Periodic solutions to parameter-dependent equations with a ϕ-Laplacian type operator

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Abstract. We study the periodic boundary value problem associated with the ϕ-Laplacian equation of the form $(\phi(u'))' + f(u)u' + g(t,u) = s$, where s is a real parameter, f and g are continuous functions, and g is T-periodic in the variable t. The interest is in Ambrosetti–Prodi type alternatives which provide the existence of zero, one or two solutions depending on the choice of the parameter s. We investigate this problem for a broad family of nonlinearities, under non-uniform type conditions on $g(t,u)$ as $u \to \pm\infty$. We generalize, in a unified framework, various classical and recent results on parameter-dependent nonlinear equations.

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1. Introduction

In this paper, we study the problem of existence, non-existence, and multiplicity of T-periodic solutions of periodic boundary value problems associated with the ϕ-Laplacian generalized Liénard equation

$$(\phi(u'))' + f(u)u' + g(t,u) = s, \tag{$\mathscr{E}_s$}$$

where s is a real parameter. Our aim is to unify, in a generalized setting, different classical and recent results obtained in this area concerning the trichotomy given by zero/one/two solutions by varying the parameter s. Such kind of alternative can be traced back to the pioneering work by Ambrosetti and Prodi [2]. In more detail, we discuss several configurations of g related to the works by Fabry, Mawhin, Nkashama [7], and Bereanu, Mawhin [5].

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Looking at [7], the Authors considered the parameter-dependent Liénard equation

$$u'' + f(u)u' + g(t,u) = s, \tag{1.1}$$

with g a continuous function, T-periodic in t, and satisfying

$$\lim_{|u|\to+\infty} g(t,u) = +\infty, \quad \text{uniformly in } t,$$

In this framework, they proved the existence of a value $s_0 \in \mathbb{R}$ such that the T-periodic problem associated with (1.1) satisfies

> Alternative by Ambrosetti–Prodi (AP): *there exist zero, at least one or at least two solutions, provided that $s < s_0$, $s = s_0$ or $s > s_0$.*

Actually, this kind of theorems has been extended to more general equations. In particular, the study in [15] concerns nonlinear differential operators such as the ϕ-Laplacians and considers $(\mathscr{E}_s)$ as well. A typical application of the results in [15] can be written for the weighted equation

$$(\mathscr{W\!E}_s) \qquad (\phi(u'))' + f(u)u' + a(t)q(u) = s + e(t),$$

for q satisfying $\lim_{|u|\to+\infty} q(u) = +\infty$, and $a, e \colon \mathbb{R} \to \mathbb{R}$ continuous T-periodic functions with $\min a > 0$.

It is interesting to observe that phenomena similar to the one in (AP) have been discovered also for different kinds of nonlinearities q. In this regard, we refer to the work by Bereanu and Mawhin in [5] for the equation

$$(\phi(u'))' + q(u) = s + e(t), \tag{1.2}$$

with $\int_0^T e = 0$ and q satisfying $q(u) > 0$ for all u, along with

$$\lim_{|u|\to+\infty} q(u) = 0.$$

Indeed, in [5] the Authors, extending a previous work by Ward in [23], proved the existence of a value $s_0 \in \mathbb{R}$ such that the T-periodic problem associated with (1.2) satisfies

> Alternative by Bereanu–Mawhin (BM): *there exist zero, at least one or at least two solutions, provided that $s > s_0$, $s = s_0$ or $0 < s < s_0$.*

Moreover, they proved that *there are no solutions also for $s < 0$.* The same conclusion in (BM) was obtained in [3] for the p-Laplacian Liénard equation

$$(\phi_p(u'))' + f(u)u' + q(u) = s + e(t),$$

where $\phi_p(\xi) := |\xi|^{p-2}\xi$, with $p > 1$.

At this point, we observe that such a kind of results suggests the fact that an Ambrosetti–Prodi type alternative of the form *zero*, *at least one* or *at least two solutions*, may occur for a broad class of nonlinearities which reflect the behavior of parabola/bell-shaped functions. To be more precise, we consider nonlinearities $g(t,u)$ which include, as special cases, functions of the form

$$g(t,u) = a(t)q(u) - e(t), \tag{1.3}$$

with q having the following behavior:

Nonlinearity of type I. There exist $\omega_\pm := \lim_{u\to\pm\infty} q(u)$ with $q(u) < \omega_\pm$ for u in a neighborhood of $\pm\infty$.

Nonlinearity of type II. There exist $\omega_\pm := \lim_{u\to\pm\infty} q(u)$ with $q(u) > \omega_\pm$ for u in a neighborhood of $\pm\infty$.

We notice that if $\omega_\pm = +\infty$ then q is a nonlinearity of type I, while, if $\omega_\pm = -\infty$ then q is a nonlinearity of type II.

Moreover, when g has the form as in (1.3), we allow the weight $a(t)$ to be non-negative but possibly vanishing on sets of positive measure, so that the uniform condition $\min a > 0$ is no longer required. As a consequence, we deal with situations involving non-uniform conditions on $g(t,u)$ as $u \to \pm\infty$.

From now on, we focus our attention on the periodic boundary value problem associated with $(\mathscr{E}_s)$ where $\phi\colon \mathbb{R} \to \phi(\mathbb{R}) = \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, the map $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, and the function $g\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions (cf. [12, p. 28]). By a *solution* to $(\mathscr{E}_s)$ we mean a function $u\colon [0,T] \to \mathbb{R}$ of class $\mathcal{C}^1$ such that $\phi(u')$ is an absolutely continuous function and the equation $(\mathscr{E}_s)$ is satisfied for almost every t. Moreover, when $u(0) = u(T)$ and $u'(0) = u'(T)$, we say that u is a *T-periodic solution*. One could equivalently consider the function $g(t,u)$ defined for a.e. $t \in \mathbb{R}$ and T-periodic in t. In this case, one looks for solutions $u\colon \mathbb{R} \to \mathbb{R}$ which are T-periodic and satisfy $(\mathscr{E}_s)$, as described above.

It is worth noting that equation $(\mathscr{E}_s)$ concerns the *ϕ-Laplacian operator* which includes all the qualitative properties of the classical *p-Laplacian operator ϕ_p* or even some more general differential operators, such as the *(p,q)-Laplacian operator* defined as $\phi_{p,q}(\xi) := (|\xi|^{p-2} + |\xi|^{q-2})\xi$, with $1 < p < q$. Such kinds of differential operators are extensively studied in the literature for their relevance in many physical and mechanical models (cf. [13, 14, 19]).

We present now some new results concerning the T-periodic BVP associated with equation $(\mathscr{W}\mathscr{E}_s)$. In this introductory summary, for sake of convenience, we assume that $a, e \in L^\infty(0,T)$ are such that $a(t) \geq 0$ for a.e. $t \in [0,T]$ with $\bar{a} := \frac{1}{T}\int_0^T a(t)\,dt = 1$ and $\bar{e} := \frac{1}{T}\int_0^T e(t)\,dt = 0$.

Theorem 1.1. *Assume that $\omega_\pm = +\infty$. Then, there exists $s_0 \in \mathbb{R}$ such that:*

- *for $s < s_0$, equation $(\mathscr{W}\mathscr{E}_s)$ has no T-periodic solutions;*
- *for $s = s_0$, equation $(\mathscr{W}\mathscr{E}_s)$ has at least one T-periodic solution;*
- *for $s_0 < s$, equation $(\mathscr{W}\mathscr{E}_s)$ has at least two T-periodic solutions.*

The above theorem extends the recent result in [21] to the case of ϕ-Laplacian operators. Actually, Theorem 1.1 follows from a more general result dealing with equation $(\mathscr{E}_s)$, which extends some results in [15] to locally coercive nonlinearities.

Theorem 1.2. *Assume that $\omega_\pm = \omega \in \mathbb{R}$ and $q(u) > \omega$ for all $|u|$ sufficiently large. Then, there exists $s_0 \in \,]\omega, +\infty[$ such that:*

- *for $s > s_0$, equation $(\mathscr{W}\mathscr{E}_s)$ has no T-periodic solutions;*

- *for $s = s_0$, equation ($\mathscr{WE}_s$) has at least one T-periodic solution;*
- *for $\omega < s < s_0$, equation ($\mathscr{WE}_s$) has at least two T-periodic solutions.*

Moreover, if $q(u) > \omega$ for all u, then, for $s \leq \omega$, equation ($\mathscr{WE}_s$) has no T-periodic solutions.

The above theorem allows to consider the situation of [3, 5] in a nonlocal setting.

Theorem 1.3. *Assume that $\omega_- = +\infty$ and $q(u) \nearrow \omega_+ \in \mathbb{R}$ for $u \to +\infty$. Then, there exists $s_0 \in \,]-\infty, \omega_+[$ such that:*

- *for $s < s_0$, equation ($\mathscr{WE}_s$) has no T-periodic solutions;*
- *for $s = s_0$, equation ($\mathscr{WE}_s$) has at least one T-periodic solution;*
- *for $s_0 < s < \omega_+$, equation ($\mathscr{WE}_s$) has at least two T-periodic solutions;*
- *for $s \geq \omega_+$, equation ($\mathscr{WE}_s$) has at least one T-periodic solution.*

As far as we know, the above theorem covers some situations which are not treated in the literature from the point of view of the Ambrosetti–Prodi type alternatives.

We notice that Theorem 1.1 and Theorem 1.3 concern nonlinearities of type I, instead Theorem 1.2 is about nonlinearities of type II. These theorems can be considered as a model to produce different related results by means of symmetries or change of variables (see the foregoing sections). Moreover, they are all consequences of a general theorem given in Section 3.

The plan of the paper is as follows. In Section 2 we introduce some preliminary results based on continuation theorems and topological degree tools developed by Manásevich and Mawhin in [14]. Moreover, taking into account [21], we adapt Villari's type conditions to our setting. Section 3 is devoted to our main results for the parameter-dependent equation ($\mathscr{E}_s$). The key ingredient for the proofs is Theorem 2.2 in Section 2, combined with arguments inspired from [5, 7, 15]. In the same section, following [16, 18], we also recall a result of Amann, Ambrosetti and Mancini type on bounded nonlinearities (cf. [1]). In Section 4, we illustrate some applications of the main results achieved in Section 3 to the weighted Liénard equation ($\mathscr{WE}_s$) and Neumann problems for radially symmetric solutions.

