omega(\mathbf{s}(1,1)) \star \cdots \star \Omega(\mathbf{s}(t,a_t)) \\ &\subseteq \mathcal{B}_{p^{n_1}}\big(M(\mathbf{s}(1,1))\big) \star \cdots \star \mathcal{B}_{p^{n_t}}\big(M(\mathbf{s}(t,a_t))\big) = \mathcal{B}_n(M).\end{aligned}$$

Thus $M(\underline{\mathbf{s}}) \leq M$.

On the other hand, let $\lambda^{(i,j)} \in \Omega(\mathbf{s}(i,j))$ be such that $\lambda_1^{(i,j)} = M(\mathbf{s}(i,j))$ for each (i,j) (this is possible since $\Omega(\mathbf{s}(i,j))$ is closed under conjugation). Set $\lambda = \lambda^{(1,1)} + \cdots + \lambda^{(t,a_t)}$, so the iterated Littlewood–Richardson coefficient $c^{\lambda}_{\lambda^{(1,1)},\ldots,\lambda^{(t,a_t)}} = 1$. Hence $\lambda \in \Omega(\mathbf{s}(1,1)) \star \cdots \star \Omega(\mathbf{s}(t,a_t)) = \Omega(\underline{\mathbf{s}})$, but also $\lambda_1 = \sum_{(i,j)} \lambda_1^{(i,j)} = M$, so $M(\underline{\mathbf{s}}) \geq M$. □

The rest of this section is devoted to the determination of $m(\phi)$ for all $\phi \in \mathrm{Lin}(P_n) \setminus \{\mathbb{1}_{P_n}\}$, since the result for $\phi = \mathbb{1}_{P_n}$ follows from Theorem 5.1. To simplify notation, we let $R = \sum_{j=1}^{t} a_j$ and let $\{s_1,\ldots,s_R\} = \{\mathbf{s}(i,j) \mid i \in [t],\ j \in [a_i]\}$ as multisets. We let k_j be the length of s_j, so $\{k_1,\ldots,k_R\} = \{n_1,\ldots,n_t\}$ and $|\{j \in [R] \mid k_j = n_i\}| = a_i$. Where $\phi = \phi(\underline{\mathbf{s}})$ and $\underline{\mathbf{s}}$ is identified with $\{s_1,\ldots,s_R\}$ as above, we also denote $m(\phi)$ or $m(\underline{\mathbf{s}})$ by $m(s_1,\ldots,s_R)$. Note that the order of $s_1,\ldots,s_R$ does not matter in determining $m(\phi)$ by Lemma 4.5, since if two linear characters of P_n are $N_{\mathfrak{S}_n}(P_n)$–conjugate then their inductions to $\mathfrak{S}_n$ are equal (Lemma 4.2). Thus we may without loss of generality permute the s_i freely in our arguments.

Since P_n is trivial whenever $n < p$, from now on we may assume that $n \geq p$. Moreover, we may assume that $R \geq 2$ since the case of $R = 1$ is treated in Section 6.2.

Fix some $\phi \in \mathrm{Lin}(P_n)$ with corresponding sequences $\{s_1, \ldots, s_R\}$ as described above. Furthermore, we assume for the rest of this section that there exists some $i \in [R]$ such that $\tau(s_i) \neq 1$, since $\phi \neq \mathbb{1}_{P_n}$. We wish to express $m(\phi)$ in terms of the quantities $m(s_1), m(s_2), \ldots, m(s_R)$ that we determined in Section 6.2. In order to do this, we give the following definition.

Definition 6.18. *Let $k \in \mathbb{N}_0$ and $s \in [\overline{p}]^k$. The integer $N(s)$ is defined as follows:*

$$N(s) = \begin{cases} p^k & \text{if } \tau(s) = 1, \\ m(s) + 1 & \text{if } \tau(s) = 2, \\ m(s) & \text{if } \tau(s) \in \{3, 4\}. \end{cases}$$

(Note that if $k = 0$, then s is the empty sequence and $N(s) = p^k = 1$.)

For $\phi \in \mathrm{Lin}(P_n)$ as described above, let $N(\phi)$ be defined as follows:

$$N(\phi) = \sum_{j=1}^{R} N(s_j).$$

We are now ready to describe $m(\phi)$. This is done in the following two theorems, whose proofs appear in the next and final section of this chapter.

Theorem 6.19. *Let $n \in \mathbb{N}$ and $\phi \in \mathrm{Lin}(P_n)$ be as described above. Suppose that $\tau(s_i) = 4$ for some $i \in [R]$. Then*

(i) $m(\phi) = N(\phi)$, and

(ii) $\Omega(\phi) \setminus \mathcal{B}_n(m(\phi))$ contains no thin partitions.

If no sequence s_i is of type 4, then we can in fact completely describe $\Omega(\phi)$.

Theorem 6.20. *Let $n \in \mathbb{N}$ and $\phi \in \mathrm{Lin}(P_n)$ be as described above. Suppose that $\tau(s_i) \neq 4$ for all $i \in [R]$. Then*

$$\Omega(\phi) = \mathcal{B}_n(N(\phi)),$$

unless

$$|\{i \in [R] \mid \tau(s_i) = j\}| = \begin{cases} R - 1 & \text{if } j = 1, \\ 1 & \text{if } j = 2, \\ 0 & \text{if } j \in \{3, 4\}, \end{cases}$$

in which case

$$\Omega(\phi) = \mathcal{B}_n(N(\phi) - 1) \sqcup \{(N(\phi), \mu) \mid \mu \in \Omega(\mathbb{1}_{P_{p^{k_i - f(s_i)}}})\}^{\circ},$$

where i is the unique element of $[R]$ such that $\tau(s_i) = 2$.

We remark that Theorem 6.20 in fact holds for $\phi = \mathbb{1}_{P_n}$ as well, since $\Omega(\mathbb{1}_{P_n}) = \mathcal{P}(n)$ by Theorem 5.1 and $N(\mathbb{1}_{P_n}) = n$. We illustrate the results of Theorems 6.19 and 6.20 in Example 6.25 below. A corollary of our description of $m(\phi)$ is the following:

Corollary 6.21. *Let $n \in \mathbb{N}$ and let $\phi \in \mathrm{Lin}(P_n)$. Then $\mathcal{B}_n(\frac{n}{2}) \subseteq \Omega(\phi)$.*

Proof. Recall from Remark 6.15 that for all $k \in \mathbb{N}$ and $s \in [\overline{p}]^k$ of all types, we have $m(s) > \frac{p^k}{2}$. Thus the claim follows when n is a power of p. Otherwise, letting ϕ correspond to $s_1, \dots, s_R$ for some $R \geq 2$, we see from Theorems 6.19 and 6.20 that $m(\phi) \geq \sum_{i=1}^R m(s) > \frac{n}{2}$. □

Proof of Corollary 6.2. This follows immediately from Corollary 6.21. □

Remark 6.22. We remark that the growth of the partition function $|\mathcal{P}(n)|$ is well-known, given by the celebrated asymptotic formula of Hardy and Ramanujan [34]:

$$|\mathcal{P}(n)| \sim \frac{1}{4n\sqrt{3}} \exp(c\sqrt{n}),$$

where $c = \pi\sqrt{\frac{2}{3}}$. Of course, we did not require its full power in order to deduce Corollary 6.2, though we have included it as well as the following classical result of Erdős and Lehner [24, (1.4)] to highlight that $|\mathcal{B}_n(\frac{n}{2})|$ is in fact extremely close to $|\mathcal{P}(n)|$ when n is large: if $f(n)$ is any function such that $f(n) \to \infty$ as $n \to \infty$, then for all but $o(|\mathcal{P}(n)|)$ partitions λ of n, the quantities λ_1 and $l(\lambda)$ lie between $\sqrt{n} \cdot (\frac{\log n}{c} \pm f(n))$. ◊

Before we conclude this chapter by proving Theorems 6.19 and 6.20 below, we remark that the situation when $p \in \{2, 3\}$ is more complex.

It is not hard to verify that our determination of $M(\phi)$ holds also for the prime 3. However, crucially for $m(\phi)$, Lemma 6.12 is not true as stated when $p = 3$, and a number of our Littlewood–Richardson results also cannot be applied directly when $p = 3$. (For instance, if we wish to apply Lemma 5.8 with $n = 3$, then the result changes to $\mathcal{B}_m(t) \star \{(3)\}^\circ = \mathcal{B}_{m+3}(t+3) \setminus \{(t+1, t+1)\}^\circ$ in the special case where $m = 2t - 1$.)

The sets $\Omega(\phi)$ in the case of $p = 2$ exhibit less regular patterns still. As already remarked in Chapter 5, the sign character $\chi^{(1^n)}$ of $\mathfrak{S}_n$ restricts irreducibly and non-trivially to a Sylow 2-subgroup of $\mathfrak{S}_n$. Since $\chi^{\lambda'} = \chi^\lambda \cdot \chi^{(1^n)}$, the sets $\Omega(\phi)$ themselves are no longer closed under conjugation in general. On the other hand, sets of partitions of the form $\mathcal{B}_n(m)$ *are* always closed under conjugation, so approximating $\Omega(\phi)$ using sets of the form $\mathcal{B}_n(m)$ is less informative when $p = 2$. Nevertheless, it turns out for $k \in \mathbb{N}$ and $P_{2^k} \in \mathrm{Syl}_2(\mathfrak{S}_{2^k})$ that

$$\chi^{(1^{2^k})}\big\downarrow_{P_{2^k}} = \phi(1, 0, \dots, 0) = \mathcal{X}(\phi_1; \mathbb{1}_{P_{2^{k-1}}}) \in \mathrm{Irr}(P_2 \wr P_{2^{k-1}}),$$

and so for instance, for $s \in [\overline{2}]^k$ we have that $\Omega(t) = \Omega(s)'$ where $t = (t_1, s_2, \dots, s_k)$ and $t_1 = s_1 + 1 \pmod 2$.

6.3.1 Proofs of Theorems 6.19 and 6.20

Let $n \in \mathbb{N}$ and $\phi \in \mathrm{Lin}(P_n)$ be as described immediately following Theorem 6.17.

Lemma 6.23. *Suppose that*

$$|\{i \in [R] \mid \tau(s_i) = j\}| = \begin{cases} R-1 & \text{if } j = 1, \\ 1 & \text{if } j = 2, \\ 0 & \text{if } j \in \{3,4\}, \end{cases}$$

and let $i \in [R]$ be such that $\tau(s_i) = 2$. Then

$$\Omega(\phi) = \mathcal{B}_n(N(\phi)-1) \sqcup \{(N(\phi), \mu) \mid \mu \in \Omega(\mathbb{1}_{P_{p^{k_i - f(s_i)}}})\}^\circ.$$

Proof. Let $m = n - p^{k_i}$, so $\phi = \phi(s_1) \times \cdots \times \phi(s_R) = \mathbb{1}_{P_m} \times \phi(s_i)$. Since $R \geq 2$ and $\tau(s_i) = 2$, we have that $m \in \mathbb{N}$ and $k_i \geq 2$. To ease the notation, let $k = k_i$, $s = s_i$ and $f = f(s_i)$.

By Lemma 6.16, we have that $\Omega(\phi) = \Omega(\mathbb{1}_{P_m}) \star \Omega(s)$. By Theorem 5.1 and Remark 6.15, we have that $\Omega(\mathbb{1}_{P_m}) = \mathcal{P}(p^l) \setminus \{(p^l-1,1)\}^\circ$ if $m = p^l$ for some $l \in \mathbb{N}$, and $\Omega(\mathbb{1}_{P_m}) = \mathcal{P}(m)$ otherwise. Moreover,

$$\Omega(s) = \mathcal{B}_{p^k}(p^k - p^{k-f} - 1) \sqcup \{(p^k - p^{k-f}, \mu) \mid \mu \in \Omega(\mathbb{1}_{P_{p^{k-f}}})\}^\circ,$$

so in particular

$$\mathcal{B}_{p^k}(m(s)) \subseteq \Omega(s) \subseteq \mathcal{B}_{p^k}(m(s)+1). \tag{6.2}$$

Case 1: if $\Omega(\mathbb{1}_{P_m}) = \mathcal{P}(m)$. Since $m(s) > \frac{p^k}{2}$, applying $\mathcal{P}(m)\star$ to (6.2) gives

$$\mathcal{B}_n\big(N(\phi)-1\big) \subseteq \Omega(\phi) \subseteq \mathcal{B}_n\big(N(\phi)\big)$$

by Proposition 5.7, as $N(\phi) = m + p^k - p^{k-f} = n - p^{k-f}$. Since $\Omega(\phi)^\circ = \Omega(\phi)$, it suffices to find which partitions $\lambda \vdash n$ with $\lambda_1 = N(\phi)$ satisfy $\lambda \in \Omega(\phi)$, noting that $\lambda \in \Omega(\phi) = \mathcal{P}(m) \star \Omega(s)$ if and only if $c^\lambda_{\alpha\beta} > 0$ for some $\alpha \vdash m$ and $\beta \in \Omega(s)$.

So fix a partition $\lambda \vdash n$ such that $\lambda_1 = N(\phi)$. If $c^\lambda_{\alpha\beta} > 0$ for some $\alpha \vdash m$ and $\beta \in \Omega(s)$, then $\lambda_1 \leq \alpha_1 + \beta_1 \leq m + (p^k - p^{k-f}) = N(\phi)$, so in fact this holds with equality. Thus $\alpha = (m)$ and $\beta = (\beta_1, \mu)$ where $\mu \in \Omega(\mathbb{1}_{P_{p^{k-f}}})$, since $\beta \in \Omega(s)$ and $\beta_1 = p^k - p^{k-f}$. Moreover, $\lambda_1 = \alpha_1 + \beta_1$ and $\alpha = (m)$ together imply that $\beta = (\lambda_1 - m, \lambda_2, \lambda_3, \dots)$, that is, $\lambda = (N(\phi), \mu)$.

Conversely, if $\lambda = (N(\phi), \mu)$ for some $\mu \in \Omega(\mathbb{1}_{P_{p^{k-f}}})$, then clearly $\lambda \in \mathcal{P}(m) \star \Omega(s) = \Omega(\phi)$ since $\chi^{(m)} \times \chi^{(p^k - p^{k-f}, \mu)} \mid \chi^\lambda\big\downarrow_{\mathfrak{S}_m \times \mathfrak{S}_{p^k}}$, and thus the set $\Omega(\phi)$ is as claimed.

Case 2: if $m = p^l$ for some $l \in \mathbb{N}$ and $\Omega(\mathbb{1}_{P_m}) = \mathcal{P}(p^l) \setminus \{(p^l-1,1)\}^\circ$. We have that

$$\Omega(\phi) \subseteq \mathcal{P}(p^l) \star \mathcal{B}_{p^k}(m(s)+1) = \mathcal{B}_n\big(N(\phi)\big)$$

by Proposition 5.7. On the other hand,

$$\begin{aligned}\Omega(\phi) &= \big(\mathcal{P}(p^l) \setminus \{(p^l-1,1)\}^\circ\big) \star \Omega(s)\\ &\supseteq \big(\mathcal{P}(p^l) \setminus \{(p^l-1,1)\}^\circ\big) \star \mathcal{B}_{p^k}(p^k - p^{k-f} - 1) = \mathcal{B}_n\big(N(\phi)-1\big)\end{aligned}$$

by Lemma 5.8. We find by the same argument as in Case 1 that $\Omega(\phi) = \mathcal{B}_n(N(\phi)-1) \sqcup \{(N(\phi),\mu) \mid \mu \in \Omega(\mathbb{1}_{P_{p^{k-f}}})\}^\circ$, as required. □

Proof of Theorem 6.20. We may now assume that $s_1, \dots, s_R$ do not satisfy the hypothesis of Lemma 6.23. That is, $s_1, \dots, s_R$ are such that *either* there exists $i \in [R]$ with $\tau(s_i) = 3$, *or* there exists $i \neq j \in [R]$ with $\tau(s_i) = \tau(s_j) = 2$ and $\tau(s_l) \in \{1,2\}$ for all $l \in [R]$. We proceed by induction on R.

We begin with the base case $R = 2$. Recall from the exact description of $\Omega(s_i)$ from Remark 6.15 and that $\Omega(\phi) = \Omega(s_1) \star \Omega(s_2)$ from Lemma 6.16. Since we may reorder the s_i without loss of generality, we may assume that

$$\big(\tau(s_1), \tau(s_2)\big) \in \{(1,3),(2,3),(3,3),(2,2)\}.$$

The arguments in each case are similar, but for clarity we will treat each one separately. To ease the notation we let $k = k_1$, $f = f(s_1)$ (if $\tau(s_1) \neq 1$), $l = k_2$ and $e = f(s_2)$.

