Optimal disturbance compensation for constrained linear systems operating in stationary conditions: a scenario-based approach

Alessandro Falsone, Luca Deori, Daniele Ioli, Simone Garatti, Maria Prandini [a]

[a] *Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, Via Ponzio 34/5, 20133 Milano, Italy*

Abstract

We consider the problem of optimizing the stationary performance of a discrete time linear system affected by a disturbance and subject to probabilistic input and state constraints. More precisely, the goal is to design a disturbance compensator which optimally shapes the stationary state distribution so as to best satisfy the given control specifications. To this purpose, we formulate a chance-constrained program with the compensator parametrization as optimization vector. Chance-constrained programs are generally hard to solve and a possible way to tackle them is resorting to the so-called scenario approach. In our set-up, however, the scenario approach is not directly applicable since the stationary state process depends on disturbance realizations of infinite extent. Our contribution is then to provide a new scenario-based methodology, where the stationary state process is approximated and constraints are suitably tightened so as to retain the chance-constrained feasibility guarantees of the scenario solution. Design of a periodic compensator of a cyclostationary disturbance can be embedded in our framework, as illustrated in an energy management numerical example.

Key words: Optimal constrained control, disturbance compensation, stochastic linear systems.

1 Introduction

We consider a discrete time linear time invariant system affected by a stationary additive disturbance. Our goal is to set its control input so as to *optimize performance*, while *satisfying some probabilistic constraint on the state/input* when the system is operating *in stationary conditions*. This involves characterizing and optimally shaping the distribution of the stationary state process. In the same vein, in minimum variance control, (Åström 1970), generalized minimum variance control, (Clarke & Hastings-James 1971, Peterka 1972, Shaked & Kumar 1986, Grimble 1988, Huang 2002, Gawthrop 2004), and H_2 control, (Sinha 2007, pag. 273), the state distribution is optimally shaped so as to minimize the stationary variance of some suitably defined output signal. In these approaches, however, it is very difficult to include state/input constraints and/or actuation constraints; these are typically accounted for only indirectly by introducing a control penalization term to the variance in the cost function, see e.g., (Grimble 2002). In this paper, in contrast with the previously mentioned approaches, we are not restricted to the variance as cost criterion and we can include explicitly joint state and input constraints, which are imposed in probability. On the other hand, we assume that disturbance measurements are available to shape the stationary state distribution through a compensator. This set-up is particularly appealing in those applications in which either the state is hardly accessible, or there are no sensors in place to measure it and adding them would be excessively costly, whereas disturbance measurements are easy to obtain.

Computing the compensator parameters involves solving a chance-constrained optimization program. This is a hard problem, in general, due to the presence of probabilistic constraints. In our set-up, the problem is even harder since the probability appearing in the constraints is associated with the system operating in stationary conditions and, hence, it involves the whole disturbance process, which makes analytic methods to treat the probabilistic constraints, (Zhou & Cogill 2013, Cinquemani, Agarwal, Chatterjee & Lygeros 2011, Bertsimas & Brown 2007), as well as randomized methods,

⋆ Corresponding author A. Falsone. Tel. +39-02-23993542. Fax +39-02-23993412.

Email address: name.surname@polimi.it (Alessandro Falsone, Luca Deori, Daniele Ioli, Simone Garatti, Maria Prandini).

(Campi, Garatti & Prandini 2009, Campi & Garatti 2008, Campi & Garatti 2011, Alamo, Tempo, Luque & Ramirez 2015), not directly applicable.

In this paper, we are able to extend the randomized method known as scenario approach and its guarantees on feasibility, (Calafiore & Campi 2005, Calafiore & Campi 2006, Campi & Garatti 2008), to the case of chance-constraints involving the stationary state process. This is achieved by suitably approximating such a process and compensating the introduced approximation error via constraint tightening.

In our approach, the computational effort involved in the compensator design is entirely off-line, and on-line operation does not require any (re)computation of its parameters. When the state of the controlled system reaches the steady-state conditions, *the resulting stationary state process is guaranteed by construction to satisfy the probabilistic constraints and performance is optimal.*

Other infinite horizon approaches that account for probabilistic constraints are proposed within the stochastic model predictive control (SMPC) framework (see (Mesbah 2016) for a survey). In SMPC, a feedback control policy is implemented by adopting a receding horizon, which, however, typically implies on-line computations. Moreover, to the best of our knowledge, no results are available in the literature on the optimality of the SMPC solution and on the satisfaction of probabilistic state/input constraints in the long run. More importantly, SMPC is not applicable to our set-up where disturbance measurements are available whereas state measurements are not.

This paper significantly extends its preliminary version (Falsone, Deori, Ioli, Garatti & Prandini 2017), by providing the proofs and presenting a more extensive numerical example section.

Structure of the paper: The addressed problem is precisely described in Section 2, while the new scenario-based resolution approach is presented in Section 3 together with the main result statement. Proofs are deferred to Section 4. A numerical case study is illustrated in Section 5. Some concluding remarks are drawn in Section 6.

Notations: Given a discrete time process $\{v_k,\ k \in \mathbb{Z}\}$, we denote it as $\boldsymbol{v}$ and the probability distribution of $\boldsymbol{v}$ as $\mathbb{P}_{\boldsymbol{v}}$. Correspondingly, the expected value operator with respect to $\mathbb{P}_{\boldsymbol{v}}$ is denoted as $\mathbb{E}_{\boldsymbol{v}}[\cdot]$. I denotes the identity matrix (a subscript denotes the order of I when it is not obvious from the context), $\mathbb{1}$ is a vector containing all ones, $J = \mathrm{blkdiag}(J_1, \dots, J_m)$ is the block diagonal matrix built from the square matrices $J_1, \dots, J_m$, and $X^\top$ denotes the conjugate transpose of X. We denote by $\rho_X = \max\{|\lambda| : \det(\lambda I - X) = 0\}$ the spectral radius of a square matrix X, and by $\|\cdot\|_p$ the matrix norm induced by the standard p-norm for vectors, i.e., $\|X\|_p = \sup_{v:\|v\|_p=1} \|Xv\|_p$. The symbols $\xrightarrow{L^1}$ and $\xrightarrow{L^2}$ denote the convergence in mean and in mean square of random vectors, respectively, and $\xrightarrow{P}$ the convergence in probability. Given the integers n and k, the binomial coefficient $\binom{n}{k}$ is assumed to be zero if $n < k$.

2 Problem formulation

Consider a linear system where the state $x_k \in \mathbb{R}^{n_x}$ evolves according to the discrete time equation

$$x_{k+1} = Ax_k + Bu_k + Wd_k, \tag{1}$$

affected by the control input $u_k \in \mathbb{R}^{n_u}$ and an additive stochastic disturbance $d_k \in \mathbb{R}^{n_d}$, A, B, and W being matrices of appropriate dimensions.

We make the following assumptions.

Assumption 1 (Asymptotic stability) *The spectral radius of matrix A satisfies $\rho_A < 1$.* □

Assumption 2 (Disturbance) *The stochastic process $\boldsymbol{d}$ is strictly stationary with zero mean* [1] *and well-defined and known second order moments.* □

Throughout the paper we will assume that the value taken by d_k at any time $k \in \mathbb{Z}$ is available for compensation purposes. However, note that it is typically not possible to cancel out the contribution of d_k on the state dynamics (1) by setting $Bu_k = -Wd_k$, because B could be not invertible and u_k may be subject to constraints.

