Analytical solutions for two-dimensional singly periodic Stokes flow singularity arrays near walls

Darren Crowdy · Elena Luca

Abstract New analytical representations of the Stokes flows due to periodic arrays of point singularities in a two-dimensional no-slip channel and in the half-plane near a no-slip wall are derived. The analysis makes use of a conformal mapping from a concentric annulus (or a disc) to a rectangle and a complex variable formulation of Stokes flow to derive the solutions. The form of the solutions is amenable to fast and accurate numerical computation without the need for Ewald summation or other fast summation techniques.

Keywords Stokes flow · singly periodic · fundamental singularity

1 Introduction

There has been a resurgence of interest in the mathematical theory of Stokes flows as a result of the multifarious problems arising in burgeoning fields such as microfluidics, nonlinear electrokinetics, low-Reynolds-number swimming, superhydrophobic surfaces and the study of active suspensions. In many situations singularity theory is used to model complicated bodies, such as swimming microorganisms or colloidal particles: a low-Reynolds-number swimmer, which is force- and torque-free, is often modelled as a point stresslet [1]; Jeong & Moffatt [5] modelled a pair of counter-rotating rollers beneath a free surface in a viscous fluid by an irrotational point dipole. Suspensions, or swarms of microorganisms, require the introduction of many singularities to model the complicated interaction mechanisms and it is common to consider flows comprising periodic arrays of the fundamental singularities of Stokes flow.

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Hasimoto [6] was the first to consider doubly and triply periodic arrays of Stokes singularities, recognising even then that the numerical computation of such flows is not without difficulty. He introduced the idea of using Ewald summation techniques to improve the numerical efficacy of computing such periodic flows. Pozrikidis [14] reconsidered similar problems from the perspective of constructing the basic Green's functions needed in formulating boundary integral methods for spatially periodic Stokes flows; in some cases, closed-form expressions for flows near solid boundaries had been previously found [15]. In his monograph, Pozrikidis [13] used standard Fourier transform techniques to solve for a periodic array of point singularities in a two-dimensional channel and in the half-plane. Davis [3] used Fourier transform techniques to solve various problems involving periodic arrays of Stokeslets in a two-dimensional channel and in the half-plane, and investigated the blocking properties of periodic arrays of wall-attached barriers. Work on improving numerical methods for Stokes flows in multi-particle settings continues [9, 16] and extensions of these ideas to multi-particle interactions in confined geometries have also been made [7]. Many of these methods rely on a splitting of the flow into a local contribution for which rapidly decaying free-space analytical solutions can be employed together with a global contribution whose effect is determined numerically in an efficient way (either spectrally, or using an iterative scheme).

The present authors [2] have recently given analytical representations that are amenable to fast numerical evaluation of the flows associated with doubly periodic arrangements of point singularities of two-dimensional Stokes flow. They used analytic function theory, a conformal mapping from a concentric annulus and the so-called Schottky–Klein prime function associated with that annulus to derive their new form of the solutions.

The present paper makes a basic theoretical contribution to the study of *singly* periodic two-dimensional Stokes flows in a no-slip channel and in the half-plane near a single no-slip wall. This paper can be viewed as a sequel to [2] where doubly periodic arrays of Stokes flow singularities are considered. Here we focus on singly periodic flows and, specifically, those generated by arrays of fundamental singularities, i.e. the Stokeslet and the higher order singularities. By employing an approach combining conformal mapping with a complex variable formulation of Stokes flow, we show how a fast and accurate representation of such flows can be derived without the need for the aforementioned "splitting" of the flow into separate local and global contributions or any Ewald summation techniques. While our results are limited to two dimensions, we believe they are valuable additions to the basic mathematical theory of Stokes flow.

2 Stokes flows in two dimensions

Consider a region of incompressible fluid of viscosity η governed by the Stokes equations

$$\nabla p = \eta \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u} = (u, v)$ is the two-dimensional velocity field and p is the fluid pressure. It is well-known [8] that an incompressible solution of the Stokes equations for the velocity field (u, v) can be written in terms of a stream function $\psi(x, y)$ with

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{2}$$

The stream function satisfies the biharmonic equation

$$\nabla^4 \psi = 0, \tag{3}$$

where ∇^2 is the two-dimensional Laplacian. The general solution of (3) can be written

$$\psi = \operatorname{Im}[\bar{z} f(z) + g(z)], \tag{4}$$

where $f(z)$ and $g(z)$ are analytic functions (which can have isolated singularities) in the fluid region and are often referred to as *Goursat functions* [8]. These analytic functions are related to physical quantities via

$$4 f'(z) = \frac{p}{\eta} - \mathrm{i}\omega, \qquad -\overline{f(z)} + \bar{z} f'(z) + g'(z) = u - \mathrm{i}v, \tag{5}$$

where ω is the fluid vorticity.

This connection with analytic functions is important because we propose to harness the powerful mathematical results of analytic function theory to find expressions for the fundamental singularities of Stokes flows which can be evaluated efficiently. Since we will consider singly periodic configurations with period l in the x-direction and channel width h it is natural to non-dimensionalise lengths with respect to the period width l, say.

3 Fundamental singularities of Stokes flow

In this section we present the representations for the Goursat functions for the fundamental singularities of Stokes flow. We focus here on the Stokeslet, stresslet, force quadrupole, along with some irrotational (or "source") singularities and higher-order singularities can be treated similarly. More details on the derivation of these expressions, as well as the associated velocity fields are presented in the recent paper [2] by the authors. By the linearity of the Stokes equations the singularities below can be superposed to form singularities of more complicated type.

3.1 Free-space Stokeslet

A Stokeslet at z_0 corresponds to $f(z)$ and $g'(z)$ locally having the form

$$f(z) = \mu \log(z - z_0) + \text{analytic function},$$
$$g'(z) = -\bar{\mu} \log(z - z_0) - \frac{\mu \overline{z_0}}{z - z_0} + \text{analytic function}, \tag{6}$$

where $\mu \in \mathbb{C}$ and the strength of the Stokeslet is $-8\pi\eta\mu$.

3.2 Free-space stresslet

A stresslet at z_0 corresponds to $f(z)$ and $g'(z)$ locally having the form

$$f(z) = \frac{\mu}{z - z_0} + \text{analytic function},$$
$$g'(z) = \frac{\mu \overline{z_0}}{(z - z_0)^2} + \text{analytic function}, \tag{7}$$

where $\mu \in \mathbb{C}$ determines the strength of the stresslet.

3.3 Free-space force quadrupole

A force quadrupole at z_0 corresponds to $f(z)$ and $g'(z)$ locally having the form

$$f(z) = \frac{\mu}{(z - z_0)^2} + \text{analytic function},$$

$$g'(z) = \frac{2\mu\overline{z_0}}{(z - z_0)^3} + \text{analytic function}, \tag{8}$$

where $\mu \in \mathbb{C}$ determines the strength of the quadrupole.

3.4 Source singularities

If $f(z)$ and $g'(z)$ have the local form near z_0 given by

$$f(z) = \text{analytic function},$$

$$g'(z) = \frac{\mu}{z - z_0} + \text{analytic function}, \tag{9}$$

then, if $\mu \in \mathbb{R}$ we say that there is a source/sink at z_0 (with an associated mass flux given by $2\pi\mu$). If $\mu \in i\mathbb{R}$ we say that there is a rotlet at z_0 (which exerts a torque on the fluid of strength dictated by the modulus of μ).

A (source) dipole at z_0 corresponds to

$$f(z) = \text{analytic function},$$

$$g'(z) = -\frac{\mu}{(z - z_0)^2} + \text{analytic function}, \tag{10}$$

where the parameter μ determines the strength of the dipole.

A (source) quadrupole at z_0 corresponds to

$$f(z) = \text{analytic function},$$

$$g'(z) = -\frac{2\mu}{(z - z_0)^3} + \text{analytic function}, \tag{11}$$

where the parameter μ determines the strength of the quadrupole.

