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Designs for graphs with six vertices and ten edges

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Abstract

The design spectrum has been determined for two of the 15 graphs with six vertices and ten edges. In this paper we completely solve the design spectrum problem for a further eight of these graphs.

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1. Introduction

Throughout this paper all graphs are simple. Let G be a graph. If the edge set of a graph K can be partitioned into edge sets of graphs each isomorphic to G, we say that there exists a *decomposition* of K into G. In the case where K is the complete graph K_n we refer to the decomposition as a G-*design* of order n. The *design spectrum* of G is the set of non-negative integers n for which there exists a G-design of order n. For completeness, we remark that the empty set is a G-design of order 0 as well as 1; these trivial cases are usually assumed henceforth. A complete solution of the spectrum problem often seems to be difficult. However it has been achieved in many cases, especially amongst the smaller graphs. We refer the reader to the survey article of Adams, Bryant and Buchanan, [2] and, for more up to date results, the Web site maintained by Bryant and McCourt, [5]. If the graph G has v vertices, e edges, and if d is the greatest common divisor of the

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vertex degrees, then a G-design of order n can exist only if the following conditions hold:

$$\begin{array}{ll} \text{(i)} & n \leq 1 \text{ or } n \geq v, \\ \text{(ii)} & n-1 \equiv 0 \pmod{d}, \\ \text{(iii)} & n(n-1) \equiv 0 \pmod{2e}. \end{array} \qquad (1)$$

Except where (i) of (1) applies, adding an isolated vertex to a graph does not affect its design spectrum.

The problem for small graphs has attracted attention. As far as the authors are aware, the design spectrum problem has been completely solved for the following.

(i) All graphs with at most five vertices. For details and references, see [2] and [5]. For more recent results, see [11], [17] and [9].

(ii) All graphs with six vertices and at most eight edges. Again, see [2] and [5] for details and references. The two cases left undecided in [2] were successfully resolved by Forbes, Griggs & Forbes, [7].

(iii) All graphs with six vertices and nine edges; See [2], [5], [1], [13], [19], [16], and the recent paper of Forbes & Griggs, [6], where the spectrum problem for 6-vertex, 9-edge graphs is completely solved.

In Table 1 we list the 15 graphs with six vertices and ten edges. The numbering in the first column corresponds to the ordering of the ten-edge graphs within the list of all 156 graphs with six vertices, available at [18]. The second column identifies the graphs as they appear in *An Atlas of Graphs* by Read & Wilson, [20]. The third column contains the edge sets which we use in our computations; the vertices are labelled in non-increasing order of degree.

Graph n_9 is K_5 with an additional, isolated vertex, and its spectrum is that of K_5 except that there is no design of order 5. Hence it follows from the classic result of Hanani, [14], that an n_9 design of order n exists if and only if $n \equiv 1$ or $5 \pmod{20}$ and $n \neq 5$. Graph n_{11} is a closed 10-trail and its design spectrum, determined by Adams, Bryant & Khodkar, [3], is the same as that of n_9. For the remaining thirteen graphs of Table 1, it is easily established that designs of order n exist only if $n \equiv 0,\ 1,\ 5,\ 16 \pmod{20}$ and $n \neq 5$. In this paper we show that these conditions are sufficient for eight of the graphs. We prove the following.

Theorem 1.1. *Designs of order n exist for graphs n_1, n_2, n_4, n_5, n_7, n_{12}, n_{14} and n_{15} if and only if $n \equiv 0,\ 1,\ 5,\ 16 \pmod{20}$ and $n \neq 5$.*

We also have partial results for the other graphs. Generally, we can say that we have solved the design spectrum problem for all 6-vertex, 10-edge graphs with at most 16 possible exceptions. Specifically, we can prove that designs for the remaining five graphs exist for two of the four admissible residue classes given above, namely (i) the 'easy' case, $n \equiv 1 \pmod{20}$, and (ii) $n \equiv 5 \pmod{20}$, $n \neq 5$, as well as various subclasses of $n \equiv 0, 16 \pmod{20}$, namely (iii) $n \equiv 0,\ 40,\ 100 \pmod{120}$ and (iv) $n \equiv 36,\ 56,\ 76,\ 116 \pmod{120}$. The results presented in this paper are for the eight graphs where we have the entire design spectrum. In all cases there exist decompositions of the complete graphs K_{16} and K_{20}, which allows for a uniform treatment of all

Table 1. The 15 graphs with 6 vertices and 10 edges

n_1	G179	{{4,3},{4,2},{4,1},{6,2},{6,1},{5,2},{5,1},{3,2},{3,1},{2,1}}
n_2	G180	{{4,3},{4,2},{4,1},{6,3},{6,1},{5,2},{5,1},{3,2},{3,1},{2,1}}
n_3	G177	{{5,3},{5,2},{5,1},{4,3},{4,2},{4,1},{6,1},{3,2},{3,1},{2,1}}
n_4	G182	{{5,3},{5,2},{5,1},{4,3},{4,2},{4,1},{6,2},{6,1},{3,1},{2,1}}
n_5	G186	{{5,3},{5,2},{5,1},{4,3},{4,2},{4,1},{6,3},{6,2},{3,1},{2,1}}
n_6	G189	{{6,2},{6,3},{6,1},{5,2},{5,3},{5,1},{4,2},{4,3},{4,1},{2,1}}
n_7	G183	{{5,3},{5,2},{5,1},{4,6},{4,2},{4,1},{3,2},{3,1},{6,1},{2,1}}
n_8	G190	{{6,4},{6,2},{6,1},{5,3},{5,2},{5,1},{4,2},{4,1},{3,2},{3,1}}
n_9	G176	{{5,4},{5,3},{5,2},{5,1},{4,3},{4,2},{4,1},{3,2},{3,1},{2,1}}
n_{10}	G178	{{4,3},{4,2},{4,5},{4,1},{6,1},{3,2},{3,5},{3,1},{2,5},{2,1}}
n_{11}	G181	{{4,3},{4,5},{4,2},{4,1},{6,2},{6,1},{3,5},{3,2},{3,1},{2,1}}
n_{12}	G185	{{3,2},{3,5},{3,4},{3,1},{6,4},{6,1},{2,5},{2,4},{2,1},{5,1}}
n_{13}	G187	{{6,4},{6,3},{6,1},{5,3},{5,2},{5,1},{4,2},{4,1},{3,1},{2,1}}
n_{14}	G184	{{3,6},{3,4},{3,2},{3,1},{5,4},{5,2},{5,1},{6,2},{4,1},{2,1}}
n_{15}	G188	{{2,5},{2,4},{2,3},{2,1},{6,4},{6,3},{6,1},{5,3},{5,1},{4,1}}

eight graphs. Proofs of results for the remaining five graphs are likely to be more involved and the details are deferred to a future paper.

Theorem 1.1 is proved in Section 2. For our computations and in the presentation of our results, we represent the labelled graph n_i by a subscripted ordered 6-tuple $(z_1, z_2, \ldots, z_6)_i$, where $z_1 = 1$, $z_2 = 2$, …, $z_6 = 6$ give the edge sets in Table 1 and the illustrations in Figure 1. For a graph G with 10 edges, the numbers of occurrences of G in a decomposition into G of the complete graph K_n, the complete r-partite graph K_{n^r} and the complete $(r+1)$-partite graph $K_{n^r m^1}$ are respectively

$$\frac{n(n-1)}{20}, \quad \frac{n^2 r(r-1)}{20} \quad \text{and} \quad \frac{nr(n(r-1)+2m)}{20}.$$

2. Proof of Theorem 1.1

We use Wilson's construction involving group divisible designs. Recall that a K-GDD of type $g_1^{t_1} \ldots g_r^{t_r}$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where V is a base set of cardinality $v = t_1 g_1 + \cdots + t_r g_r$, $\mathcal{G}$ is a partition of V into t_i subsets of cardinality g_i, $i = 1, \ldots, r$, called *groups* and $\mathcal{B}$ is a collection of subsets of cardinalities $k \in K$, called *blocks*, which collectively have the property that each pair of elements from different groups occurs in precisely one block but no pair of elements from the same group occurs at all. A $\{k\}$-GDD is also called a k-GDD. As is well known, if there exist $k-2$ mutually orthogonal Latin squares (MOLS) of side q, then there exists a k-GDD of type q^k. So when q is a prime power there exists a q-GDD of type q^q and a $(q+1)$-GDD of type q^{q+1} (obtained from affine and projective planes of order q respectively). A *parallel class* in a group divisible design is a subset of the block set in which each element of the base set appears exactly

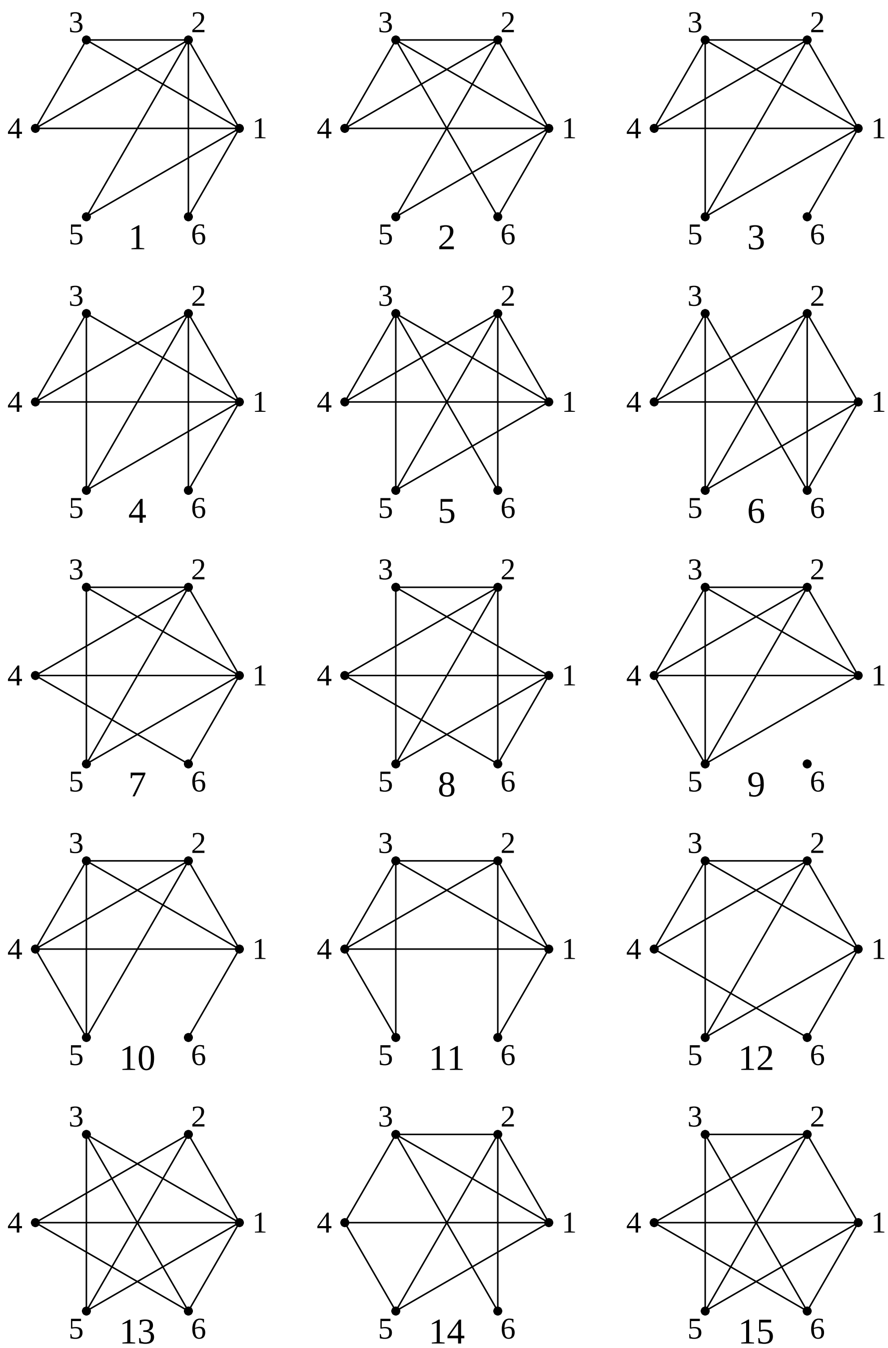

Figure 1. Graphs with 6 vertices and 10 edges.

once. A k-GDD is called *resolvable*, and denoted by k-RGDD, if the entire set of blocks can be partitioned into parallel classes.

Proposition 2.1. *Let i, t, p, q be positive integers. Let w, x, y be non-negative integers such that*

$x+y=w$ and $w \le 4t$. Let $e=0$ or 1. Suppose there exist decompositions into the graph G of the complete graphs K_{4i+e} and $K_{xp+yq+e}$ as well as the complete multipartite graphs $K_{i,i,i,i}$, $K_{i,i,i,i,p}$ and $K_{i,i,i,i,q}$. Then, there exists a G-design of order $12it+4i+xp+yq+e$.

Proof. There exists a 4-RGDD of type 4^{3t+1} for $t \ge 1$, [15], see also [12], and a simple computation establishes that it has $4t$ parallel classes of $3t+1$ blocks each. If $w > 0$, add a new group of w points, associate with each new point a distinct parallel class and extend each of its blocks by adjoining the point to them, thus creating $w(3t+1)$ five-element blocks. This is possible since w does not exceed the number of parallel classes of the 4-RGDD. Thus we either have a 4-GDD of type 4^{3t+1}, or we have created a $\{4,5\}$- or 5-GDD of type $4^{3t+1}w^1$.

Inflate x and y points in the new group by a factor of p and q respectively. Inflate points in the original groups by a factor of i. Thus the blocks of the GDD are replaced by graphs $K_{i,i,i,i}$, $K_{i,i,i,i,p}$ and $K_{i,i,i,i,q}$. If $e=1$, add an extra point. Overlay each inflated group, together with the extra point if $e=1$, with either K_{4i+e} or $K_{xp+yq+e}$, as appropriate. Since we assume the existence of decompositions into G of all of the component graphs, the construction yields a G-design of order $12it+4i+xp+yq+e$ whenever $t \ge 1$ and $t \ge w/4$. □

Before applying Proposition 2.1 we establish the existence of the various decompositions that we require to make the construction work and for handling the sporadic design orders where the construction fails.

Lemma 2.1.

(i) Designs of orders 16, 20, 21, 25, 36, 40, 41, 45, 56 and 65, exist for graphs n_1, n_2, n_4, n_5, n_7, n_{12}, n_{14} and n_{15}.

(ii) Designs of orders 60 and 61 exist for graphs n_1, n_2, n_7 and n_{12}.

(iii) A design of order 85 exists for graph n_{15}.

Proof. The decompositions are presented in the Appendix. □

Lemma 2.2.

(i) There exist decompositions of complete multipartite graphs $K_{10,10,10,10}$, $K_{20,20,20,20}$, $K_{5,5,5,5,5}$, $K_{10,10,10,10,10}$, $K_{15,15,15,15,15}$, $K_{10,10,10,10,15}$, $K_{10,10,10,10,20}$, K_{16^6} and $K_{20^5,16}$ into each of graphs n_1, n_2, n_4, n_5, n_7, n_{12}, n_{14} and n_{15}.