2. Preliminary results

In this section we deal with the differential equation

$$(\phi(u'))' + f(u)u' + h(t,u) = 0, \tag{2.1}$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We denote by $\mathcal{C}^1_T := \{u \in \mathcal{C}^1([0,T])\colon u(0) = u(T),\, u'(0) = u'(T)\}$, and by $\mathfrak{AC}$ the set of absolutely continuous functions. By a T-periodic solution to (2.1) we mean a function

$$u \in \mathfrak{D} := \bigl\{u \in \mathcal{C}^1_T\colon \phi(u') \in \mathfrak{AC}\bigr\},$$

satisfying equation (2.1) for a.e. t. Our purpose is to introduce the main tools for the discussion in the subsequent section.

2.1. Definitions and technical lemmas

We start by introducing two concepts: the Villari's type conditions, which are inspired by [22] (cf. also [4, 14, 17]), and the upper/lower solutions (cf. [15]).

Definition 2.1. *A Carathéodory function $h(t,u)$ satisfies the Villari's condition at $+\infty$ (at $-\infty$, respectively) if there exists $\delta = \pm 1$ and $d_0 > 0$ such that*

$$\delta \int_0^T h(t,u(t))\,dt > 0 \tag{2.2}$$

for each $u \in \mathcal{C}^1_T$ such that $u(t) \geq d_0$ ($u(t) \leq -d_0$, respectively) for every $t \in [0,T]$.

Definition 2.2. *Let $\alpha, \beta \in \mathfrak{D}$. We say that α is a strict lower solution to* (2.1)*, if*

$$(\phi(\alpha'(t)))' + f(\alpha(t))\alpha'(t) + h(t,\alpha(t)) > 0, \quad \text{for a.e. } t \in [0,T], \tag{2.3}$$

and if u is any T-periodic solution to (2.1) *with $u(t) \geq \alpha(t)$ for all $t \in [0,T]$, then $u(t) > \alpha(t)$ for all $t \in [0,T]$. We say that β is a strict upper solution to* (2.1)*, if*

$$(\phi(\beta'(t)))' + f(\beta(t))\beta'(t) + h(t,\beta(t)) < 0, \quad \text{for a.e. } t \in [0,T], \tag{2.4}$$

and if u is any T-periodic solution to (2.1) *with $u(t) \leq \beta(t)$ for all $t \in [0,T]$, then $u(t) < \beta(t)$ for all $t \in [0,T]$.*

Following [6, Chapter 3, Proposition 1.5], we present now a useful criterion that guarantees when a function α satisfying (2.3) is a strict lower solution.

Lemma 2.1. *Let $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

(A_0) *for all $t_0 \in [0,T]$, $u_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t - t_0| < \delta$, $|u - u_0| < \delta$, then $|h(t,u) - h(t,u_0)| < \varepsilon$ for a.e. t.*

Let $a > 0$ and $\alpha \in \mathfrak{D}$ be such that

$$(\phi(\alpha'(t)))' + f(\alpha(t))\alpha'(t) + h(t,\alpha(t)) \geq a, \quad \text{for a.e. } t \in [0,T],$$

then α is a strict lower solution to (2.1).

Proof. Let u be a T-periodic solution to (2.1) with $u(t) \geq \alpha(t)$ for every $t \in [0,T]$. Since α is not a solution to equation (2.1), there exists $t_0 \in [0,T]$ such that $u(t_0) > \alpha(t_0)$. Suppose by contradiction that there exists a maximal interval $[t_1,t_2]$ in $[0,T]$ containing t_0 such that $u(t) > \alpha(t)$ for all $t \in]t_1,t_2[$ with $u(t_1) = \alpha(t_1)$ or $u(t_2) = \alpha(t_2)$. Since $u(t) - \alpha(t) \geq 0$ for all $t \in [0,T]$, then $u'(t_1) = \alpha'(t_1)$ or $u'(t_2) = \alpha'(t_2)$. First, let us suppose that $u(t_1) = \alpha(t_1)$. In this manner, for a.e. $t \in [0,T]$, we have

$$\begin{aligned}&(\phi(u'(t)))' - (\phi(\alpha'(t)))' \leq \\ &\leq -a - f(u(t))u'(t) - h(t,u(t)) + f(\alpha(t))\alpha'(t) + h(t,\alpha(t)).\end{aligned}$$

From condition (A_0), for $\varepsilon = a/4$, there exists $\delta > 0$ such that, if $|t - t_1| < \delta$, $|u - u(t_1)| < \delta$, then $|h(t,u) - h(t,u(t_1))| < a/4$ for a.e. t. Furthermore, by continuity, let $\eta < \delta$ be such that $|u(t) - u(t_1)| < \delta$, $|\alpha(t) - \alpha(t_1)| < \delta$ and $|f(u(t))u'(t) - f(\alpha(t))\alpha'(t)| < a/2$, for all $t \in [t_1, t_1 + \eta]$. Then, we have $(\phi(u'(t)))' - (\phi(\alpha'(t)))' \leq 0$ and, by an integration on $[t_1, t] \subseteq [t_1, t_1 + \eta]$, we obtain $\phi(u'(t)) - \phi(\alpha'(t)) \leq 0$ for a.e. $t \in [t_1, t_1 + \eta]$. It follows that $u'(t) \leq \alpha'(t)$ and so $u(t) \leq \alpha(t)$, for all $t \in [t_1, t_1 + \eta]$. Then a contradiction with the definition of the interval $[t_1, t_2]$ occurs. Lastly, if $u(t_2) = \alpha(t_2)$, then a contradiction is reached in the same way. Finally, α is a strict lower solution to (2.1). □

A similar result holds for strict upper solutions.

Lemma 2.2. *Let $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying condition (A_0). Let $b > 0$ and $\beta \in \mathfrak{D}$ be such that*

$$(\phi(\beta'(t)))' + f(\beta(t))\beta'(t) + h(t,\beta(t)) \leq -b, \quad \text{for a.e. } t \in [0,T],$$

then β is a strict upper solution to (2.1).

Our approach is based on continuation theorems, hence we focus our attention on the parameter depended equation

$$(\phi(u'))' + \lambda f(u)u' + \lambda h(t,u) = 0, \tag{2.5}$$

with $\lambda \in \,]0,1]$. In particular, the detection of some a priori bounds for solutions to (2.5) leads to the following.

Lemma 2.3. *Let $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. If there exists $d_1 > 0$ such that*

$$\int_0^T h(t,u(t))\,dt \neq 0 \tag{2.6}$$

for each $u \in \mathcal{C}^1_T$ such that $u(t) \geq d_1$ for every $t \in [0,T]$, then any T-periodic solution u of (2.5) *with $\lambda \in \,]0,1]$ satisfies* $\min u < d_1$*. If there exists $d_2 > 0$ such that* (2.6) *holds for each $u \in \mathcal{C}^1_T$ such that $u(t) \leq -d_2$ for every $t \in [0,T]$, then any T-periodic solution u of* (2.5) *with $\lambda \in \,]0,1]$ satisfies* $\max u > -d_2$.

Proof. Let u be a T- periodic solution to (2.5) with $\lambda \in \,]0,1]$. By integrating, we have

$$0 = \int_0^T \bigl[(\phi(u'(t)))' + \lambda f(u(t))u'(t) + \lambda h(t,u(t))\bigr]\,dt = \lambda \int_0^T h(t,u(t))\,dt.$$

Suppose by contradiction that either $u(t) \geq d_1$ or $u(t) \leq -d_2$ for all $t \in [0,T]$, then a contradiction follows with respect to (2.6). □

Lemma 2.4. *Let $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following property:*

(A_1^{I}) *there exists $\gamma \in L^1([0,T],\mathbb{R}^+)$ such that $h(t,u) \geq -\gamma(t)$ for a.e. $t \in [0,T]$ and for all $u \in \mathbb{R}$.*

Then, there exists a constant $K_0 = K_0(\gamma)$ such that any T-periodic solution u to (2.5) *with $\lambda \in \,]0,1]$ satisfies*

$$\max u - \min u \leq K_0 \quad \text{and} \quad \|u'\|_{L^1} \leq K_0.$$

Moreover, for any $\ell_1, \ell_2 \in \mathbb{R}$ with $\ell_1 < \ell_2$ there exists $K_1 > 0$ such that any T-periodic solution u to (2.5) *with $\lambda \in \,]0,1]$ such that $\ell_1 \leq u(t) \leq \ell_2$ for all $t \in [0,T]$ satisfies $\|u'\|_\infty \leq K_1$.*

Proof. Let $\lambda \in \,]0,1]$ and let u be a T-periodic solution to (2.5). Let t^* be such that $u(t^*) = \max u$ and define $v(t) := \max u - u(t)$, which satisfies $v' = -u'$. By hypothesis (A_1^{I}), we deduce that for almost every $t \in [0,T]$

$$(\phi(u'(t)))' = -\lambda f(u(t))u'(t) - \lambda h(t,u(t)) \leq \lambda f(u(t))v'(t) + \gamma(t). \tag{2.7}$$

Up to an extension of $h(\cdot,u)$ by T-periodicity on $\mathbb{R}$, we notice that

$$\int_{t^*}^{t^*+T} f(u(\xi))v(\xi)v'(\xi)\,d\xi = 0.$$

Multiplying (2.7) by $v(t) \geq 0$ and integrating on $[t^*, t^*+T]$, we obtain

$$\int_{t^*}^{t^*+T} (\phi(u'(\xi)))'v(\xi)\,d\xi \leq \|\gamma\|_{L^1}\|v\|_\infty.$$

At this point, from the properties of ϕ it follows straightway that for every $b > 0$ there exists $K_b > 0$ such that $\phi(\xi)\xi \geq b|\xi| - K_b$, for every $\xi \in \mathbb{R}$. In this manner, via an integration by parts, it follows that

$$\begin{aligned}\int_{t^*}^{t^*+T} (\phi(u'(\xi)))'v(\xi)\,d\xi &= \int_{t^*}^{t^*+T} \phi(u'(\xi))u'(\xi)\,d\xi = \int_0^T \phi(u'(\xi))u'(\xi)\,d\xi \\ &\geq \int_0^T \big(b|u'(\xi)| - K_b\big)\,d\xi = b\|u'\|_{L^1} - K_bT = b\|v'\|_{L^1} - K_bT.\end{aligned}$$

Finally, we obtain

$$b\|v'\|_{L^1} \leq \|\gamma\|_{L^1}\|v\|_\infty + K_bT \leq \|\gamma\|_{L^1}\|v'\|_{L^1} + K_bT = \|\gamma\|_{L^1}\|u'\|_{L^1} + K_bT.$$

Then, taking $b > \|\gamma\|_{L^1}$ and $K_0 := K_bT/(b - \|\gamma\|_{L^1})$, we have $\|u'\|_{L^1} \leq K_0$ and hence $\max u - \min u \leq K_0$.

