PHYSICAL REVIEW A **96**, 012109 (2017)

Momentum coupling in non-Markovian quantum Brownian motion

Luca Ferialdi*

Department of Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia
and Department of Physics, University of Trieste, Strada costiera 11, 34151 Trieste, Italy

Andrea Smirne†

Institute of Theoretical Physics, Universität Ulm, Albert-Einstein-Allee 11D-89069 Ulm, Germany

(Received 15 May 2017; published 11 July 2017)

We consider a model of non-Markovian quantum Brownian motion that consists of a harmonic oscillator bilinearly coupled to a thermal bath, both via its position and via its momentum operators. We derive the master equation for such a model, and we solve the equations of motion for a generic Gaussian system state. We then investigate the resulting evolution of the first and second moments for both an Ohmic and a super-Ohmic spectral density. In particular, we show that, irrespective of the specific form of the spectral density, the coupling with the momentum enhances the dissipation experienced by the system, accelerating its relaxation to the equilibrium as well as modifying the asymptotic state of the dynamics. Eventually, we characterize explicitly the non-Markovianity of the evolution using a general criterion which relies on the positivity of the master equation coefficients.

DOI: 10.1103/PhysRevA.96.012109

I. INTRODUCTION

Every quantum device unavoidably interacts with the surrounding environment, which affects its dynamics. In general, such open systems are described by non-Markovian dynamics, which account for the memory effects involved in the influence of the environment on the relevant system [1,2]. These dynamics constitute a very large class of open-system evolutions (see the recent reviews [3–5]), and in order to investigate them it can be thus useful to focus on specific models. A commonly used model is provided by a system bilinearly coupled to a bath of harmonic oscillators [6–8]. This model is at the same time physically meaningful and mathematically treatable in detail. One of the most important results of this model is the so-called non-Markovian Brownian motion [8–10] where one considers a harmonic oscillator bilinearly coupled to a thermal bath via its position. In their seminal paper Hu *et al.* [8] provided the exact master equation for the non-Markovian Brownian motion and analyzed its properties.

Thanks to a recent paper [11], exact results have been extended to a wider class of systems, including a more general form of the coupling between the system and the environment; interestingly, the same analytical approach provides approximate results for finite-dimensional systems [12]. The aim of this paper is to exploit these results to take a step forward in the understanding of non-Markovian dynamics by investigating a nonstandard model for non-Markovian Brownian motion. We consider a harmonic oscillator bilinearly coupled to a thermal bath, both via its position and via its momentum. Since the non-Markovian behavior is connected strictly to memory features of the bath, it is interesting to understand how a "dynamical" system-bath coupling affects the dynamics of the system. In particular, we compare this extended model with the standard non-Markovian quantum Brownian motion, focusing on the new features of the dynamics provided by the momentum coupling. We derive the master equation fixing the open-system evolution, and we describe the corresponding evolutions for the position and momentum expectation values and variances and for the position-momentum covariance; indeed, since the dynamics preserves the Gaussian form of the reduced states, this fully characterizes the solution of the master equation for this class of states. Finally, we show explicitly the non-Markovian nature of the dynamics using the criterion for open quantum system dynamics introduced in Ref. [13].

Let us mention that the coupling with the system's momentum has been considered in phenomenological models based on Lindblad equations [14,15] and stochastic Schrödinger equations (both for Markovian [16] and for non-Markovian systems [17]). Moreover, the dissipative effects due to the coupling with the momentum instead of the position (the so-called "anomalous dissipation") has been investigated within the context of tunneling in Ref. [18], whereas the resulting thermodynamical properties have been treated in Ref. [19]; eventually, the coupling of both the system position and the momentum to the bath has been considered in Ref. [20] to characterize the dynamics of the relative phase in a Josephson junction, including both the fluctuations of the radiation field and the quasiparticle tunneling. These models indeed provide some significant examples of specific physical systems to which the analysis of the present paper may be applied.

The rest of the paper is organized as follows: In Sec. II we introduce the model, we derive the exact master equation and the evolution of relevant physical quantities. In Sec. III we provide a detailed analysis of the model under study for different spectral densities, and we compare its features with the standard quantum Brownian motion. In Sec. IV we write down the semigroup limit of the dynamics for a δ-like correlation function of the bath and discuss the non-Markovian nature of the dynamics in the other cases. In Sec. V we draw the conclusions.

*ferialdi@ts.infn.it
†andrea.smirne@uni-ulm.de

II. THE MODEL AND ITS SOLUTION

We investigate the dynamics of a harmonic oscillator bilinearly coupled to a bosonic thermal bath via a linear combination of its position and momentum operators as described by the total Hamiltonian $\hat{H} = \hat{H}_S + \hat{H}_I + \hat{H}_E$ with

$$\hat{H}_S = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_S^2\hat{q}^2, \tag{1}$$

$$\hat{H}_I = (\hat{q} - \mu\hat{p})\sum_k c_k\hat{q}_k, \tag{2}$$

$$\hat{H}_E = \sum_k \frac{\hat{p}_k^2}{2m_k} + \frac{1}{2}\omega_k^2\hat{q}_k^2, \tag{3}$$

where ω_S is the free frequency of the harmonic oscillator, m is its mass, whereas ω_k and m_k are the frequency and mass, respectively, of the kth bath mode; indeed, $\hat{q}$ and $\hat{p}$ ($\hat{q}_k$ and $\hat{p}_k$) are the system (kth bath mode) position and momentum operators. Furthermore, μ is the parameter providing us with the relative strength of the coupling with the system momentum with respect to the coupling with the system position; as said, the effects induced by a nonzero value of the coupling μ will be one of the main focuses of our following analysis. The bath is assumed to have a Gaussian (thermal) initial state,

$$\rho_E(0) = \frac{e^{-\beta\hat{H}_E}}{Z}, \quad Z = \mathrm{Tr}_E[e^{-\beta\hat{H}_E}], \tag{4}$$

and its action on the open system is characterized completely by the spectral density,

$$J(\omega) = \sum_k \frac{c_k^2}{2m_k\omega_k}\delta(\omega - \omega_k), \tag{5}$$

or, equivalently, by the two-point correlation function [1],

$$D(t-s) = \hbar\int_0^\infty d\omega\, J(\omega)\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\omega(t-s) - i\sin\omega(t-s)\right]. \tag{6}$$

Before presenting the master equation and its solution for the model, let us note that with the canonical change in variables $(\hat{q},\hat{p}) \mapsto (\hat{x} = \hat{q} - \mu\hat{p},\hat{p})$, one equivalently can describe the equations of motion using the global Hamiltonian with the same $\hat{H}_E$ but where only the system operator $\hat{x}$ is coupled to the bath operator $\sum_k c_k\hat{q}_k$, whereas the system-free Hamiltonian is given by

$$\hat{H}'_S = \frac{\hat{p}^2}{2m'} + V(\hat{x},\hat{p}), \tag{7}$$

with

$$m' = \left(\frac{1}{m} + m\omega_S^2\mu^2\right)^{-1},$$
$$V(\hat{x},\hat{p}) = \frac{m\omega_S^2}{2}\hat{x}^2 + \frac{m\omega_S^2\mu}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}). \tag{8}$$

We stress that, although one can recover a position-position coupling by means of a unitary transformation, the system we consider here is fundamentally different from the standard quantum Brownian motion [8], the difference being now enclosed in a momentum-dependent-free Hamiltonian of the system.

