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Publisher:
Elsevier Ltd

Published version:
DOI: 10.1016/j.aml.2019.105996

On the Convergence of the Rescaled Localized Radial Basis Function Method

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July 30, 2019

Abstract

The rescaled localized RBF method was introduced in [2] for scattered data interpolation. It is a rational approximation method based on interpolation with compactly supported radial basis functions. It requires the solution of two linear systems with the same sparse matrix, which has a small condition number, due to the scaling of the basis function. Hence, it can be computed using an unpreconditioned conjugate gradient method in linear time. Numerical evidence provided in [2] shows that the method produces good approximations for many examples but no theoretical results were provided. In this paper, we discuss the convergence of the rescaled localized RBF method in the case of quasi-uniform data and stationary scaling. As the method is not only interpolatory but also reproduces constants exactly, linear convergence is expected. We can show this linear convergence up to a certain conjecture.

1 Introduction

We start this note by introducing the necessary notations and the motivations that inspired this work.
Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a compactly supported radial basis function with support in the unit ball $B_1(\mathbf{0})$, which is a reproducing kernel of $H^\sigma(\mathbb{R}^d)$ for a given $\sigma > d/2$, i.e. it has a Fourier transform behaving like

$$c_1(1+\|\boldsymbol{\omega}\|_2^2)^{-\sigma} \le \widehat{\Phi}(\boldsymbol{\omega}) \le c_2(1+\|\boldsymbol{\omega}\|_2^2)^{-\sigma}, \qquad \boldsymbol{\omega} \in \mathbb{R}^d. \tag{1}$$

For $\delta > 0$, we define the scaled function

$$\Phi_\delta := \Phi(\cdot/\delta). \tag{2}$$

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq \Omega$ be a discrete subset of distinct points with fill distance and separation radius given, as usual, by

$$h_{X,\Omega} := \sup_{\mathbf{x}\in\Omega} \min_{\mathbf{x}_j \in X} \|\mathbf{x}-\mathbf{x}_j\|_2, \qquad q_X := \frac{1}{2}\min_{j\neq k} \|\mathbf{x}_j - \mathbf{x}_k\|_2,$$

respectively. We will assume that the set is *quasi-uniform*, i.e. that there is a constant c_{qu} such that

$$q_X \leq h_{X,\Omega} \leq c_{qu} q_X. \tag{3}$$

We will further assume that there are constants $\gamma, c_\gamma \in (0,1)$ such that

$$\gamma c_\gamma h_{X,\Omega} \leq \delta \leq c_\gamma h_{X,\Omega}, \tag{4}$$

which also means that δ is proportional to q_X, as we have

$$\gamma c_\gamma q_X \leq \delta \leq c_\gamma c_{qu} q_X. \tag{5}$$

With these ingredients, we can define an interpolant s_f to a function $f \in C(\Omega)$ as follows. Let

$$s_f(\mathbf{x}) = s_{f,X,\Phi_\delta}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \Phi_\delta(\mathbf{x} - \mathbf{x}_j), \qquad \mathbf{x} \in \Omega, \tag{6}$$

be the unique interpolant to a function $f \in C(\Omega)$, i.e. the coefficients α_j are determined by the interpolation condition $s_f(\mathbf{x}_i) = f(\mathbf{x}_i)$, $1 \leq i \leq N$. The interpolant or, to be more precise, the coefficients $\boldsymbol{\alpha} = (\alpha_j)$ are determined by $\boldsymbol{\alpha} = A^{-1} f|_X$ with the symmetric and positive definite interpolation matrix $A = (\Phi_\delta(\mathbf{x}_i - \mathbf{x}_j)) \in \mathbb{R}^{N \times N}$.
It is well-known that in the *stationary* situation described above, where the support radius δ of the basis functions is proportional to the fill distance $h_{X,\Omega}$, the interpolation matrix A is well-conditioned with a condition number which only depends on the constants γ, c_γ, c_{qu}. To be more precise there is a constant $C > 0$ such that we have (cf. [5])

$$\begin{aligned} \|A^{-1}\|_2 &\leq C \left(\frac{\delta}{q_X}\right)^{2\sigma - d} \leq C(c_\gamma c_{qu})^{2\sigma-d}, && (7) \\ \text{cond}_2(A) &\leq C(c_\gamma c_{qu})^{2\sigma-d}(1 + c_\gamma c_{qu})^d \Phi(0). && (8) \end{aligned}$$