4 Conformal mapping

To study singly periodic arrays of singularities, we will exploit the conformal mapping

$$z = \mathcal{Z}(\zeta) \equiv -i \log \zeta. \tag{12}$$

This map transplants the annulus $\rho < |\zeta| < 1$ to a period rectangle in the $z = x + iy$ plane occupying the region

$$0 \leq x \leq 2\pi = l, \qquad 0 \leq y \leq -\log \rho = h, \tag{13}$$

as illustrated in Fig. 1. Changing the x-period simply requires multiplication of (12) by the appropriate real factor. Under this conformal map, the unit circle $|\zeta| = 1$ corresponds to $0 \leq x \leq l$, $y = 0$, the inner circle $|\zeta| = \rho$ corresponds to
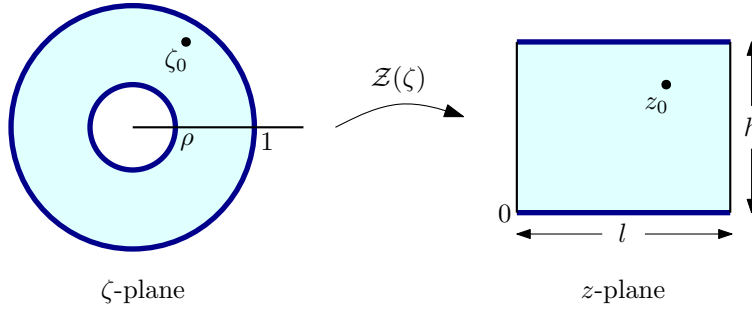
Fig. 1: Conformal mapping from the annulus $\rho < |\zeta| < 1$ in ζ-plane to the period rectangle in the physical z-plane occupying the region $0 \leq x \leq 2\pi = l$, $0 \leq y \leq -\log\rho = h$.

$0 \leq x \leq l$, $y = h$ and the interval $[\rho, 1]$ (in ζ-plane) to the vertical sides of the period window $x = 0, l$, $0 \leq y \leq h$.

The key idea of our approach is to show that analytical expressions, with fast convergence properties, for the fundamental singularities of periodic Stokes flow can be derived in terms of the variable ζ. It is then a simple matter, if required, to re-express the final results as functions of z using the relation

$$\zeta = \mathrm{e}^{\mathrm{i}z} \tag{14}$$

which follows from (12).

The representative point singularity in the fundamental period rectangle (Fig. 1) is located at z_0 and has a preimage at ζ_0 such that

$$z_0 = \mathcal{Z}(\zeta_0) = -\mathrm{i}\log\zeta_0. \tag{15}$$

5 Periodic Stokes singularity arrays in a channel

Consider a two-dimensional channel $-\infty < x < \infty$, $0 \leq y \leq h$ and a periodic array of point singularities at $z = z_0 + nl$, $n \in \mathbb{Z}$, with $0 < \mathrm{Re}[z_0] < l$, $0 < \mathrm{Im}[z_0] < h$. Figure 2 shows a schematic of the configuration. This problem can be solved using standard Fourier transform techniques (Pozrikidis [13]), but we will give an alternative derivation, and form, of the solution by extending the ideas given in the previous sections.

5.1 Periodic array of Stokeslets in a channel

Consider a periodic array of Stokeslets each of strength given by $-8\pi\eta\mu$ for some $\mu \in \mathbb{C}$. The representative Stokeslet in the fundamental period rectangle shown in Fig. 1 is located at z_0 and has a preimage at ζ_0 related to z_0 via (15). An important step is to introduce the functions

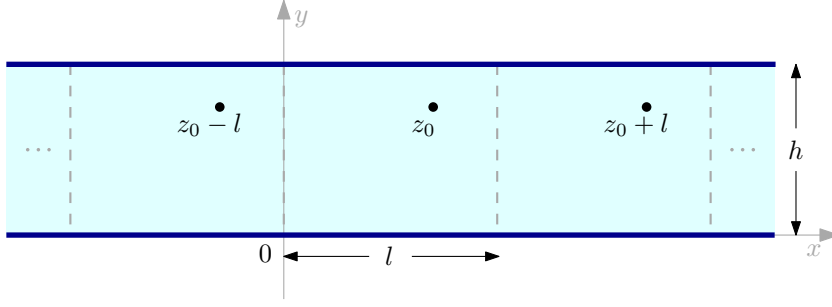$$F(\zeta) \equiv f(\mathcal{Z}(\zeta)), \qquad G(\zeta) \equiv g'(\mathcal{Z}(\zeta)), \tag{16}$$

Fig. 2: A periodic array of point singularities at points $z = z_0 + nl$, $n \in \mathbb{Z}$, with $0 < \mathrm{Re}[z_0] < l$, $0 < \mathrm{Im}[z_0] < h$, in a two-dimensional channel $-\infty < x < \infty$, $0 \le y \le h$.

where $'$ here denotes differentiation with respect to the argument of the function. Now let

$$F(\zeta) = a \log \zeta + \mu \log(\zeta - \zeta_0) + \hat{F}(\zeta),$$

$$G(\zeta) = -\overline{a} \log \zeta - \overline{\mu} \log(\zeta - \zeta_0) + \frac{\lambda}{\zeta - \zeta_0} + \mathrm{i} \log \zeta \left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)} \right] + \hat{G}(\zeta), \tag{17}$$

where $\hat{F}(\zeta)$ and $\hat{G}(\zeta)$ are analytic and single-valued in the annulus $\rho < |\zeta| < 1$ and the constants λ and a are to be found. With this ansatz, we have

$$\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)} = \mathrm{i}\zeta \left[\frac{a}{\zeta} + \frac{\mu}{\zeta - \zeta_0} + \hat{F}'(\zeta) \right] = \mathrm{i}a + \frac{\mathrm{i}\mu\zeta}{\zeta - \zeta_0} + \mathrm{i}\zeta \hat{F}'(\zeta) \tag{18}$$

and, hence, the complex velocity field expressed in terms of variable ζ is

$$u - \mathrm{i}v = -\overline{a} \log |\zeta|^2 - \overline{\mu} \log |\zeta - \zeta_0|^2 + \mathrm{i} \log |\zeta|^2 \left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)} \right]$$
$$- \overline{\hat{F}(\zeta)} + \hat{G}(\zeta) + \frac{\lambda}{\zeta - \zeta_0}, \tag{19}$$

where we have used that $\overline{z} f'(z) = \mathrm{i} \log \overline{\zeta} [F'(\zeta)/\mathcal{Z}'(\zeta)]$. By the choice of the ansatz, $u - \mathrm{i}v$ is invariant as $\zeta \mapsto \zeta \mathrm{e}^{2\pi \mathrm{i}}$ and this corresponds to the periodicity of the velocity field along the channel. A local analysis of the singularity of $u - \mathrm{i}v$ necessitates that we choose

$$\lambda = \mu \zeta_0 \log |\zeta_0|^2, \tag{20}$$

in order to ensure that the singularity at ζ_0 has the required form.

