Nondivisibility versus backflow of information in understanding revivals of quantum correlations for continuous-variable systems interacting with fluctuating environments

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We address the dynamics of quantum correlations for a bipartite continuous-variable quantum system interacting with its fluctuating environment. In particular, we consider two independent quantum oscillators initially prepared in a Gaussian state, e.g., a squeezed thermal state, and compare the dynamics resulting from local noise, i.e., oscillators coupled to two independent external fields, to that originating from common noise, i.e., oscillators interacting with a single common field. We prove non-Markovianity (nondivisibility) of the dynamics in both regimes and analyze the connections between nondivisibility, backflow of information, and revivals of quantum correlations. Our main results may be summarized as follows: (i) revivals of quantumness are present in both scenarios, however, the interaction with a common environment better preserves the quantum features of the system; (ii) the dynamics is always nondivisible but revivals of quantum correlations are present only when backflow of information is present as well. We conclude that nondivisibility in its own is not a resource to preserve quantum correlations in our system, i.e., it is not sufficient to observe recoherence phenomena. Rather, it represents a necessary prerequisite to obtain backflow of information, which is the true ingredient to obtain revivals of quantumness.

I. INTRODUCTION

Decoherence is a distinctive sign of the detrimental influence of the environment on a quantum system. In quantum information and technology, decoherence is the main obstacle to reliable quantum processing of information. More generally, decoherence is a widely accepted explanation for the loss of nonclassicality of quantum systems and for their transition to the classical realm [1,2]. In recent years, it has been recognized that the action of an environment on a system may also have some nondetrimental effects, at least for a transient. Indeed, nontrivial spectral structures and memory effects, usually leading to non-Markovian dynamics for the quantum system [3–6], may induce recoherence and revivals of quantum features.

The environment of a quantum system is usually made of several (classical and quantum) units with an overall complex structure. As a consequence, a full quantum treatment of the interaction between a system and its environment is often challenging, or even unfeasible in a closed form. On the other hand, it is often the case that the overall action of the environment may be conveniently described as an external random force acting on the system, i.e., a classical stochastic field (CSF) [7]. In general, one might expect that the modeling of a quantum environment as a classical one leads to an incomplete description, i.e., such a description cannot capture the full quantum features of the dynamics. On the contrary, it has been proved that several system-environment interactions have a classical equivalent description [8–13]. In addition, there are experimental situations in which quantum systems interact with an inherently classical Gaussian noise [14–17], or where the environment can be effectively simulated classically [18].

In this framework, the first goal of this paper is to analyze in details the dynamics of a bipartite system made of two independent quantum harmonic oscillators interacting with its classical fluctuating environments. In particular, we compare the dynamics of correlations in two different environmental situations. On the one hand, we consider a local noise model, where each oscillator interacts with its own classical environment. On the other hand, we also consider the situation where both the oscillators interact with a common environment, described by a single stochastic field. A similar analysis has been performed for qubit systems [19,20] revealing the existence of a rich phenomenology.

Our work is also aimed at better analyzing the connections between the dynamics of quantumness, e.g., revivals of quantum correlations and the quantum-to-classical transition, and the non-Markovian features of the dynamical map. In particular, we want to investigate the role of non-Markovianity itself (i.e., nondivisibility of the quantum dynamical map) against the role of the backflow of information, which is a sufficient (but not necessary) condition to prove non-Markovianity and, in turn, often used to witness its presence.

In order to introduce the subject, we recall that directly proving [21–23] the non-Markovian character of a dynamics is not always possible, as in many situations the full analytic form of the time-dependent quantum dynamical map is missing. When the direct verification is not possible, one may exploit witnesses of non-Markovianity, i.e., quantities that vanish in case of a Markovian dynamics. Even though the witnesses may successfully capture the memory feature of a non-Markovian

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process in many situations, they possess different physical meaning and may be ineffective in some specific situation. As an example, the BLP measure [24], and its continuous variable analog based on fidelity [25], have both a clear physical interpretation in terms of information flow from the environment back to the system. Used as a witness, the BLP measure has been proved useful and effective to quantify memory effects in several situations but it may also fail to detect non-Markovianity [26–28]. On the other hand, recent results [29] suggest that information backflow is the essential element in addressing non-Markovian dynamics of a quantum system and for this reason the BLP measure has been proposed as the definition of quantum non-Markovianity itself.

In this context, more information may be often extracted from the study of the quantum correlations of the system, e.g., the entanglement or the quantum discord. In fact, revivals of quantum features, especially in noninteracting bipartite systems, may be a signature of a non-Markovianity. This is what happens, e.g., for noninteracting qubits [19,30]. On the other hand, the connection between revivals of correlations and backflow of information appears quite natural, especially in open quantum systems: a temporary and partial restoration of quantum coherence, previously lost during the interaction, is a sign of a memory effect, possibly of the environment, which is supposed to be storing correlations and sending them back to the system. Remarkably, revivals of correlations may also be found when the quantum system is interacting with a classical environment, suggesting that these feature reflects a property of the map, rather than a property of the environment. For qubit systems revivals have been detected in presence of Gaussian noise [31], and may even be found in the case of noninteracting qubits [32,33]. For a single oscillator, classical memory effects have been found to increase the survival time of quantum coherence [34] and, in particular, it has been proved that a detuning between the natural frequency of the system and the central frequency of the classical field induces revivals of quantum coherence.

For the system under investigation in this paper, non-Markovianity needs not to be witnessed, as it can be easily proven in a direct way. The stochastic description of the environment, in fact, allows us to determine the analytic form of the quantum dynamical map and to check straightforwardly its nondivisibility. On the other hand, information backflow may be revealed using fidelity witness [25]. Our results indicate that the dynamics is not divisible for both local and common noise, and that also revivals of quantumness appear in both cases, with the common case better preserving the quantum features of the system. At the same time, we found that revivals of quantum correlations are present only when non-Markovianity is present *together* with a backflow of information. We conclude that non-Markovianity itself is not the resource needed to preserve quantum correlations in our system. In other words, non-Markovianity alone is not a sufficient condition to induce revivals. Rather, it represents a necessary prerequisite to obtain backflow of information, which are the true ingredients to obtain revivals of quantumness.

The paper is organized as follows. In Sec. II we introduce the system and the stochastic modeling of both the environmental scenarios. In Sec. III we introduce the quantifiers of correlations and describe the initial preparation of the system,

focusing on its inital correlations. In Sec. IV we analyze the dynamics of the correlations, delimiting the boundaries for the existence of revivals. In Sec. V we prove the non-Markovianity of the dynamics and study the fidelity witness. Section VI summarizes the paper.

II. INTERACTION MODEL

We consider two noninteracting harmonic quantum oscillators with natural frequencies ω_1 and ω_2 and analyze the dynamics of this system in two different regimes: in the first one, each oscillator is coupled to one of two independent noninteracting stochastic fields; we dub this scenario as the *local noise* case. In the second regime, the oscillators are coupled to the same classical stochastic field, so we dub this case as *common noise*. In both cases, the Hamiltonian H is composed by a free and an interaction term. The free Hamiltonian H_0 is given by

$$H_0 = \hbar \sum_{j=1}^{2} \omega_j a_j^\dagger a_j. \tag{1}$$

Unlike the free Hamiltonian H_0, which is the same in the description of both models, the interaction term H_I differs. In the following subsections, we introduce the local and the common interaction Hamiltonian.