If $\big(\tau(s_1), \tau(s_2)\big) = (1,3)$: we have that

$$\Omega(\phi) = \begin{cases} \mathcal{P}(1) \star \mathcal{B}_{p^l}(p^l - 1) & \text{if } k = 0,\\ (\mathcal{P}(p^k) \setminus \{(p^k-1,1)\}^\circ) \star \mathcal{B}_{p^l}(p^l-1) & \text{otherwise,}\end{cases}$$

which equals $\mathcal{B}_n\big(N(\phi)\big)$ in each instance by Proposition 5.7 and Lemma 5.8 respectively, as $N(\phi) = p^k + p^l - 1$.

If $\big(\tau(s_1), \tau(s_2)\big) = (2,3)$: we have $\mathcal{B}_{p^k}(m(s_1)) \subseteq \Omega(s_1) \subseteq \mathcal{B}_{p^k}(m(s_1)+1)$, and hence

$$\mathcal{B}_n\big(N(\phi)-1\big) \subseteq \Omega(\phi) \subseteq \mathcal{B}_n\big(N(\phi)\big)$$

since $N(\phi) = m(s_1) + 1 + m(s_2) = p^k - p^{k-f} + p^l - 1$. Let $\lambda = (N(\phi), \mu)$ where μ is any partition of $p^{k-f}+1$. Since $p^{k-f}+1$ is not a power of p, $\Omega(\mathbb{1}_{P_{p^{k-f}+1}}) = \mathcal{P}(p^{k-f}+1)$ by Theorem 5.1. But $\mathbb{1}_{P_{p^{k-f}+1}} = \mathbb{1}_{P_{p^{k-f}}} \times \mathbb{1}_{P_1}$, so $\mu \in \Omega(\mathbb{1}_{P_{p^{k-f}+1}}) = \Omega(\mathbb{1}_{P_{p^{k-f}}}) \star \Omega(\mathbb{1}_{P_1})$. That is, there exists $\nu \in \Omega(\mathbb{1}_{P_{p^{k-f}}})$ such that $c^\mu_{\nu,(1)} > 0$. Then by Lemma 2.11,

$$c^{(N(\phi),\mu)}_{(p^k-p^{k-f},\nu),(p^l-1,1)} = c^\mu_{\nu,(1)} > 0,$$

and thus $\lambda \in \Omega(s_1) \star \Omega(s_2) = \Omega(\phi)$. Since $\Omega(\phi)^\circ = \Omega(\phi)$, we have that $\Omega(\phi) = \mathcal{B}_n\big(N(\phi)\big)$.

If $\big(\tau(s_1), \tau(s_2)\big) = (3,3)$: then $\Omega(\phi) = \mathcal{B}_{p^k}(p^k-1) \star \mathcal{B}_{p^l}(p^l-1) = \mathcal{B}_n\big(N(\phi)\big)$, by Proposition 5.7.

If $\big(\tau(s_1), \tau(s_2)\big) = (2,2)$: then clearly $\mathcal{B}_n\big(N(\phi)-2\big) = \mathcal{B}_{p^k}(m(s_1)) \star \mathcal{B}_{p^l}(m(s_2)) \subseteq \Omega(\phi)$ and $\Omega(\phi) \subseteq \mathcal{B}_{p^k}(m(s_1)+1) \star \mathcal{B}_{p^l}(m(s_2)+1) = \mathcal{B}_n\big(N(\phi)\big)$, by Proposition 5.7. In order to show that $\mathcal{B}_n\big(N(\phi)\big) = \Omega(\phi)$, since $\Omega(\phi)^\circ = \Omega(\phi)$ it remains to show that

$$\lambda = (N(\phi) - 2 + j, \mu) \ \in \ \Omega(\phi) \qquad \forall\ j \in \{1,2\}\ \forall\ \mu \vdash p^{k-f} + p^{l-e} + 2 - j.$$

Fix some $\mu \vdash p^{k-f} + p^{l-e} + 2 - j$ and consider $\lambda = (N(\phi) - 2 + j, \mu)$. Clearly $|\mu|$ is not a power of p, so

$$\mu \in \mathcal{P}(|\mu|) = \Omega(\mathbb{1}_{P_{|\mu|}}) = \Omega(\mathbb{1}_{P_{p^{k-f}}}) \star \Omega(\mathbb{1}_{P_{p^{l-e}+2-j}}).$$

That is, there exist $\nu \in \Omega(\mathbb{1}_{P_{p^{k-f}}})$ and $\omega \in \Omega(\mathbb{1}_{P_{p^{l-e}+2-j}})$ such that $c^{\mu}_{\nu,\omega} > 0$. Then

$$c^{(N(\phi)-2+j,\mu)}_{(p^k-p^{k-f},\nu),(p^l-p^{l-e}-2+j,\omega)} = c^{\mu}_{\nu,\omega} > 0$$

by Lemma 2.11, and thus $\lambda \in \Omega(\phi)$. Hence $\mathcal{B}_n\big(N(\phi)\big) = \Omega(\phi)$ in all cases when $R = 2$.

Now for the inductive step: let $R \geq 3$ and suppose that the statement of the theorem holds for $R-1$. Since $s_1, \ldots, s_R$ do not satisfy the hypothesis of Lemma 6.23, then there exists $i \in [R]$ such that $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_R$ also do not satisfy the hypothesis of Lemma 6.23. Without loss of generality, let $i = 1$. Let $k = k_1$, $s = s_1$, $f = f(s_1)$ (if $\tau(s_1) \neq 1$) and let $\psi \in \mathrm{Lin}(P_{n-p^k})$ be such that $\phi = \phi(s) \times \psi$. Then $\Omega(\phi) = \Omega(s) \star \Omega(\psi)$ and $\Omega(\psi) = \mathcal{B}_{n-p^k}\big(N(\psi)\big)$, by the inductive hypothesis. In order to show that $\Omega(\phi) = \mathcal{B}_n\big(N(\phi)\big)$, we split into cases depending on $\tau(s) \in \{1,2,3\}$.

If $\tau(s) = 1$: then

$$\Omega(\phi) = \begin{cases} \mathcal{P}(1) \star \mathcal{B}_{n-p^k}\big(N(\psi)\big) & \text{if } k = 0, \\ (\mathcal{P}(p^k) \setminus \{(p^k-1,1)\}^\circ) \star \mathcal{B}_{n-p^k}\big(N(\psi)\big) & \text{otherwise,} \end{cases}$$

which equals $\mathcal{B}_n\big(N(\phi)\big)$ in each instance by Proposition 5.7 and Lemma 5.8 respectively, as $N(\phi) = p^k + N(\psi)$.

If $\tau(s) = 2$: then

$$\mathcal{B}_n\big(N(\phi) - 1\big) \subseteq \Omega(\phi) \subseteq \mathcal{B}_n\big(N(\phi)\big),$$

where $N(\phi) = m(s) + 1 + N(\psi) = p^k - p^{k-f} + N(\psi)$. Since $\Omega(\phi)^\circ = \Omega(\phi)$, it suffices to show that

$$\lambda = (N(\phi), \mu) \ \in \ \Omega(\phi) \qquad \forall\ \mu \vdash n - N(\phi) = n - p^k + p^{k-f} - N(\psi).$$

Fix such a partition λ. Notice that

$$\mu \in \mathcal{P}\big(n - p^k + p^{k-f} - N(\psi)\big) = \big(\mathcal{P}(p^{k-f}) \setminus \{(p^{k-f}-1,1)\}^\circ\big) \star \mathcal{P}(n - p^k - N(\psi))$$

by Lemma 5.8 since $p^{k-f} \geq p \geq 5$. (Note $\{\tau(s_j) \mid j \geq 2\} \neq \{1\}$, so $N(\psi) \lneqq n - p^k$.) Thus there exist $\nu \in \mathcal{P}(p^{k-f}) \setminus \{(p^{k-f}-1,1)\}^\circ$ and $\omega \in \mathcal{P}(n - p^k - N(\psi))$ such that

$c^{\mu}_{\nu,\omega} > 0$. This shows by Lemma 2.11 that

$$c^{(N(\phi),\mu)}_{(p^k-p^{k-f},\nu),(N(\psi),\omega)} = c^{\mu}_{\nu,\omega} > 0,$$

and so $\lambda \in \Omega(s) \star \mathcal{B}_n\big(N(\psi)\big) = \Omega(\phi)$ as required.

If $\tau(s) = 3$: then $\Omega(\phi) = \mathcal{B}_{p^k}(p^k - 1) \star \mathcal{B}_{n-p^k}\big(N(\psi)\big) = \mathcal{B}_n\big(N(\phi)\big)$ by Proposition 5.7.

Hence $\Omega(\phi) = \mathcal{B}_n\big(N(\phi)\big)$ in all cases. □

Lemma 6.24. *For $i \in \{1, 2\}$, let $n_i, m_i \in \mathbb{N}$ be such that $\frac{n_i}{2} < m_i \leq n_i$. Furthermore, let $\Delta_i \subseteq \mathcal{P}(n_i)$ be such that $\mathcal{B}_{n_i}(m_i) \subseteq \Delta_i$ and $\Delta_i \setminus \mathcal{B}_{n_i}(m_i)$ contains no thin partitions. Then*

$$\mathcal{B}_{n_1+n_2}(m_1 + m_2) \subseteq \Delta_1 \star \Delta_2$$

and $(\Delta_1 \star \Delta_2) \setminus \mathcal{B}_{n_1+n_2}(m_1 + m_2)$ contains no thin partitions.

Proof. By Proposition 5.7, we know that $\mathcal{B}_{n_1+n_2}(m_1 + m_2) \subseteq \Delta_1 \star \Delta_2$.

First, suppose $\lambda \in (\Delta_1 \star \Delta_2) \setminus \mathcal{B}_{n_1+n_2}(m_1+m_2)$ satisfies $l(\lambda) \leq 2$. Then $\lambda_1 > m_1+m_2$. But $\lambda \in \Delta_1 \star \Delta_2$ implies that $c^{\lambda}_{\mu,\nu} > 0$ for some $\mu \in \Delta_1$ and $\nu \in \Delta_2$. Thus $\mu_1 + \nu_1 \geq \lambda_1$, giving either $\mu_1 > m_1$ or $\nu_1 > m_2$. However, $\mu, \nu \subseteq \lambda$ so $l(\mu), l(\nu) \leq l(\lambda) \leq 2$. That is, both μ and ν are thin but either $\mu \in \Delta_1 \setminus \mathcal{B}_{n_1}(m_1)$ or $\nu \in \Delta_2 \setminus \mathcal{B}_{n_2}(m_2)$, a contradiction.

Next, suppose $\lambda \in (\Delta_1 \star \Delta_2) \setminus \mathcal{B}_{n_1+n_2}(m_1+m_2)$ satisfies $\lambda_1 \leq 2$. Then $l(\lambda) > m_1+m_2$. But then similarly we find that $c^{\lambda}_{\mu,\nu} > 0$ for some $\mu \in \Delta_1$ and $\nu \in \Delta_2$, meaning $l(\lambda) \leq l(\mu) + l(\nu)$, but $\mu, \nu \subseteq \lambda$ are also thin. Thus we obtain a contradiction.

Finally if $\lambda \in (\Delta_1 \star \Delta_2) \setminus \mathcal{B}_{n_1+n_2}(m_1 + m_2)$ is a hook, then either $\lambda_1 > m_1 + m_2$ or $l(\lambda) > m_1 + m_2$. But any $\mu, \nu \subseteq \lambda$ must again be hooks, so we obtain a contradiction by a similar argument to the above. □

We are now ready to prove Theorem 6.19.

Proof of Theorem 6.19. We show that $\mathcal{B}_n\big(N(\phi)\big) \subseteq \Omega(\phi)$ and that $\Omega(\phi) \setminus \mathcal{B}_n\big(N(\phi)\big)$ contains no thin partitions, from which we also deduce that $m(\phi) = N(\phi)$.

We proceed by induction on R, beginning with the base case $R = 2$. Without loss of generality we may assume that $\tau(s_2) = 4$. Let $k = k_1$, $f = f(s_1)$ (if $\tau(s_1) \neq 1$) and let $l = k_2$. By Lemma 6.16, we know that $\Omega(\phi) = \Omega(s_1) \star \Omega(s_2)$, and recall from Remark 6.15 that $m(s_2) = \eta(s_2)$ and $\Omega(s_2) \setminus \mathcal{B}_{p^l}(\eta(s_2))$ contains no thin partitions. We split into cases according to $\tau(s_1) \in \{1, 2, 3, 4\}$.

(i) If $\tau(s_1) = 1$: we have that $N(\phi) = p^k + m(s_2)$. If $k = 0$, then

$$\Omega(\phi) = \mathcal{P}(1) \star \Omega(s_2) \supseteq \mathcal{P}(1) \star \mathcal{B}_{p^l}(m(s_2)) = \mathcal{B}_n\big(N(\phi)\big)$$

by Proposition 5.7 and $\Omega(\phi)$ contains no thin partitions, by Lemma 6.24. Otherwise if $k \geq 1$, then

$$\Omega(\phi) \supseteq \big(\mathcal{P}(p^k) \setminus \{(p^k - 1, 1)\}^{\circ}\big) \star \mathcal{B}_{p^l}(m(s_2)) = \mathcal{B}_n\big(N(\phi)\big)$$

by Lemma 5.8. Suppose $\lambda \in \Omega(\phi) \setminus \mathcal{B}_n\big(N(\phi)\big)$ satisfies $l(\lambda) \leq 2$, so $\lambda_1 > N(\phi) = p^k + m(s_2)$. Then $c^\lambda_{\mu,\nu} > 0$ for some $\mu \in \Omega(s_1)$ and $\nu \in \Omega(s_2)$, and $\lambda_1 \leq \mu_1 + \nu_1$. But $\mu_1 \leq p^k$, so $\nu_1 > m(s_2)$. However, $\nu \subseteq \lambda$ so $l(\nu) \leq 2$, contradicting $\nu \in \Omega(s_2) \setminus \mathcal{B}_{p^l}(m(s_2))$. Also there cannot be any $\lambda \in \Omega(\phi) \setminus \mathcal{B}_n\big(N(\phi)\big)$ such that $\lambda_1 \leq 2$, since $\Omega(\phi)$ and $\mathcal{B}_n\big(N(\phi)\big)$ are both closed under conjugation. A similar argument shows that there are no hooks in $\Omega(\phi) \setminus \mathcal{B}_n\big(N(\phi)\big)$.

(ii) If $\tau(s_1) = 2$**:** then by Definition 6.14 and Remark 6.15 we have that

$$\Omega(\phi) \supseteq \mathcal{B}_{p^k}(p^k - p^{k-f} - 1) \star \mathcal{B}_{p^l}(m(s_2)) = \mathcal{B}_n\big(N(\phi) - 1\big).$$

Let $\lambda = (N(\phi), \mu)$ where $\mu \vdash n - N(\phi)$. Then $\mu \in \mathcal{P}(n - N(\phi)) = \Omega(\mathbb{1}_{P_{p^{k-f}}}) \star \mathcal{P}(p^l - m(s_2))$ by Lemma 5.8, so

$$c^\lambda_{(p^k - p^{k-f}, \nu), (m(s_2), \omega)} = c^\mu_{\nu,\omega} > 0$$

for some $\nu \in \Omega(\mathbb{1}_{P_{p^{k-f}}})$ and $\omega \vdash p^l - m(s_2)$, by Lemma 2.11. Thus $\lambda \in \Omega(s_1) \star \Omega(s_2) = \Omega(\phi)$, and hence $\mathcal{B}_n\big(N(\phi)\big) \subseteq \Omega(\phi)$ since $\Omega(\phi)^\circ = \Omega(\phi)$. If $\lambda \in \Omega(\phi) \setminus \mathcal{B}_n\big(N(\phi)\big)$ satisfies $l(\lambda) \leq 2$, then $c^\lambda_{\mu,\nu} > 0$ for some $\mu \in \Omega(s_1)$ and $\nu \in \Omega(s_2)$ but $\mu_1 \leq p^k - p^{k-f}$ implies that $v_1 > m(s_2)$, as $\lambda_1 > N(\phi) = p^k - p^{k-f} + m(s_2)$. But then $\nu \in \Omega(s_2) \setminus \mathcal{B}_{p^l}(m(s_2))$ and $l(\nu) \leq 2$, a contradiction. A similar argument shows that $\Omega(\phi) \setminus \mathcal{B}_n\big(N(\phi)\big)$ contains no other thin partitions.

