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Solvability of some classes of nonlinear first-order difference equations by invariants and generalized invariants

Stevo Stević ✉

Mathematical Institute of the Serbian Academy of Sciences,
Knez Mihailova 36/III, 11000 Beograd, Serbia

Department of Medical Research, China Medical University Hospital, China Medical University,
Taichung 40402, Taiwan, Republic of China

Department of Computer Science and Information Engineering, Asia University,
500 Lioufeng Rd., Wufeng, Taichung 41354, Taiwan, Republic of China

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Abstract. We introduce notion of a generalized invariant for difference equations, which naturally generalizes notion of an invariant for the equations. Some motivations, basic examples and methods for application of invariants in the theory of solvability of difference equations are given. By using an invariant, as well as, a generalized invariant it is shown solvability of two classes of nonlinear first-order difference equations of interest, for nonnegative initial values and parameters appearing therein, considerably extending and explaining some problems in the literature. It is also explained how these classes of difference equations can be naturally obtained from some linear second-order difference equations with constant coefficients.

Keywords: difference equation, solvable equation, invariant.

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1 Introduction

Throughout the paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, stand for natural, nonnegative, integer, real and complex numbers, respectively. If $k,l \in \mathbb{Z}$, then $j = \overline{k,l}$ stands for the set of all $j \in \mathbb{Z}$ such that $k \le j \le l$.

Solvability of difference equations and systems of difference equations, and finding analytic relations for their solutions, is a very popular topic for a wide audience (see, for example, [1–43] and the related references cited therein). Due to the recent use of computers, the topic considerably reattracted some interest, although there are some issues with the new concept

✉ Email: sstevic@ptt.rs

of investigation of the equations and systems (some comments of ours on such issues can be found, for example, in [28, 37, 41, 43]; the basic issue is a lack of use of theory of difference equations in many recent papers in the topic).

1.1 Some history

One of the basic difference equations is the following linear first-order one:

$$x_{n+1} = a_n x_n + b_n, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where coefficients $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ and initial value x_0 are real or complex numbers. Of course, equation (1.1) can be considered in more abstract settings. For example, all the objects in the equation can be matrices, operators etc., but here we are interested in scalar difference equations. Equation (1.1) generalizes recurrent relations satisfied by arithmetic and geometric progressions (many problems on the progressions can be found, for example, in problem books [2, 12]).

Equation (1.1) can be solved in several ways. How the equation can be solved was known to Lagrange yet. Namely, in [13], he copied a method for solving the linear first-order differential equation, and solved the equation by searching its general solution in the following form $x_n = u_n v_n$, $n \in \mathbb{N}_0$, where u_n is a solution to the corresponding homogeneous equation $u_{n+1} = a_n u_n$, $n \in \mathbb{N}_0$. This implies that v_n satisfies the relation $v_{n+1} = v_n + \frac{b_n}{u_n a_n}$, when $u_n a_n \neq 0$, $n \in \mathbb{N}_0$. From these two relations he first found u_n and then v_n in a bit complicated and not quite correct way from the point of view of present mathematics. Laplace later in [15] found a method for solving equation (1.1) corresponding to the one for solving the linear first-order differential equation by multiplying it by integrating factor. For some later presentations of these and some other methods for solving the equation see, for example, [9, 17, 19] (book [19] explains all three methods corresponding to those for solving the linear first-order differential equation).

Equation (1.1) is one of the most useful difference equations and appears in many areas of mathematics and science, and many solvable difference equations are essentially reduced to its special cases (see, for example, [1, 4, 9, 17, 19, 25, 28, 29, 32–35, 40] for various applications of the equation). Moreover, the solvability of even more complex difference equations and systems such as those in [30, 31, 36, 38, 39, 42] is essentially consequence of the solvability of equation (1.1). This means that equation (1.1) is one of the most important difference equations, especially in the solvability theory. For some other solvable difference equations and systems of difference equations and related topics see also [2, 3, 9–18, 20] and the references therein.

If sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant, that is, $a_n = a$, $b_n = b$, $n \in \mathbb{N}_0$, for some $a, b \in \mathbb{R}$ (or $\mathbb{C}$), then, naturally, there are more ways for solving the equation, which in this case becomes

$$x_{n+1} = a x_n + b, \quad n \in \mathbb{N}_0. \tag{1.2}$$

Equation (1.2) was also solved by Lagrange in [13], where he used the formula for general solution to equation (1.1) obtained therein, summed up a geometric progression, and obtained that

$$x_n = a^n x_0 + b\frac{a^n - 1}{a - 1}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

when $a \neq 1$ (case $a = 1$ was not mentioned in [13] probably because of its simplicity).

The first nontrivial difference equation which was solved in closed form seems the linear second-order difference equation with constant coefficients, that is, the following one:

$$x_{n+2} = px_{n+1} + qx_n, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where p and q are fixed numbers such that $q \neq 0$ (if $q = 0$ then it is obviously obtained a special case of equation (1.2) defining geometric progressions).

General solution to equation (1.4) in the case $p^2 + 4q \neq 0$ was found first by de Moivre, who coined the notion *recurrent sequence* in [6]. Necessary ingredients for solving the equation can be found in [5] and [6], but the solution was presented later in [7]. By using generating functions, he showed that if $\lambda_{1,2}$ are the zeros of the polynomial $P_2(\lambda) = \lambda^2 - p\lambda - q$, then general solution to equation (1.4) can be written in terms of $\lambda_{1,2}$ and initial values x_0 and x_1 as follows

$$x_n = \frac{(x_0\lambda_2 - x_1)\lambda_1^n + (x_1 - x_0\lambda_1)\lambda_2^n}{\lambda_2 - \lambda_1}, \quad n \in \mathbb{N}_0. \tag{1.5}$$

The corresponding formula for general solution in the case $p^2 + 4q = 0$, can be found in Euler's book [8] where was given more comprehensive theory on the difference equations known up to 1748, than in books by de Moivre. For some other historical details see [37].

Equations (1.2) and (1.4) are closely related, which had been already noticed by Lagrange in [13]. Namely, if we know formula (1.3), then the de Moivre formula can be obtained by using essentially an idea from [13]. This can be found in many papers and books ([12, 19]).

1.2 Some motivations for using invariants in solvability

General solution to equation (1.2) can be obtained by using equation (1.4), which is a motivation for a method that we use in this paper.

If we look at the proof of formula (1.3) given in [12], which suggests using equation (1.2) along with the following trivial consequence of the same equation

$$x_n = ax_{n-1} + b, \quad n \in \mathbb{N}, \tag{1.6}$$

we see that the idea is to eliminate, for the moment, constant b.