A. Master equation

It has recently been shown [11,21] that the exact master equation for the model fixed by the total Hamiltonian given by Eqs. (1)–(3), obtained after averaging out the environmental degrees of freedom, reads

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}(t),\hat{\rho}] + i\,\Xi_\mu(t)[\hat{q}^2,\hat{\rho}] + i\,\Upsilon_\mu(t)[\hat{q},\{\hat{p},\hat{\rho}\}] + \Gamma_\mu(t)[\hat{q},[\hat{q},\hat{\rho}]] + \Theta_\mu(t)[\hat{q},[\hat{p},\hat{\rho}]] + \gamma_\mu(t)[\hat{p},[\hat{p},\hat{\rho}]]. \tag{9}$$

The second term on the right-hand side yields a bath-induced frequency renormalization of the oscillator, whereas the third term is a dissipative contribution since it is responsible for damping of the momentum expectation value. The terms displayed by the second line of Eq. (9) describe diffusion both in position and in momentum. The subscript μ denotes the fact that we are considering the unusual coupling (2). We remark that Eq. (9) provides us with the most general form of a time-local generator such that, at any time, the operators in the dissipator are linear in the position and momentum operators whereas the Hamiltonian term is at most quadratic with respect to them [14]. Since the expressions of the functions displayed by the master equations (9) as provided in Ref. [11] have rather complicated expressions, it is useful to re-derive them in a more convenient way. We do so by solving the Heisenberg equations of motion of the system by exploiting the Laplace transform $\mathcal{L}$. By introducing the shifted system frequency [8],

$$\omega_R = \sqrt{\omega_S^2 + \frac{2}{m}\int d\omega\frac{J(\omega)}{\omega}}, \tag{10}$$

and

$$\tilde{D}(l) = \mathcal{L}[D^{\mathrm{Im}}(t)], \tag{11}$$

$$G(t) = \mathcal{L}^{-1}\left[\frac{1}{l^2 + \omega_R^2 + \frac{2}{m'}\tilde{D}(l)}\right], \tag{12}$$

one finds that the solution of the equations of motion reads

$$\hat{q}(t) = G_1(t)\hat{q}(0) + G_2(t)\hat{p}(0) - \int_0^t G_3(t-s)\hat{\phi}(s)ds, \tag{13}$$

$$\hat{p}(t) = G_4(t)\hat{q}(0) + G_5(t)\hat{p}(0) - \int_0^t G_6(t-s)\hat{\phi}(s)ds, \tag{14}$$

where $\hat{\phi}(t)$ denotes the bath coupling operator freely evolved at time t,

$$\hat{\phi}(t) \equiv \sum_k c_k\hat{q}_k(t) = \sum_k c_k\left(\hat{q}_k(0)\cos\omega_k t + \frac{\hat{p}_k(0)}{m_k}\sin\omega_k t\right), \tag{15}$$

and the Green's functions G_i read

$$
\begin{aligned}
G_1(t) &= \dot{G}(t) - 2\mu \int_0^t D^{\mathrm{Im}}(t-s)G(s)ds, \\
G_2(t) &= \frac{G(t)}{m} + 2\mu^2 \int_0^t D^{\mathrm{Im}}(t-s)G(s)ds, \\
G_3(t) &= \frac{G(t)}{m} + \mu\dot{G}(t), \\
G_4(t) &= -m\omega^2 G(t) - 2\int_0^t D^{\mathrm{Im}}(t-s)G(s)ds, \\
G_5(t) &= \dot{G}(t) + 2\mu \int_0^t D^{\mathrm{Im}}(t-s)G(s)ds, \\
G_6(t) &= -m\omega^2\mu G(t) + \dot{G}(t).
\end{aligned} \tag{16}
$$

In order to derive the master equation (9) and its coefficients, it is convenient to introduce the characteristic operator,

$$\hat{\chi}(t) = \mathrm{Tr}_E[e^{i\lambda\hat{q}(t)+i\gamma\hat{p}(t)}\hat{\rho}_E(0)]. \tag{17}$$

We now adopt the strategy outlined in Ref. [22]: We differentiate $\hat{\chi}(t)$ with respect to t, and we replace the terms of types $\hat{q}\hat{\chi}(t)$ and $\hat{p}\hat{\chi}(t)$ by suitable combinations of $d\hat{\chi}(t)/d\lambda$ and $d\hat{\chi}(t)/d\gamma$. The equation obtained is rewritten in terms of $\hat{\chi}(0)$ by exploiting the composition property of the adjoint map for $\hat{\chi}$. After some manipulations we are able to express $d\hat{\chi}(t)/dt$ in terms of (anti)commutators of $\hat{q}$ and $\hat{p}$ with $\hat{\chi}(0)$. We exploit the following relation:

$$\mathrm{Tr}_S\left[\frac{d\hat{\chi}(t)}{dt}\hat{\rho}(0)\right] = \mathrm{Tr}_S\left[\hat{\chi}(0)\frac{d\hat{\rho}(t)}{dt}\right], \tag{18}$$

and after some lengthy calculations, this procedure eventually provides us with Eq. (9) with

$$
\begin{aligned}
\hat{H}(t) &= \hat{H}_S + \frac{\hbar\mu}{2m}\frac{H_1(t)}{F(t)}\hat{p}^2 \\
&\quad + \frac{\hbar}{2}\left(m\omega^2\mu^2\frac{H_1(t)}{F(t)} + \mu\frac{H_2(t)}{F(t)}\right)\{\hat{q},\hat{p}\}, \\
\Gamma_\mu(t) &= \frac{\dot{g}_2(t)}{\hbar^2} - K_4(t)\frac{g_3(t)}{\hbar^2} - 2K_2(t)\frac{g_2(t)}{\hbar^2}, \\
\Theta_\mu(t) &= -\frac{\dot{g}_3(t)}{\hbar^2} + 2K_1(t)\frac{g_2(t)}{\hbar^2} + K_5(t)\frac{g_3(t)}{\hbar^2}, \\
\Xi_\mu(t) &= \frac{1}{2}\frac{H_2(t)}{F(t)}, \\
\Upsilon_\mu(t) &= K_5(t), \\
\gamma_\mu(t) &= \frac{\dot{g}_1(t)}{\hbar^2} - K_1(t)\frac{g_3(t)}{\hbar^2} - 2K_3(t)\frac{g_1(t)}{\hbar^2}.
\end{aligned} \tag{19}
$$

The explicit expressions for the functions displayed by these equations are provided in Appendix A. We stress that the expressions for these functions are exact and that when $\mu = 0$ they recover those for non-Markovian Brownian motion [8,9,22] as expected.