(iii) If $\tau(s_1) \in \{3, 4\}$**:** then the assertions follow from Proposition 5.7 and Lemma 6.24.

Finally, we turn to the inductive step. Assume $R \geq 3$ and that the statement of the theorem holds for $R - 1$. Let $k = k_1$, and let $\psi \in \mathrm{Lin}(P_{n-p^k})$ be such that $\phi = \phi(s_1) \times \psi$, so ψ corresponds to $s_2, \ldots, s_R$ and $\Omega(\phi) = \Omega(s_1) \star \Omega(\psi)$. We distinguish two cases, depending on the validity of the following equation:

$$|\{i \in [R] \mid \tau(s_i) = j\}| = \begin{cases} R - 2 & \text{if } j = 1, \\ 1 & \text{if } j \in \{2, 4\}, \\ 0 & \text{if } j = 3. \end{cases} \tag{6.3}$$

First suppose that (6.3) holds. Since $R \geq 3$, we may without loss of generality assume that $\tau(s_1) = 1$. By the inductive hypothesis, $m(\psi) = N(\psi)$ and $\Omega(\psi) \setminus \mathcal{B}_{n-p^k}\big(N(\psi)\big)$ contains no thin partitions. Then $\mathcal{B}_n\big(N(\phi)\big) \subseteq \Omega(\phi)$ and that $\Omega(\phi) \setminus \mathcal{B}_n\big(N(\phi)\big)$ contains no thin partitions follows from a similar argument to case (i) above.

Otherwise, suppose that (6.3) does not hold. In this case we may without loss of generality assume that $\tau(s_1) = 4$. If $|\{i \in \{2, 3, \ldots, R\} \mid \tau(s_i) = 4\}| = 0$ then the first part of Theorem 6.20 gives us that $\Omega(\psi) = \mathcal{B}_{n-p^k}\big(N(\psi)\big)$. The required results then follow from Proposition 5.7 and Lemma 6.24. On the other hand, if $|\{i \in \{2, 3, \ldots, R\} \mid \tau(s_i) = 4\}| > 0$, then by the inductive hypothesis we have that $m(\psi) = N(\psi)$ and $\Omega(\psi) \setminus \mathcal{B}_{n-p^k}\big(N(\psi)\big)$ contains no thin partitions. The required results then also follow from Proposition 5.7 and Lemma 6.24. □

We conclude with an example illustrating the main theorems of this chapter.

Example 6.25. Let $p = 5$. We consider (i) $n = 25$, (ii) $n = 125$ and (iii) $n = 175$.

(i) $n = 25$**.** We describe $\Omega(\phi)$ completely for all $\phi \in \mathrm{Lin}(P_{25})$. This is summarised in Table 6.1 below.

By Theorem 4.1 and Lemma 4.3, it suffices to consider ϕ running over a set of orbit representatives for the conjugacy action of $N_{\mathfrak{S}_{25}}(P_{25})$ on $\mathrm{Lin}(P_{25})$. That is, we need only consider $\phi = \phi(s)$ for $s \in \{(0,0),(0,*),(*,0),(*,*)\}$, where each $*$ represents any element of $\{1,2,\ldots,p-1\}$. Let $\mathcal{P}'(m) := \mathcal{P}(m) \setminus \{(m-1,1)\}^\circ = \mathcal{B}_m(m-2) \sqcup \{(m)\}^\circ$ for $m \in \mathbb{N}_{\geq 5}$.

s	type $\tau(s)$	$f(s)$	$m(s)$	$M(s)$	$\Omega(s)$
$(0,0)$	1	n/a	23	25	$\mathcal{P}'(25)$
$(0,*)$	3	2	24	24	$\mathcal{B}_{25}(24)$
$(*,0)$	2	1	19	20	$\mathcal{B}_{25}(19) \sqcup \{(20,\mu) \mid \mu \in \mathcal{P}'(5)\}^\circ$
$(*,*)$	4	1	19	20	$\mathcal{B}_{25}(19) \sqcup \{(20,\mu) \mid \mu \in \mathcal{B}_5(4)\}^\circ$

Table 6.1: Data on $\Omega(\phi)$ for $\phi = \phi(s) \in \mathrm{Lin}(P_{25})$.

The case of $\tau(s) = 1$ follows from Theorem 5.1. For $\tau(s) \neq 1$, a precise description of $\Omega(s)$ is given in Lemma 6.4 (for $\tau(s) = 3$) and Lemma 6.12 (for $\tau(s) \in \{2,4\}$). We can similarly describe $\Omega(\phi)$ explicitly for all $\phi \in \mathrm{Lin}(P_{p^2})$, for all primes $p \geq 5$.

(ii) $n = 125$**.** There are 8 orbits under the action of $N_{\mathfrak{S}_{125}}(P_{125})$ on $\mathrm{Lin}(P_{125})$. Representatives $\phi = \phi(s)$ and their corresponding $\Omega(s)$ are summarised in Table 6.2 below.

s	$\tau(s)$	$f(s)$	$g(s)$	$\eta(s)$	$m(s)$	$M(s)$	$\Omega(s)$
$(0,0,0)$	1	n/a	n/a	n/a	123	125	$\mathcal{P}'(125)$
$(0,0,*)$	3	3	n/a	n/a	124	124	$\mathcal{B}_{125}(124)$
$(0,*,0)$	2	2	n/a	n/a	119	120	$\mathcal{B}_{125}(119) \sqcup \{(120,\mu) \mid \mu \in \mathcal{P}'(5)\}^\circ$
$(0,*,*)$	4	2	3	119	119	120	$\mathcal{B}_{125}(119) \sqcup \{(120,\mu) \mid \mu \in \mathcal{B}_5(4)\}^\circ$
$(*,0,0)$	2	1	n/a	n/a	99	100	$\mathcal{B}_{125}(99) \sqcup \{(100,\mu) \mid \mu \in \mathcal{P}'(25)\}^\circ$
$(*,0,*)$	4	1	3	99	99	100	$\mathcal{B}_{125}(99) \sqcup \{(100,\mu) \mid \mu \in \mathcal{B}_{25}(24)\}^\circ$
$(*,*,0)$	4	1	2	95	95	100	(see below)
$(*,*,*)$	4	1	2	95	95	100	(see below)

Table 6.2: Data on $\Omega(\phi)$ for $\phi = \phi(s) \in \mathrm{Lin}(P_{125})$.

Recall from Remark 6.15 that we know $\Omega(s)$ exactly whenever $\tau(s) \neq 4$. We are able to determine $\Omega(s)$ completely for $s = (0,*,*)$ and $s = (*,0,*)$ even though $\tau(s) = 4$ because $M(s) = m(s) + 1$ in these cases, so the result follows from Theorem 6.10(iii).

In the remaining instances when $\tau(s) = 4$, i.e. for $s = (*,*,0)$ and $s = (*,*,*)$, then $\mathcal{B}_{125}(95) \subseteq \Omega(s) \subseteq \mathcal{B}_{125}(100)$ and $\Omega(s) \setminus \mathcal{B}_{125}(95)$ contains no thin partitions, by

Theorem 6.10. (In other words, $\Omega(s)$ does not contain $(95+i, 30-i)$, $(95+i, 1^{30-i})$ or their conjugates for any $i \in [5]$.) Moreover,

$$\Omega(*, *, x) \cap \{(100, \mu) \mid \mu \vdash 25\}^\circ = \{(100, \mu) \mid \mu \in \Omega(*, x)\}^\circ$$

for all $x \in \{0, 1, \ldots, 4\}$ where $\Omega(*, x)$ has already been determined in (i) above.

(iii) $n = 175$**.** There are 80 orbits under the action of $N_{\mathfrak{S}_{175}}(P_{175})$ on $\mathrm{Lin}(P_{175})$: a set of orbit representatives $\phi(s)$ is given by $s = (s_1, s_2, s_3)$ where $s_1 \in \overline{[5]}^3$ (i.e. $s_1 \in \{0, *\}^3$, giving 8 choices) and $s_2, s_3 \in \overline{[5]}^2$ (i.e. $s_2, s_3 \in \{0, *\}^2$) with $\tau(s_2) \leq \tau(s_3)$ (giving 10 choices for the pair s_2, s_3).

By Theorem 6.20, the set $\Omega(s)$ is determined completely whenever $\tau(s_i) \neq 4$ for all $i \in [3]$. This comprises 24 of the 80 representatives s. Of the 56 remaining s, we can actually determine $\Omega(s)$ fully without further computation in 36 of the cases by using Lemma 6.16, because we know $\Omega(*, *)$, $\Omega(0, *, *)$ and $\Omega(*, 0, *)$ exactly from (i) and (ii) above. In the remaining 20 cases corresponding to $s_1 = (*, *, 0)$ or $s_1 = (*, *, *)$, Theorems 6.17 and 6.19 give sharp bounds $\mathcal{B}_{175}(N(\phi)) \subseteq \Omega(\phi) \subseteq \mathcal{B}_{175}(M(\phi))$, and $\Omega(\phi) \setminus \mathcal{B}_{175}(N(\phi))$ contains no thin partitions.

To give some examples, we list the exact descriptions of $\Omega(s)$ when $s_1 = (0, 0, 0)$ in Table 6.3 below. ♢

s_2, s_3	$\tau(s_i)_{i=1,2,3}$	$N(s_i)_{i=1,2,3}; N(\phi)$	$\Omega(s)$
$(0,0),(0,0)$	$1,1,1$	$125,25,25;175$	$\mathcal{P}(175)$
$(0,0),(*,0)$	$1,1,2$	$125,25,20;170$	$\mathcal{B}_{175}(169) \sqcup \{(170,\mu) \mid \mu \in \mathcal{P}'(5)\}^\circ$
$(0,0),(0,*)$	$1,1,3$	$125,25,24;174$	$\mathcal{B}_{175}(174)$
$(*,0),(*,0)$	$1,2,2$	$125,20,20;165$	$\mathcal{B}_{175}(165)$
$(*,0),(0,*)$	$1,2,3$	$125,20,24;169$	$\mathcal{B}_{175}(169)$
$(0,*),(0,*)$	$1,3,3$	$125,24,24;173$	$\mathcal{B}_{175}(173)$
$(0,0),(*,*)$	$1,1,4$	$125,25,19;169$	$\mathcal{B}_{175}(169) \sqcup \{(170,\mu) \mid \mu \in \mathcal{B}_5(4)\}^\circ$
$(*,0),(*,*)$	$1,2,4$	$125,20,19;164$	$\mathcal{B}_{175}(164) \sqcup \{(165,\mu) \mid \mu \in \mathcal{B}_{10}(9)\}^\circ$
$(0,*),(*,*)$	$1,3,4$	$125,24,19;168$	$\mathcal{B}_{175}(168) \sqcup \{(169,\mu) \mid \mu \in \mathcal{B}_6(5)\}^\circ$
$(*,*),(*,*)$	$1,4,4$	$125,19,19;163$	$\mathcal{B}'$ (defined below)

$$\mathcal{B}' := \mathcal{B}_{175}(163) \sqcup \{(164,\mu) \mid \mu \in \mathcal{B}_{11}(10)\}^\circ \sqcup \{(165,\nu) \mid \nu \in \mathcal{B}_{10}(8)\}^\circ.$$

Table 6.3: Data on $\Omega(\phi)$ for $\phi = \phi(s_1, s_2, s_3) \in \mathrm{Lin}(P_{175})$ with $s_1 = (0,0,0)$. This follows from Theorem 6.20 when $\tau(s_i) \neq 4$ for all i, and Lemma 6.16 otherwise.

Chapter 7

Ringel duality for Schur algebras

In [21], Erdmann and Henke determined the values of r for which the classical Schur algebra $S(2, r)$ is Ringel self-dual by constructing explicit Morita equivalences, for example from maps between tilting and projective modules, and considering Cartan numbers. In [13], Donkin showed that $S(n, r)$ is always Ringel self-dual for $n \geq r$ via direct calculations of certain exterior algebras which are $S(n, r)$–$S(n, r)$–bimodules. This holds in the general quantized case $S_q(n, r)$ in fact; the classical Schur algebras are exactly those where $q = 1$.

In this chapter, we determine which classical Schur algebras $S(n, r)$ are Ringel self-dual in the remaining open cases $3 \leq n < r$. Section 7.2.3 follows from work done in collaboration with Dr Karin Erdmann.

Throughout this chapter, all modules are finite-dimensional left modules and all algebras finite-dimensional unless otherwise stated. Fix K to be an algebraically closed field of characteristic $p \geq 0$ (and all fields to which we refer will be algebraically closed). We begin with the classical Schur algebras $S(n, r)$ for natural numbers n and r. After setting up the necessary notation, we present some straightforward combinatorial results on order-reversing isomorphisms of certain partition posets in Section 7.1.2. These are used to reduce the classification problem to a small number of cases, which are then considered in Sections 7.2.2 and 7.2.3 using block theory, Δ–filtrations of tilting modules and decomposition numbers for symmetric groups. The classification of Ringel self-dual Schur algebras when $3 \leq n < r$ is completed in Theorem 7.18. On a related note, it was remarked by Erdmann that if B is a block of a Schur algebra and B has finite representation type, then B is Ringel self-dual. For convenience, we record a proof of this fact in Section 7.2.4.

7.1 Combinatorial setup

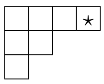
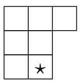
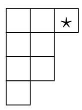
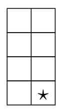
7.1.1 Notation

We say λ is a *maximal element* of a partially ordered set $(\Lambda, \leq)$ if there is no $\mu \in \Lambda$ such that $\mu > \lambda$. Throughout this chapter we will consider only finite Λ, and in this case $(\Lambda, \leq)$ has a *unique* maximal element λ if and only if $\lambda \geq \mu$ for all $\mu \in \Lambda$. Minimal elements are similarly defined and there is a corresponding characterisation of unique minimal elements. We denote the opposite (partial) ordering of $\leq$ by $\leq^{\mathrm{op}}$, so that $\lambda \leq^{\mathrm{op}} \mu$ if and only if $\mu \leq \lambda$.

For $\lambda, \mu \in \Lambda$, we say λ *covers* μ, written $\lambda \to \mu$, if $\lambda > \mu$ and there does not exist $\nu \in \Lambda$ such that $\lambda > \nu > \mu$. The Hasse diagram of $(\Lambda, \leq)$ is a graph with vertex set Λ and a directed edge from λ to μ if and only if $\lambda \to \mu$.

For $n, r \in \mathbb{N}$, let $\Lambda^+(n, r)$ be the set of partitions of r into at most n parts, partially ordered by the dominance order $\trianglelefteq$. Let $S(n, r) = S_K(n, r)$ denote the *Schur algebra* over K. Its module category $\operatorname{mod} S_K(n, r)$ is equivalent to the category $M_K(n, r)$ of homogeneous polynomial representations of degree r of the general linear group $\mathrm{GL}_n(K)$; we refer the reader to [33] for a detailed construction. $S(n, r)$ is a quasi-hereditary algebra with respect to $(\Lambda^+(n, r), \trianglelefteq)$ (see Section 7.2). Let $H(n, r)$ denote the Hasse diagram of $(\Lambda^+(n, r), \trianglelefteq)$.

Let λ be a partition and consider its Young diagram $[\lambda]$. Define a *one-box move from* λ to be the removal of one removable box of $[\lambda]$ and its addition to an addable position in a strictly lower row in $[\lambda]$ such that the result is again a partition. In particular, the addable position in question cannot be directly below (in the same column as) the removable box, although it would be in a strictly lower row. For instance, the only one-box move from $\lambda = (2, 1)$ results in the partition (1^3). We say there is a *one-box move from* λ *to* μ and write $\lambda \rightsquigarrow \mu$ if μ is the result of performing a one-box move from λ. Clearly if $\lambda \rightsquigarrow \mu$ then $\lambda \rhd \mu$. In the examples below, the 'moved' box is indicated with a $\star$:

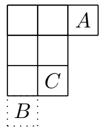
(a) $\lambda =$  $\rightsquigarrow$ $\mu =$  , (b) $\lambda =$  $\rightsquigarrow$ $\mu =$ 

We say a one-box move from λ to μ is *minimal* if, letting $A = \lambda \setminus \mu$ denote the removable box of λ and $B = \mu \setminus \lambda$ denote the addable position to which A is moved:

(i) A is the lowest removable box of λ in a row above B that could have been moved down into B; *and*

(ii) B is the highest addable position in λ in a row below A down to which the box A may be moved.