From (1.2) and (1.6), we have

$$x_{n+1} - ax_n = x_n - ax_{n-1}, \quad n \in \mathbb{N}. \tag{1.7}$$

From (1.7) we can continue in two directions. Namely, we can write the equation in the following form

$$x_{n+1} - (a+1)x_n + ax_{n-1} = 0, \quad n \in \mathbb{N},$$

and solve it by using the de Moivre formula when $a \neq 1$, or by the corresponding formula from [8] when $a = 1$, or, we can write the equation in the following form

$$x_{n+1} - x_n = a(x_n - x_{n-1}), \quad n \in \mathbb{N},$$

get easily the formula $x_n - x_{n-1} = (x_1 - x_0)a^{n-1}$, $n \in \mathbb{N}$, then apply telescoping summation and a formula for the finite sum of a geometric progression.

This method for solving equation (1.2) suggests that the fact that the expression

$$I_1(x_n, x_{n+1}) := x_{n+1} - ax_n, \quad n \in \mathbb{N}_0,$$

is constant for every solution $(x_n)_{n\in\mathbb{N}_0}$ to the equation, plays an important role in solvability of the equation.

A natural extension of equation (1.4) is the second-order linear difference equation with constant coefficients whose nonhomogeneous part is constant, that is, the following difference equation:

$$x_{n+2} = px_{n+1} + qx_n + r, \quad n \in \mathbb{N}_0, \tag{1.8}$$

where p, q, r are numbers such that $q \neq 0$ (if we allow that q can be zero it is also an obvious extension of equation (1.2)).

Motivated by the above consideration, we can define the following expression:

$$I_2(x_n, x_{n+1}, x_{n+2}) := x_{n+2} - px_{n+1} - qx_n, \quad n \in \mathbb{N}_0.$$

From (1.8) we see that

$$I_2(x_n, x_{n+1}, x_{n+2}) = r, \quad n \in \mathbb{N}_0. \tag{1.9}$$

Hence, we have

$$I_2(x_n, x_{n+1}, x_{n+2}) = I_2(x_{n-1}, x_n, x_{n+1}), \quad n \in \mathbb{N}, \tag{1.10}$$

which can be written as follows

$$x_{n+2} - (p+1)x_{n+1} - (q-p)x_n + qx_{n-1} = 0, \quad n \in \mathbb{N}, \tag{1.11}$$

which is a linear difference equation with constant coefficients. General solutions to such equations were known to de Moivre and Euler yet ([7,8]).

Now note that the initial value problem consisting of equation (1.8) with initial values x_0 and x_1 is transformed to the one consisting of equation (1.11) with initial values x_0, x_1 and

$$x_2 = px_1 + qx_0 + r. \tag{1.12}$$

Since

$$I_2(x_n, x_{n+1}, x_{n+2}) = I_2(x_0, x_1, x_2) = x_2 - px_1 - qx_0, \quad n \in \mathbb{N}_0,$$

from this and (1.12) it follows that (1.8) holds, so that these two initial value problems are equivalent.

If $p^2 + 4q \neq 0$, then the zeros λ_1 and λ_2 of polynomial $P_2(\lambda)$ are different, from which it follows that the zeros of the characteristic polynomial

$$P_3(\lambda) = \lambda^3 - (p+1)\lambda^2 - (q-p)\lambda + q$$

associated with equation (1.11) are λ_1, λ_2 and $\lambda_3 = 1$, since $P_3(\lambda) = \lambda P_2(\lambda) - P_2(\lambda)$.

Hence, in the case $\lambda_{1,2} \neq 1$, general solution to equation (1.11) has the form

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n + c_3, \quad n \in \mathbb{N}_0. \tag{1.13}$$

Constants c_j, $j=\overline{1,3}$, are found by solving the following linear system

$$\begin{aligned} c_1+c_2+c_3&=x_0,\\ \lambda_1 c_1+\lambda_2 c_2+c_3&=x_1,\\ \lambda_1^2 c_1+\lambda_2^2 c_2+c_3&=px_1+qx_0+r. \end{aligned} \tag{1.14}$$

The determinant of system (1.14) is the Vandermonde one

$$\Delta=V(\lambda_1,\lambda_2,1)=(\lambda_2-\lambda_1)(\lambda_1-1)(\lambda_2-1).$$

Hence

$$c_1=\frac{1}{\Delta}\begin{vmatrix} x_0 & 1 & 1\\ x_1 & \lambda_2 & 1\\ px_1+qx_0+r & \lambda_2^2 & 1\end{vmatrix}=\frac{(1-\lambda_2)((\lambda_2+q)x_0+(p-1-\lambda_2)x_1+r)}{(\lambda_2-\lambda_1)(\lambda_1-1)(\lambda_2-1)},$$

$$c_2=\frac{1}{\Delta}\begin{vmatrix} 1 & x_0 & 1\\ \lambda_1 & x_1 & 1\\ \lambda_1^2 & px_1+qx_0+r & 1\end{vmatrix}=\frac{(1-\lambda_1)((1+\lambda_1-p)x_1-(\lambda_1+q)x_0-r)}{(\lambda_2-\lambda_1)(\lambda_1-1)(\lambda_2-1)},$$

and

$$c_3=\frac{1}{\Delta}\begin{vmatrix} 1 & 1 & x_0\\ \lambda_1 & \lambda_2 & x_1\\ \lambda_1^2 & \lambda_2^2 & px_1+qx_0+r\end{vmatrix}=\frac{r(\lambda_2-\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_1-1)(\lambda_2-1)},$$

from which along with (1.13) it follows that general solution to equation (1.8) in this case is

$$\begin{aligned} x_n=&\frac{(1+\lambda_2-p)x_1-(\lambda_2+q)x_0-r}{(\lambda_2-\lambda_1)(\lambda_1-1)}\lambda_1^n+\frac{(\lambda_1+q)x_0+r-(1+\lambda_1-p)x_1}{(\lambda_2-\lambda_1)(\lambda_2-1)}\lambda_2^n\\ &+\frac{r}{(\lambda_1-1)(\lambda_2-1)},\quad n\in\mathbb{N}_0. \end{aligned}$$

When $\lambda_1\neq 1=\lambda_2$ (case $\lambda_1=1\neq\lambda_2$ is dual), general solution to equation (1.11) has the form

$$x_n=\widetilde{c}_1\lambda_1^n+\widetilde{c}_2 n+\widetilde{c}_3,\quad n\in\mathbb{N}_0,$$

whereas if $\lambda_1=\lambda_2=1$, then general solution to the equation has the form

$$x_n=\widehat{c}_1 n^2+\widehat{c}_2 n+\widehat{c}_3,\quad n\in\mathbb{N}_0,$$

and similarly as above $\widetilde{c}_j$ and $\widehat{c}_j$, $j=\overline{1,3}$, can be found in terms of parameters p, q, r, and initial values x_0 and x_1, which is a routine thing.