(ii) There exist decompositions of $K_{16,16,16,16,21}$ and $K_{20,20,20,20,25}$ into each of graphs n_1, n_2, n_4, n_5, n_7, n_{12} and n_{14}.

(iii) There exists a decomposition of $K_{10,10,10}$ into each of graphs n_4, n_5, n_{14} and n_{15}.

(iv) There exists a decomposition of $K_{3,3,3,3,3}$ into graph n_{15}.

Proof. The decompositions are presented in the Appendix. □

Lemma 2.3.

(i) Designs of orders 60 and 61 exist for graphs n_4, n_5, n_{14} and n_{15}.

(ii) Designs of orders 76, 80, 81, 85, 96, 100, 101, 105, 116 and 125 exist for graphs n_1, n_2, n_4, n_5, n_7, n_{12}, n_{14} and n_{15}.

Proof. We give only brief details by specifying the ingredients for Wilson's construction, namely the complete graphs, the complete multipartite graphs and the group divisible designs. It should be clear how the points of the GDDs are inflated and which GDDs are augmented by an extra point. Decompositions of the ingredients exist by Lemmas 2.1 and 2.2.

- Orders 60 and 61 for n_4, n_5, n_{14} and n_{15} are constructed from decompositions of K_{20}, K_{21} and $K_{10,10,10}$, and a 3-GDD of type 2^3 (obtained from the projective plane of order 2).
- Order 76 (for all graphs) is constructed from decompositions of K_{16} and $K_{15,15,15,15,15}$, and the trivial 5-GDD of type 1^5.
- Orders 80 and 81 are constructed from decompositions of K_{20}, K_{21} and $K_{20,20,20,20}$, and the trivial 4-GDD of type 1^4.
- Order 85 for all graphs except n_{15} is constructed from decompositions of K_{16}, K_{21} and $K_{16,16,16,16,21}$, and the trivial 5-GDD of type 1^5.
- Order 85 for graph n_{15} is given by Lemma 2.1.
- Order 96 is constructed from decompositions of K_{16} and K_{16^6}, and the trivial 6-GDD of type 1^6.
- Orders 100 and 101 are constructed from decompositions of K_{20}, K_{21} and $K_{5,5,5,5,5}$, and a 5-GDD of type 4^5 (obtained from a projective plane of order 4).
- Order 105 for all graphs except n_{15} is constructed from decompositions of K_{20}, K_{25} and $K_{20,20,20,20,25}$, and the trivial 5-GDD of type 1^5.
- Order 105 for graph n_{15} is constructed from decompositions of K_{21}, $K_{3,3,3,3,3}$, and a 5-GDD of type 7^5 obtained by removing two groups from a 7-GDD of type 7^7 (obtained from an affine plane of order 7), i.e. a set of three MOLS of side 7.
- Order 116 is constructed from decompositions of K_{16}, K_{20} and $K_{20^5,16}$, and the trivial 6-GDD of type 1^6.
- Order 125 is constructed from decompositions of K_{25} and $K_{5,5,5,5,5}$, and a 5-GDD of type 5^5 (obtained from an affine plane of order 5).

□

Lemma 2.4.

(i) There exist decompositions of complete multipartite graphs $K_{6,6,6,6,6}$, K_{4^6}, $K_{4^6,10}$ and K_{15^9} into each of graphs n_1, n_2, n_4, n_5, n_7, n_{12}, n_{14} and n_{15}.

(ii) There exist decompositions of $K_{5^5,10}$ and $K_{15^9,20}$ into each of graphs n_1, n_2, n_4, n_5, n_7, n_{12} and n_{14}.

(iii) There exists a decomposition of $K_{10,10,10,15}$ into graph n_{15}.

Table 2. The main construction

order	t	w	x	p	y	q	e	missing values
$120t + 40$	$t \geq 1$	0	0	-	0	-	0	40
$120t + 40 + 20$	$t \geq 1$	1	0	-	1	20	0	60
$120t + 40 + 40$	$t \geq 1$	2	0	-	2	20	0	80
$120t + 40 + 60$	$t \geq 1$	3	0	-	3	20	0	100
$120t + 40 + 80$	$t \geq 1$	4	0	-	4	20	0	120
$120t + 40 + 100$	$t \geq 2$	5	0	-	5	20	0	20, 140, 260
$120t + 40 + 1$	$t \geq 1$	0	0	-	0	-	1	41
$120t + 40 + 21$	$t \geq 1$	1	0	-	1	20	1	61
$120t + 40 + 41$	$t \geq 1$	2	0	-	2	20	1	81
$120t + 40 + 61$	$t \geq 1$	3	0	-	3	20	1	101
$120t + 40 + 81$	$t \geq 1$	4	0	-	4	20	1	121
$120t + 40 + 101$	$t \geq 2$	5	0	-	5	20	1	21, 141, 261
$120t + 40 + 125$	$t \geq 2$	7	3	15	4	20	0	45, 165, 285
$120t + 40 + 25$	$t \geq 1$	2	1	10	1	15	0	65
$120t + 40 + 45$	$t \geq 1$	3	3	15	0	-	0	85
$120t + 40 + 65$	$t \geq 1$	4	3	15	1	20	0	105
$120t + 40 + 85$	$t \geq 2$	5	3	15	2	20	0	5, 125, 245
$120t + 40 + 105$	$t \geq 2$	6	3	15	3	20	0	25, 145, 265
$120t + 40 + 16$	$t \geq 1$	1	1	15	0	-	1	56
$120t + 40 + 36$	$t \geq 1$	2	1	15	1	20	1	76
$120t + 40 + 56$	$t \geq 1$	3	1	15	2	20	1	96
$120t + 40 + 76$	$t \geq 1$	4	1	15	3	20	1	116
$120t + 40 + 96$	$t \geq 2$	5	1	15	4	20	1	16, 136, 256
$120t + 40 + 116$	$t \geq 2$	6	1	15	5	20	1	36, 156, 276

Proof. The decompositions are presented in the Appendix. □

We are now ready to prove Theorem 1.1. For the main construction, we use Proposition 2.1 with $i = 10$, and p and q taking values from $\{10, 15, 20\}$ as indicated in Table 2. The inflated blocks of the group divisible design become multipartite graphs $K_{10,10,10,10}$, $K_{10,10,10,10,10}$, $K_{10,10,10,10,15}$ and $K_{10,10,10,10,20}$. With the decompositions of Lemmas 2.1, 2.2 and 2.3 we obtain the designs listed in Table 2. Combining the residue classes modulo 120, we see that there exist designs of order n, $n \equiv 0$, 1, 5 and 16 (modulo 20) except for those orders listed under 'missing values'. The missing values are handled as follows.

Designs of order 5 do not exist because the graphs have more than five vertices. Orders 20, 40, 60, 80, 100, 21, 41, 61, 81, 101, 25, 45, 65, 85, 105, 125, 16, 36, 56, 76, 96 and 116 are given by Lemmas 2.1 and 2.3. For the others, we give brief details by specifying the ingredients for Wilson's construction, namely the complete graphs, the complete multipartite graphs and the group divisible designs. Unless it is clear we also indicate how the points of the GDD are inflated and whether the GDD is augmented by an extra point. Decompositions of the ingredients exist by Lemmas 2.1, 2.2 and 2.4.

- Orders 120 and 121 are constructed from decompositions of K_{20}, K_{21} and $K_{5,5,5,5,5,5}$, and a 5-GDD of type 4^6, [10], see also [8].

- Orders 140 and 141 are constructed from decompositions of K_{20}, K_{21} and $K_{10,10,10,10}$, and a 4-GDD of type 2^7, [4], [8].

- Orders 260 and 261 are constructed from decompositions of K_{20}, K_{21} and $K_{10,10,10,10}$, and a 4-GDD of type 2^{13}, [4], [8].

- Order 145 is constructed from decompositions of K_{25} and $K_{6,6,6,6,6}$, and a 5-GDD of type 4^6. The GDD is augmented with an extra point.

- Order 165 for all graphs except n_{15} is constructed from decompositions of K_{25}, K_{40}, $K_{5,5,5,5,5}$ and $K_{5^5,10}$, and a $\{5,6\}$-GDD of type $5^5 4^1$ obtained from a 6-GDD of type 5^6 (obtained from a projective plane of order 5) by removing one point from one group. The four points of the reduced group are inflated by a factor of 10, all other points by a factor of 5.

- Order 165 for graph n_{15} is constructed from decompositions of K_{40}, K_{45}, $K_{10,10,10,10}$ and $K_{10,10,10,15}$, and a 4-GDD of type 4^4. Inflate one point by a factor of 15, all other points by a factor of 10.

- Order 245 is constructed from decompositions of K_{40}, K_{45}, $K_{10,10,10,10,10}$ and $K_{10,10,10,10,15}$, and a 5-GDD of type 4^6. Inflate one point by a factor of 15, all others by a factor of 10.

- Order 265 is constructed from decompositions of K_{45} and K_{4^6}, and a 6-GDD of type 11^6 obtained by removing 5 groups from an 11-GDD of type 11^{11} (obtained from an affine plane of order 11). The GDD is augmented with an extra point.

- Order 285 is constructed from decompositions of K_{21}, K_{45}, K_{4^6} and $K_{4^6,10}$, and a $\{6,7\}$-GDD of type $11^6 2^1$ obtained by removing 4 groups and 9 points of another group from an 11-GDD of type 11^{11}. Inflate the points in the group of size 2 by a factor of 10, all other points by a factor of 4. The GDD is augmented with an extra point.

- Order 136 is constructed from decompositions of K_{16} and K_{15^9}, and the trivial 9-GDD of type 1^9.

- Order 156 for all graphs except n_{15} is constructed from decompositions of K_{16}, K_{21} and $K_{15^9,20}$, and the trivial 10-GDD of type 1^{10}.

- Order 156 for graph n_{15} is constructed from decompositions of K_{36}, K_{41}, $K_{10,10,10}$, $K_{10,10,10,10}$ and $K_{10,10,10,15}$, and a $\{3,4\}$-GDD of type $4^3 3^1$ obtained by removing one point from a 4-GDD of type 4^4. Inflate one point in the reduced group by a factor of 15, all other points by a factor of 10.

- Order 256 is constructed from an affine plane of order 16 by replacing each block with a decomposition of K_{16}.
- Order 276 is constructed from decompositions of K_{56}, $K_{5,5,5,5,5,5}$, and a 5-GDD of type 11^5 obtained by removing 6 groups from an 11-GDD of type 11^{11}, i.e. a set of three MOLS of side 11. □

3. Concluding Remarks

Most of the decompositions in the Appendix were obtained by a special computer program written in the C language. Designs where existence could not be decided by this program include order 60 for each of graphs n_1, n_2, n_7. In these cases we adapted the method outlined in [7]. We describe the construction of the n_1 design of order 60 in some detail.

We split the graph n_1 into two parts, K_4 induced by vertices 1,2,3,4, and C_4 with vertices 1,2,5,6 (see Figure 1). We assume the K_{60} to be partitioned is labelled with the elements of $N = \{0, 1, \dots, 59\}$. We create a partial Steiner system PS$(2, 4, 60)$, $\mathcal{S}$ say, with point set N, 177 blocks and automorphism $x \mapsto x + 20 \pmod{60}$. So the blocks of $\mathcal{S}$ occur in 59 orbits of size 3. For graph n_1 we have the additional requirement that $\mathcal{S}$ must have no point of even degree.

Next, we create the T-K matrix $\mathbf{M}$ for assembling the graphs C_4. Here, $T = T_1 \cup T_2$, where T_1 is the set of edge orbits (under the action of $x \mapsto x + 20 \pmod{60}$) in the leave of $\mathcal{S}$, and T_2 is the set of block orbits of $\mathcal{S}$. To each ordered 4-tuple J of integers we associate a 4-cycle graph $H(J) \cong C_4$ labelled with the elements of J. Denote by $E(J)$ the set of edge orbits corresponding to the edges of $H(J)$. We retain precisely those J where (i) $J \subset N$, (ii) the elements of J are distinct, (iii) $E(J) \subset T_1$, (iv) the elements of $E(J)$ are distinct, and (v) there is a block orbit $B(J) \in T_2$ such that $B(J)$ includes a block which contains two vertices of $H(J)$. Then K is the set of all those $E(J) \cup \{B(J)\}$ that correspond to ordered 4-tuples J satisfying (i)–(v). The matrix $\mathbf{M}$ is defined for $t \in T$ and $k \in K$ by $\mathbf{M}_{t,k} = 1$ if $t \in k$, $\mathbf{M}_{t,k} = 0$ otherwise.

We attempt to solve $\mathbf{Mv} = \mathbf{1}$ for $\mathbf{v}$ a $\{0, 1\}$ vector. If we are unsuccessful, we try another PS(2,4,60). Otherwise we recover the fifty-nine 4-tuples J corresponding to 1s in $\mathbf{v}$ and hence the orbits of the graphs K_4 and C_4 of our decomposition. Finally, we pair off orbits of K_4 graphs and orbits of C_4 graphs such that in each pair of orbits we choose a representative K_4 and a representative C_4 that have two labels in common. This is guaranteed to succeed because of condition (v) above.

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Appendix: Decompositions

Proof of Lemma 2.1

$\boldsymbol{K_{16}}$ With vertex set $Z_{15} \cup \{\infty\}$ the decompositions consist of
$(\infty, 8, 14, 5, 12, 6)_1$, $(13, 8, 1, 7, 0, 9)_1$, $(5, 2, 10, 6, 4, 7)_1$,
$(11, 9, 4, 12, 5, 6)_1$,
$(\infty, 0, 14, 7, 13, 6)_2$, $(4, 6, 9, 3, 7, 0)_2$, $(6, 11, 5, 13, 2, 10)_2$,
$(13, 7, 12, 10, 9, 8)_2$,
$(\infty, 0, 14, 6, 8, 2)_4$, $(6, 11, 4, 12, 10, 8)_4$, $(10, 7, 5, 9, 8, 2)_4$,
$(8, 12, 4, 1, 9, 3)_4$,
$(0, 4, 10, 3, 2, \infty)_5$, $(1, 12, 13, \infty, 2, 4)_5$, $(8, 10, 13, 7, 6, 9)_5$,
$(1, 6, 9, 0, 4, 12)_5$,
$(\infty, 8, 11, 9, 2, 10)_7$, $(6, 13, 9, 0, 4, 11)_7$, $(6, 2, 7, 5, 14, 8)_7$,
$(2, 10, 0, 3, 4, 13)_7$,
$(0, 11, 9, 1, 7, \infty)_{12}$, $(7, 12, 6, 0, 3, 5)_{12}$, $(5, 6, 9, 8, 13, 3)_{12}$,
$(3, 2, 9, 14, \infty, 0)_{12}$,
$(13, 0, 14, 4, 1, \infty)_{14}$, $(2, 13, 3, \infty, 6, 10)_{14}$, $(0, 4, 7, 5, 11, 10)_{14}$,
$(4, 6, 12, 2, 8, 11)_{14}$,
under the action of the mapping $x \mapsto x + 5 \pmod{15}$, $\infty \mapsto \infty$, and
$(8, 10, 5, 14, 7, 4)_{15}$, $(12, 14, 9, 2, 11, 8)_{15}$, $(0, 2, 13, 6, 15, 12)_{15}$,
$(4, 6, 1, 10, 3, 0)_{15}$, $(2, 4, 13, 9, 1, 7)_{15}$, $(6, 8, 1, 13, 5, 11)_{15}$,
$(10, 12, 5, 1, 9, 15)_{15}$, $(14, 0, 9, 5, 13, 3)_{15}$, $(2, 10, 13, 11, 3, 5)_{15}$,
$(14, 6, 9, 7, 15, 1)_{15}$, $(0, 8, 15, 3, 11, 7)_{15}$, $(4, 12, 3, 7, 15, 11)_{15}$.