Here, the height of a box or position refers to its row in the Young diagram. Note that (i) and (ii) are not equivalent: in the pictured example $\lambda = (3, 2, 2)$, B is indeed



the highest addable position below A satisfying (ii), but C is the lowest removable box above B satisfying (i) and $A \neq C$.

To abbreviate, we also call such minimal one-box moves *minimal moves*, and write $\lambda \overset{\min}{\rightsquigarrow} \mu$. In the two examples (a) and (b) above, (a) does not depict a minimal move, while (b) does. Let $G(n,r)$ be the *minimal move graph* of $\Lambda^+(n,r)$: it is the directed graph with vertex set $\Lambda^+(n,r)$, and there is a directed edge from λ to μ if and only if $\lambda \overset{\min}{\rightsquigarrow} \mu$.

For a directed graph G, let $E(G)$ denote the set of edges of G, and let the *degree* of a vertex v be its total degree, i.e. the number of edges into v plus the number of edges out of v. This denoted $\deg_G(v)$, or $\deg(v)$ when the graph is understood.

7.1.2 Reversibility

For the notion of a Ringel dual of a quasi-hereditary algebra (described in the next section), and in particular those of the Schur algebras, it will be useful to consider order-reversing isomorphisms on $\Lambda^+(n,r)$.

By an order-reversing isomorphism on a poset $(\Lambda, \leq)$ we mean an order isomorphism, or bijection of sets preserving the order relation, between $(\Lambda, \leq)$ and $(\Lambda, \leq^{\mathrm{op}})$. Since $\Lambda^+(2,r)$ is totally ordered by dominance, there is a unique order-reversing isomorphism on $(\Lambda^+(2,r), \trianglelefteq)$. When $n \geq r$, the poset $\Lambda^+(n,r)$ is simply the set of all partitions of r and $(\Lambda^+(n,r), \trianglelefteq)$ has a natural order-reversing isomorphism given by mapping a partition to its conjugate.

Proposition 7.1. *Let $3 \leq n < r$ be natural numbers and let $\Lambda = \Lambda^+(n,r)$. Then there exists an order isomorphism between $(\Lambda, \trianglelefteq)$ and $(\Lambda, \trianglelefteq^{\mathrm{op}})$ if and only if $(n,r) \in \{(3,4),(3,5),(3,7),(3,8),(4,5)\}$.*

This section is devoted to the proof of Proposition 7.1. First, we record the structure of the Hasse diagram $H(n,r)$ around the unique maximal element (r). We show that $\Lambda^+(n,r)$ has a unique minimal element α in Lemma 7.2, and conclude the desired result by describing the vertices at small distance from α in $H(n,r)$ in Propositions 7.5 and 7.7. In fact, we are able to give a combinatorial characterisation of the edges in $H(n,r)$: this is done in Propositions 7.3 and 7.4, giving a quick method to verify the structures described in Propositions 7.5 and 7.7.

Let $3 \leq n < r$. We begin with the structure of $H(n,r)$ around (r). Clearly

- (r) is the unique maximal element of $(\Lambda, \trianglelefteq)$,
- $(r-1,1)$ is the unique maximal element of $(\Lambda \setminus \{(r)\}, \trianglelefteq)$, and
- $(r-2,2)$ is the unique maximal element of $(\Lambda \setminus \{(r),(r-1,1)\}, \trianglelefteq)$.

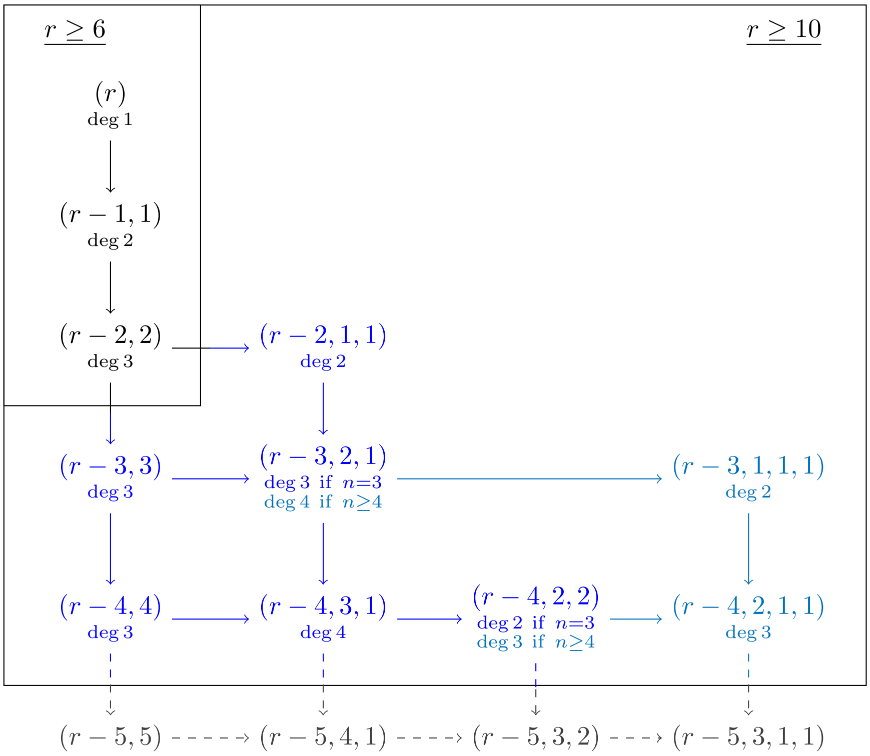


Figure 7.1: Vertices in $H(n,r)$ at small distance from (r) and their degrees, for $3 \le n < r$.

Hence $\deg_{H(n,r)}((r)) = 1$ and $\deg_{H(n,r)}((r-1,1)) = 2$. Also, $(r-2,1,1) \in \Lambda^+(n,r)$ and when $r \ge 6$, $(r-3,3) \in \Lambda^+(n,r)$, and these two are the only maximal elements of $(\Lambda \setminus \{(r),(r-1,1),(r-2,2)\}, \trianglelefteq)$. Continuing this analysis, we obtain Figure 7.1, which illustrates the vertices of $H(n,r)$ at small distance from (r), with their degrees as stated for the values of r indicated (noting that the rightmost column only exists for $n \ge 4$).

Lemma 7.2. *Let $3 \le n < r$. Write $r = nk + l$, where $l \in \{0, 1, \dots, n-1\}$ and $k \in \mathbb{N}$. Define $\alpha = (k+1, \dots, k+1, k \dots, k) = \big((k+1)^l, k^{n-l}\big) \vdash r$. Then α is the unique minimal element of $(\Lambda^+(n,r), \le)$.*

Proof. First, suppose $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n,r)$ is such that $\alpha > \lambda$. Thus $\lambda_i \le k$ for some $i \le l$. If $i < l$, then $\sum_j \lambda_j \le (l-1)(k+1) + (n-l+1)k < r$, a contradiction. Next suppose there exists $\lambda \in \Lambda^+(n,r)$ such that $\lambda \not\trianglerighteq \alpha$, so λ is incomparable to α. Clearly $\lambda_1 > \alpha_1$, and there exists $1 < m < n$ such that $\sum_{i=1}^m \lambda_i < \sum_{i=1}^m \alpha_i$ as λ and α are incomparable. Let m be minimal such that this holds, so $\lambda_m < \alpha_m$. Otherwise, $\sum_{i=1}^{m-1} \lambda_i \ge \sum_{i=1}^{m-1} \alpha_i$ and $\lambda_m \ge \alpha_m$ would give $\sum_{i=1}^m \lambda_i \ge \sum_{i=1}^m \alpha_i$, contradicting the choice of m. But then $\lambda_m < \alpha_m \le k+1$, so $\lambda_m \le k$. Then since $\lambda, \alpha \vdash r$,

$$(n-m)k \ge (n-m)\lambda_m \ge \sum_{i=m+1}^{n} \lambda_i > \sum_{i=m+1}^{n} \alpha_i \ge (n-m)k,$$

a contradiction. Thus $\lambda \trianglerighteq \alpha$ for all $\lambda \in \Lambda^+(n,r)$. $\square$

Let $\underline{G} = G(r)$, the minimal move graph of $(\underline{\Lambda}, \trianglelefteq)$ where $\underline{\Lambda} = \Lambda^+(r)$ is the set of all partitions of r. Also let $\underline{H} = H(r)$, the Hasse diagram of $(\underline{\Lambda}, \trianglelefteq)$. For $3 \leq n < r$, the directed graphs $G(n,r)$ and $H(n,r)$ are obtained from $\underline{G}$ and $\underline{H}$ respectively by removing any vertices λ with more than n parts and their incident edges.

Proposition 7.3. *Let $r \in \mathbb{N}$ and let $\lambda, \mu \vdash r$. If there is a minimal move $\lambda \overset{\min}{\rightsquigarrow} \mu$, then λ covers μ. That is, $E(\underline{G}) \subseteq E(\underline{H})$.*

Proof. Since $\lambda \rightsquigarrow \mu$, we have that

$$\lambda = (\mu_1, \mu_2, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \dots)$$

and

$$\mu = (\mu_1, \dots, \mu_{i-1}, \mu_i, \mu_{i+1}, \dots, \mu_{j-1}, \mu_j, \mu_{j+1}, \dots)$$

for some $1 \leq i < j$ with $\mu_{i-1} \geq \mu_i + 1$ and $\mu_j - 1 \geq \mu_{j+1}$. Let $A = \lambda \setminus \mu$, so A is the removable box of λ in row i, column $\mu_i + 1$, and let $B = \mu \setminus \lambda$, so B is the addable position of λ in row j, column μ_j.

Suppose $\nu \vdash r$ satisfies $\lambda \trianglerighteq \nu \trianglerighteq \mu$. Clearly $\nu_t = \mu_t$ for $1 \leq t < i$ and $j \leq t$, and $\nu_i = \mu_i$ or $\nu_i = \mu_i + 1$. If $j = i+1$, then $\nu = \mu$ or $\nu = \lambda$ respectively. Otherwise, $j \geq i+2$, so A and B are not in consecutive rows. We show that $\mu_i = \ldots = \mu_j$, whence $\nu_i = \mu_i$ or $\nu_i = \mu_i + 1$ implies $\nu = \mu$ or $\nu = \lambda$ respectively.

If $\mu_i > \mu_{i+1}$, then the highest addable position in λ in a row below A down to which A may be moved is $(i+1, \mu_{i+1}+1) \neq B$, contradicting $\lambda \overset{\min}{\rightsquigarrow} \mu$.

Suppose $\mu_i = \mu_{i+1} = \cdots = \mu_{l-1} > \mu_l \geq \cdots \geq \mu_j$ for some $i+2 \leq l \leq j$. Then the highest addable position in λ in a row below A down to which A may be moved is $(l, \mu_l + 1)$, so $\lambda \overset{\min}{\rightsquigarrow} \mu$ implies B lies in row l. Thus $l = j$, and $\mu_1 = \ldots = \mu_{j-1} > \mu_j$. On the other hand, the lowest removable box of λ in a row above B that could have been moved down into B is in row $j-1$, since $\mu_{j-1} \geq (\mu_j - 1) + 2$. Thus $\lambda \overset{\min}{\rightsquigarrow} \mu$ implies $i = j-1$, a contradiction. Hence there does not exist such l, and so $\mu_i = \ldots = \mu_j$ as claimed. $\square$

Proposition 7.4. *Let $r \in \mathbb{N}$ and $\lambda, \mu \vdash r$.*

(i) $\lambda \trianglerighteq \mu$ if and only if there exists a (possibly empty) sequence of one-box moves from λ down to μ; and

(ii) if $\lambda \rightsquigarrow \mu$ then there exists a sequence of minimal moves from λ down to μ.

Hence $E(\underline{H}) \subseteq E(\underline{G})$, and thus $\underline{G} = \underline{H}$. In particular, $G(n,r) = H(n,r)\ \forall\ n \in \mathbb{N}$.

Proof. (i) Since $\lambda \rightsquigarrow \mu \implies \lambda \triangleright \mu$, the if direction is clear. For the only if direction, suppose $\lambda \trianglerighteq \mu$. We proceed by induction on r, and then on $\lambda_1 - \mu_1$ for each fixed r. The result is clear for $r \leq 3$, and if $\lambda_1 - \mu_1 = 0$ then removing the first row of both λ and μ reduces to a smaller value of r. Now assume $\lambda_1 > \mu_1$: we exhibit a partition $\nu \vdash r$ such that $\nu \rightsquigarrow \mu$, $\lambda \trianglerighteq \nu$ and $\lambda_1 - \nu_1 < \lambda_1 - \mu_1$, whence we are done by the inductive hypothesis.

Let $e = \min\{m \mid \sum_{i=1}^m \lambda_i = \sum_{i=1}^m \mu_i\}$. By assumption, $e \geq 2$. Let the highest removable box of μ not in the first row be in row $j \geq 2$, so $\mu_1 \geq \mu_2 = \cdots = \mu_j > \mu_{j+1}$. If $j > e$, then $\mu_2 = \cdots = \mu_e = \mu_{e+1} =: c$, and

$$\sum_{i=1}^{e} \lambda_i = \sum_{i=1}^{e} \mu_i, \quad \sum_{i=1}^{e+1} \lambda_i \geq \sum_{i=1}^{e+1} \mu_i \quad \implies \quad \lambda_{e+1} \geq \mu_{e+1}.$$

Hence $\lambda_2 \geq \cdots \geq \lambda_e \geq \lambda_{e+1} \geq c$, and so $\sum_{i=1}^e \lambda_i \geq \lambda_1 + c(e-1) > \sum_{i=1}^e \mu_i$, a contradiction. Thus $j \leq e$.

Let $\nu = (\mu_1+1, \mu_2, \dots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \dots)$, so $\nu \rightsquigarrow \mu$. It suffices to show $\lambda \trianglerighteq \nu$. Since $\lambda_1 > \mu_1$, we have $\lambda_1 \geq \nu_1 = \mu_1 + 1$. Also

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m} \nu_i \quad \forall\, m \geq j$$

since $\lambda \trianglerighteq \mu$. Finally, since $j \leq e$,

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} \mu_i + 1 = \sum_{i=1}^{m} \nu_i \quad \forall\, 2 \leq m \leq j-1.$$

(ii) For a box or position C in a Young diagram, write $r(C)$ for the number of the row in which C lies. Let $A = \lambda \setminus \mu$ and $B = \mu \setminus \lambda$, so $r(A) < r(B)$. We proceed by induction on $r(B) - r(A)$: clearly if $r(B) - r(A) = 1$ then the one-box move of A down to B is minimal, and $\lambda \overset{\min}{\rightsquigarrow} \mu$.

Now suppose $r(B) - r(A) > 1$. If the move of A to B is minimal then we are done. Otherwise, either:

- the lowest removable box of λ in a row above B that could have been moved down into position B is some box $Y \neq A$, so $r(Y) > r(A)$; or
- the highest addable position of λ in a row below A down to which A could have been moved is some position $X \neq B$, so $r(X) < r(B)$.

In the latter case, let ν be the partition such that $[\nu] = ([\lambda] \setminus A) \cup X$, so $\lambda \rightsquigarrow \nu \rightsquigarrow \mu$. Since $r(A) - r(X) < r(A) - r(B)$ and $r(X) - r(B) < r(A) - r(B)$ then by the inductive hypothesis there exists chains of minimal moves from λ to ν and from ν to μ. In the former case, a similar argument holds for $[\nu] := ([\mu] \setminus Y) \cup B$.