The no-slip condition on $|\zeta| = 1$ becomes

$$-\overline{\hat{F}(\zeta)} + \hat{G}(\zeta) = \overline{\mu} \log |\zeta - \zeta_0|^2 - \frac{\mu \zeta_0 \log |\zeta_0|^2}{\zeta - \zeta_0}, \tag{21}$$

while the no-slip condition on $|\zeta| = \rho$ is

$$-\overline{\hat{F}(\zeta)} + \hat{G}(\zeta) - \log \rho^2 \left[\zeta \hat{F}'(\zeta) \right] - (a + \overline{a}) \log \rho^2$$
$$= \overline{\mu} \log |\zeta - \zeta_0|^2 - \frac{\mu \zeta_0 \log |\zeta_0|^2}{\zeta - \zeta_0} + \frac{\mu \zeta \log \rho^2}{\zeta - \zeta_0}. \tag{22}$$

For $|\zeta| = 1$, conditions (21) and (22) can be written in the form

$$-\overline{\hat{F}}(\zeta^{-1}) + \hat{G}(\zeta) = \sum_{n=-\infty}^{\infty} d_n \zeta^n,$$

$$-\overline{\hat{F}}(\rho\zeta^{-1}) + \hat{G}(\rho\zeta) - \log \rho^2 [\rho\zeta\hat{F}'(\rho\zeta)] - (a + \overline{a}) \log \rho^2 = \sum_{n=-\infty}^{\infty} e_n \zeta^n,$$

(23)

where we have introduced the following Laurent expansions of known functions:

$$\sum_{n=-\infty}^{\infty} d_n \zeta^n = \overline{\mu} \log |\zeta - \zeta_0|^2 - \frac{\mu\zeta_0 \log |\zeta_0|^2}{\zeta - \zeta_0},$$

$$\sum_{n=-\infty}^{\infty} e_n \zeta^n = \overline{\mu} \log |\rho\zeta - \zeta_0|^2 - \frac{\mu\zeta_0 \log |\zeta_0|^2}{\rho\zeta - \zeta_0} + \frac{\mu\rho\zeta \log \rho^2}{\rho\zeta - \zeta_0},$$

(24)

where the Schwarz conjugate $\overline{w}(\zeta)$ of a function $w(\zeta)$ is defined by $\overline{w}(\zeta) \equiv \overline{w(\overline{\zeta})}$. The coefficients $\{d_n, e_n | n \in \mathbb{Z}\}$ can be determined explicitly by computing the relevant Laurent expansions. Alternatively, they can be computed numerically using Fast Fourier transforms. Using local expansions, we find

$$d_n = -\overline{\mu}(\overline{\zeta_0})^n/n, \quad n \geq 1,$$
$$d_{-n} = -(\zeta_0)^n(\overline{\mu}/n + \mu \log |\zeta_0|^2), \quad n \geq 1,$$
$$d_0 = 0$$

(25)

and

$$e_n = \rho^n(\zeta_0)^{-n}[\mu(\log |\zeta_0|^2 - \log \rho^2) - \overline{\mu}/n], \quad n \geq 1,$$
$$e_{-n} = -\overline{\mu}\rho^n(\overline{\zeta_0})^{-n}/n, \quad n \geq 1,$$
$$e_0 = 2\mathrm{Re}[\mu] \log |\zeta_0|^2.$$

(26)

The Laurent expansions (24) converge rapidly as can be seen from (25)–(26): $|d_n| = |d_{-n}| = \mathcal{O}(|\zeta_0|^n)$ for $n \geq 1$, where $|\zeta_0| < 1$ and $|e_n| = |e_{-n}| = \mathcal{O}(|\rho/\zeta_0|^n)$ for $n \geq 1$, where $|\rho/\zeta_0| < 1$. Therefore, the coefficients of the series are given by powers of ζ_0 and ρ/ζ_0 (and their complex conjugates) with exponents that depend on n so the convergence of the sums is rapid.

The next step is to consider the Laurent series expansions

$$\hat{F}(\zeta) = \sum_{n=1}^{\infty} F_n \zeta^n + \sum_{n=1}^{\infty} H_n \left(\frac{\rho}{\zeta}\right)^n,$$

$$\hat{G}(\zeta) = \sum_{n=1}^{\infty} G_n \zeta^n + G_0 + \sum_{n=1}^{\infty} K_n \left(\frac{\rho}{\zeta}\right)^n.$$

(27)

Without loss of generality, the constant term in the expansion of $\hat{F}(\zeta)$ can be set equal to zero owing to an additive degree of freedom in the specification of $f(z)$. Remarkably, the Laurent series (27) can be substituted into (23) and their unknown coefficients found explicitly. The constant terms in (23) give

$$G_0 = d_0, \qquad G_0 - \log \rho^2 (a + \overline{a}) = e_0$$

(28)

which implies that

$$\text{Re}[a] = \frac{d_0 - e_0}{4 \log \rho}. \tag{29}$$

Note that the imaginary part of a can be set to zero because making the transformation $f(z) \mapsto f(z) + bz$ for some $b \in \mathbb{R}$ clearly does not affect the velocity field (5). This corresponds to adding the term $-ib \log \zeta$ to $F(\zeta)$, expressed in terms of variable ζ. It follows that we can take

$$a = \frac{d_0 - e_0}{4 \log \rho}. \tag{30}$$

Equating coefficients of the other powers of ζ produces the equations

$$\begin{aligned}
-\rho^n \overline{H_n} + G_n &= d_n, \\
-\overline{F_n} + \rho^n K_n &= d_{-n}, \\
-\overline{H_n} + \rho^n G_n - n\rho^n \log \rho^2 F_n &= e_n, \\
-\rho^n \overline{F_n} + K_n + n \log \rho^2 H_n &= e_{-n}, \quad n \geq 1.
\end{aligned} \tag{31}$$

This system can be manipulated to find

$$F_n = \frac{n\rho^n \log \rho^2 e_n + \rho^n(1 - \rho^{2n})\overline{e_{-n}} - n\rho^{2n} \log \rho^2 d_n - (1 - \rho^{2n})\overline{d_{-n}}}{(1 - \rho^{2n})^2 - n^2 \rho^{2n}(\log \rho^2)^2}, \quad n \geq 1. \tag{32}$$

The remaining coefficients needed to evaluate $\hat{F}(\zeta)$ and $\hat{G}(\zeta)$ then follow by back substitution:

$$G_n = \frac{-n\rho^{2n} \log \rho^2 F_n - \rho^n e_n + d_n}{(1 - \rho^{2n})}, \quad H_n = \frac{\overline{G_n} - \overline{d_n}}{\rho^n}, \quad K_n = \frac{\overline{F_n} + d_{-n}}{\rho^n}, \ n \geq 1. \tag{33}$$

It should be clear that substitution of the explicit form (32) into (33) will lead to explicit expressions for all these coefficients although we have avoided displaying those formulas here.

In summary, the Goursat functions for a periodic array of Stokeslets in a channel with one at z_0 in the principal period window and having strength given by $-8\pi\eta\mu$ are

$$F(\zeta) = a \log \zeta + \mu \log(\zeta - \zeta_0) + \sum_{n=1}^{\infty} F_n \zeta^n + \sum_{n=1}^{\infty} H_n \left(\frac{\rho}{\zeta}\right)^n,$$

$$G(\zeta) = -\overline{a} \log \zeta - \overline{\mu} \log(\zeta - \zeta_0) + \frac{\lambda}{\zeta - \zeta_0} + i \log \zeta \left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right]$$

$$+ \sum_{n=1}^{\infty} G_n \zeta^n + G_0 + \sum_{n=1}^{\infty} K_n \left(\frac{\rho}{\zeta}\right)^n,$$

$$\text{with} \qquad a = \frac{d_0 - e_0}{4 \log \rho}, \qquad \lambda = \mu \zeta_0 \log |\zeta_0|^2,$$