A. Local interaction

The interaction Hamiltonian H_L in the local model reads

$$H_\mathrm{L}(t) = \sum_{j=1}^{2} a_j \bar{C}_j(t) e^{i\delta_j t} + a_j^\dagger C_j(t) e^{-i\delta_j t}, \tag{2}$$

where the annihilation operators a_1, a_2 represent the oscillators, each coupled to a different local complex stochastic field $C_j(t)$ [with complex conjugate $\bar{C}_j(t)$] with $j = 1,2$, and $\delta_j = \omega_j - \omega$ is the detuning between the carrier frequency of the field and the natural frequency of the jth oscillator. In the last equation and throughout the paper, we will consider the Hamiltonian rescaled in units of a reference level of energy $\hbar\omega_0$ (for a reason to be pointed out later). Under this condition, the stochastic fields $C_1(t), C_2(t)$, their central frequency ω, the interaction time t, and the detunings all become dimensionless quantities.

The presence of fluctuating stochastic fields leads to an explicitly time-dependent Hamiltonian, whose corresponding evolution operator is given by

$$U(t) = \mathcal{T} \exp\left\{-i \int_0^t ds\, H_\mathrm{L}(s)\right\}, \tag{3}$$

where $\mathcal{T}$ is the time ordering. However, as the interaction Hamiltonian is linear in the annihilation and creation operators of the two oscillators, the two-time commutator $[H_\mathrm{L}(t_1), H_\mathrm{L}(t_2)]$ is always proportional to the identity. In particular, when the stochastic fields satisfy the conditions $C_j(t_1)\bar{C}_j(t_2) = C_j(t_2)\bar{C}_j(t_1)$, with $j = 1,2$, the two-time commutator becomes

$$[H_\mathrm{L}(t_1), H_\mathrm{L}(t_2)] = \sum_{j=1,2} 2i\, \bar{C}_j(t_1) C_j(t_2) \sin[\delta_j(t_1 - t_2)] \mathbb{I}_{12}. \tag{4}$$

This form of the two-time commutator allows us to use the Magnus expansion [35,36] to simplify the expression of the evolution operator (3) into

$$U(t) = \exp(\Xi_1 + \Xi_2), \qquad (5)$$

where Ξ_1 and Ξ_2 are given by

$$\Xi_1 = -i \int_0^t ds_1 H_I(s_1), \qquad (6)$$

$$\Xi_2 = \frac{1}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 [H_I(s_1), H_I(s_2)]. \qquad (7)$$

The specific form of Ξ_1 for the local (Ξ_1^L) scenario is given by

$$\Xi_1^L = \sum_{j=1}^2 [a_j^\dagger \phi_j(t) - a_j \phi_j^*(t)], \qquad (8)$$

where

$$\phi_j(t) = -i \int_0^t ds\, e^{-i\delta_j s} C_j(s) \quad \text{with} \quad j = 1,2. \qquad (9)$$

The evolution of the density operator of the system then reads

$$\rho_L(t) = [e^{\Xi_1^L} \rho(0) e^{(\Xi_1^L)^*}]_F = [D(\phi_a, \phi_b)\rho_0 D^\dagger(\phi_a, \phi_b)]_F, \qquad (10)$$

where $D_j(\alpha) = \exp(\alpha a_j^\dagger - \alpha^* a_j)$ is the displacement operator, $D(\alpha_1, \alpha_2) = D(\boldsymbol{\alpha}) = D_1(\alpha_1) D_2(\alpha_2)$, and $[\ldots]_F$ is the average over the realizations of the stochastic fields.

In the local scenario, we assume each CSF

$$C_j(t) = C_j^{(x)}(t) + i C_j^{(y)}(t),$$

described as a Gaussian stochastic process with zero mean $[C_j^{(x)}(t)]_F = [C_j^{(y)}(t)]_F = 0$ and autocorrelation matrix given by

$$\left[C_j^{(x)}(t_1) C_k^{(x)}(t_2)\right]_F = \left[C_j^{(y)}(t_1) C_k^{(y)}(t_2)\right]_F = \delta_{jk} K(t_1, t_2), \qquad (11)$$

$$\left[C_j^{(x)}(t_1) C_k^{(y)}(t_2)\right]_F = \left[C_j^{(y)}(t_1) C_k^{(x)}(t_2)\right]_F = 0, \qquad (12)$$

where we introduced the kernel autocorrelation function $K(t_1, t_2)$. By means of the Glauber decomposition of the initial state $\rho(0)$

$$\rho(0) = \int \frac{d^4\boldsymbol{\zeta}}{\pi^2} \chi[\rho(0)](\boldsymbol{\zeta}) D^\dagger(\boldsymbol{\zeta}), \qquad (13)$$

where $\chi[\rho](\boldsymbol{\zeta}) = \mathrm{Tr}[\rho D(\boldsymbol{\zeta})]$ is the symmetrically ordered characteristic function, the density matrix of the evolved state reads

$$\rho_L(t) = \mathcal{G}_L[\rho(0)] = \int \frac{d^4\boldsymbol{\zeta}}{\pi^2} g_L(\boldsymbol{\zeta}) D(\boldsymbol{\zeta}) \rho(0) D^\dagger(\boldsymbol{\zeta}), \qquad (14)$$

where we use the Gaussian function

$$g_L(\boldsymbol{\zeta}) = \frac{\exp\left(-\frac{1}{2} \boldsymbol{\zeta} \boldsymbol{\Omega} \boldsymbol{\sigma}_L^{-1} \boldsymbol{\Omega}^T \boldsymbol{\zeta}^T\right)}{\sqrt{\det[\boldsymbol{\sigma}_L]}}, \qquad (15)$$

where $\boldsymbol{\sigma}_L$ and the symplectic matrix $\boldsymbol{\Omega}$ are given by

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_L = \begin{pmatrix} \beta_1(t)\mathbb{I}_2 & 0 \\ 0 & \beta_2(t)\mathbb{I}_2 \end{pmatrix}. \qquad (16)$$

The matrix $\boldsymbol{\sigma}_L$ is the covariance of the noise function $g_L(\boldsymbol{\sigma})$ and its matrix elements are given by

$$\beta_j(t, t_0) = \int_{t_0}^t \int_{t_0}^t ds_1 ds_2 \cos[\delta_j(s_1 - s_2)] K(s_1, s_2). \qquad (17)$$

The map in Eq. (14) corresponds to the so-called Gaussian noise channel [37–39], i.e., a random displacement according to a Gaussian probability distribution.

B. Common interaction

The Hamiltonian H_C in the common interaction model reads

$$H_C(t) = \sum_{j=1}^2 a_j e^{i\delta_j t} \bar{C}(t) + a_j^\dagger e^{-i\delta_j t} C(t), \qquad (18)$$

where each oscillator, represented by the annihilation operators a_1, a_2, is coupled to a common stochastic field $C(t)$ which is described as a Gaussian stochastic process with zero mean $[C^{(x)}]_F = [C^{(y)}]_F = 0$ and the very same autocorrelation matrix of the local scenario.