$\boldsymbol{K_{20}}$ With vertex set Z_{20} the decompositions consist of
$(7, 19, 4, 13, 6, 16)_1$, $(5, 11, 13, 15, 12, 18)_1$, $(3, 17, 7, 11, 0, 5)_1$,
$(1, 3, 15, 19, 2, 10)_1$, $(2, 16, 4, 11, 0, 5)_1$, $(9, 19, 12, 14, 2, 17)_1$,
$(13, 16, 1, 9, 3, 12)_1$, $(8, 18, 0, 13, 2, 3)_1$, $(15, 16, 8, 17, 6, 14)_1$,
$(4, 12, 1, 8, 17, 18)_1$, $(6, 17, 2, 13, 10, 18)_1$, $(1, 14, 7, 18, 5, 17)_1$,
$(3, 6, 4, 9, 12, 14)_1$, $(7, 12, 2, 15, 0, 10)_1$, $(10, 14, 8, 11, 2, 13)_1$,
$(10, 18, 9, 15, 16, 19)_1$, $(0, 11, 1, 6, 9, 19)_1$, $(5, 8, 7, 9, 6, 19)_1$,
$(0, 4, 5, 10, 14, 15)_1$,
$(9, 10, 19, 1, 12, 7)_2$, $(4, 9, 16, 5, 15, 14)_2$, $(7, 2, 13, 1, 3, 16)_2$,
$(17, 15, 18, 19, 1, 16)_2$, $(0, 10, 16, 8, 6, 1)_2$, $(7, 0, 18, 12, 15, 6)_2$,
$(11, 3, 18, 9, 16, 1)_2$, $(5, 8, 14, 18, 19, 1)_2$, $(4, 10, 18, 13, 7, 2)_2$,
$(15, 12, 13, 8, 11, 6)_2$, $(4, 6, 11, 19, 17, 0)_2$, $(0, 13, 17, 9, 5, 2)_2$,
$(3, 14, 19, 0, 12, 13)_2$, $(5, 3, 17, 10, 15, 12)_2$, $(6, 2, 9, 8, 5, 14)_2$,
$(1, 3, 4, 8, 6, 12)_2$, $(7, 11, 17, 8, 5, 14)_2$, $(14, 10, 11, 2, 15, 13)_2$,
$(16, 2, 12, 19, 15, 6)_2$,
$(3, 17, 7, 10, 2, 12)_4$, $(7, 1, 11, 14, 6, 16)_4$, $(11, 5, 15, 18, 10, 0)_4$,
$(15, 9, 19, 2, 14, 4)_4$, $(19, 13, 3, 6, 18, 8)_4$, $(15, 1, 0, 17, 3, 8)_4$,
$(19, 5, 4, 1, 7, 12)_4$, $(3, 9, 8, 5, 11, 16)_4$, $(7, 13, 12, 9, 15, 0)_4$,
$(11, 17, 16, 13, 19, 4)_4$, $(17, 8, 18, 9, 7, 14)_4$, $(1, 12, 2, 13, 11, 18)_4$,
$(5, 16, 6, 17, 15, 2)_4$, $(9, 0, 10, 1, 19, 6)_4$, $(13, 4, 14, 5, 3, 10)_4$,

$(0,2,4,8,12,10)_4$, $(6,2,10,14,18,4)_4$, $(8,16,12,6,10,18)_4$,
$(14,16,18,0,4,12)_4$,
$(16,7,6,15,2,12)_5$, $(14,0,7,17,10,19)_5$, $(5,14,13,11,2,12)_5$,
$(1,5,16,10,17,12)_5$, $(0,13,2,8,4,15)_5$, $(0,1,16,18,9,8)_5$,
$(9,19,12,17,2,10)_5$, $(1,3,11,2,4,0)_5$, $(15,5,12,0,3,4)_5$,
$(13,6,7,1,3,0)_5$, $(19,1,18,12,14,15)_5$, $(14,3,13,9,16,19)_5$,
$(19,8,16,4,5,11)_5$, $(11,6,15,17,19,8)_5$, $(4,6,15,9,10,14)_5$,
$(18,4,8,7,17,14)_5$, $(10,8,11,3,9,12)_5$, $(10,17,18,2,13,3)_5$,
$(5,7,18,6,9,11)_5$,
$(6,19,13,7,17,16)_7$, $(13,14,0,16,3,5)_7$, $(3,19,1,9,18,16)_7$,
$(8,5,9,3,10,7)_7$, $(15,14,2,10,9,13)_7$, $(3,17,12,2,15,11)_7$,
$(4,1,15,5,16,17)_7$, $(0,17,8,9,16,1)_7$, $(7,14,17,1,18,2)_7$,
$(4,10,3,7,6,0)_7$, $(12,13,7,4,9,2)_7$, $(2,5,0,18,6,13)_7$,
$(19,2,10,8,16,15)_7$, $(11,9,4,6,18,14)_7$, $(6,8,1,18,12,15)_7$,
$(11,1,10,13,17,8)_7$, $(14,19,5,4,12,8)_7$, $(11,15,5,0,7,19)_7$,
$(12,18,0,16,10,11)_7$,
$(15,1,4,10,18,12)_{12}$, $(19,5,8,14,2,16)_{12}$, $(3,9,12,18,6,0)_{12}$,
$(7,13,16,2,10,4)_{12}$, $(11,17,0,6,14,8)_{12}$, $(15,5,7,11,6,16)_{12}$,
$(19,9,11,15,10,0)_{12}$, $(3,13,15,19,14,4)_{12}$, $(7,17,19,3,18,8)_{12}$,
$(11,1,3,7,2,12)_{12}$, $(10,18,3,16,5,6)_{12}$, $(14,2,7,0,9,10)_{12}$,
$(18,6,11,4,13,14)_{12}$, $(2,10,15,8,17,18)_{12}$, $(6,14,19,12,1,2)_{12}$,
$(1,16,0,5,8,9)_{12}$, $(1,13,17,8,5,12)_{12}$, $(4,0,13,9,12,8)_{12}$,
$(4,16,17,12,9,5)_{12}$,
$(2,10,17,6,3,0)_{14}$, $(6,14,1,10,7,4)_{14}$, $(10,18,5,14,11,8)_{14}$,
$(14,2,9,18,15,12)_{14}$, $(18,6,13,2,19,16)_{14}$, $(6,9,4,11,8,0)_{14}$,
$(10,13,8,15,12,4)_{14}$, $(14,17,12,19,16,8)_{14}$, $(18,1,16,3,0,12)_{14}$,
$(2,5,0,7,4,16)_{14}$, $(4,18,12,3,17,7)_{14}$, $(8,2,16,7,1,11)_{14}$,
$(12,6,0,11,5,15)_{14}$, $(16,10,4,15,9,19)_{14}$, $(0,14,8,19,13,3)_{14}$,
$(7,5,13,15,3,9)_{14}$, $(1,5,19,9,17,15)_{14}$, $(11,1,13,17,15,3)_{14}$,
$(11,7,19,3,9,17)_{14}$,
$(19,1,15,13,5,11)_{15}$, $(10,9,19,4,18,0)_{15}$, $(6,9,16,11,15,10)_{15}$,
$(16,17,5,13,0,2)_{15}$, $(6,14,16,0,4,7)_{15}$, $(1,12,15,16,17,8)_{15}$,
$(0,2,3,9,18,12)_{15}$, $(9,1,3,7,14,5)_{15}$, $(1,2,8,11,4,0)_{15}$,
$(5,6,1,8,18,10)_{15}$, $(6,13,5,3,12,16)_{15}$, $(3,4,13,15,7,0)_{15}$,
$(9,17,10,8,3,13)_{15}$, $(7,18,16,12,11,19)_{15}$, $(8,11,4,3,12,19)_{15}$,
$(14,17,4,11,18,5)_{15}$, $(14,15,18,7,13,8)_{15}$, $(2,17,19,7,6,10)_{15}$,
$(2,14,10,19,12,15)_{15}$.

$\boldsymbol{K_{21}}$ With vertex set Z_{21} the decompositions consist of
$(0,1,3,13,5,7)_1$, $(0,1,3,7,10,8)_2$, $(0,1,2,5,8,10)_4$,
$(0,1,2,5,8,11)_5$, $(0,1,3,10,8,4)_7$, $(0,1,3,7,14,12)_{12}$,
$(0,1,3,7,12,9)_{14}$, $(0,1,3,4,8,15)_{15}$,
under the action of the mapping $x \mapsto x+1 \pmod{21}$.

$\boldsymbol{K_{25}}$ With vertex set Z_{25} the decompositions consist of

$(0,7,20,21,23,16)_1$, $(14,20,5,9,3,22)_1$, $(12,20,17,19,13,6)_1$,
$(18,7,19,6,22,13)_1$, $(19,1,3,9,21,22)_1$, $(1,8,20,23,4,16)_1$,
$(0,20,8,17,19,7)_2$, $(3,10,11,6,0,1)_2$, $(20,22,9,1,18,11)_2$,
$(1,7,13,14,12,8)_2$, $(14,3,9,18,17,5)_2$, $(1,2,19,17,15,4)_2$,
$(0,4,17,6,8,9)_4$, $(20,8,5,2,16,15)_4$, $(18,8,13,6,19,7)_4$,
$(13,4,17,16,15,10)_4$, $(9,22,2,7,19,10)_4$, $(1,11,21,4,17,10)_4$,
$(0,13,4,23,10,8)_5$, $(14,16,0,1,5,12)_5$, $(7,16,15,23,21,17)_5$,
$(20,11,16,19,13,3)_5$, $(22,2,12,8,9,5)_5$, $(3,2,14,4,19,21)_5$,
$(0,22,2,9,10,19)_7$, $(19,8,5,22,23,16)_7$, $(14,12,16,22,21,23)_7$,
$(14,19,15,6,13,7)_7$, $(15,1,10,23,16,11)_7$, $(2,16,18,8,23,20)_7$,
$(0,14,23,18,10,16)_{12}$, $(3,14,9,16,17,6)_{12}$, $(9,19,0,18,7,6)_{12}$,
$(6,14,7,15,1,10)_{12}$, $(16,2,12,18,15,10)_{12}$, $(2,3,11,18,0,22)_{12}$,
$(0,10,17,8,19,21)_{14}$, $(20,7,16,8,9,6)_{14}$, $(7,4,17,13,12,6)_{14}$,
$(19,14,6,12,23,8)_{14}$, $(6,0,5,9,24,2)_{14}$, $(3,0,18,21,16,23)_{14}$,
$(0,17,11,1,6,4)_{15}$, $(12,16,0,5,8,14)_{15}$, $(11,21,18,19,23,12)_{15}$,
$(3,18,7,14,2,4)_{15}$, $(22,20,10,14,7,9)_{15}$, $(0,3,9,5,21,18)_{15}$,

under the action of the mapping $x \mapsto x+5 \pmod{25}$.

$\boldsymbol{K_{36}}$ With vertex set Z_{36} the decompositions consist of

$(0,11,32,26,24,5)_1$, $(8,15,16,29,3,22)_1$, $(8,33,26,25,1,24)_1$,
$(10,29,5,19,18,11)_1$, $(34,28,31,11,2,25)_1$, $(16,2,14,17,7,18)_1$,
$(5,35,7,18,3,25)_1$,
$(0,23,4,1,17,25)_2$, $(2,4,26,22,9,3)_2$, $(30,23,3,11,28,4)_2$,
$(4,12,24,15,19,34)_2$, $(32,2,5,10,25,9)_2$, $(1,10,31,29,15,27)_2$,
$(14,13,29,11,25,3)_2$,
$(0,16,29,11,26,5)_4$, $(30,23,19,13,7,29)_4$, $(22,9,31,34,23,29)_4$,
$(16,17,12,15,25,13)_4$, $(34,32,12,3,20,14)_4$, $(23,33,4,21,19,2)_4$,
$(0,6,30,21,34,23)_4$,
$(0,26,29,7,21,3)_5$, $(11,28,16,3,7,17)_5$, $(3,23,9,34,33,5)_5$,
$(3,15,1,18,17,22)_5$, $(15,6,16,14,12,29)_5$, $(28,9,34,10,12,18)_5$,
$(2,5,24,0,34,32)_5$,
$(0,14,28,1,31,7)_7$, $(12,33,23,16,0,6)_7$, $(20,26,3,11,9,22)_7$,
$(28,21,33,26,1,8)_7$, $(19,4,17,3,14,5)_7$, $(10,25,11,14,18,26)_7$,
$(3,30,21,27,29,31)_7$,
$(0,28,15,29,22,27)_{12}$, $(2,30,29,9,3,15)_{12}$, $(20,32,22,5,2,13)_{12}$,
$(15,7,32,3,31,1)_{12}$, $(28,12,33,8,9,6)_{12}$, $(6,29,23,33,18,2)_{12}$,
$(0,13,31,6,10,3)_{12}$,
$(0,22,20,30,32,17)_{14}$, $(27,13,22,6,2,19)_{14}$, $(27,33,19,9,1,10)_{14}$,
$(6,14,0,19,7,31)_{14}$, $(30,1,18,33,12,17)_{14}$, $(12,23,0,29,20,27)_{14}$,
$(0,13,15,35,1,8)_{14}$,
$(0,9,25,23,14,11)_{15}$, $(6,22,18,34,12,21)_{15}$, $(5,13,9,31,11,28)_{15}$,
$(30,29,5,8,4,12)_{15}$, $(13,6,15,8,23,16)_{15}$, $(28,19,24,30,12,7)_{15}$,
$(2,3,10,23,7,29)_{15}$,

under the action of the mapping $x \mapsto x+4 \pmod{36}$.