$$\tag{34}$$

where coefficients $\{F_n, H_n, G_n, K_n | n \in \mathbb{N}\}$ and G_0 are given by (28), (32) and (33) in terms of (25)–(26). Figures 3 and 4 show the fast decay of the Laurent expansion terms (27) and indicate that only a few terms are required in general to provide accurate values of the flow field variables.
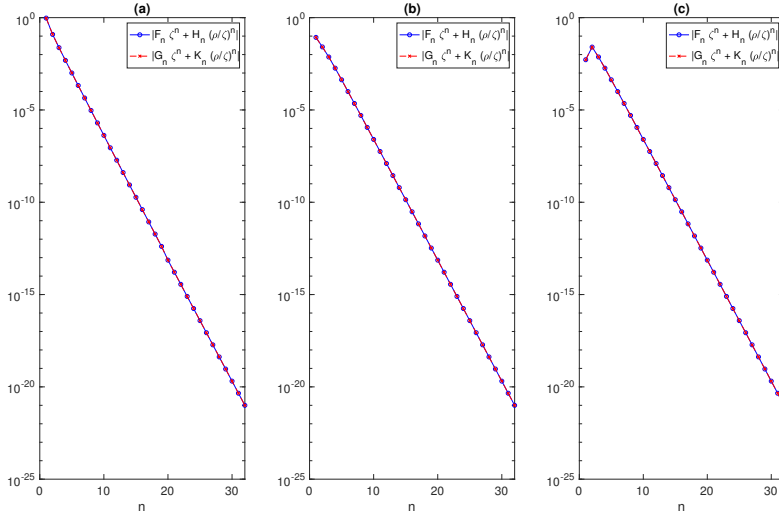
Fig. 3: Decay of the Laurent expansion terms $F_n\zeta^n + H_n(\rho/\zeta)^n$ and $G_n\zeta^n + K_n(\rho/\zeta)^n$, $n \geq 1$, given by expressions (27), for parameters $l = 2\pi$, $\zeta = \mathrm{e}^{-1}$, $\zeta_0 = 0.6$ and different channel heights: (a) $h = \pi/2$, (b) $h = \pi$ and (c) $h = 2\pi$.
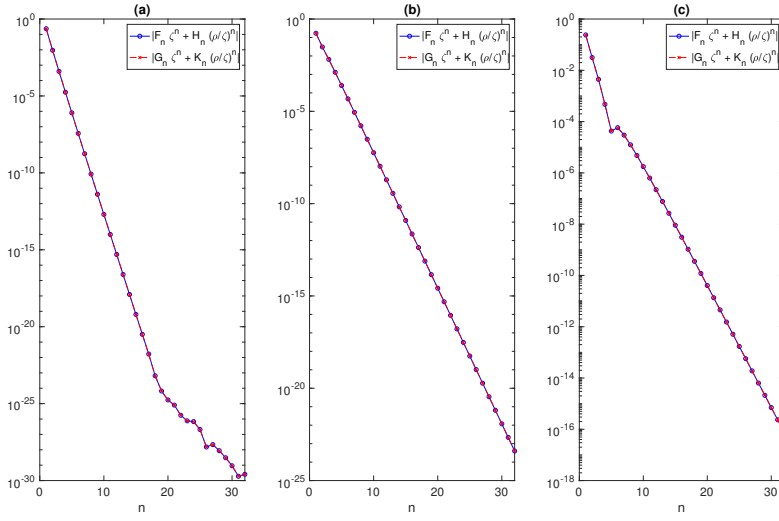

Fig. 4: Decay of the Laurent expansion terms $F_n\zeta^n + H_n(\rho/\zeta)^n$ and $G_n\zeta^n + K_n(\rho/\zeta)^n$, $n \geq 1$, given by expressions (27), for parameters $l = 2\pi$, $h = \pi$, $\zeta = \mathrm{e}^{-1}$ and different locations for the principal point Stokeslet: (a) $\zeta_0 = 0.1$, (b) $\zeta_0 = 0.5$ and (c) $\zeta_0 = 0.9$.

Davis [3] presented solutions to various problems involving distributions of Stokeslets in an unbounded domain, as well as in a two-dimensional channel, with a view to understanding the blocking properties of periodic arrays of wall-attached barriers. We have verified numerically that the associated flow field found above agrees with the solutions found by Davis [3].

As a separate check, yet another new method for solving the problem of a periodic array of point singularities in a channel geometry is summarized in Appendix A. That method is based on a novel transform approach to biharmonic boundary value problems recently described by the authors [10]; it provides a quasi-analytical solution to the problem in that it reduces determination of the Goursat functions to the solution of a small linear system whose coefficients are given by explicit integrals. The solutions presented above have been verified numerically against the latter solutions.

Figure 5 shows the streamline patterns for a singly periodic array of Stokeslets of different strengths $\mu = e^{i\phi}$, $\phi = 0, \pi/8, \pi/4, 3\pi/8, \pi/2$ and parameters $l = 2\pi$, $h = 2$, $z_0 = \pi - i\log(0.7)$ which is the location of the principal point Stokeslet. The graphs show the streamline topology transition between $\phi \to \pi/2$ and $\phi = \pi/2$ (other singularities behave similarly) and the appearance of Moffatt eddies [12]. Figure 6 shows the streamline patterns for a singly periodic array of Stokeslets of strength $\mu = i$ for different channel heights $h = 0.7, 1, 1.9, 2.1, 3$ and parameters $l = 2\pi$, $z_0 = \pi - i\log(0.7)$.

5.2 Periodic array of stresslets in a channel

Following similar steps, we find that the Goursat functions for a periodic array of torque-free stresslets in a channel with local behaviour (7) at z_0 in the principal period window are

$$F(\zeta) = a\log\zeta + \frac{i\mu\zeta_0}{\zeta - \zeta_0} + \sum_{n=1}^{\infty} F_n\zeta^n + \sum_{n=1}^{\infty} H_n\left(\frac{\rho}{\zeta}\right)^n,$$

$$G(\zeta) = -\overline{a}\log\zeta + \frac{\chi}{\zeta - \zeta_0} + \frac{\nu}{(\zeta - \zeta_0)^2} + i\log\zeta\left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right]$$

$$+ \sum_{n=1}^{\infty} G_n\zeta^n + G_0 + \sum_{n=1}^{\infty} K_n\left(\frac{\rho}{\zeta}\right)^n, \tag{35}$$

with $\quad a = \dfrac{d_0 - e_0}{4\log\rho}, \qquad \chi = i\mu\zeta_0[2\mathrm{Im}[z_0] - 1], \qquad \nu = 2i\mu\zeta_0^2\mathrm{Im}[z_0].$

Coefficients $\{F_n, H_n, G_n, K_n | n \in \mathbb{N}\}$ and G_0 are given by (28), (32) and (33) in terms of coefficients $\{d_n, e_n | n \in \mathbb{Z}\}$ which are now given by

$$d_n = -i\overline{\mu}(\overline{\zeta_0})^n, \quad n \geq 1,$$
$$d_{-n} = i\mu(\zeta_0)^n[1 - 2n\mathrm{Im}[z_0]], \quad n \geq 1, \tag{36}$$
$$d_0 = 0$$

and

$$e_n = -i\mu\rho^n(\zeta_0)^{-n}[1 + n(2\mathrm{Im}[z_0] + \log\rho^2)], \quad n \geq 1,$$
$$e_{-n} = i\overline{\mu}\rho^n(\overline{\zeta_0})^{-n}, \quad n \geq 1, \tag{37}$$
$$e_0 = 2\,\mathrm{Im}[\mu].$$