Along the same lines of the local interaction model derivation, we use the Magnus expansion in order to get to the evolution operator. By asking the stochastic field to satisfy the relation $C(t_1)\bar{C}(t_2) = C(t_2)\bar{C}(t_1)$, the two-time commutator reads

$$[H_C(t_1), H_C(t_2)] = \bar{C}(t_1) C(t_2) \sum_{j=1,2} 2i\, \sin[\delta_j(t_1 - t_2)]\mathbb{I}_{12}. \qquad (19)$$

The evolution operator for the common scenario is the same described in Eq. (5), where the specific form of Ξ_1 in the common interaction model is given by

$$\Xi_1^C = \sum_{j=1}^2 [a_j^\dagger \psi_j(t) - a_j \psi_j^*(t)], \qquad (20)$$

where

$$\psi_j(t) = -i \int_0^t ds\, e^{-i\delta_j s} C(s) \quad \text{with} \quad j = 1,2. \qquad (21)$$

The evolution of the density operator of the system then reads

$$\rho(t) = [e^{\Xi_1^C} \rho(0) e^{(\Xi_1^C)^*}]_F = [D(\psi_1, \psi_2)\rho_0 D^\dagger(\psi_1, \psi_2)]_F, \qquad (22)$$

which, following the same steps of the derivation presented in the previous subsection, leads to

$$\rho_C(t) = \mathcal{G}_C[\rho(0)] = \int \frac{d^4\boldsymbol{\zeta}}{\pi^2} g_C(\boldsymbol{\zeta}) D(\boldsymbol{\zeta}) \rho(0) D^\dagger(\boldsymbol{\zeta}), \qquad (23)$$

where we use the Gaussian function

$$g_C(\boldsymbol{\zeta}) = \frac{\exp\left(-\frac{1}{2} \boldsymbol{\zeta} \boldsymbol{\Omega} \boldsymbol{\sigma}_C^{-1} \boldsymbol{\Omega}^T \boldsymbol{\zeta}^T\right)}{\sqrt{\det[\boldsymbol{\sigma}_C]}}, \qquad (24)$$

$\boldsymbol{\sigma}_C$ being its covariance matrix, given by

$$\boldsymbol{\sigma}_C = \begin{pmatrix} \beta_1(t)\mathbb{I}_2 & \boldsymbol{R} \\ \boldsymbol{R} & \beta_2(t)\mathbb{I}_2 \end{pmatrix}, \qquad (25)$$

$$\boldsymbol{R} = \begin{pmatrix} \beta_C(t) & \gamma_C(t) \\ \gamma_C(t) & \beta_C(t) \end{pmatrix}, \tag{26}$$

with the matrix elements given by

$$\beta_c(t,t_0) = \int_{t_0}^t \int_{t_0}^t ds_1 ds_2 \cos[(\delta_1 s_1 - \delta_2 s_2)] K(s_1,s_2),$$

$$\gamma_c(t,t_0) = \int_{t_0}^t \int_{t_0}^t ds_1 ds_2 \sin[(\delta_1 s_1 - \delta_2 s_2)] K(s_1,s_2). \tag{27}$$

C. Covariance matrix dynamics in local and common interaction

The dynamical maps described by Eqs. (14) and (23) belong to the class of Gaussian channels, i.e., the evolution, in both regimes, preserves the Gaussian character of the input state. In turn, this is a useful feature, since in this case quantum correlations, entanglement and discord, may be evaluated exactly. We also recall that Gaussian channels represent the short times solution of Markovian (dissipative) master equations in the limit of high-temperature environment. In the following, this link will be exploited to analyze the limiting behavior of the two-mode dynamics.

In order to get quantitative results, we assume that fluctuations in the environment are described by Ornstein-Uhlenbeck Gaussian processes, characterized by a Lorentzian spectrum and a kernel autocorrelation function

$$K(t_1,t_2) = \tfrac{1}{2}\lambda t_E^{-1} \exp(-|t_1 - t_2|/t_E),$$

where λ is a coupling constant and t_E is the correlation time of the environment. We also assume the case of resonant oscillators ($\omega_1 = \omega_2 = \omega_0$), which implies that the oscillators are identically detuned from the central frequency of the classical stochastic field, i.e.,

$$\delta_1 = \delta_2 = \delta = 1 - \frac{\omega}{\omega_0}.$$

This assumption simplifies the expression of the state dynamics: in the local scenario, leading to $\beta_1(t) = \beta_2(t) = \beta(t)$ and, in turn,

$$\rho_L(t) = \mathcal{E}_L[\rho(0)](t)$$
$$= \int \frac{d^4\boldsymbol{\zeta}}{[\pi\beta(t)]^2} \exp\left(-\frac{|\boldsymbol{\zeta}|^2}{\beta(t)}\right) D(\boldsymbol{\zeta})\rho(0)D^\dagger(\boldsymbol{\zeta}), \tag{28}$$

where $\beta(\Delta t = t - t_0) = \beta(t,t_0)$ with the Ornstein-Uhlenbeck kernel is

$$\beta(t) = \frac{\lambda}{[1 + (\delta t_E)^2]^2} \{t - t_E + (\delta t_E)^2(t + t_E)$$
$$+ t_E e^{-t/t_E}([1 - (\delta t_E)^2]\cos\delta t - 2\delta t_E \sin\delta t)\}. \tag{29}$$

In the common noise case, the condition of resonant oscillators implies $\beta_1(t) = \beta_2(t) = \beta_c(t) = \beta(t)$ and $\gamma_c(t) = 0$, leading to simplified matrices $\boldsymbol{R}$ and $\boldsymbol{\sigma}_C$ given by

$$\boldsymbol{R} = \begin{pmatrix} \beta(t) & 0 \\ 0 & \beta(t) \end{pmatrix}, \quad \boldsymbol{\sigma}_C = \begin{pmatrix} \beta(t)\mathbb{I}_2 & \boldsymbol{R} \\ \boldsymbol{R} & \beta(t)\mathbb{I}_2 \end{pmatrix} \tag{30}$$

corresponding to the Gaussian channel

$$\rho(t) = \mathcal{E}_C[\rho(0)](t)$$
$$= \int \frac{d^2\zeta}{\pi\beta(t)} \exp\left(-\frac{|\zeta|^2}{\beta(t)}\right) D(\zeta,\zeta)\rho(0)D^\dagger(\zeta,\zeta). \tag{31}$$

As for the initial state $\rho(0)$ of the system, we assume a generic squeezed thermal state (STS), which is a zero-mean Gaussian state, described by a Gaussian characteristic function $\chi[\rho(0)](\boldsymbol{\zeta}) = \exp\left(-\frac{1}{2}\boldsymbol{\zeta}^T\boldsymbol{\sigma}_{STS}\boldsymbol{\zeta}\right)$ where the covariance matrix $\boldsymbol{\sigma}_{STS}$ of the state has the general form

$$\boldsymbol{\sigma}_{STS} = \frac{1}{2}\begin{pmatrix} A\,\mathbb{I}_2 & C\sigma_z \\ C\sigma_z & B\,\mathbb{I}_2 \end{pmatrix}, \tag{32}$$

where $\sigma_z = \mathrm{diag}(1, -1)$ is the z Pauli matrix. This covariance matrix corresponds to a density operator of the form

$$\rho = S_2(r)(\nu_1 \otimes \nu_2)S_2(r)^\dagger, \tag{33}$$

where $S_2(r) = \exp\{r(a_1^\dagger a_2^\dagger - a_1 a_2)\}$ is the two-mode squeezing operator and ν_j is a single-mode thermal state

$$\nu_j = \frac{1}{\bar{n}_j}\sum_m \left(\frac{\bar{n}_j}{\bar{n}_j + 1}\right)^m |m\rangle\langle m|. \tag{34}$$

The physical state depends on three real parameters: the squeezing parameter r and the two energies $\bar{n}_1, \bar{n}_2$, which are related to the parameters A, B, C of Eq. (32) by the relations

$$A = \cosh(2r) + 2\bar{n}_1\cosh^2 r + 2\bar{n}_2\sinh^2 r,$$
$$B = \cosh(2r) + 2\bar{n}_1\sinh^2 r + 2\bar{n}_2\cosh^2 r,$$
$$C = (1 + \bar{n}_1 + \bar{n}_2)\sinh(2r). \tag{35}$$

As the squeezed thermal state is a Gaussian state and the dynamics in both scenarios is described by a Gaussian channel, the output state at any time is Gaussian as well, so it is determined only by the covariance matrix. By evaluating the characteristic function of the evolved state, one finds that the covariance matrices of the state at time t in the local and common scenarios are

$$\boldsymbol{\beta}_L(t) = \boldsymbol{\sigma}_{STS} + 2\boldsymbol{\sigma}_L(t), \tag{36}$$

$$\boldsymbol{\beta}_C(t) = \boldsymbol{\sigma}_{STS} + 2\boldsymbol{\sigma}_C(t), \tag{37}$$

where $\boldsymbol{\sigma}_L(t)$ and $\boldsymbol{\sigma}_C(t)$ are given in Eqs. (16) and (30), respectively. As mentioned before, the Gaussian character of the states is preserved by the dynamics. Nevertheless, while the output state in the local scenario is always a STS, i.e., at any time it can be put in a diagonal form by means of a two-mode squeezing operation, the output state in the common scenario ceases to be an STS as soon as the interaction starts.