$\boldsymbol{K_{40}}$ With vertex set $Z_{39} \cup \{\infty\}$ the decompositions consist of
$(16, 6, \infty, 26, 30, 27)_1$, $(38, 1, 34, 25, 19, 17)_1$, $(35, 11, 18, 6, 38, 5)_1$,
$(30, 36, 5, 37, 27, 32)_1$, $(35, 36, 13, 1, 10, 34)_1$, $(17, 25, 6, 28, 30, 33)_1$,
$(0, 38, 28, \infty, 2, 10)_2$, $(10, 22, 25, 23, 29, 6)_2$, $(28, 22, 14, 6, 15, 37)_2$,
$(32, 23, 28, 11, 17, 27)_2$, $(7, 35, 12, 15, 9, 21)_2$, $(33, 23, 12, 27, 6, 26)_2$,
$(8, 14, \infty, 28, 3, 31)_4$, $(7, 16, 5, 8, 20, 33)_4$, $(2, 18, 13, 24, 9, 21)_4$,
$(23, 13, 33, 15, 21, 36)_4$, $(20, 6, 32, 25, 2, 21)_4$, $(6, 13, 7, 1, 31, 16)_4$,
$(0, 38, 37, \infty, 24, 14)_5$, $(20, 29, 28, 0, 8, 25)_5$, $(19, 29, 8, 36, 34, 7)_5$,
$(22, 12, 21, 28, 18, 7)_5$, $(13, 21, 31, 1, 38, 29)_5$, $(24, 3, 11, 8, 15, 5)_5$,
$(17, 4, \infty, 6, 21, 14)_7$, $(7, 27, 36, 31, 33, 8)_7$, $(4, 2, 34, 18, 7, 25)_7$,
$(19, 38, 8, 13, 18, 24)_7$, $(26, 12, 4, 24, 27, 2)_7$, $(26, 14, 22, 21, 32, 0)_7$,
$(0, 38, 30, \infty, 5, 13)_{12}$, $(2, 1, 15, 26, 38, 17)_{12}$, $(27, 16, 29, 33, 19, 21)_{12}$,
$(9, 24, 4, 6, 25, 13)_{12}$, $(23, 3, 35, 13, 6, 4)_{12}$, $(11, 16, 32, 1, 4, 7)_{12}$,
$(0, 33, 22, \infty, 35, 6)_{14}$, $(24, 16, 10, 37, 35, 6)_{14}$, $(37, 16, 13, 23, 36, 18)_{14}$,
$(16, 11, 7, 38, 17, 14)_{14}$, $(22, 11, 38, 15, 35, 21)_{14}$, $(30, 35, 27, 12, 5, 36)_{14}$,
$(0, 24, \infty, 27, 29, 1)_{15}$, $(10, 14, 6, 35, 17, 5)_{15}$, $(7, 15, 22, 38, 19, 23)_{15}$,
$(12, 3, 19, 29, 37, 10)_{15}$, $(20, 0, 2, 10, 6, 8)_{15}$, $(22, 11, 18, 28, 37, 0)_{15}$,
under the action of the mapping $x \mapsto x + 3 \pmod{39}$, $\infty \mapsto \infty$.

$\boldsymbol{K_{41}}$ With vertex set Z_{41} the decompositions consist of
$(0, 1, 3, 7, 9, 16)_1$, $(0, 10, 22, 27, 21, 23)_1$,
$(0, 1, 3, 7, 9, 16)_2$, $(0, 5, 29, 19, 20, 11)_2$,
$(0, 1, 2, 5, 8, 13)_4$, $(0, 9, 11, 27, 31, 24)_4$,
$(0, 1, 2, 5, 8, 12)_5$, $(0, 9, 12, 26, 28, 30)_5$,
$(0, 1, 3, 5, 9, 21)_7$, $(0, 7, 17, 26, 30, 12)_7$,
$(0, 1, 3, 7, 10, 15)_{12}$, $(0, 11, 16, 35, 29, 14)_{12}$,
$(0, 1, 3, 7, 12, 16)_{14}$, $(0, 6, 31, 9, 27, 14)_{14}$,
$(0, 1, 3, 4, 8, 13)_{15}$, $(0, 11, 36, 29, 17, 15)_{15}$,
under the action of the mapping $x \mapsto x + 1 \pmod{41}$.

$\boldsymbol{K_{45}}$ With vertex set $Z_{44} \cup \{\infty\}$ the decompositions consist of
$(39, 12, 29, 2, \infty, 10)_1$, $(18, 37, 7, 17, \infty, 6)_1$, $(21, 28, 20, 16, 13, 12)_1$,
$(6, 40, 31, 9, 10, 19)_1$, $(8, 10, 38, 32, 2, 19)_1$, $(30, 41, 37, 25, 6, 3)_1$,
$(6, 11, 24, 27, 3, 29)_1$, $(23, 4, 37, 22, 29, 11)_1$, $(32, 27, 29, 31, 3, 9)_1$,
$(0, \infty, 38, 11, 21, 5)_2$, $(27, 7, 23, 42, 25, 22)_2$, $(20, 24, 30, 33, 13, 17)_2$,
$(9, 18, 39, 7, 33, 8)_2$, $(3, 34, 12, 41, 20, 4)_2$, $(34, 27, 30, 4, 13, 14)_2$,
$(40, 30, 38, 13, 42, 20)_2$, $(31, 28, 16, 23, 12, 41)_2$, $(1, 5, 23, 13, 6, 29)_2$,
$(0, \infty, 9, 35, 26, 25)_4$, $(29, 26, 7, 19, 3, 41)_4$, $(13, 6, 11, 26, 40, 29)_4$,
$(43, 35, 10, 18, 12, 33)_4$, $(13, 43, 7, 27, 38, 0)_4$, $(10, 42, 19, 36, 8, 38)_4$,
$(34, 29, 33, 25, 40, 35)_4$, $(35, 16, 32, 28, 40, 38)_4$, $(0, 16, 41, 21, 30, 1)_4$,
$(0, \infty, 34, 22, 29, 31)_5$, $(19, 1, 24, 10, 5, 36)_5$, $(43, 42, 18, 38, 36, 31)_5$,
$(42, 17, 32, 6, 40, 19)_5$, $(30, 14, 3, 37, 7, 41)_5$, $(10, 25, 40, 41, 13, 20)_5$,
$(14, 20, 13, 43, 16, 33)_5$, $(8, 11, 41, 23, 35, 33)_5$, $(3, 12, 39, 5, 28, 11)_5$,
$(0, \infty, 23, 17, 26, 2)_7$, $(36, 29, 26, 8, 4, 21)_7$, $(6, 12, 15, 32, 27, 39)_7$,
$(11, 25, 7, 20, 24, 18)_7$, $(10, 17, 26, 15, 21, 42)_7$, $(24, 13, 33, 35, 21, 16)_7$,

$(1,35,7,22,15,26)_7$, $(32,2,31,38,36,18)_7$, $(1,29,2,39,3,14)_7$,
$(\infty,5,32,39,23,26)_{12}$, $(24,40,37,38,23,19)_{12}$, $(15,36,14,7,17,35)_{12}$,
$(10,38,20,0,34,19)_{12}$, $(6,29,20,2,41,0)_{12}$, $(41,4,40,36,7,2)_{12}$,
$(9,20,5,7,25,2)_{12}$, $(6,23,35,41,27,3)_{12}$, $(1,9,34,22,23,30)_{12}$,
$(\infty,30,24,27,1,7)_{14}$, $(7,42,5,9,34,33)_{14}$, $(27,34,15,37,17,14)_{14}$,
$(37,8,43,29,36,25)_{14}$, $(27,22,33,19,20,21)_{14}$, $(1,42,4,40,20,30)_{14}$,
$(26,5,10,12,16,36)_{14}$, $(7,31,36,22,18,3)_{14}$, $(0,12,31,34,21,35)_{14}$,
$(\infty,4,14,41,7,22)_{15}$, $(39,26,17,15,27,30)_{15}$, $(43,35,7,13,24,28)_{15}$,
$(14,12,10,40,20,31)_{15}$, $(31,26,25,37,33,5)_{15}$, $(1,40,8,36,29,5)_{15}$,
$(18,4,11,43,21,12)_{15}$, $(33,35,21,38,39,6)_{15}$, $(0,22,6,1,26,20)_{15}$,
under the action of the mapping $x \mapsto x+4 \pmod{44}$, $\infty \mapsto \infty$.

$\boldsymbol{K_{56}}$ With vertex set $Z_{55} \cup \{\infty\}$ the decompositions consist of
$(\infty,4,38,30,51,7)_1$, $(7,10,12,28,33,43)_1$, $(9,29,15,54,27,10)_1$,
$(40,9,49,17,30,31)_1$, $(20,25,13,21,45,36)_1$, $(50,17,35,1,31,7)_1$,
$(32,5,36,49,47,33)_1$, $(12,1,36,37,23,47)_1$, $(8,50,33,21,53,2)_1$,
$(15,19,7,1,32,13)_1$, $(47,38,54,43,21,14)_1$, $(1,8,28,54,6,22)_1$,
$(1,16,39,46,4,53)_1$, $(3,54,4,18,26,27)_1$,
$(\infty,30,19,8,27,16)_2$, $(8,3,16,49,0,17)_2$, $(1,26,6,32,12,8)_2$,
$(34,50,11,49,42,54)_2$, $(0,5,6,34,15,18)_2$, $(54,5,27,40,18,44)_2$,
$(17,10,19,42,21,3)_2$, $(41,18,38,39,50,13)_2$, $(31,45,41,49,16,22)_2$,
$(11,24,28,27,48,45)_2$, $(9,42,22,27,3,34)_2$, $(32,43,15,3,53,8)_2$,
$(2,0,12,30,36,8)_2$, $(4,33,35,11,52,54)_2$,
$(\infty,36,34,25,12,8)_4$, $(14,28,17,7,54,33)_4$, $(44,39,17,33,43,16)_4$,
$(32,47,19,28,27,35)_4$, $(12,13,40,43,21,30)_4$, $(32,46,36,7,11,49)_4$,
$(29,17,52,30,16,40)_4$, $(44,42,11,40,25,28)_4$, $(14,34,44,20,51,26)_4$,
$(41,47,23,36,38,3)_4$, $(54,25,33,1,9,20)_4$, $(5,53,11,18,51,25)_4$,
$(0,40,53,30,47,1)_4$, $(3,36,48,15,40,46)_4$,
$(33,35,27,\infty,22,43)_5$, $(40,31,24,34,32,\infty)_5$, $(18,47,37,14,35,31)_5$,
$(45,41,17,52,27,36)_5$, $(7,49,38,48,45,3)_5$, $(2,6,48,27,5,14)_5$,
$(7,47,31,4,9,29)_5$, $(17,24,44,18,13,19)_5$, $(20,52,33,30,6,26)_5$,
$(46,28,38,43,6,10)_5$, $(53,30,20,36,51,15)_5$, $(0,14,54,25,35,31)_5$,
$(1,44,46,20,26,35)_5$, $(3,19,46,33,39,4)_5$,
$(\infty,50,8,36,24,17)_7$, $(15,25,42,48,17,16)_7$, $(10,6,46,34,49,41)_7$,
$(34,35,14,26,17,53)_7$, $(20,18,7,27,14,48)_7$, $(45,12,1,16,38,28)_7$,
$(22,36,38,46,28,25)_7$, $(39,16,49,37,34,9)_7$, $(35,39,53,30,50,10)_7$,
$(0,32,6,19,12,8)_7$, $(29,37,38,47,23,52)_7$, $(1,23,6,47,36,14)_7$,
$(3,22,6,34,7,51)_7$, $(4,33,3,45,38,6)_7$,
$(6,14,43,39,\infty,4)_{12}$, $(37,17,1,0,44,\infty)_{12}$, $(1,42,15,39,47,18)_{12}$,
$(41,9,20,15,31,37)_{12}$, $(23,5,39,33,52,45)_{12}$, $(29,26,1,32,10,47)_{12}$,
$(51,15,28,41,53,48)_{12}$, $(31,25,35,9,0,16)_{12}$, $(36,1,28,6,19,7)_{12}$,
$(31,42,43,35,52,44)_{12}$, $(50,43,7,3,48,37)_{12}$, $(7,24,10,12,25,23)_{12}$,
$(3,4,23,54,0,13)_{12}$, $(4,2,27,33,49,44)_{12}$,
$(6,2,53,\infty,39,23)_{14}$, $(22,37,20,32,0,\infty)_{14}$, $(22,51,41,16,34,42)_{14}$,

$(27, 18, 53, 52, 11, 12)_{14}$, $(53, 24, 9, 22, 3, 7)_{14}$, $(37, 2, 26, 50, 9, 54)_{14}$,
$(7, 12, 15, 54, 23, 5)_{14}$, $(36, 4, 30, 49, 1, 5)_{14}$, $(13, 35, 1, 5, 16, 40)_{14}$,
$(0, 27, 49, 13, 4, 6)_{14}$, $(44, 24, 19, 3, 35, 18)_{14}$, $(21, 43, 26, 35, 6, 25)_{14}$,
$(3, 5, 48, 16, 43, 45)_{14}$, $(4, 8, 35, 14, 6, 15)_{14}$,
under the action of the mapping $x \mapsto x + 5 \pmod{55}$, $\infty \mapsto \infty$, and with vertex set Z_{56},
$(44, 1, 3, 46, 17, 25)_{15}$, $(46, 3, 5, 48, 19, 27)_{15}$, $(48, 5, 7, 50, 21, 29)_{15}$,
$(50, 7, 9, 52, 23, 31)_{15}$, $(13, 1, 10, 52, 37, 14)_{15}$, $(15, 3, 12, 54, 39, 16)_{15}$,
$(17, 5, 14, 0, 41, 18)_{15}$, $(19, 7, 16, 2, 43, 20)_{15}$, $(12, 26, 51, 13, 36, 43)_{15}$,
$(14, 28, 53, 15, 38, 45)_{15}$, $(16, 30, 55, 17, 40, 47)_{15}$, $(18, 32, 1, 19, 42, 49)_{15}$,
$(33, 37, 44, 14, 27, 36)_{15}$, $(35, 39, 46, 16, 29, 38)_{15}$, $(37, 41, 48, 18, 31, 40)_{15}$,
$(39, 43, 50, 20, 33, 42)_{15}$, $(21, 3, 8, 18, 14, 44)_{15}$, $(23, 5, 10, 20, 16, 46)_{15}$,
$(25, 7, 12, 22, 18, 48)_{15}$, $(27, 9, 14, 24, 20, 50)_{15}$, $(0, 28, 37, 44, 16, 9)_{15}$,
$(2, 30, 39, 46, 18, 11)_{15}$,
under the action of the mapping $x \mapsto x + 8 \pmod{56}$.