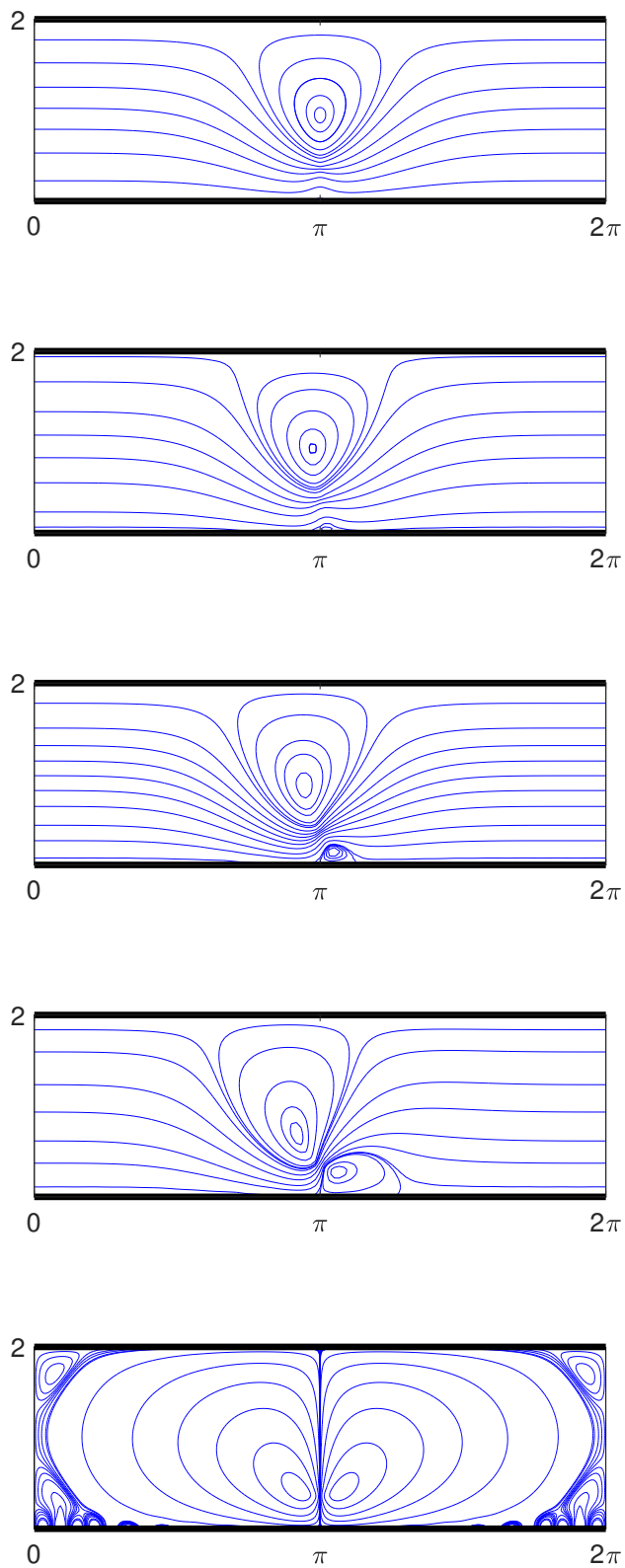Fig. 5: Streamline patterns for a periodic Stokeslet in a channel in a period window for different strengths μ. The principal period window is $0 \leq x \leq l = 2\pi$, $0 \leq y \leq h = 2$. The principal point Stokeslet is located at $z_0 = \pi - \mathrm{i}\log(0.7)$ and its strength is $\mu = \mathrm{e}^{\mathrm{i}\phi}$, (a) $\phi = 0$, (b) $\phi = \pi/8$, (c) $\phi = \pi/4$, (d) $\phi = 3\pi/8$ and (e) $\phi = \pi/2$.
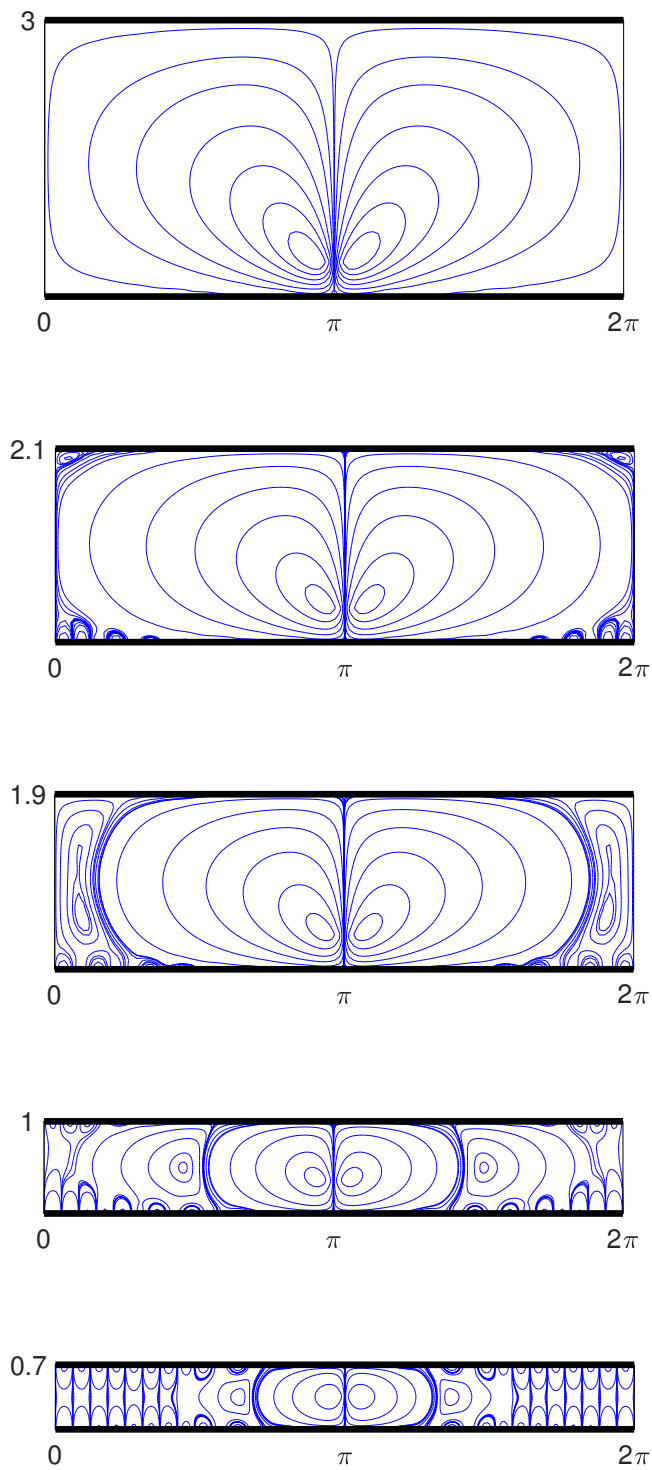
Fig. 6: Streamline patterns for a periodic Stokeslet in a channel in a period window for different channel heights $h = 0.7, 1, 1.9, 2.1, 3$. The principal period window is $0 \leq x \leq l = 2\pi, 0 \leq y \leq h$. The principal point Stokeslet is located at $z_0 = \pi - \mathrm{i} \log(0.7)$ and its strength is $\mu = \mathrm{i}$.

It turns out that the full analysis just summarized can be bypassed by noticing that the Goursat functions $f(z)$ and $g'(z)$ for the periodic array of stresslets can be derived from those of Stokeslets by computing the parametric derivatives (cf. [4]):

$$f(z) \mapsto -\frac{\partial f(z)}{\partial z_0}, \qquad g'(z) \mapsto -\frac{\partial g'(z)}{\partial z_0}. \tag{38}$$

To verify that, we apply the parametric derivatives (38) in (6) and obtain

$$
\begin{aligned}
f(z) &= \frac{\mu}{z - z_0} + \text{analytic function,} \\
g'(z) &= \frac{\mu \overline{z_0}}{(z - z_0)^2} - \frac{\overline{\mu}}{z - z_0} + \text{analytic function.}
\end{aligned}
\tag{39}
$$

The first terms in $f(z)$ and $g'(z)$ correspond to a stresslet of strength μ at z_0 given by (7), while the second term in $g'(z)$ corresponds to an additional source/sink and/or rotlet singularity. Since our aim is to find expressions for a periodic array of torque-free stresslets without any other induced flow, we can first use (38) in (34), where

$$\frac{\partial}{\partial z_0} = \frac{\partial \zeta_0}{\partial z_0}\frac{\partial}{\partial \zeta_0} = \mathrm{i}\zeta_0 \frac{\partial}{\partial \zeta_0}, \tag{40}$$

and, then, subtract terms associated to the simple pole in $g'(z)$. This alternative approach gives identical expressions to (35)–(37).

