博士学位論文

Asymptotic analysis of bending, torsional and stretching eigenfrequencies of thin elastic rods

（細長い弾性体における，曲げ・捩れ・伸び縮みモードの固有振動数の漸近挙動）

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1 Introduction

1.1 Background

Elasticity is the property of a body to resist a distorting influence and to return to its original size and shape when that influence or force is removed. Natural frequency, also known as eigenfrequency, is the frequency at which a system tends to oscillate in the absence of any driving or damping force. Resonance is a phenomenon that occurs when the frequency at which a force is periodically applied is equal or nearly equal to one of the natural frequencies of the system. This type of force causes the system to oscillate with larger amplitude than forces of other frequencies. Studies of the characteristic vibrations of elastic bodies are extremely effective methods of analysis in engineering and physics due to its numerous applications such as reduction of noise pollution of machines, efficiency of automotive tires, construction of safer buildings and so on.

We introduce some examples of how eigenfrequencies and building safety are related. The Tacoma Narrows Bridge was a suspension bridge in the state of Washington, U.S.A. that was designed to withstand winds of up to 60m/s. However in November 7, 1940 the bridge started to oscillate violently and eventually collapsed even though the speed of the wind was 19m/s. The cause was highly debated by experts who concluded that the bridge started vibrating due to a mixture of resonance and aeroelastic fluttering. Angers Bridge, also called the Basse-Chaîn Bridge, was a suspension bridge in France that collapsed in 1850 while a battalion of French soldiers was marching across it. The pace of the soldiers' march matched the natural frequency of the suspension bridge, and as a consequence, the amplitude of the oscillations started increasing. The structure could not withstand the bending and collapsed, bringing the number of casualties up to 226. Since then, the military issues orders that troops should walk ordinarily when crossing a bridge.

After these incidents, winds and other environmental phenomena are well studied and taken into account before building a new bridge. This is a measure to adjust the natural frequencies of the bridge so that it does not resonate with other influences.

Resonances can also be detected far from the epicenter of seismic activities. For example, after the 2007 Niigata Chūetsu Offshore earthquake or the 2011 earthquake off the Pacific coast of Tōhoku, long-periodic mechanical resonances caused by these earthquakes were detected in the Kantō region of Japan. In general, low-rise and middle-rise buildings as well as buildings with wide floor space do not have small eigenfrequencies. On the other hand, high-rise buildings, skyscrapers and buildings with narrow floor space have small natural frequencies. It is for this reason that tall and thin buildings are more easily affected by long-periodic seismic motions and can cause the structure to crumble. Therefore, vibrations must be thoroughly studied from different fields and point of view. These studies are applied in the safety of large-scale architectural structures in the following way. For example, they are used to develop new technologies for seismic base isolation and vibration control. Seismic base isolation, also called vibration damping, is used to change the value of the eigenfrequencies in order to avoid mechanical resonance from happening.

From all these examples, it is clear that the control of natural frequencies is crucial during the whole process of planning and construction of an architectural structure. Not only in architecture, but also in many fields of engineering, other aspects must be taken into consideration, such as the gradual change of the eigenfrequencies caused by natural processes like deterioration over time or changes on the site environment. Moreover, shape also plays an important roll in order to control the natural frequencies. We give some examples: a particular shape of a tunnel exit helps reduce the noise pollution it produces after a train passes through it, opening some holes in an industrial machine can make it more difficult to vibrate against outside influence and therefore more efficient, achieving an optimal shape for a tire can drastically reduce a vehicle interior noise, etc. Thus, the study of natural frequencies of all types of structures and shapes is of much importance in engineering. Moreover, novel contributions with studies from the point of view of mathematics gives a further advancement of the understanding of mechanical resonance and eigenfrequencies.

The description of a general shape with mathematical equations is not an easy task and the more complex the structure is, the harder it is to work with it mathematically. We explain the motivation of a usual mathematical approach using the case of buildings. Since most of the interior of a building can be thought as empty space, one can reduce its fundamental shape to a joint of walls and pillars. These simpler structures can be represented mathematically as plates and rods. As a first approach, one may assume that the thinness of the plates and rods is negligible, so that we work with 2-dimensional plates and 1-dimensional rods. However, in architecture it is well known that the thickness of the plates and rods has an influence on the eigenfrequences. Therefore a 3-dimensional model has to be considered.

There are physical 3-dimensional models that use partial differential equations to describe the deformation of an elastic body. We know that in general one cannot explicitly solve the PDE and have a solution written in terms of elementary functions. Therefore, in engineering and architecture, although they have explicit equations for the 1-dimensional model, they rely almost completely on simulations for the 3-dimensional case. However, the partial differential equation modelling the problem is also interesting from the point of view of mathematics and important information about the eigenfrequencies can be extracted, which can help to provide even better simulation methods.

We may think that the basic structure of buildings is made of thin plates and thin rods. This thinness can be represented by a small parameter. As a mathematical approach, one wants to study the asymptotic behavior of the solutions and natural frequencies as the thinness associated to the thin plates and thin rods goes to 0. In the case of the eigenfrequencies, we perform the spectral analysis of the partial differential equation that arises from the elasticity problem.

Using mathematical tools, such as asymptotic analysis, variational methods, and so on, we know that in the particular case of the linearized elasticity model of a homogeneous and isotropic rod there are several types of natural frequencies, associated to bending (or flexural), torsional and stretching modes, each with its different asymptotic behavior. In this thesis we analyze the asymptotic behavior of small eigenvalues and eigenfunctions (associated to bending mode) of the linearized elasticity eigenvalue problem of a thin rod with non-uniform cross-section (see Figure 1), as well as the asymptotic behavior of eigenvalues and eigenfunctions (associated to torsional and stretching modes) of the linearized elasticity eigenvalue problem of a thin rod with non-uniform symmetric cross-section.

1.2 Previous research

There are many works on such type of spectral problems of singularly deformed domains in these several decades (cf. Courant-Hilbert [9], Egorov-Kondratiev [13], Maz'ya-Nazarov-Plamenevskij [26]). Particularly, eigenvalue problems of vibration of thin elastic bodies like plates and rods are of much importance and interest from PDE theory and engineering point of view (see for example Antman [1], Ciarlet [6], Cioranescu-Saint Jean Paulin [8], Love [25], Nazarov [28]).

Ciarlet and Kesevan [7] pioneered ideas on elastic plates that would further be adapted to the case of thin rods. To name some previous works, Kerdid [18] studied the behavior of small eigenvalues of the linearized elasticity eigenvalue problem of a thin rod with constant cross-section. Tambača [31] gives a result on the convergence of the eigenvalues and eigenfunctions in the case of a thin curved rod. Both studies consider that the ends of the rod are clamped. Kerdid [19] and [20] also considered a joint of two rods with one of the ends without clamping.

In other similar works on linear elasticity problems that are related to the present thesis, Le Dret [21] treat the junction of two rods while Le Dret [22], [23] and [24] deals with folded plates. Griso ([14] among other works) studies the asymptotic behavior of structures made of junctions of curved rods, plates and combinations of both types. Irago-Viaño [15] obtained higher order approximations of flexural eigenvalues of a thin straight rod using an asymptotic expansion procedure. Irago-Kerdid-Viaño [16] studied the case of high-frequency vibrations related to stretching and torsional modes of thin rods. Nazarov [27], Nazarov-Slutskii [29] and Buttazzo-Cardone-Nazarov [4], [5] provide an elaborate research on asymptotic expansion methods for anisotropic and non-homogeneous elastic thin rods and plates. The study of eigenvalue problems on thin multi-structures for different equations is common and of much interest in the PDE theory. For example, works like Bunoiu-Cardone-Nazarov [2], [3] deal with the case of the Poisson equation for junctions of rods and a plate. For an extensive list of references, see Ciarlet [6].

1.3 Plan of the thesis

The present thesis contains two main results and is organized as follows. In Section 2 we explain the common mathematical setting of the problems as well as some tools and preliminaries needed throughout the proofs of the results. The purpose of Section 3 is to give similar results of the behavior of small eigenvalues associated to the bending mode in more general rods. We obtain the characterization formula, which is derived from a fourth order ordinary differential equation system on the one-dimensional limit set of the thin elastic body. We make full use of the variational characterization of the eigenvalues as well as detailed analysis of the weak formulation of the eigenfunctions. Previous works considered rods with simply connected, constant cross-section and such that its barycenter or "center of mass" is constant. We remove these restrictions and we deal with a rod that has non-uniform connected cross-section. Furthermore, we consider the case when both ends of the rod are clamped, and also the case when only one end is clamped. In Section 4 we give a result for high-frequency vibrations related to stretching and torsional modes of thin rods with axis-symmetric cross-section. We fully prove that the limit of this type of eigenfunction is non-zero, which leads to two completely independent second order ordinary differential equations, one for each vibration mode, describing the limit behavior. We provide a proof for the H^1-strong convergence and we give the idea of how to prove a more general result in curved rods. In Section A we give proof to some lemmas and further details on some computations stated in the main body of the thesis. The results of this thesis are based on a joint work with professor Shuichi Jimbo.

2 Mathematical settings and preliminaries

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth enough boundary. We want to study the oscillations of an elastic body with the shape of Ω.

We denote by $u = (u_1, u_2, u_3) : \Omega \longrightarrow \mathbb{R}^3$ the displacement vector field associated with the oscillations. Let $\lambda_1, \lambda_2 > 0$ be positive real constants corresponding to the mechanical properties of the elastic body. We define the tensors

$$e(u) = (e_{ij}(u))_{1\le i,j\le 3} = \left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)_{1\le i,j\le 3},$$
$$\sigma(u) = \lambda_1 \operatorname{tr}(e(u)) \operatorname{Id}_3 + 2\lambda_2 e(u),$$

where tr is the trace of a matrix and Id_3 is the 3×3 identity matrix. $e(u)$ is called the *linearized strain tensor* and $\sigma(u)$ is the *stress tensor* derived from Hooke's law in the case of a homogeneous isotropic elastic body (cf. Ciarlet [6]).

With this notation, the operator of the elastic equation is defined as the second order linear elliptic operator

$$L[u] = \operatorname{div}\sigma(u), \quad \text{i.e.} \quad (L[u])_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{ij}(u)) \quad (1 \le i \le 3),$$

and the oscillations of an elastic body can be described by the wave equation

$$\varrho \frac{\partial^2 u}{\partial t^2} = L[u], \tag{2.1}$$

where $\varrho > 0$ is the mass density.

We take $\varrho = 1$ and we assume that the oscillations are periodic of period $\frac{2\pi}{\omega}$ $(\omega > 0)$. In this case, we can write the displacement field as $u(x,t) = \mathrm{e}^{i\omega t}v(x)$. Thus, $\frac{\partial^2 u}{\partial t^2} = -\omega^2 u(x,t)$. Putting $\mu = \omega^2$, the wave equation (2.1) becomes the eigenvalue problem

$$L[v] + \mu v = \mathbf{0}.$$

Let Γ_1 be a subset of the boundary $\partial\Omega$ such that its 2 dimensional area is positive and let $\Gamma_2 = \partial\Omega\backslash\Gamma_1$. Denote $\boldsymbol{n}$ the unit outward normal vector on $\partial\Omega$. The main eigenvalue problem we study is as follows.

$$\begin{cases} L[u] + \mu u = \mathbf{0} & \text{in } \Omega \\ u = \mathbf{0} & \text{on } \Gamma_1 \\ \sigma(u)\,\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_2 \end{cases} \tag{2.2}$$

It is known (cf. Courant-Hilbert [9], Edmunds-Evans [12], Egorov-Kondratiev [13]) that the eigenvalues of (2.2) are a sequence of non-negative real numbers without points of accumulation, that is, the set of eigenvalues counting multiplicities is $\{\mu_k\}_{k=1}^{+\infty}$ satisfying

$$0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \dots \text{ with } \lim_{k\to+\infty} \mu_k = +\infty. \tag{2.3}$$

We introduce some common tools we need in order to prove the main results. We start with Korn's inequality (cf. Ciarlet [6], Dautray-Lions [10]).

Proposition 2.1 (Korn's inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^3$. If Γ_1 is a measurable subset of the boundary $\partial\Omega$ such that its 2 dimensional area is positive, then there exists a constant $C > 0$ such that*

$$\|v\|_{H^1(\Omega,\mathbb{R}^3)} \leq C \left(\sum_{i,j=1}^{3} \|e_{ij}(v)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

for any $v \in H^1(\Omega, \mathbb{R}^3)$ with $v|_{\Gamma_1} = \mathbf{0}$.

In order to characterize the eigenvalues of (2.2), we introduce the Rayleigh quotient.

Definition 2.2. Let $\phi, \psi \in H^1(\Omega, \mathbb{R}^3)\backslash\{\mathbf{0}\}$. We define the bilinear form

$$B[\phi, \psi] = \int_{\Omega} \left(\lambda_1 \operatorname{div} \phi \operatorname{div} \psi + 2\lambda_2 \sum_{i,j=1}^{3} e_{ij}(\phi) e_{ij}(\psi) \right) \mathrm{d}x$$

and the *Rayleigh quotient* by

$$\mathcal{R}(\phi) = \frac{B[\phi, \phi]}{\|\phi\|^2_{L^2(\Omega,\mathbb{R}^3)}}.$$

It is easy to see that the Rayleigh quotient satisfies $\mathcal{R}(c\phi) = \mathcal{R}(\phi)$ for all $c > 0$ (homogeneity condition).

Let $k \in \mathbb{N}$. We write $\mathcal{H}_{k-1}(\,\cdot\,, \mathbb{R}^3)$ the set of all linear subspaces of dimension $k-1$ of $L^2(\,\cdot\,, \mathbb{R}^3)$. We introduce the so-called *Max-Min principle*, which we use to characterize the eigenvalues of (2.2).

Proposition 2.3 (Max-Min principle)**.** *Let $\mathcal{W}$ be the function space*

$$\mathcal{W} = \{\phi \in H^1(\Omega, \mathbb{R}^3) \mid \phi = \mathbf{0} \text{ on } \Gamma_1\}$$

and let μ_k be the k-th eigenvalue of the problem (2.2). Then we have the following characterization of the eigenvalues.

$$\mu_k = \sup_{X \in \mathcal{H}_{k-1}(\Omega,\mathbb{R}^3)} \inf\{\mathcal{R}(\phi) \mid \phi \in \mathcal{W}\backslash\{\mathbf{0}\}, \phi \perp X \text{ in } L^2(\Omega, \mathbb{R}^3)\}. \tag{2.4}$$

We introduce some notation. Let $f_1^\varepsilon, f_2^\varepsilon$ be two real functions depending on a parameter $\varepsilon > 0$. Assume that there exists a constant C independent of ε, such that $f_1^\varepsilon \leq C f_2^\varepsilon$. Then we denote this relation by $f_1^\varepsilon \lesssim f_2^\varepsilon$. For a real constant $h \geq 0$, we denote $f_1^\varepsilon = O(\varepsilon^h)$ for $h \geq 0$ in a normed vector space X, whenever

$$\limsup_{\varepsilon \to 0} \frac{\|f_1^\varepsilon\|_X}{\varepsilon^h}$$

is finite.

3 Bending eigenfrequencies of a thin elastic rod with non-uniform cross-section

In this section we discuss the low-frequency eigenvalues of a thin elastic rod with non-uniform cross-section. We prepare the mathematical setting of our problem. We start presenting the domain $\Omega_\varepsilon = \Omega$, where $\varepsilon > 0$ is a small parameter corresponding to the thickness of the elastic rod. Let $l > 0$ and let $B \subseteq \mathbb{R}^2$ be a connected bounded domain such that the boundary is $\mathcal{C}^3$ with $m \in \mathbb{N}$ connected components. We consider the sets

$$\begin{aligned} S &= B \times (0,l), \qquad & s_1^{(-)} &= \overline{B} \times \{0\}, \\ s_1^{(+)} &= \overline{B} \times \{l\}, \qquad & s_2 &= \partial B \times (0,l). \end{aligned}$$

Note that $\partial S = s_1^{(-)} \cup s_1^{(+)} \cup s_2$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^3-diffeomorphism which satisfies the following properties.

i) $F(z) = (F_1(z), F_2(z), z_3) \quad (z = (z_1, z_2, z_3) \in S)$.

ii) $F_i(0,0,z_3) = 0 \quad (i = 1,2, \quad 0 \le z_3 \le l)$.

iii) The determinant of the Jacobian matrix of F is positive for all $z \in S$.

We define $F^\varepsilon(z) = (\varepsilon F_1(z), \varepsilon F_2(z), z_3)$. With this notation, we consider the following sets in $\mathbb{R}^3$.

$$\Omega_\varepsilon = F^\varepsilon(S), \quad \Gamma_{1,\varepsilon}^{(-)} = F^\varepsilon(s_1^{(-)}), \quad \Gamma_{1,\varepsilon}^{(+)} = F^\varepsilon(s_1^{(+)}), \quad \Gamma_{2,\varepsilon} = F^\varepsilon(s_2).$$

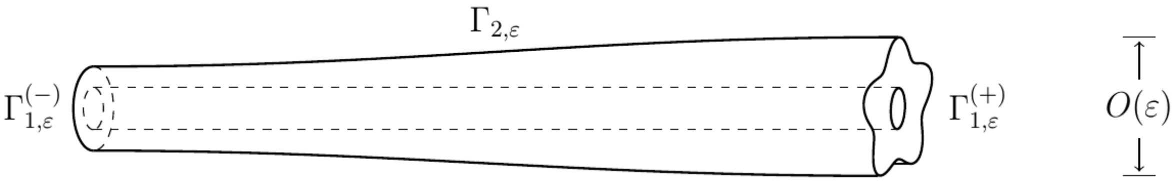

Figure 1: Example of Ω_ε

We can think of Ω_ε as a slightly smoothly deformed thin cylinder (see Figure 1). It is easy to see $\partial\Omega_\varepsilon = \Gamma_{1,\varepsilon}^{(-)} \cup \Gamma_{1,\varepsilon}^{(+)} \cup \Gamma_{2,\varepsilon}$. Moreover, we obtain Ω_1, $\Gamma_{1,1}^{(-)}$, $\Gamma_{1,1}^{(+)}$, $\Gamma_{2,1}$ just by putting $\varepsilon = 1$ in the previous definition. Note that $\Omega_1 = F(S)$.

Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ be the coordinates in the sets Ω_ε, Ω_1 and S, thus obtaining the relation between the coordinates

$$\begin{cases} (x_1, x_2, x_3) = (\varepsilon y_1, \varepsilon y_2, y_3), \\ (y_1, y_2, y_3) = (F_1(z), F_2(z), z_3), \\ (x_1, x_2, x_3) = (\varepsilon F_1(z), \varepsilon F_2(z), z_3). \end{cases} \tag{3.1}$$

We want to study the small eigenvalues (low-frequency oscillations related to bending vibrations, also called flexural vibrations) associated with the thin elastic body Ω_ε. We denote by $u = (u_1, u_2, u_3) : \Omega_\varepsilon \longrightarrow \mathbb{R}^3$ the displacement vector field associated with the oscillations.

With this notation, the main subject of this section is to study the eigenvalues and eigenfunctions when the parameter ε goes to zero of the following eigenvalue problems.

$$\begin{cases} L[u] + \mu u = \mathbf{0} & \text{in } \Omega_\varepsilon \\ u = \mathbf{0} & \text{on } \Gamma_{1,\varepsilon}^{(-)} \cup \Gamma_{1,\varepsilon}^{(+)} \\ \sigma(u)\,\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{2,\varepsilon} \end{cases} \tag{DD}$$

$$\begin{cases} L[u] + \mu u = \mathbf{0} & \text{in } \Omega_\varepsilon \\ u = \mathbf{0} & \text{on } \Gamma_{1,\varepsilon}^{(-)} \\ \sigma(u)\,\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{2,\varepsilon} \cup \Gamma_{1,\varepsilon}^{(+)} \end{cases} \tag{DN}$$

where $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega_\varepsilon$. The case (DD) corresponds to a thin rod with both ends clamped while the case (DN), to a thin rod with only one clamped end.

3.1 Main result

In order to state the main results we first introduce several notations.

Denote $\mathrm{d}y' = \mathrm{d}y_1\mathrm{d}y_2$ and define the set $\widehat{\Omega}(y_3)$ to be the cross-section of $\Omega_1 = F(S)$ at $y_3 \in [0, l]$. Furthermore, for $1 \le i, j \le 2$, we define the functions

$$H(y_3) = \int_{\widehat{\Omega}(y_3)} 1\mathrm{d}y', \quad K_i(y_3) = \int_{\widehat{\Omega}(y_3)} y_i \mathrm{d}y', \quad A_{ij}(y_3) = \int_{\widehat{\Omega}(y_3)} y_i y_j \mathrm{d}y' \quad (y_3 \in [0, l])$$

and write $Y = \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}$, known as the *Young modulus*.

We use the fact we presented in the equation (2.3) and adapt it to our thin rods. If we denote by $\{\mu_k^{DD}(\varepsilon)\}_{k=1}^{+\infty}$ and $\{\mu_k^{DN}(\varepsilon)\}_{k=1}^{+\infty}$ the eigenvalues of problem (DD) and (DN) respectively, for any $\varepsilon > 0$ there are infinite discrete sequences of positive eigenvalues

$$0 < \mu_1^{DD}(\varepsilon) \le \mu_2^{DD}(\varepsilon) \le \cdots \le \mu_k^{DD}(\varepsilon) \le \mu_{k+1}^{DD}(\varepsilon) \le \cdots \text{ with } \lim_{k\to+\infty} \mu_k^{DD}(\varepsilon) = +\infty$$
$$0 < \mu_1^{DN}(\varepsilon) \le \mu_2^{DN}(\varepsilon) \le \cdots \le \mu_k^{DN}(\varepsilon) \le \mu_{k+1}^{DN}(\varepsilon) \le \cdots \text{ with } \lim_{k\to+\infty} \mu_k^{DN}(\varepsilon) = +\infty$$

which are arranged in increasing order, counting multiplicities.

We present the main results of this section.

Theorem 3.1 (Both ends clamped). *Let $\mu_k^{DD}(\varepsilon)$ be the k-th eigenvalue of problem* (DD). *Then the following statements hold for each $k \in \mathbb{N}$.*

a) $\mu_k^{DD}(\varepsilon) = O(\varepsilon^2)$ as $\varepsilon \to 0$.

b) Moreover, we have the limit

$$\lim_{\varepsilon\to 0} \frac{\mu_k^{DD}(\varepsilon)}{\varepsilon^2} = \Lambda_k^{DD},$$

where Λ_k^{DD} *denotes the* k*-th eigenvalue of the* 4*th order ordinary differential operator*

$$\begin{cases} Y\dfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) & -K_1(\tau) \\ A_{21}(\tau) & A_{22}(\tau) & -K_2(\tau)\end{pmatrix}\begin{pmatrix}\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\end{pmatrix}\right) = \Lambda\, H(\tau)\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix} & (0 < \tau < l), \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(H(\tau)\dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\right) = \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(K_1(\tau)\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} + K_2(\tau)\dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\right) & (0 < \tau < l), \\ \eta_3(0) = \eta_i(0) = \dfrac{\mathrm{d}\eta_i}{\mathrm{d}\tau}(0) = 0 & (i = 1, 2), \\ \eta_3(l) = \eta_i(l) = \dfrac{\mathrm{d}\eta_i}{\mathrm{d}\tau}(l) = 0 & (i = 1, 2). \end{cases}$$

Theorem 3.2 (Only one end clamped). *Let* $\mu_k^{DN}(\varepsilon)$ *be the* k*-th eigenvalue of problem* (DN). *Then the following statements hold for each* $k \in \mathbb{N}$.

a) $\mu_k^{DN}(\varepsilon) = O(\varepsilon^2)$ *as* $\varepsilon \to 0$.

b) Moreover, we have the limit

$$\lim_{\varepsilon \to 0} \frac{\mu_k^{DN}(\varepsilon)}{\varepsilon^2} = \Lambda_k^{DN},$$

where Λ_k^{DN} *denotes the* k*-th eigenvalue of the* 4*th order ordinary differential operator*

$$\begin{cases} Y\dfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) & -K_1(\tau) \\ A_{21}(\tau) & A_{22}(\tau) & -K_2(\tau)\end{pmatrix}\begin{pmatrix}\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\end{pmatrix}\right) = \Lambda\, H(\tau)\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix} & (0 < \tau < l), \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(H(\tau)\dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\right) = \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(K_1(\tau)\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} + K_2(\tau)\dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\right) & (0 < \tau < l), \\ \eta_3(0) = \eta_i(0) = \dfrac{\mathrm{d}\eta_i}{\mathrm{d}\tau}(0) = 0 & (i = 1, 2), \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}(l) = \dfrac{\mathrm{d}^2\eta_i}{\mathrm{d}\tau^2}(l) = \dfrac{\mathrm{d}^3\eta_i}{\mathrm{d}\tau^3}(l) = 0 & (i = 1, 2). \end{cases}$$

Remark 3.3. There are several works which are closely related to our results, such as Irago-Viaño [15], Kerdid [19], Tambača [31]. These works assume that the cross-section is simply connected and constant, that is, the cross-section does not change along the rod. This translates to A_{ij} being constants for $1 \leq i, j \leq 2$. In this case, one can assume without loss of generality that $K_i = 0$ for $i = 1, 2$. The main novelty of Theorem 3.1 and Theorem 3.2 is studying the influence in the limit equation of A_{ij} and K_i for $1 \leq i, j \leq 2$ when they are functions of

y_3. In addition, we remove the assumption over the simply connectedness of the cross-section. Moreover, we also see that the boundary conditions of the limit equations are independent of the shape of the rod. Our method takes full advantage of variational technique by direct construction of test functions. Hence the proofs are straightforward and comprehensive.

Remark 3.4. Note that if the functions $K_i \equiv 0$ for $i = 1, 2$, then the ordinary differential equations in Theorem 3.1 and Theorem 3.2 get simpler. Using the corresponding boundary conditions, the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(H(\tau)\frac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(K_1(\tau)\frac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} + K_2(\tau)\frac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\right) \qquad (0 < \tau < l)$$

yields $\eta_3 \equiv 0$, and hence the ODE in Theorem 3.1 and Theorem 3.2 is simplified to

$$Y\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{12}(\tau) & A_{22}(\tau) \end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2} \end{pmatrix}\right) = \Lambda\, H(\tau)\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

with the respective boundary conditions.

The proofs of Theorem 3.1 and Theorem 3.2 are given in Sections 3.2 to 3.5.

3.2 Variational formulation

Recall that $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are used as the coordinates in Ω_ε and $\Omega_1 = F(S)$, respectively with the relation given in (3.1). We change the variables to transform Ω_ε into $F(S)$. We now compute the new expressions of stress and strain tensors in terms of the new variables in $F(S)$.

We begin to study the problem by variational methods. In order to consider the stress and strain tensors in terms of y, we introduce the scaling and change of variable

$$u_1 = \varepsilon U_1,\ u_2 = \varepsilon U_2,\ u_3 = \varepsilon^2 U_3.$$

We obtain the following expressions of $e_{ij}(u)$.

$$\begin{aligned} e_{ij}(u) &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{1}{\varepsilon}\frac{\partial u_i}{\partial y_j} + \frac{1}{\varepsilon}\frac{\partial u_j}{\partial y_i}\right) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_j} + \frac{\partial U_j}{\partial y_i}\right) \\ e_{i3}(u) &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_3} + \frac{1}{\varepsilon}\frac{\partial u_3}{\partial y_i}\right) = \varepsilon\frac{1}{2}\left(\frac{\partial U_i}{\partial y_3} + \frac{\partial U_3}{\partial y_i}\right) \qquad (1 \le i, j \le 2) \\ e_{33}(u) &= \frac{\partial u_3}{\partial x_3} = \frac{\partial u_3}{\partial y_3} = \varepsilon^2\frac{\partial U_3}{\partial y_3} \end{aligned}$$

We observe that after the change of variables we just introduced, we rewrote the strain tensor $e_{ij}(u)$ in terms of $U = (U_1, U_2, U_3)$. Therefore, for $1 \le i, j \le 2$ we can define

$$E_{ij}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_j} + \frac{\partial U_j}{\partial y_i}\right),\ E_{i3}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_3} + \frac{\partial U_3}{\partial y_i}\right),\quad E_{33}(U) = \frac{\partial U_3}{\partial y_3}.$$

Note also that since we have symmetry, i.e. $e_{ij}(u) = e_{ji}(u)$ $(1 \le i, j \le 3)$, we also define $E_{3i}(U) = E_{i3}(U)$ $(i = 1, 2)$. With this notation, we have the relation

$$e_{ij}(u) = E_{ij}(U), \quad e_{i3}(u) = \varepsilon E_{i3}(U) \quad (1 \le i, j \le 2), \quad e_{33}(u) = \varepsilon^2 E_{33}(U). \tag{3.2}$$

Furthermore, using (3.2), we proceed to write the divergence in terms of U.

$$\begin{aligned} \operatorname{div}(u) &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = e_{11}(u) + e_{22}(u) + e_{33}(u) \\ &= E_{11}(U) + E_{22}(U) + \varepsilon^2 E_{33}(U). \end{aligned} \tag{3.3}$$

Our next step is to rewrite the Rayleigh quotient and to describe the eigenvalues in terms of y. We distinguish between the (DD) case and the (DN) case.

3.2.1 (DD) case

We adapt Proposition 2.3 to our thin rod as follows. We define the set

$$\mathcal{W}_\varepsilon = \{\phi \in H^1(\Omega_\varepsilon, \mathbb{R}^3) \mid \phi = \mathbf{0} \text{ on } \Gamma_{1,\varepsilon}^{(-)} \cup \Gamma_{1,\varepsilon}^{(+)}\}.$$

For every $\phi \in \mathcal{W}_\varepsilon$ we set $B_\varepsilon[\phi, \phi]$ and $\mathcal{R}_\varepsilon$ analogously to Definition 2.2, that is

$$\begin{aligned} B_\varepsilon[\phi, \phi] &= \int_{\Omega_\varepsilon} \left(\lambda_1 (\operatorname{div} \phi)^2 + 2\lambda_2 \sum_{i,j=1}^{3} e_{ij}(\phi)^2 \right) \mathrm{d}x \\ \mathcal{R}_\varepsilon(\phi) &= \frac{B_\varepsilon[\phi, \phi]}{\|\phi\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^3)}}. \end{aligned}$$

We change $\phi = \phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))$ into $\Phi = \Phi(y) = (\Phi_1(y), \Phi_2(y), \Phi_3(y))$ by $\phi_i(x) = \varepsilon \Phi_i(y)$ $(i = 1, 2)$, $\phi_3(x) = \varepsilon^2 \Phi_3(y)$ according to the coordinate change $x = (\varepsilon y_1, \varepsilon y_2, y_3)$ described in (3.1). Define now the set

$$\mathcal{W}_1 = \{\Phi \in H^1(F(S), \mathbb{R}^3) \mid \Phi = \mathbf{0} \text{ on } \Gamma_{1,1}^{(-)} \cup \Gamma_{1,1}^{(+)}\}. \tag{3.4}$$

We want to describe the k-th eigenvalue $\mu_k^{DD}(\varepsilon)$ in terms of the new spaces and functions after the change of variables. Note that $\phi \in \mathcal{W}_\varepsilon$ if and only if $\Phi \in \mathcal{W}_1$. Thus, using this fact together with the relations (3.2) and (3.3), and substituting them into $B_\varepsilon[\phi, \phi]$ and $\mathcal{R}_\varepsilon(\phi)$, for every $\Phi \in \mathcal{W}_1$ we define

$$\begin{aligned} \widetilde{B}_\varepsilon[\Phi, \Phi] = \int_{F(S)} \Bigg\{ &\lambda_1 \left(E_{11}(\Phi) + E_{22}(\Phi) + \varepsilon^2 E_{33}(\Phi) \right)^2 \\ &+ 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\Phi)^2 + 2\varepsilon^2 \sum_{i=1}^{2} E_{i3}(\Phi)^2 + \varepsilon^4 E_{33}(\Phi)^2 \right) \Bigg\} \varepsilon^2 \mathrm{d}y, \end{aligned} \tag{3.5}$$

$$\widetilde{\mathcal{R}}_\varepsilon(\Phi) = \frac{\widetilde{B}_\varepsilon[\Phi, \Phi]}{\displaystyle\int_{F(S)} \left(\varepsilon^2 \Phi_1^2 + \varepsilon^2 \Phi_2^2 + \varepsilon^4 \Phi_3^2 \right) \varepsilon^2 \mathrm{d}y}. \tag{3.6}$$

Furthermore, for all $\Phi, \Psi \in \mathcal{W}_1$ we say that $\Phi \perp_\varepsilon \Psi$ if and only if

$$\int_{F(S)} (\Phi_1\Psi_1 + \Phi_2\Psi_2 + \varepsilon^2\Phi_3\Psi_3)\mathrm{d}y = 0.$$

Due to this definition, $\phi \perp \psi$ if and only if $\Phi \perp_\varepsilon \Psi$. For every $Z \in \mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$ we define the set

$$Z^{\perp_\varepsilon} = \{\Phi \in \mathcal{W}_1 \mid \Phi \perp_\varepsilon \Psi \text{ for all } \Psi \in Z\},$$

which is a closed subspace of $\mathcal{W}_1$.