To deduce that $E(\underline{H}) \subseteq E(\underline{G})$, suppose $\lambda \to \mu$. Then $\lambda \triangleright \mu$, whence there is a sequence of minimal moves $\lambda =: \nu_0 \overset{\min}{\rightsquigarrow} \nu_1 \overset{\min}{\rightsquigarrow} \cdots \overset{\min}{\rightsquigarrow} \nu_{k-1} \overset{\min}{\rightsquigarrow} \nu_k := \mu$ by (i) and (ii). But $E(G) \subseteq E(H)$ by Proposition 7.3, so $k = 1$ and $\lambda \overset{\min}{\rightsquigarrow} \mu$ as required. $\square$

Proposition 7.5. *Let $r > 3$. There exists an order-reversing isomorphism on $\Lambda^+(3, r)$ if and only if $r \in \{4, 5, 7, 8\}$.*

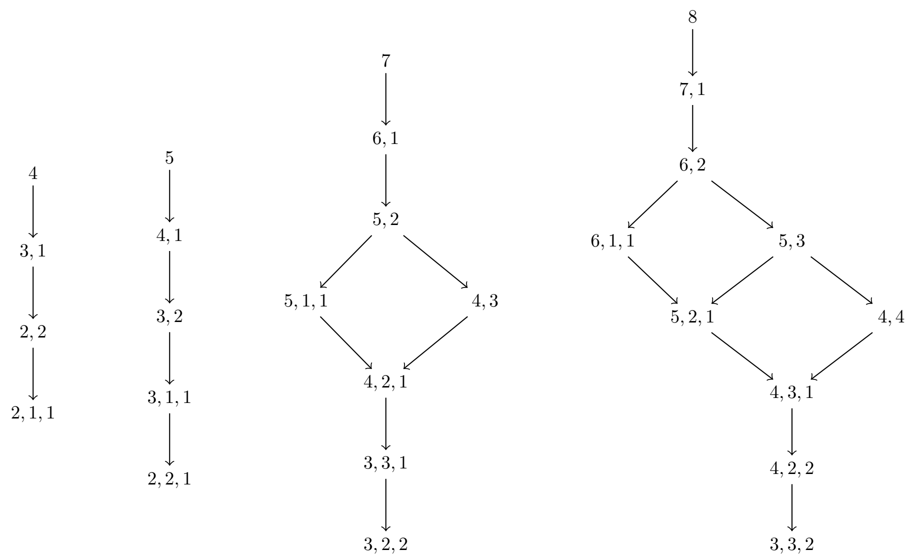


Figure 7.2: $H(3, r)$ for $r = 4, 5, 7, 8$.

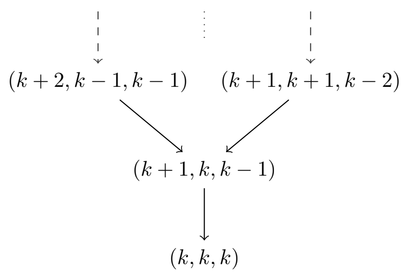


Figure 7.3: Vertices at small distance from (k, k, k) in $H(3, 3k)$, $k \geq 2$.

Proof. Let $\Lambda = \Lambda^+(3, r)$ and $H = H(3, r)$. From the Hasse diagrams in Figure 7.2, we see there is exactly one order isomorphism $(\Lambda, \trianglelefteq) \to (\Lambda, \trianglelefteq^{\mathrm{op}})$ when $r = 4, 5, 8$ and two when $r = 7$. Now let $r \notin \{4, 5, 7, 8\}$.

If $r = 3k$, $k \geq 2$: the vertices at distance ≤ 2 from $\alpha = (k, k, k)$ in H are as in Figure 7.3 (this can be verified using Proposition 7.4). Any order isomorphism $(\Lambda, \trianglelefteq) \to (\Lambda, \trianglelefteq^{\mathrm{op}})$ gives an isomorphism of (undirected) graphs $H \to H$ with $(r) \mapsto \alpha$. This must extend to $(r-1, 1) \mapsto (k+1, k, k-1)$, but $\deg((r-1, 1)) = 2 < 3 = \deg((k+1, k, k-1))$. Hence there is no order-reversing isomorphism on Λ.

If $r = 3k + 1$, $r \geq 3$: the vertices at small distance from $\alpha = (k+1, k, k)$ are as in Figure 7.4. Any graph isomorphism $H \to H$ with $(r) \mapsto \alpha$ must extend to $(r-3, 2, 1) \mapsto (k+3, k, k-2)$, but $\deg_H((r-3, 2, 1)) = 3 < 4 = \deg_H((k+3, k, k-2))$.

If $r = 3k+2$, $r \geq 3$: the vertices at small distance from $\alpha = (k+1, k+1, k)$ are as in Figure 7.5. Any graph isomorphism $H \to H$ with $(r) \mapsto \alpha$ must extend to $(r-3, 2, 1) \mapsto (k+3, k+1, k-2)$, but $\deg_H((r-3, 2, 1)) = 3 < 4 = \deg_H((k+3, k+1, k-2))$. $\square$

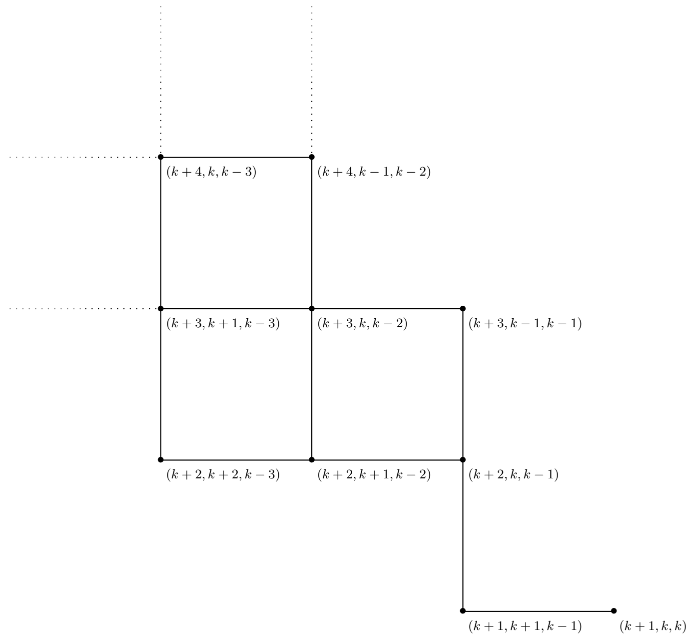


Figure 7.4: Vertices at small distance from $(k+1,k,k)$ in $H(3,3k+1)$, $k \geq 3$.

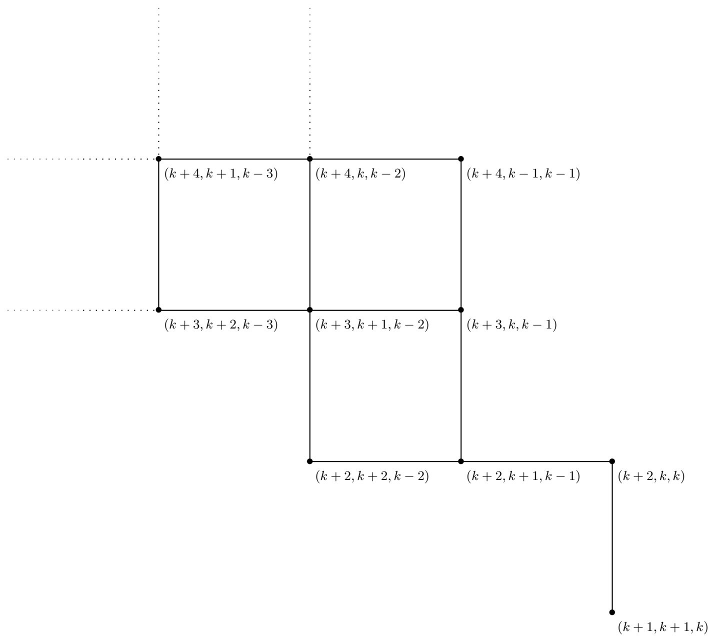


Figure 7.5: Vertices at small distance from $(k+1,k+1,k)$ in $H(3,3k+2)$, $k \geq 3$.

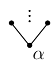
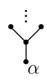
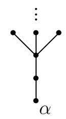
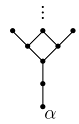
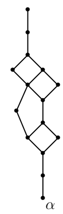
$n \geq 4,\ r \not\equiv 0, \pm 1 \pmod{n}$	$n \geq 4,\ r \equiv \pm 1 \pmod{n}$ $(n,r) \neq (4,5)$	$n \geq 6,\ r \geq 12$ $r \equiv 0 \pmod{n}$	$n \in \{4,5\},\ r \geq 10$ $r \equiv 0 \pmod{n}$	$(n,r)=(4,8)$
				

Table 7.1: Vertices at small distance from α in $H(n,r)$, for $4 \leq n < r$.

Remark 7.6. $H(3,r)$ is always a graded poset, with rank function $\mathrm{rank}(\lambda) = \lambda_1 - \lambda_3$. Note $\{\lambda_1, \lambda_3\}$ uniquely determines λ, since $\lambda_1 + \lambda_2 + \lambda_3 = r$.

Similarly, $H(r)$ is graded for $r \leq 6$ so $H(n,r)$ is graded for any n in this case. But for $n \geq 4$ and $r \geq 7$, $H(n,r)$ is never a graded poset because we have disjoint paths from $(n-3,2,1)$ to $(n-4,2,1,1)$ of length 2 and 3, namely

$$(n-3,2,1) \to (n-3,1^3) \to (n-4,2,1^2)$$

and

$$(n-3,2,1) \to (n-4,3,1) \to (n-4,2^2) \to (n-4,2,1^2).$$

$\Diamond$

Proposition 7.7. *Let $4 \leq n < r$. There exists an order-reversing isomorphism on $\Lambda^+(n,r)$ if and only if $(n,r) = (4,5)$.*

Proof. Since $\Lambda^+(4,5)$ is totally ordered under $\trianglelefteq$, there is a unique order-reversing isomorphism on $\Lambda^+(4,5)$. If $(n,r) \neq (4,5)$ then it is straightforward to verify that the vertices at small distance in $H(n,r)$ from the minimal element α of $\Lambda^+(n,r)$ are as depicted in Table 7.1, and so there cannot be an order-reversing isomorphism on $\Lambda^+(n,r)$. These have been presented in more detail in Appendix A. $\square$

Proof of Proposition 7.1. This follows immediately from Propositions 7.5 and 7.7. $\square$

7.2 Quasi-hereditary algebras

Quasi-hereditary algebras are an important class of algebras which were first introduced in the context of highest weight categories in the representation theory of semisimple complex Lie algebras and algebraic groups [5]. These include many naturally occurring algebras, such as the Schur algebras and Auslander algebras, and exhibit useful properties including finite global dimension and cellularity. We recall some standard facts about quasi-hereditary algebras and Schur algebras, following the account in [14]. For further detail, see also [13, Appendix], [20] and [48], for instance. The notion that a finite-dimensional algebra over a field is quasi-hereditary if and only if its module category is a highest weight category provides a certain combinatorial structure of which we make extensive use.

7.2.1 Background

Let S be a finite-dimensional K algebra, and denote by $\operatorname{mod} S$ the category of finite-dimensional left S–modules. Let $\{L(\lambda) : \lambda \in \Lambda\}$ be a complete set of pairwise non-isomorphic simple S–modules. For each $\lambda \in \Lambda$, let $P(\lambda)$ denote a minimal projective cover and $I(\lambda)$ a minimal injective envelope of $L(\lambda)$. Now let $\leq$ be a partial order on the set Λ.

Letting $M(\lambda)$ denote the (unique) maximal submodule of $P(\lambda)$, the module $\Delta(\lambda)$ is defined to be the quotient of $P(\lambda)$ by U, where U is minimal amongst submodules U' of $M(\lambda)$ such that all of the composition factors of $M(\lambda)/U'$ are of the form $L(\mu)$ for some $\mu < \lambda$. The module $\nabla(\lambda)$ is defined to be the submodule of $I(\lambda)$ containing $L(\lambda)$ such that $\nabla(\lambda)/L(\lambda)$ is maximal amongst submodules of $I(\lambda)/L(\lambda)$ all of whose composition factors are of the form $L(\mu)$ for some $\mu < \lambda$. A *standard* (resp. *costandard*) (also *Weyl*, resp. *dual Weyl*) S–module is one which is isomorphic to $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) for some $\lambda \in \Lambda$.

By construction, the composition multiplicities $[\Delta(\lambda) : L(\lambda)]$ and $[\nabla(\lambda) : L(\lambda)]$ are both equal to one, and it is straightforward to observe that $\{[L(\lambda)] : \lambda \in \Lambda\}$, $\{[\Delta(\lambda)] : \lambda \in \Lambda\}$ and $\{[\nabla(\lambda)] : \lambda \in \Lambda\}$ are all $\mathbb{Z}$-bases for the Grothendieck group of $\operatorname{mod} S$. For an S–module V, we define $(V : \Delta(\lambda))$ and $(V : \nabla(\lambda))$ as follows:

$$[V] = \sum_{\lambda \in \Lambda} (V : \Delta(\lambda))[\Delta(\lambda)] \quad , \quad [V] = \sum_{\lambda \in \Lambda} (V : \nabla(\lambda))[\nabla(\lambda)].$$

Let $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$) denote the full subcategory of $\operatorname{mod} S$ of those modules which have a standard or Δ-filtration (resp. costandard or ∇-filtration). If $V \in \mathcal{F}(\Delta)$, then $(V : \Delta(\lambda))$ equals the Δ-filtration multiplicity of $\Delta(\lambda)$ in every standard filtration of V, and similarly for $V \in \mathcal{F}(\nabla)$. (We reserve parentheses for Δ- and ∇-filtration multiplicities, and square brackets for composition multiplicities.)

The category $\operatorname{mod} S$ is a *highest weight category with respect to weight poset* $(\Lambda, \leq)$ if for all $\lambda, \mu \in \Lambda$,

$$P(\lambda) \in \mathcal{F}(\Delta), \quad (P(\lambda) : \Delta(\lambda)) = 1, \quad \text{and} \quad (P(\lambda) : \Delta(\mu)) > 0 \implies \mu \geq \lambda. \qquad (7.1)$$

Equivalently, S is a *quasi-hereditary algebra* (with respect to weight poset $(\Lambda, \leq)$); see [13, Appendix] or [48], for instance, though the definition just presented will be more useful for our purposes than the equivalent definition of quasi-hereditary algebras using heredity ideals. We simply say that S is quasi-hereditary when the weight poset is fixed or understood. Note that a given finite-dimensional K–algebra S may admit different quasi-hereditary structures (i.e. have different standard modules), corresponding to different partial orderings on an indexing set for simple S–modules which satisfy the conditions in (7.1).

Suppose S is quasi-hereditary with respect to $(\Lambda, \leq)$. We have Brauer–Humphreys reciprocity [14, Theorem 1.4]: for all $\lambda, \mu \in \Lambda$,

$$(P(\lambda) : \Delta(\mu)) = [\nabla(\mu) : L(\lambda)] \quad , \quad (I(\lambda) : \nabla(\mu)) = [\Delta(\mu) : L(\lambda)].$$

A *(partial) tilting module* T is one that satisfies both $T \in \mathcal{F}(\Delta)$ and $T \in \mathcal{F}(\nabla)$. A full set of pairwise non-isomorphic indecomposable tilting modules is given by $\{T(\lambda) : \lambda \in \Lambda\}$, with the properties

$$(T(\lambda) : X(\lambda)) = 1, \; (T(\lambda) : X(\mu)) > 0 \implies \mu \leq \lambda, \quad \text{for } X \in \{\Delta, \nabla\}. \tag{7.2}$$

A *full tilting module* is a tilting module T such that $T(\lambda)$ occurs as a direct summand of T for each $\lambda \in \Lambda$. Let T be a full tilting module. Then $S' = (\mathrm{End}_S(T))^{\mathrm{op}}$ is a *Ringel dual of* S (where we have taken the opposite algebra in order to consider left, not right, S'–modules). The algebra S' is determined up to Morita equivalence, and S'' is Morita equivalent to S; we denote Morita equivalence by $\sim_M$. Indeed, there exists a suitable choice of full tilting modules such that S'' is isomorphic to S as quasi-hereditary algebras [14, Theorem 1.7]. We fix $T = \oplus_{\lambda \in \Lambda} T(\lambda)$ and set $S' = (\mathrm{End}_S(\oplus_{\lambda \in \Lambda} T(\lambda)))^{\mathrm{op}}$, for convenience.

The Ringel dual S' is again quasi-hereditary, but with respect to $(\Lambda, \leq^{\mathrm{op}})$. The left exact functor $\mathrm{Hom}_S(T, -) : \mathrm{mod}\, S \longrightarrow \mathrm{mod}\, S'$ gives an equivalence between $\mathcal{F}(\nabla)$ and $\mathcal{F}(\Delta')$ [13, §A4], sending

$$\nabla(\lambda) \longmapsto \Delta'(\lambda), \qquad T(\lambda) \longmapsto P'(\lambda), \qquad I(\lambda) \longmapsto T'(\lambda)$$

for all $\lambda \in \Lambda$ and where $\Delta'(-)$, $P'(-)$ and $T'(-)$ denote the standard, indecomposable projective and indecomposable tilting modules for S' respectively. Furthermore, the filtration multiplicities satisfy

$$(T(\lambda) : \nabla(\mu)) = (P'(\lambda) : \Delta'(\mu))$$

for all $\lambda, \mu \in \Lambda$.