This method for solving equation (1.8) suggests that condition (1.9), which is similar to the condition $I_1(x_n,x_{n+1})=b$ corresponding to equation (1.2), plays an important role in solvability of the equation.

The idea for solving equation (1.8) by showing that the sequence $y_n:=x_{n+1}-x_n$ is a solution to equation (1.4), which frequently appears in the literature (see, e.g., [2]), is essentially nothing by another use of relation (1.10).

1.3 Invariants

Two simple examples presented above, suggest that some difference equations can be solved if it is possible to find functions of several variables which are constant on their solutions. Such functions are called *invariants*, and formal definition for it follows.

Consider the following difference equation

$$x_{n+s} = f(x_{n+s-1}, x_{n+s-2}, \dots, x_n), \quad n \ge -k, \tag{1.15}$$

where $s \in \mathbb{N}$ and $k \in \mathbb{Z}$. If there is a function $I : \mathbb{R}^l\,(\text{or}\,\mathbb{C}^l) \to \mathbb{R}\,(\text{or}\,\mathbb{C})$, such that for every solution $(x_n)_{n\ge -k}$ to the equation the following condition holds

$$I(x_n, x_{n+1}, \dots, x_{n+l-1}) = c, \quad \text{for } n \ge -k,$$

for some $c \in \mathbb{R}\,(\text{or}\,\mathbb{C})$, then the function I is called an *invariant* for equation (1.15).

Invariants can be useful in establishing some properties of solutions to difference equations and systems. Many invariants and their applications can be found, for example, in [21–24, 26, 27] (see also the related references therein).

Although solvability of majority difference equations and systems has been shown so far by using some suitable substitutions (see, e.g., [4, 25, 28, 29, 32, 40]), as we have shown above, invariants can also help in establishing solvability of some classes of difference equations and systems.

1.4 Some concrete motivations

Our motivation for this paper stems from two problems from student competitions.

The following problem was posed on the ninth All-Russian Olympiad in 1983 (see, e.g, [44]).

Problem 1.1. Let sequence $(x_n)_{n\in\mathbb{N}_0}$ be the solution to the difference equation

$$x_{n+1} = 5x_n + \sqrt{24x_n^2 + 1}, \quad n \in \mathbb{N}_0, \tag{1.16}$$

satisfying the initial condition $x_0 = 0$. Show that $x_n \in \mathbb{Z}$, for every $n \in \mathbb{N}_0$.

The following problem was a proposal for International Mathematical Olympiad in 1983.

Problem 1.2. Let $a \in \mathbb{N}$ and sequence $(x_n)_{n\in\mathbb{N}_0}$ be the solution to the difference equation

$$x_{n+1} = (2a+1)x_n + a + 2\sqrt{a(a+1)x_n(x_n+1)}, \quad n \in \mathbb{N}_0, \tag{1.17}$$

satisfying the initial condition $x_0 = 0$. Show that $x_n \in \mathbb{N}$, for every $n \in \mathbb{N}$.

Bearing in mind that equations (1.16) and (1.17), as well as the posed conditions are concrete, both problems can be solved in several different ways. It should be also noted that the equations are of similar form. Hence, it is a natural problem to find some general results which include the claims in the problems. Another natural problem is to try to find a method which can deal with both equations.

What is interesting is that both initial value problems are solvable in closed form, which we have noticed yet in 1983, when we tried to solve these problems for the first time.

Namely, let $\lambda_1 := 5+2\sqrt{6}$, $\lambda_2 := 5-2\sqrt{6}$, and

$$\widetilde{x}_n := \frac{\lambda_1^n - \lambda_2^n}{4\sqrt{6}}, \quad n \in \mathbb{N}_0. \tag{1.18}$$

Then, since $\lambda_1\lambda_2 = 1$, $\min\{\lambda_1, \lambda_2\} > 0$, we have

$$\begin{aligned} 5\widetilde{x}_n + \sqrt{24\widetilde{x}_n^2+1} &= 5\frac{\lambda_1^n - \lambda_2^n}{4\sqrt{6}} + \left(24\left(\frac{\lambda_1^n - \lambda_2^n}{4\sqrt{6}}\right)^2 + 1\right)^{1/2} \\ &= 5\frac{\lambda_1^n - \lambda_2^n}{4\sqrt{6}} + \frac{\lambda_1^n + \lambda_2^n}{2} = \frac{(5+2\sqrt{6})\lambda_1^n - (5-2\sqrt{6})\lambda_2^n}{4\sqrt{6}} \\ &= \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{4\sqrt{6}} = \widetilde{x}_{n+1}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{1.19}$$

On the other hand, we have $\widetilde{x}_0 = 0$, from which along with relation (1.19) it follows that $(\widetilde{x}_n)_{n\in\mathbb{N}_0}$ is a solution to the initial value problem in Problem 1.1. Since each solution to equation (1.16) is uniquely defined by initial value x_0, the sequence $(\widetilde{x}_n)_{n\in\mathbb{N}_0}$ is the solution to the initial value problem. From formula (1.18) and the binomial formula the claim in Problem 1.1 easily follows.

Further, let $\lambda_1 := \sqrt{a+1}+\sqrt{a}$, $\lambda_2 := \sqrt{a+1}-\sqrt{a}$, and

$$\widetilde{x}_n := \frac{\lambda_1^{2n} + \lambda_2^{2n}}{4} - \frac{1}{2}, \quad n \in \mathbb{N}_0. \tag{1.20}$$

Then, since $\lambda_1\lambda_2 = 1$, $\lambda_1 > \lambda_2 > 0$, after some calculation, we have

$$\begin{aligned} &\widetilde{x}_{n+1} - (2a+1)\widetilde{x}_n - a - 2\sqrt{a(a+1)\left(\left(\widetilde{x}_n + \frac{1}{2}\right)^2 - \frac{1}{4}\right)} \\ &= \frac{(\lambda_1^2 - (2a+1))\lambda_1^{2n} + (\lambda_2^2 - (2a+1))\lambda_2^{2n}}{4} - 2\left(a(a+1)\left(\left(\frac{\lambda_1^{2n}+\lambda_2^{2n}}{4}\right)^2 - \frac{1}{4}\right)\right)^{1/2} \\ &= \sqrt{a(a+1)}\frac{\lambda_1^{2n} - \lambda_2^{2n}}{2} - \sqrt{a(a+1)}\left(\left(\frac{\lambda_1^{2n}-\lambda_2^{2n}}{2}\right)^2\right)^{1/2} = 0, \quad n \in \mathbb{N}_0. \end{aligned} \tag{1.21}$$

From (1.21) and since we have $\widetilde{x}_0 = 0$, we see that the sequence defined in (1.20) is the solution to the initial value problem in Problem 1.2. From formula (1.20) and the binomial formula the claim in Problem 1.2 easily follows.