The matrix A is also a sparse matrix such that the system $A\boldsymbol{\alpha} = f|_X$ can be solved extremely efficiently. Moreover, the evaluation of s_f at a point $\mathbf{x} \in \Omega$ or even $\mathbf{x} \in \mathbb{R}^d$ requires not to form the entire sum in (6). Instead, we only need to sum over those indices k for which $\|\mathbf{x} - \mathbf{x}_k\|_2 \leq \delta$. This number is bounded by

$$\#\{k : \|\mathbf{x} - \mathbf{x}_k\|_2 \leq \delta\} \leq \frac{(\delta + q_X)^d}{q_X^d} \leq (1 + c_\gamma c_{qu})^d, \tag{9}$$

which is also used for deriving (8).
However, it is also well-known that there is no convergence for s_f to f with $h_{X,\Omega} \to 0$ (cf. e.g. [6, §11.3])
Because of this trade-off principle, a variation called *rescaled localized radial basis function* has been proposed in [2] . The idea is simply to compute the interpolant s_f as above, as well as the interpolant s_1 to the constant function 1 and form their quotient. The new approximation $S_h f$ is then

$$S_h f := \frac{s_f}{s_1} = \frac{s_{f,X,\Phi_\delta}}{s_{1,X,\Phi_\delta}}. \tag{10}$$

Of course, to make this well-defined, we must assume that s_1 does not vanish on Ω. Numerical evidence provided in [2] shows that s_1 does indeed not vanish under reasonable assumptions on γ, c_γ, c_{qu} and that $S_h f$ converges to f. Moreover, we have the following obvious observation (see also [4]).

Lemma 1.1 *As long as $S_h f$ is well-defined, it interpolates f at the data sites $\mathbf{x}_j \in X$ and it reproduces constants exactly.*

It is the goal of this paper to prove linear convergence of $S_h f$ to f. To show this, in Section 2 we rewrite $S_h f$ as a quasi-interpolant by using cardinal functions while in Section 3 we prove the main result. While the result itself is primarily of theoretical nature, it has impact to applications, as well, as it guarantees linear convergence in the quasi-uniform setting, indicating how many data sites are required for a desired accuracy. Moreover, as the proof uses mainly local arguments, it might be possible to extend the result to non-uniform data sets.

2 Cardinal Functions

Recall that we always have cardinal or Lagrange functions $\chi_j \in V_{X,\Phi_\delta}$, i.e. functions satisfying $\chi_j(\mathbf{x}_i) = \delta_{ij}$, where

$$V_{X,\Phi_\delta} = \operatorname{span}\{\Phi_\delta(\cdot - \mathbf{x}) : \mathbf{x} \in X\}.$$

These Lagrange functions can be simultaneously computed via

$$A \begin{pmatrix} \chi_1(\mathbf{x}) \\ \vdots \\ \chi_N(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \Phi_\delta(\mathbf{x} - \mathbf{x}_1) \\ \vdots \\ \Phi_\delta(\mathbf{x} - \mathbf{x}_N) \end{pmatrix}$$

or simply

$$\chi_j(\mathbf{x}) = \sum_{k=1}^{N} A^{-1}_{jk} \Phi_\delta(\mathbf{x} - \mathbf{x}_k) = \sum_{k:\|\mathbf{x}-\mathbf{x}_k\|_2 \le \delta} A^{-1}_{jk} \Phi_\delta(\mathbf{x} - \mathbf{x}_k), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{11}$$

where A^{-1}_{jk} denotes the entries of the inverse A^{-1}. With these cardinal functions we can rewrite $S_h f$ as

$$S_h f(\mathbf{x}) = \frac{\sum_{j=1}^{N} f(\mathbf{x}_j)\chi_j(\mathbf{x})}{\sum_{k=1}^{N} \chi_k(\mathbf{x})} =: \sum_{j=1}^{N} f(\mathbf{x}_j) u_j(\mathbf{x})$$

with the new weight functions

$$u_j(\mathbf{x}) := \frac{\chi_j(\mathbf{x})}{\sum_{k=1}^{N} \chi_k(\mathbf{x})}, \qquad \mathbf{x} \in \mathbb{R}^d.$$

Hence, to understand this approximation process it is necessary to understand the cardinal functions χ_j better. We start by showing that they decay exponentially. To prove this, we need to recall some material from [1] in the way it was also used in [5]. We start by indexing the matrix $A = (\Phi_\delta(\mathbf{x}_i - \mathbf{x}_j))$ differently. We will index our matrix entries not using pairs $(i, j) \in \mathbb{N}_0^2$ but pairs of multi-indices $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Z}^d \times \mathbb{Z}^d$.
Hence, a *multivariate matrix* A is a finitely supported function $A : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$. If we still write $A(\boldsymbol{\alpha}, \boldsymbol{\beta}) = A_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then it is easy to see that concepts like symmetry and positive definiteness carry easily over to multivariate matrices. We also have the following definition from [1].