$\boldsymbol{K_{60}}$ With vertex set Z_{60} the decompositions consist of
$(23, 53, 0, 24, 28, 43)_1$, $(4, 27, 28, 57, 7, 35)_1$, $(1, 31, 8, 32, 17, 30)_1$,
$(5, 35, 12, 36, 15, 21)_1$, $(16, 39, 9, 40, 19, 42)_1$, $(14, 26, 28, 29, 8, 13)_1$,
$(32, 33, 18, 30, 20, 45)_1$, $(36, 37, 22, 34, 24, 49)_1$, $(26, 40, 38, 41, 6, 58)_1$,
$(30, 42, 44, 45, 7, 24)_1$, $(2, 8, 12, 47, 39, 56)_1$, $(6, 51, 12, 16, 0, 59)_1$,
$(10, 20, 16, 55, 52, 53)_1$, $(20, 24, 14, 59, 8, 15)_1$, $(3, 28, 18, 24, 0, 41)_1$,
$(0, 27, 22, 40, 14, 37)_1$, $(31, 44, 4, 26, 0, 14)_1$, $(30, 35, 8, 48, 38, 43)_1$,
$(39, 52, 12, 34, 25, 29)_1$, $(38, 56, 16, 43, 12, 24)_1$, $(24, 43, 5, 7, 12, 50)_1$,
$(11, 47, 9, 28, 5, 46)_1$, $(13, 32, 15, 51, 21, 27)_1$, $(17, 55, 19, 36, 5, 30)_1$,
$(21, 40, 23, 59, 29, 56)_1$, $(44, 49, 52, 54, 9, 33)_1$, $(53, 56, 48, 58, 33, 45)_1$,
$(2, 57, 0, 52, 23, 49)_1$, $(1, 6, 4, 56, 27, 43)_1$, $(5, 10, 0, 8, 38, 40)_1$,
$(12, 57, 21, 46, 3, 26)_1$, $(16, 50, 1, 25, 7, 49)_1$, $(5, 29, 20, 54, 45, 46)_1$,
$(24, 58, 9, 33, 52, 57)_1$, $(13, 28, 2, 37, 42, 54)_1$, $(49, 53, 32, 47, 3, 15)_1$,
$(36, 57, 51, 53, 7, 8)_1$, $(55, 57, 1, 40, 13, 38)_1$, $(1, 59, 5, 44, 7, 34)_1$,
$(9, 48, 3, 5, 21, 51)_1$, $(27, 39, 11, 53, 30, 38)_1$, $(31, 43, 15, 57, 51, 54)_1$,
$(1, 19, 35, 47, 13, 24)_1$, $(5, 39, 23, 51, 18, 22)_1$, $(43, 55, 9, 27, 18, 34)_1$,
$(36, 47, 14, 43, 6, 54)_1$, $(47, 51, 18, 40, 25, 38)_1$, $(22, 55, 44, 51, 49, 52)_1$,
$(26, 48, 55, 59, 18, 25)_1$, $(52, 59, 3, 30, 8, 17)_1$, $(30, 53, 14, 21, 19, 26)_1$,
$(34, 57, 18, 25, 5, 17)_1$, $(29, 38, 1, 22, 19, 58)_1$, $(26, 42, 5, 33, 23, 43)_1$,
$(37, 46, 9, 30, 10, 59)_1$, $(1, 11, 2, 21, 14, 18)_1$, $(11, 30, 22, 50, 16, 49)_1$,
$(2, 22, 15, 26, 54, 58)_1$, $(34, 54, 26, 35, 30, 58)_1$,
$(12, 18, 43, 7, 29, 32)_2$, $(10, 24, 3, 18, 45, 46)_2$, $(29, 8, 31, 19, 27, 49)_2$,
$(8, 55, 12, 56, 10, 26)_2$, $(35, 54, 14, 3, 31, 33)_2$, $(11, 58, 10, 22, 28, 30)_2$,
$(45, 27, 36, 41, 43, 58)_2$, $(26, 18, 47, 36, 45, 55)_2$, $(3, 55, 42, 37, 6, 36)_2$,
$(34, 38, 29, 20, 26, 46)_2$, $(19, 26, 5, 11, 35, 50)_2$, $(25, 48, 24, 32, 20, 0)_2$,
$(6, 36, 7, 57, 31, 47)_2$, $(2, 52, 27, 20, 16, 19)_2$, $(3, 15, 41, 29, 20, 50)_2$,
$(40, 10, 0, 52, 19, 34)_2$, $(42, 10, 5, 15, 27, 35)_2$, $(11, 56, 52, 25, 53, 55)_2$,
$(53, 2, 21, 10, 49, 37)_2$, $(54, 39, 42, 24, 41, 57)_2$, $(42, 6, 22, 53, 20, 23)_2$,
$(12, 13, 39, 44, 50, 53)_2$, $(56, 14, 58, 24, 41, 0)_2$, $(5, 34, 31, 27, 52, 57)_2$,

$(47,34,57,33,59,32)_2$, $(56,45,35,13,19,39)_2$, $(0,45,29,57,47,50)_2$,
$(59,58,8,21,3,5)_2$, $(11,47,24,8,20,23)_2$, $(29,59,17,28,56,16)_2$,
$(20,23,4,8,39,56)_2$, $(28,50,37,23,34,54)_2$, $(49,50,46,14,57,30)_2$,
$(8,49,43,33,7,36)_2$, $(12,58,2,48,1,41)_2$, $(21,5,22,48,24,27)_2$,
$(35,59,18,53,0,4)_2$, $(16,8,50,54,35,36)_2$, $(1,51,30,24,15,36)_2$,
$(2,36,34,24,29,32)_2$, $(23,46,45,17,36,3)_2$, $(37,31,24,59,56,57)_2$,
$(1,43,52,29,19,31)_2$, $(29,14,25,5,21,22)_2$, $(51,29,53,40,7,23)_2$,
$(26,24,44,49,22,1)_2$, $(58,35,57,55,29,38)_2$, $(59,12,46,57,10,19)_2$,
$(6,13,28,8,1,21)_2$, $(20,57,28,22,21,35)_2$, $(50,24,27,15,17,18)_2$,
$(0,26,53,4,6,16)_2$, $(12,9,35,24,19,34)_2$, $(58,5,53,13,41,20)_2$,
$(7,13,34,3,10,39)_2$, $(22,31,38,14,51,59)_2$, $(7,33,41,37,24,35)_2$,
$(52,18,51,46,57,13)_2$, $(40,11,1,3,57,21)_2$,
$(15,19,3,0,27,14)_7$, $(34,29,16,22,28,11)_7$, $(27,25,9,49,20,2)_7$,
$(48,11,25,51,59,13)_7$, $(51,41,34,59,56,23)_7$, $(47,37,50,48,53,33)_7$,
$(47,16,0,56,51,13)_7$, $(32,11,30,24,47,29)_7$, $(49,24,13,40,53,51)_7$,
$(30,55,31,29,41,56)_7$, $(42,36,40,39,46,48)_7$, $(21,23,9,43,15,38)_7$,
$(25,33,23,51,55,42)_7$, $(34,6,40,1,43,7)_7$, $(53,40,39,52,44,25)_7$,
$(30,46,21,28,49,50)_7$, $(45,37,33,22,46,36)_7$, $(52,34,2,9,5,55)_7$,
$(44,23,6,34,10,42)_7$, $(12,42,3,8,11,21)_7$, $(52,37,24,19,26,47)_7$,
$(27,5,46,11,47,4)_7$, $(42,28,33,1,38,37)_7$, $(55,32,13,39,21,58)_7$,
$(0,43,17,46,36,10)_7$, $(28,6,36,45,52,20)_7$, $(21,42,47,4,59,7)_7$,
$(36,35,4,51,50,32)_7$, $(39,14,4,26,19,38)_7$, $(13,2,30,1,38,43)_7$,
$(53,59,22,14,30,54)_7$, $(34,53,0,55,38,37)_7$, $(26,35,2,58,42,34)_7$,
$(47,26,15,55,36,4)_7$, $(18,2,3,50,47,27)_7$, $(36,29,2,25,44,58)_7$,
$(29,3,8,52,51,21)_7$, $(38,31,1,58,46,4)_7$, $(20,40,2,10,15,17)_7$,
$(49,59,6,10,29,34)_7$, $(3,40,7,9,17,53)_7$, $(37,18,25,29,32,17)_7$,
$(18,49,17,39,31,30)_7$, $(53,51,26,20,46,21)_7$, $(41,45,0,1,39,4)_7$,
$(52,16,13,4,39,46)_7$, $(14,57,1,27,25,52)_7$, $(32,12,38,50,40,34)_7$,
$(37,8,6,36,39,13)_7$, $(36,23,5,30,19,1)_7$, $(24,30,5,35,45,57)_7$,
$(35,48,45,18,55,16)_7$, $(57,51,19,30,55,48)_7$, $(44,43,24,8,28,20)_7$,
$(17,23,42,56,52,1)_7$, $(58,8,1,49,40,47)_7$, $(54,17,24,38,51,56)_7$,
$(25,3,30,38,34,24)_7$, $(8,54,28,35,47,52)_7$,
$(27,45,7,35,34,29)_{12}$, $(6,49,48,33,19,59)_{12}$, $(3,22,45,40,23,42)_{12}$,
$(30,4,47,59,38,26)_{12}$, $(14,19,38,27,52,17)_{12}$, $(16,29,53,12,0,6)_{12}$,
$(30,29,44,7,22,13)_{12}$, $(3,46,14,0,44,31)_{12}$, $(17,15,29,56,8,12)_{12}$,
$(25,39,33,1,10,48)_{12}$, $(45,12,5,38,1,44)_{12}$, $(50,56,11,24,45,39)_{12}$,
$(40,39,18,3,53,36)_{12}$, $(37,24,48,41,0,54)_{12}$, $(48,39,45,55,21,42)_{12}$,
$(9,31,46,15,6,13)_{12}$, $(56,36,27,37,48,14)_{12}$, $(26,25,53,13,34,27)_{12}$,
$(49,17,59,48,2,34)_{12}$, $(49,10,29,47,41,12)_{12}$, $(43,45,9,20,51,18)_{12}$,
$(43,30,27,21,57,49)_{12}$, $(36,59,42,22,14,41)_{12}$, $(50,26,16,18,23,37)_{12}$,
$(15,50,14,31,20,4)_{12}$, $(44,2,47,51,48,58)_{12}$, $(17,6,38,45,22,26)_{12}$,
$(8,38,55,15,33,27)_{12}$, $(21,41,35,31,38,54)_{12}$, $(1,59,57,25,40,27)_{12}$,
$(9,44,23,33,28,0)_{12}$, $(2,12,11,20,28,5)_{12}$, $(6,11,27,51,41,13)_{12}$,

$(49,56,52,7,44,39)_{12}$, $(14,58,49,38,54,48)_{12}$, $(51,54,53,56,48,59)_{12}$, $(4,55,25,29,56,37)_{12}$, $(53,57,11,32,23,50)_{12}$, $(57,35,26,14,4,22)_{12}$, $(1,11,0,35,4,43)_{12}$, $(36,58,57,13,21,2)_{12}$, $(43,55,14,21,12,44)_{12}$, $(0,27,40,2,23,32)_{12}$, $(32,10,28,50,45,52)_{12}$, $(32,33,41,2,43,6)_{12}$, $(50,6,36,53,1,2)_{12}$, $(57,37,31,59,45,16)_{12}$, $(56,35,32,8,42,31)_{12}$, $(21,37,40,55,43,22)_{12}$, $(55,19,32,44,23,13)_{12}$, $(38,26,7,15,2,10)_{12}$, $(11,9,38,25,42,16)_{12}$, $(8,36,18,0,23,10)_{12}$, $(34,52,13,37,47,30)_{12}$, $(24,52,6,21,40,33)_{12}$, $(39,38,51,50,32,35)_{12}$, $(2,14,4,24,45,30)_{12}$, $(39,14,30,40,23,19)_{12}$, $(40,48,46,23,8,47)_{12}$,
under the action of the mapping $x \mapsto x+20 \pmod{60}$.

$\boldsymbol{K_{61}}$ With vertex set Z_{61} the decompositions consist of
$(0,2,26,53,50,43)_1$, $(35,6,2,23,12,13)_1$, $(12,28,3,59,9,13)_1$, $(0,58,26,9,57,50)_2$, $(30,36,15,46,38,58)_2$, $(8,28,33,47,1,46)_2$, $(0,47,32,25,38,20)_7$, $(21,47,5,3,55,34)_7$, $(0,3,7,2,40,12)_7$, $(0,23,6,30,26,22)_{12}$, $(33,6,58,1,18,0)_{12}$, $(0,2,18,32,50,42)_{12}$,
under the action of the mapping $x \mapsto x+1 \pmod{61}$.

$\boldsymbol{K_{65}}$ With vertex set Z_{65} the decompositions consist of
$(64,5,33,46,14,4)_1$, $(0,6,34,47,15,5)_1$, $(1,7,35,48,16,6)_1$, $(2,8,36,49,17,7)_1$, $(3,9,37,50,18,8)_1$, $(49,26,52,1,28,59)_1$, $(50,27,53,2,29,60)_1$, $(51,28,54,3,30,61)_1$, $(52,29,55,4,31,62)_1$, $(53,30,56,5,32,63)_1$, $(64,15,37,44,45,11)_1$, $(0,16,38,45,46,12)_1$, $(1,17,39,46,47,13)_1$, $(24,46,53,8,54,35)_1$, $(53,7,18,64,0,61)_1$, $(2,18,10,14,40,47)_1$,
$(62,6,43,60,11,42)_2$, $(63,7,44,61,12,43)_2$, $(64,8,45,62,13,44)_2$, $(0,9,46,63,14,45)_2$, $(1,10,47,64,15,46)_2$, $(14,10,44,17,54,36)_2$, $(15,11,45,18,55,37)_2$, $(16,12,46,19,56,38)_2$, $(17,13,47,20,57,39)_2$, $(18,14,48,21,58,40)_2$, $(6,12,24,48,22,39)_2$, $(7,13,25,49,23,40)_2$, $(8,14,26,50,24,41)_2$, $(9,27,51,15,42,61)_2$, $(0,13,39,52,26,55)_2$, $(0,18,42,6,33,16)_2$,
$(7,53,37,28,35,59)_4$, $(8,54,38,29,36,60)_4$, $(9,55,39,30,37,61)_4$, $(10,56,40,31,38,62)_4$, $(11,57,41,32,39,63)_4$, $(40,54,23,39,62,64)_4$, $(41,55,24,40,63,0)_4$, $(42,56,25,41,64,1)_4$, $(43,57,26,42,0,2)_4$, $(44,58,27,43,1,3)_4$, $(34,54,5,1,0,57)_4$, $(35,55,6,2,1,58)_4$, $(36,56,7,3,2,59)_4$, $(8,42,3,15,35,62)_4$, $(9,2,47,5,14,38)_4$, $(4,38,8,11,31,58)_4$,
$(39,25,61,30,29,14)_5$, $(40,26,62,31,30,15)_5$, $(41,27,63,32,31,16)_5$, $(42,28,64,33,32,17)_5$, $(43,29,0,34,33,18)_5$, $(64,29,7,44,6,45)_5$, $(0,30,8,45,7,46)_5$, $(1,31,9,46,8,47)_5$, $(2,32,10,47,9,48)_5$, $(3,33,11,48,10,49)_5$, $(44,0,42,3,25,13)_5$, $(45,1,43,4,26,14)_5$, $(46,2,44,5,27,15)_5$, $(0,6,53,59,12,47)_5$, $(53,5,28,7,41,31)_5$, $(4,21,44,23,57,47)_5$,
$(21,1,63,25,14,51)_7$, $(22,2,64,26,15,52)_7$, $(23,3,0,27,16,53)_7$, $(24,4,1,28,17,54)_7$, $(25,5,2,29,18,55)_7$, $(33,0,55,48,28,34)_7$,