B. Time evolution of the position and momentum first and second moments

The advantage of having solved the equations of motion in the Heisenberg picture is that they easily allow us to compute the expected values of relevant operators. The expectation values for $\hat{q}$ and $\hat{p}$ follow straightforwardly from Eqs. (13) and (14) by observing that the expectation of $\hat{\phi}$ is null:

$$
\begin{aligned}
q_a(t) &= G_1(t)q_a + G_2(t)p_a, \\
p_a(t) &= G_4(t)q_a + G_5(t)p_a,
\end{aligned} \tag{20}
$$

where we defined $q_a(t) \equiv \mathrm{Tr}[\hat{q}(t)\rho]$ and $p_a(t) \equiv \mathrm{Tr}[\hat{p}(t)\rho]$ with ρ as initial state of the system (the initial time argument will be implied from now on). The evolution of the position variance $\sigma_{q^2}(t) \equiv \mathrm{Tr}[\hat{q}(t)^2\rho] - q_a(t)^2$ is obtained by squaring Eq. (13) and taking the expectation value and similarly for the momentum variance $\sigma_{p^2}(t) \equiv \mathrm{Tr}[\hat{p}(t)^2\rho] - p_a(t)^2$ and the position-momentum covariance $\sigma_{qp}(t) \equiv \mathrm{Tr}[\{\hat{q}(t),\hat{p}(t)\}\rho]/2 - q_a(t)p_a(t)$. In conclusion, one has that the elements of the covariance matrix are given by

$$
\begin{aligned}
\sigma_{q^2}(t) &= G_1^2(t)\sigma_{q^2} + G_2^2(t)\sigma_{p^2} + 2G_1(t)G_2(t)\sigma_{qp} - 2g_1(t), \\
\sigma_{p^2}(t) &= G_4^2(t)\sigma_{q^2} + G_5^2(t)\sigma_{p^2} + 2G_4(t)G_5(t)\sigma_{qp} - 2g_2(t), \\
\sigma_{qp}(t) &= G_1(t)G_4(t)\sigma_{q^2} + G_2(t)G_5(t)\sigma_{p^2}, \\
&\quad + [G_1(t)G_5(t) + G_2(t)G_4(t)]\sigma_{qp} - g_3(t).
\end{aligned} \tag{21}
$$

By virtue of these equations we can determine the position and momentum expectation values and covariance matrix at any time t and hence any observable associated with the system's evolution as long as one restricts to a Gaussian initial state. Indeed, a crucial feature of the model at hand is that the Gaussianity is preserved by the dynamics as a consequence of the bilinear structure of the global Hamiltonian.

III. EXAMPLES OF TIME EVOLUTIONS FOR AN OHMIC AND A SUPER-OHMIC SPECTRAL DENSITY

In this section, we provide some examples of the evolution of the position and momentum expectation values and variances as well as the position-momentum covariance focusing on the features which trace back to the introduction of the coupling to the system's momentum, i.e., to $\mu \neq 0$.

To get an explicit expression of the functions $G_i(t)$ and $g_i(t)$ in Eqs. (20) and (21), we need to specify the form of the spectral density which encloses the effects of the interaction with the environment on the system dynamics. We will consider the standard case given by [8]

$$J(\omega) = \frac{2m\gamma}{\pi}\omega\left(\frac{\omega}{\Omega}\right)^{s-1} e^{-\omega^2/\Omega^2}, \tag{22}$$

where Ω is the cutoff frequency, γ fixes the global coupling strength, whereas s determines the low-frequency behavior and often is referred to as the Ohmicity parameter: For $s = 1$ one says that $J(\omega)$ in an Ohmic spectral density, whereas for $s > 1$ ($s < 1$) one speaks about super-Ohmic (sub-Ohmic) spectral density.

A. Ohmic spectral density

We start by taking into account the Ohmic case, i.e., $s = 1$. This spectral density is known to provide the semigroup description of the open-system dynamics in the infinite temperature and infinite cutoff limits [1,6,23], and then it provides us with a natural reference case. Note that the

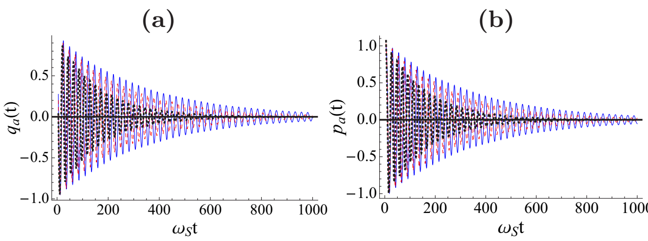

FIG. 1. Evolution in time of the expectation value of (a) position and (b) momentum [see Eq. (20)] under an Ohmic spectral density $s = 1$ in Eq. (22). The different lines correspond to different values of the coupling strength with the system momentum $\mu = 0$ (blue solid line), $m\mu\omega_S = 0.5$ (red dashed line), $m\mu\omega_S = 1$ (black dotted line); the other parameters are $\gamma/\omega_S = 3\times10^{-3}$, $\Omega/\omega_S = 20$, and $\hbar\beta\omega_S = 10^{-2}$, whereas as initial conditions we set $\sqrt{m\omega_S/\hbar}q_a = 1$, $p_a/\sqrt{m\omega_S\hbar} = 10^{-2}$, and $(m\omega_S/\hbar)\sigma_{q^2} = 0.5$; the expectation values of position and momentum are expressed in units of $\sqrt{\hbar/(m\omega_S)}$ and $\sqrt{m\omega_S\hbar}$, respectively.

mentioned semigroup limit is obtained also for $\mu \neq 0$ as stated in Ref. [23] and explicitly shown later on.

First, in Figs. 1(a) and 1(b) we see the time evolution of the expectation values of position and momentum, respectively, for different values of the coupling parameter μ. In both cases, and for any value of μ, we have decaying oscillations to the asymptotic value zero. On the other hand, the introduction of a coupling with the system momentum accelerates the relaxation process of both quantities, which is faster the higher the value of μ. The coupling with the momentum brings along a further contribution to the friction experienced by the open system due to its coupling with the environment so that the damping of the momentum itself is enhanced. Indeed, referring to the master equation (9), it is clear how this phenomenon can be traced back to the changes in the friction coefficient $\Upsilon_\mu(t)$, which now depends on the coupling μ (all the other terms vanish when one takes the expectation value with the momentum operator).