Our goal is to optimize the system performance in stationary conditions by suitably designing a disturbance compensator of the following form:

$$u_k = \gamma + \vartheta d_k, \tag{2}$$

where the control input u_k is parameterized as a static function of d_k, and γ and ϑ are the compensator parameters taking values in the convex and compact sets $\Gamma \subset \mathbb{R}^{n_u}$ and $\Theta \subset \mathbb{R}^{n_u \times n_d}$, respectively.

By plugging the disturbance compensator (2) into (1), we obtain the controlled system equation

$$x_{k+1} = Ax_k + B\gamma + (B\vartheta + W)d_k. \tag{3}$$

[1] The zero mean assumption is without loss of generality, since if this is not the case, we can introduce $\bar{x}_{k+1} = A\bar{x}_k + W\bar{d}$, where $\bar{d} = \mathbb{E}_{\boldsymbol{d}}[d_k]$, and reformulate the problem in terms of $\Delta x_k = x_k - \bar{x}_k$ which evolves according to $\Delta x_{k+1} = A\Delta x_k + Bu_k + W\Delta d_k$, and is affected by the zero mean strictly stationary disturbance process $\Delta d_k = d_k - \bar{d}$.

Under Assumptions 1 and 2, for any $k \in \mathbb{Z}$ there exists a measurable function $x_{k,\infty}$ of the process $\boldsymbol{d_{k-1}} = \{\dots, d_{k-2}, d_{k-1}\}$ such that the process $\boldsymbol{x_\infty} = \{x_{k,\infty},\ k \in \mathbb{Z}\}$ satisfies (3) and is strictly stationary with finite first and second order moments, (Caines 1988, Theorem 1.4, pag. 80). This $x_{k,\infty}$ is unique, (Caines 1988, Theorem 3.2, pag. 101), and is given by $x_{k,\infty} = (I-A)^{-1}B\gamma + \sum_{s=0}^{\infty} A^s(B\vartheta+W)d_{k-1-s}$.

Our goal is to choose the compensator parameters γ and ϑ so as to optimize some performance criterion while satisfying state and input constraints. This corresponds to suitably shaping the distribution of the stationary state $x_{k,\infty}$. Specifically, suppose that a function $\ell(x,u,d)$: $\mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \to \mathbb{R}$ is given, associating a cost to the state/control input pair (x,u) when the disturbance value is d. Moreover, a joint constraint on (x,u) is defined by requiring the non positivity of a given function $f(x,u)$: $\mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$.[2] Functions $\ell(x,u,d)$ and $f(x,u)$ are required to be convex as specified in the following.

Assumption 3 (Convexity) *The cost function $\ell(x,u,d)$ and the constraint function $f(x,u)$ are convex with respect to $(x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$.* □

Then, our design problem is formulated as the following chance-constrained optimization program:

$$\begin{aligned} \min_{\gamma\in\Gamma,\vartheta\in\Theta,h} \quad & h \\ \text{subject to:} \quad & \mathbb{P}_{\boldsymbol{d_k}}\Big\{\ell(x_{k,\infty}, \gamma+\vartheta d_k, d_k) \le h, \\ & \qquad \wedge\, f(x_{k,\infty}, \gamma+\vartheta d_k) \le 0\Big\} \ge 1-\varepsilon, \end{aligned} \tag{4}$$

where $\varepsilon \in (0,1)$ is a user-chosen probability level and $\mathbb{P}_{\boldsymbol{d_k}}$ is the probability distribution of process $\boldsymbol{d_k} = \{\dots, d_{k-1}, d_k\}$. If we set $\varepsilon = 0$ in (4), then we are minimizing the worst-case cost value while satisfying the joint state and input constraint over all disturbance realizations. Instead, by setting $0 < \varepsilon < 1$, we require that the cost is minimized and the joint state and input constraint is satisfied over a set of disturbance realizations of measure at least $1-\varepsilon$, thus allowing for the solution to (4) to have a larger cost and/or violate the constraint over the disturbance realizations in the remaining set of measure at most ε.

Notice that functions $\ell(x,u,d)$ and $f(x,u)$ are evaluated in stationary conditions, i.e., with x set equal to the stationary state $x_{k,\infty}$ and with u given by the disturbance compensator in (2). In stationary conditions, thus, the solution to (4) is optimal and satisfies the probabilistic constraint in (4) for every time instant k. In practice, thanks to Assumption 1, stationarity is always reached in the long run, with a convergence rate that depends on ρ_A (see also Remark 3 in Section 4.2). This makes the proposed approach particularly appealing because the control law is computed once (off-line) and then applied at each time step without solving any further on-line optimization problem.

Chance-constrained problems like (4) are generally challenging to solve because of the presence of the probabilistic constraint, which is not easy to express analytically as a function of the optimization variables and can be non convex even under Assumption 3, (Prèkopa 1995, Dentcheva 2006). One could then head for an approximate solution to (4) by adopting a randomization of the probabilistic constraint according to the scenario approach, (Calafiore & Campi 2005, Calafiore & Campi 2006, Campi & Garatti 2008, Campi et al. 2009). This would involve using a set $\{\boldsymbol{d_k}^{(i)} = \{\dots, d_{k-2}^{(i)}, d_k^{(i)}\}\}_{i=1}^N$ of N independent realizations of the disturbance process $\boldsymbol{d_k}$ ("scenarios") and solving the convex scenario program

$$\begin{aligned} \min_{\gamma\in\Gamma,\vartheta\in\Theta,h} \quad & h \\ \text{subject to:} \quad & \ell(x_{k,\infty}^{(i)}, \gamma+\vartheta d_k^{(i)}, d_k^{(i)}) \le h \\ & f(x_{k,\infty}^{(i)}, \gamma+\vartheta d_k^{(i)}) \le 0 \\ & x_{k,\infty}^{(i)} = (I-A)^{-1}B\gamma + \sum_{s=0}^{\infty} A^s(B\vartheta+W)d_{k-1-s}^{(i)} \\ & i = 1,\dots,N. \end{aligned} \tag{5}$$

Let n be the number of scalar variables in the controller parametrization (γ,ϑ). Given a confidence parameter $\beta \in (0,1)$, if (5) is feasible and N is selected so as to satisfy

$$\sum_{i=0}^{n} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i} \le \beta,, \tag{6}$$

then, under Assumption 3, (Campi & Garatti 2008, Theorem 2.4) guarantees that, with probability at least $1-\beta$, the optimal solution to (5) is feasible for (4).

The catch here is that (5) cannot be solved in practice since it involves N realizations of the process $\boldsymbol{d_k}$, which is an *infinite* sequence of random vectors. Since in practice one can have finite-length realizations of $\boldsymbol{d_k}$ only, this renders the standard scenario approach inapplicable as it is. The idea then is to turn (5) into a solvable problem by approximating $x_{k,\infty}$ with a truncated version. However, as shown in the next section, a suitable tightening of the constraints is needed to account for the truncation error and obtain feasibility guarantees with respect to the original problem (4) for the resulting approximate scenario-based solution.

[2] The fact that $f(\cdot)$ is a single constraint function is without loss of generality because if multiple constraint functions $f_1(\cdot),\dots,f_m(\cdot)$ are present, then, we can redefine $f(\cdot)$ as the point-wise maximum of these $f_1(\cdot),\dots,f_m(\cdot)$.