III. QUANTUM CORRELATIONS

Systems possessing quantum features have proved themselves useful in many fields, e.g., quantum information, quantum computation, and quantum estimation, increasing the performances of computation protocols and precision measurements. In the case of bipartite systems, the quantumness lies both in the nonclassicality of the state and in the correlations between the different parts of the system. In this

section, we introduce two well-known markers of correlations, entanglement and quantum discord.

A. Entanglement

While classical multipartite systems, though being correlated, are always described in a quantum picture by separable states, quantum multipartite systems may show nonclassical correlations which require a description in terms of nonseparable density operators. Aiming at quantifying these quantum correlations, the entanglement measures the degree of nonseparability of a quantum system. For a bipartite Gaussian state ρ with covariance matrix σ, the entanglement is given by the logarithmic negativity, which is defined as

$$\mathcal{N}(\rho) = \max\{0, -\ln(2\tilde{d}_-)\}, \tag{38}$$

where $\tilde{d}_-$ is the smallest symplectic eigenvalue of the partially transposed covariance matrix $\sigma^{\text{PT}} = \Delta\sigma\Delta$ with $\Delta = \text{diag}(1, -1, 1, 1)$. As stated by Simon [40], a two-mode Gaussian state is separable if and only if the symplectic eigenvalue satisfies the relation $\tilde{d}_- \geqslant \frac{1}{2}$. Any violation of the latter implies that the state is nonseparable (or entangled) and leads to a positive measure of the entanglement given in (38).

B. Quantum discord

The total amount of the correlations possessed by a bipartite quantum state ρ is called mutual information and is given by

$$\mathcal{I}(\rho) = S(\rho_1) + S(\rho_2) - S(\rho), \tag{39}$$

where $S(\rho_j)$ is the von Neumann entropy of the jth subsystem. Usually, the mutual information can be divided into two parts: a classical part $\mathcal{C}(\rho)$ and a quantum part $\mathcal{D}(\rho)$, which takes the name of quantum discord. The classical correlations, defined as the maximum amount of information extractable from one subsystem by performing local operations on the other, are given by

$$\mathcal{C}(\rho) = \max_{\Pi_i}\left\{S(\rho_1) - \sum_i p_i S\big(\rho_{1|2}^{\Pi_i}\big)\right\}, \tag{40}$$

where $\rho_{1|2}^{\Pi_i} = \text{Tr}_2(\rho\mathbb{I} \otimes \Pi_i)$ is the state after the measurement on system 2 with probability $p_i = \text{Tr}_{1,2}(\rho\mathbb{I} \otimes \Pi_i)$. The quantum discord is defined as the difference between the total correlations and the classical correlations:

$$\mathcal{D}(\rho) = \mathcal{I}(\rho) - \mathcal{C}(\rho). \tag{41}$$

The quantum discord then measures the amount of correlations whose origin cannot be addressed to the action of local operations or classical communication. However, computing the quantum discord may be challenging as it usually implies finding the POVM that maximizes the classical correlations. In the case of Gaussian states, the form of the POVM maximizing the classical correlations is known [41,42] and the quantum discord depends only on the covariance matrix by the relation

$$\mathcal{D}(\rho) = h(\sqrt{I_2}) - h(d_-) - h(d_+) + h(\sqrt{E_{\min}}), \tag{42}$$

where d_- and d_+ are the symplectic eigenvalues of the covariance matrix, I_1, I_2, I_3, I_4 are the so-called symplectic

invariants, $h(x) = (x + \frac{1}{2})\ln(x + \frac{1}{2}) - (x - \frac{1}{2})\ln(x - \frac{1}{2})$, and [42]

$$E_{\min} = \begin{cases} \left[\frac{2|I_3| + \sqrt{4I_3^2 + (4I_2-1)(4I_4-1)}}{4I_2 - 1}\right]^2 & \text{if } R_\sigma \leqslant 1, \\ \frac{I_1 I_2 + I_4 - I_3^2 - \sqrt{\left(I_1 I_2 + I_4 - I_3^2\right)^2 - 4I_1 I_2 I_4}}{2I_2} & \text{if } R_\sigma > 1, \end{cases} \tag{43}$$

where

$$R_\sigma = \frac{4(I_1 I_2 - I_4)^2}{(I_1 + 4I_4)(1 + 4I_2)I_3^2}.$$

For Gaussian states satisfying the second condition, the maximum amount of extractable information is achieved by measuring a canonical variable (e.g., by homodyne detection in optical systems [43]). On the other hand, for states falling in the first set, the optimal measurement is more general, and coincides with the projection over coherent states for STSs. For a generic Gaussian state, with covariance matrix σ written in a block form

$$\sigma = \begin{pmatrix} \mathbb{A} & \mathbb{C} \\ \mathbb{C}^T & \mathbb{B} \end{pmatrix}, \tag{44}$$

the symplectic invariants are $I_1 = \det\mathbb{A}$, $I_2 = \det\mathbb{B}$, $I_3 = \det\mathbb{C}$, and $I_4 = \det\sigma$.

C. STS correlations

In order to assess the dynamics of entanglement and discord in the presence of noise, we briefly review the static properties of quantum correlations [44] for a squeezed thermal state. We consider the case of identical thermal states ($\bar{n}_1 = \bar{n}_2 = \bar{n}$) and use a convenient representation of STSs, built upon reparametrizing the covariance matrix by means of its total energy $\epsilon = 2(\bar{n} + n_s + 2\bar{n}\,n_s)$, with $n_s = \sinh^2 r$, and a normalized squeezing parameter $\gamma \in [0, 1]$, such that

$$n_s = \gamma\epsilon, \quad \bar{n} = \frac{(1-\gamma)\epsilon}{1 + 2\gamma\epsilon}.$$

Note that, for $\gamma = 0$, the state has only thermal energy ($\epsilon = \bar{n}$) while for $\gamma = 1$ the total amount of energy comes from the two-mode squeezing operation ($\epsilon = \sinh^2 r$).

Figure 1 shows the quantum correlations of a STS as a function of the energy ϵ and the squeezing parameter γ. The left panel shows that the STS is entangled as long as γ overtakes a threshold value which depends on the total amount of energy. Conversely, the quantum discord of a STS is always positive, unless the state is purely thermal, i.e., with zero squeezing ($\gamma = 0$). Notice that the states considered in this paper belong to the class of Gaussian states for which the Gaussian discord equals the full quantum discord [45].