Now, let us move our numerical analysis to the elements of the system covariance matrix, which, as said, completes the description of the reduced observables if we restrict to Gaussian states. In Figs. 2(a)–2(c), we report the evolution of $\sigma_{q^2}(t)$, $\sigma_{p^2}(t)$, and $\sigma_{qp}(t)$, respectively, for different values of μ. Once again, we note how the relaxation toward the asymptotic value is faster the higher the strength of the momentum coupling. However, now the asymptotic values themselves of $\sigma_{p^2}(t)$ and $\sigma_{qp}(t)$ are drastically changed by a nonzero value of μ: The former is decreased, whereas the latter is increased. The asymptotic expectation value of the system kinetic-energy $\hat{p}^2/(2m)$ and, as a consequence, the asymptotic expectation value of the overall system free energy $\hat{H}_S$ in Eq. (1) is progressively decreased by an increasing value of μ: The coupling with the momentum intensifies and accelerates the dissipation of the open system. In addition, the whole evolution of $\sigma_{qp}(t)$ is modified qualitatively: We have a (nonmonotonic, see the inset) relaxation to the 0 value for $\mu = 0$, whereas there is a monotonically increasing evolution to a nonzero asymptotic value for $\mu \neq 0$; note that such monotonicity can be lost for different initial conditions (see below). The coupling with the momentum and the subsequent new terms in the master equation (9) imply that the Gibbs state

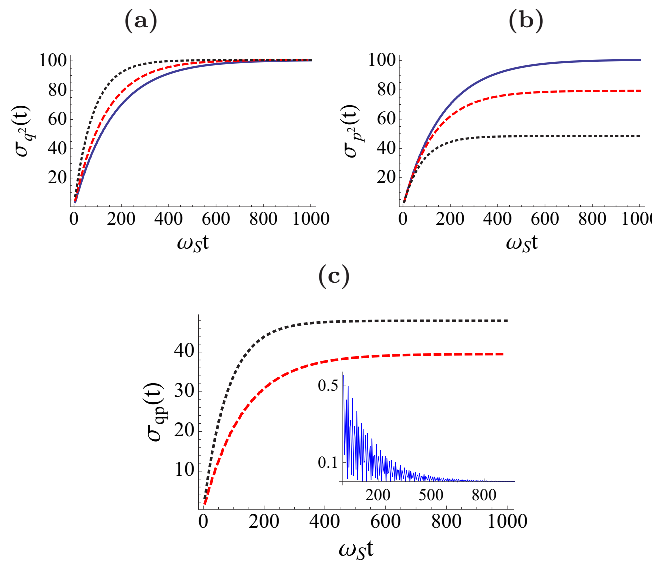

FIG. 2. Evolution in time of the elements of the covariance matrix, see Eq. (21), under an Ohmic spectral density $s = 1$ in Eq. (22): variance of the position $\sigma_{q^2}(t)$ in (a), variance of the momentum $\sigma_{p^2}(t)$ in (b), and position-momentum covariance $\sigma_{qp}(t)$ in (c). The different lines correspond to $\mu = 0$ (blue solid line), $m\mu\omega_S = 0.5$ (red dashed line), and $m\mu\omega_S = 1$ (black dotted line). The other parameters are as in Fig. 1; the position variance is expressed in units of $\hbar/(m\omega_S)$, the momentum variance is expressed in units of $m\omega_S\hbar$, and the position-momentum covariance is expressed in units of $\hbar$. The inset in (c) magnifies the case of $\mu = 0$.

is no longer the equilibrium state of the reduced dynamics, which, instead, exhibits a nonzero value of σ_{qp} [14]. Overall, the introduction of $\mu \neq 0$ squeezes the momentum uncertainty of the asymptotic state and adds a nontrivial correlation among the momentum and position statistics.

Until now, we have considered the evolution of the momentum and position expectation values and covariances for a fixed initial condition. Additionally, we verified numerically that the discussed asymptotic values do not depend on the initial conditions (at least, as long as one stays within the set of initial Gaussian states). Representative examples are given in Figs. 3(a) and 3(b) for the evolution of $\sigma_{p^2}(t)$ with $\mu = 0$ and $\mu \neq 0$, respectively, and in Fig. 3(c) for the position variance with $\mu = 0$; fully analogous results hold for the other elements of the covariance matrix and for the expectation values (for the considered values of the model parameters). Thus, the system relaxes to a unique asymptotic state, both for $\mu = 0$ and for $\mu \neq 0$; indeed, as previously shown, such a state will be different in the two cases.

Moreover, from Figs. 3(a)–3(c) we can observe that, for certain initial conditions, the position and momentum variances also relax to the asymptotic value in a nonmonotonic way as we already observed for the expectation values. Each variance can show even strong oscillations when its initial value is high enough and the oscillations are wider the higher such an initial value is. Comparing Figs. 3(a) and 3(b), one can see how the feature is present both for $\mu = 0$ and for $\mu \neq 0$. The only effect of the coupling to the system momentum is the

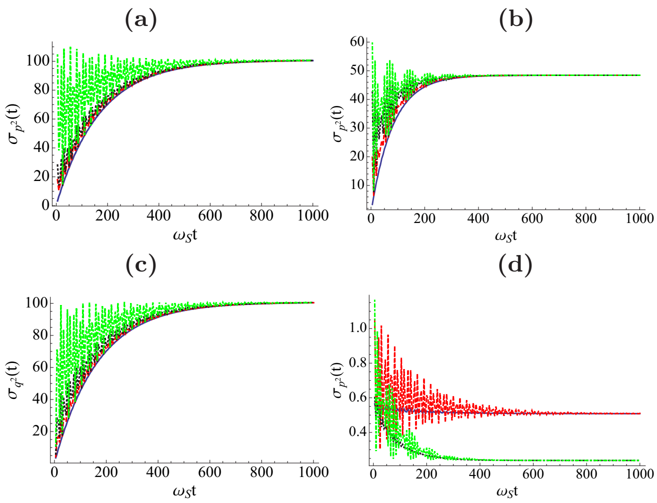

FIG. 3. Relaxation to the equilibrium of the momentum variance for (a) $\mu = 0$ and (b) $m\mu\omega_S = 1$ and relaxation to the equilibrium of the position variance for (c) $\mu = 0$ for an Ohmic spectral density. The different lines correspond to different initial Gaussian states. (d) Relaxation to the equilibrium of the momentum variance at zero-temperature $T = 0$ for an Ohmic spectral density and $\mu = 0$ [black (solid line) and red (dashed line)] and $m\mu\omega_S = 1$ [black (dotted line) and green (dot-dashed line)]; for each value of μ, the two lines correspond to different initial Gaussian states; the other parameters are the same as in Fig. 2.