Using the Max-Min principle (Proposition 2.3), after the change of variables, the characterization (2.4) of $\mu_k^{DD}(\varepsilon)$ can be rewritten as

$$\mu_k^{DD}(\varepsilon) = \sup_{Z \in \mathcal{H}_{k-1}(F(S),\mathbb{R}^3)} \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1 \backslash \{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\}. \tag{3.7}$$

3.2.2 (DN) case

For the case of the eigenvalues $\mu_k^{DN}(\varepsilon)$, we can similarly characterize $\mu_k^{DN}(\varepsilon)$ with

$$\mu_k^{DN}(\varepsilon) = \sup_{Z \in \mathcal{H}_{k-1}(F(S),\mathbb{R}^3)} \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1' \backslash \{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\}$$

where

$$\mathcal{W}_1' = \{\Phi \in H^1(F(S), \mathbb{R}^3) \mid \Phi = \mathbf{0} \text{ on } \Gamma_{1,1}^{(-)}\}. \tag{3.8}$$

3.3 Order of the eigenvalues

3.3.1 (DD) case

We show that $\mu_k^{DD}(\varepsilon) = O(\varepsilon^2)$ as $\varepsilon \to 0$. In order to do so, we will find an upper bound of the eigenvalue $\mu_k^{DD}(\varepsilon)$ using the Max-Min principle and (3.7).

Let us take test functions $\Upsilon^{(s)} = \Upsilon^{(s)}(y) = \left(\Upsilon_1^{(s)}(y), \Upsilon_2^{(s)}(y), \Upsilon_3^{(s)}(y)\right)$ $(s \in \mathbb{N})$ as follows:

$$\begin{aligned}
&\Upsilon_1^{(s)}(y) = \eta_1^{(s)}(y_3),\\
&\Upsilon_2^{(s)}(y) = \eta_2^{(s)}(y_3),\\
&\Upsilon_3^{(s)}(y) = \eta_3^{(s)}(y_3) - y_1\frac{\mathrm{d}\eta_1^{(s)}}{\mathrm{d}y_3} - y_2\frac{\mathrm{d}\eta_2^{(s)}}{\mathrm{d}y_3},
\end{aligned}$$

where $\left\{\eta_1^{(s)}, \eta_2^{(s)}, \eta_3^{(s)}\right\}_{s\in\mathbb{N}}$ is a linearly independent system satisfying

$$\begin{aligned}
&\eta_1^{(s)}, \eta_2^{(s)} \in H^2((0,l)), \eta_3^{(s)} \in H^1((0,l)),\\
&\eta_i^{(s)}(0) = \eta_i^{(s)}(l) = 0 \quad (i = 1,2,3),\\
&\frac{\mathrm{d}\eta_i^{(s)}}{\mathrm{d}z_3}(0) = \frac{\mathrm{d}\eta_i^{(s)}}{\mathrm{d}z_3}(l) = 0 \quad (i = 1,2).
\end{aligned}$$

Choose an arbitrary $Z \in \mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$ and let $\widetilde{Z} = L.H.\left[\Upsilon^{(1)}, \Upsilon^{(2)}, \dots, \Upsilon^{(k)}\right]$ denote the minimal linear space that contains the set $\{\Upsilon^{(1)}, \Upsilon^{(2)}, \dots, \Upsilon^{(k)}\}$. Since each $\Upsilon^{(s)} \in \mathcal{W}_1$ (for all $s \in \mathbb{N}$), we have that $\widetilde{Z} \subseteq \mathcal{W}_1$. Since $\dim Z < \dim \widetilde{Z}$, there exist a function $\Psi \in \widetilde{Z} \cap Z^{\perp_\varepsilon}$ and a vector $(c_1, \dots, c_k) = (c_1(\varepsilon), \dots, c_k(\varepsilon)) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ such that

$$\Psi = \sum_{s=1}^{k} c_s(\varepsilon) \Upsilon^{(s)}. \tag{3.9}$$

Note that since both $\widetilde{Z}$ and $Z^{\perp_\varepsilon}$ are subsets of $\mathcal{W}_1$, we have also that $\Psi \in \mathcal{W}_1$ and due the fact that $(c_1, \dots, c_k) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ we deduce that $\Psi \in \mathcal{W}_1 \backslash \{\mathbf{0}\}$, so we can apply $\widetilde{\mathcal{R}}_\varepsilon$ to Ψ (cf. (3.6)).

Using the definition of $\Upsilon^{(s)}$ we compute

$$E_{ij}(\Upsilon^{(s)}) = 0, \tag{3.10}$$

$$E_{i3}(\Upsilon^{(s)}) = \frac{1}{2}\left(\frac{\partial \Upsilon_i^{(s)}}{\partial y_3} + \frac{\partial \Upsilon_3^{(s)}}{\partial y_i}\right) = \frac{1}{2}\left(\frac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3} - \frac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3}\right) = 0 \qquad (1 \leq i, j \leq 2). \tag{3.11}$$

Now we want to calculate $\widetilde{\mathcal{R}}_\varepsilon(\Psi)$. Using the linearity of the operator E_{ij}, (3.10) and (3.11), we see that

$$E_{ij}(\Psi) = \sum_{s=1}^{k} c_s(\varepsilon) E_{ij}(\Upsilon^{(s)}) = 0, \quad E_{i3}(\Psi) = \sum_{s=1}^{k} c_s(\varepsilon) E_{i3}(\Upsilon^{(s)}) = 0 \quad (1 \leq i, j \leq 2). \tag{3.12}$$

Hence, using (3.12) and the definition in (3.5), we get

$$\begin{aligned}
\widetilde{B}_\varepsilon[\Psi, \Psi] &= \int_{F(S)} \Bigg\{ \lambda_1 \left(E_{11}(\Psi) + E_{22}(\Psi) + \varepsilon^2 E_{33}(\Psi)\right)^2 \\
&\qquad + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\Psi)^2 + 2\varepsilon^2 \sum_{i=1}^{2} E_{i3}(\Psi)^2 + \varepsilon^4 E_{33}(\Psi)^2 \right) \Bigg\} \varepsilon^2 \mathrm{d}y \\
&= \int_{F(S)} \left(\lambda_1 \left(\varepsilon^2 E_{33}(\Psi)\right)^2 + 2\lambda_2 \left(\varepsilon^4 E_{33}(\Psi)^2\right) \right) \varepsilon^2 \mathrm{d}y \\
&= \varepsilon^6 \int_{F(S)} (\lambda_1 + 2\lambda_2) E_{33}(\Psi)^2 \mathrm{d}y.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\widetilde{\mathcal{R}}_\varepsilon(\Psi) &= \frac{\varepsilon^6 \displaystyle\int_{F(S)} (\lambda_1 + 2\lambda_2) E_{33}(\Psi)^2 \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\varepsilon^2 \Psi_1^2 + \varepsilon^2 \Psi_2^2 + \varepsilon^4 \Psi_3^2\right) \varepsilon^2 \mathrm{d}y} = \frac{\varepsilon^6}{\varepsilon^4} \frac{\displaystyle\int_{F(S)} (\lambda_1 + 2\lambda_2) E_{33}(\Psi)^2 \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\Psi_1^2 + \Psi_2^2 + \varepsilon^2 \Psi_3^2\right) \mathrm{d}y} \\
&\leq \varepsilon^2 \frac{\displaystyle\int_{F(S)} (\lambda_1 + 2\lambda_2) E_{33}(\Psi)^2 \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\Psi_1^2 + \Psi_2^2\right) \mathrm{d}y}.
\end{aligned}$$

Now substitute the definition (3.9) into the previous equation to obtain

$$\widetilde{\mathcal{R}}_\varepsilon(\Psi) \leq \varepsilon^2 \frac{\displaystyle\int_{F(S)} (\lambda_1 + 2\lambda_2) \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) E_{33}(\Upsilon^{(p)}) E_{33}(\Upsilon^{(q)}) \mathrm{d}y}{\displaystyle\int_{F(S)} \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \left(\Upsilon_1^{(p)} \Upsilon_1^{(q)} + \Upsilon_2^{(p)} \Upsilon_2^{(q)} \right) \mathrm{d}y}. \tag{3.13}$$

Let us put

$$\gamma_{pq} = \int_{F(S)} E_{33}(\Upsilon^{(p)}) E_{33}(\Upsilon^{(q)}) \mathrm{d}y, \quad \widehat{\gamma}_{pq} = \int_{F(S)} \left(\Upsilon_1^{(p)} \Upsilon_1^{(q)} + \Upsilon_2^{(p)} \Upsilon_2^{(q)} \right) \mathrm{d}y.$$

Note that since we chose the system $\left\{ \eta_1^{(s)}, \eta_2^{(s)}, \eta_3^{(s)} \right\}_{s \in \mathbb{N}}$ to be linearly independent and by the symmetry $\gamma_{pq} = \gamma_{qp}$, $\widehat{\gamma}_{pq} = \widehat{\gamma}_{qp}$, we have that $(\gamma_{pq})_{1 \leq p,q \leq k}$ and $(\widehat{\gamma}_{pq})_{1 \leq p,q \leq k}$ are positive definite matrices. Therefore, all of its eigenvalues are positive. Let γ_* be the biggest eigenvalue of $(\gamma_{pq})_{1 \leq p,q \leq k}$ and $\widehat{\gamma}_*$, the smallest eigenvalue of $(\widehat{\gamma}_{pq})_{1 \leq p,q \leq k}$. With this notation, we have the bounds

$$\sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \gamma_{pq} \leq \gamma_* (c_1(\varepsilon)^2 + \cdots + c_k(\varepsilon)^2),$$
$$\sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widehat{\gamma}_{pq} \geq \widehat{\gamma}_* (c_1(\varepsilon)^2 + \cdots + c_k(\varepsilon)^2).$$

Therefore, (3.13) becomes

$$\begin{aligned} \widetilde{\mathcal{R}}_\varepsilon(\Psi) &\leq \varepsilon^2 \frac{\displaystyle(\lambda_1 + 2\lambda_2) \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \gamma_{pq}}{\displaystyle\sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widehat{\gamma}_{pq}} \leq \varepsilon^2 \frac{(\lambda_1 + 2\lambda_2) \gamma_* (c_1(\varepsilon)^2 + \cdots + c_k(\varepsilon)^2)}{\widehat{\gamma}_* (c_1(\varepsilon)^2 + \cdots + c_k(\varepsilon)^2)} \\ &= \varepsilon^2 \frac{(\lambda_1 + 2\lambda_2)\gamma_*}{\widehat{\gamma}_*}. \end{aligned}$$

Put $C = \frac{(\lambda_1 + 2\lambda_2)\gamma_*}{\widehat{\gamma}_*}$. We obtained that for a certain $\Psi \in \mathcal{W}_1$ there exists a positive constant C independent of ε and independent of the choice of Z such that $\widetilde{\mathcal{R}}_\varepsilon(\Psi) \leq \varepsilon^2 C$. Thus, taking the infimum, we have

$$\inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1 \backslash \{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\} \leq \widetilde{\mathcal{R}}_\varepsilon(\Psi) \leq \varepsilon^2 C.$$

Since Z was arbitrary and C does not depend on the choice of Z, we can take the supremum on both sides over $\mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$ to obtain

$$0 \leq \mu_k^{DD}(\varepsilon) = \sup_{Z \in \mathcal{H}_{k-1}(F(S), \mathbb{R}^3)} \left\{ \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1 \backslash \{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\} \right\} \leq \varepsilon^2 C.$$

Here we used the characterization (3.7) deduced in the previous section. Therefore we obtain

$$\mu_k^{DD}(\varepsilon) = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0$$

which proves Theorem 3.1-a).

3.3.2 (DN) case

For the case of the eigenvalues $\mu_k^{DN}(\varepsilon)$, note that due to the definition of the sets $\mathcal{W}_1$ and $\mathcal{W}_1'$ (see (3.4) and (3.8)), we see that $\mathcal{W}_1 \subseteq \mathcal{W}_1'$, therefore, the infimum over $\mathcal{W}_1'$ is not greater than over $\mathcal{W}_1$. Thus $0 \leq \mu_k^{DN}(\varepsilon) \leq \mu_k^{DD}(\varepsilon)$ and Theorem 3.2-a) also holds.

3.4 Weak formulation and deduction of the limit ODE

The weak formulation of the equation of (DD) and (DN) is

$$\int_{\Omega_\varepsilon} \left(\lambda_1 \operatorname{div} u \operatorname{div} v + 2\lambda_2 \sum_{i,j=1}^{3} e_{ij}(u) e_{ij}(v) \right) \mathrm{d}x = \mu \int_{\Omega_\varepsilon} \sum_{i=1}^{3} u_i v_i \mathrm{d}x.$$

Here μ is an eigenvalue, u is the corresponding eigenfunction and $v = (v_1, v_2, v_3) \in \mathcal{W}_\varepsilon$ (or $\mathcal{W}_\varepsilon'$) is a test function. By the change of the variable given in (3.1) together with $u_i = \varepsilon U_i$, $v_i = \varepsilon V_i$ $(i = 1, 2)$ and $u_3 = \varepsilon^2 U_3$, $v_3 = \varepsilon^2 V_3$, the previous weak formulation is rewritten in terms of y as follows.

$$\begin{aligned} &\int_{F(S)} \Bigg\{ \lambda_1 \left(E_{11}(U) + E_{22}(U) + \varepsilon^2 E_{33}(U) \right) \left(E_{11}(V) + E_{22}(V) + \varepsilon^2 E_{33}(V) \right) \\ &+ 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(U) E_{ij}(V) + 2\varepsilon^2 \sum_{i=1}^{2} E_{i3}(U) E_{i3}(V) + \varepsilon^4 E_{33}(U) E_{33}(V) \right) \Bigg\} \mathrm{d}y \\ &= \mu \int_{F(S)} \left(\varepsilon^2 U_1 V_1 + \varepsilon^2 U_2 V_2 + \varepsilon^4 U_3 V_3 \right) \mathrm{d}y. \end{aligned} \tag{3.14}$$

3.4.1 (DD) case

The proofs for the (DD) case and the (DN) case are very similar. Therefore, for simplicity, we will analyze the (DD) case and explain the main differences afterwards. From now on, to simplify the notation, we write $\mu_k(\varepsilon)$ instead of $\mu_k^{DD}(\varepsilon)$.

Let $\{\Phi_\varepsilon^{(k)}\}_{k=1}^{+\infty} = \{(\Phi_{1,\varepsilon}^{(k)}, \Phi_{2,\varepsilon}^{(k)}, \Phi_{3,\varepsilon}^{(k)})\}_{k=1}^{+\infty}$ be the corresponding eigenfunctions of the eigenvalues $\{\mu_k(\varepsilon)\}_{k=1}^{+\infty}$ and such that

$$\int_{F(S)} \left((\Phi_{1,\varepsilon}^{(k)})^2 + (\Phi_{2,\varepsilon}^{(k)})^2 + (\Phi_{3,\varepsilon}^{(k)})^2 \right) \mathrm{d}y = 1.$$

Now we put $U = V = \Phi_\varepsilon^{(k)}$ in (3.14) so that we get

$$\int_{F(S)} \Bigg\{ \lambda_1 \left(E_{11}(\Phi_\varepsilon^{(k)}) + E_{22}(\Phi_\varepsilon^{(k)}) + \varepsilon^2 E_{33}(\Phi_\varepsilon^{(k)}) \right)^2 + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\Phi_\varepsilon^{(k)})^2 + 2\varepsilon^2 \sum_{i=1}^{2} E_{i3}(\Phi_\varepsilon^{(k)})^2 + \varepsilon^4 E_{33}(\Phi_\varepsilon^{(k)})^2 \right) \Bigg\} \mathrm{d}y = \mu_k(\varepsilon) \int_{F(S)} \left(\varepsilon^2 (\Phi_{1,\varepsilon}^{(k)})^2 + \varepsilon^2 (\Phi_{2,\varepsilon}^{(k)})^2 + \varepsilon^4 (\Phi_{3,\varepsilon}^{(k)})^2 \right) \mathrm{d}y. \tag{3.15}$$

Note that by the choice of the $\{\Phi_\varepsilon^{(k)}\}_{k=1}^{+\infty}$ and by Theorem 3.1-a), i.e. $\mu_k(\varepsilon) = O(\varepsilon^2)$ as $\varepsilon \to 0$, we see that the right-hand side of (3.15) is $O(\varepsilon^4)$ as $\varepsilon \to 0$. Therefore, the left-hand side must also satisfy the same condition and we conclude that

$$E_{ij}(\Phi_\varepsilon^{(k)}) = O(\varepsilon^2), \quad E_{i3}(\Phi_\varepsilon^{(k)}) = O(\varepsilon), \quad E_{33}(\Phi_\varepsilon^{(k)}) = O(1) \tag{3.16}$$

in the $L^2(F(S), \mathbb{R}^3)$ sense for $1 \le i, j \le 2$. Combining this fact with Korn's inequality (Proposition 2.1), we can see that $\Phi_\varepsilon^{(k)}$ is bounded in $H^1(F(S), \mathbb{R}^3)$. Let $\{\varepsilon_p\}_{p=1}^{+\infty}$ be any positive sequence such that $\varepsilon_p \to 0$ as $p \to +\infty$. Then, using the previous facts, there exists a subsequence $\{\varepsilon_{p(q)}\}_{q=1}^{+\infty}$ such that

$$\lim_{q \to +\infty} \Phi_{\varepsilon_{p(q)}}^{(k)} = \Phi^{(k)} \text{ weakly in } H^1(F(S), \mathbb{R}^3).$$

Moreover, from Rellich's theorem, we have

$$\lim_{q \to +\infty} \Phi_{\varepsilon_{p(q)}}^{(k)} = \Phi^{(k)} \text{ in } L^2(F(S), \mathbb{R}^3) \text{ with } \|\Phi^{(k)}\|_{L^2(F(S),\mathbb{R}^3)} = 1,$$

so we have non-trivial limit functions $\{\Phi^{(k)}\}_{k=1}^{+\infty} = \{(\Phi_1^{(k)}, \Phi_2^{(k)}, \Phi_3^{(k)})\}_{k=1}^{+\infty}$, which form an orthonormal basis of $L^2(F(S), \mathbb{R}^3)$. For $1 \le i, j \le 2$, we now set

$$\kappa_{ij}^\varepsilon = \frac{1}{\varepsilon^2} E_{ij}(\Phi_\varepsilon^{(k)}), \quad \kappa_{i3}^\varepsilon = \frac{1}{\varepsilon} E_{i3}(\Phi_\varepsilon^{(k)}), \quad \kappa_{33}^\varepsilon = E_{33}(\Phi_\varepsilon^{(k)}).$$

Furthermore, we define $\kappa_{3i}^\varepsilon = \kappa_{i3}^\varepsilon$. We remark that for $1 \le i, j \le 3$, each κ_{ij}^ε depends also on k. Due to (3.16) we have that $\kappa_{ij}^\varepsilon = O(1)$ $(1 \le i, j, \le 3)$ as $\varepsilon \to 0$ in the $L^2(F(S), \mathbb{R}^3)$ sense, that is, κ_{ij}^ε are bounded in $L^2(F(S), \mathbb{R}^3)$. Therefore, there exists a further subsequence $\{\varepsilon_{p(q(n))}\}_{n=1}^{+\infty}$ such that

$$\lim_{n \to +\infty} \kappa_{ij}^{\varepsilon_{p(q(n))}} = \kappa_{ij} \text{ weakly in } L^2(F(S), \mathbb{R}^3) \quad (1 \le i, j \le 3).$$

Note again, that each κ_{ij} still depends on k. Furthermore, in virtue of Theorem 3.1.a) there exists a constant c such that $\frac{\mu_k(\varepsilon)}{\varepsilon^2} \le c$ and we conclude that there exist an even further subsequence $\{\zeta_r\}_{r=1}^{+\infty} \subseteq \{\varepsilon_{p(q(n))}\}_{n=1}^{+\infty}$ and a constant $\widetilde{\Lambda}_k$ that satisfy

$$\lim_{r \to +\infty} \frac{\mu_k(\zeta_r)}{\zeta_r^2} = \widetilde{\Lambda}_k. \tag{3.17}$$

This proves the existence of the limit for a subsequence of $\{\varepsilon_p\}_{p=1}^{+\infty}$.

We characterize $\{\widetilde{\Lambda}_k\}_{k=1}^{+\infty}$. We take particular test functions and deduce several conditions for the limit functions $\Phi^{(k)}$ and κ_{ij}. We put $U = \Phi^{(k)}_{\zeta_r}$, $\varepsilon = \zeta_r$, substitute them into (3.14) and after dividing both sides by ζ_r^2 we obtain

$$\begin{aligned}\int_{F(S)} \Bigg\{ &\lambda_1(\kappa_{11}^{\zeta_r} + \kappa_{22}^{\zeta_r} + \kappa_{33}^{\zeta_r}) \left(E_{11}(V) + E_{22}(V) + \zeta_r^2 E_{33}(V)\right) \\ &+ 2\lambda_2 \left(\sum_{i,j=1}^{2} \kappa_{ij}^{\zeta_r} E_{ij}(V) + 2\zeta_r \sum_{i=1}^{2} \kappa_{i3}^{\zeta_r} E_{i3}(V) + \zeta_r^2 \kappa_{33}^{\zeta_r} E_{33}(V) \right) \Bigg\} \mathrm{d}y \\ &= \mu_k(\zeta_r) \int_{F(S)} \left(\Phi^{(k)}_{1,\zeta_r} V_1 + \Phi^{(k)}_{2,\zeta_r} V_2 + \zeta_r^2 \Phi^{(k)}_{3,\zeta_r} V_3 \right) \mathrm{d}y \end{aligned} \tag{3.18}$$

for any test function $V = (V_1, V_2, V_3) \in \mathcal{W}_1$. By letting $r \to +\infty$ in (3.18), we get

$$\int_{F(S)} \left(\lambda_1(\kappa_{11} + \kappa_{22} + \kappa_{33}) \left(E_{11}(V) + E_{22}(V)\right) + 2\lambda_2 \sum_{i,j=1}^{2} \kappa_{ij} E_{ij}(V) \right) \mathrm{d}y = 0. \tag{3.19}$$

Next we choose $V_2 = 0$. We see that $E_{22}(V) = 0$, and since $\kappa_{12} = \kappa_{21}$, (3.19) becomes

$$\int_{F(S)} \left\{ \lambda_1 \sum_{p=1}^{3} \kappa_{pp} \frac{\partial V_1}{\partial y_1} + 2\lambda_2 \left(\kappa_{11} \frac{\partial V_1}{\partial y_1} + \kappa_{12} \frac{\partial V_1}{\partial y_2} \right) \right\} \mathrm{d}y = 0$$

$$\int_{F(S)} \left\{ \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) \frac{\partial V_1}{\partial y_1} + 2\lambda_2 \kappa_{12} \frac{\partial V_1}{\partial y_2} \right\} \mathrm{d}y = 0. \tag{3.20}$$

By integration by parts in (3.20) we obtain

$$-\int_{F(S)} \left\{ \frac{\partial}{\partial y_1} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) V_1 + \frac{\partial}{\partial y_2} \left(2\lambda_2 \kappa_{12}\right) V_1 \right\} \mathrm{d}y = 0$$

$$-\int_{F(S)} \left\{ \frac{\partial}{\partial y_1} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) + \frac{\partial}{\partial y_2} \left(2\lambda_2 \kappa_{12}\right) \right\} V_1 \mathrm{d}y = 0.$$

In fact, due to the arbitrariness of V_1 we have

$$\frac{\partial}{\partial y_1} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) + \frac{\partial}{\partial y_2} \left(2\lambda_2 \kappa_{12}\right) = 0 \tag{3.21}$$

in the distribution sense. Similarly, letting $V_1 = 0$ we also deduce that

$$\int_{F(S)} \left\{ (2\lambda_2\kappa_{12}) \frac{\partial V_2}{\partial y_1} + \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2\kappa_{22} \right) \frac{\partial V_2}{\partial y_2} \right\} \mathrm{d}y = 0, \tag{3.22}$$

$$\frac{\partial}{\partial y_1}(2\lambda_2\kappa_{12}) + \frac{\partial}{\partial y_2}\left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2\kappa_{22} \right) = 0. \tag{3.23}$$

We write

$$\begin{aligned} \alpha_1 &= \lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2\kappa_{11}, & \alpha_2 &= 2\lambda_2\kappa_{12}, \\ \beta_1 &= 2\lambda_2\kappa_{12}, & \beta_2 &= \lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2\kappa_{22}, \end{aligned} \tag{3.24}$$

so that (3.20), (3.21), (3.22) and (3.23) become

$$\int_{F(S)} \left(\alpha_1 \frac{\partial V_1}{\partial y_1} + \alpha_2 \frac{\partial V_1}{\partial y_2} \right) \mathrm{d}y = 0, \quad \int_{F(S)} \left(\beta_1 \frac{\partial V_2}{\partial y_1} + \beta_2 \frac{\partial V_2}{\partial y_2} \right) \mathrm{d}y = 0, \tag{3.25}$$

$$\frac{\partial \alpha_1}{\partial y_1} = -\frac{\partial \alpha_2}{\partial y_2}, \quad \frac{\partial \beta_1}{\partial y_1} = -\frac{\partial \beta_2}{\partial y_2}. \tag{3.26}$$

Note however that the functions V_1 and V_2 in (3.25) are arbitrary test functions. Therefore, for every $\phi \in H^1(F(S))$ with $\phi = 0$ on $\Gamma_{1,1}^{(+)} \cup \Gamma_{1,1}^{(-)}$, we have

$$\int_{F(S)} \left(\alpha_1 \frac{\partial \phi}{\partial y_1} + \alpha_2 \frac{\partial \phi}{\partial y_2} \right) \mathrm{d}y = 0, \quad \int_{F(S)} \left(\beta_1 \frac{\partial \phi}{\partial y_1} + \beta_2 \frac{\partial \phi}{\partial y_2} \right) \mathrm{d}y = 0. \tag{3.27}$$

We use the following lemma.

Lemma 3.5. *Assume that properties (3.26) and (3.27) are satisfied. Then the following statements hold.*

a) There exist functions $h_1, h_2 \in L^2(F(S))$ such that $\frac{\partial h_p}{\partial y_j} \in L^2(F(S))$ for $1 \le j, p \le 2$ and

$$\frac{\partial h_1}{\partial y_1} = -\alpha_2, \; \frac{\partial h_1}{\partial y_2} = \alpha_1, \; \frac{\partial h_2}{\partial y_1} = -\beta_2, \; \frac{\partial h_2}{\partial y_2} = \beta_1. \tag{3.28}$$

Moreover, h_1, h_2 take values on the boundary and $h_p|_{\Gamma_{2,1}} \in L^2(\Gamma_{2,1})$ for $p = 1, 2$.

b) Write $\Gamma_{2,1} = g_1 \cup \cdots \cup g_m$ where each g_i is the i-th connected component of $\Gamma_{2,1}$ ($m \in \mathbb{N}$, $i = 1, \ldots, m$). Then, for $i = 1, \ldots, m$ the functions $h_1|_{g_i}, h_2|_{g_i}$ do not depend on (y_1, y_2) along g_i.

For the proof of this lemma see Section A Appendix. Let us use the functions h_1 and h_2 given by this lemma. From (3.24) and (3.28), we note

$$\frac{\partial h_1}{\partial y_1} + \frac{\partial h_2}{\partial y_2} = \beta_1 - \alpha_2 = 0, \tag{3.29}$$

$$\frac{\partial h_1}{\partial y_2} - \frac{\partial h_2}{\partial y_1} = \alpha_1 + \beta_2 = 2\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2(\kappa_{11} + \kappa_{22}). \tag{3.30}$$

For brevity, let us write

$$Q = \frac{\partial h_1}{\partial y_2} - \frac{\partial h_2}{\partial y_1}.$$

We rewrite the equality (3.30) with Q and we calculate

$$Q = 2\left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + \lambda_2(\kappa_{11} + \kappa_{22})\right) = 2\left((\lambda_1 + \lambda_2) \sum_{p=1}^{3} \kappa_{pp} - \lambda_2 \kappa_{33}\right)$$

$$\lambda_1 Q = 2\left(\lambda_1(\lambda_1 + \lambda_2) \sum_{p=1}^{3} \kappa_{pp} - \lambda_1\lambda_2\kappa_{33}\right)$$

$$\lambda_1 Q + 2\lambda_2(3\lambda_1 + 2\lambda_2)\kappa_{33} = 2(\lambda_1 + \lambda_2)\left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2\kappa_{33}\right).$$

Eventually, we obtain

$$\frac{\lambda_1}{2(\lambda_1 + \lambda_2)} Q + \frac{\lambda_2(3\lambda_1 + 2\lambda_2)}{\lambda_1 + \lambda_2}\kappa_{33} = \lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2\kappa_{33}. \tag{3.31}$$

This computation will be useful afterwards.