IV. DYNAMICS OF QUANTUM CORRELATIONS

Before proceeding with a detailed analysis of the dynamics of the quantum correlations, let us focus on the function $\beta(t)$, in order to understand which parameters affect the dynamics of the output state. As a matter of fact, the function $\beta(t)$ in (29) depends only on two parameters (besides the time t), as it can be rescaled in units of t_{E} by assuming $\tilde{\delta} = \delta t_{\text{E}}$, $\tilde{\lambda} = \lambda t_{\text{E}}$, and $\tilde{t} = t/t_{\text{E}}$, leading to the expression (in which tildes have
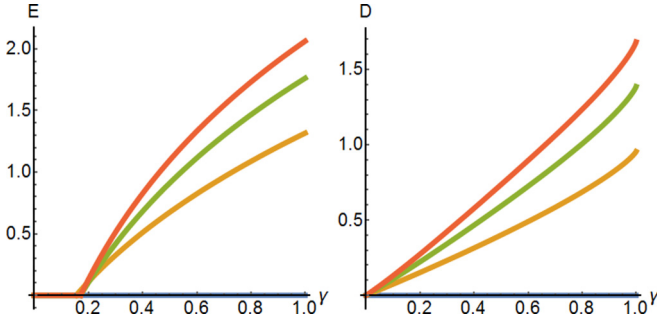
FIG. 1. Quantum correlations of STS for different values of the energy ϵ. Left panel: entanglement of a STS as a function of squeezing parameter γ. The STS is entangled as long as γ overtakes a threshold value that depends on the energy ϵ. Right panel: discord of a STS as a function of squeezing parameter γ. The STS is always a discordant state unless $\gamma = 0$. In both panels, from bottom to top, $\epsilon = 0$ (blue line), $\epsilon = 1$ (yellow line), $\epsilon = 2$ (green line), and $\epsilon = 3$ (red line).

already been dropped)

$$\beta(t) = \frac{\lambda}{[1 + \delta^2]^2} \{ t - 1 + \delta^2(t + 1)$$
$$+ e^{-t}[(1 - \delta^2)\cos\delta\, t - 2\delta\,\sin\delta\, t] \}. \qquad (45)$$

Intuitively, given the form of the covariance matrices in Eqs. (36) and (37) one realizes that when $\beta(t)$ shows a monotonous behavior, the system cannot gain any quantum features, or go back to the initial state at any value of the interaction time. Conversely, an oscillating $\beta(t)$ would let the system orbit in the phase space, which means that the quantum features of the output state may have a chance to be restored.

Formally, imposing the condition $d\beta(t)/dt = 0$ leads to the following equation:

$$\lambda\, \frac{1 - e^{-t}(\cos\delta t - \delta\sin\delta t)}{1 + \delta^2} = 0, \qquad (46)$$

which cannot be analytically solved. The left panel of Fig. 2 contains a numerical plot of the solutions of (46) and shows the existence of a lower bound on the rescaled detuning δ for
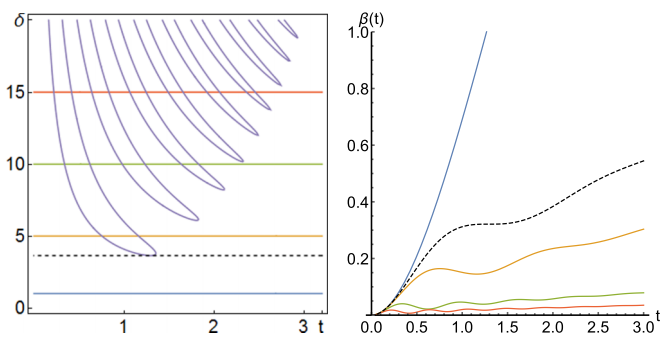
the oscillations of $\beta(t)$. The lower bound is represented by the black dashed line, corresponding to

$$\delta_0 = \frac{3\pi}{2}\left[\text{ProductLog}\left(\frac{3\pi}{2}\right)\right]^{-1} \simeq 3.644,$$

independent of λ. The existence of a threshold value of δ may be interpreted as a sign of the competition between dissipation and recoherence phenomena: indeed, the detuning always weakens the dissipative dynamics, but not any value of δ is sufficient to induce revivals. In addition, it is worth noting that a pattern similar to the left panel of Fig. 2 can be found in [24], even though in that case the threshold value distinguishes Markovian and non-Markovian regimes of interaction.

With this in mind, we now examine the dynamics of quantum correlations of initially maximally entangled squeezed thermal states ($\gamma = 1$) and two-mode thermal states ($\gamma = 0$) in the presence of local and common stochastic environments. In order to be able to compare the results of the different scenarios, we recall we already limited the analysis to resonant oscillators and that we assume the identical rescaled coupling constant $\lambda^{(c)}$ for the common scenario and $\lambda^{(1)}, \lambda^{(2)}$ for the local scenario, $\lambda^{(c)} = \lambda^{(1)} = \lambda^{(2)} = \lambda$.

Let us start by addressing the dynamics of correlations of an initially entangled STS: the upper panels in Fig. 3 show how the classical stochastic fields, whether they be local or common, induce loss of correlations in time. However, the decay rate of correlations is not the same in both scenarios: indeed, the presence of a common stochastic field is less detrimental, i.e., the interaction with the same environment leads to a slower loss of correlations. This effect may be seen as the consequence of the fact the interaction of a two-mode systems with a common environment may be rewritten as the nonsymmetric interaction of two collective modes with separate enviroments. In particular, decoherence strongly affects only one of the collective modes, and this mechanism is physically responsible for a slower loss of correlations. In all the four panels, the green line corresponds to $\delta = \delta_0$, the threshold value over which $\beta(t)$ shows an oscillating behavior. As it is possible to see, $\delta = \delta_0$ plays the role of the threshold value also in the case of the correlations. In fact, detunings bigger than δ_0 induce revivals of entanglement (upper right) and discord (lower left and right). The entanglement dynamics of the upper panel allows us to point out an important issue: $\delta > \delta_0$ is a necessary condition for an oscillating $\beta(t)$, though revivals of entanglement also depend on the rescaled coupling λ. In other words, when $\delta > \delta_0$, the symplectic eigenvalue $\tilde{d}_-$ of (38) flows in time in unison with $\beta(t)$, without necessarily violating the separability condition $\tilde{d}_- \geq \frac{1}{2}$. This explains the presence of a plateau in the entanglement of the common scenario with $\delta = \delta_0$.

Let us now focus on the discord dynamics (see the lower panels in Fig. 3). While the entanglement shows a vanishing behavior in both scenarios in any setup of parameters, the same cannot be said for the quantum discord. While in the local scenario the initial discord tends to vanish, the common interaction introduces some correlations which clearly arise after the drop of the initial discord [46]. The effect of the common stochastic field on the dynamics of the quantum discord is even clearer in the case of thermal input states (squeezing parameter $\gamma = 0$). The upper left panel of Fig. 4

FIG. 2. Left panel: contour plot of $d\beta(t)/dt = 0$ as a function of t and δ. The purple curve represents the solution of $d\beta(t)/dt = 0$. The black dashed line represents the maximum value of $\delta = \delta_0$ for which $\beta(t)$ does not oscillate. Right panel: $\beta(t)$ for different values of δ. From bottom to top, the other lines are for $\delta = 1$ (blue), $\delta = 5$ (yellow), $\delta = 10$ (green), and $\delta = 15$ (red). An oscillating behavior is present only if $\delta > \delta_0$.
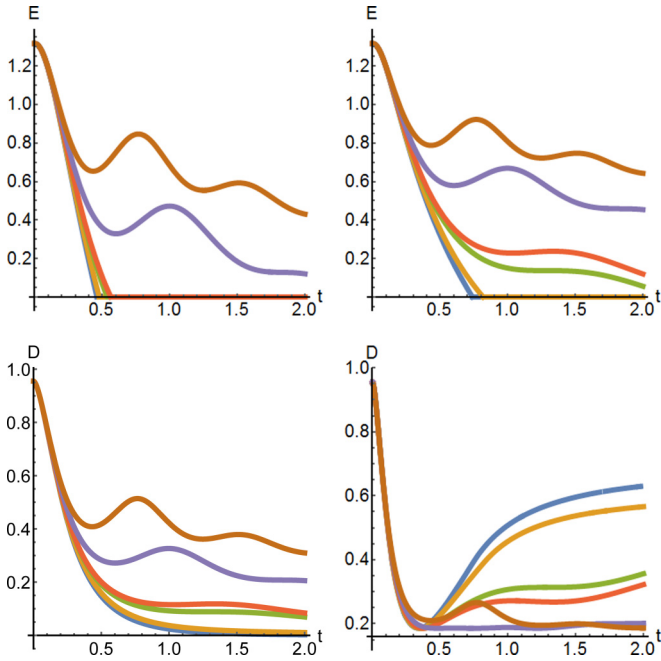
FIG. 3. Dynamics of correlations in presence of CSFs for different values of rescaled detuning δ. Upper left panel: entanglement dynamics in local scenario as a function of time t. Upper right panel: entanglement dynamics in common scenario as a function of time t. In both scenarios, the entanglement possessed by the system may revive. Lower left panel: discord dynamics in local scenario as a function of time t. The initial discord decreases in time. Right left panel: discord dynamics in common scenario as a function of time t. The initial discord decreases, reaches a minimum, and then increases monotonically as a consequence of the interaction. In all panels, we set $\epsilon = 1, \gamma = 1, \lambda = 1$, and, from bottom to top, $\delta = 0$ (blue line), $\delta = 2$ (yellow line), $\delta = \delta_0$ (green line), $\delta = 4$ (red line), $\delta = 6$ (purple line), and $\delta = 8$ (brown line).