Hence, another question is to explain theoretically solvability of equations (1.16) and (1.17), and to generalize these solvability results by finding some classes of difference equations including equations (1.16) and (1.17), which are solvable on a "large" domain, for example, for positive initial values and parameters.

Our aim here is to present some answers to above posed questions, and to suggest using the method of invariants in dealing with solvability of difference equations.

1.5 Generalized invariants

Before we formulate and prove our main results, we introduce a notion which is a generalization of (standard) invariants (many standard invariants can be found in [21–24,26,27]).

Definition 1.3. Consider difference equation (1.15), where $s \in \mathbb{N}$ and $k \in \mathbb{Z}$. If there is a function $I : \mathbb{R}^l \,(\text{or}\, \mathbb{C}^l) \to \mathbb{R}\,(\text{or}\, \mathbb{C})$, such that for every solution $(x_n)_{n\geq -k}$ to the difference equation the following condition holds

$$I(x_{n+1}, x_{n+2}, \dots, x_{n+l}) = bI(x_n, x_{n+1}, \dots, x_{n+l-1}), \tag{1.22}$$

for every $n \geq -k$ and for some constant $b \in \mathbb{R}(\text{or}\, \mathbb{C})$. Then the function is called the *generalized invariant* for the difference equation.

2 Main results

In this section we prove our main results which extend solvability results regarding equations (1.16) and (1.17).

2.1 An extension of difference equation (1.16)

Our first main theorem generalizes the solvability result concerning solutions to equation (1.16) mentioned in the previous section. The proof essentially uses a generalized invariant. Namely, we find a generalized invariant which helps in finding a closed-form formula for general solution to a difference equation of first order, whose special case is difference equation (1.16).

Theorem 2.1. *Consider the following difference equation*

$$x_{n+1} = ax_n + \sqrt{(a^2 - b)x_n^2 + cb^n}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where parameters a, b, c are positive real numbers, such that $a^2 > b$.

Then, for $x_0 \in [0, \infty)$ the difference equation is solvable in closed form.

Proof. By using the assumptions

$$\min\{a, b, c, a^2 - b\} > 0 \quad \text{and} \quad x_0 \geq 0, \tag{2.2}$$

we have

$$x_1 = ax_0 + \sqrt{(a^2 - b)x_0^2 + c} > \sqrt{c} > 0. \tag{2.3}$$

By using (2.1), (2.2), (2.3), and a simple inductive argument we obtain that

$$x_n > 0, \quad n \in \mathbb{N}. \tag{2.4}$$

We also have

$$x_{n+1} - ax_n = \sqrt{(a^2 - b)x_n^2 + cb^n} \geq \sqrt{c}\,(\sqrt{b})^n > 0, \tag{2.5}$$

for $n \in \mathbb{N}_0$.

By squaring both sides of the equality in (2.5), and after some simple calculation, we obtain

$$x_{n+1}^2 - 2ax_{n+1}x_n + bx_n^2 = cb^n, \tag{2.6}$$

for $n \in \mathbb{N}_0$.

Let

$$I_3(x_n, x_{n+1}) := x_{n+1}^2 - 2ax_{n+1}x_n + bx_n^2,$$

for $n \in \mathbb{N}_0$.

Then, from (2.6), we have

$$I_3(x_n, x_{n+1}) = bI_3(x_{n-1}, x_n), \quad n \in \mathbb{N}, \tag{2.7}$$

which means that the function $I_3(u, v) := u^2 - 2auv + bv^2$ is a generalized invariant for equation (2.1).

Relation (2.7) can be written as follows

$$x_{n+1}^2 - 2ax_{n+1}x_n + 2abx_{n-1}x_n - b^2x_{n-1}^2 = 0, \quad n \in \mathbb{N},$$

which can be further written in the following form

$$(x_{n+1} - bx_{n-1})(x_{n+1} - 2ax_n + bx_{n-1}) = 0, \tag{2.8}$$

for $n \in \mathbb{N}$.

Now note that form (2.1), (2.2) and (2.4), we have

$$x_{n+1} = ax_n + \sqrt{(a^2 - b)x_n^2 + cb^n} > (a + \sqrt{a^2 - b})x_n, \tag{2.9}$$

for $n \in \mathbb{N}$.

By iterating inequality (2.9), using (2.2) and (2.4), we obtain

$$\begin{aligned} x_{n+1} &> (a + \sqrt{a^2 - b})^2 x_{n-1} \\ &= (2a^2 - b + 2a\sqrt{a^2 - b})x_{n-1} \\ &> (2a^2 - b)x_{n-1} \\ &> bx_{n-1}, \end{aligned} \tag{2.10}$$

for $n \geq 2$.

If $x_0 > 0$, then inequality (2.9) also holds for $n = 0$, and consequently inequality (2.10) holds for $n = 1$.

If $x_0 = 0$, then we have $x_1 = \sqrt{c}$, from which it follows that

$$x_2 = a\sqrt{c} + \sqrt{(a^2 - b)c + cb} = 2a\sqrt{c} > 0 = bx_0,$$

so that inequality (2.10) also holds for $n = 1$ in this case.

From this analysis and inequality (2.10) we see that

$$x_{n+1} \neq bx_{n-1}, \tag{2.11}$$

for every $n \in \mathbb{N}$.

From (2.8) and (2.11) we obtain that it must be

$$x_{n+1} - 2ax_n + bx_{n-1} = 0, \tag{2.12}$$

for $n \in \mathbb{N}$.

The characteristic polynomial

$$\widetilde{P}_2(\lambda) = \lambda^2 - 2a\lambda + b \tag{2.13}$$

associated with equation (2.12) has the following two zeros

$$\lambda_1 = a + \sqrt{a^2 - b} \quad \text{and} \quad \lambda_2 = a - \sqrt{a^2 - b}.$$

These zeros are different if $a^2 \neq b$, which is the case here.

By using de Moivre formula (1.5), we have

$$x_n = \frac{(x_1 - \lambda_2 x_0)\lambda_1^n + (x_0\lambda_1 - x_1)\lambda_2^n}{2\sqrt{a^2 - b}}, \tag{2.14}$$

for $n \in \mathbb{N}_0$.

Now note that

$$x_1 = a x_0 + \sqrt{(a^2 - b)x_0^2 + c}. \tag{2.15}$$

Using (2.15) in (2.14), we obtain that general solution to equation (2.12) in this case is

$$x_n = \frac{\left(x_0\sqrt{a^2 - b} + \sqrt{(a^2 - b)x_0^2 + c}\right)\lambda_1^n + \left(x_0\sqrt{a^2 - b} - \sqrt{(a^2 - b)x_0^2 + c}\right)\lambda_2^n}{2\sqrt{a^2 - b}}, \tag{2.16}$$

for $n \in \mathbb{N}_0$.