Let $\ell_1, \ell_2 \in \mathbb{R}$ with $\ell_1 < \ell_2$. Let u be such that $\ell_1 \leq u(t) \leq \ell_2$ for all $t \in [0,T]$. By the Carathéodory condition on h and the boundedness of u, there exists a constant $c > 0$ (independent of u and $\lambda \in \,]0,1]$) such that $u'(t) \in \phi^{-1}([-c,c])$, and so there exists $K_1 > 0$ such that $\|u'\|_\infty \leq K_1$. □

Lemma 2.5. *Let $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following property:*

(A_1^{II}) *there exists $\gamma \in L^1([0,T],\mathbb{R}^+)$ such that $h(t,u) \leq \gamma(t)$ for a.e. $t \in [0,T]$ and for all $u \in \mathbb{R}$.*

Then, there exists a constant $K_0 = K_0(\gamma)$ such that any T-periodic solution u to (2.5) *with $\lambda \in \,]0,1]$ satisfies*

$$\max u - \min u \leq K_0 \quad \text{and} \quad \|u'\|_{L^1} \leq K_0.$$

Moreover, for any $\ell_1, \ell_2 \in \mathbb{R}$ *with* $\ell_1 < \ell_2$ *there exists* $K_1 > 0$ *such that any* T*-periodic solution* u *to* (2.5) *with* $\lambda \in \,]0,1]$ *such that* $\ell_1 \leq u(t) \leq \ell_2$ *for all* $t \in [0,T]$ *satisfies* $\|u'\|_\infty \leq K_1$.

Proof. Let $\lambda \in \,]0,1]$ and let u be a T-periodic solution to (2.5). We enter in the same setting of Lemma 2.4 via the change of variable $x := -u$ which leads to the study of

$$(\tilde{\phi}(x'))' + \lambda \tilde{f}(x)x' + \lambda \tilde{h}(t,x) = 0,$$

where $\tilde{\phi}(\xi) = -\phi(-\xi)$, $\tilde{f}(\xi) = f(-\xi)$, and $\tilde{h}(t,\xi) = -h(t,-\xi)$. $\square$

2.2. Continuation theorem and abstract results

We introduce the fixed point operator and the continuation theorem for the more general periodic boundary value problem

$$(\phi(u'))' + F(t,u,u') = 0, \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{2.8}$$

where $F\colon [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We consider the following Banach spaces $X := \mathcal{C}^1_T$, endowed with the norm $\|u\|_X := \|u\|_\infty + \|u'\|_\infty$, and $Z := L^1(0,T)$, with the standard norm $\|\cdot\|_{L^1}$. In the same spirit of [14], we define the continuous projectors $P\colon X \to X$ by $Pu := u(0)$, and $Q\colon Z \to Z$ by $Qu := \frac{1}{T}\int_0^T u(t)\,dt$. In the sequel, we also denote by Q the mean value operator defined on subspaces of Z. We introduce the following Nemytskii operator $N\colon X \to Z$ by $(Nu)(t) := -F(t,u(t),u'(t))$ for $t \in [0,T]$.

At this point, following [14], one has that u is a solution of problem (2.8) if and only if u is a fixed point of the completely continuous operator $\mathcal{G}\colon X \to X$ defined as

$$\mathcal{G}u := Pu + QNu + \mathcal{K}Nu, \quad u \in X,$$

where $\mathcal{K}\colon Z \to X$ is the map which, to any $w \in Z$, associates the unique T-periodic solution $u(t)$ of the problem

$$(\phi(u'))' = w(t) - \frac{1}{T}\int_0^T w(t)\,dt, \quad u(0) = 0.$$

Let us consider the periodic parameter-dependent problem

$$(\phi(u'))' + \lambda F(t,u,u') = 0, \quad u(0) = u(T), \quad u'(0) = u'(T), \quad \lambda \in \,]0,1]. \tag{2.9}$$

We are now ready to state the following continuation theorem, adapted from [14], where by $\mathrm{d_{LS}}(Id-\mathcal{G},\Omega,0)$ we denote the *Leray–Schauder degree* of $Id-\mathcal{G}$ in Ω, with $\Omega \in X$ an open bounded set, and by $\mathrm{d_B}$ we indicate the finite-dimensional *Brouwer degree*. We refer to [14, Theorem 3.1] for the proof of the following theorem (see also [8, Section 3]).

Theorem 2.1. *Let* Ω *be an open bounded set in* X *such that the following conditions hold:*

- *for each* $\lambda \in \,]0,1]$ *problem* (2.9) *has no solution on* $\partial\Omega$*;*
- *the equation* $F^{\#}(\xi) := \frac{1}{T}\int_0^T F(t,\xi,0)\,dt = 0$ *has no solution on* $\partial\Omega \cap \mathbb{R}$.

Then, $\mathrm{d_{LS}}(Id-\mathcal{G},\Omega,0) = \mathrm{d_B}(F^{\#},\Omega\cap\mathbb{R},0)$*. Moreover, if the Brouwer degree* $\mathrm{d_B}(F^{\#},\Omega\cap\mathbb{R},0) \neq 0$*, then problem* (2.1) *has a solution in* Ω.

Dealing with equation (2.1), we consider now the special form of the Carathéodory function $F(t,u,u') = f(u)u' + h(t,u)$ and the following result holds.

Theorem 2.2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $h\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (A_1^{I}) and the Villari's condition at $-\infty$ with $\delta = 1$. Suppose there exists $\beta \in \mathfrak{D}$ which is a strict upper solution for equation* (2.1)*. Then,* (2.1) *has at least a T-periodic solution $\tilde{u}$ such that $\tilde{u} < \beta$. Moreover, there exist $R_0 \geq d_0$ and $K_1 > 0$, such that for each $R > R_0$ and $K > K_1$, we have*

$$\mathrm{d_{LS}}(Id - \mathcal{G}, \Omega, 0) = -1$$

for $\Omega = \Omega^{\mathrm{I}}(R,\beta,K) := \{u \in \mathcal{C}^1_T \colon -R < u(t) < \beta(t),\ \forall\, t \in [0,T],\ \|u'\|_\infty < K\}$.

Proof. First of all, we introduce the truncated function

$$\hat{h}(t,u) := \begin{cases} h(t,u), & \text{if } u \leq \beta(t), \\ h(t,\beta(t)), & \text{if } u \geq \beta(t), \end{cases}$$

and consider the parameter-dependent equation

$$(\phi(u'))' + \lambda f(u)u' + \lambda \hat{h}(t,u) = 0, \quad \lambda \in \,]0,1]. \tag{2.10}$$

By the assumptions on h, it is easy to prove that $\hat{h}$ satisfies condition (A_1^{I}). Then, we can apply Lemma 2.4 and obtain that any T-periodic solution u to (2.10) with $\lambda \in \,]0,1]$ satisfies $\max u - \min u \leq K_0$ and $\|u'\|_{L^1} \leq K_0$, for some constant K_0. Let $d_2 > \max\{d_0, \|\beta\|_\infty\}$. By Lemma 2.3 we deduce that $\max u > -d_2$ and so $\min u > -K_0 - d_2 =: -R_0$.

We claim that, for any T-periodic solution u to (2.10) with $\lambda \in \,]0,1]$, there exists $\hat{t} \in [0,T]$ such that $u(\hat{t}) < \beta(\hat{t})$. Indeed, if by contradiction $u(t) \geq \beta(t)$ for all $t \in [0,T]$, then u is a T-periodic solution to

$$(\phi(u'))' + \lambda f(u)u' + \lambda h(t,\beta(t)) = 0, \quad \lambda \in \,]0,1].$$

By an integration, we have $\int_0^T h(t,\beta(t))\,dt = 0$. The strict upper solution β is T-periodic and satisfies (2.4), then we obtain $\int_0^T h(t,\beta(t))\,dt < 0$, a contradiction. Therefore $\min u < \|\beta\|_\infty$ and so $\max u < \|\beta\|_\infty + K_0 < R_0$.

An application of Lemma 2.4 in the framework of (2.10) (with $\ell_1 = -R_0$, $\ell_2 := \|\beta\|_\infty + K_0$) guarantees the existence of a constant $\hat{K}_0$ such that $\|u'\|_\infty \leq \hat{K}_0$, for any T-periodic solution u to (2.10) with $\lambda \in \,]0,1]$.

We deduce that the Leray–Schauder degree $\mathrm{d_{LS}}(Id - \hat{\mathcal{G}}, \Gamma, 0)$ is well-defined on any open and bounded set

$$\Gamma := \bigl\{u \in \mathcal{C}^1_T \colon -R < u(t) < C,\ \forall\, t \in [0,T],\ \|u'\|_\infty < \hat{K}\}$$

for any $R > R_0$, $C \geq \|\beta\|_\infty + K_0$ and $\hat{K} > \hat{K}_0$.

Now we introduce the average scalar map $\hat{h}^{\#}\colon \mathbb{R} \to \mathbb{R}$, defined by $\hat{h}^{\#}(\xi) := \frac{1}{T}\int_0^T \hat{h}(t,\xi)\,dt$, for $\xi \in \mathbb{R}$. We notice that $\hat{h}^{\#}(-R) > 0$, by the

Villari's condition at $-\infty$, and $\hat{h}^{\#}(c) < 0$, taking $c \geq \|\beta\|_\infty$. As a consequence of Theorem 2.1, we have

$$d_{\mathrm{LS}}(Id - \hat{\mathcal{G}}, \Gamma, 0) = d_{\mathrm{B}}(\hat{F}^{\#}, \Gamma \cap \mathbb{R}, 0) = -1,$$

and so problem (2.10) with $\lambda = 1$ has at least a solution $\tilde{u}$ in Γ, more precisely $\tilde{u}$ satisfies $-R < \tilde{u}(t) < C$, for all $t \in [0,T]$, and $\|\tilde{u}'\|_\infty < \hat{K}$.

We claim that $\tilde{u}(t) \leq \beta(t)$ for all $t \in [0,T]$. We have already proved that there exists $t_* \in [0,T]$ such that $\tilde{u}(t_*) < \beta(t_*)$. Suppose by contradiction that there exists $t^* \in [0,T]$ such that $\tilde{u}(t^*) > \beta(t^*)$. Let $]t_1, t_2[$ be the maximal open interval containing t^* such that $\tilde{u} > \beta$. Then $\tilde{u}(t_1) = \beta(t_1)$ and $\tilde{u}(t_2) = \beta(t_2)$. Moreover $\tilde{u}'(t_1) \geq \beta'(t_1)$ and $\tilde{u}'(t_2) \leq \beta'(t_2)$, and so $\phi(\tilde{u}'(t_1)) \geq \phi(\beta'(t_1))$ and $\phi(\tilde{u}'(t_2)) \leq \phi(\beta'(t_2))$, due to the monotonicity of the homeomorphism ϕ. Next, by an integration and recalling the definition of $\hat{h}$, we have

$$\begin{aligned} 0 &\geq \int_{t_1}^{t_2} \big[(\phi(\tilde{u}'(t)))' - (\phi(\beta'(t)))'\big]\,dt \\ &> \int_{t_1}^{t_2} \big[f(\tilde{u}(t))\tilde{u}'(t) - f(\beta(t))\beta'(t)\big]\,dt + \int_{t_1}^{t_2} \big[\hat{h}(t,\tilde{u}(t)) - \hat{h}(t,\beta(t))\big]\,dt = 0 \end{aligned}$$

and a contradiction is found. Then $\tilde{u}(t) \leq \beta(t)$ for all $t \in [0,T]$. Hence, $\tilde{u}$ is a solution of (2.1) and, since β is a strict upper solution, $\tilde{u}(t) < \beta(t)$ for all $t \in [0,T]$.