FIG. 4. Upper panels: correlations of a two-mode thermal state. Left panel: discord dynamics in time t for different values of δ. The initially zero-discord state becomes a discord state because of the interaction. Right panel: dynamics of symplectic eigenvalue $\tilde{d}_-$ for different values of δ. The state always remains separable, though becoming a discord state. In both panels we have $\epsilon = 1, \gamma = 0, \lambda = 1$ and, from top to bottom, $\delta = 0$ (blue line), $\delta = 2$ (yellow line), $\delta = \delta_0$ (green line), $\delta = 4$ (red line), $\delta = 6$ (purple line), and $\delta = 8$ (brown line). Lower left panel: region plot of the POVM minimizing the quantum discord. We set $\epsilon = 1, \delta = 3, \lambda = 1$. Lower right panel: contour plot of the asymptotic value of the quantum discord as a function the input state parameters ϵ and γ. We set $\delta = 3, \lambda = 1$.

shows the discord evolution of the state $\rho = \nu_1 \otimes \nu_2$ in the common scenario. The interaction transforms the initial zero-discord state into a discord state without affecting the separability of the input state (the symplectic eigenvalue $\tilde{d}_-$ always satisfies the condition $\tilde{d}_- \geqslant \frac{1}{2}$, as is apparent from the upper right panel of Fig. 4). Furthermore, the quantum discord tends to an asymptotic value which depends on both the energy ϵ and the squeezing parameter γ of the input Gaussian state, but it is not affected by the parameters of the environment λ and δ. Indeed, this can be seen as a consequence of the non-Markovianity of the quantum map, as the long-time dynamics is influenced by the input state. A contour plot of the asymptotic value of the discord as a function of ϵ and γ is shown in the lower right panel of Fig. 4. Finally, we want to mention that the POVM that minimizes the quantum discord changes in time preserving the continuity of the discord itself. As an example, we report one particular scenario in the lower left panel of Fig. 4 where the regions corresponding to the two POVMs are colored differently.

V. NONDIVISIBILITY VS INFORMATION BACKFLOW

The presence of revivals of correlations might be interpreted as a signature of some form of information backflow between
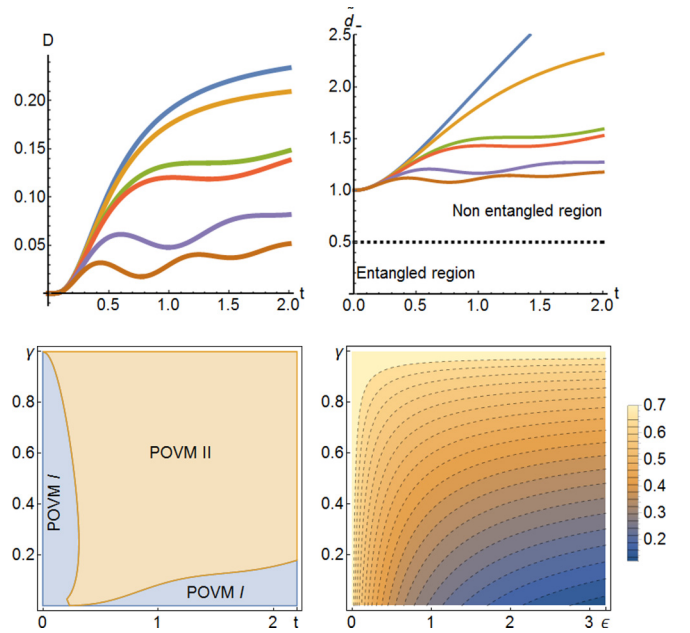
the system and the environment, a phenomenon typically associated to non-Markovian effects. It is the purpose of this section to reveal the non-Markovian character of the Gaussian maps of both scenarios and explore the link between nondivisibility and information backflow, analyzing the evolution of the fidelity of two input states.

A completely positive map $\mathcal{E}_{(t,t_0)}$ describing the dynamical evolution of a quantum system is said to be *divisible* if it satisfies the decomposition rule $\mathcal{E}_{(t_2,t_0)} = \mathcal{E}_{(t_2,t_1)}\mathcal{E}_{(t_1,t_0)}$ for any $t_2 \geqslant t_1 \geqslant t_0$. Divisibility is often assumed to be the key concept to characterize non-Markovianity [47] in the quantum regime and a completely positive map is said to be non-Markovian if it violates the decomposition rule for some set of times.

In our system, it is straightforward to prove that the Gaussian maps (14) and (23) do not satisfy the divisibility conditions, i.e., they cannot describe a Markovian dynamics. In order to prove these results, we notice that the composition of maps $\mathcal{E}_L(\Delta t_2)\mathcal{E}_L(\Delta t_1)$ corresponds to a convolution, leading to $\mathcal{E}_L(\Delta t_2)\mathcal{E}_L(\Delta t_1) = \mathcal{E}_L(\Delta t_1 + \Delta t_2)$. This condition is satisfied if and only if

$$\beta(\Delta t_1 + \Delta t_2) = \beta(\Delta t_1) + \beta(\Delta t_2), \qquad (47)$$

which is not satisfied for any choice of the parameters δ and λ, thus implying that the map is always non-Markovian. A
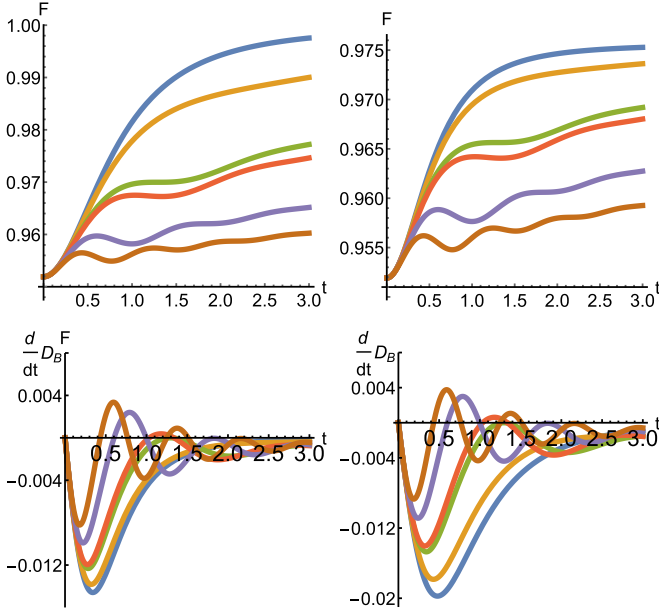
FIG. 5. Dynamics of fidelity and Bures distance derivative for different values of rescaled detuning δ. Upper left panel: dynamics of fidelity in a local scenario. Upper right panel: dynamics of fidelity in a common scenario. In both situations, fidelity ceases to oscillate when δ is lower than the threshold value δ_0. The distinguishability of the input states diminishes monotonically in time and no backflow of information is detected. Lower left panel: derivative of Bures distance in local scenario. Lower right panel: derivative of Bures distance in common scenario. The curves with a positive part of derivative of Bures distance have $\delta > \delta_0$ and correspond to the curves in the upper panels where oscillations of fidelity are shown. In all panels, we set $\epsilon_1 = 2, \epsilon_2 = 1, \gamma_1 = \gamma_2 = 1, \lambda = 1$ and, from top to bottom, $\delta = 0$ (blue line), $\delta = 2$ (yellow line), $\delta = \delta_0$ (green line), $\delta = 4$ (red line), $\delta = 6$ (purple line), and $\delta = 8$ (brown line).