We go back to (3.18) with some particular test functions. Take functions $\rho_1 = \rho_1(y_3)$, $\rho_2 = \rho_2(y_3), \rho_3 = \rho_3(y_3)$ such that

$$\begin{aligned}
&\rho_1, \rho_2 \in H^2((0,l)), \quad \rho_3 \in H^1((0,l)), \\
&\rho_i(0) = \rho_i(l) = 0 && (i = 1,2,3), \\
&\frac{\mathrm{d}\rho_i}{\mathrm{d}y_3}(0) = \frac{\mathrm{d}\rho_i}{\mathrm{d}y_3}(l) = 0 && (i = 1,2),
\end{aligned}$$

and put a test function $V = (V_1, V_2, V_3) \in \mathcal{W}_1$ by

$$\begin{aligned}
V_1(y) &= \rho_1(y_3), \\
V_2(y) &= \rho_2(y_3), \\
V_3(y) &= \rho_3(y_3) - y_1 \frac{\mathrm{d}\rho_1}{\mathrm{d}y_3} - y_2 \frac{\mathrm{d}\rho_2}{\mathrm{d}y_3}.
\end{aligned}$$

For this test function we note that $E_{ij}(V) = 0$, $E_{i3}(V) = 0$ for $1 \leq i, j \leq 2$ (see the computations in (3.10) and (3.11)). Substituting the new test function into (3.18), dividing both sides by ζ_r^2, letting $r \to +\infty$ and using (3.17) we deduce

$$\int_{F(S)} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{33} \right) E_{33}(V) \mathrm{d}y = \widetilde{\Lambda}_k \int_{F(S)} \left(\Phi_1^{(k)} \rho_1 + \Phi_2^{(k)} \rho_2 \right) \mathrm{d}y. \tag{3.32}$$

Now we begin the next step to characterize the behavior of the eigenvalue limit. We substitute (3.31) into (3.32) to get

$$\int_{F(S)} \left(\frac{\lambda_1}{2(\lambda_1 + \lambda_2)} Q + \frac{\lambda_2(3\lambda_1 + 2\lambda_2)}{\lambda_1 + \lambda_2} \kappa_{33} \right) E_{33}(V) \mathrm{d}y = \widetilde{\Lambda}_k \int_{F(S)} \left(\Phi_1^{(k)} \rho_1 + \Phi_2^{(k)} \rho_2 \right) \mathrm{d}y. \tag{3.33}$$

Using the above test function V, we have

$$E_{33}(V) = \frac{\partial V_3}{\partial y_3} = \frac{\mathrm{d}\rho_3}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2 \rho_1}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2 \rho_2}{\mathrm{d}y_3^2}. \tag{3.34}$$

Define $\mathrm{d}y' = \mathrm{d}y_1 \mathrm{d}y_2$ and let $\widehat{\Omega}(y_3)$ be the the cross-section of $F(S)$ at $y_3 \in [0, l]$. We now look into equation (3.33) and we rewrite

$$\begin{aligned}
\int_{F(S)} Q E_{33}(V) \mathrm{d}y &= \int_0^l \int_{\widehat{\Omega}(y_3)} Q \left(\frac{\mathrm{d}\rho_3}{\mathrm{d}z_3} - y_1 \frac{\mathrm{d}^2 \rho_1}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2 \rho_2}{\mathrm{d}y_3^2} \right) \mathrm{d}y' \mathrm{d}y_3 \\
&= \int_0^l \left(\int_{\widehat{\Omega}(y_3)} Q \frac{\mathrm{d}\rho_3}{\mathrm{d}y_3} \mathrm{d}y' + \int_{\widehat{\Omega}(y_3)} Q y_1 \frac{\mathrm{d}^2 \rho_1}{\mathrm{d}y_3^2} \mathrm{d}y' + \int_{\widehat{\Omega}(y_3)} Q y_2 \frac{\mathrm{d}^2 \rho_2}{\mathrm{d}z_3^2} \mathrm{d}y' \right) \mathrm{d}y_3 \\
&= \int_0^l \left(\frac{\mathrm{d}\rho_3}{\mathrm{d}y_3} \int_{\widehat{\Omega}(y_3)} Q \mathrm{d}y' + \frac{\mathrm{d}^2 \rho_1}{\mathrm{d}y_3^2} \int_{\widehat{\Omega}(y_3)} Q y_1 \mathrm{d}y' + \frac{\mathrm{d}^2 \rho_2}{\mathrm{d}y_3^2} \int_{\widehat{\Omega}(y_3)} Q y_2 \mathrm{d}y' \right) \mathrm{d}y_3. \qquad (3.35)
\end{aligned}$$

We use the following lemma (see the proof in Section A Appendix).

Lemma 3.6. *With the same notation as above, for every $y_3 \in [0, l]$ it holds that*

$$\int_{\widehat{\Omega}(y_3)} Q \mathrm{d}y' = 0, \qquad \int_{\widehat{\Omega}(y_3)} Q y_i \mathrm{d}y' = 0 \quad (i = 1, 2).$$

Using this lemma, we see that (3.35) becomes

$$\int_{F(S)} Q E_{33}(V) \mathrm{d}y = 0.$$

As a consequence, (3.33) simplifies to

$$\int_{F(S)} \frac{\lambda_2(3\lambda_1 + 2\lambda_2)}{\lambda_1 + \lambda_2} \kappa_{33} E_{33}(V) \mathrm{d}y = \widetilde{\Lambda}_k \int_{F(S)} \left(\Phi_1^{(k)} \rho_1 + \Phi_2^{(k)} \rho_2 \right) \mathrm{d}y. \tag{3.36}$$

We now proceed to compute κ_{33}. Recall that $\kappa_{33} = E_{33}(\Phi^{(k)})$. We know by (3.16) that $E_{ij}(\Phi^{(k)}) = E_{i3}(\Phi^{(k)}) = 0$ for $1 \leq i,j \leq 2$. This will help us find a more explicit form of the functions $\Phi^{(k)}$. In order to solve the partial differential equation in the weak sense for $\Phi^{(k)}$, we first write

$$E_{ij}(\Phi^{(k)}) = \frac{1}{2}\left(\frac{\partial \Phi_i^{(k)}}{\partial y_j} + \frac{\partial \Phi_j^{(k)}}{\partial y_i}\right), \qquad E_{i3}(\Phi^{(k)}) = \frac{1}{2}\left(\frac{\partial \Phi_i^{(k)}}{\partial y_3} + \frac{\partial \Phi_3^{(k)}}{\partial y_i}\right).$$

For $i = 1,2$, from $E_{ii}(\Phi^{(k)}) = 0$ we have $\frac{\partial \Phi_i^{(k)}}{\partial y_i} = 0$ and therefore we deduce that $\Phi_i^{(k)}$ does not depend on y_i. By $E_{12}(\Phi^{(k)}) = 0$, we see

$$\frac{\partial \Phi_1^{(k)}}{\partial y_2} + \frac{\partial \Phi_2^{(k)}}{\partial y_1} = 0 \quad \text{and thus} \quad \frac{\partial \Phi_1^{(k)}}{\partial y_2} = -\frac{\partial \Phi_2^{(k)}}{\partial y_1} \quad \text{in } F(S).$$

Note that since $\Phi_i^{(k)}$ does not depend on y_i, $\frac{\partial \Phi_1^{(k)}}{\partial y_2}$ does not depend on y_1 and $\frac{\partial \Phi_2^{(k)}}{\partial y_1}$ does not depend on y_2. Due to the relation we found in the previous equation, we conclude that there exists a function $\xi^{(k)}(y_3) \in L^2((0,l))$ depending only on y_3 such that

$$\frac{\partial \Phi_1^{(k)}}{\partial y_2} = -\frac{\partial \Phi_2^{(k)}}{\partial y_1} = -\xi^{(k)}(y_3).$$

For further details see Section A Proposition A.1. Hence, there exist functions $\eta_1^{(k)}(y_3)$, $\eta_2^{(k)}(z_3) \in H^1((0,l))$ that depend only on y_3 such that

$$\Phi_1^{(k)}(y) = -\xi^{(k)}(y_3)y_2 + \eta_1^{(k)}(y_3), \quad \Phi_2^{(k)}(y) = \xi^{(k)}(y_3)y_1 + \eta_2^{(k)}(y_3) \quad (i = 1,2).$$

Applying the boundary conditions, we see $\xi^{(k)}(0) = 0$. Moreover, due to $E_{i3}(\Phi^{(k)}) = 0$,

$$\frac{\partial \Phi_3^{(k)}}{\partial y_1} = -\frac{\partial \Phi_1^{(k)}}{\partial y_3} = y_2\frac{\mathrm{d}\xi^{(k)}}{\mathrm{d}y_3} - \frac{\mathrm{d}\eta_1^{(k)}}{\mathrm{d}y_3}, \qquad \frac{\partial \Phi_3^{(k)}}{\partial y_2} = -\frac{\partial \Phi_2^{(k)}}{\partial y_3} = -y_1\frac{\mathrm{d}\xi^{(k)}}{\mathrm{d}y_3} - \frac{\mathrm{d}\eta_2^{(k)}}{\mathrm{d}y_3}.$$

Differentiating the first equation with respect to y_2 and the second equation with respect to y_1 and comparing the two results, we see that $\frac{\mathrm{d}\xi^{(k)}}{\mathrm{d}y_3} = 0$, and therefore, $\xi^{(k)}$ is a constant. However, by the boundary condition we know that $\xi^{(k)}(0) = 0$, thus we see that in fact $\xi^{(k)} \equiv 0$. Hence,

$$\frac{\partial \Phi_3^{(k)}}{\partial y_1} = -\frac{\mathrm{d}\eta_1^{(k)}}{\mathrm{d}y_3}, \qquad \frac{\partial \Phi_3^{(k)}}{\partial y_2} = -\frac{\mathrm{d}\eta_2^{(k)}}{\mathrm{d}y_3}.$$

Since $\frac{\partial}{\partial y_2}\left(-\frac{\mathrm{d}\eta_1^{(k)}}{\mathrm{d}y_3}\right) = \frac{\partial}{\partial y_1}\left(-\frac{\mathrm{d}\eta_2^{(k)}}{\mathrm{d}y_3}\right) = 0$ we can solve for $\Phi_3^{(k)}$, and we get the solution

$$\begin{aligned}
\Phi_1^{(k)}(y) &= \eta_1^{(k)}(y_3),\\
\Phi_2^{(k)}(y) &= \eta_2^{(k)}(y_3),\\
\Phi_3^{(k)}(y) &= \eta_3^{(k)}(y_3) - y_1\frac{\mathrm{d}\eta_1^{(k)}}{\mathrm{d}y_3} - y_2\frac{\mathrm{d}\eta_2^{(k)}}{\mathrm{d}y_3}.
\end{aligned} \tag{3.37}$$

Now we are able to compute

$$\kappa_{33} = E_{33}(\Phi^{(k)}) = \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}y_3^2}. \tag{3.38}$$

For commodity, let us put $Y = \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}$. We substitute (3.34) and (3.38) into (3.36), so it becomes

$$\begin{aligned} &\int_{F(S)} Y \left(\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}y_3^2} \right) \left(\frac{\mathrm{d}\rho_3}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2\rho_1}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2\rho_2}{\mathrm{d}y_3^2} \right) \mathrm{d}y \\ &\qquad = \widetilde{\Lambda}_k \int_{F(S)} \left(\eta_1^{(k)}\rho_1 + \eta_2^{(k)}\rho_2 \right) \mathrm{d}y. \end{aligned} \tag{3.39}$$

Let us now analyze the integrals of (3.39). For $1 \le i,j \le 2$ let us define the following functions.

$$\begin{aligned} H = H(y_3) = \int_{\widehat{\Omega}(y_3)} 1\mathrm{d}y', \qquad K_i = K_i(y_3) = \int_{\widehat{\Omega}(y_3)} y_i \mathrm{d}y', \\ A_{ij} = A_{ij}(y_3) = \int_{\widehat{\Omega}(y_3)} y_i y_j \mathrm{d}y' \quad (y_3 \in [0,l]) \,. \end{aligned} \tag{3.40}$$

With this notation and using integration by parts accordingly, we have

$$\begin{aligned} &\int_{F(S)} \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3}\frac{\mathrm{d}\rho_3}{\mathrm{d}y_3}\mathrm{d}y = \int_0^l H \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\frac{\mathrm{d}\rho_3}{\mathrm{d}z_3}\mathrm{d}z_3 = -\int_0^l \frac{\mathrm{d}}{\mathrm{d}z_3}\left(H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} \right)\rho_3 \mathrm{d}z_3, \\ &\int_{F(S)} y_i \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3}\frac{\mathrm{d}^2\rho_i}{\mathrm{d}y_3^2}\mathrm{d}y = \int_0^l K_i \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\frac{\mathrm{d}^2\rho_i}{\mathrm{d}z_3^2}\mathrm{d}z_3 = \int_0^l \frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(K_i\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} \right)\rho_i \mathrm{d}z_3, \\ &\int_{F(S)} y_i \frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}y_3^2}\frac{\mathrm{d}\rho_3}{\mathrm{d}y_3}\mathrm{d}y = \int_0^l K_i \frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}\rho_3}{\mathrm{d}z_3}\mathrm{d}z_3 = -\int_0^l \frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_i\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2} \right)\rho_3 \mathrm{d}z_3, \\ &\int_{F(S)} y_i y_j \frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}y_3^2}\frac{\mathrm{d}^2\rho_j}{\mathrm{d}y_3^2}\mathrm{d}y = \int_0^l A_{ij} \frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}^2\rho_j}{\mathrm{d}z_3^2}\mathrm{d}z_3 = \int_0^l \frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{ij}\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2} \right)\rho_j \mathrm{d}z_3, \\ &\widetilde{\Lambda}_k \int_{F(S)} \left(\eta_1^{(k)}\rho_1 + \eta_2^{(k)}\rho_2 \right)\mathrm{d}y = \widetilde{\Lambda}_k \int_0^l H\left(\eta_1^{(k)}\rho_1 + \eta_2^{(k)}\rho_2 \right)\mathrm{d}z_3. \end{aligned}$$

Plugging this into (3.39) and rearranging it we obtain

$$\begin{aligned} Y\int_0^l \Bigg\{ &\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{11}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{12}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - K_1\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} \right)\rho_1 \\ &+\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{12}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{22}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - K_2\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} \right)\rho_2 \\ &+\frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + K_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} \right)\rho_3 \Bigg\}\mathrm{d}z_3 = \widetilde{\Lambda}_k \int_0^l H\left(\eta_1^{(k)}\rho_1 + \eta_2^{(k)}\rho_2 \right)\mathrm{d}z_3. \end{aligned} \tag{3.41}$$

Choosing $\rho_1, \rho_2 = 0$, we see that

$$Y\int_0^l \frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + K_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_3\mathrm{d}z_3 = 0. \tag{3.42}$$

Note now that (3.42) holds for all $\rho_3 \in H_0^1((0,l))$, so we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + K_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right) = 0,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}z_3}\left(H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right) = \frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + K_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}\right). \tag{3.43}$$

Plugging (3.42) into (3.41), we get

$$\begin{aligned} Y\int_0^l \Bigg\{&\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{11}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{12}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - K_1\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_1 \\ &+\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{12}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{22}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - K_2\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_2\Bigg\}\mathrm{d}z_3 = \widetilde{\Lambda}_k\int_0^l H\left(\eta_1^{(k)}\rho_1 + \eta_2^{(k)}\rho_2\right)\mathrm{d}z_3. \end{aligned} \tag{3.44}$$

Now taking $\rho_2 = 0$ in (3.44), we see

$$Y\int_0^l \frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{11}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{12}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - K_1\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_1\mathrm{d}z_3 = \widetilde{\Lambda}_k\int_0^l H\eta_1^{(k)}\rho_1\mathrm{d}z_3.$$

Since ρ_1 is arbitrary, we conclude that

$$Y\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{11}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{12}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - K_1\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right) = \widetilde{\Lambda}_k H\eta_1^{(k)}. \tag{3.45}$$

Similarly, with the same argument but taking $\rho_1 = 0$, we get

$$Y\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{21}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{22}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - K_2\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right) = \widetilde{\Lambda}_k H\eta_2^{(k)}. \tag{3.46}$$

Combining the equations (3.43), (3.45) and (3.46) we obtain the system of differential equations

$$\begin{cases} Y\dfrac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(\begin{pmatrix} A_{11} & A_{12} & -K_1 \\ A_{21} & A_{22} & -K_2 \end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}z_3^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}z_3^2} \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}z_3} \end{pmatrix}\right) = \widetilde{\Lambda}\, H\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} & (0 < z_3 < l), \\ \dfrac{\mathrm{d}}{\mathrm{d}z_3}\left(H\dfrac{\mathrm{d}\eta_3}{\mathrm{d}z_3}\right) = \dfrac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}z_3^2} + K_2\dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}z_3^2}\right) & (0 < z_3 < l). \end{cases} \tag{3.47}$$

We now discuss the boundary conditions of the functions $\eta_i^{(k)}$ for $i = 1, 2, 3$ for the (DD) case, that is, the case with both ends clamped. Then, we know that $\Phi^{(k)}(y_1, y_2, 0) = 0$ and $\Phi^{(k)}(y_1, y_2, l) = 0$. From (3.37) we can deduce that

$$\begin{cases} \eta_3^{(k)}(0) = \eta_i^{(k)}(0) = \dfrac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3}(0) = 0 \\ \eta_3^{(k)}(l) = \eta_i^{(k)}(l) = \dfrac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3}(l) = 0 \end{cases} \qquad (i = 1, 2). \tag{dd}$$

Let $\{\Lambda_{k^*}\}_{k^*=1}^{+\infty}$ be the set of eigenvalues of problem (3.47) with (dd) boundary conditions. Then, we have proved that $\widetilde{\Lambda}_k \in \{\Lambda_{k^*}\}_{k^*=1}^{+\infty}$, and more generally $\{\widetilde{\Lambda}_k\}_{k=1}^{+\infty} \subseteq \{\Lambda_{k^*}\}_{k^*=1}^{+\infty}$. Thus, we can assure that

$$\widetilde{\Lambda}_k \geq \Lambda_k \qquad (k \in \mathbb{N}). \tag{3.48}$$

It still remains to prove that $\widetilde{\Lambda}_k \leq \Lambda_k$ for $k \in \mathbb{N}$ (cf. Section 3.5).

3.4.2 (DN) case

We now cover the case of $\mu_k^{DN}(\varepsilon)$. The proof is pretty similar to the case of $\mu_k^{DD}(\varepsilon)$ with some minor changes, specially on the boundary.

The function space $\mathcal{W}_1$ changes to

$$\mathcal{W}_1' = \{\phi \in H^1(F(S), \mathbb{R}^3) \mid \phi = \mathbf{0} \text{ on } \Gamma_{1,1}^{(-)}\},$$

and the test functions chosen during the proof, now only vanish on $\Gamma_{1,1}^{(-)}$. In particular, $\rho_i(0) = 0$ for $i = 1, 2, 3$ and $\frac{\mathrm{d}\rho_i}{\mathrm{d}z_3}(0) = 0$ for $i = 1, 2$. Let us now discuss the boundary conditions of the functions $\eta_i^{(k)}$ for $i = 1, 2, 3$. With the same argument as before, on the clamped end, we easily see that $\eta_i^{(k)}(0) = 0$ for $i = 1, 2, 3$ and $\frac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3}(0) = 0$ for $i = 1, 2$. We go back to (3.39) and put $\rho_2 = 0$ and $\rho_3 = 0$, to obtain

$$Y \int_{F(S)} \left(-\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3} + y_1 \frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}y_3^2} + y_2 \frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}y_3^2} \right) y_1 \frac{\mathrm{d}^2\rho_1}{\mathrm{d}y_3^2} \mathrm{d}y = \widetilde{\Lambda}_k \int_{F(S)} \eta_1^{(k)} \rho_1 \mathrm{d}y.$$

Using the definition (3.40) of the functions H, K_i and A_{ij} for $1 \leq i, j \leq 2$, we transform the previous equation into

$$Y \int_0^l \left(-K_1 \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} + A_{11} \frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{12} \frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} \right) \frac{\mathrm{d}^2\rho_1}{\mathrm{d}z_3^2} \mathrm{d}z_3 = \widetilde{\Lambda}_k \int_0^l H \eta_1^{(k)} \rho_1 \mathrm{d}z_3. \tag{3.49}$$

To simplify notation we write

$$P_i(z_3) = -K_i(z_3) \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} + A_{i1}(z_3) \frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{i2}(z_3) \frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} \qquad (i = 1, 2),$$

$$P_3(z_3) = H(z_3) \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} - K_1(z_3) \frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} - K_2(z_3) \frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}.$$

We use integration by parts two times in (3.49) to obtain

$$Y\left(\left[P_1(z_3)\frac{\mathrm{d}\rho_1}{\mathrm{d}z_3}\right]_0^l - \left[\frac{\mathrm{d}P_1}{\mathrm{d}z_3}\rho_1(z_3)\right]_0^l + \int_0^l \frac{\mathrm{d}^2 P_1}{\mathrm{d}z_3^2}\rho_1 \mathrm{d}z_3\right) = \widetilde{\Lambda}_k \int_0^l H\eta_1^{(k)}\rho_1 \mathrm{d}z_3.$$

Using (3.45), we see that the previous equation becomes

$$Y\left(\left[P_1(z_3)\frac{\mathrm{d}\rho_1}{\mathrm{d}z_3}\right]_0^l - \left[\frac{\mathrm{d}P_1}{\mathrm{d}z_3}\rho_1(z_3)\right]_0^l\right) = 0$$

Note that in the (DD) case, we can see that all terms above vanish. However, in the (DN) case we have that $\rho_1(0) = 0$ and $\frac{\mathrm{d}\rho_1}{\mathrm{d}z_3}(0) = 0$. Therefore

$$P_1(l)\frac{\mathrm{d}\rho_1}{\mathrm{d}z_3}(l) - \frac{\mathrm{d}P_1}{\mathrm{d}z_3}(l)\rho_1(l) = 0$$

Using proper test functions ρ_1, we conclude $P_1(l) = 0$ and $\frac{\mathrm{d}P_1}{\mathrm{d}z_3}(l) = 0$. In a similar fashion, choosing $\rho_1 = 0$ and $\rho_3 = 0$, we deduce $P_2(l) = 0$ and $\frac{\mathrm{d}P_2}{\mathrm{d}z_3}(l) = 0$. Finally, taking $\rho_1 = 0$ and $\rho_2 = 0$, we get $P_3(l) = 0$. Moreover, from (3.43), we also get $\frac{\mathrm{d}P_3}{\mathrm{d}z_3}(l) = 0$. Thus, we have seen that $P_i(l) = 0$ and $\frac{\mathrm{d}P_i}{\mathrm{d}z_3}(l) = 0$ for $i = 1, 2, 3$ and therefore solving the systems we obtain

$$\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}(l) = \frac{\mathrm{d}^3\eta_i^{(k)}}{\mathrm{d}z_3^3}(l) = 0 \quad (i = 1, 2), \qquad \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}(l) = \frac{\mathrm{d}^2\eta_3^{(k)}}{\mathrm{d}z_3^2}(l) = 0.$$

To sum up, we have the boundary conditions

$$\begin{cases} \eta_3^{(k)}(0) = \eta_i^{(k)}(0) = \dfrac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3}(0) = 0 \\ \dfrac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}(l) = \dfrac{\mathrm{d}^2\eta_3^{(k)}}{\mathrm{d}z_3^2}(l) = \dfrac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}(l) = \dfrac{\mathrm{d}^3\eta_i^{(k)}}{\mathrm{d}z_3^3}(l) = 0 \end{cases} \qquad (i = 1, 2).$$

Remark 3.7. It can be shown that the condition $\frac{\mathrm{d}^2\eta_3^{(k)}}{\mathrm{d}z_3^2} = 0$ is not independent and can be deduced from the other conditions and equations. Thus we can drop it when stating the main result of this section.

3.5 Upper bound for the limit eigenvalues

In Section (3.4) we have seen that $\widetilde{\Lambda}_k \geq \Lambda_k$, where $\widetilde{\Lambda}_k$ is the limit $\widetilde{\Lambda}_k = \lim_{r\to+\infty} \frac{1}{\zeta_r^2}\mu_k(\zeta_r)$ (see (3.17)) and Λ_k is the k-th eigenvalue of the ordinary differential equation (3.47) with (dd) boundary condition. We now start to prove that $\widetilde{\Lambda}_k \leq \Lambda_k$. Consider the system of ordinary differential equations

$$
\begin{cases}
Y\dfrac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(\begin{pmatrix} A_{11} & A_{12} & -K_1 \\ A_{21} & A_{22} & -K_2 \end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}z_3^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}z_3^2} \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}z_3} \end{pmatrix}\right) = \Lambda\, H\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} & (0 < z_3 < l), \\
\dfrac{\mathrm{d}}{\mathrm{d}z_3}\left(H\dfrac{\mathrm{d}\eta_3}{\mathrm{d}z_3}\right) = \dfrac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}z_3^2} + K_2\dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}z_3^2}\right) & (0 < z_3 < l).
\end{cases} \tag{3.50}
$$

where $Y = \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}$. In a very similar fashion as before, we first consider the (DD) case, so we assume the functions satisfy the (dd) boundary condition.

Let Λ_k be the k-th eigenvalue of the problem (3.50) with (dd) boundary condition and $\eta^{(k)} = (\eta_1^{(k)}, \eta_2^{(k)}, \eta_3^{(k)})$ its associated eigenfunction. By the relation we have in (3.50), $\eta_3^{(k)}$ satisfies $\frac{\mathrm{d}}{\mathrm{d}z_3}\left(H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right) = \frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + K_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}\right)$.

We recall that $\widetilde{\Lambda}_k = \lim_{r\to+\infty}\frac{1}{\zeta_r^2}\mu_k(\zeta_r)$ and the eigenvalue $\mu_k(\varepsilon)$ can be characterized by Rayleigh's quotient via

$$
\mu_k(\varepsilon) = \sup_{Z\in\mathcal{H}_{k-1}(F(S),\mathbb{R}^3)} \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1\backslash\{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\}.
$$

(see (3.7)). We want to show that $\widetilde{\Lambda}_k \leq \Lambda_k$.

We multiply the system (3.50) by (η_1, η_2) and integrate over the interval $(0, l)$. Applying the integration by parts we obtain

$$
Y\int_0^l \left(\sum_{i,j=1}^2 A_{ij}\frac{\mathrm{d}^2\eta_i}{\mathrm{d}z_3^2}\frac{\mathrm{d}^2\eta_j}{\mathrm{d}z_3^2} - \sum_{i=1}^2 K_i\frac{\mathrm{d}^2\eta_i}{\mathrm{d}z_3^2}\frac{\mathrm{d}\eta_3}{\mathrm{d}z_3}\right)\mathrm{d}z_3 = \Lambda\int_0^l H\left(\eta_1^2 + \eta_2^2\right)\mathrm{d}z_3.
$$

Using the relationship between η_3 and (η_1, η_2) we have in (3.47), we deduce that

$$
Y\int_0^l \left(\sum_{i,j=1}^2 A_{ij}\frac{\mathrm{d}^2\eta_i}{\mathrm{d}z_3^2}\frac{\mathrm{d}^2\eta_j}{\mathrm{d}z_3^2} - 2\sum_{i=1}^2 K_i\frac{\mathrm{d}^2\eta_i}{\mathrm{d}z_3^2}\frac{\mathrm{d}\eta_3}{\mathrm{d}z_3} + H\left(\frac{\mathrm{d}\eta_3}{\mathrm{d}z_3}\right)^2\right)\mathrm{d}z_3 = \Lambda\int_0^l H\left(\eta_1^2 + \eta_2^2\right)\mathrm{d}z_3.
$$

Therefore, if $\eta^{(k)} = (\eta_1^{(k)}, \eta_2^{(k)})$ is the k-th eigenfunction of the ordinary differential equation (3.50), we have that

$$
\Lambda_k = \frac{Y\displaystyle\int_0^l \left(\sum_{i,j=1}^2 A_{ij}\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}^2\eta_j^{(k)}}{\mathrm{d}z_3^2} - 2\sum_{i=1}^2 K_i\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} + H\left(\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)^2\right)\mathrm{d}z_3}{\displaystyle\int_0^l H\left(\left(\eta_1^{(k)}\right)^2 + \left(\eta_2^{(k)}\right)^2\right)\mathrm{d}z_3}. \tag{3.51}
$$

We recall Rayleigh's quotient $\widetilde{\mathcal{R}}_\varepsilon$ introduced in (3.6). We try new test functions $\Theta(y) = \Theta = (\Theta_1, \Theta_2, \Theta_3)$, $\phi(y) = \phi = (\phi_1, \phi_2, \phi_3)$ given by

$$\begin{aligned}
\Theta_i &= \eta_i + \varepsilon^2 \phi_i \qquad (i = 1, 2),\\
\Theta_3 &= \eta_3 - y_1 \frac{\mathrm{d}\eta_1}{\mathrm{d}y_3} - y_2 \frac{\mathrm{d}\eta_2}{\mathrm{d}y_3} + \varepsilon \phi_3,
\end{aligned}$$

where the functions η_i for $i = 1, 2, 3$ depend only on y_3. The choice of these test functions comes from the fact that we want $E_{ij}(\Theta)$ to satisfy (3.16). Indeed, since for $1 \le i, j \le 2$ we have $E_{ij}(\eta) = 0$ and $E_{i3}(\eta) = 0$, we calculate

$$\begin{aligned}
E_{ij}(\Theta) &= \varepsilon^2 E_{ij}(\phi),\\
E_{i3}(\Theta) &= \frac{1}{2}\left(\varepsilon^2 \frac{\partial \phi_i}{\partial y_3} + \varepsilon \frac{\partial \phi_3}{\partial y_i}\right) \qquad (1 \le i, j \le 2),\\
E_{33}(\Theta) &= \frac{\mathrm{d}\eta_3}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2} + \varepsilon \frac{\partial \phi_3}{\partial y_3}.
\end{aligned}$$

For brevity we write $N = \frac{\mathrm{d}\eta_3}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}$. With this notation, we compute $\widetilde{\mathcal{R}}_\varepsilon(\Theta)$.

$$\begin{aligned}
\widetilde{\mathcal{R}}_\varepsilon(\Theta) &= \frac{\displaystyle\int_{F(S)} \left(\lambda_1 \left(\varepsilon^2 \frac{\partial \phi_1}{\partial y_1} + \varepsilon^2 \frac{\partial \phi_2}{\partial y_2} + \varepsilon^2 N + \varepsilon^3 \frac{\partial \phi_3}{\partial y_3}\right)^2 + 2\lambda_2 \sum_{i,j=1}^{2} \varepsilon^4 E_{ij}(\phi)^2\right) \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\varepsilon^2(\eta_1 + \varepsilon^2\phi_1)^2 + \varepsilon^2(\eta_2 + \varepsilon^2\phi_2)^2 + \varepsilon^4(\eta_3 - y_1 \tfrac{\mathrm{d}\eta_1}{\mathrm{d}y_3} - y_2 \tfrac{\mathrm{d}\eta_2}{\mathrm{d}y_3} + \varepsilon\phi_3)^2\right) \mathrm{d}y}\\
&+ \frac{\displaystyle\int_{F(S)} 2\lambda_2 \left(2\varepsilon^2 \sum_{i=1}^{2} \frac{1}{4}\left(\varepsilon^2 \frac{\partial \phi_i}{\partial y_3} + \varepsilon \frac{\partial \phi_3}{\partial y_i}\right)^2 + \varepsilon^4 \left(N + \varepsilon \frac{\partial \phi_3}{\partial y_3}\right)^2\right) \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\varepsilon^2(\eta_1 + \varepsilon^2\phi_1)^2 + \varepsilon^2(\eta_2 + \varepsilon^2\phi_2)^2 + \varepsilon^4(\eta_3 - y_1 \tfrac{\mathrm{d}\eta_1}{\mathrm{d}y_3} - y_2 \tfrac{\mathrm{d}\eta_2}{\mathrm{d}y_3} + \varepsilon\phi_3)^2\right) \mathrm{d}y}.
\end{aligned}$$

Multiplying by $\frac{1}{\varepsilon^2}$ and taking the limit $\varepsilon \to 0$, we see

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \widetilde{\mathcal{R}}_\varepsilon(\Theta) &= \frac{\displaystyle\int_{F(S)} \lambda_1 \left(\frac{\partial \phi_1}{\partial y_1} + \frac{\partial \phi_2}{\partial y_2} + N\right)^2 \mathrm{d}y}{\displaystyle\int_{F(S)} (\eta_1^2 + \eta_2^2) \mathrm{d}y}\\
&+ \frac{\displaystyle\int_{F(S)} 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\phi)^2 + \frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \phi_3}{\partial y_i}\right)^2 + N^2\right) \mathrm{d}y}{\displaystyle\int_{F(S)} (\eta_1^2 + \eta_2^2) \mathrm{d}y}. \qquad (3.52)
\end{aligned}$$

We want to find the $\phi = (\phi_1, \phi_2, \phi_3)$ that minimizes the numerator in (3.52)

$$\mathcal{M}(\phi) = \int_{F(S)} \left(\lambda_1 \left(\frac{\partial \phi_1}{\partial y_1} + \frac{\partial \phi_2}{\partial y_2} + N\right)^2 + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\phi)^2 + \frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \phi_3}{\partial y_i}\right)^2 + N^2\right)\right) \mathrm{d}y.$$

In order to minimize $\mathcal{M}$, we put the test function ϕ as follows.

$$\begin{aligned}\phi_i(y) &= \sum_{p,q=1}^{2} \alpha_{pq}^{(i)} y_p y_q + \sum_{p=1}^{2} \beta_p^{(i)} y_p \qquad (i=1,2), \\ \phi_3(y) &= 0\end{aligned} \tag{3.53}$$

where $\alpha_{pq}^{(i)}$ and $\beta_p^{(i)}$ depend only on y_3 for $1 \le p,q,i \le 2$ and satisfy $\alpha_{12}^{(i)} = \alpha_{21}^{(i)}$ for $i = 1,2$.