5.3 Periodic array of force quadrupoles in a channel

Consider now the problem of a periodic array of force quadrupoles in a channel. In this problem too the full analysis can be bypassed by noticing that the Goursat functions $f(z)$ and $g'(z)$ for this problem can be derived from those associated with an array of torque-free stresslets by computing the parametric derivatives:

$$f(z) \mapsto \frac{\partial f(z)}{\partial z_0}, \qquad g'(z) \mapsto \frac{\partial g'(z)}{\partial z_0}. \tag{41}$$

Following similar steps as previously, or using the parametric derivatives (41), we find that the Goursat functions for a periodic array of force quadrupoles in a channel with local behaviour (8) at z_0 in the principal window to are given by

$$
\begin{aligned}
F(\zeta) &= \frac{\beta}{\zeta - \zeta_0} + \frac{\gamma}{(\zeta - \zeta_0)^2} + \sum_{n=1}^{\infty} F_n \zeta^n + \sum_{n=1}^{\infty} H_n \left(\frac{\rho}{\zeta}\right)^n, \\
G(\zeta) &= \frac{\delta}{\zeta - \zeta_0} + \frac{\epsilon}{(\zeta - \zeta_0)^2} + \frac{\kappa}{(\zeta - \zeta_0)^3} + \mathrm{i}\log\zeta\left[\frac{F'(\zeta)}{\mathscr{Z}'(\zeta)}\right] \\
&\quad + \sum_{n=1}^{\infty} G_n \zeta^n + G_0 + \sum_{n=1}^{\infty} K_n \left(\frac{\rho}{\zeta}\right)^n,
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
\text{with} \quad & \beta = -\mu\zeta_0, \qquad \gamma = -\mu\zeta_0^2, \qquad \delta = 2\mu\zeta_0[1 - \mathrm{Im}[z_0]], \\
& \epsilon = 2\mu\zeta_0^2[1 - 3\mathrm{Im}[z_0]], \qquad \kappa = -4\mu\zeta_0^3\mathrm{Im}[z_0].
\end{aligned}
$$

The coefficients $\{F_n, H_n, G_n, K_n | n \in \mathbb{N}\}$ and G_0 are again given by (28), (32) and (33) in terms of coefficients $\{d_n, e_n | n \in \mathbb{Z}\}$ which, in this case, are

$$
\begin{aligned}
d_n &= -n\overline{\mu}(\overline{\zeta_0})^n, \quad n \geq 1, \\
d_{-n} &= 2n\mu(\zeta_0)^n[n\text{Im}[z_0] - 1], \quad n \geq 1, \\
d_0 &= 0
\end{aligned}
\tag{43}
$$

and

$$
\begin{aligned}
e_n &= -n\mu\rho^n(\zeta_0)^{-n}[2 + n(2\text{Im}[z_0] + \log\rho^2)], \quad n \geq 1, \\
e_{-n} &= -n\overline{\mu}\rho^n(\overline{\zeta_0})^{-n}, \quad n \geq 1, \\
e_0 &= 0.
\end{aligned}
\tag{44}
$$

5.4 Periodic array of dipoles and quadrupoles in a channel

The Goursat functions $f(z)$ and $g'(z)$ for a periodic array of dipoles can be derived from those of Stokeslets (34) by computing the mixed parametric derivatives:

$$
f(z) \mapsto \frac{\partial^2 f(z)}{\partial z_0 \partial \overline{z_0}}, \qquad g'(z) \mapsto \frac{\partial^2 g'(z)}{\partial z_0 \partial \overline{z_0}}.
\tag{45}
$$

where

$$
\frac{\partial^2}{\partial z_0 \partial \overline{z_0}} = \zeta_0 \overline{\zeta_0} \frac{\partial^2}{\partial \zeta_0 \partial \overline{\zeta_0}}.
\tag{46}
$$

The Goursat functions for a periodic array of dipoles in a channel with local behaviour (10) at z_0 in the principal period window are

$$
F(\zeta) = \sum_{n=1}^{\infty} F_n \zeta^n + \sum_{n=1}^{\infty} H_n \left(\frac{\rho}{\zeta}\right)^n,
$$

$$
G(\zeta) = \frac{\mu\zeta_0}{\zeta - \zeta_0} + \frac{\mu\zeta_0^2}{(\zeta - \zeta_0)^2} + i\log\zeta\left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right] + \sum_{n=1}^{\infty} G_n \zeta^n + G_0 + \sum_{n=1}^{\infty} K_n \left(\frac{\rho}{\zeta}\right)^n.
\tag{47}
$$

Again, the coefficients $\{F_n, H_n, G_n, K_n | n \in \mathbb{N}\}$ and G_0 are given by (28), (32) and (33) in terms of coefficients $\{d_n, e_n | n \in \mathbb{Z}\}$ which are given by

$$
\begin{aligned}
d_n &= 0, \quad n \geq 1, \\
d_{-n} &= -n\mu(\zeta_0)^n, \quad n \geq 1, \\
d_0 &= 0
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
e_n &= -n\mu\rho^n(\zeta_0)^{-n}, \quad n \geq 1, \\
e_{-n} &= 0, \quad n \geq 1, \\
e_0 &= 0.
\end{aligned}
\tag{49}
$$

Similarly the Goursat functions $f(z)$ and $g'(z)$ for a periodic array of quadrupoles can be derived from those of torque-free stresslets (35) by computing the mixed parametric derivatives:

$$f(z) \mapsto -\frac{\partial^2 f(z)}{\partial z_0 \partial \overline{z_0}}, \qquad g'(z) \mapsto -\frac{\partial^2 g'(z)}{\partial z_0 \partial \overline{z_0}}. \tag{50}$$

The Goursat functions for a periodic array of quadrupoles in a channel with local behaviour (11) at z_0 in the principal period window are

$$F(\zeta) = \sum_{n=1}^{\infty} F_n \zeta^n + \sum_{n=1}^{\infty} H_n \left(\frac{\rho}{\zeta}\right)^n,$$

$$G(\zeta) = \frac{\mathrm{i}\mu\zeta_0}{\zeta - \zeta_0} + \frac{3\mathrm{i}\mu\zeta_0^2}{(\zeta - \zeta_0)^2} + \frac{2\mathrm{i}\mu\zeta_0^3}{(\zeta - \zeta_0)^3} + \mathrm{i}\log\zeta \left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right] \tag{51}$$

$$+ \sum_{n=1}^{\infty} G_n \zeta^n + G_0 + \sum_{n=1}^{\infty} K_n \left(\frac{\rho}{\zeta}\right)^n$$

with coefficients $\{F_n, H_n, G_n, K_n | n \in \mathbb{N}\}$ and G_0 given by (28), (32) and (33) in terms of coefficients $\{d_n, e_n | n \in \mathbb{Z}\}$:

$$d_n = 0, \quad n \geq 1,$$
$$d_{-n} = -\mathrm{i}n^2\mu(\zeta_0)^n, \quad n \geq 1, \tag{52}$$
$$d_0 = 0$$

and

$$e_n = \mathrm{i}n^2\mu\rho^n(\zeta_0)^{-n}, \quad n \geq 1,$$
$$e_{-n} = 0, \quad n \geq 1, \tag{53}$$
$$e_0 = 0.$$

6 Periodic singularity arrays in the half-plane

In the limit where the upper wall of the channel becomes distant from the lower wall we encounter the situation of a periodic array of Stokes singularities in the upper half plane near an infinite straight wall and it is useful to consider this situation too since it arises in many physical modelling situations (e.g. [11]).

Consider a periodic array of point singularities at $z = z_0 + nl$, $n \in \mathbb{Z}$, with $0 < \mathrm{Re}[z_0] < l$, $\mathrm{Im}[z_0] > 0$, in the upper half-plane $-\infty < x < \infty$, $y \geq 0$. Figure 7 shows a schematic of the configuration. While the problem can be solved using standard Fourier transform techniques [13], we give an alternative derivation, and form, of the solution by extending the ideas given in the previous sections.
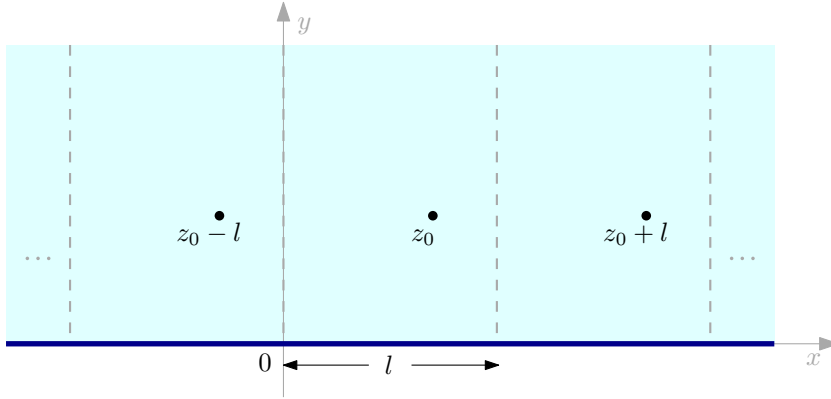
Fig. 7: A periodic array of point singularities at points $z = z_0 + nl$, $n \in \mathbb{Z}$ in the upper half-plane $-\infty < x < \infty$, $y \geq 0$.