As a final step, we apply Lemma 2.4 in the framework of (2.1) and we obtain a constant $K_1 > 0$ such that $\|u'\|_\infty \leq K_1$, for any T-periodic solution u to (2.1). We reach the thesis via the excision property of the topological degree. □

Analogously we obtain the following result.

Theorem 2.3. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $h \colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (A_1^{II}) and the Villari's condition at $+\infty$ with $\delta = -1$. Suppose there exists $\alpha \in \mathfrak{D}$ which is a strict lower solution for equation (2.1). Then, (2.1) has at least a T-periodic solution $\tilde{u}$ such that $\tilde{u} > \alpha$. Moreover, there exist $R_0 \geq d_0$ and $K_1 > 0$, such that for each $R > R_0$ and $K > K_1$, we have*

$$d_{\mathrm{LS}}(Id - \mathcal{G}, \Omega, 0) = -1$$

for $\Omega = \Omega^{\mathrm{II}}(R, \alpha, K) := \{u \in \mathcal{C}^1_T \colon \alpha(t) < u(t) < R,\ \forall\, t \in [0,T],\ \|u'\|_\infty < K\}$.

Remark 2.1. Assuming (A_0) and given $\beta \in \mathfrak{D}$ an upper solution to (2.1) (or in other words satisfying the weaker form of (2.4)), one can still prove the existence of a T-periodic solution $\tilde{u}$ with $\tilde{u} \leq \beta$ under the weaker inequality in (2.2), namely $\int_0^T h(t,u(t))\,dt \geq 0$ for each $u \leq -c_0$. To this purpose, we introduce the auxiliary function

$$h^\varepsilon(t,u) := h(t,u) + \varepsilon \min\{1, \max\{-1, -u - \|\beta\|_\infty - 1\}\}, \quad \text{for } \varepsilon > 0.$$

Now β becomes a strict upper solution for the modified equation $(\phi(u'))' + f(u)u' + h^{\varepsilon}(t,u) = 0$ (for each $\varepsilon > 0$). Moreover, the Villari's condition holds in the original strict form (for $u \le -d_0$ with $d_0 > \max\{c_0, \|\beta\|_\infty + 1\}$). It is easy to check that Theorem 2.2 can be applied to obtain the existence of a T-periodic solution $\tilde{u}^{\varepsilon}$ to $(\phi(u'))' + f(u)u' + h^{\varepsilon}(t,u) = 0$ such that $-R_0 < \tilde{u}^{\varepsilon} \le \beta$. The constants R_0 and K_1 can be chosen uniformly with respect to ε due to particular form of the (bounded) perturbation. An application of Ascoli–Arzelà theorem leads to the existence of a solution $\tilde{u}$ for (2.1). An analogous weaker formulation of Theorem 2.3 holds too. $\triangleleft$

3. Main results

In this section we present our main results concerning T-periodic solutions to the parameter-dependent equation

$$(\mathscr{E}_s) \qquad (\phi(u'))' + f(u)u' + g(t,u) = s.$$

Along the section, we assume that $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, the map $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, the function $g\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and $s \in \mathbb{R}$. Furthermore, we introduce the following condition.

(H_0) *For all $t_0 \in [0,T]$, $u_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t - t_0| < \delta$, $|u - u_0| < \delta$, then $|g(t,u) - g(t,u_0)| < \varepsilon$ for a.e. t.*

In the first result, the following hypotheses will be considered as well.

(H_1^{I}) *There exists $\gamma_0 \in L^1([0,T], \mathbb{R}^+)$ such that $g(t,u) \ge -\gamma_0(t)$, for all $u \in \mathbb{R}$ and a.e. $t \in [0,T]$.*

(H_2^{I}) *There exist u_0, $g_0 \in \mathbb{R}$ such that $g(t,u_0) \le g_0$ for a.e. $t \in [0,T]$.*

(H_3^{I}) *There exist $\sigma > \max\{0, g_0\}$ and $d > 0$ such that $\frac{1}{T}\int_0^T g(t,u(t))\,dt > \sigma$ for each $u \in \mathcal{C}^1_T$ such that $u(t) \le -d$ for all $t \in [0,T]$.*

(H_4^{I}) *There exist $\sigma > \max\{0, g_0\}$ and $d > 0$ such that $\frac{1}{T}\int_0^T g(t,u(t))\,dt > \sigma$ for each $u \in \mathcal{C}^1_T$ such that $u(t) \ge d$ for all $t \in [0,T]$.*

We are now in position to state and prove our first main result.

Theorem 3.1. *Assume (H_0), (H_1^{I}), (H_2^{I}) and (H_3^{I}) and let*

$$\sigma^* := \sup\bigl\{\sigma \in \,]g_0, +\infty[\colon (H_3^{\mathrm{I}}) \text{ is satisfied}\bigr\}. \tag{3.1}$$

Then, there exists $s_0 \in \,]-\infty, \sigma^*[$ such that:

- *for $s_0 < s < \sigma^*$, equation ($\mathscr{E}_s$) has at least one T-periodic solution;*
- *for $s < s_0$, equation ($\mathscr{E}_s$) has no T-periodic solutions.*

Moreover, if (H_4^{I}) holds, then for

$$\sigma^{**} := \sup\bigl\{\sigma \in \,]g_0, \sigma^*]\colon (H_4^{\mathrm{I}}) \text{ is satisfied}\bigr\}$$

it follows that:

- *for $s = s_0$, equation ($\mathscr{E}_s$) has at least one T-periodic solution;*
- *for $s_0 < s < \sigma^{**}$, equation ($\mathscr{E}_s$) has at least two T-periodic solutions.*

Proof. We split the proof into two steps. In the first one, we prove that there are no solutions to ($\mathscr{E}_s$) if the parameter s is sufficiently small. Moreover, we show that the set of the parameters s for which ($\mathscr{E}_s$) has at least one T-periodic solution is an interval. In the second one, we discuss the existence and the multiplicity of solutions to ($\mathscr{E}_s$) in dependence of the parameter s. We follow the approach in [7], by adapting also some arguments from [21]. We consider $h_s(t,u) := g(t,u) - s$ to deal with an equation of the form (2.1).

Step 1. If u is a T-periodic solution to ($\mathscr{E}_s$), then we have $\frac{1}{T}\int_0^T g(t,u(t))\,dt = s$, taking the average of the equation on $[0,T]$. Hence, from condition (H_1^{I}), it follows that equation ($\mathscr{E}_s$) has no T-periodic solutions for

$$s < s^{\#} := -\frac{1}{T}\int_0^T \gamma_0(t)\,dt. \tag{3.2}$$

It is worth noting that assumption (H_0) implies that the function h_s satisfies (A_0). For each $s > g_0$, the constant function $\beta \equiv u_0$ is a strict upper solution to ($\mathscr{E}_s$). Indeed, we observe that

$$(\phi(\beta'(t)))' + f(\beta(t))\beta'(t) + h_s(t,\beta(t)) = g(t,u_0) - s \leq g_0 - s < 0$$

and so, by Lemma 2.2 we have the claim.

Let σ_1 satisfying assumption (H_3^{I}) so that the Villari's condition at $-\infty$ with $\delta = 1$ holds. Therefore, we are in position to apply Theorem 2.2 and we obtain the existence of at least one T-periodic solution u of ($\mathscr{E}_s$) for $s = \sigma_1$ with $u < u_0$.

We claim now that if w is a T-periodic solution to ($\mathscr{E}_s$) for some $s = \tilde{\sigma} < \sigma_1$, then ($\mathscr{E}_s$) has a T-periodic solution for each $s \in [\tilde{\sigma}, \sigma_1]$. Indeed, let $s \in \,]\tilde{\sigma}, \sigma_1[$, then by applying Lemma 2.2, we notice that w is a strict upper solution to ($\mathscr{E}_s$), since

$$(\phi(w'(t)))' + f(w(t))w'(t) + g(t,w(t)) - s = \tilde{\sigma} - s < 0.$$

Moreover, as observed above, for $\sigma \in [\tilde{\sigma}, \sigma_1]$, assumption ($H_3^{\mathrm{I}}$) implies again the Villari's condition at $-\infty$ with $\delta = 1$. In this manner, by Theorem 2.2 there exists at least one T-periodic solution u of ($\mathscr{E}_s$) for $s = \sigma$ with $u < w$.

Recalling (3.2), we have deduced that the set of the parameters $s \leq \sigma_1$ for which equation ($\mathscr{E}_s$) has T-periodic solutions is an interval bounded from below (by $s^{\#}$). Let

$$s_0 := \inf\bigl\{ s \in \mathbb{R} \colon (\mathscr{E}_s) \text{ has at least one } T\text{-periodic solution} \bigr\}.$$

By the arbitrary choice of σ_1 and the definition of σ^*, we conclude that there exists at least a T-periodic solution to ($\mathscr{E}_s$) for each $s \in \,]s_0, \sigma^*[$.

Step 2. Let N_s the Nemytskii operator associated with $f(u)u' + h_s(t,u)$, namely $(N_s u)(t) := f(u(t))u'(t) + h_s(t,u(t))$, $u \in X$. Defining

$$\mathcal{G}_{\lambda,s} u := Pu + \lambda Q N_s u + \lambda \mathcal{K} N_s u, \quad u \in X,\ \lambda \in \,]0,1].$$

we obtain that problem (2.9) is equivalent to $u = \mathcal{G}_{\lambda,s} u$.

Let σ_1 satisfy assumptions (H_3^{I}) and (H_4^{I}). We claim that there exists a positive constant $\Lambda = \Lambda(\sigma_1)$ such that for each $s \leq \sigma_1$ any solution of $u = \mathcal{G}_{\lambda,s}u$, with $0 < \lambda \leq 1$, satisfies $\|u\|_\infty < \Lambda$.