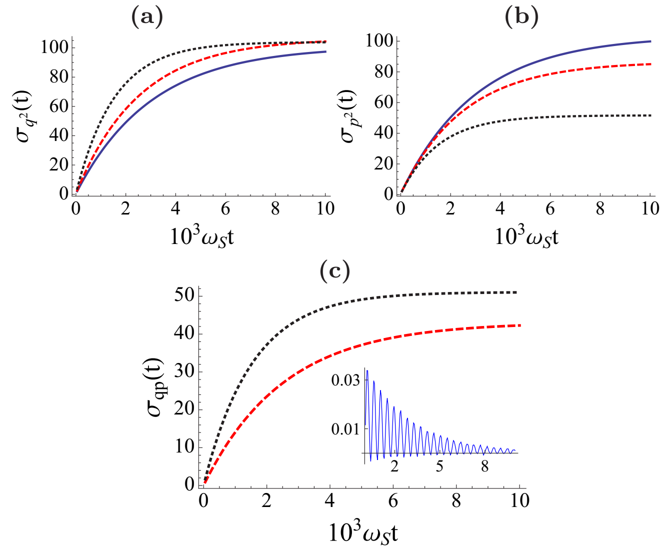

FIG. 4. Evolution in time of the elements of the covariance matrix, see Eq. (21), under a super-Ohmic spectral density $s = 2$ in Eq. (22): variance of the position $\sigma_{q^2}(t)$ in (a), variance of the momentum $\sigma_{p^2}(t)$ in (b), and position-momentum correlation $\sigma_{qp}(t)$ in (c). The different lines correspond to $\mu = 0$ (blue solid line), $m\mu\omega_S = 0.5$ (red dashed line), and $m\mu\omega_S = 1$ (black dotted line); the other parameters are as in Fig. 1 apart from $\gamma/\omega_S = 3.4\times10^{-3}$. The inset in (c) magnifies the case of $\mu = 0$.

appearance of some beats in the oscillating evolutions of the variances.

Note that all the previous examples concern the high-T regime. Nevertheless, the results in Eqs. (20) and (21) are referred to a completely generic temperature. In particular, one readily can see how the evolution of the momentum and position expectation values is not affected by a change in T [since the two quantities do not depend on $D^{\mathrm{Re}}(t)$, see Eqs. (12), (17), and (20)]. On the other hand, the temperature influences the evolution of the elements of the covariance matrix and, especially, their asymptotic values. In Fig. 3(d), we study the relaxation to the equilibrium of the momentum variance for different initial conditions and different values of μ at $T = 0$. Of course, the zero-temperature environment makes the system's momentum variance relax to a smaller value, compared to the high-T regime, whereas the qualitative behavior of the whole time evolution is rather similar for the two temperature regimes. Importantly for our purposes, we note that also for $T = 0$, as previously described for the high-T regime, introducing a nonzero value of μ affects the relaxation process by accelerating it and changing the asymptotic values; Fig. 3(d) shows how the asymptotic value of $\sigma_{p^2}(t)$ for $\mu \neq 0$ is decreased with respect to the case of $\mu = 0$. Finally, we also recover that a nonzero value of μ may induce some beats in the oscillating evolution of $\sigma_{p^2}(t)$.

B. Super-Ohmic spectral density

Here, we examine the behavior of the system's first and second moments for a non-Ohmic spectral density in order to show that the conclusions we drew previously about the effects of the coupling $\mu \neq 0$ do not depend on the peculiar case given by the Ohmic spectral density. Besides s, the other parameters are the same as those of the previous paragraph with the exception of the coupling constant γ, which has been set so to keep the overall strength of the coupling to the bath unchanged as quantified by $\int d\omega\, J(\omega)$. Note also that the renormalized frequency changes due to the different spectral density, see Eq. (10).

In particular, we considered the case of $s = 2$, i.e., a super-Ohmic spectral density. The most relevant effect due to the transition from an Ohmic to a super-Ohmic spectral density is that the dynamics is slowed down strongly. This can be observed from the plots in Fig. 4 (note the different scales in the time axis compared to the plots in Fig. 2) where we reported the evolution of the position and momentum variances and covariance; indeed, the same behavior could be observed looking at the momentum and position expectation values. The slowing down of the system dissipation, which already is well known [8] in the case of $\mu = 0$, remains essentially unaltered, i.e., on the same time scales, also in the presence of the coupling with the system momentum. On the other hand, one can see how a nonzero value of μ introduces some changes in the system dynamics, which are essentially the same as for the Ohmic case. The relaxation process is accelerated with respect to $\mu = 0$ due to the further contributions to friction and dissipation: The asymptotic values are approached in a shorter time, and the asymptotic value of the system free energy is smaller the higher μ. Moreover, as for the Ohmic case, the evolution of $\sigma_{qp}(t)$ also is modified qualitatively, leading to an asymptotic nonzero value.

The asymptotic values are increased slightly by the super-Ohmicity of the spectral density; nevertheless, the effects of

TABLE I. Ratio among the asymptotic values for the momentum variance $\sigma^{\infty}_{p^2}(m\mu\omega_S)$ and the position-momentum covariance $\sigma^{\infty}_{qp}(m\mu\omega_S)$ (the position variance does not change) for different values of μ for the Ohmic (left column) and the super-Ohmic (right column) spectral densities.

Asymptotic ratio	$s=1$	$s=2$
$\sigma^{\infty}_{p^2}(0)/\sigma^{\infty}_{p^2}(0.5)$	1.20	1.27
$\sigma^{\infty}_{p^2}(0)/\sigma^{\infty}_{p^2}(1)$	2.00	2.07
$\sigma^{\infty}_{qp}(0.5)/\sigma^{\infty}_{qp}(1)$	0.84	0.82

$\mu \neq 0$ are even quantitatively very close to the Ohmic case: The ratio among the asymptotic values for different values of μ is approximately the same for the Ohmic and the super-Ohmic cases as shown in Table I.

Finally, we also checked the relaxation to a unique asymptotic state within the set of initial Gaussian conditions. In Figs. 5(a) and 5(b), we reported the evolution of the momentum variance for $\mu = 0$ and $\mu \neq 0$, respectively. In both cases, one has a convergence to the same asymptotic value on longer time scales. Moreover, we note that, also in the super-Ohmic case, high enough initial values of the variance lead to an oscillating behavior for both $\mu = 0$ and $\mu \neq 0$. A nonzero value of μ now increases the amplitude of the oscillations for certain initial conditions but without leading to the appearance of the beats as in the Ohmic case.