$(34, 1, 56, 49, 29, 35)_7$, $(35, 2, 57, 50, 30, 36)_7$, $(36, 3, 58, 51, 31, 37)_7$,
$(37, 4, 59, 52, 32, 38)_7$, $(36, 27, 15, 38, 61, 44)_7$, $(37, 28, 16, 39, 62, 45)_7$,
$(38, 29, 17, 40, 63, 46)_7$, $(39, 30, 18, 41, 64, 47)_7$, $(10, 54, 1, 18, 35, 12)_7$,
$(2, 20, 38, 49, 56, 31)_7$,
$(12, 57, 62, 51, 14, 48)_{12}$, $(13, 58, 63, 52, 15, 49)_{12}$, $(14, 59, 64, 53, 16, 50)_{12}$,
$(15, 60, 0, 54, 17, 51)_{12}$, $(16, 61, 1, 55, 18, 52)_{12}$, $(8, 42, 54, 14, 41, 4)_{12}$,
$(9, 43, 55, 15, 42, 5)_{12}$, $(10, 44, 56, 16, 43, 6)_{12}$, $(11, 45, 57, 17, 44, 7)_{12}$,
$(12, 46, 58, 18, 45, 8)_{12}$, $(52, 11, 29, 38, 3, 8)_{12}$, $(53, 12, 30, 39, 4, 9)_{12}$,
$(54, 13, 31, 40, 5, 10)_{12}$, $(5, 21, 29, 56, 47, 63)_{12}$, $(54, 40, 47, 26, 61, 3)_{12}$,
$(3, 42, 50, 26, 12, 17)_{12}$,
$(41, 26, 35, 58, 28, 54)_{14}$, $(42, 27, 36, 59, 29, 55)_{14}$, $(43, 28, 37, 60, 30, 56)_{14}$,
$(44, 29, 38, 61, 31, 57)_{14}$, $(45, 30, 39, 62, 32, 58)_{14}$, $(37, 34, 29, 17, 13, 7)_{14}$,
$(38, 35, 30, 18, 14, 8)_{14}$, $(39, 36, 31, 19, 15, 9)_{14}$, $(40, 37, 32, 20, 16, 10)_{14}$,
$(41, 38, 33, 21, 17, 11)_{14}$, $(5, 21, 60, 6, 39, 35)_{14}$, $(6, 22, 61, 7, 40, 36)_{14}$,
$(7, 23, 62, 8, 41, 37)_{14}$, $(27, 56, 20, 34, 63, 49)_{14}$, $(63, 14, 53, 64, 32, 39)_{14}$,
$(4, 5, 38, 20, 59, 63)_{14}$,
$(35, 57, 37, 44, 39, 61)_{15}$, $(36, 58, 38, 45, 40, 62)_{15}$, $(37, 59, 39, 46, 41, 63)_{15}$,
$(38, 60, 40, 47, 42, 64)_{15}$, $(39, 61, 41, 48, 43, 0)_{15}$, $(24, 47, 57, 18, 32, 45)_{15}$,
$(25, 48, 58, 19, 33, 46)_{15}$, $(26, 49, 59, 20, 34, 47)_{15}$, $(27, 50, 60, 21, 35, 48)_{15}$,
$(28, 51, 61, 22, 36, 49)_{15}$, $(42, 12, 46, 58, 9, 47)_{15}$, $(43, 13, 47, 59, 10, 48)_{15}$,
$(44, 14, 48, 60, 11, 49)_{15}$, $(14, 21, 35, 7, 28, 0)_{15}$, $(46, 43, 11, 15, 57, 16)_{15}$,
$(1, 17, 54, 50, 20, 55)_{15}$,

under the action of the mapping $x \mapsto x + 5 \pmod{65}$.

$\boldsymbol{K_{85}}$ With vertex set Z_{85} the decomposition consists of

$(8, 78, 69, 18, 28, 42)_{15}$, $(9, 79, 70, 19, 29, 43)_{15}$, $(10, 80, 71, 20, 30, 44)_{15}$,
$(11, 81, 72, 21, 31, 45)_{15}$, $(12, 82, 73, 22, 32, 46)_{15}$, $(60, 29, 32, 46, 62, 28)_{15}$,
$(61, 30, 33, 47, 63, 29)_{15}$, $(62, 31, 34, 48, 64, 30)_{15}$, $(63, 32, 35, 49, 65, 31)_{15}$,
$(64, 33, 36, 50, 66, 32)_{15}$, $(57, 36, 43, 20, 62, 49)_{15}$, $(58, 37, 44, 21, 63, 50)_{15}$,
$(59, 38, 45, 22, 64, 51)_{15}$, $(60, 39, 46, 23, 65, 52)_{15}$, $(61, 40, 47, 24, 66, 53)_{15}$,
$(20, 65, 54, 43, 8, 7)_{15}$, $(21, 66, 55, 44, 9, 8)_{15}$, $(22, 67, 56, 45, 10, 9)_{15}$,
$(23, 68, 57, 46, 11, 10)_{15}$, $(70, 28, 29, 27, 71, 69)_{15}$, $(1, 2, 59, 14, 48, 37)_{15}$,

under the action of the mapping $x \mapsto x + 5 \pmod{85}$. □

Proof of Lemma 2.2

$\boldsymbol{K_{10,10,10}}$ Let the vertex set be Z_{30} partitioned according to residue class modulo 3. The decompositions consist of

$(0, 1, 28, 5, 20, 14)_4$, $(0, 1, 4, 14, 23, 6)_5$, $(0, 1, 11, 13, 8, 15)_{14}$,
$(0, 1, 3, 5, 23, 16)_{15}$,

under the action of the mapping $x \mapsto x + 1 \pmod{30}$.

$\boldsymbol{K_{10,10,10,10}}$ Let the vertex set be Z_{40} partitioned into $\{3j + i : j = 0, 1, \ldots, 9\}$, $i = 0, 1, 2$, and $\{30, 31, \ldots, 39\}$. The decompositions consist of

$(0, 2, 7, 30, 10, 39)_1$, $(0, 35, 13, 29, 4, 11)_1$,
$(0, 1, 8, 30, 34, 10)_2$, $(30, 5, 19, 24, 22, 23)_2$,

$(0,1,2,30,33,35)_4$, $(0,11,17,7,25,37)_4$,
$(0,1,2,30,33,37)_5$, $(0,13,16,5,20,30)_5$,
$(0,1,14,30,32,5)_7$, $(0,34,7,8,11,10)_7$,
$(0,30,2,24,7,4)_{12}$, $(0,13,34,2,29,39)_{12}$,
$(0,1,5,7,30,15)_{14}$, $(0,8,32,11,36,25)_{14}$,
$(0,1,3,5,11,30)_{15}$, $(0,36,7,13,14,31)_{15}$,
under the action of the mapping $x \mapsto x+1 \pmod{30}$ for $x<30$, $x \mapsto (x+1 \pmod{10})+30$ for $x \geq 30$.

$\boldsymbol{K_{20,20,20,20}}$ Let the vertex set be Z_{80} partitioned according to residue class modulo 4. The decompositions consist of
$(0,18,55,29,63,15)_1$, $(8,27,29,22,17,14)_1$, $(42,41,11,64,0,8)_1$,
$(0,3,50,41,65,19)_2$, $(65,14,39,60,28,38)_2$, $(65,8,78,71,30,67)_2$,
$(0,11,30,25,13,57)_4$, $(67,5,66,8,40,20)_4$, $(19,50,28,9,57,13)_4$,
$(0,13,70,19,59,67)_5$, $(18,49,13,56,40,63)_5$, $(43,25,41,8,66,26)_5$,
$(0,15,1,69,62,71)_7$, $(29,19,2,6,60,35)_7$, $(54,11,4,8,49,29)_7$,
$(0,10,17,36,59,11)_{12}$, $(68,31,45,18,46,23)_{12}$, $(62,12,59,30,53,28)_{12}$,
$(0,19,37,10,65,54)_{14}$, $(73,0,50,72,67,41)_{14}$, $(66,15,69,55,17,36)_{14}$,
$(0,27,5,29,42,46)_{15}$, $(3,9,19,70,12,8)_{15}$, $(3,2,47,57,33,24)_{15}$,
under the action of the mapping $x \mapsto x+1 \pmod{80}$.

$\boldsymbol{K_{3,3,3,3,3}}$ Let the vertex set be Z_{15} partitioned according to residue class modulo 5. The decomposition consists of
$(0,6,12,4,8,3)_{15}$, $(0,1,9,13,2,12)_{15}$, $(0,11,3,7,14,9)_{15}$,
under the action of the mapping $x \mapsto x+5 \pmod{15}$.

$\boldsymbol{K_{5,5,5,5,5}}$ Let the vertex set be Z_{25} partitioned according to residue class modulo 5. The decompositions consist of
$(0,1,3,7,9,12)_1$, $(0,1,3,9,12,7)_2$, $(0,1,6,8,22,12)_4$,
$(0,1,4,18,23,13)_5$, $(0,1,3,12,19,4)_7$, $(0,1,3,14,9,18)_{12}$,
$(0,1,3,21,12,9)_{14}$, $(0,1,3,4,19,17)_{15}$,
under the action of the mapping $x \mapsto x+1 \pmod{25}$.

$\boldsymbol{K_{10,10,10,10,10}}$ Let the vertex set be Z_{50} partitioned according to residue class modulo 5. The decompositions consist of
$(0,6,34,47,2,27)_1$, $(0,7,19,33,8,18)_1$,
$(0,19,13,11,1,4)_2$, $(0,12,26,33,28,3)_2$,
$(0,31,6,47,2,8)_4$, $(0,17,36,24,49,28)_4$,
$(0,3,34,7,1,12)_5$, $(0,6,13,24,42,25)_5$,
$(0,28,44,31,7,4)_7$, $(0,1,9,12,33,14)_7$,
$(0,38,6,4,7,21)_{12}$, $(0,3,27,49,14,8)_{12}$,
$(0,47,29,33,6,31)_{14}$, $(0,1,43,24,13,15)_{14}$,
$(0,36,2,42,3,4)_{15}$, $(0,18,5,7,27,26)_{15}$,
under the action of the mapping $x \mapsto x+1 \pmod{50}$.

$\boldsymbol{K_{15,15,15,15,15}}$ Let the vertex set be Z_{75} partitioned according to residue class modulo 5. The decompositions consist of
$(0,37,34,66,58,53)_1$, $(10,43,17,74,16,62)_1$, $(0,12,4,51,13,14)_1$,

$(0,48,12,41,56,61)_2$, $(70,57,13,19,54,12)_2$, $(0,11,32,9,33,4)_2$,
$(0,24,17,13,36,27)_4$, $(61,0,2,8,23,18)_4$, $(0,2,74,28,33,9)_4$,
$(0,57,61,28,34,73)_5$, $(57,54,1,3,50,10)_5$, $(0,17,38,6,39,25)_5$,
$(0,41,73,62,37,48)_7$, $(3,6,12,22,29,34)_7$, $(0,29,11,1,53,8)_7$,
$(0,52,71,38,3,36)_{12}$, $(55,44,28,6,72,12)_{12}$, $(0,1,13,42,67,18)_{12}$,
$(0,49,13,64,66,6)_{14}$, $(31,10,9,12,4,43)_{14}$, $(0,4,18,47,16,41)_{14}$,
$(0,18,11,56,12,42)_{15}$, $(65,13,67,16,24,14)_{15}$, $(0,9,1,13,48,59)_{15}$,
under the action of the mapping $x \mapsto x+1 \pmod{75}$.

$\boldsymbol{K_{10,10,10,10,15}}$ Let the vertex set be Z_{55} partitioned into $\{4j+i : j=0,1,\ldots,9\}$, $i=0,1,2,3$, and $\{40,41,\ldots,54\}$. The decompositions consist of
$(0,37,39,46,42,41)_2$, $(28,5,39,47,52,9)_2$, $(0,9,22,45,14,7)_2$,
$(0,34,31,13,53,48)_4$, $(23,9,0,40,54,41)_4$, $(0,1,5,30,38,52)_4$,
$(0,18,31,46,45,41)_5$, $(29,18,30,15,23,48)_5$, $(26,5,13,3,44,49)_5$,
$(0,18,31,16,46,42)_{12}$, $(18,24,17,7,53,44)_{12}$, $(0,3,29,8,45,49)_{12}$,
$(0,34,11,52,19,47)_{14}$, $(6,11,33,41,20,42)_{14}$, $(0,2,3,33,51,52)_{14}$,
under the action of the mapping $x \mapsto x+1 \pmod{40}$ for $x<40$, $x \mapsto (x-40+3 \pmod{15})+40$ for $x \geq 40$, and
$(0,49,27,1,5,3)_1$, $(25,43,0,34,12,28)_1$, $(25,4,42,19,50,23)_1$,
$(32,39,10,45,51,9)_1$, $(5,12,44,23,10,53)_1$, $(35,26,12,41,36,48)_1$,
$(0,29,26,19,40,45)_7$, $(46,29,35,32,18,31)_7$, $(51,37,36,35,2,20)_7$,
$(21,14,7,47,50,16)_7$, $(39,8,21,18,51,49)_7$, $(0,47,18,23,27,38)_7$,
$(0,40,3,11,9,48)_{15}$, $(23,5,38,4,42,49)_{15}$, $(28,22,11,50,53,25)_{15}$,
$(30,42,2,11,4,49)_{15}$, $(15,53,39,13,16,22)_{15}$, $(0,10,28,5,13,15)_{15}$,
under the action of the mapping $x \mapsto x+2 \pmod{40}$ for $x<40$, $x \mapsto (x-40+3 \pmod{15})+40$ for $x \geq 40$.