An application of Lemma 2.3 ensures that $\max u > -d$ and $\min u < d$, for any possible T-periodic solution u to

$$(\phi(u'(t)))' + \lambda f(u(t))u'(t) + \lambda h_s(t, u(t)) = 0, \quad \lambda \in \,]0, 1]. \tag{3.3}$$

Moreover, by (H_1^{I}) and $s \leq \sigma_1$, it follows that $h_s(t, u) \geq -\gamma_0(t) - \sigma_1$, for a.e. $t \in [0, T]$. Now we apply Lemma 2.4 with $\gamma(t) := \gamma_0(t) + |\sigma_1|$, and so there exists a positive constant $K_0 = K_0(\sigma_1)$ such that $\max u - \min u \leq K_0$, for any possible T-periodic solution u to (3.3). Thus the claim follows, since by the above inequalities, we have that $\|u\|_\infty < \Lambda(\sigma_1) := K_0(\sigma_1) + d$.

Let us fix now a constant $\sigma_2 < s^{\#}$. Let also ρ_g be a non-negative L^1-function such that $|h_s(t, u)| \leq \rho_g(t) + \max\{\sigma_1, |\sigma_2|\}$, for a.e. $t \in [0, T]$, for all $s \in [\sigma_2, \sigma_1]$, for all $u \in [-\Lambda(\sigma_1), \Lambda(\sigma_1)]$. From Lemma 2.4 there exists a constant $K_1 = K_1(\sigma_1, \sigma_2) > 0$ such that, for each $s \in [\sigma_2, \sigma_1]$, any solution of $u = \mathcal{G}_{\lambda,s}u$ with $0 < \lambda \leq 1$ satisfies $\|u'\|_\infty < K_1$.

By considering the homotopic parameter $s \in [\sigma_2, \sigma_1]$ and defining

$$\Omega_1 = \Omega_1(R_0, R_1) := \bigl\{u \in \mathcal{C}^1_T : \|u\|_\infty < R_0,\ \|u'\|_\infty < R_1\bigr\},$$

we obtain that

$$\mathrm{d_{LS}}(Id - \mathcal{G}_{1,\sigma_1}, \Omega_1, 0) = \mathrm{d_{LS}}(Id - \mathcal{G}_{1,\sigma_2}, \Omega_1, 0) = 0, \quad \forall\, R_0 \geq \Lambda(\sigma_1), \forall\, R_1 \geq K_1. \tag{3.4}$$

From the conclusions achieved in Step 1, $(\mathscr{E}_s)$ has a T-periodic solution for every $s \in \,]s_0, \sigma^{**}[\, \subseteq \,]s_0, \sigma^*[$.

Let $\tilde{u}_1$ be a T-periodic solution to $(\mathscr{E}_s)$ for some $s = \tilde{\sigma}_1 \in \,]s_0, \sigma^{**}[$. Let us fix $s \in \,]\tilde{\sigma}_1, \sigma^{**}[$ and claim that a second solution to $(\mathscr{E}_s)$ exists. Clearly, since $s > \tilde{\sigma}_1$, it follows that $\tilde{u}_1$ is a strict upper solution to $(\mathscr{E}_s)$.

By the validity of the Villari's condition at $-\infty$ with $\delta = 1$ and an application of Theorem 2.2 we have

$$\mathrm{d_{LS}}(Id - \mathcal{G}_{1,s}, \Omega^{\mathrm{I}}(R_0, w, R_1), 0) = -1, \tag{3.5}$$

where $R_0 \geq \Lambda(\sigma_1) + 1$ and $R_1 \geq K_1$.

Now, from (3.4), (3.5) and $\Omega^{\mathrm{I}}(R_0, w, R_1) \subseteq \Omega_1$, we obtain that there exists also a second solution to $(\mathscr{E}_s)$ contained in $\Omega_1 \setminus \overline{\Omega^{\mathrm{I}}(R_0, w, R_1)}$, via the additivity property of the topological degree.

We conclude the proof by showing that for $s = s_0$ there is at least one T-periodic solution.

Let us fix σ_1, σ_2 with $\sigma_2 < s_0 < \sigma_1 < \sigma^{**}$. Let $(s_n)_n \subseteq \,]s_0, \sigma_1]$ be a decreasing sequence with $s_n \to s_0$. By the above estimates, for each n there exists at least one T-periodic solution w_n to

$$(\phi(w_n'(t)))' + f(w_n(t))w_n'(t) + g(t, w_n(t)) = s_n$$

with $\|w_n\|_\infty \leq \Lambda(\sigma_1)$ and $\|w_n'\|_\infty \leq K_1(\sigma_1, \sigma_2)$. Passing to the limit as $n \to \infty$ and applying Ascoli–Arzelà theorem, we achieve the existence of at least one T-periodic solution to $(\mathscr{E}_s)$ for $s = s_0$, concluding the proof. $\square$

The following hypotheses will be assumed in the next result.

(H_1^{II}) *There exists* $\gamma_0 \in L^1([0,T],\mathbb{R}^+)$ *such that* $g(t,u) \leq \gamma_0(t)$, *for all* $u \in \mathbb{R}$ *and a.e.* $t \in [0,T]$.

(H_2^{II}) *There exist* u_0, $g_0 \in \mathbb{R}$ *such that* $g(t,u_0) \geq g_0$ *for a.e.* $t \in [0,T]$.

(H_3^{II}) *There exist* $\nu < \min\{0,g_0\}$ *and* $d > 0$ *such that* $\frac{1}{T}\int_0^T g(t,u(t))\,dt < \nu$ *for each* $u \in \mathcal{C}^1_T$ *such that* $u(t) \leq -d$ *for all* $t \in [0,T]$.

(H_4^{II}) *There exist* $\nu < \min\{0,g_0\}$ *and* $d > 0$ *such that* $\frac{1}{T}\int_0^T g(t,u(t))\,dt < \nu$ *for each* $u \in \mathcal{C}^1_T$ *such that* $u(t) \geq d$ *for all* $t \in [0,T]$.

Our second main result is the following, which can be viewed as a "dual" version of Theorem 3.1.

Theorem 3.2. *Assume* (H_0), (H_1^{II}), (H_2^{II}) *and* (H_4^{II}) *and let*

$$\nu^* := \inf\bigl\{\nu \in \,]-\infty, g_0[\,\colon (H_4^{\mathrm{II}}) \text{ is satisfied}\bigr\}.$$

Then, there exists $s_0 \in \,]\nu^*, +\infty[$ *such that:*

- *for* $\nu^* < s < s_0$, *equation* ($\mathscr{E}_s$) *has at least one* T*-periodic solution;*
- *for* $s > s_0$, *equation* ($\mathscr{E}_s$) *has no* T*-periodic solutions.*

Moreover, if (H_3^{II}) *holds, then for*

$$\nu^{**} := \sup\bigl\{\nu \in [\nu^*, g_0[\,\colon (H_3^{\mathrm{II}}) \text{ is satisfied}\bigr\}$$

it follows that:

- *for* $s = s_0$, *equation* ($\mathscr{E}_s$) *has at least one* T*-periodic solution;*
- *for* $\nu^{**} < s < s_0$, *equation* ($\mathscr{E}_s$) *has at least two* T*-periodic solutions.*

Proof. As in the proof of Lemma 2.5, the change of variable $x := -u$ transforms ($\mathscr{E}_s$) to

$$(\tilde{\phi}(x'))' + \tilde{f}(x)x' + \tilde{g}(t,x) = -s, \tag{3.6}$$

where $\tilde{\phi}(\xi) = -\phi(-\xi)$, $\tilde{f}(\xi) = f(-\xi)$, and $\tilde{g}(t,\xi) = -g(t,-\xi)$. Then we apply Theorem 3.1 to the T-periodic problem associated with (3.6). Precisely, there exists $\tilde{s}_0 \in \,]-\infty, \sigma^*[$ such that equation (3.6) has: no T-periodic solutions, for $s < \tilde{s}_0$; at least one T-periodic solution, for $s = \tilde{s}_0$; at least one T-periodic solution, for $\tilde{s}_0 < s < \sigma^*$; at least two T-periodic solutions, for $\tilde{s}_0 < s < \sigma^{**} \leq \sigma^*$. Defining $s_0 := -\tilde{s}_0$ and observing that $\nu^* = -\sigma^*$ and $\nu^{**} = -\sigma^{**}$, the thesis follows. □

Remark 3.1. We stress that conditions (H_2^{I}) and (H_2^{II}) ensure the existence of a strict upper/lower solution to ($\mathscr{E}_s$), respectively, which is given by the constant function u_0. In place of those conditions, one can assume the following.

($H_2^{\bigstar}$) *There exist a function* $u_0 \in \mathfrak{D}$ *and* $g_0 \in \mathbb{R}$ *such that*

$$(\phi(u_0'))' + f(u_0)u_0' + g(t,u_0) = g_0.$$

Indeed, by assuming condition ($H_2^{\bigstar}$), we immediately have the existence of a T-periodic solution u_0 to ($\mathscr{E}_s$) for $s = g_0$, which in turns is a strict upper/lower to ($\mathscr{E}_s$) for $s > g_0$ and for $s < g_0$, respectively. ◁

We conclude the section by proving that $(H_2^\bigstar)$ holds for semi-bounded nonlinearities g (see Proposition 3.2).

As a first step, recalling the definitions of the Banach spaces Z and $\mathfrak{D}$, and of the projector Q given in Section 2.2, we introduce the following subspaces

$$\tilde{Z} := \{w \in Z \colon Qz = 0\}, \quad \tilde{\mathfrak{D}} := \left\{u \in \mathfrak{D} \colon \int_0^T u(t)\,dt = 0\right\}.$$

We state the following result, which is a minor variant of [14, Lemma 2.1], where the operator $\tilde{\mathcal{K}}$ in our context takes the form $\tilde{\mathcal{K}} = \mathcal{K} - Q\mathcal{K}$ (with the notation introduced in Section 2.2).

Lemma 3.1. *For every $w \in \tilde{Z}$ there exists unique $u \in \tilde{\mathfrak{D}}$ such that*

$$(\phi(u'))' = w. \tag{3.7}$$

Furthermore, let $\tilde{\mathcal{K}} \colon \tilde{Z} \to \tilde{\mathfrak{D}}$ be the operator which associates to w the unique solution u to (3.7). Then, $\tilde{\mathcal{K}}$ is continuous, maps bounded sets on bounded sets, and sends equi-integrable sets into relatively compact sets.