If S and T are quasi-hereditary algebras with respect to some given weight posets, i.e. partial orderings on the indexing sets, then a Morita equivalence between S and T is a *Morita equivalence of quasi-hereditary algebras* if the resulting bijection between the set of simple S–modules and the set of simple T–modules respects the partial orders. A quasi-hereditary algebra S (w.r.t. $(\Lambda, \leq)$) is *Ringel self-dual* if S is Morita equivalent to S' as quasi-hereditary algebras. Letting $\{L'(\lambda) : \lambda \in \Lambda\}$ denote a complete set of simple S'–modules, the Morita equivalence maps $L(\lambda)$ to $L'(\hat{\lambda})$ with $\lambda \longmapsto \hat{\lambda}$ being an order-reversing isomorphism on $(\Lambda, \leq)$.

For all natural numbers n and r, the Schur algebra $S(n, r)$ is quasi-hereditary with respect to $(\Lambda^+(n,r), \trianglelefteq)$. We remark that Donkin's result of Ringel self-duality of $S(n,r)$ when $n \geq r$ gives the conjugation map on partitions $\lambda \longmapsto \lambda'$, which is indeed order-reversing on $\Lambda^+(n,r)$, in fact the set of all partitions of r in this case [13]. Moreover, for Schur algebras there is a contravariant duality $^\circ$ (see [33, §2.7]) such that for all $\lambda \in \Lambda$,

$$L(\lambda)^\circ \cong L(\lambda), \qquad \Delta(\lambda)^\circ \cong \nabla(\lambda), \qquad T(\lambda)^\circ \cong T(\lambda).$$

Hence we have identities such as $[\Delta(\lambda) : L(\mu)] = [\nabla(\lambda) : L(\mu)]$ and $(T(\lambda) : \Delta(\mu)) = (T(\lambda) : \nabla(\mu))$ by taking $^\circ$.

Corollary 7.8. *Let $3 \leq n < r$. For $(n,r) \notin \{(3,4),(3,5),(3,7),(3,8),(4,5)\}$, the Schur algebra $S(n,r)$ is not Ringel self-dual.*

Proof. Ringel self-duality would imply a Morita equivalence of quasi-hereditary algebras $S(n,r) \to S(n,r)'$ such that the permutation on the indexing set $\Lambda = \Lambda^+(n,r)$ of irreducible modules is an order-reversing map on $(\Lambda, \trianglelefteq)$. The assertion then follows from Proposition 7.1. □

Remark 7.9. Our current definition of Ringel self-duality (see [21, (1.1)]) depends crucially on the poset structure $(\Lambda, \leq)$. This dependence has the advantage of allowing us to exploit combinatorial features of $(\Lambda, \leq)$ as in Corollary 7.8, for instance, but one may also ask similar questions for a definition of Ringel duality 'intrinsic' to the quasi-hereditary algebra. In [6, Theorem 2.1.1], Coulembier shows that if a finite-dimensional algebra A has a simple-preserving duality, then up to equivalence there is only one possible quasi-hereditary structure $(\Lambda, \leq^e)$ on A, where this 'essential' partial order $\leq^e$ is derived from the unique choice of standard modules for A. Nevertheless, for now we investigate only the specific dominance ordering $\trianglelefteq$ on partitions, ubiquitous in the study of the representation theory of symmetric groups. ♢

7.2.2 Blocks of Schur algebras

The Ext^1 quiver of an algebra S is a directed graph with vertices labelled by the simple S–modules, say $\{S_i\}_{i\in I}$, and the number of arrows $S_i \to S_j$ is equal to the dimension of $\mathrm{Ext}^1_S(S_i, S_j)$. A (finite-dimensional) algebra S is indecomposable if and only if its Ext^1 quiver is connected, and Ext^1 quivers are preserved under Morita equivalence. Hence if $S \sim_M T$ are algebras and S is indecomposable, then T is also indecomposable. We begin with two easy results, whose proofs have been included for convenience.

Lemma 7.10. *Let S be a quasi-hereditary algebra. If $S = \oplus_i B_i$ is the block decomposition of S, then each $(B_i)'$ is indecomposable and $\oplus_i (B_i)' \cong S'$.*

Proof. Let the weight poset for S be $(\Lambda, \leq)$. For each $\lambda \in \Lambda$, clearly the simple module $L(\lambda)$ lies in some block, and the modules $\Delta(\lambda)$, $\nabla(\lambda)$, $P(\lambda)$, $I(\lambda)$ and $T(\lambda)$ all lie in that same block since they are indecomposable and have a non-zero map to $L(\lambda)$. We say λ lies in the block B if $L(\lambda)$ lies in B. In particular, the blocks of S inherit a quasi-hereditary structure from S and their weight posets form a partition of $(\Lambda, \leq)$, since a block B_j acts as zero on any modules lying in a block B_i whenever $i \neq j$. Since $\mathrm{Hom}_S(T(\lambda), T(\mu)) = 0$ whenever $\lambda, \mu \in \Lambda$ do not lie in the same block, we have that

$$(S')^{\mathrm{op}} \cong \bigoplus_i \mathrm{End}_S(\oplus_{\lambda \in B_i} T(\lambda)) = \bigoplus_i \mathrm{End}_{B_i}(\oplus_{\lambda \in B_i} T(\lambda)) = \bigoplus_i ((B_i)')^{\mathrm{op}}.$$

It remains to observe that $(B_i)'$ is again indecomposable, for B_i a block of S: if $(B_i)'$ decomposes into blocks $b_1 \oplus \cdots \oplus b_l$, then by a similar argument we find that $(B_i)'' \cong (b_1)' \oplus \cdots \oplus (b_l)'$ is a decomposition into (non-zero) algebras. But B_i is Morita equivalent to $(B_i)''$, so $(B_i)''$ is indecomposable and $l = 1$ as required. □

Proposition 7.11. *Let S be a quasi-hereditary algebra with weight poset $(\Lambda, \leq)$. Suppose that S is semisimple and that there exists an order-reversing isomorphism t on $(\Lambda, \leq)$. Then S is Ringel self-dual.*

Proof. Since S is semisimple, in fact $L(\lambda) = \Delta(\lambda) = P(\lambda)$ for all $\lambda \in \Lambda$ and each $L(\lambda)$ lies in its own block. Each block of S is isomorphic to a matrix algebra $\mathrm{Mat}_m(K)$, which in turn is Morita equivalent to K for any $m \in \mathbb{N}$, and the algebra K has a unique quasi-hereditary structure. For $\lambda \in \Lambda$, let B_λ denote the block of S containing $L(\lambda)$. Using Lemma 7.10, we may construct a Morita equivalence between S and S' which sends B_λ to $(B_{t(\lambda)})'$, and hence $L(\lambda)$ to $L'(t(\lambda))$, for every $\lambda \in \Lambda$. □

Doty and Nakano have determined exactly when $S(n,r)$ is semisimple in [17].

Theorem 7.12 ([17, Theorem 2])**.** *Let K be an infinite field of characteristic $p \geq 0$, and let $n, r \in \mathbb{N}$. Then $S_K(n,r)$ is semisimple if and only if (i) $p = 0$, (ii) $p > r$, or (iii) $p = n = 2$ and $r = 3$.*

By Corollary 7.8 and Proposition 7.11, it remains to investigate whether $S(n,r)$ is Ringel self-dual in the following cases, where $p = \mathrm{char}(K)$:

$$\begin{gathered} S(3,4),\ p \in \{2,3\}, \quad S(3,5),\ p \in \{2,3,5\}, \quad S(3,7),\ p \in \{2,3,5,7\}, \\ S(3,8),\ p \in \{2,3,5,7\}, \quad S(4,5),\ p \in \{2,3,5\} \end{gathered} \tag{7.3}$$

In the rest of this and the following section we show that none of the Schur algebras in (7.3) are Ringel self-dual, so the only $S(n,r)$ with $3 \leq n < r$ which are self-dual are those which are semisimple with $(n,r) \in \{(3,4),(3,5),(3,7),(3,8),(4,5)\}$.

We record how the simple modules are partitioned into blocks for the Schur algebras in (7.3), expressed as a partition of $\Lambda^+(n,r)$, from which it will be immediately clear why most of these $S(n,r)$ cannot be self-dual.

Proposition 7.13. *Let $(n,r) \in \{(3,4),(3,5),(3,7),(3,8),(4,5)\}$ and consider a Schur algebra $S(n,r)$ defined over a field K of characteristic $0 < p < r$.*

1. *For $(n,r) = (3,4)$:*

(i) *All four simple modules lie in the same block when* $p = 2$.

(ii) *There are three blocks when* $p = 3$*:* $\{(4), (2, 2)\}$, $\{(3, 1)\}$ *and* $\{(2, 1, 1)\}$.

2. *For* $(n, r) = (3, 5)$*:*

(i) *There are two blocks when* $p = 2$*:* $\{(4, 1)\}$ *and* $\Lambda^+(3, 5) \setminus \{(4, 1)\}$.

(ii) *There are three blocks when* $p = 3$*:* $\{(5), (2^2, 1)\}$, $\{(4, 1), (3, 2)\}$ *and* $\{(3, 1^2)\}$.

(iii) *There are three blocks when* $p = 5$*:* $\{(5), (4, 1), (3, 1^2)\}$, $\{(3, 2)\}$ *and* $\{(2^2, 1)\}$.

3. *For* $(n, r) = (3, 7)$*:*

(i) *There are two blocks when* $p = 2$*:* $\{(6, 1), (4, 3)\}$ *and* $\Lambda^+(3, 7) \setminus \{(6, 1), (4, 3)\}$.

(ii) *There are three blocks when* $p = 3$*:* $\{(7), (5, 2), (4, 3), (4, 2, 1)\}$, $\{(6, 1), (3, 2^2)\}$ *and* $\{(5, 1^2), (3^2, 1)\}$.

(iii) *There are four blocks when* $p = 5$*:* $\{(5, 1^2)\}$, $\{(4, 2, 1)\}$, $\{(6, 1), (5, 2), (3, 2^2)\}$ *and* $\{(7), (4, 3), (3^2, 1)\}$.

(iv) *There are six blocks when* $p = 7$*:* $\{(7), (6, 1), (5, 1, 1)\}$ *and each of the remaining* λ *lies in its own block.*

4. *For* $(n, r) = (3, 8)$*:*

(i) *There are two blocks when* $p = 2$*:* $\{(5, 2, 1)\}$ *and* $\Lambda^+(3, 8) \setminus \{(5, 2, 1)\}$.

(ii) *There are three blocks when* $p = 3$*:* $\{(6, 1^2), (3^2, 2)\}$, $\{(7, 1), (6, 2), (4^2), (4, 2^2)\}$ *and* $\{(8), (5, 3), (5, 2, 1), (4, 3, 1)\}$.

(iii) *There are five blocks when* $p = 5$*:* $\{(6, 1^2), (5, 2, 1), (4, 2^2)\}$, $\{(4, 3, 1)\}$, $\{(8), (4^2)\}$, $\{(6, 2)\}$ *and* $\{(7, 1), (5, 3), (3^2, 2)\}$.

(iv) *There are eight blocks when* $p = 7$*:* $\{(8), (6, 2), (5, 2, 1)\}$ *and each of the remaining* λ *lies in its own block.*

5. *For* $(n, r) = (4, 5)$*:*

(i) *There are two blocks when* $p = 2$*:* $\{(5), (3, 2), (3, 1^2), (2^2, 1)\}$ *and* $\{(4, 1), (2, 1^3)\}$.

(ii) *There are three blocks when* $p = 3$*:* $\{(4, 1), (3, 2)\}$, $\{(5), (2^2, 1), (2, 1^3)\}$ *and* $\{(3, 1^2)\}$.

(iii) *There are three blocks when* $p = 5$*:* $\{(5), (4, 1), (3, 1, 1), (2, 1^3)\}$, $\{(3, 2)\}$ *and* $\{(2, 2, 1)\}$.

Proof. This can be calculated directly from Donkin's description of blocks in [11]. □

Since a Morita equivalence between algebras preserves the Ext^1 quiver, blocks must be mapped to blocks, being determined by (the existence of) non-split extensions between simple modules. Thus if a Schur algebra $S = S(n, r)$ is Ringel self-dual, then the Morita equivalence $S \sim_M S'$ gives an order-reversing isomorphism t on $\Lambda^+(n, r)$ such

(n,r,p)	λ	$\|B_i\|$	$t(\lambda)$	$\|B_j\|$
$(3,4,3)$	(4)	2	$(2,1,1)$	1
$(3,5,2)$	$(3,1,1)$	4	$(4,1)$	1
$(3,5,3)$	$(4,1)$	2	$(3,1,1)$	1
$(3,5,5)$	(5)	3	$(2,2,1)$	1
$(3,7,2)$	$(3,3,1)$	6	$(6,1)$	2
$(3,7,3)$	(7)	4	$(3,2,2)$	2
$(3,7,5)$	$(5,2)$	3	$(4,2,1)$	1
$(3,7,7)$	(7)	3	$(3,2,2)$	1
$(3,8,2)$	$(5,3)$	9	$(5,2,1)$	1
$(3,8,3)$	(8)	4	$(3,3,2)$	2
$(3,8,5)$	$(3,3,2)$	3	(8)	2
$(3,8,7)$	(8)	3	$(3,3,2)$	1
$(4,5,2)$	(5)	4	$(2,1^3)$	2
$(4,5,3)$	$(3,2)$	2	$(3,1,1)$	1
$(4,5,5)$	$(3,1,1)$	4	$(3,2)$	1

Table 7.2: Example λ and $t(\lambda)$ showing that $S(n,r)$ over a field of characteristic p is not Ringel self-dual for all $(n,r,p) \neq (3,4,2)$ in (7.3).

that λ and μ lie in the same block of S if and only if $t(\lambda)$ and $t(\mu)$ lie in the same block of S', for all $\lambda, \mu \in \Lambda^+(n,r)$. Let $|B|$ denote the number of simple modules lying in the block B. Let $S = \oplus_{i \in I} B_i$ be a block decomposition of S. Given some block B_i of S, suppose it is mapped under t to some block of S': it is isomorphic to $(B_j)'$ for some $j \in I$, by Lemma 7.10, so

$$|B_i| = |(B_j)'| = |B_j|.$$

Then we obtain a contradiction to the assumption of Ringel self-duality of S if in fact there is some $i \in I$ such that there exists $\lambda \in B_i$ and $t(\lambda) \in B_j$ with $|B_i| \neq |B_j|$.

Example 7.14. Consider a Schur algebra $S(3,4)$ over a field of characteristic 3. The blocks of $S(3,4)$ are $\{(4),(2,2)\}$, $\{(3,1)\}$ and $\{(2,1,1)\}$. The map t being order-reversing implies $B_i \ni (4) \mapsto (2,1,1) \in (B_j)'$, but $|B_i| = 2$ and $|B_j| = 1$. Hence $S(3,4)$ over a field of characteristic 3 is not Ringel self-dual. ♢

Proposition 7.15. *All of the Schur algebras $S(n,r)$ in (7.3), except $S(3,4)$ over fields K of characteristic 2, are not Ringel self-dual.*

Proof. See Table 7.2. □

Remark 7.16. This argument is inconclusive for $S(3,4)$ over a field of characteristic 2 as the Schur algebra itself is indecomposable (i.e. a single block) in this case. A different argument which covers this case is given in the next section. ♢

7.2.3 Tilting matrices and decomposition numbers

We compare Δ–filtration multiplicities of tilting modules with decomposition numbers for symmetric groups to see that $S(3,4)$ over a field of characteristic 2 is not Ringel self-dual.

Let $n, r \in \mathbb{N}$ and let K be an algebraically closed field of characteristic $p > 0$. Suppose that $S = S_K(n,r)$ is Ringel self-dual, so we have a Morita equivalence $S' \sim_M S$ of quasi-hereditary algebras, giving an order-reversing function $t : \Lambda \to \Lambda$. For convenience we write $t\lambda$ for $t(\lambda)$. Since $\Delta(\lambda)^\circ \cong \nabla(\lambda)$ and $T(\lambda) \cong T(\lambda)^\circ$, we have

$$(P(t\lambda) : \Delta(t\mu)) = (P'(\lambda) : \Delta'(\mu)) = (T(\lambda) : \nabla(\mu)) = (T(\lambda) : \Delta(\mu)).$$

By Brauer-Humphreys reciprocity,

$$(P(t\lambda) : \Delta(t\mu)) = [\nabla(t\mu) : L(t\lambda)],$$

which is equal to the decomposition number $[\Delta(t\mu) : L(t\lambda)]$ for $S(n,r)$ as $^\circ$ fixes simple modules. Therefore

$$(T(\lambda) : \Delta(\mu)) = [\Delta(t\mu) : L(t\lambda)].$$

Further, if λ is p-regular then $(T(\lambda) : \Delta(\mu)) = [S^\mu : D^\lambda]$, a decomposition number for the symmetric group $\mathfrak{S}_r$, i.e. the multiplicity of the modular irreducible module D^λ as a composition factor of the Specht module S^μ, by [20, Lemma 4.5]. Note for p-regular λ, S^λ has simple top isomorphic to D^λ, and all other composition factors are isomorphic to D^ν with $\nu \rhd \lambda$.