Definition 2.1 *A multivariate matrix $A : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is called R-banded if $R > 0$ and if $A_{\alpha,\beta} = 0$ whenever $\|\alpha - \beta\|_2 > R$.*

The advantage of this definition over the classical definition of banded matrices is that it reflects the given geometrical background in a natural way.

Lemma 2.2 *[1, Theorem 3.3] Let $A : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ be a symmetric and positive definite R-banded matrix. Then,*

$$|A^{-1}_{\boldsymbol{\alpha},\boldsymbol{\beta}}| \le 2\|A^{-1}\|_2 \mu^{\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|_2}$$

with

$$\mu = \left(\frac{\sqrt{\operatorname{cond}_2(A)} - 1}{\sqrt{\operatorname{cond}_2(A)} + 1} \right)^{1/R}. \tag{12}$$

To interpret the interpolation matrix $A = (\Phi_\delta(\mathbf{x}_i - \mathbf{x}_j))$ as a banded multivariate matrix, we relabel it such that A can be viewed as a finitely supported function $A : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$. To do this we uniquely assign a multi-index $\mathbf{y}_i \in \mathbb{Z}^d$ to each data point $\mathbf{x}_i \in X \subseteq \mathbb{R}^d$ with the property that

$$A(\mathbf{y}_j, \mathbf{y}_k) = \Phi_\delta(\mathbf{x}_j - \mathbf{x}_k).$$

As shown in ([5]) we can achieve this by setting $\mathbf{z}_j := \frac{\sqrt{d}}{2q_X}\mathbf{x}_j \in \mathbb{R}^d$ and defining $\mathbf{y}_j \in \mathbb{Z}^d$ by

$$\mathbf{y}_j = \left(\lfloor z_j^1 \rfloor, \ldots, \lfloor z_j^d \rfloor \right), \qquad 1 \le j \le N.$$

Then we have $\|\mathbf{z}_j - \mathbf{z}_k\|_2 \ge \sqrt{d}$ for $j \ne k$ such that [1, Lemma 3.7] guarantees that all the $\mathbf{y}_j$ are pairwise distinct. Moreover, with $R := \max(c_{qu}c_\gamma, 4)\sqrt{d}$ the condition $\|\mathbf{y}_j - \mathbf{y}_k\|_2 \ge R \ge 4\sqrt{d}$ implies $\|\mathbf{z}_j - \mathbf{z}_k\|_2 \ge R/2$ by the same lemma and this means in our situation

$$\|\mathbf{x}_j - \mathbf{x}_k\|_2 \ge \frac{2q_X}{\sqrt{d}}\frac{R}{2} = \frac{2q_X}{\sqrt{d}}\frac{c_{qu}c_\gamma\sqrt{d}}{2} = c_{qu}c_\gamma q_X \ge \delta,$$

Hence, Lemma 2.2 implies

$$|A^{-1}_{jk}| = |A^{-1}_{\mathbf{y}_j,\mathbf{y}_k}| \le 2\|A^{-1}\|_2 \mu^{\|\mathbf{y}_j - \mathbf{y}_k\|_2} \le C\mu^{\|\mathbf{y}_j - \mathbf{y}_k\|_2}, \tag{13}$$

where we have used (7). Moreover, using (8) in the definition of μ in (12) and the fact that the function $f(x) := (x-1)/(x+1)$ is monotonically increasing, we see that $\mu \in (0,1)$ is indeed a constant independent of X. This is the basis for the proof of the following result.