When we substitute this test function into $\mathcal{M}$ we obtain an expression that can be written as a polynomial of degree 2 on the variables $\alpha_{pq}^{(i)}$ and $\beta_p^{(i)}$ for $1 \le i,p,q \le 2$ (in total there are 10 variables). Thus, it can be further rewritten as $\int_0^l \left(\alpha^T \mathcal{X}\alpha + \mathcal{Y}\alpha\right) \mathrm{d}y$ for a certain matrix valued function $\mathcal{X}$ and a certain vector valued function $\mathcal{Y}$ (for the explicit forms of $\mathcal{X}$ and $\mathcal{Y}$ see Appendix Remark A.2) with

$$\alpha = (\alpha_{11}^{(1)}, \alpha_{12}^{(1)}, \alpha_{22}^{(1)}, \alpha_{11}^{(2)}, \alpha_{12}^{(2)}, \alpha_{22}^{(2)}, \beta_1^{(1)}, \beta_2^{(1)}, \beta_1^{(2)}, \beta_2^{(2)})^T.$$

Since we want the minimum, we differentiate the expression $\int_0^l \left(\alpha^T \mathcal{X}\alpha + \mathcal{Y}\alpha\right) \mathrm{d}y$ with respect to α and solve the linear system $2\mathcal{X}\alpha + \mathcal{Y} = 0$ for α. After long but simple calculations we obtain

$$\begin{aligned}
&\alpha_{11}^{(1)} = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2}, \quad \alpha_{12}^{(1)} = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}, \quad \alpha_{22}^{(1)} = -\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2}, \\
&\alpha_{11}^{(2)} = -\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}, \quad \alpha_{12}^{(2)} = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2}, \quad \alpha_{22}^{(2)} = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}, \\
&\beta_1^{(1)} = -\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3}, \quad \beta_2^{(2)} = -\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3}.
\end{aligned}$$

In fact, the matrix $\mathcal{X}$ in the system is degenerate and we additionally obtain the condition $\beta_1^{(2)} + \beta_2^{(1)} = 0$. It can also be checked that the minimum obtained is always the same, so to simplify, we put $\beta_1^{(2)} = 0$ and $\beta_2^{(1)} = 0$. Therefore, recalling (3.53), we obtain

$$\begin{aligned}
\phi_1(y) &= \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2}y_1^2 + 2\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}y_1y_2 - \frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2}y_2^2 - 2\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3}y_1\right), \\
\phi_2(y) &= \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(-\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}y_1^2 + 2\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2}y_1y_2 + \frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}y_2^2 - 2\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3}y_2\right), \\
\phi_3(y) &= 0.
\end{aligned} \tag{3.54}$$

Substituting (3.54) into (3.52) and after long but elementary computations we obtain the minimum

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\widetilde{\mathcal{R}}_\varepsilon(\Theta) = \frac{\displaystyle\int_{F(S)} \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}N^2\mathrm{d}y}{\displaystyle\int_{F(S)} (\eta_1^2+\eta_2^2)\mathrm{d}y}. \tag{3.55}$$

Substituting $(\eta_1, \eta_2, \eta_3) = (\eta_1^{(k)}, \eta_2^{(k)}, \eta_3^{(k)})$ and the definition of N into (3.55) and integrating over the cross-section we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \widetilde{\mathcal{R}}_\varepsilon(\Theta) = \frac{Y\displaystyle\int_0^l \left(\sum_{i,j=1}^{2} A_{ij} \frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2} \frac{\mathrm{d}^2\eta_j^{(k)}}{\mathrm{d}z_3^2} - 2\sum_{i=1}^{2} K_i \frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2} \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} + H \left(\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} \right)^2 \right) \mathrm{d}z_3}{\displaystyle\int_0^l H \left(\left(\eta_1^{(k)}\right)^2 + \left(\eta_2^{(k)}\right)^2 \right) \mathrm{d}z_3},$$

which, from (3.51), turns out to be

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \widetilde{\mathcal{R}}_\varepsilon(\Theta) = \Lambda_k.$$

These computations are a motivation of how to choose proper test functions for our next goal, which is to use the Max-Min method in order to prove the inequality $\widetilde{\Lambda}_k \le \Lambda_k$. First, we consider the eigenfunction $\eta^{(k)} = (\eta_1^{(k)}, \eta_2^{(k)}, \eta_3^{(k)})$ corresponding to the eigenvalue Λ_k of problem (3.50) with (dd) boundary condition. We also choose the functions $\eta^{(k)}$ so that

$$\int_{F(S)} \left(\eta_1^{(k)}\eta_1^{(k')} + \eta_2^{(k)}\eta_2^{(k')} \right) \mathrm{d}y = \delta(k,k'), \tag{3.56}$$

where δ is the Kronecker delta. We define

$$N_k = \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}y_3^2}.$$

Using the weak formulation of (3.50) we know that

$$Y \int_{F(S)} N_k N_{k'} \mathrm{d}y = \Lambda_k \delta(k,k'). \tag{3.57}$$

Let us consider the test functions

$$\begin{aligned} \Phi_i^{(s)} &= \eta_i^{(s)} + \varepsilon^2 \phi_i^{(s)} \qquad (i = 1, 2), \\ \Phi_3^{(s)} &= \eta_3^{(s)} - y_1 \tfrac{\mathrm{d}\eta_1^{(s)}}{\mathrm{d}y_3} - y_2 \tfrac{\mathrm{d}\eta_2^{(s)}}{\mathrm{d}y_3}, \end{aligned}$$

with $s \in \mathbb{N}$ and

$$\begin{aligned} \phi_1^{(s)} &= \frac{1}{4} \frac{\lambda_1}{\lambda_1 + \lambda_2} \left(\frac{\mathrm{d}^2\eta_1^{(s)}}{\mathrm{d}y_3^2} y_1^2 + 2 \frac{\mathrm{d}^2\eta_2^{(s)}}{\mathrm{d}y_3^2} y_1 y_2 - \frac{\mathrm{d}^2\eta_1^{(s)}}{\mathrm{d}y_3^2} y_2^2 - 2 \frac{\mathrm{d}\eta_3^{(s)}}{\mathrm{d}y_3} y_1 \right), \\ \phi_2^{(s)} &= \frac{1}{4} \frac{\lambda_1}{\lambda_1 + \lambda_2} \left(-\frac{\mathrm{d}^2\eta_2^{(s)}}{\mathrm{d}y_3^2} y_1^2 + 2 \frac{\mathrm{d}^2\eta_1^{(s)}}{\mathrm{d}y_3^2} y_1 y_2 + \frac{\mathrm{d}^2\eta_2^{(s)}}{\mathrm{d}y_3^2} y_2^2 - 2 \frac{\mathrm{d}\eta_3^{(s)}}{\mathrm{d}y_3} y_2 \right). \end{aligned}$$

Choose an arbitrary $Z \in \mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$ and let $\widetilde{Z} = L.H. \left[\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(k)}\right]$ be the minimal linear space that contains the set $\{\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(k)}\}$. Note that $\dim \widetilde{Z} = k$ and that each

$\Phi^{(s)} \in \mathcal{W}_1$ (for all $s \in \mathbb{N}$), so we have that $\widetilde{Z} \subseteq \mathcal{W}_1$. Since $\dim Z < \dim \widetilde{Z}$, we know that there exist a function $\Psi = (\Psi_1, \Psi_2, \Psi_3) \in \widetilde{Z} \cap Z^{\perp_\varepsilon}$ and a vector $(c_1, \dots, c_k) = (c_1(\varepsilon), \dots, c_k(\varepsilon)) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ such that

$$\Psi = \sum_{s=1}^{k} c_s(\varepsilon) \Phi^{(s)}.$$

Note that since both $\widetilde{Z}$ and $Z^{\perp_\varepsilon}$ are subsets of $\mathcal{W}_1$, we have also that $\Psi \in \mathcal{W}_1$ and due the fact that $(c_1(\varepsilon), \dots, c_k(\varepsilon)) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ we deduce that $\Psi \in \mathcal{W}_1 \backslash \{\mathbf{0}\}$, so we can apply $\widetilde{\mathcal{R}}_\varepsilon$ to Ψ. We compute

$$E_{ii}(\Psi) = \varepsilon^2 \sum_{s=1}^{k} c_s(\varepsilon) \tfrac{1}{2} \tfrac{\lambda_1}{\lambda_1 + \lambda_2} N_s, \qquad E_{i3}(\Psi) = \varepsilon^2 \sum_{s=1}^{k} c_s(\varepsilon) E_{i3}(\phi) \qquad (1 \leq i, j \leq 2),$$
$$E_{12}(\Psi) = E_{21}(\Psi) = 0, \qquad E_{33}(\Psi) = \sum_{s=1}^{k} c_s(\varepsilon) N_s.$$

Using these computations, the numerator of the Rayleigh quotient $\widetilde{\mathcal{R}}_\varepsilon(\Psi)$ is

$$\begin{aligned} &\int_{F(S)} \left(\lambda_1 \left(\varepsilon^2 \left(\sum_{s=1}^{k} c_s(\varepsilon) \tfrac{\lambda_2}{\lambda_1+\lambda_2} N_s \right) \right)^2 + 2\lambda_2 \left(\sum_{i=1}^{2} \varepsilon^4 \left(\sum_{s=1}^{k} c_s(\varepsilon) \tfrac{1}{2} \tfrac{\lambda_1}{\lambda_1+\lambda_2} N_s \right)^2 \right) \right) \mathrm{d}y \\ &\quad + \int_{F(S)} 2\lambda_2 \left(2\varepsilon^4 \sum_{i=1}^{2} \frac{1}{4} \left(\varepsilon^2 \sum_{s=1}^{k} c_s(\varepsilon) E_{i3}(\phi) \right)^2 + \varepsilon^4 \left(\sum_{s=1}^{k} c_s(\varepsilon) N_s \right)^2 \right) \mathrm{d}y \\ &\quad = \varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \int_{F(S)} Y N_p N_q \mathrm{d}y + \varepsilon^6 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p, q, \varepsilon) \end{aligned} \tag{3.58}$$

for some functions $\widetilde{\kappa}(p, q, \varepsilon) = O(1)$ as $\varepsilon \to 0$. Note that these functions $\widetilde{\kappa}(p, q, \varepsilon)$ do not depend on the choice of Z. Due to (3.57), it follows that (3.58) becomes

$$\begin{aligned} &\varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \int_{F(S)} Y N_p N_q \mathrm{d}y + \varepsilon^6 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p, q, \varepsilon) \\ &\qquad = \varepsilon^4 \sum_{p=1}^{k} c_p(\varepsilon)^2 \Lambda_p + \varepsilon^6 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p, q, \varepsilon). \end{aligned} \tag{3.59}$$

Note also that the denominator of $\mathcal{R}_\varepsilon(\Psi)$ satisfies

$$
\begin{aligned}
&\varepsilon^2 \int_{F(S)} \left(\Psi_1^2 + \Psi_2^2 + \varepsilon^2 \Psi_3^2\right) \mathrm{d}y \geq \varepsilon^2 \int_{F(S)} \left(\Psi_1^2 + \Psi_2^2\right) \mathrm{d}y \\
&= \varepsilon^2 \int_{F(S)} \left(\left(\sum_{s=1}^{k} c_k(\varepsilon)(\eta_1^{(s)} + \varepsilon^2 \phi_1^{(s)}) \right)^2 + \left(\sum_{s=1}^{k} c_k(\varepsilon)(\eta_2^{(s)} + \varepsilon^2 \phi_2^{(s)}) \right)^2 \right) \mathrm{d}y \\
&= \varepsilon^2 \int_{F(S)} \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \left(\sum_{n=1}^{2} (\eta_n^{(p)} + \varepsilon^2 \phi_n^{(p)})(\eta_n^{(q)} + \varepsilon^2 \phi_n^{(q)}) \right) \mathrm{d}y \\
&= \varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \int_{F(S)} (\eta_1^{(p)} \eta_1^{(q)} + \eta_2^{(p)} \eta_2^{(q)}) \mathrm{d}y + \varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widehat{\kappa}(p,q,\varepsilon)
\end{aligned}
\tag{3.60}
$$

for certain functions $\widehat{\kappa}(p,q,\varepsilon) = O(1)$ as $\varepsilon \to 0$. Note again that the functions $\widehat{\kappa}(p,q,\varepsilon)$ do not depend on the choice of Z. By the homogeneity property of Rayleigh's quotient we may assume without loss of generality that $\sum_{p=1}^{k} c_p(\varepsilon)^2 = 1$. Thus we have $|c_p(\varepsilon)| \leq 1$ for $1 \leq p \leq k$. Combining this fact with the orthogonality in (3.56), we get

$$
\begin{aligned}
&\varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \int_{F(S)} (\eta_1^{(p)} \eta_1^{(q)} + \eta_2^{(p)} \eta_2^{(q)}) \mathrm{d}y + \varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widehat{\kappa}(p,q,\varepsilon) \\
&= \varepsilon^2 \sum_{p=1}^{k} c_p(\varepsilon)^2 + \varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widehat{\kappa}(p,q,\varepsilon) = \varepsilon^2 + \varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widehat{\kappa}(p,q,\varepsilon) \\
&\geq \varepsilon^2 - \varepsilon^4 \sum_{p,q=1}^{k} |c_p(\varepsilon)||c_q(\varepsilon)||\widehat{\kappa}(p,q,\varepsilon)| \geq \varepsilon^2 - \varepsilon^4 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|.
\end{aligned}
\tag{3.61}
$$

Therefore, with (3.60) and (3.61), we deduce that

$$
\varepsilon^2 \int_{F(S)} \left(\Psi_1^2 + \Psi_2^2 + \varepsilon^2 \Psi_3^2\right) \mathrm{d}y \geq \varepsilon^2 - \varepsilon^4 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|.
\tag{3.62}
$$

Using (3.59), the bound (3.62) and the fact that $\Lambda_k \leq \Lambda_{k+1}$ for $k \in \mathbb{N}$, we obtain

$$
\begin{aligned}
\frac{1}{\varepsilon^2} \widetilde{\mathcal{R}}_\varepsilon(\Psi) &\leq \frac{1}{\varepsilon^2} \frac{\varepsilon^4 \sum_{p=1}^{k} c_p(\varepsilon)^2 \Lambda_p + \varepsilon^6 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p,q,\varepsilon)}{\varepsilon^2 - \varepsilon^4 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|} \\
&\leq \frac{\Lambda_k \sum_{p=1}^{k} c_p(\varepsilon)^2 + \varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p,q,\varepsilon)}{1 - \varepsilon^2 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|} \leq \frac{\Lambda_k + \varepsilon^2 \sum_{p,q=1}^{k} |\widetilde{\kappa}(p,q,\varepsilon)|}{1 - \varepsilon^2 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|}
\end{aligned}
\tag{3.63}
$$

provided that the denominator is positive (this is possible because ε is a small real parameter). Let us denote the right hand side of the previous inequality

$$\mathfrak{L}_k(\varepsilon) = \frac{\Lambda_k + \varepsilon^2 \sum_{p,q=1}^{k} |\widetilde{\kappa}(p,q,\varepsilon)|}{1 - \varepsilon^2 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|}.$$

Note once again that $\mathfrak{L}_k(\varepsilon)$ does not depend on the choice of Z. We know from (3.63) that

$$\frac{1}{\varepsilon^2} \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1 \backslash \{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\} \leq \frac{1}{\varepsilon^2} \widetilde{\mathcal{R}}_\varepsilon(\Psi) \leq \mathfrak{L}_k(\varepsilon).$$

Since $Z \in \mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$ was arbitrary, we take the supremum over $\mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$, so we obtain the upper estimate

$$\frac{1}{\varepsilon^2} \mu_k(\varepsilon) \leq \mathfrak{L}_k(\varepsilon).$$

Taking the limit $\varepsilon \to 0$ and using (3.17), we have

$$\widetilde{\Lambda}_k \leq \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mu_k(\varepsilon) \leq \limsup_{\varepsilon \to 0} \mathfrak{L}_k(\varepsilon) = \Lambda_k,$$

which agrees to the desired inequality $\widetilde{\Lambda}_k \leq \Lambda_k$ $(k \in \mathbb{N})$. We combine this fact together with (3.48) to conclude that

$$\widetilde{\Lambda}_k = \Lambda_k \qquad (k \in \mathbb{N}).$$

We proved $\lim\limits_{r \to +\infty} \frac{\mu_k(\zeta_r)}{\zeta_r^2} = \widetilde{\Lambda}_k$ only for a certain subsequence $\{\zeta_r\}_{r=1}^{+\infty} \subseteq \{\varepsilon_p\}_{p=1}^{+\infty}$, but note that we have shown that $\widetilde{\Lambda}_k = \Lambda_k$ independently of the first chosen sequence $\{\varepsilon_p\}_{p=1}^{+\infty}$. Since this sequence was arbitrary, we can see that in fact for every $k \in \mathbb{N}$ we have

$$\lim_{\varepsilon \to 0} \frac{\mu_k(\varepsilon)}{\varepsilon^2} = \widetilde{\Lambda}_k.$$

Similarly, we prove the same result in the case (DN).

4 Torsional and stretching modes

In this section we discuss high-frequency eigenvalues of a thin elastic rod with axis-symmetric cross-section that varies along the rod. We prepare the mathematical setting of our problem. Let $l > 0$, let $a : [0, l] \to (0, +\infty)$ be a $\mathcal{C}^2$ positive function and $\varepsilon > 0$, a small parameter related to the thinness of the rod. We consider the following domain.

$$\begin{aligned}
\Omega_{\varepsilon,a} &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < \varepsilon^2 a(x_3)^2,\ 0 < x_3 < l\},\\
\Gamma_{1,\varepsilon,a} &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < \varepsilon^2 a(x_3)^2,\ x_3 = 0 \text{ or } x_3 = l\},\\
\Gamma_{2,\varepsilon,a} &= \partial\Omega_{\varepsilon,a} \backslash \Gamma_{1,\varepsilon,a}.
\end{aligned}$$

We study the following eigenvalue problem.

$$\begin{cases} L[u] + \mu u = \mathbf{0} & \text{in } \Omega_{\varepsilon,a},\\ u = \mathbf{0} & \text{on } \Gamma_{1,\varepsilon,a},\\ \sigma(u)\,\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{2,\varepsilon,a}. \end{cases} \tag{4.1}$$

Here $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega_{\varepsilon,a}$. Note that this eigenvalue problem is a particular case of (DD), studied in Section 3, so we may use the results we obtained in the previous section. We denote by $\{\mu_k(\varepsilon)\}_{k=1}^{+\infty}$ the set of eigenvalues of problem (4.1) and we recall that for any $\varepsilon > 0$ there is an infinite discrete sequence of positive eigenvalues

$$0 < \mu_1(\varepsilon) \le \mu_2(\varepsilon) \le \cdots \le \mu_k(\varepsilon) \le \mu_{k+1}(\varepsilon) \le \cdots \text{ with } \lim_{k\to+\infty} \mu_k(\varepsilon) = +\infty,$$

which are arranged in increasing order, counting multiplicities. Moreover, by Theorem 3.1, we know that for each $k \in \mathbb{N}$ we have $\mu_k(\varepsilon) = O(\varepsilon^2)$. Thus, we have

$$\begin{aligned}
\lim_{k\to+\infty} \mu_k(\varepsilon) &= +\infty \text{ for each } \varepsilon > 0,\\
\lim_{\varepsilon\to 0} \mu_k(\varepsilon) &= 0 \text{ for each } k \in \mathbb{N}.
\end{aligned}$$

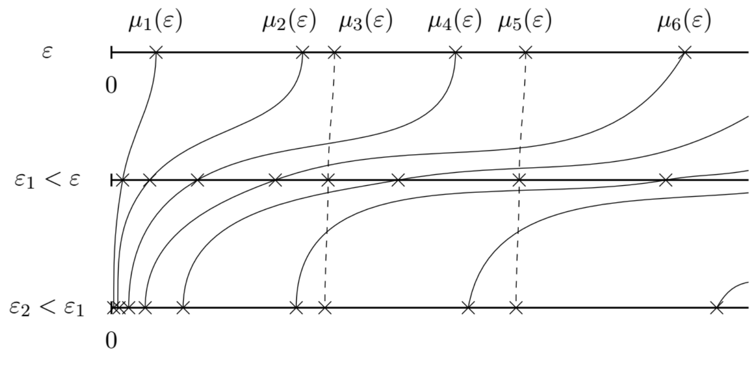

Figure 2: General behavior of the eigenvalues $\mu_k(\varepsilon)$ as ε becomes smaller

These eigenvalues are low-frequency eigenvalues corresponding to flexural vibrations (bending mode). However, it is known ([16] or [17]) that there exist high-frequency eigenvalues corresponding to stretching and torsional vibrations, which do not tend to zero as the thinness gets smaller. These high-frequency eigenvalues cannot be analyzed when the subindex k is fixed. As an example, we can focus on the eigenvalue $\mu_3(\varepsilon)$ of Figure 2. If we assume that this eigenvalue does not indeed converge to 0 as ε becomes smaller, then low-frequency eigenvalues will progressively "overtake" it, as seen in Figure 2. Due to the convergence $\lim_{\varepsilon\to 0}\mu_k(\varepsilon) = 0$, we see that in order to "catch" this eigenvalues, we have to vary the subindex k and the parameter ε simultaneously. In this section we study the eigenvalues related to torsional and stretching modes.

4.1 Main result

We present the main result of this section. Recall that λ_1, λ_2 are the Lamé constants and $Y = \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}$.

Theorem 4.1. *Let $k \in \mathbb{N}$ and let $\mu_k(\varepsilon)$ be the k-th eigenvalue of (4.1). Then the following statements hold.*

a) For every $k \in \mathbb{N}$, there exists a sequence $(q(k,\varepsilon))_{\varepsilon>0}$ with $q(k,\varepsilon) \in \mathbb{N}$, $q(k,\varepsilon) < q(k+1,\varepsilon)$ and $q(k,\varepsilon) \to +\infty$ as $\varepsilon \to 0$ and a constant $\xi_k > 0$ such that

$$\lim_{\varepsilon\to 0}\mu_{q(k,\varepsilon)}(\varepsilon) = \xi_k.$$

b) There exists a subset $\{\mu_k^S(\varepsilon)\}_{k=1}^{+\infty} \cup \{\mu_k^T(\varepsilon)\}_{k=1}^{+\infty} = \{\mu_{q(k,\varepsilon)}(\varepsilon)\}_{k=1}^{+\infty} \subseteq \{\mu_k(\varepsilon)\}_{k=1}^{+\infty}$ such that for every $k \in \mathbb{N}$,

$$\lim_{\varepsilon\to 0}\mu_k^S(\varepsilon) = \mu_k^S, \qquad \lim_{\varepsilon\to 0}\mu_k^T(\varepsilon) = \mu_k^T,$$

with $\{\mu_k^S\}_{k=1}^{+\infty} \cup \{\mu_k^T\}_{k=1}^{+\infty} = \{\xi_k\}_{k=1}^{+\infty}$. Here μ_k^S and μ_k^T are the respective k-th eigenvalues of the following eigenvalue problems.

$$\begin{cases} -Y\frac{\mathrm{d}}{\mathrm{d}y_3}\left(a(y_3)^2\frac{\mathrm{d}\tau}{\mathrm{d}y_3}\right) = \mu^S a(y_3)^2\tau \quad (0 < y_3 < l), \\ \tau(0) = \tau(l) = 0. \end{cases}$$
$$\begin{cases} -\lambda_2\frac{\mathrm{d}}{\mathrm{d}y_3}\left(a(y_3)^4\frac{\mathrm{d}\rho}{\mathrm{d}y_3}\right) = \mu^T a(y_3)^4\rho \quad (0 < y_3 < l), \\ \rho(0) = \rho(l) = 0. \end{cases}$$

c) Let $v_\varepsilon^{(k)}(x)$ and $w_\varepsilon^{(k)}(x)$, $x \in \Omega_{\varepsilon,a}$, be the eigenfunctions associated to $\mu_k^S(\varepsilon)$ and $\mu_k^T(\varepsilon)$ respectively. Then,

$$\begin{aligned} v_\varepsilon^{(k)}(x) &= (x_1\chi_\varepsilon^{(k)}(s,x_3), x_2\chi_\varepsilon^{(k)}(s,x_3), \tau_\varepsilon^{(k)}(s,x_3)), \\ w_\varepsilon^{(k)}(x) &= (-x_2\rho_\varepsilon^{(k)}(s,x_3), x_1\rho_\varepsilon^{(k)}(s,x_3), 0), \end{aligned}$$

where $s = \sqrt{x_1^2 + x_2^2}$, for some functions $\chi_\varepsilon^{(k)}, \tau_\varepsilon^{(k)}, \rho_\varepsilon^{(k)} \in H^1(\Omega_{\varepsilon,a})$ with $\chi_\varepsilon^{(k)}, \tau_\varepsilon^{(k)}, \rho_\varepsilon^{(k)} = 0$ on $\Gamma_{1,\varepsilon,a}$.

d) If we denote $\overline{\chi}_\varepsilon^{(k)}(\widetilde{s}, y_3) = \chi_\varepsilon^{(k)}(\varepsilon\widetilde{s}, y_3)$*,* $\overline{\tau}_\varepsilon^{(k)}(\widetilde{s}, y_3) = \tau_\varepsilon^{(k)}(\varepsilon\widetilde{s}, y_3)$ *and* $\overline{\rho}_\varepsilon^{(k)}(\widetilde{s}, y_3) = \rho_\varepsilon^{(k)}(\varepsilon\widetilde{s}, y_3)$ *with* $0 < y_3 < l$*,* $0 \le \widetilde{s} < a(y_3)$*, then*

$$\begin{aligned} \overline{\chi}_\varepsilon^{(k)} &\rightharpoonup 0 && \textit{weakly in } H^1(\Omega_{1,a}), \\ \overline{\tau}_\varepsilon^{(k)} &\longrightarrow \tau^{(k)} && \textit{strongly in } H^1(\Omega_{1,a}), \\ \overline{\rho}_\varepsilon^{(k)} &\longrightarrow \rho^{(k)} && \textit{strongly in } H^1(\Omega_{1,a}), \end{aligned}$$

as $\varepsilon \to 0$*, where* $\tau^{(k)}$ *and* $\rho^{(k)}$ *are the eigenfunctions associated to* μ_k^S *and* μ_k^T *respectively.*

The proof of Theorem 4.1 is given in Sections 4.2 to 4.5. In Section 4.6 we provide a sufficient condition and an example of the case that the convergence of $\overline{\chi}_\varepsilon^{(k)}$ in Theorem 4.1.d) is strong in $H^1(\Omega_{1,a})$. In Section 4.7 we try to generalize Theorem 4.1 to curved rods and we give a conjecture about the stretching and torsional modes on a curved rod with non-uniform cross-section.