As a second step, for $u \in \mathfrak{D}$ let $\bar{u} := \frac{1}{T}\int_0^T u(t)\,dt$. Then, we have that $u = \bar{u} + \tilde{u}$, with $\tilde{u} \in \tilde{\mathfrak{D}}$. We deal now with the problem

$$\begin{cases} (\phi(\tilde{u}'))' = \lambda\left(F(t, \bar{u} + \tilde{u}, \tilde{u}') - \dfrac{1}{T}\displaystyle\int_0^T F(\xi, \bar{u} + \tilde{u}(\xi), \tilde{u}'(\xi))\,d\xi\right), \\ \tilde{u} \in \tilde{\mathfrak{D}}, \quad \lambda \in [0,1], \end{cases} \tag{3.8}$$

which can be equivalently written as a fixed point problem of the form

$$\tilde{u} = \tilde{\mathcal{N}}(\bar{u}, \tilde{u}; \lambda) := \tilde{\mathcal{K}}(\lambda\mathcal{F}(\bar{u} + \tilde{u})), \quad \tilde{u} \in \tilde{\mathfrak{D}},\ \lambda \in [0,1],$$

where $(\mathcal{F}u)(t) = F(t, u(t), u'(t)) - QF(\cdot, u, u')(t)$, for $t \in [0,T]$. In this setting the following result adapted from [18] holds (see also [1, 10, 16]).

Lemma 3.2. *Suppose that there exists $\bar{u}_0 \in \mathbb{R}$ such that for $\bar{u} = \bar{u}_0$ the set of solutions $\tilde{u}$ to (3.8) with $\lambda \in [0,1]$ is bounded. Moreover, assume that for any $M > 0$ there exists $M' > 0$ such that if $|\bar{u}| \leq M$ then $\|\tilde{u}\|_{\mathcal{C}^1} \leq M'$, where $(\bar{u}, \tilde{u})$ is a solution pair to (3.8) for $\lambda = 1$. Then, there exists a closed and connected set $\mathscr{C} \subseteq \mathbb{R} \times \mathcal{C}^1_T$ of solutions pairs $(\bar{u}, \tilde{u})$ to (3.8) for $\lambda = 1$ such that $\{\bar{u} \in \mathbb{R} \colon (\bar{u}, \tilde{u}) \in \mathscr{C}\} = \mathbb{R}$.*

As a third step, we present an application of Lemma 3.2 for problem

$$\begin{cases} (\phi(\tilde{u}'))' + f(\bar{u} + \tilde{u})\tilde{u}' + \lambda g(t, \bar{u} + \tilde{u}) - \dfrac{\lambda}{T}\displaystyle\int_0^T g(\xi, \bar{u} + \tilde{u}(\xi))\,d\xi = 0, \\ \tilde{u} \in \tilde{\mathfrak{D}}, \quad \lambda \in [0,1]. \end{cases} \tag{3.9}$$

Proposition 3.1. *Assume that there exists a function $\rho \in L^1([0,T], \mathbb{R}^+)$ such that $|g(t,u)| \leq \rho(t)$ for a.e. $t \in [0,T]$ and for all $u \in \mathbb{R}$. Then, the following results hold.*

(i) *There exists $K = K(\phi,\rho) > 0$ such that any solution pair $(\bar{u},\tilde{u})$ to (3.9) satisfies $\|\tilde{u}'\|_{L^1} \leq K$ and $\|\tilde{u}\|_\infty \leq K$.*

(ii) *For any $M > 0$ there exists $M' = M'(\phi,f,\rho) > 0$ such that if $|\bar{u}| \leq M$ then $\|\tilde{u}\|_{\mathcal{C}^1} \leq M'$, where $(\bar{u},\tilde{u})$ is a solution pair to (3.9) for $\lambda = 1$.*

Proof. Let $\lambda \in [0,1]$ and let $(\bar{u},\tilde{u})$ be a solution pair to (3.9). We proceed similarly as in the proof of Lemma 2.4. Multiplying (3.9) by $\tilde{u}$ and integrating on $[0,T]$, we obtain

$$\int_0^T (\phi(\tilde{u}'(\xi)))'\tilde{u}(\xi)\,d\xi \leq \|\rho\|_{L^1}\|\tilde{u}\|_\infty.$$

We notice that for every $b > 0$ there exists $K_b > 0$ such that $\phi(\xi)\xi \geq b|\xi| - K_b$, for every $\xi \in \mathbb{R}$. Hence, via an integration by parts, it follows that

$$\begin{aligned}\int_0^T (\phi(\tilde{u}'(\xi)))'\tilde{u}(\xi)\,d\xi &= \int_0^T \phi(\tilde{u}'(\xi))\tilde{u}'(\xi)\,d\xi \\ &\geq \int_0^T \big(b|\tilde{u}'(\xi)| - K_b\big)\,d\xi = b\|\tilde{u}'\|_{L^1} - K_bT.\end{aligned}$$

Finally, we obtain

$$b\|\tilde{u}'\|_{L^1} \leq \|\rho\|_{L^1}\|\tilde{u}\|_\infty + K_bT \leq \|\rho\|_{L^1}\|\tilde{u}'\|_{L^1} + K_bT.$$

Then, taking $b > \|\rho\|_{L^1}$ and $K = K(\phi,\rho) := K_bT/(b - \|\rho\|_{L^1})$, we have $\|\tilde{u}'\|_{L^1} \leq K$ and so $\|\tilde{u}\|_\infty \leq K$. Hence, (i) is proved.

Let $(\bar{u},\tilde{u})$ be a solution pair to (3.9) for $\lambda = 1$. Let $M > 0$ and suppose that $|\bar{u}| \leq M$. By the assumptions on f and g, and the above remarks, we have that $f(\bar{u}+\tilde{u})\tilde{u}' + g(t,\bar{u}+\tilde{u}) - \frac{1}{T}\int_0^T g(\xi,\bar{u}+\tilde{u}(\xi))\,d\xi$ is bounded in L^1. Next, proceeding as in the last step of the proof of Lemma 2.4, we have $\|\tilde{u}'\|_\infty \leq K_1$ and so, from (i), $\|\tilde{u}\|_{\mathcal{C}^1} \leq M' := K + K_1$. The proof of (ii) is completed. □

In the next proposition we combine the results of Proposition 3.1 with an argument exploited in [5].

Proposition 3.2. *Assume that there exists a function $\rho \in L^1([0,T],\mathbb{R}^+)$ such that $|g(t,u)| \leq \rho(t)$ for a.e. $t \in [0,T]$ and for all $u \geq 0$ (or for all $u \leq 0$, respectively). Then, for every $d > 0$ there exists $g_0 \in \mathbb{R}$ such that equation $(\mathscr{E}_s)$ for $s = g_0$ has a T-periodic solution u_0 with $u_0(t) \geq d$ for all $t \in [0,T]$ (or with $u_0(t) \leq -d$ for all $t \in [0,T]$, respectively).*

Proof. Let us suppose that $|g(t,u)| \leq \rho(t)$ for a.e. $t \in [0,T]$ and for all $u \geq 0$. Let us define the nonlinearity $\hat{g}\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ as follows

$$\hat{g}(t,u) := \begin{cases} g(t,0), & \text{if } u \leq 0, \\ g(t,u), & \text{if } u \geq 0. \end{cases}$$

We notice that $|\hat{g}(t,u)| \leq \rho(t)$ for a.e. $t \in [0,T]$ and for all $u \in \mathbb{R}$. Let us consider the equation

$$(\hat{\mathscr{E}}_s) \qquad (\phi(u'))' + f(u)u' + \hat{g}(t,u) = s.$$

As a direct application of Lemma 3.2 and Proposition 3.1, we obtain that there exists a continuum $\mathscr{C} \subseteq \mathbb{R} \times \tilde{\mathfrak{D}}$ of solution pairs $(\bar{u}, \tilde{u})$ to (3.9) for $\lambda = 1$ such that $\{\bar{u} \in \mathbb{R} \colon (\bar{u}, \tilde{u}) \in \mathscr{C}\} = \mathbb{R}$. As a consequence, for every $\bar{u} \in \mathbb{R}$ there exists a solution $\tilde{u} \in \tilde{\mathfrak{D}}$ to (3.9) for $\lambda = 1$ and satisfying conditions (i) and (ii). Let $\bar{u}_0 \geq d + K$ and let $\tilde{u}_0$ be the corresponding solution to (3.9) for $\lambda = 1$. Let us define $u_0 := \bar{u}_0 + \tilde{u}_0$. We notice that u_0 is a T-periodic solution to $(\hat{\mathscr{E}}_s)$ for $s = g_0 := \frac{1}{T}\int_0^T \hat{g}(t, u_0(t))\,dt$. Moreover, $u_0(t) \geq d + K - \|\tilde{u}_0\|_\infty > d$, for all $t \in [0, T]$. Then u_0 is a T-periodic solution to $(\mathscr{E}_s)$ for $s = g_0$ with $u_0(t) \geq d$, for all $t \in [0, T]$.

If we assume that $|g(t, u)| \leq \rho(t)$ for a.e. $t \in [0, T]$ and for all $u \leq 0$, one can proceed in a similar manner. The theorem is thus proved. $\square$

4. Applications

In this final section, we present two consequences of the theorems illustrated in Section 3. More precisely, first we show some results in the framework of T-periodic forced Liénard-type equations for which theorems illustrated in the introduction are straightforward corollaries. Secondly, we analyse Neumann problems in the framework of radially symmetric solutions to partial differential equations.

4.1. Weighted periodic problems

We deal with the T-periodic forced Liénard-type equation

$$(\mathscr{W}\mathscr{E}_s) \qquad (\phi(u'))' + f(u)u' + a(t)q(u) = s + e(t),$$

where $s \in \mathbb{R}$ is a parameter, $\phi \colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, the functions $f, q \colon \mathbb{R} \to \mathbb{R}$ are continuous. We also assume $a \in L^\infty(0, T)$ and $e \in L^1(0, T)$. Moreover, we suppose $a(t) \geq 0$ for a.e. $t \in [0, T]$ with $\bar{a} := \frac{1}{T}\int_0^T a(t)\,dt > 0$. When the limits of the continuous function q at $\pm\infty$ exist, we set

$$\lim_{u \to -\infty} q(u) = \omega_-, \qquad \lim_{u \to +\infty} q(u) = \omega_+. \tag{4.1}$$

In the sequel we apply the general results achieved in Section 3 to the broadest class of nonlinear terms q. In order to do this, we observe that it is not restrictive to assume that $\bar{e} := \frac{1}{T}\int_0^T e(t)\,dt = 0$, and, moreover, that $\min\{\omega_-, \omega_+\} > 0$, if q is bounded from below, or that $\max\{\omega_-, \omega_+\} < 0$, if q is bounded from above. Indeed, if necessary, one can include in the forcing term $e(t)$ the function $a(t)(-\inf q + \varepsilon)$ (or $a(t)(-\sup q - \varepsilon)$, respectively) for some $\varepsilon > 0$, and, next, add the mean value $\bar{e}$ in the parameter s.