3 Proposed solution and main results

Given an integer $M \in \mathbb{N}$, let

$$x_{k,M} = (I-A)^{-1}B\gamma + \sum_{s=0}^{M-1} A^s(B\vartheta + W)d_{k-1-s} \quad (7)$$

be a truncated version of $x_{k,\infty}$, which satisfies $x_{k,M} \xrightarrow{L^2} x_{k,\infty}$, and hence, approximates $x_{k,\infty}$ with an accuracy that increases as M grows to infinity. Let also define the process $\boldsymbol{x_M} = \{x_{k,M},\ k \in \mathbb{Z}\}$. Consider now a set $\{d_{k-j}^{(i)},\ j = 0, \dots, M\}_{i=1}^N$ of N independent realizations of length $M+1$ of the disturbance process $\boldsymbol{d_k}$. Then, we can formulate the following approximated version of (5):

$$\begin{aligned}
&\min_{\gamma\in\Gamma,\vartheta\in\Theta,h} \quad h && (8)\\
&\text{subject to: } \ell(x_{k,M}^{(i)}, \gamma + \vartheta d_k^{(i)}, d_k^{(i)}) \le h - \delta\\
&\qquad f(x_{k,M}^{(i)}, \gamma + \vartheta d_k^{(i)}) \le -\delta\\
&\qquad x_{k,M}^{(i)} = (I-A)^{-1}B\gamma + \sum_{s=0}^{M-1} A^s(B\vartheta + W)d_{k-1-s}^{(i)}\\
&\qquad i = 1, \dots, N,
\end{aligned}$$

in which we used $x_{k,M}$ in place of $x_{k,\infty}$, thus requiring only finite-length realizations of the disturbance. Note also that, to account for the introduced approximation error, we introduced a $\delta \ge 0$ to tighten the constraints in which $x_{k,\infty}$ appears.

Under Assumption 3, the scenario optimization problem (8) is convex and can be efficiently solved via standard convex optimization techniques, (Boyd & Vandenberghe 2004). For instance, when $\ell(\cdot)$ and $f(\cdot)$ are linear function of their first two arguments, and Γ and Θ are box-sets, (8) reduces to a linear program.

Let $(\gamma^\star, \vartheta^\star, h^\star)$ be the optimal solution to (8). Our main result, Theorem 1 below, is that $(\gamma^\star, \vartheta^\star, h^\star)$ is with high confidence feasible for the original problem (4) under the following two assumptions.

Assumption 4 (Feasibility and uniqueness) *For any N, for any sample of disturbance realizations, the constrained optimization problem (8) is feasible and its feasibility set has a nonempty interior. Moreover, its solution exists and is unique.* □

Assumption 5 (Lipschitz continuity) *The cost function $\ell(x, u, d)$ and the constraint function $f(x, u)$ are Lipschitz continuous in $x \in \mathbb{R}^{n_x}$, with Lipschitz constant L, for any $(u, d) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_d}$.* □

Assumption 4 is quite standard in scenario-based optimization (see, e.g., (Campi & Garatti 2008, Campi & Garatti 2011, Caré, Garatti & Campi 2015)). The uniqueness part of Assumption 4 can be relaxed by considering suitable convex tie-break rules to single out a unique solution (see (Campi & Garatti 2008)). Also the feasibility part of Assumption 4 can be relaxed, see e.g. (Calafiore 2010). Assumption 5 is instead a regularity condition on the cost function $\ell(\cdot)$ and the constraint function $f(\cdot)$. The main theorem follows.

Theorem 1 (Guarantees) *Fix a confidence parameter $\beta \in (0, 1)$, and let $M, N \in \mathbb{N}$ and $\delta > 0$ be such that*

$$\sum_{i=0}^{n} \binom{N}{i} \tilde{\varepsilon}^i (1-\tilde{\varepsilon})^{N-i} \le \beta, \quad (9)$$

$$\tilde{\varepsilon} = \varepsilon - \frac{\mathcal{K}_M}{\delta} > 0, \quad (10)$$

n being the number of scalar variables in the controller parametrization (γ, ϑ) and

$$\mathcal{K}_M = L\,\eta\,\sigma\,\|T\|_1\,\|T^{-1}\|_1\,\rho_A^M \sum_{i=0}^{\overline{m}-1}\sum_{s=0}^{i}\left((-1)^s\binom{M}{i}\binom{i}{s}\frac{M-i}{M+s-i}\frac{\rho_A^{s-i}}{(1-\rho_A)^{i+1}}\right), \quad (11)$$

where L is the Lipschitz constant in Assumption 5, $\eta = \max_{\vartheta\in\Theta}\|B\vartheta + W\|_1$, $\sigma = \sum_{j=1}^{n_d}\sqrt{[V_d]_{j,j}}$ with $V_d = \mathbb{E}_{\boldsymbol{d_k}}[d_k d_k^\top]$, T is a nonsingular transformation matrix such that the matrix $J = T^{-1}AT$ is the Jordan normal form of A, $\overline{m}$ is the maximum eigenvalue index, and ρ_A is the spectral radius of the dynamic matrix A.[3] *Then, if Assumptions 1-5 hold, the solution $(\gamma^\star, \vartheta^\star, h^\star)$ of the scenario program (8) is feasible for the original chance-constrained problem (4) with probability larger than or equal to $1-\beta$, i.e.,*

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{d_k}}^N\{\mathbb{P}_{\boldsymbol{d_k}}\{&\ell(x_{k,\infty}^\star, \gamma^\star + \vartheta^\star d_k, d_k) \le h^\star && (12)\\
&\wedge f(x_{k,\infty}^\star, \gamma^\star + \vartheta^\star d_k) \le 0\} \ge 1-\varepsilon\} \ge 1-\beta,
\end{aligned}$$

where $x_{k,\infty}^\star$ is the stationary process defined as the mean square limit of $x_{k,M}$ with (γ, ϑ) set equal to $(\gamma^\star, \vartheta^\star)$. □

The solution $(\gamma^\star, \vartheta^\star, h^\star)$ to the scenario program (8) is a random quantity and, hence, the feasibility result in Theorem 1 holds with a certain probability $1-\beta$. Since the dependence of N on β is logarithmic, very small values of β such as 10^{-6} can be enforced without affecting N too much, thus providing a result that in practice holds beyond any reasonable doubt.

[3] L, η, ρ_A can be replaced by upper bounds without compromising the result, the only constraint being that the upper bound on ρ_A must be smaller than 1 to preserve the exponential decay to zero of $\mathcal{K}_M$ as a function of M. In particular, one can replace η with $\|B\|_1 \max_{\vartheta\in\Theta}\|\vartheta\|_1 + \|W\|_1$, which is typically easier to calculate.

Theorem 1 provides a similar result to the standard scenario theory in (Campi & Garatti 2008, Theorem 2.4). Indeed, condition (9) on the multi-sample size is the same as (6) except that $\tilde{\varepsilon}$ is used in place of ε, thus resulting in a bigger value for N. We actually need to tighten the probability level from ε to $\tilde{\varepsilon}$ by subtracting $\frac{\mathcal{K}_M}{\delta}$ from ε to account for the approximation of $x_{k,\infty}$ by its truncated version $x_{k,M}$. The presence of $\delta > 0$ thus introduces conservatism twice, because it gives rise to: (i) a tightening of the constraints, and (ii) a reduction of the value of $\tilde{\varepsilon}$ with respect to the violation parameter ε, which determines an increase of the multi-sample size N. The modulation of δ has an opposite impact on (i) and (ii): to push $\tilde{\varepsilon}$ closer to ε, higher values of δ are needed, leading eventually to infeasibility of problem (8) because of the tightening of the constraints; on the other hand, δ cannot be taken too small because condition $\tilde{\varepsilon} > 0$ in (10) has to be satisfied and, moreover, N grows to infinity as $\tilde{\varepsilon}$ goes to zero. In a given problem, various values of δ can be explored to find the best trade-off between (i) and (ii).