IV. NON-MARKOVIANITY OF THE DYNAMICS

In this section we show explicitly that the dynamics of the model we are describing is generally non-Markovian, according to one of the definite notions of quantum Markovianity which have been widely discussed in the literature (see Refs. [3–5] and references therein). In particular, we will adopt the definition which identifies quantum Markovian dynamics with those dynamics characterized by a time-local master equation with (possibly time-dependent) positive coefficients [13]. We first briefly recall the definition for finite-dimensional systems, and then we apply it to the system we are dealing with here.

Hence, consider the open-system dynamics described by the one-parameter family of completely positive (CP) maps

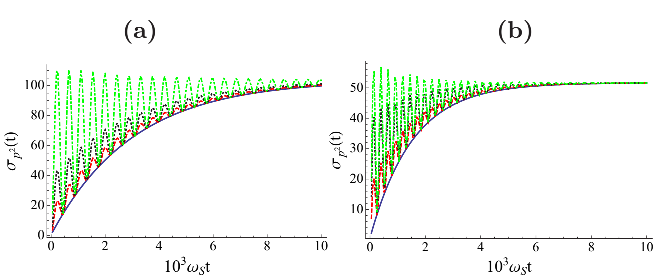

FIG. 5. Relaxation to the equilibrium of the momentum variance for (a) $\mu = 0$ and (b) $\mu = 10^{-2}$ for a super-Ohmic spectral density with $s = 2$; the different lines correspond to different initial Gaussian states. The other parameters are the same as in Fig. 4.

$\{\Lambda(t)\}_{t\geqslant 0}$ and the associated time-local master equation,

$$\frac{d}{dt}\rho(t) = \mathcal{K}(t)\rho(t); \tag{23}$$

the possible presence of times where the time-local generator $\mathcal{K}(t)$ does not exist would not affect the following discussion. Now, given a system associated with the finite-dimensional Hilbert space $\mathbb{C}^N$, the time-local generator $\mathcal{K}(t)$ can always be written in the form

$$\mathcal{K}(t)\rho = -i[\hat{H},\rho] + \sum_{ij=1}^{N^2} a_{ij}(t)\left(\hat{G}_i\rho\hat{G}_j^{\dagger} - \frac{1}{2}\{\hat{G}_j^{\dagger}\hat{G}_j,\rho\}\right), \tag{24}$$

as a consequence of trace and Hermiticity preservation [24]. Here, $\hat{H}$ is a Hermitian operator, $\{G_i\}_{i=1,\dots,N^2}$ is a generic basis in the set of linear operators on $\mathbb{C}^N$, and the coefficients $a_{ij}(t)$ define a Hermitian matrix, the so-called Kossakowski matrix, at any time t: Set $[A(t)]_{ij} \equiv a_{ij}(t)$, and one has $A^{\dagger}(t) = A(t)$. Therefore, one can always diagonalize $A(t)$ via the unitary matrix $V(t)$ so that $A(t) = V(t)D(t)V^{\dagger}(t)$ with $D(t) = \mathrm{diag}\{d_1(t),\dots,d_{N^2}(t)\}$ and the $d_i(t)$'s are real functions of time. As a consequence, introducing the time-dependent (Lindblad) operators $L_i(t) = \sum_j U_{ji}(t)G_j$, one gets the canonical diagonal form [25] of the time-local generator,

$$\mathcal{K}(t)\rho = -i[\hat{H},\rho] + \sum_{i=1}^{N^2} d_i(t)\left(\hat{L}_i(t)\rho\hat{L}_i^{\dagger}(t) - \frac{1}{2}\{\hat{L}_i^{\dagger}(t)\hat{L}_i(t),\rho\}\right). \tag{25}$$

Now, the definition introduced in Ref. [13] identifies Markovian dynamics with those dynamics where $d_i(t) \geqslant 0$ for any i and for any time t. In the special case of constant positive coefficients, we thus recover the Lindblad master equation [1,24,26], which corresponds to the case of a Markovian time-homogeneous dynamics. The mentioned definition further identifies Markovian time-inhomogenous dynamics with those given by a master equation with time-dependent positive coefficients. Finally, the presence of time intervals where some coefficient is negative is equivalent to the occurrence of a non-Markovian dynamics. Indeed, the condition about the positivity of the coefficients of the diagonal form of the time-local generator in Eq. (25) can be expressed equivalently in terms of the positive definitiveness of the Kossakowski matrix $A(t)$ in Eq. (24).

In order to extend the previous definition to the open-system dynamics we are studying here, which involves a master equation for an infinite-dimensional space and with unbounded operators, we simply can proceed as follows. We rewrite our master equation in the nondiagonal form:

$$\frac{d\hat{\rho}}{dt} = -i[\tilde{\hat{H}},\hat{\rho}] + \sum_{i,j} a_{ij}(t)\left(\hat{F}_i\hat{\rho}\hat{F}_j - \frac{1}{2}\{\hat{F}_j\hat{F}_i,\hat{\rho}\}\right), \tag{26}$$

with $\hat{F}_1 = \hat{q}$, $\hat{F}_2 = \hat{p}$,

$$\tilde{\hat{H}} = \hat{H}(t) - \hbar\,\Xi_{\mu}(t)\hat{q}^2 - \frac{\hbar}{2}\Upsilon_{\mu}(t)\{\hat{q},\hat{p}\}, \tag{27}$$

and $a_{ij}(t)$ matrix elements of

$$A(t) = \begin{pmatrix} -2\Gamma_\mu(t) & -\Theta_\mu(t) + i\Upsilon_\mu(t) \\ -\Theta_\mu(t) - i\Upsilon_\mu(t) & -2\gamma_\mu(t) \end{pmatrix}. \quad (28)$$

As is common in the literature we will still call this the Kossakowski matrix, although it is not referred to as a basis in the linear space of operators on the (infinite-dimensional) Hilbert space associated with our system. Now, we can identify Markovian dynamics precisely with those dynamics where the Kossakowski matrix $A(t)$ is positive definite, and hence the resulting diagonal time-local master equation is fixed by positive coefficients.