6.1 Periodic array of Stokeslets in the upper half-plane

Consider a periodic array of Stokeslets each of strength given by $-8\pi\eta\mu$ for some $\mu \in \mathbb{C}$. The representative Stokeslet in the fundamental period semi-strip shown in Fig. 7 is located at z_0 and has a preimage at ζ_0 with $z_0 = z(\zeta_0) = -\mathrm{i}\log\zeta_0$. The same conformal mapping (12) is employed, but now we take the limit $\rho \to 0$ so that the preimage domain in the unit disc in the parametric ζ plane. The unit circle $|\zeta| = 1$ is transplanted to the no-slip wall along the x axis. In this section the relevant functions $F(\zeta)$ and $G(\zeta)$ can all be expressed in closed form as log-rational functions.

Again we introduce the functions $F(\zeta)$ and $G(\zeta)$ defined in (16). Let

$$F(\zeta) = \mu\log(\zeta - \zeta_0) + \epsilon\log(1 - \overline{\zeta_0}\zeta) + \frac{\kappa}{1 - \overline{\zeta_0}\zeta} + \lambda, \tag{54}$$

where constants ϵ, κ and λ are to be found. In terms of functions of the ζ variable the no-slip boundary condition on the wall $\overline{z} = z$ takes the form

$$-\overline{F}(1/\zeta) + \mathcal{Z}(\zeta)\left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right] + G(\zeta) = 0, \tag{55}$$

where we have that $\overline{\zeta} = 1/\zeta$ on $|\zeta| = 1$; this implies that

$$G(\zeta) = \overline{F}(1/\zeta) - \mathcal{Z}(\zeta)\left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right]. \tag{56}$$

It remains to find the unknown constants ϵ, κ and λ. These follow by insisting that $u - \mathrm{i}v$ is invariant as $\zeta \mapsto \zeta\mathrm{e}^{2\pi\mathrm{i}}$ which corresponds to the periodicity of the velocity field and by ensuring that the singularity of $u - \mathrm{i}v$ at ζ_0 has the required form. It is found that

$$\epsilon = -\mu, \qquad \kappa = \overline{\mu}\log|\zeta_0|^2, \qquad \lambda = -\kappa. \tag{57}$$

In summary, the Goursat functions for a periodic array of Stokeslets in the upper half-plane with one at z_0 in the principal period window and having strength given by $-8\pi\eta\mu$ are

$$F(\zeta) = \mu \log(\zeta - \zeta_0) - \mu \log(1 - \overline{\zeta_0}\zeta) + \frac{\kappa}{1 - \overline{\zeta_0}\zeta} - \kappa,$$

$$G(\zeta) = -\overline{\mu} \log(\zeta - \zeta_0) + \overline{\mu} \log(1 - \overline{\zeta_0}\zeta) + \frac{\overline{\kappa}\zeta_0}{\zeta - \zeta_0} + \mathrm{i} \log \zeta \left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right]. \tag{58}$$

As expected, these expressions coincide with those obtained by taking $\rho \to 0$ in the solutions presented in §5 (which corresponds to the limit $h \to \infty$).

6.2 Periodic array of stresslets in the upper half-plane

Following similar steps or taking parametric derivatives as shown in §5, we find that the Goursat functions for a periodic array of torque-free stresslets in the upper half-plane with local behaviour (7) at z_0 in the principal period window are

$$F(\zeta) = \frac{\mathrm{i}\mu\zeta_0}{\zeta - \zeta_0} + \frac{\beta}{1 - \overline{\zeta_0}\zeta} + \frac{\gamma}{(1 - \overline{\zeta_0}\zeta)^2} - \mathrm{i}\overline{\mu},$$

$$G(\zeta) = \frac{\chi}{\zeta - \zeta_0} + \frac{\nu}{(\zeta - \zeta_0)^2} - \frac{\mathrm{i}\overline{\mu}}{1 - \overline{\zeta_0}\zeta} + \mathrm{i} \log \zeta \left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right] + \mathrm{i}\overline{\mu}, \tag{59}$$

$$\text{with} \quad \beta = \mathrm{i}\overline{\mu}[2\mathrm{Im}[z_0] + 1], \qquad \gamma = -2\mathrm{i}\overline{\mu}\mathrm{Im}[z_0],$$

$$\chi = \mathrm{i}\mu\zeta_0[2\mathrm{Im}[z_0] - 1], \qquad \nu = 2\mathrm{i}\mu\zeta_0^2\mathrm{Im}[z_0].$$

6.3 Periodic array of force quadrupoles in the upper half-plane

The Goursat functions for a periodic array of force quadrupoles in the upper half-plane with local behaviour (8) at z_0 in the principal period window are

$$F(\zeta) = -\frac{\mu\zeta_0}{\zeta - \zeta_0} - \frac{\mu\zeta_0^2}{(\zeta - \zeta_0)^2} + \frac{\beta}{1 - \overline{\zeta_0}\zeta} + \frac{\gamma}{(1 - \overline{\zeta_0}\zeta)^2} + \frac{\delta}{(1 - \overline{\zeta_0}\zeta)^3},$$

$$G(\zeta) = \frac{\epsilon}{\zeta - \zeta_0} + \frac{\kappa}{(\zeta - \zeta_0)^2} + \frac{\lambda}{(\zeta - \zeta_0)^3} + \frac{\overline{\mu}}{1 - \overline{\zeta_0}\zeta} - \frac{\overline{\mu}}{(1 - \overline{\zeta_0}\zeta)^2} + \mathrm{i} \log \zeta \left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right],$$

$$\text{with} \quad \beta = -2\overline{\mu}[1 + \mathrm{Im}[z_0]], \qquad \gamma = 2\overline{\mu}[1 + 3\mathrm{Im}[z_0]], \qquad \delta = -4\overline{\mu}\mathrm{Im}[z_0],$$

$$\epsilon = 2\mu\zeta_0[1 - \mathrm{Im}[z_0]], \qquad \kappa = 2\mu\zeta_0^2[1 - 3\mathrm{Im}[z_0]], \qquad \lambda = -4\mu\zeta_0^3\mathrm{Im}[z_0]. \tag{60}$$

6.4 Periodic array of dipoles and quadrupoles in the upper half-plane

The Goursat functions for a periodic array of dipoles in the upper half-plane with local behaviour (10) at z_0 in the principal period window are

$$F(\zeta) = -\frac{\overline{\mu}}{1 - \overline{\zeta_0}\zeta} + \frac{\overline{\mu}}{(1 - \overline{\zeta_0}\zeta)^2},$$

$$G(\zeta) = \frac{\mu\zeta_0}{\zeta - \zeta_0} + \frac{\mu\zeta_0^2}{(\zeta - \zeta_0)^2} + \mathrm{i} \log \zeta \left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right]. \tag{61}$$

The Goursat functions for a periodic array of quadrupoles in the upper half-plane with local behaviour (11) at z_0 in the principal period window are

$$
\begin{aligned}
F(\zeta) &= -\frac{\mathrm{i}\overline{\mu}}{1 - \overline{\zeta_0}\zeta} + \frac{3\mathrm{i}\overline{\mu}}{(1 - \overline{\zeta_0}\zeta)^2} - \frac{2\mathrm{i}\overline{\mu}}{(1 - \overline{\zeta_0}\zeta)^3}, \\
G(\zeta) &= \frac{\mathrm{i}\mu\zeta_0}{\zeta - \zeta_0} + \frac{3\mathrm{i}\mu\zeta_0^2}{(\zeta - \zeta_0)^2} + \frac{2\mathrm{i}\mu\zeta_0^3}{(\zeta - \zeta_0)^3} + \mathrm{i}\log\zeta\left[\frac{F'(\zeta)}{\mathcal{Z}'(\zeta)}\right].
\end{aligned}
\tag{62}
$$

7 Discussion

New analytical representations for a wide range of Stokes flows due to periodic arrays of point singularities in a two-dimensional no-slip channel and in a half-plane near a wall have been given. The solutions are given explicitly as functions of a parametric ζ variable. If preferred, in all cases, the solutions can be expressed directly in terms of the variable $z = x + \mathrm{i}y$ on use of (14).