Let

$$\widetilde{x}_n := c_1\lambda_1^n + c_2\lambda_2^n, \tag{2.17}$$

for $n \in \mathbb{N}_0$, where

$$c_1 := \frac{x_0\sqrt{a^2 - b} + \sqrt{(a^2 - b)x_0^2 + c}}{2\sqrt{a^2 - b}} \quad \text{and} \quad c_2 := \frac{x_0\sqrt{a^2 - b} - \sqrt{(a^2 - b)x_0^2 + c}}{2\sqrt{a^2 - b}}.$$

Then, by using the facts that $\lambda_1\lambda_2 = b$, $c_1c_2 = -\frac{c}{4(a^2-b)}$, $c_1 > 0 > c_2$ and $\min\{\lambda_1, \lambda_2\} > 0$, the assumption $c > 0$, as well as some calculation, we have

$$\begin{aligned}
&\widetilde{x}_{n+1} - a\widetilde{x}_n - \sqrt{(a^2 - b)\widetilde{x}_n^2 + cb^n} \\
&\quad = c_1(\lambda_1 - a)\lambda_1^n + c_2(\lambda_2 - a)\lambda_2^n - \sqrt{(a^2 - b)(c_1^2\lambda_1^{2n} + 2c_1c_2(\lambda_1\lambda_2)^n + c_2^2\lambda_2^{2n}) + cb^n} \\
&\quad = \sqrt{a^2 - b}\,(c_1\lambda_1^n - c_2\lambda_2^n) - \sqrt{(a^2 - b)(c_1^2\lambda_1^{2n} + 2c_1c_2(\lambda_1\lambda_2)^n + c_2^2\lambda_2^{2n} - 4c_1c_2(\lambda_1\lambda_2)^n)} \\
&\quad = \sqrt{a^2 - b}\big(c_1\lambda_1^n - c_2\lambda_2^n - |c_1\lambda_1^n - c_2\lambda_2^n|\big) = 0,
\end{aligned} \tag{2.18}$$

for $n \in \mathbb{N}_0$.

From (2.18) and since it obviously holds $\widetilde{x}_0 = x_0$, we see that the sequence defined in (2.17) (i.e. in (2.16)) is the solution to equation (2.1) with the initial value x_0. □

Remark 2.2. If $a^2 = b$, then the zeros of characteristic polynomial (2.13) are

$$\lambda_1 = \lambda_2 = a.$$

Hence, general solution to equation (2.12) has the following form

$$x_n = (c_1 + c_2 n)a^n, \quad n \in \mathbb{N}_0.$$

By using initial values x_0 and x_1, it is easily obtained that general solution to equation (2.12) in this case is given by the following formula

$$x_n = (ax_0 + (x_1 - ax_0)n)a^{n-1}, \quad n \in \mathbb{N}_0. \tag{2.19}$$

In this case, we also have

$$x_1 = ax_0 + \sqrt{c}. \tag{2.20}$$

Employing (2.20) in (2.19), we obtain

$$x_n = (ax_0 + \sqrt{c}n)a^{n-1}, \quad n \in \mathbb{N}_0. \tag{2.21}$$

However, in this case equation (2.1) becomes

$$x_{n+1} = ax_n + \sqrt{c}a^n, \quad n \in \mathbb{N}_0, \tag{2.22}$$

which is a special case of equation (1.1), and can be solved by using one of above mentioned ways.

For example, by dividing equation (2.22) by a^{n+1}, we get

$$\frac{x_{n+1}}{a^{n+1}} = \frac{x_n}{a^n} + \frac{\sqrt{c}}{a}, \quad n \in \mathbb{N}_0. \tag{2.23}$$

By telescoping summation of the equalities which are obtained when in (2.23), n is replaced by $0, 1, \ldots, n-1$, respectively, we obtain

$$\frac{x_n}{a^n} = x_0 + n\frac{\sqrt{c}}{a}, \quad n \in \mathbb{N}_0,$$

from which is also obtained formula (2.21), in the case $a \neq 0$.

If $a = 0$, then equation (2.22) is trivial.

2.2 An extension of difference equation (1.17)

Our second main theorem generalizes the solvability result concerning solutions to equation (1.17) mentioned in the previous section. This time the proof essentially uses an invariant (generalized invariant with $b = 1$ in the definition), for finding a closed-form formula for general solution to a difference equation of first order.

Theorem 2.3. *Consider the following difference equation*

$$x_{n+1} = ax_n + b + \sqrt{cx_n^2 + dx_n + f}, \quad n \in \mathbb{N}_0, \tag{2.24}$$

where parameters $a, b, c, d, f \in [0, +\infty)$, are such that $b > 0$,

$$a^2 = c + 1 \quad \text{and} \quad 2b(a+1) = d. \tag{2.25}$$

Then, for $x_0 \in [0, \infty)$ the difference equation is solvable in closed form.

Proof. By using the assumptions $a, c, d, f \in [0, +\infty)$, $b > 0$ and $x_0 \geq 0$, we have

$$x_1 = ax_0 + b + \sqrt{cx_0^2 + dx_0 + f} > b > 0,$$

from which, along with equation (2.24) and by a simple inductive argument, it follows that

$$x_n > 0, \quad n \in \mathbb{N}. \tag{2.26}$$

From (2.25), we have $a \geq 1$, from which along with the assumptions $a, c, d, f \in [0, +\infty)$, $b > 0$, (2.24) and (2.26), it follows that

$$x_{n+1} - x_n = (a-1)x_n + b + \sqrt{cx_n^2 + dx_n + f} > b > 0,$$

for $n \in \mathbb{N}_0$, that is, sequence $(x_n)_{n\in\mathbb{N}_0}$ is strictly increasing.

We also have

$$x_{n+1} - ax_n - b = \sqrt{cx_n^2 + dx_n + f} \geq 0, \quad n \in \mathbb{N}_0. \tag{2.27}$$

By squaring both sides of the equality in (2.27), and after some simple calculation, we obtain

$$x_{n+1}^2 + (a^2 - c)x_n^2 - 2ax_{n+1}x_n - 2bx_{n+1} + (2ab - d)x_n = f - b^2, \quad n \in \mathbb{N}_0.$$

Let

$$I_4(x_n, x_{n+1}) := x_{n+1}^2 + (a^2 - c)x_n^2 - 2ax_{n+1}x_n - 2bx_{n+1} + (2ab - d)x_n, \quad n \in \mathbb{N}_0.$$

Then, clearly we have

$$I_4(x_n, x_{n+1}) = I_4(x_{n-1}, x_n), \quad n \in \mathbb{N},$$

which means that the function $I_4(u, v) = u^2 + (a^2 - c)v^2 - 2auv - 2bu + (2ab - d)v$ is an invariant for equation (2.24), and that

$$\begin{aligned} &x_{n+1}^2 + (a^2 - c)x_n^2 - 2ax_{n+1}x_n - 2bx_{n+1} + (2ab - d)x_n \\ &\quad = x_n^2 + (a^2 - c)x_{n-1}^2 - 2ax_nx_{n-1} - 2bx_n + (2ab - d)x_{n-1}, \end{aligned} \tag{2.28}$$

for $n \in \mathbb{N}$.