Example 7.17. Let $(n,r,p) = (3,4,2)$. Since $\Lambda^+(3,4)$ is totally ordered under $\unlhd$, the map t is uniquely determined. Consider the *tilting matrix* $(T(\lambda) : \Delta(\mu))$ for $S(3,4)$ over characteristic 2:

	$T(4)$	$T(3,1)$	$T(2,2)$	$T(2,1^2)$
$\Delta(4)$	1			
$\Delta(3,1)$	1	1		
$\Delta(2,2)$	0	1	1	
$\Delta(2,1^2)$	1	1	a	1

(For convenience, we write $T\lambda$ for $T(\lambda)$, and so on, when the meaning is clear from context.) The matrix as indexed is lower unitriangular. The columns corresponding to the 2-regular partitions (4) and $(3,1)$ contain certain decomposition numbers for $\mathfrak{S}_4$ when $p = 2$ as described above, and these values are known: see for example [41]. The only unknown value thus far is $a := (T(2,2) : \Delta(2,1^2))$; to show $S(3,4)$ is not self-dual we will not need to know its exact value (but nevertheless it is calculated in Remark 7.20 below). In certain cases there are ad hoc methods to calculate such multiplicities. In general it remains a central open problem in representation theory to compute decomposition numbers.

The *reverse decomposition matrix* $[\Delta(t\mu) : L(t\lambda)]$ for $S(3,4)$ over characteristic 2, indexed by λ, μ in the same order as the tilting matrix above, is also lower unitriangular:

	$L(2,1^2)$	$L(2,2)$	$L(3,1)$	$L(4)$
$\Delta(2,1^2)$	1			
$\Delta(2,2)$	$*$	1		
$\Delta(3,1)$	$*$	$*$	1	
$\Delta(4)$	$*$	$*$	$*$	1

We can use results of Erdmann and Kovács [22] on the structure of the symmetric power $S^rE = \nabla(r) \cong \Delta(r)^\circ$ of the natural n-dimensional GL_n–module E to find the last row of decomposition numbers $[\Delta(r) : L(t\lambda)]$. Note here S^rE is viewed as a $S_K(n,r)$–module via the equivalence of $M_K(n,r)$ and mod $S_K(n,r)$ (see, for example, [33, (2.4d)]).

By [22, Lemma 4.6], S^rE has a filtration whose quotients are isomorphic to

$$\overline{S^r}, \quad \overline{S^{r-p}} \otimes (S^1E)^F, \quad \overline{S^{r-2p}} \otimes (S^2E)^F, \quad \ldots, \quad \overline{S^{r-kp}} \otimes (S^kE)^F$$

where $k = \lfloor r/p \rfloor$ and F is the Frobenius functor (the first few quotients may vanish when $n(p-1) < r$). Here $\overline{S^l} = L(\lambda)$ where $\lambda = (p-1, p-1, \ldots, p-1, b) \vdash l$ with $0 \le b < p-1$. So we can calculate the composition factors of S^rE inductively.

Returning to $(n,r,p) = (3,4,2)$: $S^4(E)$ has filtration quotients $\overline{S^4}$, $\overline{S^2} \otimes E^F$ and $\overline{S^0} \otimes (S^2E)^F$.

- The factor $\overline{S^4} = L(1^4)$ vanishes when $n = 3$;
- $\overline{S^2} = L(1^2)$ and $E = L(1)$;
- $\overline{S^0} = L(0)$, the trivial module, and S^2E has filtration quotients $\overline{S^2} = L(1^2)$ and $\overline{S^0} \otimes E^F = L(1)^F$.

Simplifying these using the Steinberg tensor product formula, $S^4(E)$ has the following three composition factors:

- $L(1^2) \otimes L(1)^F = L(1^2) \otimes L(2) = L(3,1)$,
- $(\overline{S^2})^F = L(1^2)^F = L(2^2)$, and
- $E^{F^2} = L(1)^{F^2} = L(4)$,

giving

	$L(2,1^2)$	$L(2,2)$	$L(3,1)$	$L(4)$
$\Delta(2,1^2)$	1			
$\Delta(2,2)$	$*$	1		
$\Delta(3,1)$	$*$	$*$	1	
$\Delta(4)$	0	1	1	1

So the bottom left entries $(T(4) : \Delta(2,1^2))$ and $[\Delta(4) : L(2,1^2)]$ of the tilting matrix and decomposition matrix respectively are not equal, implying that $S(3,4)$ over a field of characteristic 2 cannot be Ringel self-dual. $\Diamond$

We have at last shown the following:

Theorem 7.18. *Let $3 \leq n < r$ and $\mathrm{char}(K) = p$. Then the Schur algebra $S_K(n,r)$ is Ringel self-dual if and only if*

(i) $(n,r) \in \{(3,4),(3,5),(3,7),(3,8),(4,5)\}$, *and*

(ii) $p > r$ *or* $p = 0$.

Remark 7.19. By considering tilting matrices and decomposition numbers, one can show that the Schur algebras in (7.3) are not Ringel self-dual, giving a second proof of Theorem 7.18. We need only consider the last row of the reverse decomposition matrix, as in Example 7.17 above, when $(n,r,p) \neq (3,5,3)$. These calculations are given in Appendix B.

When $(n,r,p) = (3,5,3)$, the tilting matrix is

	$T(5)$	$T(4,1)$	$T(3,2)$	$T(3,1^2)$	$T(2^2,1)$
$\Delta(5)$	1				
$\Delta(4,1)$	0	1			
$\Delta(3,2)$	0	1	1		
$\Delta(3,1^2)$	0	0	0	1	
$\Delta(2^2,1)$	1	0	0	0	1

The composition factors of $\Delta(5)$ are $L(2^2,1)$ and $L(5)$, so thus far the reverse decomposition matrix is

	$L(2^2,1)$	$L(3,1^2)$	$L(3,2)$	$L(4,1)$	$L(5)$
$\Delta(2^2,1)$	1				
$\Delta(3,1^2)$	b	1			
$\Delta(3,2)$	$*$	c	1		
$\Delta(4,1)$	$*$	$*$	$*$	1	
$\Delta(5)$	1	0	0	0	1

and we need to compare further entries. In [39] James proved first row and first column removal theorems for the decomposition numbers of GL_n of the form $[\Delta(\mu) : L(\lambda)]$, and hence of Schur algebras; he further obtained these removal theorems for the symmetric groups via the use of Schur functors. By first column removal,

$$b := [\Delta(3,1^2) : L(2^2,1)] = [\Delta(2) : L(1^2)]$$

and by first row removal,

$$c := [\Delta(3,2) : L(3,1^2)] = [\Delta(2) : L(1^2)]$$

also. But the entries corresponding to b and c in the tilting matrix above are not equal, whence $S(3,5)$ in characteristic 3 cannot be Ringel self-dual. ◊

Remark 7.20. When $(n, r, p) = (3, 4, 2)$, we can in fact show that

$$a := (T_3(2, 2, 0) : \Delta_3(2, 1, 1)) = 1$$

where we have added the subscript $_m$ to $T(\lambda)$ and $\Delta(\mu)$ to indicate that they are $S(m, r)$–modules (for fixed r).

In general, (noting that it will sometimes be convenient to allow partitions to have trailing zeros; the meaning should always be clear from context)

$$(T_n(\lambda) : \Delta_n(\mu)) = (T_{n-1}(\lambda^*) : \Delta_{n-1}(\mu^*))$$

if both λ and μ in $\Lambda^+(n, r)$ have at least one trailing zero and * means to remove a trailing zero. We can similarly remove trailing zeros for multiplicities involving L, Δ, ∇, T. This follows directly from $S(n, r) \cong eS(N, r)e$ where $N \geq n$ and e is an appropriate idempotent: see [33, §6.5] or [20, (3.9), (1.6), (1.7)]. Furthermore, if $N \geq r$ and $\lambda, \mu, \lambda', \mu' \vdash r$ all have at most N parts, then

$$(T(\lambda') : \nabla(\mu')) = [\nabla'(\mu') : L'(\lambda')] = [\nabla(\mu) : L(\lambda)].$$

The first equality follows from [12, Lemma 3.1], while the second follows from the Ringel self-duality of $S(N, r)$, sending $\lambda \mapsto \lambda'$ from $\operatorname{mod} S(N, r)$ to $\operatorname{mod} S(N, r)'$. Hence we may pass from $n = 3$ to $N = 4 = r$ to see that

$$a = (T_4(2, 2, 0, 0) : \Delta_4(2, 1, 1, 0)) = [\Delta_4(3, 1, 0, 0) : L_4(2, 2, 0, 0)] = [\Delta_2(3, 1) : L_2(2, 2)]$$

which equals $[\Delta_2(2, 0) : L_2(1, 1)]$ by first column removal. But $S^2(E) = \nabla(2)$ has composition factors $\overline{S}^2 = L(1, 1)$ and $\overline{S}^0 \otimes E^F = L(2)$, so

$$[\Delta_2(2, 0) : L_2(1, 1)] = [\nabla_2(2, 0) : L_2(1, 1)] = 1$$

as claimed.

This also completes the tilting matrix when $(n, r, p) = (3, 5, 2)$ for instance, where the value $(T_3(3, 1, 1, 0) : \Delta_3(2, 2, 1, 0))$ can be seen to equal

$$[\Delta_3(3, 2, 0) : L_3(3, 1, 1)] = [\Delta_2(2, 0) : L_2(1, 1)] = 1.$$

$\Diamond$

7.2.4 Ringel self-duality of blocks of finite type

Of independent interest are the Ringel duals of blocks of Schur algebras. An algebra A has *finite representation type* if it has only finitely many isomorphism classes of indecomposable modules in $\operatorname{mod} A$. It is straightforward to observe that if B is a block of a Schur algebra and B has finite representation type, then B is Ringel self-dual. For convenience, we include a proof. We then have, for example when $\mathrm{char}(K) = 3$, that all of the blocks of $S(3,4)$ are self-dual (since $S(3,4)$ itself has finite type [16]) but the whole algebra $S(3,4)$ is not self-dual via some global function t in the sense of [21], by Theorem 7.18.

In fact, the blocks of finite type were classified completely by Donkin and Reiten in [15], in terms of n, r, the characteristic p of the field and p-weights of blocks. For our purposes, we need only observe that such blocks are Morita equivalent to certain basic algebras.

A (finite-dimensional) algebra A over an algebraically closed field is *basic* if all irreducible A–modules are 1-dimensional. Given an algebra A, there is a unique basic algebra which is Morita equivalent to A (see [18, Corollary I.2.7], for example). Indeed, to construct the basic algebra of A, we essentially perform an idempotent truncation on A, taking a subset of idempotents in an orthogonal primitive idempotent decomposition of $1 \in A$ corresponding to pairwise non-isomorphic indecomposable projective A–modules. Moreover, if A is quasi-hereditary then its basic algebra is Morita equivalent to A as quasi-hereditary algebras.

For each $m \in \mathbb{N}$, define $\mathscr{A}_m$ to be the algebra KQ/I where Q is the following quiver with m vertices:

$$\overset{1}{\bullet} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \overset{2}{\bullet} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \bullet \cdots \bullet \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} \overset{m}{\bullet}$$

and the ideal I is generated by relations (reading arrows in the order of function composition)

$$\alpha_i\alpha_{i-1} = 0, \quad \beta_{i-1}\beta_i = 0, \quad \alpha_{m-1}\beta_{m-1} = 0, \quad \beta_i\alpha_i = \alpha_{i-1}\beta_{i-1} \quad \forall\ 2 \leq i \leq m-1.$$

Then $\mathscr{A}_m$ is quasi-hereditary with respect to $(\{1, 2, \ldots, m\}, \leq)$, where $\leq$ denotes the usual ordering on integers.

Proposition 7.21 ([15, Theorem 2.1], [19])**.** *Let B be a block of a Schur algebra $S(n,r)$. If B has finite type, then the block algebra $S_B(n,r)$ is Morita equivalent to the basic algebra $\mathscr{A}_m$ where $m = |B|$.*

For B of finite type, the simple B–modules may be linearly ordered (see [19, Proposition 4.1], for example), and the Morita equivalence above is in fact one of quasi-hereditary algebras. Indeed, by considering the form of the decomposition matrix for B, we find that B only has two possible quasi-hereditary structures: that given by the stated linear order, and another given by its reverse. That B of finite type is Ringel self-dual follows immediately from the Ringel self-duality of $\mathscr{A}_m$:

Proposition 7.22. *The quasi-hereditary algebra $\mathscr{A}_m$ is Ringel self-dual.*

Proof. Let $L_1, L_2, \ldots, L_m$ denote the simple $\mathscr{A}_m$–modules, corresponding to the vertices $1, 2, \ldots, m$ respectively. By [18, I.5.6], we can calculate the Loewy layers of the corresponding indecomposable projective modules P_i, which are as follows:

$$P_1 = \substack{L_1\\L_2\\L_1}, \qquad P_i = \substack{L_i\\L_{i-1}\oplus L_{i+1}\\L_i}, \qquad P_m = \substack{L_m\\L_{m-1}} \qquad (2 \leq i \leq m-1)$$

By (7.1), [15, Corollary 1.3] and Brauer–Humphreys reciprocity, we then deduce that

$$\Delta_1 = \nabla_1 = L_1, \qquad \Delta_i = \substack{L_i\\L_{i-1}}, \qquad \nabla_i = \substack{L_{i-1}\\L_i}, \qquad (2 \leq i \leq m)$$

where Δ_i (resp. ∇_i) denotes the standard (resp. costandard) $\mathscr{A}_m$–modules corresponding to L_i, for $1 \leq i \leq m$. Finally, observe that $P_1, P_2, \ldots, P_{m-1}$ are indecomposable tilting modules. Thus by (7.2), we have the following:

$$T_1 = L_1, \qquad T_2 = \substack{L_1\\L_2\\L_1}, \qquad T_i = \substack{L_{i-1}\\L_i\oplus L_{i-2}\\L_i}, \qquad (3 \leq i \leq m)$$

From this it is also clear that $\dim_K \operatorname{Hom}_{\mathscr{A}_m}(T_i, T_j) = 1$ whenever $|i-j| = 1$. Hence by [21, Proposition 3.2], $\mathscr{A}_m' := \operatorname{End}_{\mathscr{A}_m}(\oplus_{i=1}^m T_i) \cong \mathscr{A}_m$ as quasi-hereditary algebras.. □

Corollary 7.23. *Let B be a block of a Schur algebra. If B has finite representation type, then B is Ringel self-dual.*

7.2.5 Quantized Schur algebras

Finally, we conclude with some remarks on quantized Schur algebras. Quantized Schur algebras $S_q(n, r)$, or q-Schur algebras, were introduced by Dipper and James in [10] as a generalisation or quantization (deformation) of the classical Schur algebras. This is in analogy with the relationship between symmetric group algebras and their deformations, known as the Hecke algebras of type A, coming from the general linear groups. We refer the reader to [13] for a detailed account of the q-Schur algebras, their representation theory, and connections with the representation theory of Hecke algebras and quantum general linear groups.

For all natural numbers n and r and non-zero elements $q \in K$, the q-Schur algebra $S_q(n, r)$ is also quasi-hereditary with respect to $(\Lambda^+(n, r), \trianglelefteq)$. Thus, a natural extension of the question considered in the first part of this chapter is the following:

Question 7.24. *Which q-Schur algebras are Ringel self-dual?*

Since q-Schur algebras have the same indexing posets for their irreducible modules as their corresponding classical Schur algebra, Proposition 7.1 implies that if $S_q(n, r)$ is Ringel self-dual then $n \leq 2$, $n \geq r$ or $(n, r) \in \{(3, 4), (3, 5), (3, 7), (3, 8), (4, 5)\}$. Donkin has proved self-duality in the case $n \geq r$ in [13, §4.1], and the remaining cases are still open.