Theorem 2.3 *Suppose Φ is a compactly supported kernel with Fourier transform satisfying (1). Let Φ_δ be defined by (2). Let $X \subseteq \mathbb{R}^d$ be quasi-uniform with (3) and let δ be proportional to $h_{X,\Omega}$ as in (4). Then, there is a $C > 0$ and $\nu > 0$ such that, for $1 \le j \le N$,*

$$|\chi_j(\mathbf{x})| \le Ce^{-\nu\|\mathbf{x}-\mathbf{x}_j\|_2/q_X}, \qquad \mathbf{x} \in \mathbb{R}^d.$$

Proof: As we have for two numbers $x, y \in \mathbb{R}$ the relation $|\lfloor x \rfloor - \lfloor y \rfloor| \ge |x - y| - 1$, we see that, with the definitions above,

$$\|\mathbf{y}_j - \mathbf{y}_k\|_2 \ge \frac{1}{\sqrt{d}}\|\mathbf{y}_j - \mathbf{y}_k\|_1 \ge \frac{1}{\sqrt{d}}\|\mathbf{z}_j - \mathbf{z}_k\|_2 - \sqrt{d} = \frac{1}{2q_X}\|\mathbf{x}_j - \mathbf{x}_k\|_2 - \sqrt{d}$$

Hence, (13) implies

$$|A_{jk}^{-1}| \le C\mu^{\frac{1}{2}\frac{\|\mathbf{x}_j-\mathbf{x}_k\|_2}{q_X}}\mu^{-\sqrt{d}} \le Ce^{-\nu\frac{\|\mathbf{x}_j-\mathbf{x}_k\|_2}{q_X}}$$

with $\nu := -\frac{1}{2}\log\mu > 0$. Using $\|\mathbf{x}_k-\mathbf{x}_j\|_2 \ge \|\mathbf{x}-\mathbf{x}_j\|_2 - \|\mathbf{x}-\mathbf{x}_k\|_2 \ge \|\mathbf{x}-\mathbf{x}_j\|_2 - \delta$ in the representation (11), together with (9) and (5), this yields

$$\begin{aligned}|\chi_j(\mathbf{x})| &\le \sum_{\|\mathbf{x}-\mathbf{x}_k\|_2\le\delta} |A_{jk}^{-1}||\Phi_\delta(\mathbf{x}-\mathbf{x}_j)| \le C\|\Phi\|_{L_\infty(\mathbb{R}^d)} \sum_{\|\mathbf{x}-\mathbf{x}_k\|_2\le\delta} e^{-\nu\|\mathbf{x}_j-\mathbf{x}_k\|_2/q_X}\\ &\le C\sum_{\|\mathbf{x}-\mathbf{x}_k\|_2\le\delta} e^{-\nu\|\mathbf{x}-\mathbf{x}_j\|_2/q_X}e^{\delta\nu/q_X}\\ &\le C(1+c_\gamma c_{qu})^d e^{c_\gamma c_{qu}\nu}e^{-\nu\|\mathbf{x}-\mathbf{x}_j\|_2/q_X}.\end{aligned}$$

□

This result has several immediate consequences, which we want to list now.

Corollary 2.4 *Under the assumptions of Theorem 2.3, the cardinal functions are Lipschitz continuous, i.e. there is a constant $C_L > 0$ such that*

$$|\chi_j(\mathbf{x}) - \chi_j(\mathbf{y})| \le C_L\frac{\|\mathbf{x}-\mathbf{y}\|_2}{q_X}, \qquad \mathbf{x},\mathbf{y}\in\mathbb{R}^d.$$

Proof: Differentiating the representation (11), using that $\Phi \in C^1(\mathbb{R}^d)$ and using the exponential decay of the Lagrangian shows

$$|\partial_i\chi_j(\mathbf{x})| = \frac{1}{\delta}\sum_{k=1}^N |A_{jk}^{-1}||\partial_i\Phi((\mathbf{x}-\mathbf{x}_k)/\delta)| \le \frac{C}{\delta}\sum_{k:\|\mathbf{x}-\mathbf{x}_k\|\le\delta} e^{-\nu\|\mathbf{x}_j-\mathbf{x}_k\|_2/q_X}.$$

Using again $\|\mathbf{x}_k-\mathbf{x}_j\|_2 \ge \|\mathbf{x}-\mathbf{x}_j\|_2 - \|\mathbf{x}-\mathbf{x}_k\|_2 \ge \|\mathbf{x}-\mathbf{x}_j\|_2 - \delta$, this yields, together with (9) and (5),

$$|\partial_i\chi_j(\mathbf{x})| \le \frac{C}{q_X}e^{-\nu\|\mathbf{x}-\mathbf{x}_j\|_2/q_X}.$$

With this, the intermediate value theorem finally shows

$$|\chi_j(\mathbf{x})-\chi_j(\mathbf{y})| \le \|\nabla\chi_j(\boldsymbol{\xi})\|_2\,\|\mathbf{x}-\mathbf{y}\|_2 \le C\frac{\|\mathbf{x}-\mathbf{y}\|_2}{q_X}e^{-\nu\|\boldsymbol{\xi}-\mathbf{x}_j\|_2} \le C\frac{\|\mathbf{x}-\mathbf{y}\|_2}{q_X}.$$

□

Another consequence is that the Lebesgue functions are uniformly bounded. We can even show the following more general result.