By using (2.25) in (2.28), we obtain

$$x_{n+1}^2 - 2ax_{n+1}x_n - 2bx_{n+1} - x_{n-1}^2 + 2ax_nx_{n-1} + 2bx_{n-1} = 0, \tag{2.29}$$

for $n \in \mathbb{N}$, that is,

$$(x_{n+1} - x_{n-1})(x_{n+1} - 2ax_n + x_{n-1} - 2b) = 0, \tag{2.30}$$

for $n \in \mathbb{N}$.

Using the strict monotonicity of the sequence x_n in (2.30), we have

$$x_{n+1} - 2ax_n + x_{n-1} = 2b, \quad n \in \mathbb{N}. \tag{2.31}$$

The characteristic polynomial $\widehat{P}_2(\lambda) = \lambda^2 - 2a\lambda + 1$ associated with equation

$$x_{n+1} - 2ax_n + x_{n-1} = 0, \quad n \in \mathbb{N}, \tag{2.32}$$

has the following two zeros

$$\lambda_1 = a + \sqrt{a^2 - 1} \quad \text{and} \quad \lambda_2 = a - \sqrt{a^2 - 1},$$

which are different when $a > 1$.

Case $a > 1$. When $a > 1$ general solution to equation (2.32) is

$$x_n^h = c_1(a + \sqrt{a^2 - 1})^n + c_2(a - \sqrt{a^2 - 1})^n, \quad n \in \mathbb{N}_0.$$

In this case a solution to equation (2.31) can be found in the form

$$x_n^p := c, \quad n \in \mathbb{N}_0$$

where c is a real constant.

Putting it in (2.31), we easily obtain

$$x_n^p = \frac{b}{1-a}, \quad n \in \mathbb{N}_0.$$

Hence, when $a > 1$, the general solution to equation (2.31) is

$$x_n = c_1(a + \sqrt{a^2 - 1})^n + c_2(a - \sqrt{a^2 - 1})^n + \frac{b}{1-a}, \tag{2.33}$$

for $n \in \mathbb{N}_0$.

To find constants c_1 and c_2 in terms of initial values, we need to solve the linear system

$$\begin{aligned} c_1 + c_2 &= x_0 + \frac{b}{a-1}, \\ \lambda_1 c_1 + \lambda_2 c_2 &= x_1 + \frac{b}{a-1}, \end{aligned}$$

from which it follows that

$$\begin{aligned} c_1 &= \frac{x_1 + \frac{b}{a-1} - \lambda_2(x_0 + \frac{b}{a-1})}{\lambda_1 - \lambda_2} \\ &= \frac{\sqrt{a^2-1}((a-1)x_0 + b) + (a-1)\sqrt{(a^2-1)x_0^2 + 2b(a+1)x_0 + f}}{2(a-1)\sqrt{a^2-1}}, \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} c_2 &= \frac{\lambda_1(x_0 + \frac{b}{a-1}) - (x_1 + \frac{b}{a-1})}{\lambda_1 - \lambda_2} \\ &= \frac{\sqrt{a^2-1}((a-1)x_0 + b) - (a-1)\sqrt{(a^2-1)x_0^2 + 2b(a+1)x_0 + f}}{2(a-1)\sqrt{a^2-1}}. \end{aligned} \tag{2.35}$$

By using (2.34) and (2.35) in (2.33), general solution to equation (2.31) is

$$\begin{aligned} x_n = &\frac{\sqrt{a^2-1}((a-1)x_0 + b) + (a-1)\sqrt{(a^2-1)x_0^2 + 2b(a+1)x_0 + f}}{2(a-1)\sqrt{a^2-1}} \lambda_1^n \\ &+ \frac{\sqrt{a^2-1}((a-1)x_0 + b) - (a-1)\sqrt{(a^2-1)x_0^2 + 2b(a+1)x_0 + f}}{2(a-1)\sqrt{a^2-1}} \lambda_2^n \\ &+ \frac{b}{1-a}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{2.36}$$

Let

$$\widetilde{x}_n = \frac{\sqrt{a^2-1}((a-1)x_0+b)+(a-1)\sqrt{(a^2-1)x_0^2+2b(a+1)x_0+f}}{2(a-1)\sqrt{a^2-1}}\lambda_1^n$$
$$+\frac{\sqrt{a^2-1}((a-1)x_0+b)-(a-1)\sqrt{(a^2-1)x_0^2+2b(a+1)x_0+f}}{2(a-1)\sqrt{a^2-1}}\lambda_2^n$$
$$+\frac{b}{1-a}, \quad n \in \mathbb{N}_0. \tag{2.37}$$

Then, since $\lambda_1\lambda_2 = 1$, $c_1 > c_2$, $\lambda_1 > \lambda_2 > 0$, and by some calculation, we have

$$\begin{aligned}
&\widetilde{x}_{n+1} - a\widetilde{x}_n - b - \sqrt{(a^2-1)\widetilde{x}_n^2 + 2b(a+1)\widetilde{x}_n + f}\\
&= c_1(\lambda_1 - a)\lambda_1^n + c_2(\lambda_2 - a)\lambda_2^n\\
&\quad - \Bigg((a^2-1)\left(c_1^2\lambda_1^{2n} + 2c_1c_2 + c_2^2\lambda_2^{2n} + \frac{b^2}{(1-a)^2} + \frac{2b(c_1\lambda_1^n + c_2\lambda_2^n)}{1-a}\right)\\
&\qquad + 2b(a+1)\left(c_1\lambda_1^n + c_2\lambda_2^n + \frac{b}{1-a}\right) + f\Bigg)^{1/2}\\
&= \sqrt{a^2-1}\left(c_1\lambda_1^n - c_2\lambda_2^n - \left((c_1\lambda_1^n - c_2\lambda_2^n)^2 + 4c_1c_2 - \frac{b^2}{(a-1)^2} + \frac{f}{a^2-1}\right)^{1/2}\right)\\
&= 0,
\end{aligned} \tag{2.38}$$

where in the last equality, we have used the fact that

$$4c_1c_2 = \frac{b^2}{(a-1)^2} - \frac{f}{a^2-1},$$

which is checked by some calculation.