4.2 Characterization of torsional and stretching eigenvalues

We adapt to our domain $\Omega_{\varepsilon,a}$ the Rayleigh quotient and the Max-Min characterization of the eigenvalues that we introduced in Section 2. Let $\phi, \psi \in H^1(\Omega_{\varepsilon,a}, \mathbb{R}^3)\backslash\{\mathbf{0}\}$. We define

$$\begin{aligned} B_{\varepsilon,a}[\phi, \psi] &= \int_{\Omega_{\varepsilon,a}} \left(\lambda_1 \operatorname{div} \phi \operatorname{div} \psi + 2\lambda_2 \sum_{i,j=1}^{3} e_{ij}(\phi) e_{ij}(\psi) \right) \mathrm{d}x, \\ \mathcal{R}_{\varepsilon,a}(\phi) &= \frac{B_{\varepsilon,a}[\phi, \phi]}{\|\phi\|^2_{L^2(\Omega_{\varepsilon,a}, \mathbb{R}^3)}}. \end{aligned} \tag{4.2}$$

Furthermore, we set the function space

$$\mathcal{W}_{\varepsilon,a} = \{\phi \in H^1(\Omega_{\varepsilon,a}, \mathbb{R}^3) \mid \phi = \mathbf{0} \text{ on } \Gamma_{1,\varepsilon,a}\}.$$

Then the k-th eigenvalue is characterized as follows:

$$\mu_k(\varepsilon) = \sup_{X \in \mathcal{H}_{k-1}(\Omega_{\varepsilon,a}, \mathbb{R}^3)} \inf\{\mathcal{R}_{\varepsilon,a}(\phi) \mid \phi \in \mathcal{W}_{\varepsilon,a}\backslash\{\mathbf{0}\}, \phi \perp X \text{ in } L^2(\Omega_{\varepsilon,a}, \mathbb{R}^3)\}.$$

We begin to study the torsional and stretching modes by variational methods. We rewrite the Rayleigh quotient and make sure that we exclude the bending mode. We put the test function $v_\varepsilon = (v_{1,\varepsilon}, v_{2,\varepsilon}, v_{3,\varepsilon}) \in \mathcal{W}_{\varepsilon,a}$ as follows.

$$\begin{aligned} v_{1,\varepsilon}(x) &= -x_2 \rho_\varepsilon(s, x_3) + x_1 \chi_\varepsilon(s, x_3), \\ v_{2,\varepsilon}(x) &= x_1 \rho_\varepsilon(s, x_3) + x_2 \chi_\varepsilon(s, x_3), \\ v_{3,\varepsilon}(x) &= \tau_\varepsilon(s, x_3), \end{aligned} \tag{4.3}$$

where $s = \sqrt{x_1^2 + x_2^2}$. We compute $\mathcal{R}_{\varepsilon,a}(v_\varepsilon)$. After the change of variables $x_1 = s\cos\vartheta$, $x_2 = s\sin\vartheta$, $0 \le \vartheta < 2\pi$, $0 < s < \varepsilon a(x_3)$, the numerator $B_{\varepsilon,a}[v_\varepsilon, v_\varepsilon]$ becomes

$$\int_{\Omega_{\varepsilon,a}} \Bigg\{ \lambda_1 \left(s\frac{\partial \chi_\varepsilon}{\partial s} + 2\chi_\varepsilon + \frac{\partial \tau_\varepsilon}{\partial x_3} \right)^2 \tag{4.4}$$
$$+ 2\lambda_2 \left(\left(\frac{\partial \tau_\varepsilon}{\partial x_3} \right)^2 + \frac{1}{2}\left(s\frac{\partial \chi_\varepsilon}{\partial x_3} + \frac{\partial \tau_\varepsilon}{\partial s} \right)^2 + \left(s\frac{\partial \chi_\varepsilon}{\partial s} + \chi_\varepsilon \right)^2 + \chi_\varepsilon^2 + \frac{s^2}{2}\left(\left(\frac{\partial \rho_\varepsilon}{\partial s} \right)^2 + \left(\frac{\partial \rho_\varepsilon}{\partial x_3} \right)^2 \right) \right) \Bigg\} s \,\mathrm{d}s\mathrm{d}x_3,$$

and the denominator $\|v_\varepsilon\|^2_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}$ becomes

$$\int_{\Omega_{\varepsilon,a}} \left(s^2\rho_\varepsilon^2 + s^2\chi_\varepsilon^2 + \tau_\varepsilon^2 \right) s \,\mathrm{d}s\mathrm{d}x_3.$$

We use variational methods in $\mathcal{R}_{\varepsilon,a}(v_\varepsilon)$ and obtain functions $(\rho_\varepsilon, \tau_\varepsilon, \chi_\varepsilon)$ that satisfy the PDEs

$$\begin{cases}
\left.\begin{aligned}
\lambda_2 \left(\frac{3}{s}\frac{\partial \rho_\varepsilon}{\partial s} + \frac{\partial^2 \rho_\varepsilon}{\partial s^2} + \frac{\partial^2 \rho_\varepsilon}{\partial x_3^2} \right) + \xi_\varepsilon \rho_\varepsilon = 0 \\
(\lambda_1 + 2\lambda_2)\frac{\partial^2 \tau_\varepsilon}{\partial x_3^2} + \lambda_2 \frac{1}{s}\frac{\partial \tau_\varepsilon}{\partial s} + \lambda_2 \frac{\partial^2 \tau_\varepsilon}{\partial s^2} + 2(\lambda_1 + \lambda_2)\frac{\partial \chi_\varepsilon}{\partial x_3} + (\lambda_1 + \lambda_2)s\frac{\partial^2 \chi_\varepsilon}{\partial s \partial x_3} + \xi_\varepsilon \tau_\varepsilon = 0 \\
\lambda_2 \frac{\partial^2 \chi_\varepsilon}{\partial x_3^2} + (\lambda_1 + \lambda_2)\frac{1}{s}\frac{\partial^2 \tau_\varepsilon}{\partial s \partial x_3} + 3(\lambda_1 + 2\lambda_2)\frac{1}{s}\frac{\partial \chi_\varepsilon}{\partial s} + (\lambda_1 + 2\lambda_2)\frac{\partial^2 \chi_\varepsilon}{\partial s^2} + \xi_\varepsilon \chi_\varepsilon = 0
\end{aligned}\right\} \text{ in } \Omega_{\varepsilon,a}, \\
\left.\begin{aligned}
\rho = 0 \\
\tau = 0 \\
\chi = 0
\end{aligned}\right\} \text{ on } \Gamma_{1,\varepsilon,a}, \\
\left.\begin{aligned}
\frac{\partial \rho_\varepsilon}{\partial s} - \varepsilon a'(x_3)\frac{\partial \rho_\varepsilon}{\partial x_3} = 0 \\
\lambda_2 \left(s\frac{\partial \chi_\varepsilon}{\partial x_3} + \frac{\partial \tau_\varepsilon}{\partial s} \right) - \varepsilon a'(x_3) \left(\lambda_1 s\frac{\partial \chi_\varepsilon}{\partial s} + 2\lambda_1 \chi_\varepsilon + (\lambda_1 + 2\lambda_2)\frac{\partial \tau_\varepsilon}{\partial x_3} \right) = 0 \\
(\lambda_1 + 2\lambda_2)s\frac{\partial \chi_\varepsilon}{\partial s} + 2(\lambda_1 + \lambda_2)\chi_\varepsilon + \lambda_1 \frac{\partial \tau_\varepsilon}{\partial x_3} - \varepsilon a'(x_3)\lambda_2 \left(s\frac{\partial \chi_\varepsilon}{\partial x_3} + \frac{\partial \tau_\varepsilon}{\partial s} \right) = 0
\end{aligned}\right\} \text{ on } \Gamma_{2,\varepsilon,a}.
\end{cases} \tag{4.5}$$

Note that ρ_ε is independent of the pair $(\tau_\varepsilon, \chi_\varepsilon)$. Moreover, (4.5) is an elliptic equation so we can assure the existence of a sequence $\{\xi_k(\varepsilon)\}_{k=1}^{+\infty}$ of positive eigenvalues.

However, these arguments do not prove that v_ε is an eigenfunction of (4.1). It is left to prove that there are eigenfunctions v_ε of (4.1) with the same structure as defined in (4.3). In other words, for $\zeta > 0$ we want to see that

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} \left(\mathcal{R}_{\varepsilon,a}(v_\varepsilon + \zeta\Phi_\varepsilon) \right) |_{\zeta=0} = 0, \tag{4.6}$$

for every $\Phi_\varepsilon \in \mathcal{W}_{\varepsilon,a}$ such that $(v_\varepsilon, \Phi_\varepsilon)_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)} = 0$, so that v_ε is a critical point of Rayleigh's quotient and thus v_ε is in fact an eigenfunction of (4.1). Since $B_{\varepsilon,a}$ is symmetric and bilinear, we note that

$$\begin{aligned}\mathcal{R}_{\varepsilon,a}(v_\varepsilon+\zeta\Phi_\varepsilon) &= \frac{B_{\varepsilon,a}[v_\varepsilon+\zeta\Phi_\varepsilon, v_\varepsilon+\zeta\Phi_\varepsilon]}{\|v_\varepsilon+\zeta\Phi_\varepsilon\|^2_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}}\\ &= \frac{B_{\varepsilon,a}[v_\varepsilon,v_\varepsilon]+2\zeta B_{\varepsilon,a}[v_\varepsilon,\Phi_\varepsilon]+\zeta^2 B_{\varepsilon,a}[\Phi_\varepsilon,\Phi_\varepsilon]}{\|v_\varepsilon\|^2_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}+2\zeta(v_\varepsilon,\Phi_\varepsilon)_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}+\zeta^2\|\Phi_\varepsilon\|^2_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}}\\ &= \frac{B_{\varepsilon,a}[v_\varepsilon,v_\varepsilon]+2\zeta B_{\varepsilon,a}[v_\varepsilon,\Phi_\varepsilon]+\zeta^2 B_{\varepsilon,a}[\Phi_\varepsilon,\Phi_\varepsilon]}{\|v_\varepsilon\|^2_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}+\zeta^2\|\Phi_\varepsilon\|^2_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}}.\end{aligned}$$

In order to prove (4.6), we only have to see that $B_{\varepsilon,a}[v_\varepsilon,\Phi_\varepsilon]=0$. For that purpose, let $\mathcal{H}_{\mathrm{TS}}$ denote the closed subspace of $\mathcal{W}_{\varepsilon,a}$ made of functions v_ε defined as in (4.3). By definition we have $v_\varepsilon\in\mathcal{H}_{\mathrm{TS}}$. We write $\Phi_\varepsilon=\widehat{\Psi}_\varepsilon+\Psi_\varepsilon$ with $\widehat{\Psi}_\varepsilon\in\mathcal{H}_{\mathrm{TS}}$ and $\Psi_\varepsilon\in\mathcal{H}_{\mathrm{TS}}^{\perp}$. It is clear that $(\widehat{\Psi}_\varepsilon,\Psi_\varepsilon)_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}=0$. We consider the case when $\widehat{\Psi}_\varepsilon$ is not a multiple of v_ε. Then from the construction of $\mathcal{H}_{\mathrm{TS}}$ we have $(v_\varepsilon,\widehat{\Psi}_\varepsilon)_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}=0$ and $B_{\varepsilon,a}[v_\varepsilon,\widehat{\Psi}_\varepsilon]=0$. Moreover, since $v_\varepsilon\in\mathcal{H}_{\mathrm{TS}}$ and $\Psi_\varepsilon\in\mathcal{H}_{\mathrm{TS}}^{\perp}$, we see that $(v_\varepsilon,\Psi_\varepsilon)_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)}=0$. Thus,

$$B_{\varepsilon,a}[v_\varepsilon,\Phi_\varepsilon]=B_{\varepsilon,a}[v_\varepsilon,\widehat{\Psi}_\varepsilon+\Psi_\varepsilon]=B_{\varepsilon,a}[v_\varepsilon,\Psi_\varepsilon]. \tag{4.7}$$

We write $\Psi_\varepsilon=(\Psi_{1,\varepsilon},\Psi_{2,\varepsilon},\Psi_{3,\varepsilon})$. We compute

$$B_{\varepsilon,a}[v_\varepsilon,\Psi_\varepsilon]=\int_{\Omega_{\varepsilon,a}}\left(\lambda_1\,\mathrm{div}\,v_\varepsilon\ \mathrm{div}\,\Psi_\varepsilon+2\lambda_2\sum_{i,j=1}^{3}e_{ij}(v_\varepsilon)e_{ij}(\Psi_\varepsilon)\right)\mathrm{d}x.$$

We perform the integral by parts and pass all the derivatives to v_ε. Therefore, we obtain

$$\begin{aligned}B_{\varepsilon,a}[v_\varepsilon,\Psi_\varepsilon]=&-\int_{\Omega_{\varepsilon,a}}\sum_{i,j=1}^{3}\left(\lambda_1\frac{\partial^2 v_{j,\varepsilon}}{\partial x_i\partial x_j}+\lambda_2\left(\frac{\partial^2 v_{i,\varepsilon}}{\partial^2 x_j}+\frac{\partial^2 v_{j,\varepsilon}}{\partial x_i\partial x_j}\right)\right)\Psi_{i,\varepsilon}\,\mathrm{d}x\\ &+\int_{\partial\Omega_{\varepsilon,a}}\sum_{i,j=1}^{3}\left(\lambda_1\frac{\partial v_{j,\varepsilon}}{\partial x_j}n_{x_i}+\lambda_2\left(\frac{\partial v_{i,\varepsilon}}{\partial x_j}+\frac{\partial v_{j,\varepsilon}}{\partial x_i}\right)n_{x_j}\right)\Psi_{i,\varepsilon}\,\mathrm{d}A.\end{aligned}$$

We compute $B_{\varepsilon,a}[v_\varepsilon,\Psi_\varepsilon]$ using (4.3) and the boundary conditions of (4.5), and after long and elementary computations the previous equation becomes

$$\begin{aligned}-\int_{\Omega_{\varepsilon,a}}\Bigg\{&\Bigg[-x_2\left(\lambda_2\left(\tfrac{3}{s}\tfrac{\partial\rho_\varepsilon}{\partial s}+\tfrac{\partial^2\rho_\varepsilon}{\partial s^2}+\tfrac{\partial^2\rho_\varepsilon}{\partial x_3^2}\right)\right)\\ &\quad+x_1\left(\lambda_2\tfrac{\partial^2\chi_\varepsilon}{\partial x_3^2}+(\lambda_1+\lambda_2)\tfrac{1}{s}\tfrac{\partial^2\tau_\varepsilon}{\partial s\partial x_3}+3(\lambda_1+2\lambda_2)\tfrac{1}{s}\tfrac{\partial\chi_\varepsilon}{\partial s}+(\lambda_1+2\lambda_2)\tfrac{\partial^2\chi_\varepsilon}{\partial s^2}\right)\Bigg]\Psi_{1,\varepsilon}\\ &+\Bigg[x_1\left(\lambda_2\left(\tfrac{3}{s}\tfrac{\partial\rho_\varepsilon}{\partial s}+\tfrac{\partial^2\rho_\varepsilon}{\partial s^2}+\tfrac{\partial^2\rho_\varepsilon}{\partial x_3^2}\right)\right)\\ &\quad+x_2\left(\lambda_2\tfrac{\partial^2\chi_\varepsilon}{\partial x_3^2}+(\lambda_1+\lambda_2)\tfrac{1}{s}\tfrac{\partial^2\tau_\varepsilon}{\partial s\partial x_3}+3(\lambda_1+2\lambda_2)\tfrac{1}{s}\tfrac{\partial\chi_\varepsilon}{\partial s}+(\lambda_1+2\lambda_2)\tfrac{\partial^2\chi_\varepsilon}{\partial s^2}\right)\Bigg]\Psi_{2,\varepsilon}\\ &+\left((\lambda_1+2\lambda_2)\tfrac{\partial^2\tau_\varepsilon}{\partial x_3^2}+\lambda_2\tfrac{1}{s}\tfrac{\partial\tau_\varepsilon}{\partial s}+\lambda_2\tfrac{\partial^2\tau_\varepsilon}{\partial s^2}+2(\lambda_1+\lambda_2)\tfrac{\partial\chi_\varepsilon}{\partial x_3}+(\lambda_1+\lambda_2)s\tfrac{\partial^2\chi_\varepsilon}{\partial s\partial x_3}\right)\Psi_{3,\varepsilon}\Bigg\}\mathrm{d}x.\end{aligned}$$

We use the equations in (4.5) to deduce

$$\begin{aligned} B_{\varepsilon,a}[v_\varepsilon,\Psi_\varepsilon] &= -\int_{\Omega_{\varepsilon,a}} \{(-x_2\xi_\varepsilon\rho_\varepsilon + x_1\xi_\varepsilon\chi_\varepsilon)\,\Psi_{1,\varepsilon} + (x_1\xi_\varepsilon\rho_\varepsilon + x_2\xi_\varepsilon\chi_\varepsilon)\,\Psi_{2,\varepsilon} + \xi_\varepsilon\tau_\varepsilon\Psi_{3,\varepsilon}\}\,\mathrm{d}x \\ &= -\xi_\varepsilon \int_{\Omega_{\varepsilon,a}} \{(-x_2\rho_\varepsilon + x_1\chi_\varepsilon)\,\Psi_{1,\varepsilon} + (x_1\rho_\varepsilon + x_2\chi_\varepsilon)\,\Psi_{2,\varepsilon} + \tau_\varepsilon\Psi_{3,\varepsilon}\}\,\mathrm{d}x \\ &= -\xi_\varepsilon \int_{\Omega_{\varepsilon,a}} (v_{\varepsilon,1}\Psi_{1,\varepsilon} + v_{\varepsilon,2}\Psi_{2,\varepsilon} + v_{\varepsilon,3}\Psi_{3,\varepsilon})\,\mathrm{d}x = -\xi_\varepsilon (v_\varepsilon,\Psi_\varepsilon)_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)} = 0 \end{aligned}$$

since $(v_\varepsilon,\Psi_\varepsilon)_{L^2(\Omega_{\varepsilon,a},\mathbb{R}^3)} = 0$. Thus we have seen that $B_{\varepsilon,a}[v_\varepsilon,\Psi_\varepsilon] = 0$. Hence, from (4.7) we see $B_{\varepsilon,a}[v_\varepsilon,\Phi_\varepsilon] = 0$ and we conclude

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\mathcal{R}_{\varepsilon,a}(v_\varepsilon + \zeta\Phi_\varepsilon)\right)|_{\zeta=0} = 0,$$

and v_ε is indeed an eigenfunction of (4.1).

Using the characterization of the eigenvalues, we know that $\{\xi_k(\varepsilon)\}_{k=1}^{+\infty} \subseteq \{\mu_k(\varepsilon)\}_{k=1}^{+\infty}$ and that the associated functions $v_\varepsilon^{(k)} = (v_{1,\varepsilon}^{(k)}, v_{2,\varepsilon}^{(k)}, v_{3,\varepsilon}^{(k)})$ with

$$\begin{aligned} v_{1,\varepsilon}^{(k)} &= -x_2\rho_\varepsilon^{(k)}(s,x_3) + x_1\chi_\varepsilon^{(k)}(s,x_3), \\ v_{2,\varepsilon}^{(k)} &= x_1\rho_\varepsilon^{(k)}(s,x_3) + x_2\chi_\varepsilon^{(k)}(s,x_3), \\ v_{3,\varepsilon}^{(k)} &= \tau_\varepsilon^{(k)}(s,x_3), \end{aligned} \tag{4.8}$$

are eigenfunctions of (4.1).

However, these eigenfunctions are not related (and in fact are orthogonal) to the bending mode eigenfunctions (see Section 3). From the Max-Min characterization we prove that the eigenvalues $\xi_k(\varepsilon)$ are bounded for $\varepsilon > 0$ and each fixed $k \in \mathbb{N}$. For that purpose, we note that we can split the eigenvalue problem (4.5) into two different eigenvalue problems, since there is no equation involving ρ_ε and $(\chi_\varepsilon,\tau_\varepsilon)$ (or its derivatives) at the same time. Thus, we write

$$\{\xi_k(\varepsilon)\}_{k=1}^{+\infty} = \{\mu_k^S(\varepsilon)\}_{k=1}^{+\infty} \cup \{\mu_k^T(\varepsilon)\}_{k=1}^{+\infty} \subseteq \{\mu_k(\varepsilon)\}_{k=1}^{+\infty},$$

We perform the following change of variables. For $x = (x_1,x_2,x_3) \in \Omega_{\varepsilon,a}$, let $s = \sqrt{x_1^2 + x_2^2}$, and $(x_1,x_2,x_3) = (\varepsilon y_1, \varepsilon y_2, y_3)$, so that $s = \varepsilon\widetilde{s}$ with $\widetilde{s} = \sqrt{y_1^2 + y_2^2}$. We write $y = (y_1,y_2,y_3)$. We define the following function spaces.

$$\begin{aligned} \mathcal{W}_{1,a} &= \{\phi \in H^1(\Omega_{1,a},\mathbb{R}^3) \mid \phi = \mathbf{0} \text{ on } \Gamma_{1,1,a}\}, \\ W_{1,a} &= \{\phi \in H^1(\Omega_{1,a}) \mid \phi = 0 \text{ on } \Gamma_{1,1,a}\}. \end{aligned}$$

We define $\mathfrak{H}_{\mathrm{S}}$ to be the smallest closed subspace of $\mathcal{W}_{1,a}$ such that if $v_\varepsilon^S = (v_{1,\varepsilon}^S, v_{2,\varepsilon}^S, v_{3,\varepsilon}^S) \in \mathfrak{H}_{\mathrm{S}}$, then

$$(v_{1,\varepsilon}^S, v_{2,\varepsilon}^S, v_{3,\varepsilon}^S) = (y_1\chi_\varepsilon(\widetilde{s},y_3), y_2\chi_\varepsilon(\widetilde{s},y_3), \tau_\varepsilon(\widetilde{s},y_3)),$$

for some $\chi_\varepsilon, \tau_\varepsilon \in W_{1,a}$. Let $v_\varepsilon^S \in \mathfrak{H}_\mathrm{S}$. The Rayleigh quotient for the stretching mode is rewritten as

$$\widetilde{\mathcal{R}}_\varepsilon^S(v_\varepsilon^S) = \frac{1}{\displaystyle\int_{\Omega_{1,a}} \left(\widetilde{\chi}_\varepsilon^2 + \tau_\varepsilon^2\right) \widetilde{s}\, \mathrm{d}\widetilde{s}\mathrm{d}y_3} \times \int_{\Omega_{1,a}} \Bigg\{ \lambda_1 \left(\tfrac{1}{\varepsilon} \tfrac{\partial \widetilde{\chi}_\varepsilon}{\partial \widetilde{s}} + \tfrac{1}{\varepsilon} \tfrac{1}{\widetilde{s}} \widetilde{\chi}_\varepsilon + \tfrac{\partial \tau_\varepsilon}{\partial y_3} \right)^2$$

$$+ 2\lambda_2 \left(\left(\tfrac{\partial \tau_\varepsilon}{\partial y_3} \right)^2 + \tfrac{1}{2} \left(\tfrac{\partial \widetilde{\chi}_\varepsilon}{\partial y_3} + \tfrac{1}{\varepsilon} \tfrac{\partial \tau_\varepsilon}{\partial \widetilde{s}} \right)^2 + \left(\tfrac{1}{\varepsilon} \tfrac{\partial \widetilde{\chi}_\varepsilon}{\partial \widetilde{s}} \right)^2 + \left(\tfrac{1}{\varepsilon} \tfrac{1}{\widetilde{s}} \widetilde{\chi}_\varepsilon \right)^2 \right) \Bigg\} \widetilde{s}\, \mathrm{d}\widetilde{s}\mathrm{d}y_3, \tag{4.9}$$

with $\widetilde{\chi}_\varepsilon(\widetilde{s}, y_3) = \varepsilon \widetilde{s} \chi_\varepsilon(\widetilde{s}, y_3)$. Using the Max-Min principle (Proposition 2.3), after the change of variables, $\mu_k^S(\varepsilon)$ is rewritten as

$$\mu_k^S(\varepsilon) = \sup_{Z \in \mathcal{H}_{k-1}(\Omega_{1,a}, \mathbb{R}^3)} \inf\{\widetilde{\mathcal{R}}_\varepsilon^S(\Phi) \mid \Phi \in \mathfrak{H}_\mathrm{S} \backslash \{\mathbf{0}\}, \Phi \perp Z \text{ in } L^2(\Omega_{1,a})\}.$$

Let $\{\tau^{(n)}(y_3)\}_{n=1}^{+\infty}$ be a linearly independent system satisfying $\tau^{(n)} = \tau^{(n)}(y_3) \in W_{1,a}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $v_S^{(n)} = (0, 0, \tau^{(n)}) \in \mathfrak{H}_\mathrm{S}$. We fix $k \in \mathbb{N}$. Choose an arbitrary $Z \in \mathcal{H}_{k-1}(\Omega_{1,a}, \mathbb{R}^3)$ and let $\widetilde{Z} = L.H. \left[v_S^{(1)}, v_S^{(2)}, \ldots, v_S^{(k)} \right]$ be the minimal linear space that contains the set $\{v_S^{(1)}, v_S^{(2)}, \ldots, v_S^{(k)}\}$. Note that $\dim \widetilde{Z} = k$ and that each $v_S^{(n)} \in \mathcal{W}_{1,a}$ (for all $n \in \mathbb{N}$), so we have that $\widetilde{Z} \subseteq \mathcal{W}_{1,a}$. Since $\dim Z < \dim \widetilde{Z}$, we know that there exist a function $\Psi \in \widetilde{Z} \cap Z^\perp$ and a vector $(c_1(\varepsilon), \ldots, c_k(\varepsilon)) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$, $\varepsilon > 0$, such that

$$\Psi = \sum_{n=1}^{k} c_s(\varepsilon) v_S^{(n)}.$$

Note that since both $\widetilde{Z}$ and $Z^\perp$ are subsets of $\mathcal{W}_{1,a}$, we have also that $\Psi \in \mathcal{W}_{1,a}$ and due the fact that $(c_1(\varepsilon), \ldots, c_k(\varepsilon)) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ we deduce that $\Psi \in \mathcal{W}_{1,a} \backslash \{\mathbf{0}\}$, so we can apply $\widetilde{\mathcal{R}}_\varepsilon^S$ to Ψ. We compute $\widetilde{\mathcal{R}}_\varepsilon^S(\Psi)$ and obtain

$$\begin{aligned} \widetilde{\mathcal{R}}_\varepsilon^S(\Psi) &= \frac{\displaystyle\int_{\Omega_{1,a}} (\lambda_1 + 2\lambda_2) \left(\sum_{n=1}^{k} c_s(\varepsilon) \frac{\mathrm{d}\tau^{(n)}}{\mathrm{d}y_3} \right)^2 \widetilde{s}\, \mathrm{d}\widetilde{s}\mathrm{d}y_3}{\displaystyle\int_{\Omega_{1,a}} \left(\sum_{n=1}^{k} c_s(\varepsilon) \tau^{(n)} \right)^2 \widetilde{s}\, \mathrm{d}\widetilde{s}\mathrm{d}y_3} \\ &= \frac{\displaystyle\int_{\Omega_{1,a}} (\lambda_1 + 2\lambda_2) \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \frac{\mathrm{d}\tau^{(p)}}{\mathrm{d}y_3} \frac{\mathrm{d}\tau^{(q)}}{\mathrm{d}y_3} \widetilde{s}\, \mathrm{d}\widetilde{s}\mathrm{d}y_3}{\displaystyle\int_{\Omega_{1,a}} \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \tau^{(p)} \tau^{(q)} \widetilde{s}\, \mathrm{d}\widetilde{s}\mathrm{d}y_3} \end{aligned} \tag{4.10}$$

Let us put

$$\gamma_{pq} = \int_{\Omega_{1,a}} \frac{\mathrm{d}\tau^{(p)}}{\mathrm{d}y_3} \frac{\mathrm{d}\tau^{(q)}}{\mathrm{d}y_3} \widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}y_3, \quad \widehat{\gamma}_{pq} = \int_{\Omega_{1,a}} \tau^{(p)}\tau^{(q)}\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}y_3.$$

Note that since we chose the system $\left\{\tau^{(n)}\right\}_{n=1}^{+\infty}$ to be linearly independent and by the symmetry $\gamma_{pq} = \gamma_{qp}$, $\widehat{\gamma}_{pq} = \widehat{\gamma}_{qp}$, we have that $(\gamma_{pq})_{1\le p,q\le k}$ and $(\widehat{\gamma}_{pq})_{1\le p,q\le k}$ are positive definite matrices. Therefore, all of its eigenvalues are positive. Let γ_* be the biggest eigenvalue of $(\gamma_{pq})_{1\le p,q\le k}$ and $\widehat{\gamma}_*$, the smallest eigenvalue of $(\widehat{\gamma}_{pq})_{1\le p,q\le k}$. With this notation, we have the bounds

$$\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\gamma_{pq} \le \gamma_*(c_1(\varepsilon)^2 + \cdots + c_k(\varepsilon)^2),$$
$$\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widehat{\gamma}_{pq} \ge \widehat{\gamma}_*(c_1(\varepsilon)^2 + \cdots + c_k(\varepsilon)^2).$$

Therefore, (4.10) becomes

$$\widetilde{\mathcal{R}}^S_\varepsilon(\Psi) = \frac{(\lambda_1 + 2\lambda_2)\displaystyle\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\gamma_{pq}}{\displaystyle\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widehat{\gamma}_{pq}} \le \frac{(\lambda_1 + 2\lambda_2)\gamma_*(c_1(\varepsilon)^2 + \cdots + c_k(\varepsilon)^2)}{\widehat{\gamma}_*(c_1(\varepsilon)^2 + \cdots + c_k(\varepsilon)^2)} = \frac{(\lambda_1 + 2\lambda_2)\gamma_*}{\widehat{\gamma}_*}.$$

Put $C = \frac{(\lambda_1+2\lambda_2)\gamma_*}{\widehat{\gamma}_*}$. We obtained that for a certain $\Psi \in \mathcal{W}_{1,a}$ there exists a positive constant C independent of ε and independent of the choice of Z such that $\widetilde{\mathcal{R}}^S_\varepsilon(\Psi) \le C$. Thus, taking the infimum, we have

$$\inf\{\widetilde{\mathcal{R}}^S_\varepsilon(\Phi) \mid \Phi \in \mathfrak{H}_{\mathrm{S}}\backslash\{\mathbf{0}\}, \Phi \perp Z \text{ in } L^2(\Omega_{1,a})\} \le \widetilde{\mathcal{R}}^S_\varepsilon(\Psi) \le C.$$

Since Z was arbitrary and C does not depend on the choice of Z, we can take the supremum on both sides over $\mathcal{H}_{k-1}(\Omega_{1,a}, \mathbb{R}^3)$ to obtain

$$\mu^S_k(\varepsilon) = \sup_{Z\in\mathcal{H}_{k-1}(\Omega_{1,a},\mathbb{R}^3)} \inf\{\widetilde{\mathcal{R}}^S_\varepsilon(\Phi) \mid \Phi \in \mathfrak{H}_{\mathrm{S}}\backslash\{\mathbf{0}\}, \Phi \perp Z \text{ in } L^2(\Omega_{1,a})\} \le C.$$

Here we used the characterization of $\mu^S_k(\varepsilon)$. Therefore we obtain

$$\mu^S_k(\varepsilon) = O(1) \quad \text{as } \varepsilon \to 0, \tag{4.11}$$

that is, $\mu^S_k(\varepsilon)$ is bounded for $\varepsilon > 0$ and for each fixed $k \in \mathbb{N}$. Thus, there exists a subsequence, still denoted by ε, such that $\lim_{\varepsilon\to 0} \mu^S_k(\varepsilon) = \widetilde{\mu}^S_k$.

Remark 4.2. Since bending eigenvalues satisfy $\lim_{\varepsilon\to 0} \mu_k(\varepsilon) = 0$ (Theorem 3.1), if we see that $\widetilde{\mu}^S_k \ne 0$, that would mean that for every $k \in \mathbb{N}$ there exists a sequence $q(k, \varepsilon) \to +\infty$ as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} \mu_{q(k,\varepsilon)}(\varepsilon) = \widetilde{\mu}^S_k.$$

We will see a similar result for torsional eigenvalues.