The introduced conservatism can be effectively reduced by increasing the length M of the extracted disturbance realizations, which can be performed in most cases at low computational effort. Since $\mathcal{K}_M$ in (11) decreases to zero exponentially fast as M grows, larger values of M result in values of $\mathcal{K}_M$ closer to zero, which, in turn, allow for lower values of δ. Therefore, by increasing M, we can reduce the constraint tightening without deteriorating the value of $\tilde{\varepsilon}$.

Remark 1 (Guidelines on the use of Theorem 1) *Typically, ε and β are given and we then need to choose $\tilde{\varepsilon}$, δ, N, and M. From* (10) *we have that $\tilde{\varepsilon} \in (0, \varepsilon)$. Since the complexity of* (8) *depends on N, and N is proportional to $1/\tilde{\varepsilon}$, it is better to set $\tilde{\varepsilon}$ as close as possible to ε, to keep N as low as possible. If we fix a tentative value for $\tilde{\varepsilon}$ and a threshold for δ, then we can use* (11) *to choose M such that $\delta = \frac{\mathcal{K}_M}{\varepsilon - \tilde{\varepsilon}}$ (from* (10)*) is less than the threshold, and this is always possible since $\mathcal{K}_M$ in* (11) *is exponentially decaying to zero as M grows to infinity. If the value for M is too high, we can increase the threshold on δ and repeat the procedure. If a suitable pair (δ, M) cannot be found, then we can go back and select a lower $\tilde{\varepsilon}$ at the price of growing N.*
In the case when N and M are given (e.g., when a dataset is provided), we need to compute ε, β, and δ. We can start by setting β to a desired confidence level and use (9) *to compute the corresponding $\tilde{\varepsilon}$. Given M, we can then compute $\mathcal{K}_M$ from* (11)*. Finally, from* (10) *we can let δ vary and determine the corresponding ε. If obtaining a reasonable ε requires a too high value for δ, then we can either increase β or collect more data.* □

4 Proofs

4.1 Preparatory results on matrix norm bounds

Let $A = TJT^{-1}$ where J is the Jordan normal form of A and T is a suitable nonsingular transformation matrix. J is known to have a block-diagonal structure $J = \text{blkdiag}(J_1, \dots, J_{n_b})$, whose b-th block is given by

$$J_b = \begin{bmatrix} \lambda_b & 1 & & \\ & \lambda_b & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_b \end{bmatrix},$$

λ_b being the b-th eigenvalue of A. By the Jordan form of A we can compute $A^s = TJ^sT^{-1}$, where

$$J^s = \text{blkdiag}(J_1^s, \dots, J_{n_b}^s). \tag{13}$$

For a Jordan block J_b of order m_b, the (i,j)-th element of J_b^s is given by

$$[J_b^s]_{i,j} = \begin{cases} \binom{s}{j-i} \lambda_b^{s-(j-i)} & 0 \le j - i \le s \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

with $1 \le i \le m_b$ and $1 \le j \le m_b$.

Lemma 1 (Bound on the matrix norm) *Set $\overline{m} = \max_b m_b$. The norm $\|A^s\|_1$ satisfies the following bound:*

$$\|A^s\|_1 \le \|T\|_1 \|T^{-1}\|_1 \sum_{i=0}^{\overline{m}-1} \binom{s}{i} {\rho_A}^{s-i}. \tag{15}$$

Proof. By the sub-multiplicativity of $\|\cdot\|_1$, we have that

$$\|A^s\|_1 \le \|T\|_1 \|T^{-1}\|_1 \|J^s\|_1, \tag{16}$$

and the block-diagonal structure of $\|J^s\|_1$ gives

$$\begin{aligned} \|J^s\|_1 &= \max_{\substack{1 \le b \le n_b \\ 1 \le j \le m_b}} \sum_{i=1}^{m_b} |[J_b^s]_{i,j}| \\ &= \max_{1 \le b \le n_b} \sum_{i=1}^{m_b} |[J_b^s]_{i,m_b}| \\ &= \max_{1 \le b \le n_b} \sum_{i=1}^{m_b} \binom{s}{m_b - i} |\lambda_b|^{s-(m_b-i)} \\ &\le \sum_{i=0}^{\overline{m}-1} \binom{s}{i} {\rho_A}^{s-i}, \end{aligned} \tag{17}$$

where the first inequality holds because, given the structure of J_b^s, the maximum over j is achieved when $j = m_b$, the second equality is obtained using the definition of $[J_b^s]_{i,j}$ in (14) with $j = m_b$, and the last inequality is obtained by changing the sum index, by setting $\overline{m} = \max_b m_b$, and by using $|\lambda_b| \le \rho_A$ for any b. Inequality (15) follows by plugging (17) into (16), thus concluding the proof. □

Lemma 2 (Sum of the matrix norm) *Under Assumption 1, we have*

$$\sum_{s=M}^{\infty} \|A^s\|_1 \le \mathcal{H}_M, \tag{18}$$

where we set

$$\mathcal{H}_M = \|T\|_1 \|T^{-1}\|_1 \rho_A^M \sum_{i=0}^{\overline{m}-1} \sum_{s=0}^{i} \left((-1)^s \binom{M}{i} \binom{i}{s} \frac{M-i}{M+s-i} \frac{\rho_A^{s-i}}{(1-\rho_A)^{i+1}} \right). \tag{19}$$

Proof. By (15) in Lemma 1

$$\begin{aligned}\sum_{s=M}^{\infty} \|A^s\|_1 &\le \sum_{s=M}^{\infty} \|T\|_1 \|T^{-1}\|_1 \sum_{i=0}^{\overline{m}-1} \binom{s}{i} \rho_A^{s-i} \\ &= \|T\|_1 \|T^{-1}\|_1 \sum_{i=0}^{\overline{m}-1} \sum_{s=M}^{\infty} \binom{s}{i} \rho_A^{s-i} \\ &= \|T\|_1 \|T^{-1}\|_1 \sum_{i=0}^{\overline{m}-1} \sum_{s=0}^{\infty} \binom{s+M}{i} \rho_A^{s+M-i}\end{aligned} \tag{20}$$

where the first equality is the exchange between the two summations and the second equality is due to a shift in the inner summation index. Using the fact that

$$\binom{s+M}{i} = \binom{M}{i} \frac{\binom{s+M}{M}}{\binom{s+M-i}{s}},$$

and that $\rho_A < 1$ by Assumption 1, we have

$$\begin{aligned}&\sum_{s=0}^{\infty} \binom{s+M}{i} \rho_A^{s+M-i} \\ &\quad = \rho_A^{M-i} \binom{M}{i} \sum_{s=0}^{\infty} \frac{\binom{s+M}{M}}{\binom{s+M-i}{s}} \rho_A^s \\ &\quad = \rho_A^{M-i} \binom{M}{i} {}_2F_1(1, M+1, M+1-i, \rho_A),\end{aligned} \tag{21}$$

where ${}_2F_1(a,b,c,z)$ is the Gaussian Hypergeometric function, (Olde Daalhuis 2010). Using the Euler's transformation, ${}_2F_1(a,b,c,z)$ can be expressed in the following closed form:

$$\begin{aligned}&{}_2F_1(1, M+1, M+1-i, \rho_A) \\ &\quad = \frac{M-i}{(1-\rho_A)^{i+1}} \sum_{s=0}^{i} (-1)^s \binom{i}{s} \frac{\rho_A^s}{M+s-i}\end{aligned} \tag{22}$$

Combining (20), (21), and (22) yields (18), thus concluding the proof. □

Remark 2 (Tightness) *Note that the bounds given in Lemmas 1 and 2 are tight. If matrix A is equal to a single Jordan block, then, equality holds in both* (15) *and* (18). *In fact, matrix T would be the identity, and* (16) *and* (17) *would hold with the equality.* □

4.2 Preparatory results on the truncation error bounds

We now formally assess the quality of $\boldsymbol{x_M}$ in (7) as an approximation of $\boldsymbol{x_\infty}$ in Proposition 1 and Corollary 1. Results in this subsection hold for any choice of the controller parameters $(\gamma, \vartheta) \in (\Gamma, \Theta)$.