With this definition at hand, we first note that, if the bath correlation function is proportional to a Dirac δ, $D(t-s) = C\delta(t-s)$, $C > 0$, the master equation (9) reduces to

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - C[\hat{q},[\hat{q},\hat{\rho}]] + 2\mu C[\hat{q},[\hat{p},\hat{\rho}]] - \mu^2 C[\hat{p},[\hat{p},\hat{\rho}]], \quad (29)$$

with $\hat{H} = \hat{H}_0 - \hbar C\hat{q}^2 - \hbar C\mu^2\hat{p}^2 + \hbar C\mu\{\hat{q},\hat{p}\}$, which can be cast in the Lindblad form

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \gamma\left(\hat{L}\hat{\rho}\hat{L} - \frac{1}{2}\{\hat{L}^2,\hat{\rho}\}\right),$$
$$\gamma \equiv 2C, \quad \hat{L} \equiv (\hat{q} - \mu\hat{p}). \quad (30)$$

A δ-correlated (or uncorrelated) two-point function for the bath can be obtained by considering an Ohmic spectral density and taking the limits for temperature and cutoff to infinity (see, e.g., Ref. [6]); in this case the constant C is proportional to the temperature itself $C = 2m\gamma/\hbar^2\beta$. Moreover, for $\mu = 0$ one recovers the Joos and Zeh master equation [27] as one expects from a nondissipative Markovian dynamics (see Ref. [23] for further comments on this issue).

For all the other bath correlation functions, the Kossakowski matrix in Eq. (28) is not positive definite: One of its eigenvalues is always negative as can be shown by evaluating the determinant of $A(t)$. Actually, to do that it is convenient to exploit the coefficients of the master equation as derived with the method of Ref. [11], being the expressions in Eq. (19) are rather involved. In Appendix B we evaluate explicitly the determinant of $A(t)$ getting

$$\det[a(t)] \equiv 4\Gamma_\mu(t)\gamma_\mu(t) - [\Theta_\mu(t)^2 + \Upsilon_\mu(t)^2]$$
$$= -\{[\Theta_\mu(t) + \mu\Gamma_\mu(t)]^2 + \Upsilon_\mu(t)^2\}, \quad (31)$$

which is negative for any nonsingular bath correlation function [whereas for a singular bath correlation function, it is equal to 0, see also Eq. (30)]. Accordingly, the dynamics of the system, apart from the special case of a δ-correlated bath, is always non-Markovian. We conclude that, as was argued in Ref. [23], the master equation (9) with coefficients as in Eq. (19) can describe a time-homogeneous Markovian (i.e., semigroup) dynamics as a singular limiting case, but it never yields a time-inhomogeneous Markovian dynamics.

As a final remark, let us note that the connection between the positivity of the coefficients of the diagonal time-local master equation and the other definitions of quantum Markovianity becomes more subtle in the infinite-dimensional case. In particular, let us mention CP divisibility, i.e., the property of the dynamical maps of being not only CP, but also decomposable into CP terms, according to $\Lambda(t) = \Phi(t,s)\Lambda(s)$, where $\Phi(t,s)$'s are CP maps for any $t \geqslant s$. This property has been identified with quantum Markovianity in Ref. [28], and, in the finite-dimensional case, one can show quite straightforwardly that the dynamics is CP divisible if and only if the coefficients of the master equation (25) are non-negative at any time [29,30]. Such an equivalence is not *a priori* guaranteed in the infinite-dimensional case due to the lack of a general theorem about the generator of CP semigroups involving unbounded operators [31]. On the other hand, in the presence of Gaussian-preserving dynamics and if one restricts to Gaussian states, CP divisibility can be formulated by means of definite conditions, possibly expressed in terms of the matrices fixing the evolution of the expectation values and covariance matrix [32,33]. Moreover, also in infinite-dimensional systems non-Markovianity can be traced back to a nonmonotonic time evolution of proper quantities [34–36].

V. CONCLUSIONS

We have investigated a model for non-Markovian quantum Brownian motion where the system is bilinearly coupled to a bosonic bath not only via its position, but also via its momentum. By means of the exact master equation along with the solution of the equations of motion for the first and second momenta of the position and momentum operators, we have studied the contributions to friction and dissipation induced by such an unusual momentum coupling. The latter induces a faster relaxation to the asymptotic steady state, characterized by a smaller average free energy, along with the appearance of a significant correlation between the position and the momentum statistics. These results hold for different spectral densities (Ohmic and super-Ohmic) as well as for different bath temperatures and system initial states.

In addition, we also have clarified the non-Markovian nature of the dynamics. We have shown that the exact model at hand includes as a limiting case the time-homogeneous Markovian (i.e., semigroup) dynamics, but it never describes a time-inhomogeneous Markovian dynamics.

ACKNOWLEDGMENTS

The authors wish to thank M. Carlesso for fruitful discussions. The work of L.F. was supported by the TALENTS[3] Fellowship Programme, CUP Code No. J26D15000050009 and FP Code No.1532453001, managed by AREA Science Park through the European Social Fund. The work of A.S. was supported by QUCHIP Project (GA No. 641039).

APPENDIX A: ANALYTIC EXPRESSIONS FOR THE FUNCTIONS DEFINING THE COEFFICIENTS OF THE TIME-LOCAL MASTER EQUATION

In this Appendix we provide the analytic expressions for the functions displayed by Eqs. (19). It is useful to introduce the "average Green's function" $\bar{G}$,

$$\bar{G}(t) = \int_0^t D(t-s)G(s)ds, \quad (A1)$$

and its suitable combinations with the Green's function and its derivatives (the dot over a symbol denotes differentiation with respect to time),

$$F(t) = \hbar[\dot{\bar{G}}(t)\dot{G}(t) - \ddot{G}(t)G(t)], \tag{A2}$$

$$H_1(t) = \bar{G}(t)\dot{G}(t) - \dot{\bar{G}}(t)G(t), \tag{A3}$$

$$H_2(t) = \dot{\bar{G}}(t)\dot{G}(t) - \ddot{G}(t)\bar{G}(t). \tag{A4}$$

These functions are the building blocks of the coefficients of the master equation (9). The lengthy procedure described in Sec. II A eventually provides us with the following functions displayed by Eqs. (19):

$$K_1(t) = \frac{1}{m} + \frac{\hbar\mu}{m}\frac{H_1(t)}{F(t)},$$

$$K_2(t) = \frac{\hbar}{m}\frac{H_1(t)}{F(t)} - \hbar\mu\frac{H_2(t)}{F(t)},$$

$$K_3(t) = \hbar m\omega_S^2\mu^2\frac{H_1(t)}{F(t)} + \hbar\mu\frac{H_2(t)}{F(t)},$$

$$K_4(t) = -m\omega_S^2 + \hbar\frac{H_2(t)}{F(t)},$$

$$K_5(t) = \left(\frac{1}{2m} + \frac{m\omega_S^2\mu^2}{2}\right)\frac{H_1(t)}{F(t)},$$

$$g_1(t) = -\frac{1}{4}\int_0^t ds\int_0^t dl\, D^{\mathrm{Im}}(s-l)G_3(t-s)G_3(t-l),$$

$$g_2(t) = -\frac{1}{4}\int_0^t ds\int_0^t dl\, D^{\mathrm{Im}}(s-l)G_6(t-s)G_6(t-l),$$

$$g_1(t) = -\frac{1}{2}\int_0^t ds\int_0^t dl\, D^{\mathrm{Im}}(s-l)G_3(t-s)G_6(t-l).$$