similar proof can be obtained for the common noise map $\mathcal{E}_C(\Delta t)$.

We are now ready to discuss the connections between revivals of correlations, nondivisibility, and information backflow [47–49]. As we already mentioned in the Introduction, non-Markovianity may be revealed by some witnesses as the BLP measure or the analog measure based on fidelity for CV systems. Both techniques are based on the contractive property (valid for Markovian dynamics) of the trace distance and the Bures distance, respectively. Therefore, a nonmonotonous behavior of the trace distance or the fidelity is a signature of non-Markovianity. Furthermore, both these witnesses possess physical meaning: the trace distance is directly related to the probability of discriminating two states in time, whereas the Bures distance may be used to evaluate upper and lower bounds of the very same error probability defined by the trace distance. Therefore, a nonmonotonous dynamics also implies a partial clawback of distinguishability of two input states, which has been interpreted as a sign of a backflow of information [50]. A measure of non-Markovianity $\mathcal{N}_\mathcal{F}$ can be constructed by the violation of the contractive property of the

fidelity,

$$\mathcal{N}_\mathcal{F} = \frac{1}{2} \int \frac{1}{2} \int \left(\left| \frac{d}{dt} D_B(\rho_1, \rho_2) \right| + \frac{d}{dt} D_B(\rho_1, \rho_2) \right) dt, \tag{48}$$

where we used the Bures distance

$$D_B(\rho_1, \rho_2) = \sqrt{2[1 - \sqrt{\mathcal{F}(\rho_1, \rho_2)}]}. \tag{49}$$

The quantity $\mathcal{N}_\mathcal{F}$ is nonzero only when the derivative of the Bures distance is positive, i.e., the contractive property is violated and the fidelity has a nonmonotonous behavior. In our system, the fidelity between any pair of two-mode Gaussian states

$$\mathcal{F}(\rho_1, \rho_2) = [\text{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}] \tag{50}$$

may be evaluated analytically [51], though its expression is cumbersome and will not be reported here.

In Fig. 5 we show the time evolution of the fidelity and the derivative of the Bures distance between a pair of two-mode squeezed vacuum states ($\gamma = 1$) with different energies ($\epsilon_1 \neq \epsilon_2$). The existence of sets of parameters leading to a nonmonotonous behavior of the fidelity and a region of positive derivative of Bures distance is enough to confirm the already proven non-Markovianity of both maps. However, non-Markovianity is not detected when $\delta \leqslant \delta_0$, where δ_0 is the very same threshold obtained in the previous section, i.e., the threshold to observe revivals of correlations. The same behavior is observed for any choice of the involved parameters, confirming that revivals of correlations are connected to the backflow of information revealed by the fidelity measure, rather than a feature related to nondivisibility of the map itself.

VI. CONCLUSIONS

In conclusion, we have investigated the evolution of entanglement and quantum discord for two harmonic oscillators interacting with classical stochastic fields. We analyzed two different regimes: in the first one, the two modes interact with two separate environments describing local noise, whereas in the second case we consider a single environment describing a situation where the two oscillators are exposed to a common source of noise.

We have obtained the analytic form of the quantum map for both the local and the common noise model and analyzed the dynamics of quantum correlations for initial states ranging from maximally entangled to zero discord states. Our results show that the interaction with a classical environment always induces a loss of entanglement, while the quantum discord shows a vanishing behavior in the local scenario but may exhibit a nonzero asymptotic value in the common scenario, independently on the initial value of the discord. We have also shown that the interaction with a common environment is, in general, less detrimental than the interaction with separate ones.

Finally, we have proved the nondivisibility of the maps and found some structural boundaries on the existence of revivals of correlations in terms of a threshold value of the detuning between the natural frequency of the system and the central frequency of the noise. The same threshold

determines the presence of backflow of information, associated to oscillations of the fidelity between a pair of initial states. Overall, this suggests that nondivisibility in itself is not a resource to preserve quantum correlations in our system, i.e., it is not sufficient to observe recoherence phenomena. Rather, it represents a necessary prerequisite to obtain backflow of information, which is the true ingredient to obtain revivals of quantumness and, in turn, the physically relevant resource. In this framework, our findings support some recent results [29,52] about the definition of quantum non-Markovianity, which emphasize the fundamental role of information back-

flow, as opposed to divisibility, as a key concept for the characterization of non-Markovian dynamics in the quantum regime.

[1] W. H. Zurek, Decoherence and the transition from quantum to classical, Phys. Today **44**(10), 36 (1991).

[2] J. P. Paz, S. Habib, and W. H. Zurek, Reduction of the wave packet: Preferred observable and decoherence time scale, Phys. Rev. D **47**, 488 (1993).

[3] B.-H. Liu, L. Li, Y.-F. Huang, C.-F. Li, G.-C. Guo, E.-M. Laine, H.-P. Breuer, and J. Piilo, Experimental control of the transition from Markovian to non-Markovian dynamics of open quantum systems, Nat. Phys. **7**, 931 (2011).

[4] A. Smirne, D. Brivio, S. Cialdi, B. Vacchini, and M. G. A. Paris, Experimental investigation of initial system-environment correlations via trace-distance evolution, Phys. Rev. A **84**, 032112 (2011).

[5] J. Piilo, S. Maniscalco, K. Harkonen, and K.-A. Suominen, Non-Markovian Quantum Jumps, Phys. Rev. Lett. **100**, 180402 (2008).

[6] A. Smirne, S. Cialdi, G. Anelli, M. G. A. Paris, and B. Vacchini, Quantum probes to experimentally assess correlations in a composite system, Phys. Rev. A **88**, 012108 (2013).

[7] C. W. Gardiner, *Harndbook of Stochastic Methods* (Springer, Berlin, 1983).

[8] J. Helm and W. T. Strunz, Quantum decoherence of two qubits, Phys. Rev. A **80**, 042108 (2009).

[9] J. Helm, W. T. Strunz, S. Rietzler, and L. E. Würflinger, Characterization of decoherence from an environmental perspective, Phys. Rev. A **83**, 042103 (2011).

[10] D. Crow and R. Joynt, Classical simulation of quantum dephasing and depolarizing noise, Phys. Rev. A **89**, 042123 (2014).

[11] W. M. Witzel, K. Young, and S. Das Sarma, Converting a real quantum spin bath to an effective classical noise acting on a central spin, Phys. Rev. B **90**, 115431 (2014).

[12] W. T. Strunz, L. Díosi, and N. Gisin, Open System Dynamics with Non-Markovian Quantum Trajectories, Phys. Rev. Lett. **82**, 1801 (1999).