We are now in position to present some corollaries of Theorem 3.1, Theorem 3.2 and their variants. In more detail, we are interested in applications which always involve (AP) or (BM) alternatives, where the existence of at least two T-periodic solutions to $(\mathscr{W}\mathscr{E}_s)$ is considered. Beside these results, we warn that even partial alternatives, concerning only the existence of at least one or non-existence of T-periodic solutions, could be performed within our framework.

For nonlinearities q bounded from below, the following result holds true.

Theorem 4.1. *Assume that there exists a number $u_0 \in \mathbb{R}$ such that*

$$0 \leq q(u_0) < \min\{\omega_-, \omega_+\}$$

and

$$\bar{a} \min\{\omega_-, \omega_+\} > \|a\|_\infty q(u_0) + \|e^-\|_\infty. \tag{4.2}$$

Then, there exists $s_0 \in \,]-\infty, \bar{a}\omega_-[$ such that:

- *for $s < s_0$, equation $(\mathscr{W}\mathscr{E}_s)$ has no T-periodic solutions;*
- *for $s = s_0$, equation $(\mathscr{W}\mathscr{E}_s)$ has at least one T-periodic solution;*
- *for $s_0 < s < \bar{a}\omega_-$, equation $(\mathscr{W}\mathscr{E}_s)$ has at least one T-periodic solution;*
- *for $s_0 < s < \bar{a}\min\{\omega_-, \omega_+\}$, equation $(\mathscr{W}\mathscr{E}_s)$ has at least two T-periodic solutions.*

Proof. We apply Theorem 3.1 for $g(t,u) := a(t)q(u) - e(t)$. We notice that, since q is continuous and a is an L^∞-function, condition (H_0) is satisfied. Moreover, since $q(u_0) < \min\{\omega_-, \omega_+\}$, we obtain that $q_0 := \min q \in \mathbb{R}$ is well defined. Then, defining γ_0 as the negative part of $a(t)q_0 - e(t)$, condition (H_1^{I}) holds. Furthermore, for $g_0 = \|a\|_\infty q(u_0) + \|e^-\|_\infty$, (H_2^{I}) is satisfied too.

Lastly, we need that both Villari's type conditions (H_3^{I}) and (H_4^{I}) are satisfied as well. Let $\sigma \in \,]g_0, \bar{a}\omega_-[$ and notice that the interval is well defined and non-empty, since by hypothesis we have $g_0 < \bar{a}\omega_-$. From (4.1) it follows that there exists $\kappa_\sigma > 0$ such that $\bar{a}q(\xi) > \sigma$, for all $\xi \leq -\kappa_\sigma$. Let $d = \kappa_\sigma$ and let $u \in \mathcal{C}^1_T$ be such that $u(t) \leq -d$ for every $t \in [0,T]$. From the generalized mean value theorem, there exists $\tilde{t} \in [0,T]$ such that the following holds

$$\frac{1}{T}\int_0^T g(t,u(t))\,dt = \frac{1}{T}\int_0^T a(t)q(u(t))\,dt = \bar{a}q(u(\tilde{t})) > \sigma$$

and so (H_3^{I}) is satisfied. By the arbitrary choice of σ we have that (H_3^{I}) is satisfied for every $\sigma \in \,]g_0, \bar{a}\omega_-[$. Recalling the definition of σ^* in (3.1), we claim that $\sigma^* = \bar{a}\omega_-$. Indeed, if $\omega_- = +\infty$, then the claim is straightforward verified. If $\omega_- < +\infty$, the claim is reached by noticing that (H_3^{I}) is not true for $\sigma > \bar{a}\omega_-$. Analogously, one can prove that (H_4^{I}) holds for every $\sigma \in \,]g_0, \sigma^{**}[$, with $\sigma^{**} = \bar{a}\min\{\omega_-, \omega_+\}$. The thesis follows. □

Applying the variant of Theorem 3.1 introduced in Remark 3.1 we have the following result.

Theorem 4.2. *Assume that there exists $D > 0$ such that*

$$q(u) < \omega_-, \quad \text{for } u \leq -D, \qquad \text{and} \qquad q(u) < \omega_+, \quad \text{for } u \geq D. \tag{4.3}$$

Moreover, suppose that $\min\{\omega_-, \omega_+\} < +\infty$. Then, the conclusion of Theorem 4.1 holds.

Proof. Let us suppose that $\min\{\omega_-, \omega_+\} = \omega_- < +\infty$. We apply Theorem 3.1 for $g(t,u) := a(t)q(u) - e(t)$. The conditions (H_0) and (H_1^{I}) are verified as in the proof of Theorem 4.1. Furthermore, an application of Proposition 3.2 ensures that condition $(H_2^{\bigstar})$ is satisfied for some $g_0 \in \mathbb{R}$ and $u_0 \in \mathcal{C}^1_T$. Indeed, since $\omega_- < +\infty$, it follows that there exist $g_0 \in \mathbb{R}$ and $u_0 \in \mathcal{C}^1_T$ such that $u_0(t) \leq -D$ for every $t \in [0,T]$, with $D > 0$ as in the statement. Then, on the light of Remark 3.1, one has only to verify the Villari's type conditions (H_3^{I}) and (H_4^{I}). In order to do this, we first observe that

$$g_0 = \frac{1}{T}\int_0^T a(t)q(u_0(t))\,dt < \bar{a}\omega_- \leq \bar{a}\omega_+.$$

Then, as in the proof of Theorem 4.1, we have that (H_3^{I}) is satisfied for every $\sigma \in \,]g_0, \bar{a}\omega_-[$ and (H_4^{I}) is verified too for every $\sigma \in \,]g_0, \bar{a}\min\{\omega_-, \omega_+\}[$.

On the other hand, if we suppose that $\min\{\omega_-, \omega_+\} = \omega_+ < +\infty$, we achieve the thesis in a similar way. □

Analogously, the following results for nonlinearities q bounded from above can be obtained as an application of Theorem 3.2 (cf. also Remark 3.1).

Theorem 4.3. *Assume that there exists a number $u_0 \in \mathbb{R}$ such that*

$$0 \geq q(u_0) > \max\{\omega_+, \omega_-\}$$

and

$$\bar{a}\max\{\omega_-, \omega_+\} < \|a\|_\infty q(u_0) - \|e^+\|_\infty. \tag{4.4}$$

Then, there exists $s_0 \in \,]\bar{a}\omega_-, +\infty[$ such that:

- *for $s > s_0$, equation $(\mathscr{W}\!\mathscr{E}_s)$ has no T-periodic solutions;*
- *for $s = s_0$, equation $(\mathscr{W}\!\mathscr{E}_s)$ has at least one T-periodic solution;*
- *for $\bar{a}\omega_- < s < s_0$, equation $(\mathscr{W}\!\mathscr{E}_s)$ has at least one T-periodic solution;*
- *for $\bar{a}\max\{\omega_-, \omega_+\} < s < s_0$, equation $(\mathscr{W}\!\mathscr{E}_s)$ has at least two T-periodic solutions.*

Theorem 4.4. *Assume that there exists $D > 0$ such that*

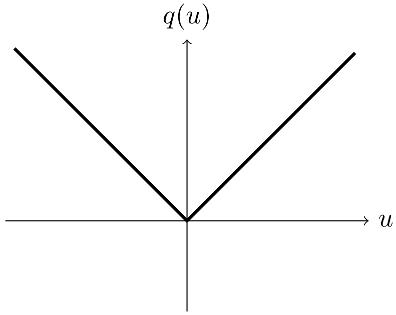
$$q(u) > \omega_-, \ \text{ for } u \leq -D, \quad \text{ and } \quad q(u) > \omega_+, \ \text{ for } u \geq D. \tag{4.5}$$

Moreover, suppose that $\max\{\omega_-, \omega_+\} > -\infty$*. Then, the conclusion of Theorem* 4.3 *holds.*

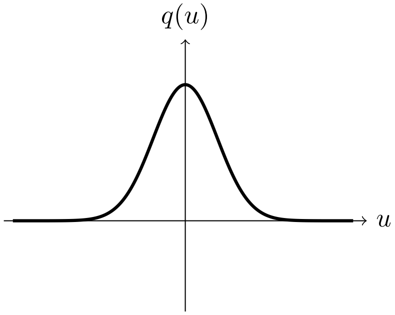
We conclude the discussion concerning the T-periodic forced Liénard-type equation $(\mathscr{W}\!\mathscr{E}_s)$ by presenting some examples. In this manner, we highlight the potentiality of the results proposed in this paper that, acting in a unified framework, lead to a generalization of some classical theorems. Furthermore, our approach allows us to treat more general situations by considering several types of nonlinearities (cf. Table 1).

Example 4.1. Let us consider $q\colon \mathbb{R} \to \mathbb{R}$ defined as $q(u) := |u|$. We notice that $\omega^\pm = +\infty$. This nonlinearity is of type I and characterizes the classical Ambrosetti–Prodi periodic problem. An application of Theorem 4.1 ensures the existence of $s_0 \in \mathbb{R}$ such that $(\mathscr{W}\!\mathscr{E}_s)$ has zero, at least one or at least two T-periodic solutions according to $s < s_0$, $s = s_0$ or $s > s_0$.

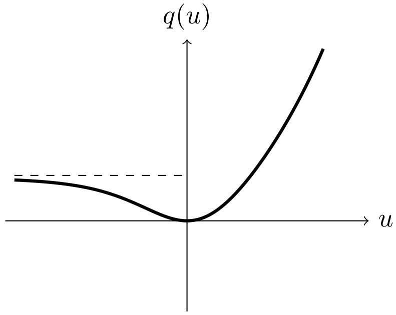
Table 1. Illustration of the graphs of the nonlinearities q considered in the examples presented in Section 4.1.