As in the classical case, a semisimple q-Schur algebra $S_q(n, r)$ is Ringel self-dual (for our current definition) if and only if $\Lambda^+(n, r)$ is reversible. Thus in order to classify the Ringel self-dual q-Schur algebras, it remains to investigate those $S_q(n, r)$ which are not semisimple. Semisimplicity of q-Schur algebras was determined by Erdmann and Nakano in [23].

Moreover, the blocks of q-Schur algebras are described by Cox in [7, Theorem 5.3]. Assuming that a non-semisimple q-Schur algebra $S_q(n,r)$ is Ringel self-dual further imposes combinatorial restrictions on the posets indexing irreducible modules in each block, as in Section 7.2.2, and we may then similarly consider decomposition numbers for the corresponding Hecke algebras and filtration multiplicities for tilting modules of $S_q(n,r)$.

Extending our techniques from this chapter to the case of q-Schur algebras is a first step towards tackling our primary goal along this line of research, which is to answer Question 7.24 and provide a classification of the Ringel self-dual q-Schur algebras.

Appendix A

Structure of partition posets

The vertices at small distance from the minimal element of $H(n, r)$ when $4 \leq n < r$ (excluding $(n, r) = (4, 5)$) are as follows:

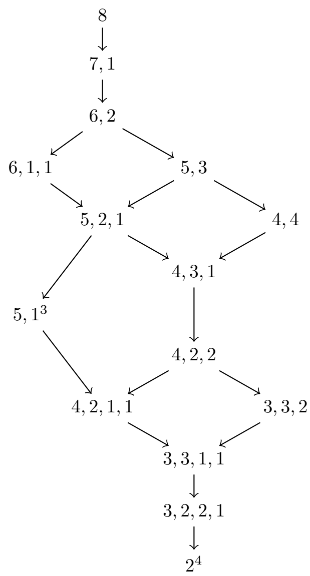


Figure A.1: $H(4, 8)$.

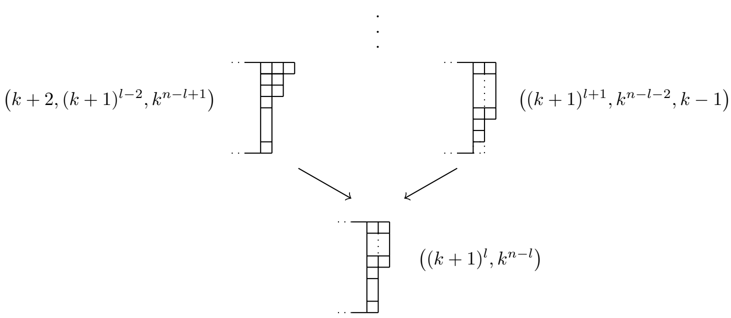


Figure A.2: $n \geq 4$, $r = nk + l$, $l \in \{2, 3, \ldots, n-2\}$.

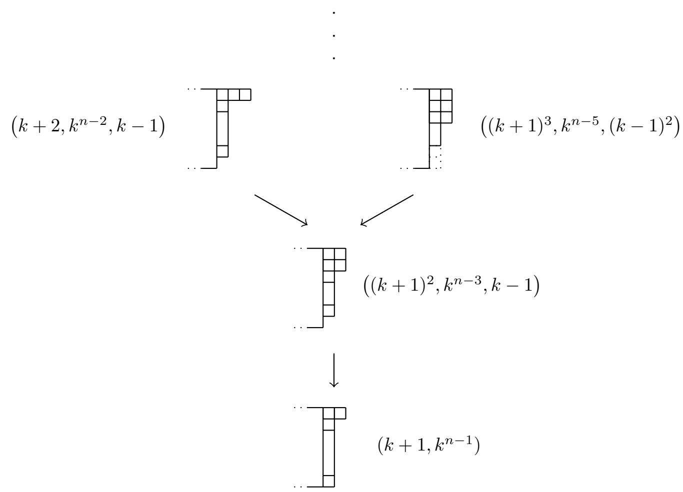


Figure A.3: $n \geq 5$, $r = nk + 1$.

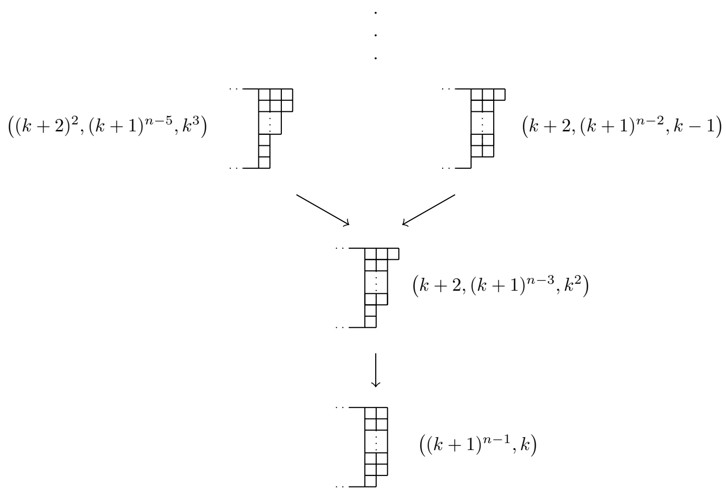


Figure A.4: $n \geq 5$, $r = nk + n - 1$.

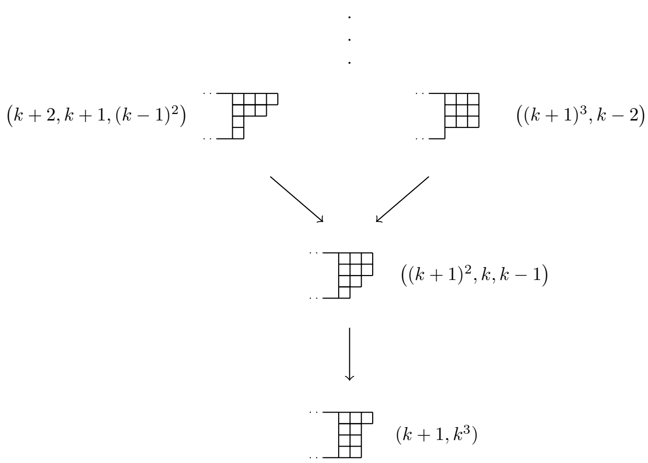


Figure A.5: $n = 4$, $r = 4k + 1$, $k \geq 2$.

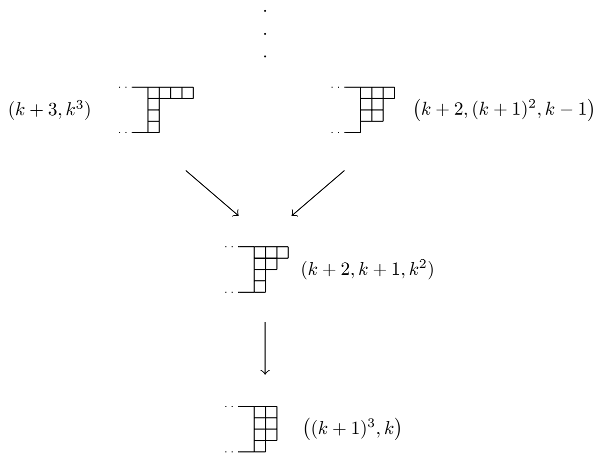


Figure A.6: $n = 4$, $r = 4k + 3$.

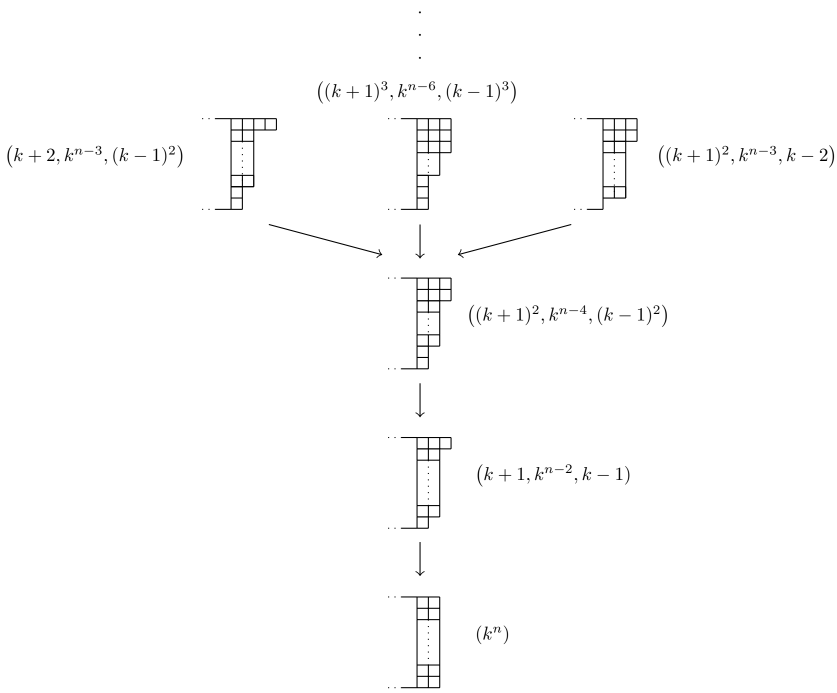


Figure A.7: $n \geq 6$, $r = nk$.

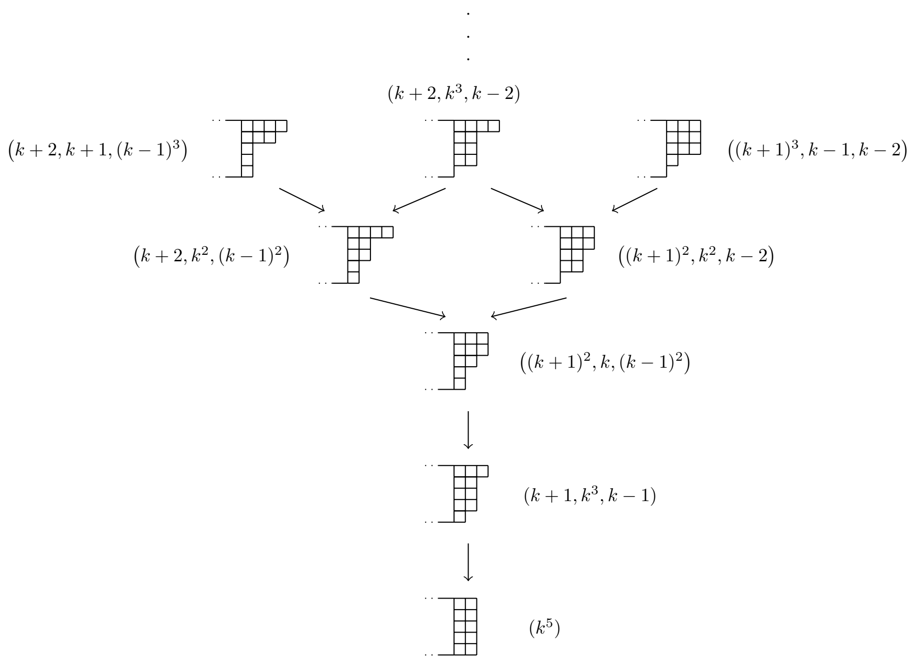


Figure A.8: $n = 5$, $r = 5k$.

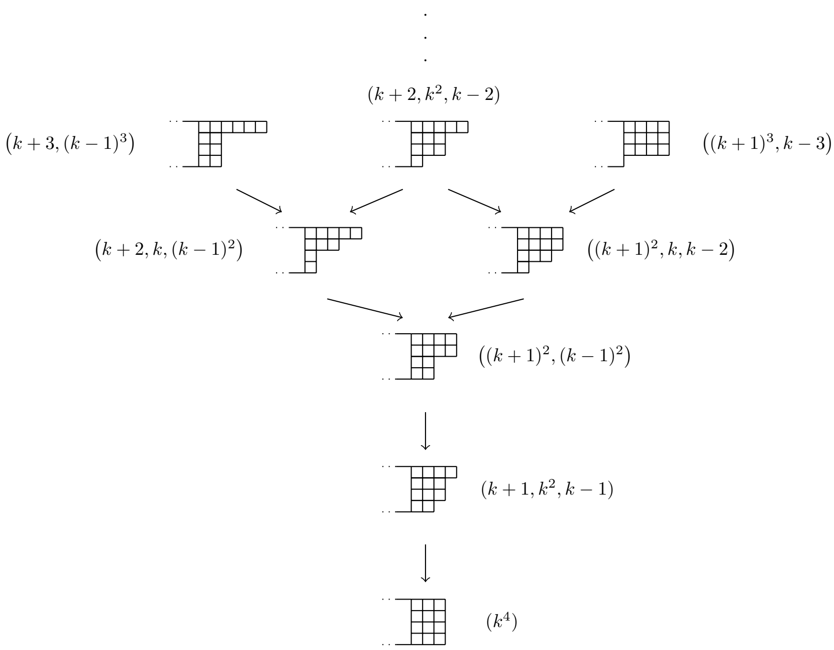


Figure A.9: $n = 4$, $r = 4k$, $k \geq 3$.

Appendix B

Tilting matrices and decomposition numbers

For the tilting matrices considered, entries in the columns $T(\lambda)$ for p-regular λ equal certain decomposition numbers of symmetric groups, which may be found in [40, Appendix] or calculated using [40, Theorems 21.11, 24.1], for instance. Also, composition factors of $\Delta(r)$ always occur with multiplicity one, by [**?**].

(n,r,p)	Composition factors of $\Delta(r)$	Example $\lambda,\mu \vdash r$ s.t. $(T(\lambda):\Delta(\mu))$	$\neq$	$[\Delta(t\mu):L(t\lambda)]$
$(3,4,3)$	$L(2^2), L(4)$	$(T(3,1):\Delta(2,1^2))=0$	,	$[\Delta(4):L(2^2)]=1$
$(3,5,2)$	$L(3,1^2), L(3,2), L(5)$	$(T(5):\Delta(2^2,1))=1$	,	$[\Delta(5):L(2^2,1)]=0$
$(3,5,5)$	$L(4,1), L(5)$	$(T(3,1^2):\Delta(2^2,1))=0$	,	$[\Delta(5):L(4,1)]=1$
$(3,7,2)$	$L(3^2,1), L(5,1^2),$ $L(3,2^2), L(7)$	$(T(6,1):\Delta(3,2^2))=0$	,	$[\Delta(7):L(3^2,1)]=1$
$(3,7,3)$	$L(5,2), L(7)$	$(T(6,1):\Delta(3,2^2))=1$	,	$[\Delta(7):L(3^2,1)]=0$
$(3,7,7)$	$L(6,1), L(7)$	$(T(3^2,1):\Delta(3,2^2))=0$	,	$[\Delta(7):L(6,1)]=1$
$(3,8,2)$	$L(3^2,2), L(7,1),$ $L(6,2), L(4^2), L(8)$	$(T(8):\Delta(3^2,2))=2$	,	$[\Delta(8):L(3^2,2)]=1$
$(3,8,3)$	$L(5,2,1), L(8)$	$(T(5,3):\Delta(3^2,2))=0$	,	$[\Delta(8):L(5,2,1)]=1$
$(3,8,5)$	$L(4^2), L(8)$	$(T(6,1^2):\Delta(3^2,2))=0$	,	$[\Delta(8):L(4^2)]=1$
$(3,8,7)$	$L(6,2), L(8)$	$(T(4,3,1):\Delta(3^2,2))=0$	,	$[\Delta(8):L(6,2)]=1$
$(4,5,2)$	$L(3,1^2), L(3,2), L(5)$	$(T(4,1):\Delta(2,1^3))=1$	,	$[\Delta(5):L(2^2,1)]=0$
$(4,5,3)$	$L(2^2,1), L(5)$	$(T(4,1):\Delta(2,1^3))=0$	,	$[\Delta(5):L(2^2,1)]=1$
$(4,5,5)$	$L(4,1), L(5)$	$(T(3,1^2):\Delta(2,1^3))=1$	,	$[\Delta(5):L(3,2)]=0$

Table B.1: Data giving a second proof of Theorem 7.18; see Remark 7.19.

For $(n,r,p)=(3,7,5)$, the composition factors of $\Delta(7)$ are $L(4,3)$ and $L(7)$. While the order-reversing isomorphism t on $\Lambda^+(3,7)$ is not unique (t either fixes $(5,1^2)$ or $t(5,1^2) = (4,3)$), we have that $(T(4,3) : \Delta(3,2^2)) = (T(5,1^2) : \Delta(3,2^2)) = 0$ but $[\Delta(7) : L(4,3)] = 1$.

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