We have shown that the associated Laurent expansion terms (for the channel geometry) decay rapidly which suggests that only a few terms are, in general, required to provide accurate and efficient computations. The analytical expressions presented here can be used to model Stokes flow problems in singly periodic geometries; for example, one can model a flow past a periodic array of wall-attached straight barriers in a channel or in the half-plane [3], or the flow induced by cilia sheets [11].

A Transform method

We briefly discuss how to extend a novel transform approach to biharmonic boundary value problems for both polygonal and circular domains recently described by the authors in [10] to reappraise the problem of a periodic array of point singularities in a channel geometry thereby offering an alternative to the solutions given in §5.

The Goursat functions can be represented by

$$
f(z) = f_s(z) + f_R(z), \qquad g'(z) = g_s'(z) + g_R'(z),
\tag{63}
$$

where $f_s(z)$, $g_s'(z)$ are related to the point singularity at z_0 (for example, for a point stresslet at z_0, they are given by (7)) and $f_R(z)$, $g_R'(z)$ are the correction functions to be found. The functions $f_R(z)$, $g_R'(z)$ are analytic and single-valued in the fluid region and they have the following integral representations [10]:

$$
f_R(z) = \frac{1}{2\pi}\left[\sum_{j=1}^{4}\int_{L_j}\rho_j(k)\mathrm{e}^{\mathrm{i}kz}\mathrm{d}k\right], \qquad g_R'(z) = \frac{1}{2\pi}\left[\sum_{j=1}^{4}\int_{L_j}\hat{\rho}_j(k)\mathrm{e}^{\mathrm{i}kz}\mathrm{d}k\right],
\tag{64}
$$

where L_j, $j = 1, 2, 3, 4$ are oriented rays from 0 in the spectral k-plane [10] and $\rho_j(k), \hat{\rho}_j(k)$, $j = 1, 2, 3, 4$ are the spectral functions defined by

$$
\begin{aligned}
\rho_1(k) &= \int_0^l f_R(z)\mathrm{e}^{-\mathrm{i}kz}\mathrm{d}z, & \rho_2(k) &= \int_l^{l+\mathrm{i}h} f_R(z)\mathrm{e}^{-\mathrm{i}kz}\mathrm{d}z, \\
\rho_3(k) &= \int_{l+\mathrm{i}h}^{\mathrm{i}h} f_R(z)\mathrm{e}^{-\mathrm{i}kz}\mathrm{d}z, & \rho_4(k) &= \int_{\mathrm{i}h}^0 f_R(z)\mathrm{e}^{-\mathrm{i}kz}\mathrm{d}z,
\end{aligned}
\tag{65}
$$

and

$$\hat{\rho}_1(k) = \int_0^l g_R'(z)\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \qquad \hat{\rho}_2(k) = \int_l^{l+\mathrm{i}h} g_R'(z)\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z,$$

$$\hat{\rho}_3(k) = \int_{l+\mathrm{i}h}^{\mathrm{i}h} g_R'(z)\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \qquad \hat{\rho}_4(k) = \int_{\mathrm{i}h}^0 g_R'(z)\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z. \tag{66}$$

The spectral functions satisfy the so-called global relations:

$$\sum_{j=1}^4 \rho_j(k) = 0, \qquad \sum_{j=1}^4 \hat{\rho}_j(k) = 0, \qquad \text{for } k \in \mathbb{C}. \tag{67}$$

The analysis of the boundary and periodicity conditions allows us to deduce relations between the spectral functions. We omit the details and report the key expressions; these are:

$$-\overline{\rho_1}(-k) - \frac{\partial[k\rho_1(k)]}{\partial k} + \hat{\rho}_1(k) + lf_R(l)\mathrm{e}^{-\mathrm{i}kl} = R_1(k), \tag{68}$$

$$\rho_4(k) + \mathrm{e}^{\mathrm{i}kl}\rho_2(k) + d\ q(k) = R_2(k), \tag{69}$$

$$-\mathrm{e}^{2kh}\overline{\rho_3}(-k) - \frac{\partial[k\rho_3(k)]}{\partial k} + 2kh\rho_3(k) + \hat{\rho}_3(k) + r(k) = R_3(k), \tag{70}$$

$$-\mathrm{i}kl\rho_4(k) + \hat{\rho}_4(k) + \mathrm{e}^{\mathrm{i}kl}\hat{\rho}_2(k) - lf_R(0) + lf_R(\mathrm{i}h)\mathrm{e}^{kh} + \overline{d}\ q(k) = R_4(k), \tag{71}$$

where $d \in \mathbb{C}$ is a constant and

$$R_1(k) \equiv \int_0^l [\overline{f_s(z)} - zf_s'(z) - g_s'(z)]\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \tag{72}$$

$$R_2(k) \equiv \int_{\mathrm{i}h}^0 [-f_s(z) + f_s(z+l)]\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \tag{73}$$

$$R_3(k) \equiv \int_{l+\mathrm{i}h}^{\mathrm{i}h} [\overline{f_s(z)} - (z-2\mathrm{i}h)f_s'(z) - g_s'(z)]\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \tag{74}$$

$$R_4(k) \equiv \int_{\mathrm{i}h}^0 [lf_s'(z) - g_s'(z) + g_s'(z+l)]\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z \tag{75}$$

and

$$q(k) \equiv \int_{\mathrm{i}h}^0 \mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \qquad r(k) = -\mathrm{i}hf_R(\mathrm{i}h)\mathrm{e}^{kh} - (l-\mathrm{i}h)f_R(l+\mathrm{i}h)\mathrm{e}^{-\mathrm{i}k(l+\mathrm{i}h)}. \tag{76}$$

Addition of (68) and (70) and use of (67),(69),(71) gives, after some algebra,

$$\rho_1(k) = \frac{2khW(k) - (\mathrm{e}^{2kh}-1)\overline{W}(-k)}{4[\sinh^2(kh) - k^2h^2]}, \tag{77}$$

where $W(k)$ contains $\rho_4(k), \hat{\rho}_4(k), f_R(0), f_R(\mathrm{i}h), d$ and known quantities. The spectral function $\rho_1(k)$ is analytic everywhere in the complex k-plane which means that its numerator in (77) must vanish at zeros of its denominator in the k-plane, i.e. we must require

$$2khW(k) - (\mathrm{e}^{2kh}-1)\overline{W}(-k) = 0, \qquad \text{for} \quad k \in \Sigma_1 \equiv \{k \in \mathbb{C}|\sinh^2(kh) - k^2h^2] = 0\}, \tag{78}$$

together with conditions at $k = 0$ following from (77). Next, we use the series representations

$$f_R(z) = \sum_m a_m T_m(z), \quad g_R(z) = \sum_m b_m T_m(z), \quad \text{along} \quad z = \mathrm{i}y, y \in [0,h], \tag{79}$$

where $a_m, b_m \in \mathbb{C}$ are unknown coefficients and $T_m(z)$ are basis functions (e.g. Fourier, Chebyshev), and truncate the sums in (79) to finite number of terms. Then, we formulate a linear system for the unknown coefficients a_m, b_m, parameter d and their complex conjugates. The linear system comprises conditions (78) evaluated at as many points in the set Σ_1 as needed, together with conditions at $k = 0$. Once the unknowns are computed from the solution of the truncated linear system, the spectral functions $\rho_4(k)$ and $\hat{\rho}_4(k)$ can be found. The remaining spectral functions can be found by back substitution into (68)–(71), and therefore the correction functions $f_R(z)$ and $g_R'(z)$ can be computed.

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