Analogously, we define $\mathfrak{H}_{\mathrm{T}}$ to be the smallest closed subspace of $\mathcal{W}_{1,a}$ such that if $v_\varepsilon^T = (v_{1,\varepsilon}^T, v_{2,\varepsilon}^T, v_{3,\varepsilon}^T) \in \mathfrak{H}_{\mathrm{T}}$, then

$$(v_{1,\varepsilon}^T, v_{2,\varepsilon}^T, v_{3,\varepsilon}^T) = (-y_2\rho_\varepsilon(\widetilde{s}, x_3), y_1\rho_\varepsilon(\widetilde{s}, y_3), 0),$$

for some $\rho_\varepsilon \in W_{1,a}$. Let $v_\varepsilon^T \in \mathfrak{H}_{\mathrm{T}}$. The Rayleigh quotient for the torsional mode is rewritten as

$$\widetilde{\mathcal{R}}_\varepsilon^T(v_\varepsilon^T) = \frac{\displaystyle\int_{\Omega_{1,a}} \lambda_2 \widetilde{s}^3 \left(\frac{1}{\varepsilon^2} \left(\frac{\partial \rho_\varepsilon}{\partial \widetilde{s}} \right)^2 + \left(\frac{\partial \rho_\varepsilon}{\partial y_3} \right)^2 \right) \mathrm{d}\widetilde{s}\mathrm{d}y_3}{\displaystyle\int_{\Omega_{1,a}} \widetilde{s}^3 \rho_\varepsilon^2 \, \mathrm{d}\widetilde{s}\mathrm{d}y_3}. \tag{4.12}$$

After the change of variables, $\mu_k^T(\varepsilon)$ is rewritten as

$$\mu_k^T(\varepsilon) = \sup_{Z \in \mathcal{H}_{k-1}(\Omega_{1,a}, \mathbb{R}^3)} \inf\{\widetilde{\mathcal{R}}_\varepsilon^T(\Phi) \mid \Phi \in \mathfrak{H}_{\mathrm{T}} \backslash \{\mathbf{0}\}, \Phi \perp Z \text{ in } L^2(\Omega_{1,a})\}.$$

Let $\{\rho^{(n)}(y_3)\}_{n=1}^{+\infty}$ be a linearly independent system satisfying $\rho^{(n)} = \rho^{(n)}(y_3) \in W_{1,a}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $v_T^{(n)} = (-y_2\rho_\varepsilon, y_1\rho_\varepsilon, 0) \in \mathfrak{H}_{\mathrm{T}}$. For each fixed $k \in \mathbb{N}$, we repeat a similar process as before, to obtain that

$$\mu_k^T(\varepsilon) = O(1) \quad \text{ as } \varepsilon \to 0,$$

that is, $\mu_k^T(\varepsilon)$ is bounded for $\varepsilon > 0$ and for each fixed $k \in \mathbb{N}$. Thus, there exists a subsequence, still denoted by ε, such that $\lim_{\varepsilon \to 0} \mu_k^T(\varepsilon) = \widetilde{\mu}_k^T$.

Our goal is to study the limit eigenvalue problem of the sequence $\{\widetilde{\mu}_k^S\}_{k=1}^{+\infty} \cup \{\widetilde{\mu}_k^T\}_{k=1}^{+\infty}$, so we restrict ourselves to torsional and stretching eigenvalues.

4.3 L^2 convergence of the eigenfunctions

We prove that the eigenfunctions related to torsional and stretching modes of (4.1) have a non-zero limit. Note that in Rayleigh's quotient in (4.4) there are no cross-terms involving ρ_ε and $(\chi_\varepsilon, \tau_\varepsilon)$ (or its derivatives). Thus, we split the computations into two parts.

4.3.1 Stretching mode

We deal with the stretching mode. We fix $k \in \mathbb{N}$. Let $\mu_k^S(\varepsilon)$ be the k-th stretching eigenvalue of the problem (4.1) and let $v_\varepsilon^{(k)}(x) = v_\varepsilon^{(k)} = (v_{1,\varepsilon}^{(k)}, v_{2,\varepsilon}^{(k)}, v_{3,\varepsilon}^{(k)})$, $x \in \Omega_{\varepsilon,a}$ with

$$v_\varepsilon^{(k)} = (x_1\chi_\varepsilon^{(k)}(s, x_3), x_2\chi_\varepsilon^{(k)}(s, x_3), \tau_\varepsilon^{(k)}(s, x_3)),$$

be its associated stretching eigenfunction. Here $s = \sqrt{x_1^2 + x_2^2}$. For commodity we write $s\chi_\varepsilon^{(k)} = \widetilde{\chi}_\varepsilon^{(k)}$. We change the domain to $\Omega_{1,a}$, hence we perform the change of variable $s = \varepsilon\widetilde{s}$. The previous functions in the new domain become $\overline{v}_\varepsilon^{(k)}(\widetilde{s}, y_3) = v_\varepsilon^{(k)}(\varepsilon\widetilde{s}, y_3)$, $\overline{\widetilde{\chi}}_\varepsilon^{(k)}(\widetilde{s}, y_3) = \widetilde{\chi}_\varepsilon^{(k)}(\varepsilon\widetilde{s}, y_3)$ and $\overline{\tau}_\varepsilon^{(k)}(\widetilde{s}, y_3) = \tau_\varepsilon^{(k)}(\varepsilon\widetilde{s}, y_3)$ with $0 < y_3 < l$, $0 < \widetilde{s} < a(y_3)$. However, for commodity purposes, we abuse notation and we still denote the new functions in $W_{1,a}$ by $\widetilde{\chi}_\varepsilon^{(k)}, \tau_\varepsilon^{(k)}$. With this notation, the weak formulation for the stretching eigenfunction $v_\varepsilon^{(k)}$ is

$$\begin{aligned}\int_{\Omega_{1,a}} \Bigg\{ &\lambda_1 \left(\frac{1}{\varepsilon} \frac{\partial \widetilde{\chi}_\varepsilon^{(k)}}{\partial \widetilde{s}} + \frac{1}{\varepsilon} \frac{1}{\widetilde{s}} \widetilde{\chi}_\varepsilon^{(k)} + \frac{\partial \tau_\varepsilon^{(k)}}{\partial x_3} \right) \left(\frac{1}{\varepsilon} \frac{\partial \widetilde{\psi}}{\partial \widetilde{s}} + \frac{1}{\varepsilon} \frac{1}{\widetilde{s}} \widetilde{\psi} + \frac{\partial \varphi}{\partial x_3} \right) \\ &+ 2\lambda_2 \left(\frac{\partial \tau_\varepsilon^{(k)}}{\partial x_3} \frac{\partial \varphi}{\partial x_3} + \frac{1}{2} \left(\frac{\partial \widetilde{\chi}_\varepsilon^{(k)}}{\partial x_3} + \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon^{(k)}}{\partial \widetilde{s}} \right) \left(\frac{\partial \widetilde{\psi}}{\partial x_3} + \frac{1}{\varepsilon} \frac{\partial \varphi}{\partial \widetilde{s}} \right) + \frac{1}{\varepsilon^2} \frac{\partial \widetilde{\chi}_\varepsilon^{(k)}}{\partial \widetilde{s}} \frac{\partial \widetilde{\psi}}{\partial \widetilde{s}} + \frac{1}{\varepsilon^2} \frac{1}{\widetilde{s}^2} \widetilde{\chi}_\varepsilon^{(k)} \widetilde{\psi} \right) \Bigg\} \widetilde{s} \,\mathrm{d}\widetilde{s} \mathrm{d}x_3 \\ &= \mu_k^S(\varepsilon) \int_{\Omega_{1,a}} \left(\left(\widetilde{\chi}_\varepsilon^{(k)} \right)^2 + \left(\tau_\varepsilon^{(k)} \right)^2 \right) \widetilde{s} \,\mathrm{d}\widetilde{s} \mathrm{d}x_3, \end{aligned} \tag{4.13}$$

where $\widetilde{\psi}, \varphi \in W_{1,a}$ are test functions. We put $\widetilde{\psi} = \widetilde{\chi}_\varepsilon^{(k)}$, $\varphi = \tau_\varepsilon^{(k)}$ in (4.13) and obtain

$$\begin{aligned}\int_{\Omega_{1,a}} \Bigg\{ &\lambda_1 \left(\frac{1}{\varepsilon} \frac{\partial \widetilde{\chi}_\varepsilon^{(k)}}{\partial \widetilde{s}} + \frac{1}{\varepsilon} \frac{1}{\widetilde{s}} \widetilde{\chi}_\varepsilon^{(k)} + \frac{\partial \tau_\varepsilon^{(k)}}{\partial x_3} \right)^2 \\ &+ 2\lambda_2 \left(\left(\frac{\partial \tau_\varepsilon^{(k)}}{\partial x_3} \right)^2 + \frac{1}{2} \left(\frac{\partial \widetilde{\chi}_\varepsilon^{(k)}}{\partial x_3} + \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon^{(k)}}{\partial \widetilde{s}} \right)^2 + \left(\frac{1}{\varepsilon} \frac{\partial \widetilde{\chi}_\varepsilon^{(k)}}{\partial \widetilde{s}} \right)^2 + \left(\frac{1}{\varepsilon} \frac{1}{\widetilde{s}} \widetilde{\chi}_\varepsilon^{(k)} \right)^2 \right) \Bigg\} \widetilde{s} \,\mathrm{d}\widetilde{s} \mathrm{d}x_3 \\ &= \mu_k^S(\varepsilon) \int_{\Omega_{1,a}} \left(\left(\widetilde{\chi}_\varepsilon^{(k)} \right)^2 + \left(\tau_\varepsilon^{(k)} \right)^2 \right) \widetilde{s} \,\mathrm{d}\widetilde{s} \mathrm{d}x_3. \end{aligned} \tag{4.14}$$

We fix the norm

$$\int_{\Omega_{1,a}} \left(\left(\widetilde{\chi}_\varepsilon^{(k)} \right)^2 + \left(\tau_\varepsilon^{(k)} \right)^2 \right) \widetilde{s} \,\mathrm{d}\widetilde{s} \mathrm{d}x_3 = 1. \tag{4.15}$$

Let $(\varepsilon_p)_{p=1}^{+\infty}$ be any positive sequence such that $\varepsilon_p \to 0$ as $p \to +\infty$. From (4.11), we know that the eigenvalue $\mu_k^S(\varepsilon_p)$ is bounded. Therefore, from (4.14) we deduce that

$$\frac{\partial \tau_{\varepsilon_p}^{(k)}}{\partial x_3} = O(1), \tag{4.16}$$

$$\frac{\partial \widetilde{\chi}_{\varepsilon_p}^{(k)}}{\partial x_3} + \frac{1}{\varepsilon_p} \frac{\partial \tau_{\varepsilon_p}^{(k)}}{\partial \widetilde{s}} = O(1), \tag{4.17}$$

$$\frac{\partial \widetilde{\chi}_{\varepsilon_p}^{(k)}}{\partial \widetilde{s}} = O(\varepsilon_p), \tag{4.18}$$

$$\frac{1}{\widetilde{s}} \widetilde{\chi}_{\varepsilon_p}^{(k)} = O(\varepsilon_p), \tag{4.19}$$

as $p \to +\infty$ in the $L^2(\Omega_{1,a})$ sense. From (4.18) we know that $\frac{\partial \widetilde{\chi}_{\varepsilon_p}^{(k)}}{\partial \widetilde{s}}$ is bounded in $L^2(\Omega_{1,a})$, so $\frac{\partial^2 \widetilde{\chi}_{\varepsilon_p}^{(k)}}{\partial x_3 \partial \widetilde{s}}$ is bounded in $H^{-1}(\Omega_{1,a})$. On the other hand, from (4.17) we see that $\frac{\partial^2 \widetilde{\chi}_{\varepsilon_p}^{(k)}}{\partial x_3 \partial s} + \frac{1}{\varepsilon_p} \frac{\partial^2 \tau_{\varepsilon_p}^{(k)}}{\partial \widetilde{s}^2}$ is bounded in $H^{-1}(\Omega_{1,a})$. Combining this two facts, we get that $\frac{\partial^2 \tau_{\varepsilon_p}^{(k)}}{\partial \widetilde{s}^2}$ is bounded in $H^{-1}(\Omega_{1,a})$. From (4.16) we know that $\frac{\partial \tau_{\varepsilon_p}^{(k)}}{\partial x_3}$ is bounded in $L^2(\Omega_{1,a})$, so $\frac{\partial^2 \tau_{\varepsilon_p}^{(k)}}{\partial x_3 \partial \widetilde{s}}$ is bounded in $H^{-1}(\Omega_{1,a})$.

Moreover, since $\tau_{\varepsilon_p}^{(k)}$ is bounded in $L^2(\Omega_{1,a})$, we see that $\frac{\partial\tau_{\varepsilon_p}^{(k)}}{\partial\widetilde{s}}$ is bounded in $H^{-1}(\Omega_{1,a})$. To sum up, $\frac{\partial\tau_{\varepsilon_p}^{(k)}}{\partial\widetilde{s}}$, $\frac{\partial^2\tau_{\varepsilon_p}^{(k)}}{\partial\widetilde{s}^2}$ and $\frac{\partial^2\tau_{\varepsilon_p}^{(k)}}{\partial x_3\partial\widetilde{s}}$ are all bounded in $H^{-1}(\Omega_{1,a})$, so we deduce that $\frac{\partial\tau_{\varepsilon_p}^{(k)}}{\partial\widetilde{s}}$ is bounded in $L^2(\Omega_{1,a})$. Note that $\tau_{\varepsilon_p}^{(k)}$, $\frac{\partial\tau_{\varepsilon_p}^{(k)}}{\partial\widetilde{s}}$ and $\frac{\partial\tau_{\varepsilon_p}^{(k)}}{\partial x_3}$ are bounded in $L^2(\Omega_{1,a})$, hence $\tau_{\varepsilon_p}^{(k)}$ is bounded in $H^1(\Omega_{1,a})$. Therefore, there exists a subsequence $(\varepsilon_{p(q)})_{q=1}^{+\infty} \subseteq (\varepsilon_p)$ such that

$$\tau_{\varepsilon_{p(q)}}^{(k)} \rightharpoonup \tau^{(k)} \text{ weakly in } H^1(\Omega_{1,a}) \text{ as } q \to +\infty. \tag{4.20}$$

From Rellich's theorem, we moreover know that

$$\tau_{\varepsilon_{p(q)}}^{(k)} \to \tau^{(k)} \text{ strongly in } L^2(\Omega_{1,a}) \text{ as } q \to +\infty.$$

We combine this with (4.15) and (4.19) to see that $\|\tau^{(k)}\|_{L^2(\Omega_{1,a})} = 1$ and we conclude that $\tau_{\varepsilon_{p(q)}}^{(k)}$ converges strongly in $L^2(\Omega_{1,a})$ to a non-zero function $\tau^{(k)}$. From (4.17) we deduce that $\frac{\partial\tau^{(k)}}{\partial\widetilde{s}} = 0$, so we have $\tau^{(k)} = \tau^{(k)}(x_3)$. Furthermore, in virtue of the boundedness of $\mu_k^S(\varepsilon_p)$, there exists an even further subsequence $(\zeta_r)_{r=1}^{+\infty} \subseteq (\varepsilon_{p(q)})$ and a constant $\widetilde{\mu}_k^S$ such that

$$\lim_{r\to+\infty} \mu_k^S(\zeta_r) = \widetilde{\mu}_k^S.$$

We deduce the limit equation for the stretching mode. To simplify notation, we define $\phi_\varepsilon^{(k)}$, $\Phi_\varepsilon^{(k)}$, $\Phi_\varepsilon^{(k)}$ by the following relations.

$$\begin{aligned}
\phi_\varepsilon^{(k)} &= \frac{1}{\varepsilon\widetilde{s}}\widetilde{\chi}_\varepsilon^{(k)},\\
\Phi_\varepsilon^{(k)} &= \widetilde{s}\frac{\partial\phi_\varepsilon^{(k)}}{\partial\widetilde{s}} + 2\phi_\varepsilon^{(k)} + \frac{\partial\tau_\varepsilon^{(k)}}{\partial x_3},\\
\Psi_\varepsilon^{(k)} &= \varepsilon\widetilde{s}\frac{\partial\phi_\varepsilon^{(k)}}{\partial x_3} + \frac{1}{\varepsilon}\frac{\partial\tau_\varepsilon^{(k)}}{\partial\widetilde{s}}.
\end{aligned} \tag{4.21}$$

From (4.16)-(4.19), we note that $\phi_{\zeta_r}^{(k)} = O(1)$, $\Phi_{\zeta_r}^{(k)} = O(1)$ and $\Psi_{\zeta_r}^{(k)} = O(1)$ in $L^2(\Omega_{1,a})$ as $r \to +\infty$. We write $\phi^{(k)}$, $\Phi^{(k)}$, $\Psi^{(k)}$ their respective (weak-$L^2(\Omega_{1,a})$) limit. The weak formulation is then written as follows.

$$\begin{aligned}
&\int_{\Omega_{1,a}} \left\{ \lambda_1\Phi_\varepsilon^{(k)}\left(\tfrac{1}{\varepsilon}\widetilde{s}\tfrac{\partial\psi}{\partial\widetilde{s}} + \tfrac{2}{\varepsilon}\psi + \tfrac{\partial\varphi}{\partial x_3}\right)\right.\\
&\quad \left. +2\lambda_2\left(\tfrac{\partial\tau_\varepsilon^{(k)}}{\partial x_3}\tfrac{\partial\varphi}{\partial x_3} + \tfrac{1}{2}\Psi_\varepsilon^{(k)}\left(\widetilde{s}\tfrac{\partial\psi}{\partial x_3} + \tfrac{1}{\varepsilon}\tfrac{\partial\varphi}{\partial\widetilde{s}}\right) + \tfrac{1}{\varepsilon}\left(\widetilde{s}\tfrac{\partial\phi_\varepsilon^{(k)}}{\partial\widetilde{s}} + \phi_\varepsilon^{(k)}\right)\left(\widetilde{s}\tfrac{\partial\psi}{\partial\widetilde{s}} + \psi\right) + \tfrac{1}{\varepsilon}\phi_\varepsilon^{(k)}\psi\right)\right\}\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3\\
&= \mu_k^S(\varepsilon)\int_{\Omega_{1,a}} \left(\widetilde{s}^2\varepsilon\phi_\varepsilon^{(k)}\psi + \tau_\varepsilon^{(k)}\varphi\right)\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3,
\end{aligned} \tag{4.22}$$

where $\psi, \varphi \in W_{1,a}$ are test functions. We put $\varepsilon = \zeta_r$ in (4.22). We let $\psi = 0$ and (4.22) becomes

$$\int_{\Omega_{1,a}} \left\{ \lambda_1\Phi_{\zeta_r}^{(k)}\frac{\partial\varphi}{\partial x_3} + 2\lambda_2\left(\frac{\partial\tau_{\zeta_r}^{(k)}}{\partial x_3}\frac{\partial\varphi}{\partial x_3} + \frac{1}{2}\Psi_{\zeta_r}^{(k)}\frac{1}{\zeta_r}\frac{\partial\varphi}{\partial\widetilde{s}}\right)\right\}\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3 = \mu_k^S(\zeta_r)\int_{\Omega_{1,a}} \tau_{\zeta_r}^{(k)}\varphi\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3.$$

We let $\varphi = \varphi(x_3)$ and take the limit $r \to +\infty$. Recall that $\frac{\partial \tau^{(k)}}{\partial \widetilde{s}} = 0$ and $\mu_k^S(\zeta_r) \to \widetilde{\mu}_k^S$. We obtain

$$\int_{\Omega_{1,a}} \left(\lambda_1 \Phi^{(k)} \frac{\mathrm{d}\varphi}{\mathrm{d}x_3} + 2\lambda_2 \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} \frac{\mathrm{d}\varphi}{\mathrm{d}x_3} \right) \widetilde{s} \, \mathrm{d}\widetilde{s} \mathrm{d}x_3 = \widetilde{\mu}_k^S \int_{\Omega_{1,a}} \tau^{(k)} \varphi \widetilde{s} \, \mathrm{d}\widetilde{s} \mathrm{d}x_3.$$

Using the definition of $\Phi_\varepsilon^{(k)}$, we know that $\Phi^{(k)} = \widetilde{s}\frac{\partial \phi^{(k)}}{\partial \widetilde{s}} + 2\phi^{(k)} + \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}$. We substitute this expression in the previous equation and we get

$$\int_{\Omega_{1,a}} \left(\lambda_1 \left(\widetilde{s} \left(\widetilde{s} \frac{\partial \phi^{(k)}}{\partial \widetilde{s}} + 2\phi^{(k)} + \frac{\mathrm{d}\tau}{\mathrm{d}x_3} \right) \frac{\mathrm{d}\varphi}{\mathrm{d}x_3} \right) + 2\lambda_2 \widetilde{s} \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} \frac{\mathrm{d}\varphi}{\mathrm{d}x_3} \right) \mathrm{d}\widetilde{s} \mathrm{d}x_3 = \widetilde{\mu}_k^S \int_{\Omega_{1,a}} \tau^{(k)} \varphi \widetilde{s} \, \mathrm{d}\widetilde{s} \mathrm{d}x_3.$$

We integrate respect to $\widetilde{s}$ to see

$$\int_0^l \left(\lambda_1 a(x_3)^2 \phi^{(k)}(a(x_3), x_3) + (\lambda_1 + 2\lambda_2) \frac{a(x_3)^2}{2} \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} \right) \frac{\mathrm{d}\varphi}{\mathrm{d}x_3} \, \mathrm{d}x_3 = \widetilde{\mu}_k^S \int_0^l \frac{a(x_3)^2}{2} \tau^{(k)} \varphi \, \mathrm{d}x_3. \tag{4.23}$$

We go back to equation (4.22), and we put $\varphi = 0$ and $\psi = \zeta_r \eta$ so it becomes

$$\int_{\Omega_{1,a}} \left\{ \lambda_1 \Phi_{\zeta_r}^{(k)} \left(\widetilde{s} \frac{\partial \eta}{\partial \widetilde{s}} + 2\eta \right) + 2\lambda_2 \left(\frac{\zeta_r}{2} \Psi_{\zeta_r}^{(k)} \frac{\partial \eta}{\partial x_3} + \left(\widetilde{s} \frac{\partial \phi_{\zeta_r}^{(k)}}{\partial \widetilde{s}} + \phi_{\zeta_r}^{(k)} \right) \left(\widetilde{s} \frac{\partial \eta}{\partial \widetilde{s}} + \eta \right) + \phi_{\zeta_r}^{(k)} \eta \right) \right\} \widetilde{s} \, \mathrm{d}\widetilde{s} \mathrm{d}x_3$$
$$= \widetilde{\mu}_k^S(\zeta_r) \int_{\Omega_{1,a}} \widetilde{s}^2 \zeta_r^2 \phi_{\zeta_r}^{(k)} \eta \widetilde{s} \, \mathrm{d}\widetilde{s} \mathrm{d}x_3.$$

We put $\eta = \eta(x_3)$. Then the previous equation becomes

$$\int_{\Omega_{1,a}} \left\{ \lambda_1 \Phi_{\zeta_r}^{(k)} \cdot 2\eta + 2\lambda_2 \left(\frac{\zeta_r}{2} \Psi_{\zeta_r}^{(k)} \frac{\partial \eta}{\partial x_3} + \left(\widetilde{s} \frac{\partial \phi_{\zeta_r}^{(k)}}{\partial \widetilde{s}} + \phi_{\zeta_r}^{(k)} \right) \eta + \phi_{\zeta_r}^{(k)} \eta \right) \right\} \widetilde{s} \, \mathrm{d}\widetilde{s} \mathrm{d}x_3$$
$$= \mu_k^S(\zeta_r) \int_{\Omega_{1,a}} \widetilde{s}^2 \zeta_r^2 \phi_{\zeta_r}^{(k)} \eta \widetilde{s} \, \mathrm{d}s \mathrm{d}x_3.$$

We take the limit $r \to +\infty$ and we see

$$\int_{\Omega_{1,a}} \left(2\lambda_1 \Phi^{(k)} \eta \widetilde{s} + 2\lambda_2 \left(\widetilde{s} \frac{\partial \phi^{(k)}}{\partial \widetilde{s}} + 2\phi^{(k)} \right) \eta \widetilde{s} \right) \mathrm{d}\widetilde{s} \mathrm{d}x_3 = 0,$$
$$\int_{\Omega_{1,a}} \left(2\lambda_1 \left(\frac{\partial}{\partial \widetilde{s}} \left(\widetilde{s}^2 \phi^{(k)} \right) \eta + \widetilde{s} \frac{\mathrm{d}\tau}{\mathrm{d}x_3} \eta \right) + 2\lambda_2 \frac{\partial}{\partial \widetilde{s}} \left(\widetilde{s}^2 \phi^{(k)} \right) \eta \right) \mathrm{d}\widetilde{s} \mathrm{d}x_3 = 0,$$
$$\int_{\Omega_{1,a}} \left((2\lambda_1 + 2\lambda_2) \frac{\partial}{\partial \widetilde{s}} \left(\widetilde{s}^2 \phi^{(k)} \right) + 2\lambda_1 \widetilde{s} \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} \right) \eta \, \mathrm{d}\widetilde{s} \mathrm{d}x_3 = 0.$$

We integrate with respect to $\widetilde{s}$ and get

$$\int_0^l \left(2(\lambda_1 + \lambda_2) a(x_3)^2 \phi^{(k)}(a(x_3), x_3) + 2\lambda_1 \frac{a(x_3)^2}{2} \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} \right) \eta \, \mathrm{d}x_3 = 0.$$

Since η is an arbitrary test function, we deduce

$$2(\lambda_1 + \lambda_2) a(x_3)^2 \phi^{(k)}(a(x_3), x_3) + \lambda_1 a(x_3)^2 \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} = 0.$$

We know that $a(x_3)$ is a positive function so we obtain the relation

$$\phi^{(k)}(a(x_3), x_3) = -\frac{1}{2} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}. \tag{4.24}$$

We substitute this relation into (4.23) and conclude

$$Y \int_0^l a(x_3)^2 \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} \frac{\mathrm{d}\varphi}{\mathrm{d}x_3} \mathrm{d}x_3 = \widetilde{\mu}_k^S \int_0^l a(x_3)^2 \tau^{(k)} \varphi \, \mathrm{d}x_3. \tag{4.25}$$

We recall that $Y = \frac{\lambda_2(3\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}$. The test function $\varphi(x_3) \in W_{1,a}$ satisfies $\varphi(0) = \varphi(l) = 0$, so we use the integration by parts and deduce

$$-Y \int_0^l \frac{\mathrm{d}}{\mathrm{d}x_3} \left(a(x_3)^2 \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} \right) \varphi \, \mathrm{d}x_3 = \widetilde{\mu}_k^S \int_0^l a(x_3)^2 \tau^{(k)} \varphi \, \mathrm{d}x_3.$$

Since φ is an arbitrary test function, we get

$$-Y \frac{\mathrm{d}}{\mathrm{d}x_3} \left(a(x_3)^2 \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} \right) = \widetilde{\mu}_k^S a(x_3)^2 \tau^{(k)}. \tag{4.26}$$

Moreover, note that the eigenfunction $v_\varepsilon^{(k)}$ satisfies the Dirichlet conditions $v_\varepsilon^{(k)} = \mathbf{0}$ on $\Gamma_{1,\varepsilon,a}$. Since $v_{3,\varepsilon}^{(k)} = \tau_\varepsilon^{(k)}$, we have that the limit function $\tau^{(k)}$ satisfies $\tau^{(k)}(0) = \tau^{(k)}(l) = 0$. We combine these boundary conditions with (4.26) and conclude that the pair $(\widetilde{\mu}_k^S, \tau^{(k)})$ satisfies the eigenvalue problem

$$\begin{cases} -Y \frac{\mathrm{d}}{\mathrm{d}x_3} \left(a(x_3)^2 \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3} \right) = \widetilde{\mu}_k^S a(x_3)^2 \tau^{(k)} \quad (0 < x_3 < l), \\ \tau^{(k)}(0) = \tau^{(k)}(l) = 0. \end{cases}$$

Note however, that these eigenvalues $\{\widetilde{\mu}_k^S\}_{k=1}^{+\infty}$ might not be all the eigenvalues of the previous eigenvalue problem, in other words, if we let $\{\mu_k^S\}_{k=1}^{+\infty}$ be the set of all eigenvalues of the eigenvalue problem

$$\begin{cases} -Y \frac{\mathrm{d}}{\mathrm{d}y_3} \left(a(y_3)^2 \frac{\mathrm{d}\tau}{\mathrm{d}y_3} \right) = \mu_k^S a(y_3)^2 \tau \quad (0 < y_3 < l), \\ \tau(0) = \tau(l) = 0, \end{cases}$$

then we proved $\{\widetilde{\mu}_k^S\}_{k=1}^{+\infty} \subseteq \{\mu_k^S\}_{k=1}^{+\infty}$. Since the sets are discrete, alternatively, we see that for every $k \in \mathbb{N}$, the inequality

$$\widetilde{\mu}_k^S \geq \mu_k^S \tag{4.27}$$

holds. Moreover, we note that $\mu_k^S \neq 0$, hence $\widetilde{\mu}_k^S > 0$. In Section 4.4.1 we will prove that the inequality $\widetilde{\mu}_k^S \leq \mu_k^S$ also holds.

Remark 4.3. The fact $\widetilde{\mu}_k^S > 0$ contrasts with the nature of bending eigenvalues. If $\mu_k(\varepsilon)$ is the k-th eigenvalue (without restricting to any mode) of (4.1), then we know from Section (3) that $\lim_{\varepsilon\to 0} \mu_k(\varepsilon) = 0$. However if we restrict ourselves to *stretching* eigenvalues, then $\lim_{\varepsilon\to 0} \mu_k^S(\varepsilon) \neq 0$. Therefore, there exists a sequence $q(k,\varepsilon) \to +\infty$ as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} \mu_{q(k,\varepsilon)}(\varepsilon) = \widetilde{\mu}_k^S.$$

The same statement holds for torsional eigenvalues, as we will see in the next part.

4.3.2 Torsional mode

We repeat a similar procedure for torsional vibrations. Let $\mu_k^T(\varepsilon)$ be the k-th torsional eigenvalue of the problem (4.1) and let $w_\varepsilon^{(k)} = (-x_2\rho_\varepsilon^{(k)}(s,x_3), x_1\rho_\varepsilon^{(k)}(s,x_3), 0)$ be its associated eigenfunction. As before, we perform the change of variables $s = \varepsilon\widetilde{s}$ and the weak formulation of the problem becomes

$$\lambda_2 \int_{\Omega_{1,a}} \widetilde{s}^2 \left(\frac{1}{\varepsilon^2} \frac{\partial\rho_\varepsilon^{(k)}}{\partial\widetilde{s}} \frac{\partial\theta}{\partial\widetilde{s}} + \frac{\partial\rho_\varepsilon^{(k)}}{\partial x_3} \frac{\partial\theta}{\partial x_3} \right) \widetilde{s}\, \mathrm{d}\widetilde{s}\mathrm{d}x_3 = \mu_k^T(\varepsilon) \int_{\Omega_{1,a}} \widetilde{s}^3 \rho_\varepsilon^{(k)} \theta\, \mathrm{d}\widetilde{s}\mathrm{d}x_3, \tag{4.28}$$

where $\theta \in W_{1,a}$ is a test function. We fix the norm

$$\int_{\Omega_{1,a}} \widetilde{s}^3 (\rho_\varepsilon^{(k)})^2 \mathrm{d}\widetilde{s}\mathrm{d}x_3 = 1.$$

As before, let $(\varepsilon_p)_{p=1}^{+\infty}$ be any positive sequence such that $\varepsilon_p \to$ as $p \to +\infty$. Since $\mu_k^T(\varepsilon_p)$ is bounded, from (4.28) with $\varepsilon = \varepsilon_p$ and $\theta = \rho_{\varepsilon_p}^{(k)}$, we have that $\frac{\partial\rho_{\varepsilon_p}^{(k)}}{\partial\widetilde{s}} = O(\varepsilon_p)$, $\frac{\partial\rho_{\varepsilon_p}^{(k)}}{\partial x_3} = O(1)$ in the $L^2(\Omega_{1,a})$ sense. Therefore, since $\rho_{\varepsilon_p}^{(k)}$, $\frac{\partial\rho_{\varepsilon_p}^{(k)}}{\partial\widetilde{s}}$ and $\frac{\partial\rho_{\varepsilon_p}^{(k)}}{\partial x_3}$ are bounded in $L^2(\Omega_{1,a})$, we have that $\rho_{\varepsilon_p}^{(k)}$ is bounded in $H^1(\Omega_{1,a})$. Therefore, there exists a subsequence $(\varepsilon_{p(q)})_{q=1}^{+\infty} \subseteq (\varepsilon_p)$ such that

$$\rho_{\varepsilon_{p(q)}}^{(k)} \rightharpoonup \rho^{(k)} \text{ weakly in } H^1(\Omega_{1,a}) \text{ as } q \to +\infty.$$

By Rellich's theorem, we furthermore know that

$$\rho_{\varepsilon_{p(q)}}^{(k)} \to \rho^{(k)} \text{ strongly in } L^2(\Omega_{1,a}) \text{ as } q \to +\infty.$$

From the strong convergence and $\|\rho^{(k)}\|_{L^2(\Omega_{1,a})} = 1$, we have that $\rho^{(k)}$ is a non-zero function. Note moreover that $\frac{\partial\rho^{(k)}}{\partial\widetilde{s}} = 0$ and in particular $\rho^{(k)} = \rho^{(k)}(x_3)$. Furthermore, in virtue of the boundedness of $\mu_k^T(\varepsilon_p)$, there exists an even further subsequence $(\zeta_r)_{r=1}^{+\infty} \subseteq (\varepsilon_{p(q)})$ and a constant $\widetilde{\mu}_k^T$ such that

$$\lim_{r\to+\infty} \mu_k^T(\zeta_r) = \widetilde{\mu}_k^T.$$

We remark that the subsequences $(\varepsilon_{p(q)})$ and (ζ_r) might not be the same as in the stretching case. However, with abuse of notation, we denote them the same way.