$\boldsymbol{K_{10,10,10,10,20}}$ Let the vertex set be Z_{60} partitioned into $\{4j+i : j=0,1,\ldots,9\}$, $i=0,1,2,3$, and $\{40,41,\ldots,59\}$. The decompositions consist of
$(1,12,42,18,51,55)_1$, $(9,0,10,27,52,40)_1$, $(35,4,2,40,9,50)_1$,
$(38,33,41,20,54,56)_1$, $(15,36,59,21,44,41)_1$, $(25,15,18,53,4,51)_1$,
$(0,1,26,47,3,50)_1$,
$(26,44,27,37,16,55)_2$, $(25,27,50,30,8,7)_2$, $(54,20,5,19,11,0)_2$,
$(4,33,10,50,49,48)_2$, $(15,46,12,34,21,47)_2$, $(50,26,0,13,19,15)_2$,
$(1,32,34,55,48,50)_2$,
$(34,23,8,13,56,51)_4$, $(39,24,30,43,42,56)_4$, $(10,50,42,23,33,17)_4$,
$(4,10,58,1,5,43)_4$, $(16,15,26,37,48,38)_4$, $(8,1,10,35,47,50)_4$,
$(1,48,52,38,39,27)_4$,
$(4,25,56,2,14,27)_5$, $(57,9,1,12,8,36)_5$, $(28,46,53,6,15,23)_5$,
$(6,15,40,9,0,25)_5$, $(16,35,2,46,49,57)_5$, $(20,56,19,5,37,28)_5$,
$(0,7,35,44,47,48)_5$,
$(24,14,42,56,7,38)_7$, $(49,17,23,18,14,37)_7$, $(1,52,14,15,39,55)_7$,
$(33,12,45,42,23,32)_7$, $(9,2,54,48,4,34)_7$, $(42,2,39,31,8,13)_7$,
$(0,13,18,48,43,17)_7$,
$(34,16,15,18,54,49)_{12}$, $(33,50,30,0,35,42)_{12}$, $(37,19,30,25,53,54)_{12}$,

$(50,11,37,10,18,1)_{12}$, $(47,39,26,58,9,19)_{12}$, $(51,14,8,55,23,29)_{12}$,
$(0,25,43,8,14,53)_{12}$,
$(48,12,21,23,1,6)_{14}$, $(48,9,35,30,32,40)_{14}$, $(56,3,30,31,32,8)_{14}$,
$(9,28,44,26,50,35)_{14}$, $(13,16,19,51,3,42)_{14}$, $(39,58,20,48,6,30)_{14}$,
$(1,16,54,10,52,30)_{14}$,
$(4,11,28,40,55,19)_{15}$, $(38,46,30,15,12,47)_{15}$, $(5,0,43,49,34,8)_{15}$,
$(26,49,9,35,28,36)_{15}$, $(56,1,12,15,7,37)_{15}$, $(1,41,15,14,22,44)_{15}$,
$(0,3,33,1,44,42)_{15}$,

under the action of the mapping $x \mapsto x+2 \pmod{40}$ for $x<40$, $x \mapsto (x+1 \pmod{20})+40$ for $x \geq 40$.

$\boldsymbol{K_{16,16,16,16,21}}$ For graphs n_2, n_5, n_7, n_{12} and n_{14}, let the vertex set be Z_{85} partitioned into $\{3j+i : j=0,1,\ldots,15\}$, $i=0,1,2$, $\{48,49,\ldots,63\}$ and $\{64,65,\ldots,84\}$. The decompositions consist of

$(0,22,65,50,48,35)_2$, $(26,9,68,25,19,63)_2$, $(30,78,34,57,2,76)_2$,
$(32,34,66,49,54,24)_2$, $(49,20,77,9,76,40)_2$, $(0,5,67,51,19,25)_2$,
$(0,49,56,1,19,5)_5$, $(20,45,12,54,25,14)_5$, $(62,76,67,19,42,17)_5$,
$(68,11,36,49,4,25)_5$, $(5,65,70,9,43,49)_5$, $(0,64,66,22,63,2)_5$,
$(0,19,44,49,60,67)_7$, $(56,80,5,16,45,33)_7$, $(40,78,50,29,45,72)_7$,
$(16,69,17,52,30,18)_7$, $(30,68,46,59,8,72)_7$, $(0,70,7,20,54,74)_7$,
$(0,68,60,14,7,61)_{12}$, $(35,22,54,5,45,27)_{12}$, $(45,41,70,4,49,73)_{12}$,
$(33,71,38,10,60,67)_{12}$, $(47,15,76,49,1,67)_{12}$, $(48,6,65,5,25,73)_{12}$,
$(0,49,70,31,2,23)_{14}$, $(79,58,40,33,17,18)_{14}$, $(29,56,81,43,42,24)_{14}$,
$(82,21,26,1,56,71)_{14}$, $(69,56,20,30,34,78)_{14}$, $(66,0,61,17,28,8)_{14}$,

under the action of the mapping $x \mapsto x+1 \pmod{48}$ for $x<48$, $x \mapsto (x+1 \pmod{16})+48$ for $48 \leq x < 64$, $x \mapsto (x-64+7 \pmod{21})+64$ for $x \geq 64$.

For graphs n_1 and n_4, let the vertex set be Z_{85} partitioned into $\{4j+i : j=0,1,\ldots,15\}$, $i=0,1,2,3$, and $\{64,65,\ldots,84\}$. The decompositions consist of

$(63,40,69,45,84,38)_1$, $(10,16,27,78,71,9)_1$, $(35,69,41,14,4,34)_1$,
$(63,18,53,65,37,77)_1$, $(32,1,22,35,14,58)_1$, $(48,68,25,34,43,37)_1$,
$(32,7,75,10,67,62)_1$, $(19,4,71,33,41,72)_1$, $(0,49,47,76,65,77)_1$,
$(45,14,22,27,23,84)_4$, $(52,43,67,13,46,22)_4$, $(74,0,57,7,18,14)_4$,
$(14,52,17,66,68,7)_4$, $(71,29,17,2,52,39)_4$, $(77,53,28,15,6,51)_4$,
$(55,12,79,22,61,10)_4$, $(48,19,21,65,83,68)_4$, $(65,1,7,12,54,2)_4$,

under the action of the mapping $x \mapsto x+2 \pmod{64}$ for $x<64$, $x \mapsto (x-64+5 \pmod{20})+64$ for $64 \leq x < 84$, $84 \mapsto 84$.

$\boldsymbol{K_{20,20,20,20,25}}$ Let the vertex set be Z_{105} partitioned into $\{4j+i : j=0,1,\ldots,19\}$, $i=0,1,2,3$, and $\{80,81,\ldots,104\}$. The decompositions consist of

$(54,104,67,65,11,73)_1$, $(55,84,68,66,12,74)_1$, $(11,98,64,25,2,8)_1$,
$(12,103,65,26,3,9)_1$, $(58,97,1,75,7,24)_1$, $(59,102,2,76,8,25)_1$,
$(85,55,54,33,6,17)_1$, $(90,56,55,34,7,18)_1$, $(56,23,101,30,38,96)_1$,
$(38,43,13,48,8,53)_1$, $(0,7,33,81,25,86)_1$,
$(104,77,31,24,40,78)_2$, $(84,78,32,25,41,79)_2$, $(86,50,76,9,67,58)_2$,
$(91,51,77,10,68,59)_2$, $(87,17,36,74,3,25)_2$, $(92,18,37,75,4,26)_2$,

$(55, 40, 34, 5, 45, 93)_2$, $(56, 41, 35, 6, 46, 98)_2$, $(90, 56, 25, 34, 53, 27)_2$,
$(47, 25, 88, 16, 22, 72)_2$, $(1, 0, 2, 100, 88, 90)_2$,
$(84, 78, 6, 32, 0, 29)_4$, $(89, 79, 7, 33, 1, 30)_4$, $(81, 68, 65, 27, 46, 9)_4$,
$(86, 69, 66, 28, 47, 10)_4$, $(54, 57, 59, 72, 24, 47)_4$, $(55, 58, 60, 73, 25, 48)_4$,
$(92, 1, 57, 0, 28, 44)_4$, $(97, 2, 58, 1, 29, 45)_4$, $(19, 8, 10, 83, 95, 33)_4$,
$(76, 80, 93, 13, 65, 67)_4$, $(0, 98, 85, 14, 55, 63)_4$,
$(64, 51, 63, 6, 97, 102)_5$, $(65, 52, 64, 7, 102, 82)_5$, $(44, 69, 27, 58, 42, 84)_5$,
$(45, 70, 28, 59, 43, 89)_5$, $(79, 29, 76, 58, 70, 96)_5$, $(0, 30, 77, 59, 71, 101)_5$,
$(24, 58, 66, 103, 99, 85)_5$, $(25, 59, 67, 83, 104, 90)_5$, $(44, 51, 18, 80, 101, 61)_5$,
$(33, 59, 76, 80, 101, 66)_5$, $(83, 1, 43, 48, 76, 62)_5$,
$(101, 27, 6, 5, 9, 58)_7$, $(81, 28, 7, 6, 10, 59)_7$, $(100, 23, 10, 49, 52, 56)_7$,
$(80, 24, 11, 50, 53, 57)_7$, $(32, 88, 55, 79, 41, 104)_7$, $(33, 93, 56, 0, 42, 84)_7$,
$(102, 48, 37, 49, 31, 30)_7$, $(82, 49, 38, 50, 32, 31)_7$, $(30, 45, 40, 75, 79, 32)_7$,
$(69, 32, 67, 99, 84, 14)_7$, $(1, 6, 40, 56, 71, 103)_7$,
$(0, 93, 79, 14, 1, 21)_{12}$, $(20, 43, 62, 8, 84, 53)_{12}$, $(94, 61, 19, 81, 8, 33)_{12}$,
$(34, 90, 17, 56, 68, 61)_{12}$, $(96, 58, 21, 84, 64, 26)_{12}$, $(41, 74, 72, 101, 102, 35)_{12}$,
$(70, 79, 4, 21, 29, 88)_{12}$, $(93, 57, 75, 82, 6, 48)_{12}$, $(19, 26, 5, 80, 100, 9)_{12}$,
$(0, 58, 102, 49, 77, 15)_{12}$, $(1, 14, 27, 76, 24, 26)_{12}$,
$(50, 77, 90, 44, 101, 8)_{14}$, $(51, 78, 95, 45, 81, 9)_{14}$, $(12, 51, 9, 87, 61, 28)_{14}$,
$(13, 52, 10, 92, 62, 29)_{14}$, $(10, 28, 65, 36, 93, 87)_{14}$, $(11, 29, 66, 37, 98, 92)_{14}$,
$(41, 56, 34, 32, 11, 103)_{14}$, $(42, 57, 35, 33, 12, 83)_{14}$, $(54, 55, 8, 84, 41, 86)_{14}$,
$(67, 21, 68, 54, 99, 101)_{14}$, $(84, 0, 5, 22, 17, 85)_{14}$,

under the action of the mapping $x \mapsto x + 2 \pmod{80}$ for $x < 80$, $x \mapsto (x - 80 + \lambda \pmod{25}) + 80$ for $x \geq 80$, where $\lambda = 5$ for graph n_{12} and $\lambda = 10$ otherwise.

$\boldsymbol{K_{16^6}}$ Let the vertex set be Z_{96} partitioned according to residue class modulo 6. The decompositions consist of

$(0, 39, 37, 64, 43, 38)_1$, $(10, 21, 62, 29, 0, 66)_1$,
$(71, 49, 40, 56, 3, 54)_1$, $(0, 47, 67, 70, 13, 61)_1$,
$(0, 2, 52, 25, 88, 49)_2$, $(3, 58, 79, 2, 86, 18)_2$,
$(11, 56, 27, 18, 70, 44)_2$, $(0, 39, 43, 65, 5, 11)_2$,
$(0, 86, 34, 3, 7, 87)_4$, $(32, 3, 60, 17, 64, 11)_4$,
$(14, 59, 55, 36, 39, 19)_4$, $(0, 11, 94, 37, 44, 49)_4$,
$(0, 87, 57, 67, 56, 13)_5$, $(39, 86, 76, 18, 13, 35)_5$,
$(8, 61, 58, 42, 69, 44)_5$, $(0, 5, 7, 3, 32, 20)_5$,
$(0, 45, 14, 20, 94, 5)_7$, $(86, 76, 54, 79, 1, 27)_7$,
$(63, 28, 5, 90, 24, 7)_7$, $(0, 41, 8, 67, 87, 68)_7$,
$(0, 88, 23, 43, 1, 64)_{12}$, $(46, 12, 39, 67, 2, 92)_{12}$,
$(70, 55, 66, 26, 17, 13)_{12}$, $(0, 2, 19, 35, 5, 61)_{12}$,
$(0, 13, 94, 39, 76, 2)_{14}$, $(34, 78, 20, 42, 51, 25)_{14}$,
$(25, 74, 24, 86, 46, 43)_{14}$, $(0, 3, 67, 16, 26, 74)_{14}$,
$(0, 59, 64, 45, 68, 26)_{15}$, $(83, 7, 51, 90, 22, 91)_{15}$,
$(38, 71, 54, 61, 22, 11)_{15}$, $(0, 34, 9, 55, 31, 94)_{15}$,

under the action of the mapping $x \mapsto x + 1 \pmod{96}$.

$\boldsymbol{K_{20,20,20,20,20,16}}$ Let the vertex set be Z_{116} partitioned into $\{4j+i : j = 0,1,\dots,19\}$, $i = 0,1,2,3$, $\{80,81,\dots,99\}$ and $\{100,101,\dots,115\}$. The decompositions consist of

$(0,82,53,22,41,37)_1$, $(62,110,98,20,97,28)_1$, $(72,82,109,25,100,49)_1$,
$(50,53,39,68,97,51)_1$, $(69,110,6,79,24,19)_1$, $(77,89,23,18,10,6)_1$,
$(14,20,39,101,88,113)_1$,
$(0,13,101,87,51,69)_2$, $(113,40,89,30,34,29)_2$, $(38,43,89,17,29,73)_2$,
$(18,40,59,83,3,105)_2$, $(58,59,28,96,56,75)_2$, $(106,31,81,8,77,20)_2$,
$(7,104,14,89,60,32)_2$,
$(0,113,80,47,53,84)_4$, $(112,44,74,3,90,81)_4$, $(88,14,60,73,108,17)_4$,
$(15,58,34,41,20,96)_4$, $(95,23,30,53,52,25)_4$, $(114,26,73,27,8,15)_4$,
$(23,33,29,78,108,82)_4$,
$(0,22,114,45,1,4)_5$, $(85,112,13,68,42,55)_5$, $(80,104,50,0,1,91)_5$,
$(49,98,110,62,3,47)_5$, $(95,7,71,10,1,54)_5$, $(74,60,76,69,33,82)_5$,
$(80,13,102,2,67,28)_5$,
$(0,87,109,30,46,19)_7$, $(107,50,83,15,19,17)_7$, $(18,112,44,59,21,81)_7$,
$(21,87,79,16,12,58)_7$, $(67,86,77,101,14,34)_7$, $(64,49,43,50,113,80)_7$,
$(80,15,22,66,40,105)_7$,
$(0,43,114,95,50,19)_{12}$, $(57,85,68,58,34,110)_{12}$, $(0,21,95,12,15,78)_{12}$,
$(48,26,43,87,61,17)_{12}$, $(31,6,112,35,5,93)_{12}$, $(6,3,105,92,45,110)_{12}$,
$(0,27,105,74,93,99)_{12}$,
$(0,27,91,105,62,76)_{14}$, $(55,18,80,48,1,101)_{14}$, $(62,49,92,112,72,63)_{14}$,
$(58,103,3,0,59,54)_{14}$, $(1,40,111,47,81,25)_{14}$, $(82,65,74,76,46,92)_{14}$,
$(3,78,109,82,9,40)_{14}$,
$(0,81,38,55,3,104)_{15}$, $(104,44,94,15,34,1)_{15}$, $(78,107,40,59,61,95)_{15}$,
$(92,48,82,10,100,33)_{15}$, $(60,54,47,59,45,29)_{15}$, $(8,93,104,114,45,35)_{15}$,
$(0,13,24,39,46,91)_{15}$,

under the action of the mapping $x \mapsto x+1 \pmod{80}$ for $x < 80$, $x \mapsto (x+1 \pmod{20}) + 80$ for $80 \le x < 100$, $x \mapsto (x-100+1 \pmod{16}) + 100$ for $x \ge 100$. □

Proof of Lemma 2.4

$\boldsymbol{K_{10,10,10,15}}$ Let the vertex set be Z_{45} partitioned into $\{3j+i : j = 0,1,\dots,9\}$, $i = 0,1,2$, and $\{30,31,\dots,44\}$. The decomposition consists of

$(0,35,26,7,19,30)_{15}$, $(9,10,24,39,43,19)_{15}$, $(18,8,21,37,10,19)_{15}$,
$(18,38,27,22,5,35)_{15}$, $(1,38,11,26,15,34)_{15}$,

under the action of the mapping $x \mapsto x+2 \pmod{30}$ for $x < 30$, $x \mapsto (x+1 \pmod{15}) + 30$ for $x \ge 30$.