From (2.38) and since it obviously holds $\widetilde{x}_0 = x_0$, we see that the sequence defined in (2.37) (i.e. in (2.36)) is the solution to equation (2.24) with the initial value x_0, finishing the proof in this case.

Case $a = 1$. When $a = 1$, then the zeros of the characteristic polynomial $\widehat{P}_2$ are

$$\lambda_1 = \lambda_2 = 1,$$

and consequently general solution to equation (2.32) in this case is

$$x_n^h = c_1 + c_2 n, \quad n \in \mathbb{N}_0. \tag{2.39}$$

A solution to equation (2.31) can be found in the following form

$$x_n^p := \widetilde{c}n^2, \quad n \in \mathbb{N}_0,$$

for some $\widetilde{c} \in \mathbb{R}$.

Putting it in (2.31), we easily obtain

$$x_n^p = bn^2, \quad n \in \mathbb{N}_0. \tag{2.40}$$

Hence, from (2.39) and (2.40), we have that general solution to equation (2.31) in this case, has the following form

$$x_n = c_1 + c_2 n + bn^2, \quad n \in \mathbb{N}_0. \tag{2.41}$$

To find constants c_1 and c_2 in terms of initial values x_0 and x_1, we need to solve the linear system

$$c_1 = x_0,$$
$$c_1 + c_2 = x_1 - b,$$

from which it follows that

$$c_1 = x_0 \quad \text{and} \quad c_2 = x_1 - x_0 - b. \tag{2.42}$$

By using (2.42) in (2.41), general solution to equation (2.31) in this case, is

$$x_n = x_0 + (x_1 - x_0 - b)n + bn^2, \quad n \in \mathbb{N}_0. \tag{2.43}$$

From (2.43) and since

$$x_1 = x_0 + b + \sqrt{4bx_0 + f},$$

we have

$$x_n = x_0 + \sqrt{4bx_0 + f}\, n + bn^2, \quad n \in \mathbb{N}_0. \tag{2.44}$$

Let

$$\widetilde{x}_n = x_0 + \sqrt{4bx_0 + f}\, n + bn^2, \quad n \in \mathbb{N}_0. \tag{2.45}$$

Then, by using the fact $\lambda_1\lambda_2 = 1$ and some calculation, we have

$$\begin{aligned}
&\widetilde{x}_{n+1} - \widetilde{x}_n - b - \sqrt{4b\widetilde{x}_n + f} \\
&\quad = x_0 + \sqrt{4bx_0 + f}(n+1) + b(n+1)^2 - (x_0 + \sqrt{4bx_0 + f}\, n + bn^2) - b \\
&\qquad - \sqrt{4b(x_0 + \sqrt{4bx_0 + f}\, n + bn^2) + f} \\
&\quad = \sqrt{4bx_0 + f} + 2bn - \sqrt{(\sqrt{4bx_0 + f} + 2bn)^2} = 0,
\end{aligned} \tag{2.46}$$

since

$$\sqrt{4bx_0 + f} + 2bn \geq 0, \quad n \in \mathbb{N}_0.$$

From (2.46) and since it obviously holds $\widetilde{x}_0 = x_0$, we see that the sequence defined in (2.45) (i.e. in (2.44)) is the solution to equation (2.24) with the initial value x_0, finishing the proof of the theorem. □

Remark 2.4. When $a = 1$ and $d = 4b$, then equation (2.24), can be solved in another natural way.

Let us conduct an analysis of equation (2.24) in the case when $c = 0$ and $d > 0$. In this case the equation becomes

$$x_{n+1} = ax_n + b + \sqrt{dx_n + f}, \quad n \in \mathbb{N}_0. \tag{2.47}$$

where $f, x_0 \geq 0$, and $\min\{a,b,d\} > 0$.

By using a simple inductive argument it is easy to see that $x_n > 0$, $n \in \mathbb{N}$. Let

$$y_n = \sqrt{dx_n + f}, \quad n \in \mathbb{N}_0. \tag{2.48}$$

Then, we have

$$x_n = \frac{y_n^2 - f}{d}, \quad n \in \mathbb{N}_0. \tag{2.49}$$

By using (2.49) in (2.47), we have

$$\frac{y_{n+1}^2 - f}{d} = a\frac{y_n^2 - f}{d} + b + y_n, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$\begin{aligned} y_{n+1}^2 &= ay_n^2 + dy_n + db + f - af \\ &= a\left(y_n + \frac{d}{2a}\right)^2 + db + f - af - \frac{d^2}{4a}, \end{aligned} \tag{2.50}$$

for $n \in \mathbb{N}_0$.

Hence, if

$$4a(bd + f(1-a)) = d^2, \tag{2.51}$$

from (2.50), we obtain

$$y_{n+1}^2 = a\left(y_n + \frac{d}{2a}\right)^2, \tag{2.52}$$

for $n \in \mathbb{N}_0$.

Since y_n is obviously a nonnegative sequence, from (2.52) we have

$$y_{n+1} = \sqrt{a}y_n + \frac{d}{2\sqrt{a}}, \quad n \in \mathbb{N}_0.$$

By the Langrage formula it follows that

$$y_n = (\sqrt{a})^n y_0 + \frac{d}{2\sqrt{a}}\frac{(\sqrt{a})^n - 1}{\sqrt{a} - 1}, \quad n \in \mathbb{N}_0, \tag{2.53}$$

when $a \neq 1$, and

$$y_n = y_0 + \frac{d}{2}n, \quad n \in \mathbb{N}_0, \tag{2.54}$$

when $a = 1$.

Using (2.53) in (2.49) we have

$$x_n = \frac{1}{d}\left(\left((\sqrt{a})^n\sqrt{dx_0 + f} + \frac{d}{2\sqrt{a}}\frac{(\sqrt{a})^n - 1}{\sqrt{a} - 1}\right)^2 - f\right), \quad n \in \mathbb{N}_0, \tag{2.55}$$

when $a \neq 1$, while when $a = 1$, then by using (2.54) in (2.49), we obtain

$$x_n = \frac{1}{d}\left(\left(\sqrt{dx_0 + f} + \frac{d}{2}n\right)^2 - f\right), \quad n \in \mathbb{N}_0. \tag{2.56}$$

From the above analysis we see that the following result holds.

Theorem 2.5. *Consider difference equation (2.47), where $f, x_0 \geq 0$, and $\min\{a, b, d\} > 0$. Then the following statements hold.*

(a) If $a \neq 1$, then general solution to the equation is given by formula (2.55).

(b) If $a = 1$, then general solution to the equation is given by formula (2.56).

Remark 2.6. Note that if $a = 1$ and the second condition in (2.25) holds, that is, $d = 4b$, then it is easy to see that condition (2.51) is satisfied. Hence, formula (2.56) presents general solution to equation (2.24) in this case, but this is nothing but formula (2.45) when $d = 4b$.