We now deduce the limit equation for the torsional mode. We put $\varepsilon = \zeta_r$, $\theta = \theta(x_3)$ in (4.28) and take the limit $r \to +\infty$. Recall that $\frac{\partial \rho^{(k)}}{\partial \widetilde{s}} = 0$ and $\mu_k^T(\zeta_r) \to \widetilde{\mu}_k^T$. We obtain

$$\lambda_2 \int_{\Omega_{1,a}} \widetilde{s}^3 \frac{\mathrm{d}\rho^{(k)}}{\mathrm{d}x_3} \frac{\mathrm{d}\theta}{\mathrm{d}x_3} \mathrm{d}\widetilde{s}\mathrm{d}x_3 = \widetilde{\mu}_k^T \int_{\Omega_{1,a}} \widetilde{s}^3 \rho^{(k)} \theta \, \mathrm{d}\widetilde{s}\mathrm{d}x_3,$$

$$\lambda_2 \int_0^l \frac{a(x_3)^4}{4} \frac{\mathrm{d}\rho^{(k)}}{\mathrm{d}x_3} \frac{\mathrm{d}\theta}{\mathrm{d}x_3} \mathrm{d}x_3 = \widetilde{\mu}_k^T \int_0^l \frac{a(x_3)^4}{4} \rho^{(k)} \theta \, \mathrm{d}x_3.$$

The test function $\theta(x_3) \in W_{1,a}$ satisfies $\theta(0) = \theta(l) = 0$, so we perform the integration by parts to see

$$-\lambda_2 \int_0^l \frac{\mathrm{d}}{\mathrm{d}x_3} \left(a(x_3)^4 \frac{\mathrm{d}\rho^{(k)}}{\mathrm{d}x_3} \right) \theta \, \mathrm{d}x_3 = \widetilde{\mu}_k^T \int_0^l a(x_3)^4 \rho^{(k)} \theta \, \mathrm{d}x_3.$$

Since θ is an arbitrary test function we deduce

$$-\lambda_2 \frac{\mathrm{d}}{\mathrm{d}x_3} \left(a(x_3)^4 \frac{\mathrm{d}\rho^{(k)}}{\mathrm{d}x_3} \right) = \widetilde{\mu}_k^T a(x_3)^4 \rho^{(k)}. \tag{4.29}$$

Moreover, note that the eigenfunction $w_\varepsilon^{(k)}$ satisfies the Dirichlet condition $w_\varepsilon^{(k)} = \mathbf{0}$ on $\Gamma_{1,\varepsilon,a}$. Since $(w_{1,\varepsilon}^{(k)}, w_{2,\varepsilon}^{(k)}) = (-x_2\rho_\varepsilon^{(k)}, x_1\rho_\varepsilon^{(k)})$, we have that the limit function $\rho^{(k)}$ satisfies $\rho^{(k)}(0) = \rho^{(k)}(l) = 0$. We combine these boundary conditions with (4.29) and conclude that the pair $(\widetilde{\mu}_k^T, \rho^{(k)})$ satisfies the eigenvalue problem

$$\begin{cases} -\lambda_2 \frac{\mathrm{d}}{\mathrm{d}x_3} \left(a(x_3)^4 \frac{\mathrm{d}\rho^{(k)}}{\mathrm{d}x_3} \right) = \widetilde{\mu}_k^T a(x_3)^4 \rho^{(k)} & (0 < x_3 < l), \\ \rho^{(k)}(0) = \rho^{(k)}(l) = 0. \end{cases}$$

Note however, that these eigenvalues $\{\widetilde{\mu}_k^T\}_{k=1}^{+\infty}$ might not be all the eigenvalues of the previous eigenvalue problem, in other words, if we let $\{\mu_k^T\}_{k=1}^{+\infty}$ be the set of all eigenvalues of the eigenvalue problem

$$\begin{cases} -\lambda_2 \frac{\mathrm{d}}{\mathrm{d}y_3} \left(a(y_3)^4 \frac{\mathrm{d}\rho}{\mathrm{d}y_3} \right) = \mu_k^T a(y_3)^4 \rho & (0 < y_3 < l), \\ \rho(0) = \rho(l) = 0, \end{cases}$$

then we proved $\{\widetilde{\mu}_k^T\}_{k=1}^{+\infty} \subseteq \{\mu_k^T\}_{k=1}^{+\infty}$. Since the sets are discrete, alternatively, we see that for every $k \in \mathbb{N}$, the inequality

$$\widetilde{\mu}_k^T \geq \mu_k^T$$

holds. Moreover, we note that $\mu_k^T \neq 0$, hence $\widetilde{\mu}_k^T$. In Section 4.4.2 we will prove that the inequality $\widetilde{\mu}_k^T \leq \mu_k^T$ also holds.

4.4 Upper bound for the limit eigenvalues

From Section 4.2 we know that the problem (4.1) has eigenvalues $\{\mu_k^S(\varepsilon)\}_{k=1}^{+\infty}$ and $\{\mu_k^T(\varepsilon)\}_{k=1}^{+\infty}$ associated to stretching and torsional vibrations. We let $\{\mu_k^S\}_{k=1}^{+\infty}$ and $\{\mu_k^T\}_{k=1}^{+\infty}$ be the set of eigenvalues of the following eigenvalue problems respectively.

$$\begin{cases} -Y\frac{\mathrm{d}}{\mathrm{d}x_3}\left(a(x_3)^2\frac{\mathrm{d}\tau}{\mathrm{d}x_3}\right) = \mu^S a(x_3)^2\tau \quad (0 < x_3 < l), \\ \tau(0) = \tau(l) = 0. \end{cases}$$

$$\begin{cases} -\lambda_2\frac{\mathrm{d}}{\mathrm{d}x_3}\left(a(x_3)^4\frac{\mathrm{d}\rho}{\mathrm{d}x_3}\right) = \mu^T a(x_3)^4\rho \quad (0 < x_3 < l), \\ \rho(0) = \rho(l) = 0. \end{cases}$$

We have proved in Section 4.3 that for every sequence $(\varepsilon_p)_{p=1}^{+\infty}$ with $\varepsilon_p \to 0$ as $p \to +\infty$ there exists a subsequence $(\zeta_r) \subseteq (\varepsilon_p)$ such that

$$\lim_{r\to+\infty}\mu_k^S(\zeta_r) = \widetilde{\mu}_k^S, \qquad \lim_{r\to+\infty}\mu_k^T(\zeta_r) = \widetilde{\mu}_k^T,$$

with $\{\widetilde{\mu}_k^S\}_{k=1}^{+\infty} \subseteq \{\mu_k^S\}_{k=1}^{+\infty}$ and $\{\widetilde{\mu}_k^T\}_{k=1}^{+\infty} \subseteq \{\mu_k^T\}_{k=1}^{+\infty}$. As a consequence, we have the inequalities $\widetilde{\mu}_k^S \geq \mu_k^S$ and $\widetilde{\mu}_k^T \geq \mu_k^T$. In this section, our aim is to prove the inequalities $\widetilde{\mu}_k^S \leq \mu_k^S$ and $\widetilde{\mu}_k^T \leq \mu_k^T$. Furthermore, we prove that the previous convergence does not depend on the sequence (ε_p).

We perform the change of variables $(x_1, x_2, x_3) = (\varepsilon y_1, \varepsilon y_2, y_3)$ and we rewrite in terms of $y = (y_1, y_2, y_3)$ the Rayleigh quotient introduced in (4.2).

$$\widetilde{\mathcal{R}}_{\varepsilon,a}(U) = \frac{1}{\|U\|^2_{L^2(\Omega_{1,a},\mathbb{R}^3)}} \times \int_{\Omega_{1,a}} \Bigg\{\lambda_1\left(\frac{1}{\varepsilon}E_{11}(U) + \frac{1}{\varepsilon}E_{22}(U) + E_{33}(U)\right)^2 + 2\lambda_2\left(\frac{1}{\varepsilon^2}\sum_{i,j=1}^{2}E_{ij}(U)^2 + 2\sum_{i=1}^{2}E_{i3}(U)^2 + E_{33}(U)^2\right)\Bigg\}\mathrm{d}y$$

where $U = (U_1, U_2, U_3) \in \mathcal{W}_{1,a}\backslash\{\mathbf{0}\}$ and for $1 \leq i, j \leq 2$

$$E_{ij}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_j} + \frac{\partial U_j}{\partial y_i}\right), \; E_{i3}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_3} + \frac{1}{\varepsilon}\frac{\partial U_3}{\partial y_i}\right), \quad E_{33}(U) = \frac{\partial U_3}{\partial y_3}.$$

4.4.1 Stretching eigenvalues

We first work with the stretching mode. Consider the eigenvalue problem of the ordinary differential equation

$$\begin{cases} -Y\frac{\mathrm{d}}{\mathrm{d}y_3}\left(a(y_3)^2\frac{\mathrm{d}\tau}{\mathrm{d}y_3}\right) = \mu^S a(y_3)^2\tau \quad (0 < y_3 < l), \\ \tau(0) = \tau(l) = 0. \end{cases} \tag{4.30}$$

Let μ_k^S be the k-th eigenvalue of the problem (4.30) and $\tau^{(k)}$ its associated eigenfunction. We recall that for stretching eigenvalues we have $\widetilde{\mu}_k^S = \lim_{r\to+\infty}\mu_k^S(\zeta_r)$.

We want to show that $\widetilde{\mu}_k^S \leq \mu_k^S$. We multiply the system (4.30) by τ and integrate over the interval $(0,l)$. Applying the integration by parts we obtain

$$Y\int_0^l a(y_3)^2\left(\frac{\mathrm{d}\tau}{\mathrm{d}y_3}\right)^2\mathrm{d}y_3 = \mu^S\int_0^l a(y_3)^2\tau^2\mathrm{d}y_3.$$

Therefore, if $\tau^{(k)}$ is the eigenfunction associated to the μ_k^S, we have that

$$\mu_k^S = \frac{Y\displaystyle\int_0^l a(y_3)^2\left(\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}y_3}\right)^2\mathrm{d}y_3}{\displaystyle\int_0^l a(x_3)^2\left(\tau^{(k)}\right)^2\mathrm{d}y_3}. \tag{4.31}$$

We try new test functions $\Theta(y) = \Theta = (\Theta_1, \Theta_2, \Theta_3)$, $\phi(y) = \phi = (\phi_1, \phi_2, \phi_3)$ given by

$$\begin{aligned}&\Theta_i = \varepsilon\phi_i \qquad (i = 1, 2),\\ &\Theta_3 = \tau + \varepsilon\phi_3,\end{aligned}$$

where the function $\tau = \tau(y_3)$ depends only on y_3. We calculate

$$\begin{aligned}&E_{ij}(\Theta) = \varepsilon E_{ij}(\phi),\\ &E_{i3}(\Theta) = \frac{1}{2}\left(\varepsilon\frac{\partial\phi_i}{\partial y_3} + \frac{\partial\phi_3}{\partial y_i}\right) \qquad (1 \leq i, j \leq 2),\\ &E_{33}(\Theta) = \frac{\mathrm{d}\tau}{\mathrm{d}y_3} + \varepsilon\frac{\partial\phi_3}{\partial y_3}.\end{aligned}$$

Knowing this, we compute $\widetilde{\mathcal{R}}_{\varepsilon,a}(\Theta)$.

$$\begin{aligned}\widetilde{\mathcal{R}}_{\varepsilon,a}(\Theta) = &\frac{\displaystyle\int_{\Omega_{1,a}}\left(\lambda_1\left(\frac{\partial\phi_1}{\partial y_1} + \frac{\partial\phi_2}{\partial y_2} + \frac{\mathrm{d}\tau}{\mathrm{d}y_3} + \varepsilon\frac{\partial\phi_3}{\partial y_3}\right)^2 + 2\lambda_2\sum_{i,j=1}^2 E_{ij}(\phi)^2\right)\mathrm{d}y}{\displaystyle\int_{\Omega_{1,a}}\left(\varepsilon^2\phi_1^2 + \varepsilon^2\phi_2^2 + (\tau + \varepsilon\phi_3)^2\right)\mathrm{d}y}\\ &+\frac{\displaystyle\int_{\Omega_{1,a}} 2\lambda_2\left(2\sum_{i=1}^2\frac{1}{4}\left(\varepsilon\frac{\partial\phi_i}{\partial y_3} + \frac{\partial\phi_3}{\partial y_i}\right)^2 + \left(\frac{\mathrm{d}\tau}{\mathrm{d}y_3} + \varepsilon\frac{\partial\phi_3}{\partial y_3}\right)^2\right)\mathrm{d}y}{\displaystyle\int_{\Omega_{1,a}}\left(\varepsilon^2\phi_1^2 + \varepsilon^2\phi_2^2 + (\tau + \varepsilon\phi_3)^2\right)\mathrm{d}y}.\end{aligned}$$

We take the limit $\varepsilon \to 0$ and we see

$$\lim_{\varepsilon\to 0} \widetilde{\mathcal{R}}_{\varepsilon,a}(\Theta) = \frac{\displaystyle\int_{\Omega_{1,a}} \lambda_1 \left(\frac{\partial\phi_1}{\partial y_1} + \frac{\partial\phi_2}{\partial y_2} + \frac{\mathrm{d}\tau}{\mathrm{d}y_3}\right)^2 \mathrm{d}y}{\displaystyle\int_{\Omega_{1,a}} \tau^2 \mathrm{d}y} + \frac{\displaystyle\int_{\Omega_{1,a}} 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\phi)^2 + \frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\phi_3}{\partial y_i}\right)^2 + \left(\frac{\mathrm{d}\tau}{\mathrm{d}y_3}\right)^2\right) \mathrm{d}y}{\displaystyle\int_{\Omega_{1,a}} \tau^2 \mathrm{d}y}. \tag{4.32}$$

We want to find $\phi = (\phi_1, \phi_2, \phi_3)$ that minimizes the numerator of (4.32)

$$\mathcal{M}_S(\phi) = \int_{\Omega_{1,a}} \left(\lambda_1 \left(\frac{\partial\phi_1}{\partial y_1} + \frac{\partial\phi_2}{\partial y_2} + \frac{\mathrm{d}\tau}{\mathrm{d}y_3}\right)^2 + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\phi)^2 + \frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\phi_3}{\partial y_i}\right)^2 + \left(\frac{\mathrm{d}\tau}{\mathrm{d}y_3}\right)^2\right)\right) \mathrm{d}y.$$

In order to minimize $\mathcal{M}_S$, we put the test function ϕ as follows.

$$\begin{aligned} \phi_i(y) &= \sum_{p=1}^{2} \beta_p^{(i)}(y_3) y_p \qquad (i = 1, 2). \\ \phi_3(y) &= 0, \end{aligned} \tag{4.33}$$

Remark 4.4. Note that we could have put the same ϕ as in (3.53). In this case we would obtain $\alpha_{pq}^{(i)} = 0$ for all $1 \le p, q, i \le 2$, so it makes sense to drop them.

If we substitute the test function ϕ into $\mathcal{M}_S$ we obtain an expression that can be written as a polynomial of degree 2 on the variables $\beta_p^{(i)}$ for $1 \le i, p \le 2$. Thus, it can be further rewritten as $\int_0^l \left(\beta^T \mathcal{X} \beta + \mathcal{Y}\beta\right) \mathrm{d}y_3$ for a certain matrix valued function $\mathcal{X}$ and a certain vector valued function $\mathcal{Y}$ with

$$\beta = (\beta_1^{(1)}, \beta_2^{(1)}, \beta_1^{(2)}, \beta_2^{(2)})^T.$$

Since we want the minimum, we differentiate the expression $\int_0^l \left(\beta^T \mathcal{X} \beta + \mathcal{Y}\beta\right) \mathrm{d}y_3$ with respect to β and solve the linear system $2\mathcal{X}\beta + \mathcal{Y} = 0$ for β. After long but simple calculations we obtain

$$\beta_1^{(1)} = -\frac{1}{2}\frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{\mathrm{d}\tau}{\mathrm{d}y_3}, \quad \beta_2^{(2)} = -\frac{1}{2}\frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{\mathrm{d}\tau}{\mathrm{d}y_3}.$$

In fact, the matrix $\mathcal{X}$ in the system is degenerate and we additionally obtain the condition $\beta_1^{(2)} + \beta_2^{(1)} = 0$. It can also be checked that the minimum obtained is always the same, so to simplify, we put $\beta_1^{(2)} = 0$ and $\beta_2^{(1)} = 0$. Therefore, recalling (4.33), we obtain

$$\begin{aligned} \phi_1(y) &= -\frac{1}{2}\frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{\mathrm{d}\tau}{\mathrm{d}y_3} y_1, \\ \phi_2(y) &= -\frac{1}{2}\frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{\mathrm{d}\tau}{\mathrm{d}y_3} y_2, \\ \phi_3(y) &= 0. \end{aligned} \tag{4.34}$$

Substituting (4.34) into (4.32) we obtain the minimum

$$\lim_{\varepsilon\to 0} \widetilde{\mathcal{R}}_{\varepsilon,a}(\Theta) = \frac{\displaystyle\int_{\Omega_{1,a}} \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}\left(\frac{\mathrm{d}\tau}{\mathrm{d}y_3}\right)^2 \mathrm{d}y}{\displaystyle\int_{\Omega_{1,a}} \tau^2 \mathrm{d}y}. \tag{4.35}$$

Substituting $\tau = \tau^{(k)}$ into (4.35) and integrating over the cross-section we have

$$\lim_{\varepsilon\to 0} \widetilde{\mathcal{R}}_{\varepsilon,a}(\Theta) = \frac{Y\displaystyle\int_0^l a(y_3)^2 \left(\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}y_3}\right)^2 \mathrm{d}y_3}{\displaystyle\int_0^l a(y_3)^2 \left(\tau^{(k)}\right)^2 \mathrm{d}y_3},$$

which, from (4.31), turns out to be

$$\lim_{\varepsilon\to 0} \widetilde{\mathcal{R}}_{\varepsilon,a}(\Theta) = \mu_k^S.$$

These computations are a motivation of how to choose proper test functions for our next goal, which is to use the Max-Min method in order to prove the inequality $\widetilde{\mu}_k^S \leq \mu_k^S$. We consider the eigenfunction $\tau^{(k)}$ corresponding to the eigenvalue μ_k^S of problem (4.30). We choose the functions $\tau^{(k)}$ so that

$$\int_{\Omega_{1,a}} \tau^{(k)}\tau^{(k')}\mathrm{d}y = \delta(k,k'), \tag{4.36}$$

where $k, k' \in \mathbb{N}$ and δ is the Kronecker delta. Using the weak formulation of (4.30) we know that

$$Y\int_{\Omega_{1,a}} \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}y_3}\frac{\mathrm{d}\tau^{(k')}}{\mathrm{d}y_3}\mathrm{d}y = \mu_k^S\delta(k,k'). \tag{4.37}$$

Let us consider the test functions

$$\begin{aligned} \Phi_i^{(n)} &= \varepsilon\cdot\left(-\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}\tau^{(n)}}{\mathrm{d}y_3}y_i\right) \qquad (i=1,2), \\ \Phi_3^{(n)} &= \tau^{(n)}, \end{aligned} \tag{4.38}$$

with $n \in \mathbb{N}$, so that $\Phi^{(n)} = (\Phi_1^{(n)}, \Phi_2^{(n)}, \Phi_3^{(n)}) \in \mathfrak{H}_\mathrm{S} \subseteq \mathcal{W}_{1,a}$. Choose an arbitrary $Z \in \mathcal{H}_{k-1}(\Omega_{1,a}, \mathbb{R}^3)$ and let $\widetilde{Z} = L.H.\left[\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(k)}\right]$ be the minimal linear space that contains the set $\{\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(k)}\}$. Note that $\dim \widetilde{Z} = k$ and that each $\Phi^{(n)} \in \mathcal{W}_{1,a}$ (for all $n \in \mathbb{N}$), so we have that $\widetilde{Z} \subseteq \mathcal{W}_{1,a}$. Since $\dim Z < \dim \widetilde{Z}$, we know that there exist a function $\Psi = (\Psi_1, \Psi_2, \Psi_3) \in \widetilde{Z} \cap Z^\perp$ and a vector $(c_1, \ldots, c_k) = (c_1(\varepsilon), \ldots, c_k(\varepsilon)) \in \mathbb{R}^k\backslash\{\mathbf{0}\}$ such that

$$\Psi = \sum_{n=1}^{k} c_s(\varepsilon)\Phi^{(n)}.$$

Note that since both $\widetilde{Z}$ and $Z^{\perp}$ are subsets of $\mathcal{W}_{1,a}$, we have also that $\Psi \in \mathcal{W}_{1,a}$ and due the fact that $(c_1(\varepsilon), \ldots, c_k(\varepsilon)) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ we deduce that $\Psi \in \mathcal{W}_{1,a} \backslash \{\mathbf{0}\}$, so we can apply $\widetilde{\mathcal{R}}_{\varepsilon,a}$ to Ψ. We compute

$$E_{ii}(\Psi) = -\varepsilon \sum_{n=1}^{k} c_n(\varepsilon) \tfrac{1}{2} \tfrac{\lambda_1}{\lambda_1+\lambda_2} \frac{\mathrm{d}\tau^{(n)}}{\mathrm{d}y_3}, \qquad E_{i3}(\Psi) = \varepsilon \frac{1}{2} \sum_{n=1}^{k} c_n(\varepsilon) \frac{\partial \widetilde{\Phi}_i^{(n)}}{\partial y_3} \qquad (1 \leq i, j \leq 2),$$
$$E_{12}(\Psi) = E_{21}(\Psi) = 0, \qquad E_{33}(\Psi) = \sum_{n=1}^{k} c_n(\varepsilon) \frac{\mathrm{d}\tau^{(n)}}{\mathrm{d}y_3},$$

where $\widetilde{\Phi}_i^{(n)} = \frac{1}{\varepsilon} \Phi_i^{(n)} = -\frac{1}{2} \frac{\lambda_1}{\lambda_1+\lambda_2} \frac{\mathrm{d}\tau^{(n)}}{\mathrm{d}y_3} y_i$ for $i = 1, 2$. Using these computations, the numerator of the Rayleigh quotient $\widetilde{\mathcal{R}}_{\varepsilon,a}(\Psi)$ is

$$\int_{\Omega_{1,a}} \left(\lambda_1 \left(\sum_{n=1}^{k} c_n(\varepsilon) \tfrac{\lambda_2}{\lambda_1+\lambda_2} \frac{\mathrm{d}\tau^{(n)}}{\mathrm{d}y_3} \right)^2 + 2\lambda_2 \sum_{i=1}^{2} \left(\sum_{n=1}^{k} c_n(\varepsilon) \tfrac{1}{2} \tfrac{\lambda_1}{\lambda_1+\lambda_2} \frac{\mathrm{d}\tau^{(n)}}{\mathrm{d}y_3} \right)^2 \right) \mathrm{d}y$$
$$+ \int_{\Omega_{1,a}} 2\lambda_2 \left(2 \sum_{i=1}^{2} \frac{1}{4} \left(\varepsilon^2 \sum_{n=1}^{k} c_n(\varepsilon) \frac{\partial \Phi_i^{(n)}}{\partial y_3} \right)^2 + \left(\sum_{n=1}^{k} c_n(\varepsilon) \frac{\mathrm{d}\tau^{(n)}}{\mathrm{d}y_3} \right)^2 \right) \mathrm{d}y$$
$$= \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \int_{\Omega_{1,a}} Y \frac{\mathrm{d}\tau^{(p)}}{\mathrm{d}y_3} \frac{\mathrm{d}\tau^{(q)}}{\mathrm{d}y_3} \mathrm{d}y + \varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p, q, \varepsilon) \tag{4.39}$$

for some functions $\widetilde{\kappa}(p, q, \varepsilon) = O(1)$ as $\varepsilon \to 0$. Note that these functions $\widetilde{\kappa}(p, q, \varepsilon)$ do not depend on the choice of Z. Due to (4.37), it follows that (4.39) becomes

$$\sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \int_{\Omega_{1,a}} Y \frac{\mathrm{d}\tau^{(p)}}{\mathrm{d}y_3} \frac{\mathrm{d}\tau^{(q)}}{\mathrm{d}y_3} \mathrm{d}y + \varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p, q, \varepsilon)$$
$$= \sum_{p=1}^{k} c_p(\varepsilon)^2 \mu_p^S + \varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p, q, \varepsilon). \tag{4.40}$$

Note also that the denominator of $\widetilde{\mathcal{R}}_{\varepsilon,a}(\Psi)$ satisfies

$$\int_{\Omega_{1,a}} \left(\Psi_1^2 + \Psi_2^2 + \Psi_3^2 \right) \mathrm{d}y \geq \int_{\Omega_{1,a}} \Psi_3^2 \mathrm{d}y = \int_{\Omega_{1,a}} \left(\sum_{n=1}^{k} c_n(\varepsilon) \tau^{(n)} \right)^2 \mathrm{d}y$$
$$= \int_{\Omega_{1,a}} \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \tau^{(p)} \tau^{(q)} \mathrm{d}y \tag{4.41}$$

By the homogeneity property of Rayleigh's quotient we may assume without loss of generality that $\sum_{p=1}^{k} c_p(\varepsilon)^2 = 1$. Combining this fact with the orthogonality in (4.36), we get

$$\int_{\Omega_{1,a}} \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \tau^{(p)} \tau^{(q)} \mathrm{d}y = \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \int_{\Omega_{1,a}} \tau^{(p)} \tau^{(q)} \mathrm{d}y = \sum_{p=1}^{k} c_p(\varepsilon)^2 = 1 \tag{4.42}$$

Therefore, with (4.41) and (4.42), we deduce that

$$\int_{\Omega_{1,a}} \left(\Psi_1^2 + \Psi_2^2 + \Psi_3^2\right) \mathrm{d}y \geq 1. \tag{4.43}$$

Using (4.40), the bound (4.43) and the fact that $\mu_k^S \leq \mu_{k+1}^S$ holds for $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \widetilde{\mathcal{R}}_{\varepsilon,a}(\Psi) &\leq \frac{\displaystyle\sum_{p=1}^{k} c_p(\varepsilon)^2 \mu_p^S + \varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p,q,\varepsilon)}{1} \\ &\leq \mu_k^S \sum_{p=1}^{k} c_p(\varepsilon)^2 + \varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon) c_q(\varepsilon) \widetilde{\kappa}(p,q,\varepsilon) \leq \mu_k^S + \varepsilon^2 \sum_{p,q=1}^{k} |\widetilde{\kappa}(p,q,\varepsilon)|. \end{aligned} \tag{4.44}$$

We denote

$$\mathfrak{L}_k^S(\varepsilon) = \mu_k^S + \varepsilon^2 \sum_{p,q=1}^{k} |\widetilde{\kappa}(p,q,\varepsilon)|.$$

Note once again that $\mathfrak{L}_k^S(\varepsilon)$ does not depend on the choice of Z. We know from (4.44) that

$$\inf\{\widetilde{\mathcal{R}}_{\varepsilon,a}(\Phi) \mid \Phi \in \mathfrak{H}_{\mathrm{S}} \backslash \{\mathbf{0}\}, \Phi \perp Z \text{ in } L^2(\Omega_{1,a})\} \leq \widetilde{\mathcal{R}}_{\varepsilon,a}(\Psi) \leq \mathfrak{L}_k^S(\varepsilon).$$

Since $Z \in \mathcal{H}_{k-1}(\Omega_{1,a}, \mathbb{R}^3)$ was arbitrary and $\mathfrak{L}_k^S(\varepsilon)$ does not depend on the choice of Z, we take the supremum over $\mathcal{H}_{k-1}(\Omega_{1,a}, \mathbb{R}^3)$, so we obtain the upper estimate

$$\mu_k^S(\varepsilon) \leq \mathfrak{L}_k^S(\varepsilon).$$

Taking the limit $\varepsilon \to 0$ and using $\mu_k^S(\varepsilon) \to \widetilde{\mu}_k^S$ (for a certain subsequence), we have

$$\widetilde{\mu}_k^S \leq \limsup_{\varepsilon \to 0} \mu_k^S(\varepsilon) \leq \limsup_{\varepsilon \to 0} \mathfrak{L}_k^S(\varepsilon) = \mu_k^S,$$

which agrees to the desired inequality $\widetilde{\mu}_k^S \leq \mu_k^S$ ($k \in \mathbb{N}$). We combine this fact together with the inequality (4.27) to conclude that for every $k \in \mathbb{N}$,

$$\widetilde{\mu}_k^S = \mu_k^S.$$

We proved $\lim_{r \to +\infty} \mu_k^S(\zeta_r) = \widetilde{\mu}_k^S$ for a certain subsequence $(\zeta_r)_{r=1}^{+\infty} \subseteq (\varepsilon_p)_{p=1}^{+\infty}$, but note that we have shown that $\widetilde{\mu}_k^S = \mu_k^S$ independently of the first chosen sequence $(\varepsilon_p)_{p=1}^{+\infty}$. Since this sequence was arbitrary, we can see that in fact for every $k \in \mathbb{N}$ we have

$$\lim_{\varepsilon \to 0} \mu_k^S(\varepsilon) = \mu_k^S. \tag{4.45}$$

4.4.2 Torsional eigenvalues

Similarly, we prove a similar result for the torsional mode. We consider the ordinary differential equation

$$\begin{cases} -\lambda_2 \dfrac{\mathrm{d}}{\mathrm{d}y_3}\left(a(x_3)^4 \dfrac{\mathrm{d}\rho}{\mathrm{d}y_3}\right) = \mu^T a(x_3)^4 \rho \quad (0 < y_3 < l), \\ \rho(0) = \rho(l) = 0. \end{cases} \tag{4.46}$$