Proposition 1 (Error bound) *Under Assumptions 1 and 2, we have that*

$$\mathbb{E}_{\boldsymbol{d_k}}\left[\|x_{k,\infty} - x_{k,M}\|_1\right] \le \eta\, \sigma\, \mathcal{H}_M, \tag{23}$$

where we set $\eta = \max_{\vartheta \in \Theta} \|B\vartheta + W\|_1$, $\sigma = \sum_{j=1}^{n_d} \sqrt{[V_d]_{j,j}}$, $V_d = \mathbb{E}_{\boldsymbol{d_k}}\left[d_k d_k^\top\right]$, *and* $\mathcal{H}_M$ *is defined in* (19).

Proof. Consider $\mathbb{E}_{\boldsymbol{d_k}}\left[\|x_{k,\infty} - x_{k,M}\|_1\right]$ and pick $M' > M$. By adding and subtracting $x_{k,M'}$ we have

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{d_k}}\left[\|x_{k,\infty} - x_{k,M}\|_1\right] \\ &\quad \le \mathbb{E}_{\boldsymbol{d_k}}\left[\|x_{k,\infty} - x_{k,M'}\|_1\right] + \mathbb{E}_{\boldsymbol{d_k}}\left[\|x_{k,M'} - x_{k,M}\|_1\right],\end{aligned} \tag{24}$$

due to sub-additivity of $\|\cdot\|_1$ and linearity of $\mathbb{E}_{\boldsymbol{d_k}}[\,\cdot\,]$. The second term on the right-hand-side of (24) can be bounded as follows

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{d_k}}\left[\|x_{k,M'} - x_{k,M}\|_1\right] \\ &\quad = \mathbb{E}_{\boldsymbol{d_k}}\left[\left\| \sum_{s=M}^{M'-1} A^s (B\vartheta + W) d_{k-1-s} \right\|_1\right] \\ &\quad \le \mathbb{E}_{\boldsymbol{d_k}}\left[\sum_{s=M}^{M'-1} \|A^s (B\vartheta + W) d_{k-1-s}\|_1\right] \\ &\quad \le \sum_{s=M}^{M'-1} \|A^s\|_1 \|(B\vartheta + W)\|_1 \mathbb{E}_{\boldsymbol{d_k}}[\|d_{k-1-s}\|_1],\end{aligned} \tag{25}$$

where the first equality is due to (7), the first inequality to sub-additivity of $\|\cdot\|_1$, and the last inequality to sub-multiplicativity of $\|\cdot\|_1$ and linearity of $\mathbb{E}_{\boldsymbol{d_k}}[\,\cdot\,]$. The right-hand-side of (25) can be upper bounded as follows

$$\begin{aligned}\sum_{s=M}^{M'-1} &\|A^s\|_1\|B\vartheta+W\|_1\mathbb{E}_{\boldsymbol{d_k}}[\|d_{k-1-s}\|_1]\\ &=\sum_{s=M}^{M'-1}\|A^s\|_1\|B\vartheta+W\|_1\sum_{j=1}^{n_d}\mathbb{E}_{\boldsymbol{d_k}}\left[|[d_{k-1-s}]_j|\right]\\ &\leq\sum_{s=M}^{M'-1}\|A^s\|_1\|B\vartheta+W\|_1\sum_{j=1}^{n_d}\sqrt{\mathbb{E}_{\boldsymbol{d_k}}\left[|[d_{k-1-s}]_j|^2\right]}\\ &\leq\eta\sigma\sum_{s=M}^{M'-1}\|A^s\|_1 \qquad (26)\end{aligned}$$

where the first equality is given by the definition of $\|\cdot\|_1$, the first inequality is the Lyapunov inequality, (Shiryaev 1996, pag. 193), and the last inequality directly follows from the stationarity of $\boldsymbol{d}$ and the definition of $\eta=\max_{\vartheta\in\Theta}\|B\vartheta+W\|_1$, $\sigma=\sum_{j=1}^{n_d}\sqrt{[V_d]_{j,j}}$, and $V_d=\mathbb{E}_{\boldsymbol{d_k}}\left[d_k d_k^\top\right]$. Using (25) and (26) in (24) we get

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{d_k}}\left[\|x_{k,\infty}-x_{k,M}\|_1\right]\\ &\quad\leq\mathbb{E}_{\boldsymbol{d_k}}\left[\|x_{k,\infty}-x_{k,M'}\|_1\right]+\eta\sigma\sum_{s=M}^{M'-1}\|A^s\|_1. \qquad (27)\end{aligned}$$

Taking the limit on both sides of (27) as $M'\to\infty$ and recalling that $x_{k,M'}\xrightarrow{L^2}x_{k,\infty}$ (see discussion below (7)) and that $x_{k,M'}\xrightarrow{L^2}x_{k,\infty}$ implies $x_{k,M'}\xrightarrow{L^1}x_{k,\infty}$, we have

$$\mathbb{E}_{\boldsymbol{d_k}}\left[\|x_{k,\infty}-x_{k,M}\|_1\right]\leq\eta\sigma\sum_{s=M}^{\infty}\|A^s\|_1. \qquad (28)$$

Combining (28) and (18) from Lemma 2, yields (23), thus concluding the proof. □

The following result follows from Proposition 1.

Corollary 1 (Convergence rate) *Under Assumptions 1 and 2, the convergence rate of $x_{k,M}\xrightarrow{L^1}x_{k,\infty}$, as $M\to\infty$ is a $O(\bar\rho^M)$ for any $\bar\rho\in(\rho_A,1)$.*

Proof. The statement directly follows from Proposition 1 by taking the limit on both sides of (23) as $M\to\infty$, while noticing that $\mathcal{H}_M$ defined in (19) is the summation of a finite number of terms where the term $\binom{M}{i}$ appears re-scaled by ρ_A^M and, hence, tends to zero. The asymptotic divergence rate $O(M^i)$ of $\binom{M}{i}$ is in fact dominated by the exponential convergence rate to zero of ρ_A^M. Overall we get a $O(\bar\rho^M)$ asymptotic rate of convergence to zero with $\bar\rho\in(\rho_A,1)$. □

Corollary 1 confirms the intuition that, since the controlled system in (3) is an asymptotically stable system fed by a stationary disturbance, the larger the value of M, the closer the truncation $x_{k,M}$ to $x_{k,\infty}$.