Corollary 2.5 *Under the assumptions of Theorem 2.3 there is a constant $C > 0$ such that, for $\ell \in \mathbb{N}_0$,*

$$\sum_{j=1}^N \|\mathbf{x}-\mathbf{x}_j\|_2^\ell|\chi_j(\mathbf{x})| \le Ch_{X,\Omega}^\ell, \qquad \mathbf{x}\in\mathbb{R}^d.$$

Proof: Similarly as in the proof of [6, Theorem 12.3], we let $E_n = \{\mathbf{y} \in \mathbb{R}^d : nq_X \le \|\mathbf{x}-\mathbf{y}\|_2 \le (n+1)q_X\}$ for $n \in \mathbb{N}_0$. Then, a volume comparison argument shows

$$\#(X \cap E_n) \le 3^d n^{d-1}.$$

Using that the union of all the E_n contains all the data sites X, and the monotonicity of the exponential function and the function $\|\cdot - \mathbf{x}_j\|_2^\ell$ allows us to derive

$$\begin{aligned}\sum_{j=1}^N \|\mathbf{x}-\mathbf{x}_j\|_2^\ell |\chi_j(\mathbf{x})| &\le \sum_{n=0}^\infty \sum_{\mathbf{x}_j \in E_n} \|\mathbf{x}-\mathbf{x}_j\|_2^\ell |\chi_j(\mathbf{x})| \le C \sum_{n=0}^\infty \sum_{\mathbf{x}_j \in E_n} \|\mathbf{x}-\mathbf{x}_j\|_2^\ell e^{-\nu\|\mathbf{x}-\mathbf{x}_j\|_2/q_X} \\ &\le C \sum_{n=0}^\infty n^{d-1}(n+1)^\ell q_X^\ell e^{-\nu n} \le C h_{X,\Omega}^\ell \sum_{n=0}^\infty (n+1)^{d+\ell-1} e^{-\nu n} \\ &\le C h_{X,\Omega}^\ell,\end{aligned}$$

as the final sum is clearly finite. □

3 Convergence of the RL-RBF Method

To show convergence of the RL-RBF method we need one additional auxiliary result.

Lemma 3.1 *Under the assumptions of Theorem 2.3, the constant c_γ from (4) can be chosen such that there is a constant $c > 0$ such that*

$$\sum_{j=1}^N \chi_j(\mathbf{x}) \ge c, \qquad \mathbf{x} \in \Omega. \tag{14}$$

Remark. Unfortunately we are presently not able to prove this Lemma. We can easily prove it for $n = 2$ by resorting to the fact that each cardinal is a ratio of two strictly positive determinants. For $n > 2$ we checked it numerically in many different instances, even in the case of standard approximation and in all cases it was confirmed. As an example in Fig. 1 we show the plots of the sum (14) on 20 Halton points. This shows the more stability we get with the rescaled RBF interpolation as the sum approximates one.

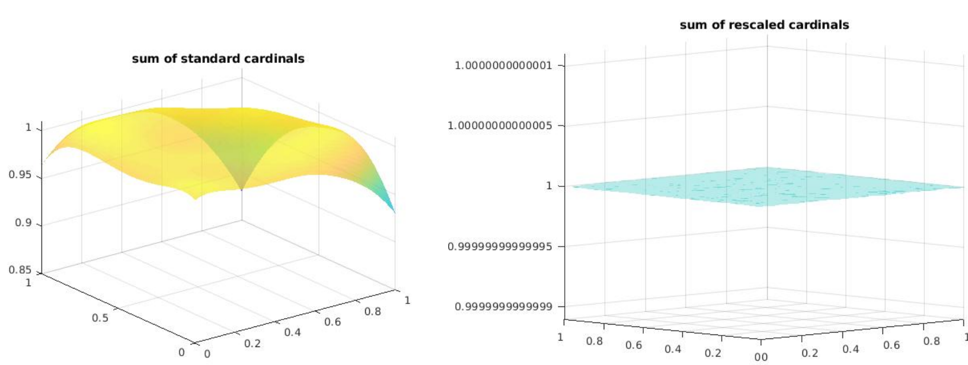

Figure 1: Sum of cardinals on 20 Halton points on $[0,1]^2$

We will express convergence using the modulus of continuity defined by

$$\omega_f(\epsilon) := \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in \Omega \text{ with } \|\mathbf{x} - \mathbf{y}\|_2 \le \epsilon\}.$$