$\boldsymbol{K_{6,6,6,6,6}}$ Let the vertex set be Z_{30} partitioned into $\{4j+i : j = 0,1,\dots,5\}$, $i = 0,1,2,3$, and $\{24,25,\dots,29\}$. The decompositions for seven of the graphs consist of

$(0,17,26,23,27,28)_1$, $(26,16,7,21,18,2)_1$, $(11,13,4,10,0,16)_1$,
$(0,28,25,2,19,23)_4$, $(22,1,13,16,12,15)_4$, $(0,11,7,5,29,26)_4$,
$(0,23,18,5,9,20)_5$, $(18,13,4,27,24,11)_5$, $(16,5,13,28,25,12)_5$,
$(0,26,7,14,13,15)_7$, $(10,5,19,3,27,28)_7$, $(14,8,25,11,19,12)_7$,
$(0,18,7,26,28,17)_{12}$, $(6,15,16,13,21,27)_{12}$, $(8,24,6,17,9,3)_{12}$,

$(0, 23, 1, 15, 24, 26)_{14}$, $(23, 4, 6, 20, 17, 11)_{14}$, $(19, 10, 27, 8, 25, 4)_{14}$,
$(0, 19, 12, 1, 22, 23)_{15}$, $(1, 22, 11, 28, 26, 20)_{15}$, $(15, 22, 16, 25, 24, 1)_{15}$,
under the action of the mapping $x \mapsto x + 2 \pmod{24}$ for $x < 24$, $x \mapsto (x + 1 \pmod{6}) + 24$ for $x \geq 24$.

For graph n_2, let the vertex set be Z_{30} partitioned according to residue class modulo 5. The decomposition consists of
$(0, 12, 21, 13, 1, 17)_2$, $(21, 15, 22, 14, 18, 24)_2$, $(2, 13, 19, 5, 14, 8)_2$,
$(5, 27, 21, 23, 24, 9)_2$, $(7, 28, 14, 5, 21, 23)_2$, $(4, 8, 10, 21, 26, 6)_2$,
under the action of the mapping $x \mapsto x + 5 \pmod{30}$.

$\boldsymbol{K_{4^6}}$ Let the vertex set be Z_{24} partitioned according to residue class modulo 6. The decompositions consist of
$(0, 1, 3, 20, 9, 11)_1$, $(0, 1, 3, 10, 5, 11)_2$, $(0, 1, 2, 5, 9, 11)_4$,
$(0, 1, 2, 5, 9, 12)_5$, $(0, 1, 3, 5, 10, 13)_7$, $(0, 1, 4, 9, 15, 2)_{12}$,
$(0, 1, 3, 11, 15, 8)_{14}$, $(0, 1, 3, 4, 8, 13)_{15}$,
under the action of the mapping $x \mapsto x + 1 \pmod{24}$.

$\boldsymbol{K_{5,5,5,5,5,10}}$ Let the vertex set be Z_{35} partitioned into $\{5j + i : j = 0, 1, 2, 3, 4\}$, $i = 0, 1, 2, 3, 4$, and $\{25, 26, \ldots, 34\}$. The decompositions consist of
$(0, 1, 3, 25, 9, 26)_1$, $(0, 7, 11, 32, 13, 31)_1$,
$(0, 1, 3, 25, 7, 11)_2$, $(0, 4, 13, 26, 32, 27)_2$,
$(15, 6, 8, 25, 2, 26)_4$, $(0, 2, 11, 3, 32, 31)_4$,
$(10, 12, 24, 32, 6, 34)_5$, $(0, 1, 12, 9, 25, 31)_5$,
$(12, 23, 20, 10, 25, 27)_7$, $(26, 0, 1, 4, 7, 13)_7$,
$(2, 0, 14, 18, 25, 26)_{12}$, $(0, 1, 17, 23, 28, 29)_{12}$,
$(11, 24, 7, 28, 0, 25)_{14}$, $(0, 3, 9, 2, 29, 28)_{14}$,
under the action of the mapping $x \mapsto x + 1 \pmod{25}$ for $x < 25$, $x \mapsto (x - 25 + 2 \pmod{10}) + 25$ for $x \geq 25$.

$\boldsymbol{K_{4^6,10}}$ Let the vertex set be Z_{34} partitioned into $\{6j + i : j = 0, 1, 2, 3\}$, $i = 0, 1, 2, 3, 4, 5$, and $\{24, 25, \ldots, 33\}$. The decompositions consist of
$(3, 20, 11, 31, 19, 0)_1$, $(19, 17, 18, 24, 15, 22)_1$, $(23, 20, 32, 13, 30, 9)_1$,
$(1, 15, 6, 27, 25, 30)_1$, $(19, 4, 26, 21, 8, 28)_1$, $(0, 5, 13, 29, 28, 33)_1$,
$(21, 12, 32, 11, 30, 1)_2$, $(5, 16, 3, 8, 24, 28)_2$, $(5, 12, 31, 10, 26, 1)_2$,
$(21, 13, 24, 18, 22, 8)_2$, $(8, 22, 25, 23, 32, 1)_2$, $(0, 10, 28, 7, 8, 20)_2$,
$(20, 10, 24, 13, 9, 3)_4$, $(20, 23, 6, 30, 31, 21)_4$, $(16, 7, 24, 5, 12, 11)_4$,
$(14, 22, 13, 32, 28, 6)_4$, $(9, 14, 0, 27, 33, 5)_4$, $(0, 13, 22, 25, 26, 3)_4$,
$(27, 8, 5, 16, 9, 24)_5$, $(29, 9, 19, 10, 6, 26)_5$, $(31, 1, 8, 3, 5, 21)_5$,
$(21, 11, 13, 30, 25, 20)_5$, $(11, 16, 3, 28, 33, 32)_5$, $(1, 6, 22, 15, 23, 8)_5$,
$(23, 19, 0, 29, 31, 22)_7$, $(8, 23, 30, 6, 16, 28)_7$, $(12, 23, 20, 4, 32, 2)_7$,
$(18, 23, 3, 26, 24, 1)_7$, $(18, 7, 21, 16, 30, 33)_7$, $(1, 4, 0, 14, 27, 33)_7$,
$(0, 15, 10, 32, 25, 14)_{12}$, $(4, 3, 14, 19, 11, 6)_{12}$, $(11, 19, 10, 31, 28, 18)_{12}$,
$(3, 7, 29, 5, 10, 24)_{12}$, $(6, 2, 3, 31, 28, 11)_{12}$, $(2, 13, 17, 32, 25, 0)_{12}$,
$(28, 17, 18, 21, 16, 22)_{14}$, $(25, 19, 21, 12, 17, 27)_{14}$, $(32, 15, 11, 14, 1, 29)_{14}$,
$(4, 15, 24, 7, 31, 5)_{14}$, $(15, 23, 16, 8, 26, 33)_{14}$, $(0, 2, 16, 7, 11, 25)_{14}$,
$(7, 16, 25, 9, 12, 17)_{15}$, $(4, 1, 9, 32, 26, 2)_{15}$, $(2, 19, 28, 29, 6, 5)_{15}$,

$(9, 10, 30, 29, 23, 0)_{15}$, $(5, 18, 15, 26, 32, 14)_{15}$, $(0, 5, 1, 10, 33, 2)_{15}$,
under the action of the mapping $x \mapsto x + 3 \pmod{24}$ for $x < 24$, $x \mapsto (x - 24 + 5 \pmod{10}) + 24$ for $x \geq 24$.

$\boldsymbol{K_{15^9}}$ Let the vertex set be Z_{135} partitioned according to residue class modulo 9. The decompositions consist of

$(0, 24, 19, 34, 21, 109)_1$, $(105, 97, 127, 3, 32, 50)_1$, $(66, 103, 8, 72, 68, 65)_1$,
$(43, 14, 103, 0, 82, 129)_1$, $(60, 64, 108, 121, 48, 8)_1$, $(0, 59, 82, 110, 17, 66)_1$,
$(0, 71, 97, 3, 124, 95)_2$, $(4, 114, 63, 92, 18, 132)_2$, $(22, 54, 21, 41, 59, 79)_2$,
$(58, 35, 70, 118, 50, 86)_2$, $(79, 123, 9, 5, 113, 33)_2$, $(0, 30, 73, 79, 80, 31)_2$,
$(0, 1, 115, 111, 105, 119)_4$, $(77, 91, 31, 80, 25, 38)_4$, $(122, 52, 114, 8, 19, 37)_4$,
$(103, 129, 116, 81, 36, 6)_4$, $(91, 40, 57, 132, 128, 98)_4$, $(100, 95, 41, 22, 39, 72)_4$,
$(0, 77, 61, 129, 49, 17)_5$, $(16, 67, 27, 30, 14, 83)_5$, $(56, 115, 126, 46, 10, 94)_5$,
$(133, 102, 32, 94, 31, 127)_5$, $(132, 32, 94, 117, 89, 120)_5$, $(130, 17, 106, 110, 123, 58)_5$,
$(0, 120, 104, 106, 80, 46)_7$, $(125, 131, 124, 24, 75, 86)_7$, $(51, 115, 54, 31, 119, 39)_7$,
$(42, 25, 130, 118, 77, 120)_7$, $(107, 132, 20, 63, 10, 74)_7$, $(14, 35, 9, 127, 112, 33)_7$,
$(0, 39, 62, 77, 64, 121)_{12}$, $(11, 45, 122, 66, 37, 5)_{12}$, $(100, 20, 96, 30, 48, 49)_{12}$,
$(28, 8, 110, 132, 21, 103)_{12}$, $(5, 97, 128, 50, 98, 91)_{12}$, $(21, 53, 56, 123, 16, 4)_{12}$,
$(0, 75, 56, 73, 80, 60)_{14}$, $(130, 46, 71, 101, 117, 128)_{14}$, $(33, 0, 77, 127, 34, 8)_{14}$,
$(84, 16, 127, 87, 64, 62)_{14}$, $(113, 91, 119, 15, 103, 93)_{14}$, $(0, 14, 114, 11, 97, 53)_{14}$,
$(0, 62, 57, 120, 95, 46)_{15}$, $(31, 41, 75, 43, 123, 18)_{15}$, $(84, 61, 29, 85, 125, 105)_{15}$,
$(108, 125, 78, 57, 59, 71)_{15}$, $(79, 57, 92, 107, 63, 37)_{15}$, $(0, 60, 86, 91, 56, 83)_{15}$,
under the action of the mapping $x \mapsto x + 1 \pmod{135}$.

$\boldsymbol{K_{15^9,20}}$ Let the vertex set be Z_{155} partitioned into $\{9j + i : j = 0, 1, \ldots, 14\}$, $i = 0, 1, \ldots, 8$, and $\{135, 136, \ldots, 154\}$. The decompositions consist of

$(0, 64, 50, 26, 21, 34)_1$, $(101, 140, 130, 64, 98, 52)_1$, $(21, 138, 37, 54, 8, 60)_1$,
$(9, 143, 87, 56, 83, 85)_1$, $(18, 83, 93, 95, 60, 43)_1$, $(114, 73, 66, 58, 92, 119)_1$,
$(111, 56, 107, 4, 67, 76)_1$, $(117, 137, 44, 111, 35, 118)_1$,
$(0, 50, 110, 112, 44, 64)_2$, $(13, 10, 80, 50, 23, 137)_2$, $(55, 89, 2, 146, 60, 116)_2$,
$(37, 96, 49, 140, 76, 48)_2$, $(99, 50, 19, 43, 116, 132)_2$, $(151, 115, 18, 101, 87, 14)_2$,
$(118, 26, 110, 153, 146, 140)_2$, $(134, 15, 108, 93, 34, 31)_2$,
$(0, 91, 148, 59, 48, 70)_4$, $(102, 35, 145, 106, 69, 128)_4$, $(73, 120, 110, 4, 79, 128)_4$,
$(106, 20, 116, 103, 76, 45)_4$, $(17, 15, 130, 68, 10, 142)_4$, $(37, 94, 97, 114, 83, 135)_4$,
$(113, 85, 97, 143, 138, 145)_4$, $(123, 122, 111, 11, 26, 152)_4$,
$(0, 10, 30, 26, 35, 113)_5$, $(81, 66, 92, 69, 94, 58)_5$, $(78, 58, 7, 102, 77, 153)_5$,
$(78, 107, 119, 5, 146, 136)_5$, $(85, 34, 46, 131, 132, 151)_5$, $(49, 6, 131, 124, 63, 75)_5$,
$(0, 76, 93, 143, 136, 45)_5$, $(0, 6, 22, 80, 137, 150)_5$,
$(0, 68, 148, 128, 44, 22)_7$, $(16, 54, 146, 82, 128, 17)_7$, $(41, 121, 74, 1, 16, 44)_7$,
$(141, 132, 115, 53, 1, 34)_7$, $(143, 79, 47, 28, 1, 70)_7$, $(98, 108, 25, 12, 96, 152)_7$,
$(37, 31, 11, 45, 42, 146)_7$, $(123, 36, 70, 23, 86, 64)_7$,
$(0, 20, 8, 67, 21, 146)_{12}$, $(24, 30, 52, 109, 141, 98)_{12}$, $(53, 3, 70, 147, 51, 77)_{12}$,
$(129, 29, 125, 54, 122, 106)_{12}$, $(70, 80, 37, 150, 110, 119)_{12}$, $(134, 30, 15, 85, 81, 152)_{12}$,
$(17, 61, 3, 35, 152, 151)_{12}$, $(6, 35, 95, 133, 1, 141)_{12}$,
$(0, 77, 33, 135, 79, 153)_{14}$, $(60, 77, 125, 128, 88, 65)_{14}$, $(128, 108, 73, 122, 107, 44)_{14}$,

$(113, 132, 4, 9, 1, 146)_{14}$, $(107, 73, 60, 22, 83, 148)_{14}$, $(87, 4, 45, 137, 41, 147)_{14}$,
$(3, 99, 69, 138, 46, 83)_{14}$, $(60, 119, 9, 82, 140, 41)_{14}$,
under the action of the mapping $x \mapsto x+1 \pmod{135}$ for $x < 135$, $x \mapsto (x-135+4 \pmod{20}) + 135$ for $x \geq 135$. $\square$