Remark 3 (Stationarity) *If system (3) is initialized at time $k=0$ with a random variable $x_0\neq x_{k,\infty}$ with finite first and second order moments, then, convergence in mean (and hence also in probability) of the resulting state process to $x_{k,\infty}$ can be proven as $k\to\infty$, with a $O(\bar\rho^k)$ rate, by following similar (yet not identical) steps to those for the proof of Corollary 1. This entails that we can optimize the controller parameters (γ,ϑ) by referring to the stationary process $\boldsymbol{x_\infty}$, and get an optimal stationary behavior in the long run.* □

Define the function

$$g(x,u,d,h)=\max\{\ell(x,u,d)-h,f(x,u)\}. \qquad (29)$$

Then, we get the following equivalence

$$\ell(x,u,d)\leq h,\ f(x,u)\leq 0\iff g(x,u,d,h)\leq 0. \qquad (30)$$

Set $g_{k,\infty}=g(x_{k,\infty},\gamma+\vartheta d_k,d_k,h)$ and $g_{k,M}=g(x_{k,M},\gamma+\vartheta d_k,d_k,h)$ for ease of notation.

In the following we exploit the result of Proposition 1 to establish the convergence $g_{k,M}$ to $g_{k,\infty}$ as $M\to\infty$.

Proposition 2 (Constraint function error bound) *Under Assumptions 1, 2, and 3, we have that $g_{k,M}\xrightarrow{P}g_{k,\infty}$ as $M\to\infty$. Moreover, if also Assumption 5 holds, we have that*

$$\mathbb{E}_{\boldsymbol{d_k}}[|g_{k,\infty}-g_{k,M}|]\leq\mathcal{K}_M, \qquad (31)$$

where $\mathcal{K}_M=L\eta\sigma\mathcal{H}_M$, and $g_{k,M}\xrightarrow{L^1}g_{k,\infty}$ as $M\to\infty$ with a convergence rate that is $O(\bar\rho^M)$ for any $\bar\rho\in(\rho_A,1)$.

Proof. The convergence in mean of a random variable implies the convergence in probability (see, e.g, (Shiryaev 1996, Theorem 2, pag. 256)), therefore $x_{k,M}\xrightarrow{L^1}x_{k,\infty}$ implies $x_{k,M}\xrightarrow{P}x_{k,\infty}$. Moreover, since $g(x,u,d,h)$ is continuous in x thanks to the convexity requirement of Assumption 3, the Continuous Mapping Theorem (Billingsley 1968, Corollary 2, pag. 31) applies, yielding $g_{k,M}\xrightarrow{P}g_{k,\infty}$, which is the first part of the proposition. Under Assumption 5, we have

$$|g_{k,\infty}-g_{k,M}|\leq L\|x_{k,\infty}-x_{k,M}\|_2\leq L\|x_{k,\infty}-x_{k,M}\|_1.$$

Applying the expected value operator on both sides and exploiting (23) in Proposition 1, we obtain (31). By taking the limit on both sides of (31), the last part of the proposition readily follows. □

4.3 Proof of Theorem 1

Consider the following optimization problem

$$\min_{\gamma\in\Gamma,\vartheta\in\Theta,h} \quad h \tag{32}$$
$$\text{subject to:} \quad \mathbb{P}_{\boldsymbol{d_k}}\{g(x_{k,M},\gamma+\vartheta d_k,d_k,h)\le -\delta\}\ge 1-\tilde{\varepsilon},$$

where $x_{k,M}$ is defined in (7) and $\tilde{\varepsilon}$ is defined in (10).

As a consequence of definition (29) and (30), thanks to Assumptions 3 and 4, it follows from (Campi & Garatti 2008, Theorem 2.4) that the solution $(\gamma^\star,\vartheta^\star,h^\star)$ to the scenario program (8) with N satisfying (9) is a feasible solution for the chance-constrained problem (32) with probability at least $1-\beta$. Equivalently,

$$\mathbb{P}^N_{\boldsymbol{d_k}}\{\mathbb{P}_{\boldsymbol{d_k}}\{g(x^\star_{k,M},\gamma^\star+\vartheta^\star d_k,d_k,h^\star)>-\delta\}\le\tilde{\varepsilon}\}\ge 1-\beta, \tag{33}$$

where $x^\star_{k,M}$ is as in (7) with $(\gamma,\vartheta)=(\gamma^\star,\vartheta^\star)$.

Consider now the original chance-constrained optimization problem (4), and notice that the left-hand side of its probabilistic constraint evaluated for $(\gamma^\star,\vartheta^\star,h^\star)$, namely, $\mathbb{P}_{\boldsymbol{d_k}}\{g(x^\star_{k,\infty},\gamma^\star+\vartheta^\star d_k,d_k,h^\star)>0\}\le\varepsilon$, can be upper bounded as follows

$$\begin{aligned}
&\mathbb{P}_{\boldsymbol{d_k}}\{g(x^\star_{k,\infty},\gamma^\star+\vartheta^\star d_k,d_k,h^\star)>0\}\\
&\quad=\mathbb{P}_{\boldsymbol{d_k}}\{g^\star_{k,M}+\delta+g^\star_{k,\infty}-g^\star_{k,M}-\delta>0\}\\
&\quad\le\mathbb{P}_{\boldsymbol{d_k}}\{g^\star_{k,M}>-\delta\ \vee\ g^\star_{k,\infty}-g^\star_{k,M}>\delta\}\\
&\quad\le\mathbb{P}_{\boldsymbol{d_k}}\{g^\star_{k,M}>-\delta\}+\mathbb{P}_{\boldsymbol{d_k}}\{g^\star_{k,\infty}-g^\star_{k,M}>\delta\},
\end{aligned}\tag{34}$$

where we adopted the shorthand notations $g^\star_{k,\infty}=g(x^\star_{k,\infty},\gamma^\star+\vartheta^\star d_k,d_k,h^\star)$ and $g^\star_{k,M}=g(x^\star_{k,M},\gamma^\star+\vartheta^\star d_k,d_k,h^\star)$. The first term in (34) is the inner probability appearing in (33). The second term, instead, can be upper bounded by $\frac{\mathcal{K}_M}{\delta}$ irrespective of the value taken by $\gamma^\star$ and $\vartheta^\star$:

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{d_k}}\{g^\star_{k,\infty}-g^\star_{k,M}>\delta\}&\le\mathbb{P}_{\boldsymbol{d_k}}\{|g^\star_{k,\infty}-g^\star_{k,M}|>\delta\}\\
&\le\frac{\mathbb{E}_{\boldsymbol{d_k}}\left[|g^\star_{k,\infty}-g^\star_{k,M}|\right]}{\delta}\le\frac{\mathcal{K}_M}{\delta},
\end{aligned}\tag{35}$$

where the second inequality is the application of Chebyshev's inequality (Shiryaev 1996, pag. 192), and the last inequality follows from (31) in Proposition 2. From (34), (35), and (33) we eventually obtain that

$$\begin{aligned}
&\mathbb{P}^N_{\boldsymbol{d_k}}\left\{\mathbb{P}_{\boldsymbol{d_k}}\{g(x^\star_{k,\infty},\gamma^\star+\vartheta^\star d_k,d_k,h^\star)>0\}\le\tilde{\varepsilon}+\frac{\mathcal{K}_M}{\delta}\right\}\\
&\quad\ge\mathbb{P}^N_{\boldsymbol{d_k}}\Big\{\mathbb{P}_{\boldsymbol{d_k}}\{g^\star_{k,M}>-\delta\}\le\tilde{\varepsilon}\ \wedge\\
&\qquad\qquad\mathbb{P}_{\boldsymbol{d_k}}\{g^\star_{k,\infty}-g^\star_{k,M}>\delta\}\le\frac{\mathcal{K}_M}{\delta}\Big\}\\
&\quad=\left[\mathbb{P}^N_{\boldsymbol{d_k}}\left\{\mathbb{P}_{\boldsymbol{d_k}}\{g^\star_{k,\infty}-g^\star_{k,M}>\delta\}\le\frac{\mathcal{K}_M}{\delta}\right\}=1\right]\\
&\quad=\mathbb{P}^N_{\boldsymbol{d_k}}\left\{\mathbb{P}_{\boldsymbol{d_k}}\{g^\star_{k,M}>-\delta\}\le\tilde{\varepsilon}\right\}\ge 1-\beta.
\end{aligned}$$