for a function $f : \Omega \to \mathbb{R}$ and with $\epsilon > 0$.
We need one final assumption on the domain $\Omega \subseteq \mathbb{R}^d$. We will, from now on assume, that there is a constant $C_\Omega \ge 1$ such that any two points $\mathbf{x}$ and $\mathbf{y}$ in Ω can be joined by a rectifiable curve Γ in Ω with length $|\Gamma| \le C_\Omega \|\mathbf{x} - \mathbf{y}\|_2$.
If Ω satisfies the above assumption, then [3, Lemma 2.2] shows for arbitrary $0 < \epsilon < \delta$ that

$$\omega_f(\delta) \le 2C_\Omega \frac{\delta}{\epsilon} \omega_f(\epsilon). \tag{15}$$

Theorem 3.2 *Under the assumptions of Theorem 2.3 and under the assumption that Lemma 3.1 holds, the RL-RBF method converges linearly for every target function $f \in C^1(\Omega)$, i.e. there is a constant $C > 0$ such that*

$$\|f - S_h f\|_{L_\infty(\Omega)} \le h_{X,\Omega} \|f\|_{C^1(\Omega)}.$$

Proof: We have

$$S_h f = \sum_{j=1}^{N} f(\mathbf{x}_j) u_j, \qquad u_j = \frac{\chi_j}{\sum_{k=1}^N \chi_k}$$

and hence $S_h 1 = 1$, i.e. constants a reproduced. Moreover, we have the bound

$$|u_j(\mathbf{x})| = \frac{|\chi_j(\mathbf{x})|}{\left|\sum_{k=1}^N \chi_k(\mathbf{x})\right|} \le C e^{-\nu \|\mathbf{x} - \mathbf{x}_j\|_2 / q_X}$$

using Lemma 3.1 and Theorem 2.3. With $h = h_{X,\Omega}$, (14), (15) and Corollary 2.5 we have

$$\begin{aligned}
|f(\mathbf{x}) - S_h f(\mathbf{x})| &= \left| \sum_{j=1}^N [f(\mathbf{x}) - f(\mathbf{x}_j)]\, u_j(\mathbf{x}) \right| \le \frac{1}{c} \sum_{j=1}^N |f(\mathbf{x}) - f(\mathbf{x}_j)|\, |\chi_j(\mathbf{x})| \\
&\le \frac{1}{c} \sum_{\|\mathbf{x}-\mathbf{x}_j\|_2 \le h} |f(\mathbf{x}) - f(\mathbf{x}_j)|\, |\chi_j(\mathbf{x})| + \frac{1}{c} \sum_{\|\mathbf{x}-\mathbf{x}_j\|_2 > h} |f(\mathbf{x}) - f(\mathbf{x}_j)| |\chi_j(\mathbf{x})| \\
&\le \frac{1}{c} \sum_{\|\mathbf{x}-\mathbf{x}_j\|_2 \le h} \omega_f(h) |\chi_j(\mathbf{x})| + \frac{1}{c} \sum_{\|\mathbf{x}-\mathbf{x}_j\|_2} \omega_f(\|\mathbf{x} - \mathbf{x}_j\|_2) |\chi_j(\mathbf{x})| \\
&\le C\omega_f(h) \left[1 + 2C_\Omega \frac{1}{h} \sum_{j=1}^N \|\mathbf{x} - \mathbf{x}_j\|_2 |\chi_j(\mathbf{x})| \right] \\
&\le C\omega_f(h),
\end{aligned}$$

and the result follows from $\omega_f(h) \le h\|f\|_{C^1(\Omega)}$ for all $f \in C^1(\Omega)$. □

We cannot expect a better convergence order in general, as the process only reproduces constants exactly.

Acknowledgements. The first author thanks the colleagues Rick Beatson (University of Canterbury, NZ) and Ana Paula Peron (University of São Paulo, BR) for fruithful discussions. This work has been supported by the Department of Mathematics "Tullio Levi-Civita", Visiting Scientists funds 2017. We thanks also the Italian Network on Approximation, RITA, and the GNCS-INdAM.

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