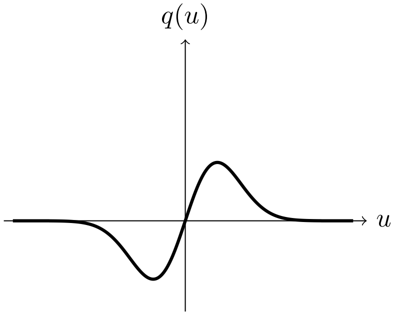
Example 4.1



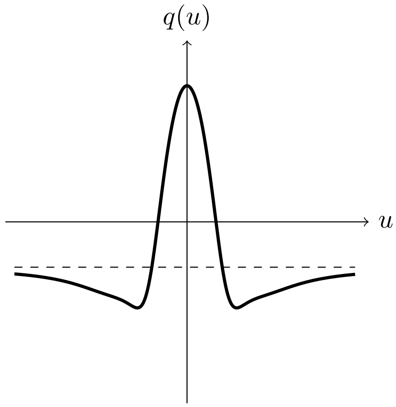
Example 4.2



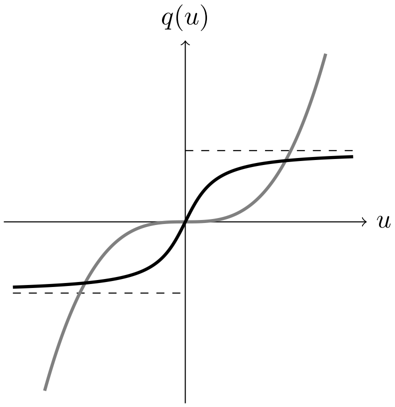
Example 4.3



Example 4.4



Example 4.5



Example 4.6

Example 4.2. Let us consider $q\colon \mathbb{R} \to \mathbb{R}$ defined as $q(u) := e^{-u^2}$. We notice that q is a Gaussian function with $\omega_\pm = 0$, so that is of type II, and corresponds to the classical problem by Ward. An application of Theorem 4.4 ensures the existence of $s_0 \in \mathbb{R}$ such that $(\mathscr{W}\mathscr{E}_s)$ has zero, at least one or at least two T-periodic solutions according to $s > s_0$, $s = s_0$ or $0 < s < s_0$. Moreover, by an integration on a period, one can observe that for $s \leq 0$ equation $(\mathscr{W}\mathscr{E}_s)$ has no T-periodic solutions.

We notice that, if we consider $q(u) := e^{-u^2} + \kappa$ with $\kappa \neq 0$, then the (BM) alternative holds without assuming a uniform condition in the limits. More precisely, there are zero, at least one or at least two T-periodic solutions according to $s > s_0$, $s = s_0$ or $\bar{a}\kappa < s < s_0$. In the same context, one could be also driven to apply Theorem 4.3. However, this an example of the difference between these results. Indeed, if for example $\kappa = -1$, then Theorem 4.3 ensures the existence of $s_0 \in \,]-\bar{a}, +\infty[$ such that $(\mathscr{W}\mathscr{E}_s)$ satisfies the above alternative under the additional hyphotesis $\bar{a} > \|e^+\|_\infty$ (cf. condition (4.4)).

Example 4.3. Let us consider $q\colon \mathbb{R} \to \mathbb{R}$ defined as

$$q(u) := \frac{(e^u + 1)u^2}{u^2 + 1},$$

where $\omega_- = 1$ and $\omega_+ = +\infty$. An application of Theorem 4.2 guarantees the existence of $s_0 \in \mathbb{R}$ such that $(\mathscr{W}\mathscr{E}_s)$ has zero, at least one or at least two T-periodic solutions according to $s < s_0$, $s = s_0$ or $s_0 \leq s < \bar{a}\omega_-$. On the other hand, since $\omega_- \in \,]0, +\infty]$, whether the terms a, e satisfy the additional condition (4.2), the same alternative can be proved via Theorem 4.1.

Example 4.4. Let us consider $q\colon \mathbb{R} \to \mathbb{R}$ defined as $q(u) := u\, e^{-u^2}$, where $\omega_\pm = 0$. We notice that either condition (4.3) or condition (4.5) are not satisfied, so that, both Theorem 4.2 and Theorem 4.4 cannot be applied in such a framework. However, one can recover an (AP) alternative through Theorem 4.1 for the auxiliary equation $(\phi(u'))' + f(u)u' + a(t)q_1(u) = \ell + e_1(t)$, where $q_1(u) = q(u) - \min q$, $e_1(t) = e(t) - \bar{e} - (a(t) - \bar{a}) \min q$, and $\ell = s + \bar{e} - \bar{a} \min q$. In this way a, e_1 satisfy (4.2). On the other hand, in a similar manner, one can recover a (BM) alternative through Theorem 4.3.

Example 4.5. Let us consider $q\colon \mathbb{R} \to \mathbb{R}$ defined as

$$q(u) := \frac{(e^{-u^2} + 2)(u^6 - u^4 - u^2 + 1)}{u^6 + 1} + 5e^{-u^2} - 3,$$

where $\omega_\pm = -1$. We notice that condition (4.3) is satisfied, so that from Theorem 4.2 we can prove that there exists $s_0 < -\bar{a}$ such that $(\mathscr{W}\mathscr{E}_s)$ has zero, at least one or at least two T-periodic solutions according to $s < s_0$, $s = s_0$ or $s_0 < s < -\bar{a}$. On the other hand, if condition (4.4) is satisfied too, then an application of Theorem 4.3 gives the existence of $s_1 > -\bar{a}$ such that $(\mathscr{W}\mathscr{E}_s)$ has zero, at least one or at least two T-periodic solutions according to $s > s_1$, $s = s_1$ or $-\bar{a} < s < s_1$. In this manner, a combination of the classical (AP) and (BM) alternatives hold simultaneously, leading to an interesting phenomenon of 0 1 2 - 2 1 0 solutions to the periodic problem associated with ($\mathscr{W}\mathscr{E}_s$).

Example 4.6. Let us consider $q\colon \mathbb{R} \to \mathbb{R}$ defined as $q(u) := u^{2n+1}$, with $n \in \mathbb{N}$, or $q(u) := \arctan(u)$. In these cases, q is a non-decreasing function, and consequently, the above theorems do not apply. However, when the nonlinearity is bounded, one could achieve the existence of at least a T-periodic solution for some ranges of the parameter s (cf. [16, 18]).

4.2. Radial Neumann problem on annular domains

Let us consider the open annular domain

$$\mathfrak{A} := \bigl\{x \in \mathbb{R}^N \colon R_i < |x| < R_e\bigr\}, \quad \text{with } 0 < R_i < R_e,$$

where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^N$, with $N \geq 2$. In the present section we study non-existence, existence and multiplicity of (classical) radially symmetric solutions to the parameter-dependent Neumann problem

$$\begin{cases} \nabla \cdot (A(|\nabla u|)\nabla u) + G(|x|, u) = s & \text{in } \mathfrak{A}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathfrak{A}, \end{cases} \tag{4.6}$$

where $A\colon \,]0,+\infty[\, \to \,]0,+\infty[$ is a continuous function such that the map $\phi(\xi) = A(|\xi|)\xi$ for $\xi \neq 0$ and $\phi(0) = 0$ is a homeomorphism on the real line, $G\colon [R_i, R_e] \times \mathbb{R} \to \mathbb{R}$ is a continuous map, and $s \in \mathbb{R}$. In this manner, we pursue the study started in [20, 21] for Neumann problems with local coercive nonlinearities.

When dealing with radially symmetric solutions to (4.6), one is led to define $r = |x|$, $v(r) = v(|x|) = u(x)$, and so to study the problem

$$\begin{cases} (r^{N-1}A(|v'|)v')' + r^{N-1}G(r, v) = r^{N-1}s, \\ v'(R_i) = v'(R_e) = 0. \end{cases} \tag{4.7}$$

We notice that the map $\xi \mapsto A(|\xi|)\xi$ is an increasing homeomorphism. Hence, looking at solutions to (4.7), we now present our result in the framework of a more general problem. Namely, we deal with a Neumann problem of the form

$$\begin{cases} (\zeta(t)\phi(u'))' + g(t, u(t)) = p(t)s, \\ u'(a) = u'(b) = 0, \end{cases} \tag{4.8}$$

where $a < b$, $\zeta, p\colon [a, b] \to \,]0,+\infty[$ are continuos functions, $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, $g\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and $s \in \mathbb{R}$.

First of all, let $X := \bigl\{u \in \mathcal{C}^1([a, b]) \colon u'(a) = u'(b) = 0\bigr\}$ be the Banach space endowed with the norm $\|u\|_X := \|u\|_\infty + \|u'\|_\infty$. Next we define the completely continuous operator $\mathcal{G}\colon X \to X$ as

$$(\mathcal{G}u)(t) := u(a) - \frac{1}{b-a}\int_a^b h(t, u(t))\,dt + \int_a^t \phi^{-1}\biggl(-\frac{1}{\zeta(t)}\int_a^s h(\xi, u(\xi))\,d\xi\biggr)\,ds,$$

where $h(t,u) := g(t,u) - p(t)s$. One can easily verify that u is a solution to problem (4.8) if and only if u is a fixed point of $\mathcal{G}$ (cf. [11, Section 2]).

In this setting, up to minimal changes in the discussion in Section 2 and Section 3, all the results presented therein hold for problem (4.8), too.

Taking into account that

$$\int_\Omega G(|x|, u(x))\, dx = \omega_{N-1} \int_{R_i}^{R_e} r^{N-1} G(|r|, u(r))\, dr,$$

where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$ (cf. [9, Section 2.7]), we obtain the following theorem for radially symmetric solutions to the Neumann problem (4.6).

Theorem 4.5. *Assume that*

(G_1) *there exists $C_0 > 0$ such that $G(|x|, u) \geq -C_0$, for all $u \in \mathbb{R}$ and $x \in \Omega$;*
(G_2) *there exists $u_0, g_0 \in \mathbb{R}$ such that $G(|x|, u_0) \leq g_0$, for all $x \in \Omega$;*
(G_3) *for each σ there exists $d_\sigma > 0$ such that $\frac{1}{|\Omega|} \int_\Omega G(|x|, u(x))\, dx > \sigma$ for each radially symmetric $u \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$ with $u(x) \leq -d_\sigma$ for all $x \in \Omega$;*
(G_4) *for each σ there exists $d_\sigma > 0$ such that $\frac{1}{|\Omega|} \int_\Omega G(|x|, u(x))\, dx > \sigma$ for each radially symmetric $u \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$ with $u(x) \geq d_\sigma$ for all $x \in \Omega$.*

Then, there exists $s_0 \in \mathbb{R}$ such that

- *for $s < s_0$, problem* (4.6) *has no radially symmetric solutions;*
- *for $s = s_0$, problem* (4.6) *has at least one radially symmetric solution;*
- *for $s > s_0$, problem* (4.6) *has at least two radially symmetric solutions.*

A direct application of Theorem 4.5 is the following one.

Corollary 4.1. *Let $a \in \mathcal{C}([R_i, R_e], \mathbb{R}^+)$ be such that $a(\xi_0) > 0$ for some $\xi_0 \in [R_i, R_e]$. Let $q \in \mathcal{C}(\mathbb{R})$ be such that $\lim_{|u| \to +\infty} q(u) = +\infty$. Let $G(|x|, u) := a(|x|)q(u)$. Then, the conclusion of Theorem* 4.5 *holds.*

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