The obtained inequality is statement (12) in Theorem 1 given the definition of $\tilde{\varepsilon}$ in (10) and the definition of $g(\cdot)$ in (29). This concludes the proof. □

5 Numerical example

We present an energy management application example to show the efficacy of the proposed scenario-based approach. In this example we solve the problem by the scenario program (8) with design parameters tuned according to Theorem 1 and the guidelines in Remark 1.

Consider a photovoltaic panel installation connected to the grid. The amount of energy produced by the panels and injected into the grid clearly depends on the amount of solar irradiation, which is unpredictable due to variabilities in the weather conditions and can vary significantly. A battery is introduced to act as a buffer between the photovoltaic panel installation and the main grid and absorb the energy fluctuations. The goal is to offer to the grid a nominal exchange profile and a given (minimal) variability around that profile, so as to ease the grid operator task of balancing electrical energy production and consumption, and, hence, to facilitate the integration of the highly varying solar energy production in the electrical grid. Due to the daily periodicity of the solar irradiation phenomenon, an effective battery management strategy can be designed with reference to a one-day time horizon. We next embed the stochastic periodic control problem in our framework by a lifting transformation and apply Theorem 1 to design (off-line, and only once!) a disturbance compensator for battery management.

Consider a one day time horizon discretized into $T_h=144$ time slots of 10 minutes duration each. Let $E_b(t)\in\mathbb{R}$ denote the energy exchanged with the battery ($E_b(t)>0$ if the battery is charged and $E_b(t)<0$ if discharged), $E_p(t)\in\mathbb{R}$ the energy produced ($E_p(t)\ge 0$) by the solar panels, and $E_g(t)\in\mathbb{R}$ the amount of energy injected into the grid, in time slot $t\in\mathbb{N}$. Then, the energy balance equation $E_g(t)=E_p(t)-E_b(t)$ must hold for any t. The evolution of the battery state of charge (SOC) $\xi(t)\in\mathbb{R}$ at the beginning of each time slot t follows the first order model $\xi(t+1) = a\xi(t) + E_b(t)$, where we set $a = 0.998$ to account for the self-discharging losses per time slot.

We assume that the solar energy production $E_p(t)$ is a strictly cyclostationary process with period T_h with known first and second order moments. Let k denote the index of the day and $\mu_p = \mathbb{E}[[E_p(kT_h)\, E_p(kT_h+1)\, \dots\, E_p((k+1)T_h-1)]^\top] \in \mathbb{R}^{T_h}$ the daily average production profile. Then, the disturbance process $\boldsymbol{d}$ with $d_k \in \mathbb{R}^{T_h}$ representing the solar energy production fluctuations with respect to the average profile μ_p, i.e., $d_k = [E_p(kT_h)\, E_p(kT_h+1)\, \dots\, E_p((k+1)T_h-1)]^\top - \mu_p$, satisfies Assumption 2. If we let $x_k = \xi(kT_h)$ be the battery SOC at the beginning of day k, by iterating the battery model for T_h time slots, the day-by-day evolution of the battery SOC is given by $x_{k+1} = Ax_k + Bu_k$, with $A = a^{T_h}$, $B = [a^{T_h-1} \cdots a\ 1]$, and $u_k = [E_b(kT_h) \cdots E_b((k+1)T_h-1)]^\top \in \mathbb{R}^{T_h}$. Clearly, A satisfies Assumption 1 since $a < 1$.

Note that the disturbance d_k does not enter directly into the battery model, but it does affect the control input $u_k = \gamma + \vartheta d_k$, where $\gamma \in \mathbb{R}^{T_h}$ and $\vartheta \in \mathbb{R}^{T_h \times T_h}$, thus allowing the battery to react to the measured solar production profile representing the disturbance to compensate. Since the value of $E_b(t)$ cannot depend on the future solar energy production $E_p(\tau)$ with $\tau \geq t$, ϑ has a strictly lower triangular structure, i.e., with zeros on and above the main diagonal. To reduce the number of optimization variables, we constrain ϑ to have only the first $p = 3$ subdiagonals different from zero and with the elements on the same subdiagonal being equal, which corresponds to using the solar production in the latest 3 time slots for compensation and weight it with time-invariant parameters.

Our goal is to design the compensator such that the energy exchanged with the grid $\mu_p + d_k - u_k$ remains within a minimum-width tube centered in $\mu_p - \gamma$ (i.e., the nominal grid exchange profile) with a pre-defined high probability. To this end, let $\mathbb{1} \in \mathbb{R}^{T_h}$ and parameterize γ and the tube half-width as $\gamma = c_\gamma \mathbb{1}$ and $c_{t,0}\mathbb{1} + c_{t,1}\mu_p$, respectively, where $c_\gamma, c_{t,0}, c_{t,1} \in \mathbb{R}$, with $c_{t,0}, c_{t,1} \geq 0$, are further optimization variables (the total number is $n = p + 3 = 6$). In this way, the tube width is tuned according to the average production profile and is possibly zero in those time slots where the production is zero with probability 1, like at night. The constraint that $\mu_p + d_k - u_k$ belongs to the tube can then be written as

$$|\mu_p + d_k - u_k - (\mu_p - \gamma)| = |(I - \vartheta)d_k| \leq c_{t,0}\mathbb{1} + c_{t,1}\mu_p,$$

where the absolute value and the inequality have to be intended component-wise. In order to minimize the deviation from the nominal profile, we set the cost function $\ell(x_k, u_k, d_k) = \mathbb{1}^\top(c_{t,0}\mathbb{1} + c_{t,1}\mu_p)$. Clearly, we also need to ensure that the battery SOC $\xi(t)$ and the battery energy exchange $E_b(t)$ both stay within their minimum and maximum values in each time slot t.

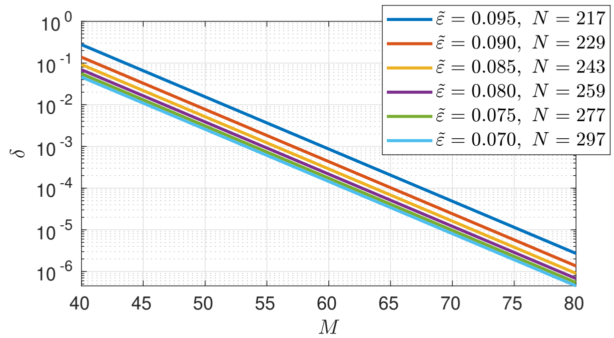

Fig. 1. Value of δ as a function of M for different values of $\tilde{\varepsilon}$ and the corresponding N, when $\varepsilon = 0.1$ and $\beta = 10^{-4}$.