2.3 A natural way for obtaining equations (1.16) and (1.17)

It is a natural question if there is a way how equations (1.16) and (1.17) can be naturally obtained from some linear second-order difference equations with constant coefficients. The answer to the question is positive. To do this we will use an idea, which essentially belongs to Euler [8].

Consider difference equation

$$x_{n+2} - 2px_{n+1} + qx_n = r, \quad n \in \mathbb{N}_0, \tag{2.57}$$

where p, q, r are reals such that $q \neq 0$.

Assume that the both roots λ_1 and λ_2 of the characteristic polynomial $P_2(\lambda) = \lambda^2 - 2p\lambda + q$, associated with the difference equation

$$x_{n+2} - 2px_{n+1} + qx_n = 0, \quad n \in \mathbb{N}_0,$$

are such that $\lambda_1 \neq 1 \neq \lambda_2 \neq \lambda_1$. This means that the following conditions must hold

$$q \neq 2p - 1 \quad \text{and} \quad p^2 \neq q.$$

Then, general solution to equation (2.57) has the following form

$$x_n = \mathfrak{A}\lambda_1^n + \mathfrak{B}\lambda_2^n + \mathfrak{C}, \quad n \in \mathbb{N}_0. \tag{2.58}$$

Note also that from Viete's formulas, we have

$$\lambda_1 + \lambda_2 = 2p \quad \text{and} \quad \lambda_1\lambda_2 = q, \tag{2.59}$$

and that

$$\lambda_1 = p + \sqrt{p^2 - q} \quad \text{and} \quad \lambda_2 = p - \sqrt{p^2 - q}. \tag{2.60}$$

By using (2.58), we have

$$x_{n+1} - \lambda_1 x_n = \mathfrak{B}(\lambda_2 - \lambda_1)\lambda_2^n + (1 - \lambda_1)\mathfrak{C}, \tag{2.61}$$
$$x_{n+1} - \lambda_2 x_n = \mathfrak{A}(\lambda_1 - \lambda_2)\lambda_1^n + (1 - \lambda_2)\mathfrak{C}, \tag{2.62}$$

for $n \in \mathbb{N}_0$.

From (2.59), (2.60), (2.61) and (2.62), we have

$$(x_{n+1} - \lambda_1 x_n + (\lambda_1 - 1)\mathfrak{C})(x_{n+1} - \lambda_2 x_n + (\lambda_2 - 1)\mathfrak{C}) = 4\mathfrak{A}\mathfrak{B}(q - p^2)q^n,$$

for $n \in \mathbb{N}_0$, from which after some calculation and use of (2.59) and (2.60), we get

$$x_{n+1}^2 - (2px_n + 2(1-p)\mathfrak{C})x_{n+1} + qx_n^2 + 2(p-q)\mathfrak{C}x_n + (1-2p+q)\mathfrak{C}^2 + 4\mathfrak{A}\mathfrak{B}(p^2-q)q^n = 0, \quad (2.63)$$

for $n \in \mathbb{N}_0$.

To find $\mathfrak{A}$ and $\mathfrak{B}$, it should be solved the following linear system

$$\mathfrak{A} + \mathfrak{B} = x_0 - \mathfrak{C}$$
$$\lambda_1\mathfrak{A} + \lambda_2\mathfrak{B} = x_1 - \mathfrak{C},$$

from which it easily follows that

$$\mathfrak{A} = \frac{\lambda_2(x_0-\mathfrak{C}) - (x_1-\mathfrak{C})}{\lambda_2-\lambda_1} \quad \text{and} \quad \mathfrak{B} = \frac{x_1-\mathfrak{C}-\lambda_1(x_0-\mathfrak{C})}{\lambda_2-\lambda_1}. \quad (2.64)$$

Multiplying the quantities in (2.64), and using (2.59) and (2.60), we obtain

$$\mathfrak{A}\mathfrak{B} = -\frac{(x_1-\mathfrak{C})^2 - 2p(x_0-\mathfrak{C})(x_1-\mathfrak{C}) + q(x_0-\mathfrak{C})^2}{4(p^2-q)}. \quad (2.65)$$

By using (2.65) in (2.63), it follows that

$$x_{n+1}^2 - (2px_n + 2(1-p)\mathfrak{C})x_{n+1} + qx_n^2 + 2(p-q)\mathfrak{C}x_n + (1-2p+q)\mathfrak{C}^2 - ((x_1-\mathfrak{C})^2 - 2p(x_0-\mathfrak{C})(x_1-\mathfrak{C}) + q(x_0-\mathfrak{C})^2)q^n = 0,$$

for $n \in \mathbb{N}_0$, which is a quadratic equation in variable x_{n+1}.

Hence, by solving the quadratic equation in x_{n+1} and after some calculations, we obtain

$$x_{n+1} = px_n + (1-p)\mathfrak{C} \pm \sqrt{(p^2-q)(x_n-\mathfrak{C})^2 + ((x_1-\mathfrak{C})^2 - 2p(x_0-\mathfrak{C})(x_1-\mathfrak{C}) + q(x_0-\mathfrak{C})^2)q^n}, \quad (2.66)$$

for $n \in \mathbb{N}_0$.

Constant $\mathfrak{C}$ can be obtained by searching a solution to equation (2.57) in the form $x_n^p := \mathfrak{C}$, from which it easily follows that

$$\mathfrak{C} = \frac{r}{1-2p+q}. \quad (2.67)$$

Employing (2.67) in (2.66) is obtained a nonlinear multi-valued first-order difference equation which is satisfied by solutions to equation (2.57).

If in equation (2.57) is taken $r = 0$, then $\mathfrak{C} = 0$. Hence, in this case equation (2.66) becomes

$$x_{n+1} = px_n \pm \sqrt{(p^2-q)x_n + (x_1^2 - 2px_0x_1 + qx_0^2)q^n},$$

for $n \in \mathbb{N}_0$, which for the case when for every n is taken positive sign before the square root, is essentially an equation of the form in (2.1).

If in equation (2.57) is taken $q = 1$, then equation (2.66) becomes

$$x_{n+1} = px_n + (1-p)\mathfrak{C} \pm \sqrt{(p^2-1)(x_n-\mathfrak{C})^2 + (x_1-\mathfrak{C})^2 - 2p(x_0-\mathfrak{C})(x_1-\mathfrak{C}) + (x_0-\mathfrak{C})^2}, \quad (2.68)$$

for $n \in \mathbb{N}_0$, where $\mathfrak{C} = \frac{r}{2(1-p)}$, which for the case when for every n is taken positive sign before the square root in (2.68), is essentially an equation of the form in (2.24).

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