APPENDIX B: DERIVATION OF EQ. (31)

The elements of the Kossakovski matrix (28) as derived with the technique of Ref. [11] read

$$\begin{aligned}
\Gamma_\mu(t) &= -\int_0^t ds\, \mathbb{D}^{\mathrm{Re}}(t,s)[\cos\omega_S(s-t) \\
&\quad + m\mu\omega_S\sin\omega_S(s-t)], \\
\Theta_\mu(t) &= \mu\int_0^t ds\, \mathbb{D}^{\mathrm{Re}}(t,s)[2\cos\omega_S(s-t) \\
&\quad - m_-\sin\omega_S(s-t)], \\
\Upsilon_\mu(t) &= -im_+\mu\int_0^t ds\, \mathbb{D}^{\mathrm{Im}}(t,s)\sin\omega_S(s-t),
\end{aligned} \tag{B1}$$

where we have introduced $m_\pm = 1/m\mu\omega_S \pm m\mu\omega_S$. The integral kernels $\mathbb{D}$ have a series structure that depends both on the bath correlation function and on the free propagator of the system. For the numerical purposes of this paper, the structure of $\mathbb{D}$ represents a drawback because one needs to truncate the series introducing systematic errors. Accordingly, the Heisenberg approach exploited in the main text is more suitable. On the other hand, Eqs. (B1) allow for calculating the determinant of $a(t)$ of Eq. (28) in an easier way. Indeed, we consider the definition of the determinant of $a(t)$,

$$\det[a(t)] = 4\Gamma_\mu(t)\gamma_\mu(t) - [\Theta_\mu(t)^2 + \Upsilon_\mu(t)^2], \tag{B2}$$

and we replace Eqs. (B1) in it. By exploiting the composition properties of trigonometric functions, after some calculations we obtain

$$\begin{aligned}
\det[a(t)] = -m_+^2\mu^2\Bigg[&\left(\int_0^t ds\, \mathbb{D}^{\mathrm{Re}}(t,s)\sin\omega_S(s-t)\right)^2 \\
&+ \left(\int_0^t ds\, \mathbb{D}^{\mathrm{Im}}(t,s)\sin\omega_S(s-t)\right)^2\Bigg].
\end{aligned} \tag{B3}$$

We then invert Eqs. (B1) and replace the result in the equation above to eventually obtain Eq. (31). One easily can check that when the bath is δ correlated the determinant of a is zero.

[1] H. P. Breuer and F. Petruccione, *Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
[2] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 2008).
[3] Á. Rivas, S. F. Huelga, and M. B. Plenio, Rep. Prog. Phys. **77**, 094001 (2014).
[4] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Rev. Mod. Phys. **88**, 021002 (2016).
[5] I. de Vega and D. Alonso, Rev. Mod. Phys. **89**, 015001 (2017).
[6] A. O. Caldeira and A. Leggett, Physica A **121**, 587 (1983).
[7] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. A **37**, 4419 (1988).
[8] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).
[9] J. J. Halliwell and T. Yu, Phys. Rev. D **53**, 2012 (1996).
[10] G. W. Ford and R. F. O'Connell, Phys. Rev. D **64**, 105020 (2001).
[11] L. Ferialdi, Phys. Rev. Lett. **116**, 120402 (2016).
[12] L. Ferialdi, Phys. Rev. A **95**, 020101(R) (2017).
[13] M. J. W. Hall, J. D. Cresser, L. Li, and E. Andersson, Phys. Rev. A **89**, 042120 (2014).
[14] A. Sandulescu and H. Scutaru, Ann. Phys. (NY) **173**, 277 (1987); A. Isar, A. Sandulescu, H. Scutaru, E. Stefanescu, and W. Scheid, Int. J. Mod. Phys. E **03**, 635 (1994).
[15] S. Gao, Phys. Rev. Lett. **79**, 3101 (1997); G. W. Ford and R. F. O'Connell, *ibid.* **82**, 3376 (1999); S. Gao, *ibid.* **82**, 3377 (1999).
[16] A. Bassi, E. Ippoliti, and B. Vacchini, J. Phys. A **38**, 8017 (2005).
[17] L. Ferialdi and A. Bassi, Phys. Rev. Lett. **108**, 170404 (2012); Phys. Rev. A **86**, 022108 (2012); Europhys. Lett. **98**, 30009 (2012).
[18] A. J. Leggett, Phys. Rev. B **30**, 1208 (1984).
[19] A. Cuccoli, A. Fubini, V. Tognetti, and R. Vaia, Phys. Rev. E **64**, 066124 (2001).
[20] H. Kohler, F. Guinea, and F. Sols, Ann. Phys. (NY) **310**, 127 (2004); H. Kohler and F. Sols, New J. Phys. **8**, 149 (2006).
[21] L. Diósi and L. Ferialdi, Phys. Rev. Lett. **113**, 200403 (2014).

[22] M. Carlesso and A. Bassi, Phys. Rev. A **95**, 052119 (2017).
[23] L. Ferialdi, Phys. Rev. A **95**, 052109 (2017).
[24] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
[25] I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, UK, 2006).
[26] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[27] E. Joos and H. D. Zeh, Z. Phys. B: Condens. Matter **59**, 223 (1985).
[28] Á. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **105**, 050403 (2010).
[29] E. M. Laine, J. Piilo, and H.-P. Breuer, Phys. Rev. A **81**, 062115 (2010).
[30] Á. Rivas and S. F. Huelga, *Open Quantum Systems* (Springer, Berlin, 2012).
[31] A. S. Holevo, in *Irreversibility and Causality*, edited by A. Bohm, H. D. Doebner and P. Kielanowski, Lecture Notes in Physics Vol. 504 (Springer, Berlin, 1997), p. 67.
[32] G. Torre, W. Roga, and F. Illuminati, Phys. Rev. Lett. **115**, 070401 (2015).
[33] P. Liuzzo-Scorpo, W. Roga, L. A. M. Souza, N. K. Bernardes, and G. Adesso, Phys. Rev. Lett. **118**, 050401 (2017).
[34] R. Vasile, S. Maniscalco, M. G. A. Paris, H.-P. Breuer, and J. Piilo, Phys. Rev. A **84**, 052118 (2011).
[35] A. Smirne and B. Vacchini, Phys. Rev. A **82**, 042111 (2010).
[36] L. A. M. Souza, H. S. Dhar, M. N. Bera, P. Liuzzo-Scorpo, and G. Adesso, Phys. Rev. A **92**, 052122 (2015).