The resulting control problem can be formulated as

$$\begin{aligned}
\min_{c_\gamma, \vartheta, c_{t,0}, c_{t,1}, h} \quad & h \\
\text{subject to:} \quad & \mathbb{P}_{\boldsymbol{d_k}}\Big\{ |(I - \vartheta)d_k| \leq c_{t,0}\mathbb{1} + c_{t,1}\mu_p, \\
& \quad \xi^{\min} \leq A_T x_{k,\infty} + B_T(c_\gamma \mu_p + \vartheta d_k) \leq \xi^{\max}, \\
& \quad |c_\gamma \mu_p + \vartheta d_k| \leq s^{\max} \Big\} \geq 1 - \varepsilon \\
& \mathbb{1}^\top(c_{t,0}\mathbb{1} + c_{t,1}\mu_p) \leq h \\
& c_{t,0} \geq 0,\ c_{t,1} \geq 0,\ \|\vartheta\|_1 \leq r,
\end{aligned}$$

where $\xi^{\min} = 5\%$ and $\xi^{\max} = 95\%$ of the battery total capacity of 60MJ, $s^{\max} = 1$MJ is the maximum energy that can be exchanged with the battery in one time slot, $A_T \in \mathbb{R}^{T_h+1}$ and $B_T \in \mathbb{R}^{(T_h+1)\times T_h}$ are such that $A_T\xi(kT_h) + B_T(c_\gamma\mu_p + \vartheta d_k) = [\xi(kT_h)\, \xi(kT_h+1) \cdots \xi((k+1)T_h)]^\top$, and $\|\vartheta\|_1 \leq r$ is used to define the set Θ.

As for $\mathcal{K}_M$ appearing in Theorem 1, the constraints are linear in x with Lipschitz constant $L = 1$, $\sigma = 11.7624$ (as it can be inferred from a historical dataset of 229 daily energy production profiles collected in the General Electric Research Center in Munich, Germany), and it is easy to see that $\eta \leq \|B\|_1 \max_{\vartheta \in \Theta} \|\vartheta\|_1 + \|W\|_1 \leq r$ since $\|B\|_1 = 1$, $W = 0$, and $\|\vartheta\|_1 \leq r$, and we can therefore set $\eta = r$. If we further notice that $T = 1$ the expression for $\mathcal{K}_M$ simplifies to $\mathcal{K}_M = \frac{r\sigma}{1-a^{T_h}} a^{T_h M} \approx 141 \cdot 0.75^M$.

Figure 1 reports the value of δ as a function of M, for different tentative values of $\tilde{\varepsilon} < \varepsilon$ and the corresponding number N of scenarios, when $\varepsilon = 0.1$ and $\beta = 10^{-4}$.

Results reported next refer to $\tilde{\varepsilon} = 0.095$, which corresponds to $N = 217$ by solving (9) via bisection. We chose $M = 60$ to have $\delta < 10^{-3}$ ($\delta = 8.67 \cdot 10^{-4}$) so as not to overtighten the constraints. For comparison purposes we also computed the tube when the battery is not present. Disturbance realizations were generated using a Gaussian model with mean and covariance estimated from the available dataset of 229 daily energy production profiles.

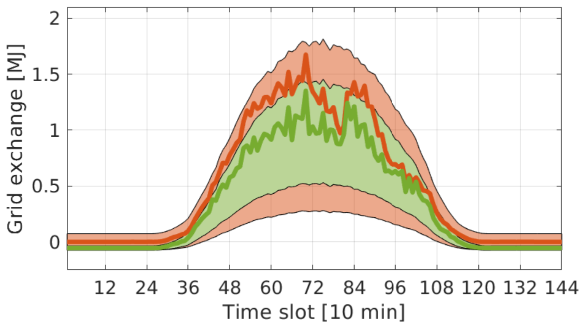

Fig. 2. A realization of $E_g(t)$ at day $M = 60$ together with the optimal tubes, with (green) and without (red) battery.

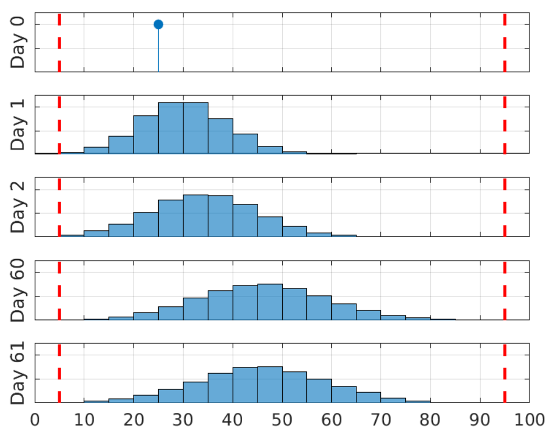

Fig. 3. Probability distribution of $\xi(kT_h)$ for days $k = 1, 2, 3$, $M, M+1$, $M = 60$. Red dashed bars denote $\xi^{\min}$ and $\xi^{\max}$.

We simulated $N_v = 10^4$ validation scenarios (different from those used for the optimization) for both strategies: with and without the battery. In Figure 2 we report a representative behavior of $E_g(t)$ during day $M = 60$ when the battery is absent (red) and when the battery is present (green) along with the corresponding tubes. Both profiles stay within the respective tube, but the tube obtained with our strategy is narrower with respect to the one without the battery (44% reduction of the tube area when using the battery).

Figure 3 represents the evolution of the probability distribution of $\xi(kT_h)$ computed over the N_v validation scenarios for days $k = 0, 1, 2, M, M+1$, with $M = 60$: the battery was initialized at 25% in all scenarios and, after a transient, its probability distribution tends to the asymptotic one. Note also that there are some realizations which go beyond $\xi^{\max}$ or $\xi^{\min}$, but the probability mass in those regions is very low.

To validate the statement in Theorem 1 we computed a posteriori the empirical violation $\hat{\varepsilon}$, which counts the percentage of realizations that violate at least one constraint over day M, with respect to the total number of

δ	M	$\hat{\varepsilon}$	$h^\star$
$8.79 \cdot 10^{0}$	28	0.0245	29.07
$8.76 \cdot 10^{-1}$	36	0.0215	25.46
$8.73 \cdot 10^{-2}$	44	0.0214	25.29
$8.70 \cdot 10^{-3}$	52	0.0214	25.27
$8.67 \cdot 10^{-4}$	60	0.0214	25.27

Table 1
Empirical violation $\hat{\varepsilon}$ and optimal cost $h^\star$ as a function of the tightening parameter δ and the truncation length M.

N_v simulations in the case where the battery is present, for different values of the tightening δ (see Table 1). As can be seen from the table, $\hat{\varepsilon}$ is always less than $\varepsilon = 0.1$, as predicted by the theory. Moreover, both $\hat{\varepsilon}$ and $h^\star$ are barely affected by δ when $\delta \leq 1$. Instead, when δ is larger, the tightening introduces some conservatism and an increase in the optimal cost is experienced.

6 Conclusions

We addressed the design of a disturbance compensator for a discrete time linear system so as to optimize its performance in stationary regime while satisfying probabilistic joint state/input constraints. We proposed an off-line one-shot design method that rests on the approximation of the stationary state process with a truncated version and on the resolution of a chance-constrained optimization program via an extended scenario approach, where a tightening of the constraints is introduced so as to preserve feasibility guarantees. The approach looks competitive in terms of ease of computation, applicability, and performance guarantees, with respect to alternative approaches to optimal constrained control when the state measurement is not available for feedback.

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