[13] J. T. Stockburger and H. Grabert, Exact c-Number Representation of Non-Markovian Quantum Dissipation, Phys. Rev. Lett. **88**, 170407 (2002).

[14] O. Astafiev, Yu. A. Pashkin, Y. Nakamura, T. Yamamoto, and J. S. Tsai, Quantum Noise in the Josephson Charge Qubit, Phys. Rev. Lett. **93**, 267007 (2004).

[15] Y. M. Galperin, B. L. Altshuler, J. Bergli, and D. V. Shantsev, Non-Gaussian Low-Frequency Noise as a Source of Qubit Decoherence, Phys. Rev. Lett. **96**, 097009 (2006).

[16] B. Abel and F. Marquardt, Decoherence by quantum telegraph noise: A numerical evaluation, Phys. Rev. B **78**, 201302(R) (2008).

[17] T. Grotz, L. Heaney, and W. T. Strunz, Quantum dynamics in fluctuating traps: Master equation, decoherence, and heating, Phys. Rev. A **74**, 022102 (2006).

[18] Q. A. Turchette, C. J. Hyatt, B. E. King, C. A. Sackett, D. Kielpinski, W. M. Itano, C. Monroe, and D. J. Wineland, Decoherence and decay of motional quantum states of a trapped atom coupled to engineered reservoirs, Phys. Rev. A **62**, 053807 (2000).

[19] B. Bellomo, R. Lo Franco, and G. Compagno, Entanglement dynamics of two independent qubits in environments with and without memory, Phys. Rev. A **77**, 032342 (2008).

[20] L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen, and B. M. Garraway, Sudden death and sudden birth of entanglement in common structured reservoirs, Phys. Rev. A **79**, 042302 (2009).

[21] A. Rivas, S. F. Huelga, and M. B. Plenio, Entanglement and Non-Markovianity of Quantum Evolutions, Phys. Rev. Lett. **105**, 050403 (2010).

[22] G. Torre, W. Roga, and F. Illuminati, Non-Markovianity of Gaussian Channels, Phys. Rev. Lett. **115**, 070401 (2015).

[23] L. A. M. Souza, H. S. Dhar, M. N. Bera, P. Liuzzo-Scorpo, and G. Adesso, Gaussian interferometric power as a measure of continuous-variable non-Markovianity, Phys. Rev. A **92**, 052122 (2015).

[24] H.-P. Breuer, E.-M. Laine, and J. Piilo, Measure for the Degree of Non-Markovian Behavior of Quantum Processes in Open Systems, Phys. Rev. Lett. **103**, 210401 (2009).

[25] R. Vasile, S. Maniscalco, M. G. A. Paris, H.-P. Breuer, and J. Piilo, Quantifying non-Markovianity of continuous-variable Gaussian dynamical maps, Phys. Rev. A **84**, 052118 (2011).

[26] J. Liu, X.-M. Lu, and X. Wang, Nonunital non-Markovianity of quantum dynamics, Phys. Rev. A **87**, 042103 (2013).

[27] P. Haikka, J. D. Cresser, and S. Maniscalco, Comparing different non-Markovianity measures in a driven qubit system, Phys. Rev. A **83**, 012112 (2011).

[28] L. Mazzola, E.-M. Laine, H.-P. Breuer, S. Maniscalco, and J. Piilo, Phenomenological memory-kernel master equations and time-dependent Markovian processes, Phys. Rev. A **81**, 062120 (2010).

[29] S. Wißmann, H.-P. Breuer, and B. Vacchini, Generalized trace distance measure connecting quantum and classical non-Markovianity, Phys. Rev. A **92**, 042108 (2015).

[30] B. Bellomo, R. Lo Franco, and G. Compagno, Non-Markovian Effects on the Dynamics of Entanglement, Phys. Rev. Lett. **99**, 160502 (2007).

[31] C. Benedetti, M. G. A. Paris, and S. Maniscalco, Non-Markovianity of colored noisy channels, Phys. Rev. A **89**, 012114 (2014).

[32] F. Benatti, R. Floreanini, and M. Piani, Environment Induced Entanglement in Markovian Dissipative Dynamics, Phys. Rev. Lett. **91**, 070402 (2003).

[33] R. Lo Franco, B. Bellomo, E. Andersson, and G. Compagno, Revival of quantum correlations without system-environment back-action, Phys. Rev. A **85**, 032318 (2012).

[34] J. Trapani, M. Bina, S. Maniscalco, and M. G. A. Paris, Collapse and revival of quantum coherence for a harmonic oscillator interacting with a classical fluctuating environment, Phys. Rev. A **91**, 022113 (2015).

[35] W. Magnus, On the exponential solution of differential equations for a linear operator, Commun. Pure Appl. Math. **7**, 649 (1954).

[36] S. Blanes, F. Casas, J. A. Oteo, and J. Ros, The Magnus expansion and some of its applications, Phys. Rep. **470**, 151 (2008).

[37] A. S. Holevo and R. F. Werner, Evaluating capacities of bosonic Gaussian channels, Phys. Rev. A **63**, 032312 (2001).

[38] J. Harrington and J. Preskill, Achievable rates for the Gaussian quantum channel, Phys. Rev. A **64**, 062301 (2001).

[39] V. Giovannetti, S. Lloyd, L. Maccone, J. H. Shapiro, and B. J. Yen, Minimum Renyi and Wehrl entropies at the output of bosonic channels, Phys. Rev. A **70**, 022328 (2004).

[40] R. Simon, Peres-Horodecki Separability Criterion for Continuous Variable Systems, Phys. Rev. Lett. **84**, 2726 (2000).

[41] P. Giorda and M. G. A. Paris, Gaussian Quantum Discord, Phys. Rev. Lett. **105**, 020503 (2010).

[42] G. Adesso and A. Datta, Quantum versus Classical Correlations in Gaussian States, Phys. Rev. Lett. **105**, 030501 (2010).

[43] R. Blandino, M. G. Genoni, J. Etesse, M. Barbieri, M. G. A. Paris, P. Grangier, and R. Tualle-Brouri, Homodyne Estimation of Gaussian Quantum Discord, Phys. Rev. Lett. **109**, 180402 (2012).

[44] M. Brunelli, C. Benedetti, S. Olivares, A. Ferraro, and M. G. A. Paris, Single- and two-mode quantumness at a beam splitter, Phys. Rev. A **91**, 062315 (2015).

[45] S. Pirandola, G. Spedalieri, S. L. Braunstein, N. J. Cerf, and S. Lloyd, Optimality of Gaussian Discord, Phys. Rev. Lett. **113**, 140405 (2014).

[46] F. Ciccarello and V. Giovannetti, Creating quantum correlations through local nonunitary memoryless channels, Phys. Rev. A **85**, 010102(R) (2012).

[47] A. Rivas, S. F. Huelga, and M. B. Plenio, Quantum non-Markovianity: characterization, quantification and detection, Rep. Prog. Phys. **77**, 094001 (2014).

[48] P. Haikka and S. Maniscalco, Non-Markovian Quantum Probes, Op. Sys. Inf. Dyn. **21**, 1440005 (2014).

[49] B. Bylicka, D. Chruściński, and S. Maniscalco, Non-Markovianity and reservoir memory of quantum channels: A quantum information theory perspective, Sci. Rep. **4**, 5720 (2014).

[50] E.-M. Laine, J. Piilo, and H.-P. Breuer, Measure for the non-Markovianity of quantum processes, Phys. Rev. A **81**, 062115 (2010).

[51] P. Marian and T. A. Marian, Uhlmann fidelity between two-mode Gaussian states, Phys. Rev. A **86**, 022340 (2012).

[52] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Non-Markovian dynamics in open quantum systems, arXiv:1505.01385.