Let μ_k^T be the k-th eigenvalue of the problem (4.46) and $\rho^{(k)}$ its associated eigenfunction. We repeat a similar process we employed in Section 4.4.1 with some minor changes. In this case we use the test functions

$$\begin{aligned} \Phi_1^{(n)} &= -y_2 \rho^{(n)}, \\ \Phi_2^{(n)} &= y_1 \rho^{(n)}, \\ \Phi_3^{(n)} &= 0, \end{aligned}$$

with $n \in \mathbb{N}$ (compare these test functions with the ones in (4.38)). Following similar steps we prove that for every $k \in \mathbb{N}$ we have

$$\widetilde{\mu}_k^T = \mu_k^T$$

and

$$\lim_{\varepsilon \to 0} \mu_k^T(\varepsilon) = \mu_k^T.$$

4.5 H^1 convergence

We now prove a result on the strong convergence in $H^1(\Omega_{1,a})$ of stretching and torsional eigenfunctions. We follow the notation in Section 4.3. We define

$$\begin{aligned} &\gamma_0^{(k)}(\varepsilon) = \frac{1}{\varepsilon}\frac{\partial \widetilde{\chi}_\varepsilon^{(k)}}{\partial \widetilde{s}} + \frac{1}{\varepsilon}\frac{1}{\widetilde{s}}\widetilde{\chi}_\varepsilon^{(k)} + \frac{\partial \tau_\varepsilon^{(k)}}{\partial x_3}, \\ &\gamma_1^{(k)}(\varepsilon) = \frac{\partial \tau_\varepsilon^{(k)}}{\partial x_3}, \quad \gamma_2^{(k)}(\varepsilon) = \frac{1}{\sqrt{2}}\left(\frac{\partial \widetilde{\chi}_\varepsilon^{(k)}}{\partial x_3} + \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon^{(k)}}{\partial \widetilde{s}}\right), \quad \gamma_3^{(k)}(\varepsilon) = \frac{1}{\varepsilon}\frac{\partial \widetilde{\chi}_\varepsilon^{(k)}}{\partial \widetilde{s}}, \\ &\gamma_4^{(k)}(\varepsilon) = \frac{1}{\varepsilon}\frac{1}{\widetilde{s}}\widetilde{\chi}_\varepsilon^{(k)}, \quad \gamma_5^{(k)}(\varepsilon) = \frac{1}{\varepsilon}\frac{\partial \rho_\varepsilon^{(k)}}{\partial \widetilde{s}}, \quad \gamma_6^{(k)}(\varepsilon) = \frac{\partial \rho_\varepsilon^{(k)}}{\partial x_3}. \end{aligned}$$

From (4.16) and the weak convergence $\tau_\varepsilon^{(k)} \rightharpoonup \tau^{(k)}(x_3)$ in $H^1(\Omega_{1,a})$ (see (4.20)), we know that $\gamma_1^{(k)}(\varepsilon) \rightharpoonup \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}$ weakly in $L^2(\Omega_{1,a})$. From (4.18), (4.19), the definitions in (4.21) and the equation (4.24), we see that $\gamma_3^{(k)}(\varepsilon) \rightharpoonup \phi^{(k)} = -\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}$, $\gamma_4^{(k)}(\varepsilon) \rightharpoonup \phi^{(k)} = -\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}$ and $\gamma_2^{(k)}(\varepsilon) \rightharpoonup \Psi^{(k)}$ weakly in $L^2(\Omega_{1,a})$. We note that $\gamma_0^{(k)}(\varepsilon) = \gamma_1^{(k)}(\varepsilon) + \gamma_3^{(k)}(\varepsilon) + \gamma_4^{(k)}(\varepsilon)$ and we write

$$\gamma_0^{(k)} = -\frac{\lambda_2}{\lambda_1+\lambda_2}\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3},$$
$$\gamma_1^{(k)} = \frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}, \quad \gamma_2^{(k)} = \frac{1}{\sqrt{2}}\Psi^{(k)}, \quad \gamma_3^{(k)} = -\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}$$
$$\gamma_4^{(k)} = -\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3},$$

so that $\gamma_i^{(k)}(\varepsilon) \rightharpoonup \gamma_i^{(k)}$ weakly in $L^2(\Omega_{1,a})$ for $i = 0, \dots, 4$. We now prove that these limits are in fact in the strong sense in $L^2(\Omega_{1,a})$. We define

$$\Lambda_\varepsilon = \int_{\Omega_{1,a}} \left(\lambda_1(\gamma_0^{(k)}(\varepsilon) - \gamma_0^{(k)})^2 + 2\lambda_2 \sum_{i=1}^{4}(\gamma_i^{(k)}(\varepsilon) - \gamma_i^{(k)})^2\right) \widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3. \tag{4.47}$$

It is clear that $\Lambda_\varepsilon \geq 0$. For $i = 0, \dots, 4$ we have that

$$\|\gamma_i^{(k)}(\varepsilon) - \gamma_i^{(k)}\|^2_{L^2(\Omega_{1,a})} \lesssim \Lambda_\varepsilon \tag{4.48}$$

by the ellipticity of the operator. We use (4.14) to compute Λ_ε in (4.47).

$$\begin{aligned}\Lambda_\varepsilon =& \mu_k^S(\varepsilon)\int_{\Omega_{1,a}} \left(\left(\widetilde{\chi}_\varepsilon^{(k)}\right)^2 + \left(\tau_\varepsilon^{(k)}\right)^2\right)\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3 \\ &+ \int_{\Omega_{1,a}} \left(\lambda_1\left((\gamma_0^{(k)})^2 - 2\gamma_0^{(k)}\gamma_0^{(k)}(\varepsilon)\right) + 2\lambda_2\sum_{i=1}^{4}\left((\gamma_i^{(k)})^2 - 2\gamma_i^{(k)}\gamma_i^{(k)}(\varepsilon)\right)\right)\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3.\end{aligned}$$

From the weak convergences $\gamma_i^{(k)}(\varepsilon) \rightharpoonup \gamma_i^{(k)}$ in $L^2(\Omega_{1,a})$ as $\varepsilon \to 0$, we deduce that there exists a constant $\Lambda \geq 0$ such that $\lim_{\varepsilon\to 0}\Lambda_\varepsilon = \Lambda$. Moreover, we compute Λ and get

$$\Lambda = \mu_k^S \int_0^l \frac{a(x_3)^2}{2}\left(\tau^{(k)}\right)^2 \mathrm{d}x_3 - \int_{\Omega_{1,a}}\left(\lambda_1(\gamma_0^{(k)})^2 + 2\lambda_2\sum_{i=1}^{4}(\gamma_i^{(k)})^2\right)\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3, \tag{4.49}$$

where we used (4.45). From (4.25) (with $\varphi = \tau^{(k)}$ and the fact that $\widetilde{\mu}_k^S = \mu_k^S$), we see that

$$\mu_k^S \int_0^l \frac{a(x_3)^2}{2}\left(\tau^{(k)}\right)^2 \mathrm{d}x_3 = Y\int_0^l \frac{a(x_3)^2}{2}\left(\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}\right)^2 \mathrm{d}x_3. \tag{4.50}$$

We compute the latter integral of the equation (4.49).

$$\begin{aligned}&\int_{\Omega_{1,a}}\left(\lambda_1(\gamma_0^{(k)})^2 + 2\lambda_2\sum_{i=1}^{4}(\gamma_i^{(k)})^2\right)\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3 \\ &= \int_{\Omega_{1,a}}\left(\left(\lambda_1\left(\tfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^2 + 2\lambda_2\left(1 + 2\left(\tfrac{1}{2}\tfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^2\right)\right)\left(\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}\right)^2 + \lambda_2\left(\Psi^{(k)}\right)^2\right)\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3 \\ &= \int_{\Omega_{1,a}} Y\left(\frac{\mathrm{d}\tau^{(k)}}{\mathrm{d}x_3}\right)^2\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3 + \lambda_2\int_{\Omega_{1,a}}\left(\Psi^{(k)}\right)^2\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3.\end{aligned}$$

We substitute this computation and (4.50) into (4.49) and we see

$$\Lambda = -\lambda_2 \|\Psi^{(k)}\|^2_{L^2(\Omega_{1,a})}.$$

However, we know that $\Lambda \geq 0$ and $\lambda_2 > 0$. Therefore, $\Lambda = 0$ and $\|\Psi^{(k)}\|_{L^2(\Omega_{1,a})} = 0$. Since $\Lambda = 0$, from (4.48) we furthermore obtain that $\gamma_i^{(k)}(\varepsilon) \to \gamma_i^{(k)}$ strongly in $L^2(\Omega_{1,a})$ as $\varepsilon \to 0$ for $i = 0, \ldots, 4$. Using a very similar argument as in the proof of the $L^2(\Omega_{1,a})$-convergence of $\tau_\varepsilon^{(k)}$, we conclude that

$$\tau_\varepsilon^{(k)} \longrightarrow \tau^{(k)} \text{ strongly in } H^1(\Omega_{1,a}) \text{ as } \varepsilon \to 0. \tag{4.51}$$

For stretching eigenfunctions we repeat a similar argument with $\rho_\varepsilon^{(k)}$. We know that $\frac{\partial \rho_\varepsilon^{(k)}}{\partial x_3} \rightharpoonup \frac{\mathrm{d}\rho^{(k)}}{\mathrm{d}x_3}$ and $\frac{1}{\varepsilon}\frac{\partial \rho_\varepsilon^{(k)}}{\partial \widetilde{s}} \rightharpoonup \varrho^{(k)}$ for a certain function $\varrho^{(k)}$ weakly in $L^2(\Omega_{1,a})$ as $\varepsilon \to 0$. We define $\gamma_5^{(k)} = \varrho^{(k)}$, $\gamma_6^{(k)} = \frac{\mathrm{d}\rho^{(k)}}{\mathrm{d}x_3}$ and

$$\widehat{\Lambda}_\varepsilon = \lambda_2 \int_{\Omega_{1,a}} \left(\left(\gamma_5^{(k)}(\varepsilon) - \gamma_5^{(k)}\right)^2 + \left(\gamma_6^{(k)}(\varepsilon) - \gamma_6^{(k)}\right)^2 \right) \widetilde{s}\, \mathrm{d}\widetilde{s} \mathrm{d}x_3.$$

We take the corresponding limit and compute

$$\begin{aligned}\widehat{\Lambda} &= \mu_k^T \int_0^l \frac{a(x_3)^4}{4} \left(\rho^{(k)}\right)^2 \mathrm{d}x_3 - \lambda_2 \int_0^l \frac{a(x_3)^4}{4} \left(\frac{\mathrm{d}\rho^{(k)}}{\mathrm{d}x_3}\right)^2 \mathrm{d}x_3 - \lambda_2 \int_{\Omega_{1,a}} \left(\varrho^{(k)}\right)^2 \widetilde{s}\, \mathrm{d}\widetilde{s} \mathrm{d}x_3 \\ &= -\lambda_2 \left\| \varrho^{(k)} \right\|^2_{L^2(\Omega_{1,a})}.\end{aligned}$$

Therefore, $\widehat{\Lambda} = 0$, $\gamma_i^{(k)}(\varepsilon) \to \gamma_i^{(k)}$ strongly in $L^2(\Omega_{1,a})$ for $i = 5, 6$ and

$$\rho_\varepsilon^{(k)} \longrightarrow \rho^{(k)} \text{ strongly in } H^1(\Omega_{1,a}) \text{ as } \varepsilon \to 0. \tag{4.52}$$

Recall that torsional eigenfunctions are written as $w_\varepsilon^{(k)}(s, x_3) = (-x_2 \rho_\varepsilon^{(k)}, x_1 \rho_\varepsilon^{(k)}, 0)$. If we write $\widetilde{w}_\varepsilon^{(k)}(\widetilde{s}, x_3) = w_\varepsilon^{(k)}(\varepsilon \widetilde{s}, x_3)$, from (4.52), we deduce that $\widetilde{w}_\varepsilon^{(k)}$ converges strongly in $H^1(\Omega_{1,a}, \mathbb{R}^3)$. Let consider stretching eigenfunctions $v_\varepsilon^{(k)}(s, x_3) = (x_1 \chi_\varepsilon^{(k)}, x_2 \chi_\varepsilon^{(k)}, \tau_\varepsilon^{(k)})$. If we write $\widetilde{v}_\varepsilon^{(k)}(\widetilde{s}, x_3) = v_\varepsilon^{(k)}(\varepsilon \widetilde{s}, x_3)$, from (4.51) we see that $\widetilde{v}_{3,\varepsilon}^{(k)}$ converges strongly in $H^1(\Omega_{1,a})$. However, we do not have information about the strong convergence of $\chi_\varepsilon^{(k)}$ (or $\widetilde{\chi}_\varepsilon^{(k)}$) in $H^1(\Omega_{1,a})$, so we cannot claim the same convergence result for the components $\widetilde{v}_{1,\varepsilon}^{(k)}$ and $\widetilde{v}_{2,\varepsilon}^{(k)}$. This finishes the proof of Theorem 4.1.

We conjecture that $\widetilde{\chi}_\varepsilon^{(k)}$ also converges strongly in $H^1(\Omega_{1,a})$ in general. In Section 4.6 we present a proof of this conjecture under some assumptions.

4.6 Korn's inequality for torsional and stretching modes

In the previous sections we have proved all items of Theorem 4.1. We now further discuss some topics about the convergence of the eigenfunctions. In Section 2 we introduced a Korn inequality. There are several versions of Korn's inequality specific for each situation. In order to prove the strong convergence of stretching eigenfunctions of (4.1), we may use a Korn inequality designed specifically for torsional and stretching vibrations. Our next goal is to prove a Korn inequality that works for these kind of high-frequency modes. Let $v_\varepsilon = (v_{1,\varepsilon}, v_{2,\varepsilon}, v_{3,\varepsilon}) \in \mathcal{W}_{\varepsilon,a}$ be such that

$$\begin{aligned} v_{1,\varepsilon} &= -x_2\rho_\varepsilon(s,x_3) + x_1\chi_\varepsilon(s,x_3),\\ v_{2,\varepsilon} &= x_1\rho_\varepsilon(s,x_3) + x_2\chi_\varepsilon(s,x_3),\\ v_{3,\varepsilon} &= \tau_\varepsilon(s,x_3). \end{aligned}$$

For commodity, we write $s\chi_\varepsilon = \widetilde{\chi}_\varepsilon$ and we do the change of variables $s = \varepsilon\widetilde{s}$. Moreover, set $\gamma^\varepsilon(v_\varepsilon) = (\gamma_1^\varepsilon(v_\varepsilon), \dots, \gamma_6^\varepsilon(v_\varepsilon))$, where

$$\begin{aligned} \gamma_1^\varepsilon(v_\varepsilon) &= \frac{\partial \tau_\varepsilon}{\partial x_3}, \quad \gamma_2^\varepsilon(v_\varepsilon) = \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} + \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}}, \quad \gamma_3^\varepsilon(v_\varepsilon) = \frac{1}{\varepsilon}\frac{\partial \widetilde{\chi}_\varepsilon}{\partial \widetilde{s}},\\ \gamma_4^\varepsilon(v_\varepsilon) &= \frac{1}{\varepsilon}\frac{1}{\widetilde{s}}\widetilde{\chi}_\varepsilon, \quad \gamma_5^\varepsilon(v_\varepsilon) = \frac{1}{\varepsilon}\frac{\partial \rho_\varepsilon}{\partial \widetilde{s}}, \quad \gamma_6^\varepsilon(v_\varepsilon) = \frac{\partial \rho_\varepsilon}{\partial x_3}. \end{aligned}$$

These are the terms that appear in Rayleigh's quotient (4.14) and (4.28) (with $\theta = \rho_\varepsilon^{(k)}$).

Proposition 4.5. *Assume*

$$\left\| \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \right\|_{H^{-\frac{1}{2}}(\Gamma_{2,1,a})} \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0. \tag{4.53}$$

Then there exists a constant $C > 0$ such that

$$\|v_\varepsilon\|_{H^1(\Omega_{1,a},\mathbb{R}^3)} \le C\|\gamma^\varepsilon(v_\varepsilon)\|_{L^2(\Omega_{1,a},\mathbb{R}^6)}.$$

Remark 4.6. Let $a(x_3) = c$ be a constant function and we assume slip conditions on the boundary $\Gamma_{2,\varepsilon,a}$, that is, $u \cdot \boldsymbol{n} = \boldsymbol{0}$ and $(e(u)\cdot \boldsymbol{n}) \times \boldsymbol{n} = \boldsymbol{0}$ on $\Gamma_{2,\varepsilon,a}$. In this case assumption (4.53) is satisfied. We conjecture that this assumption should hold in general.

Remark 4.7. By the characterization of the eigenfunctions in (4.8), we note that assumption (4.53) is not necessary for Korn's inequality for the torsional mode.

Proof of Proposition 4.5. The proof is divided in three steps.

Step 1: We assume the contrary.

Let us assume that for all C, $\varepsilon_0 > 0$ there exist $0 < \varepsilon(C,\varepsilon_0) < \varepsilon_0$ and $v_{C,\varepsilon_0} \in \mathcal{W}_{1,a}$ such that

$$\|v_{C,\varepsilon_0}\|_{H^1(\Omega_{1,a},\mathbb{R}^3)} > C\|\gamma^\varepsilon(v_{C,\varepsilon_0})\|_{L^2(\Omega_{1,a},\mathbb{R}^6)}.$$

The functions v_{C,ε_0} can be chosen with norm equal to one. For a special choice $C = m$, $\varepsilon_0 = \frac{1}{m}$, $m \in \mathbb{N}$ it follows that there exist ε_m with $0 < \varepsilon_m \le \frac{1}{m}$ and $v_{\varepsilon_m} \in \mathcal{W}_{1,a}$ with $\|v_{\varepsilon_m}\|_{H^1(\Omega_{1,a},\mathbb{R}^3)} = 1$ such that

$$\|\gamma^{\varepsilon_m}(v_{\varepsilon_m})\|_{L^2(\Omega_{1,a},\mathbb{R}^6)} < \frac{1}{m}.$$

From the boundedness of v_{ε_m} in $H^1(\Omega_{1,a},\mathbb{R}^3)$, it follows that there exist a subsequence denoted by $v_{\varepsilon_n} \in \mathcal{W}_{1,a}$ and a function v with $\|v\|_{H^1(\Omega_{1,a})^3} = 1$ such that

$$v_{\varepsilon_n} \rightharpoonup v \quad \text{weakly in } H^1(\Omega_{1,a},\mathbb{R}^3), \tag{4.54}$$
$$\gamma^{\varepsilon_n}(v_{\varepsilon_n}) \to 0 \quad \text{strongly in } L^2(\Omega_{1,a},\mathbb{R}^6), \tag{4.55}$$

when $\varepsilon_n \to 0$.

Step 2: Strong convergence of v_{ε_n} in $H^1(\Omega_{1,a},\mathbb{R}^3)$

We want to now prove that in fact v_{ε_n} converges strongly in $H^1(\Omega_{1,a},\mathbb{R}^3)$. From (4.54) and (4.55) for $\gamma_5^\varepsilon(v_\varepsilon)$, $\gamma_6^\varepsilon(v_\varepsilon)$ we see that ρ_ε, $\frac{\partial \rho_\varepsilon}{\partial x_3}$ and $\frac{\partial \rho_\varepsilon}{\partial \widetilde{s}}$ all converge strongly in $L^2(\Omega_{1,a})$ so we conclude that ρ_ε converges strongly to a function ρ in $H^1(\Omega_{1,a})$.

We prove the strong convergence of τ_ε. We compute

$$\frac{\partial \gamma_2^\varepsilon(v_\varepsilon)}{\partial \widetilde{s}} - \varepsilon \frac{\partial \gamma_3^\varepsilon(v_\varepsilon)}{\partial x_3} = \frac{\partial^2 \widetilde{\chi}_\varepsilon}{\partial \widetilde{s} \partial x_3} + \frac{1}{\varepsilon}\frac{\partial^2 \tau_\varepsilon}{\partial \widetilde{s}^2} - \varepsilon \frac{1}{\varepsilon}\frac{\partial^2 \widetilde{\chi}_\varepsilon}{\partial \widetilde{s} \partial x_3} = \frac{1}{\varepsilon}\frac{\partial^2 \tau_\varepsilon}{\partial \widetilde{s}^2}.$$

We see that $\frac{\partial^2 \tau_\varepsilon}{\partial \widetilde{s}^2}$ converges strongly in $H^{-1}(\Omega_{1,a})$. From (4.54) for $v_{3,\varepsilon}$ and (4.55) for $\gamma_1^\varepsilon(v_\varepsilon)$ we see that $\frac{\partial^2 \tau_\varepsilon}{\partial \widetilde{s}^2}$, $\frac{\partial^2 \tau_\varepsilon}{\partial x_3 \partial \widetilde{s}}$ and $\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}}$ all converge strongly in $H^{-1}(\Omega_{1,a})$. Therefore $\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}}$ converges strongly in $L^2(\Omega_{1,a})$. Finally, combining this with (4.54) for $v_{3,\varepsilon}$ and (4.55) for $\gamma_1^\varepsilon(v_\varepsilon)$ we deduce that τ_ε converges strongly to a function τ in $H^1(\Omega_{1,a})$.

To see the convergence of $\widetilde{\chi}_\varepsilon$ we do the following.

$$\int_{\Omega_{1,a}} \left(\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} + \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}} \right) \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \widetilde{s}\, \mathrm{d}\widetilde{s} \mathrm{d}x_3 = \left\| \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \right\|^2_{L^2(\Omega_{1,a})} + \int_{\Omega_{1,a}} \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}}\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \widetilde{s}\, \mathrm{d}\widetilde{s} \mathrm{d}x_3.$$

We want to prove that the $L^2(\Omega_{1,a})$ norm of $\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3}$ tends to 0. We rearrange the previous equation and get

$$\left\| \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \right\|^2_{L^2(\Omega_{1,a})} = -\int_{\Omega_{1,a}} \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}}\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \widetilde{s}\, \mathrm{d}\widetilde{s} \mathrm{d}x_3 + \int_{\Omega_{1,a}} \left(\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} + \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}} \right) \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \widetilde{s}\, \mathrm{d}\widetilde{s} \mathrm{d}x_3.$$

Using absolute values we estimate

$$\left\| \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \right\|^2_{L^2(\Omega_{1,a})} \le \left| \int_{\Omega_{1,a}} \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}}\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \widetilde{s}\, \mathrm{d}\widetilde{s} \mathrm{d}x_3 \right| + \left| \int_{\Omega_{1,a}} \left(\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} + \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}} \right) \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \widetilde{s}\, \mathrm{d}\widetilde{s} \mathrm{d}x_3 \right|. \tag{4.56}$$

Note that by (4.55) for $\gamma_2^\varepsilon(v_\varepsilon)$ and Cauchy-Schwartz inequality we can estimate

$$\left| \int_{\Omega_{1,a}} \left(\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} + \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}} \right) \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \widetilde{s}\, \mathrm{d}\widetilde{s} \mathrm{d}x_3 \right| \le \left\| \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} + \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}} \right\|_{L^2(\Omega_{1,a})} \left\| \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \right\|_{L^2(\Omega_{1,a})}.$$

We combine this equation with Young's inequality, and substitute it into (4.56) to obtain

$$\left\| \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} \right\|^2_{L^2(\Omega_{1,a})} \lesssim \left| \int_{\Omega_{1,a}} \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial s}\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3} s\, \mathrm{d}s \mathrm{d}x_3 \right| \tag{4.57}$$

We use the integration by parts on the integral of the right hand side of (4.57) and get

$$\left|\int_{\Omega_{1,a}} \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial \widetilde{s}}\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3}\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3\right| = \left|\int_{\partial\Omega_{1,a}} \frac{1}{\varepsilon}\tau_\varepsilon\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3}\boldsymbol{n}_{\widetilde{s}}\,\mathrm{d}A - \int_{\Omega_{1,a}} \frac{1}{\varepsilon}\tau_\varepsilon\frac{\partial}{\partial \widetilde{s}}\left(\widetilde{s}\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3}\right)\mathrm{d}\widetilde{s}\mathrm{d}x_3\right|.$$

Note that since τ_ε converges strongly in $H^1(\Omega_{1,a})$ we deduce, by the trace theorem, that τ_ε also converges strongly in $H^{\frac{1}{2}}(\Gamma_{2,1,a})$. Using assumption (4.53) we see that the first integral of the right-hand side of the previous equation converges to 0. We analyze the second integral of the right-hand side of the previous equation.

$$\left|\int_{\Omega_{1,a}} \frac{1}{\varepsilon}\tau_\varepsilon\frac{\partial}{\partial \widetilde{s}}\left(\widetilde{s}\frac{\partial \chi_\varepsilon}{\partial x_3}\right)\mathrm{d}\widetilde{s}\mathrm{d}x_3\right| = \left|\int_{\Omega_{1,a}} \left(\frac{1}{\varepsilon}\tau_\varepsilon\frac{\partial^2 \widetilde{\chi}_\varepsilon}{\partial \widetilde{s}\partial x_3}\widetilde{s} + \frac{1}{\varepsilon}\tau_\varepsilon\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3}\right)\mathrm{d}\widetilde{s}\mathrm{d}x_3\right| \tag{4.58}$$

Note that τ_ε converges strongly in $H^1(\Omega_{1,a})$ and $\frac{\partial^2 \widetilde{\chi}_\varepsilon}{\partial \widetilde{s}\partial x_3}$ converges strongly to 0 in $H^{-1}(\Omega_{1,a})$, so that

$$\int_{\Omega_{1,a}} \frac{1}{\varepsilon}\tau_\varepsilon\frac{\partial^2 \widetilde{\chi}_\varepsilon}{\partial \widetilde{s}\partial x_3}\widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3 \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.59}$$

On the other hand we apply integration by parts to the second integral of the right-hand side of (4.58) to see

$$\int_{\Omega_{1,a}} \frac{1}{\varepsilon}\tau_\varepsilon\frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3}\mathrm{d}s\mathrm{d}x_3 = \int_{\partial\Omega_{1,a}} \frac{1}{\varepsilon}\tau_\varepsilon\widetilde{\chi}_\varepsilon\boldsymbol{n}_{x_3}\mathrm{d}A - \int_{\Omega_{1,a}} \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial x_3}\widetilde{\chi}_\varepsilon\mathrm{d}s\mathrm{d}x_3. \tag{4.60}$$

By a similar argument as before, since τ_ε converges strongly in $H^1(\Omega_{1,a})$, it can be seen by the trace theorem that τ_ε converges strongly in $L^2(\Gamma_{2,1,a})$. It remains to see that $\frac{1}{\varepsilon}\widetilde{\chi}_\varepsilon$ converges to 0 in the $L^2(\Gamma_{2,1,a})$ sense. Let $\delta > 0$ and let $\iota(\widetilde{s})$ be a smooth function in $(0, +\infty)$ with

$$\iota(\widetilde{s}) = \begin{cases} 1 & \text{if } \widetilde{s} > \frac{2\delta}{3}, \\ 0 & \text{if } 0 \le \widetilde{s} \le \frac{\delta}{3}. \end{cases}$$

We define $\widetilde{\widetilde{\chi}}_\varepsilon(\widetilde{s}, x_3) = \iota(\widetilde{s}, x_3)\widetilde{\chi}_\varepsilon$. It is clear that $\widetilde{\widetilde{\chi}}_\varepsilon$ and $\frac{\partial \widetilde{\widetilde{\chi}}_\varepsilon}{\partial \widetilde{s}}$ converge strongly to 0 in $L^2(\Omega_{1,a})$ as $\varepsilon \to 0$, and $\widetilde{\widetilde{\chi}}_\varepsilon(a(x_3), x_3) = \widetilde{\chi}_\varepsilon(a(x_3), x_3)$. We compute

$$\left\|\frac{1}{\varepsilon}\widetilde{\widetilde{\chi}}_\varepsilon\right\|^2_{L^2(\Gamma_{2,1,a})} = 2\pi\int_0^l \left|\frac{1}{\varepsilon}\widetilde{\widetilde{\chi}}_\varepsilon(a(x_3), x_3)\right|^2 \sqrt{1 + a'(x_3)^2}\mathrm{d}x_3. \tag{4.61}$$

On the other hand

$$\left|\frac{1}{\varepsilon}\widetilde{\widetilde{\chi}}_\varepsilon(a(x_3), x_3)\right|^2 \le \left(a(x_3) - \frac{\delta}{3}\right)\int_{\frac{\delta}{3}}^{a(x_3)} \left|\frac{1}{\varepsilon}\frac{\partial \widetilde{\widetilde{\chi}}_\varepsilon}{\partial \widetilde{s}}(\widetilde{s}, x_3)\right|^2 \mathrm{d}\widetilde{s} \lesssim \int_{\frac{\delta}{3}}^{a(x_3)} \left|\frac{1}{\varepsilon}\frac{\partial \widetilde{\widetilde{\chi}}_\varepsilon}{\partial \widetilde{s}}(\widetilde{s}, x_3)\right|^2 \widetilde{s}\frac{1}{\frac{\delta}{3}}\,\mathrm{d}\widetilde{s}.$$

Substituting this into (4.61), we obtain

$$\left\|\frac{1}{\varepsilon}\widetilde{\widetilde{\chi}}_\varepsilon\right\|^2_{L^2(\Gamma_{2,1,a})} \lesssim \int_0^l \int_{\frac{\delta}{3}}^{a(x_3)} \left|\frac{1}{\varepsilon}\frac{\partial \widetilde{\widetilde{\chi}}_\varepsilon}{\partial \widetilde{s}}(\widetilde{s}, x_3)\right|^2 \widetilde{s}\,\mathrm{d}\widetilde{s}\mathrm{d}x_3 = \left\|\frac{1}{\varepsilon}\frac{\partial \widetilde{\widetilde{\chi}}_\varepsilon}{\partial \widetilde{s}}\right\|^2_{L^2(\Omega_{1,a})} \longrightarrow 0$$

as $\varepsilon \to 0$ since $\frac{1}{\varepsilon}\frac{\partial \widetilde{\widetilde{\chi}}_\varepsilon}{\partial \widetilde{s}} \to 0$ strongly in $L^2(\Omega_{1,a})$ as $\varepsilon \to 0$. Thus $\frac{1}{\varepsilon}\widetilde{\widetilde{\chi}}_\varepsilon \to 0$ in $L^2(\Gamma_{2,1,a})$ and due to $\widetilde{\widetilde{\chi}}_\varepsilon(a(x_3), x_3) = \widetilde{\chi}_\varepsilon(a(x_3), x_3)$, we get that $\frac{1}{\varepsilon}\widetilde{\chi}_\varepsilon \to 0$ in $L^2(\Gamma_{2,1,a})$. Therefore,

$$\int_{\partial\Omega_{1,a}} \frac{1}{\varepsilon}\tau_\varepsilon \widetilde{\chi}_\varepsilon \boldsymbol{n}_{x_3} \mathrm{d}A \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.62}$$

We rewrite the second integral of the right-hand side of (4.60) as follows.

$$\int_{\Omega_{1,a}} \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial x_3}\widetilde{\chi}_\varepsilon \mathrm{d}\widetilde{s}\mathrm{d}x_3 = \int_{\Omega_{1,a}} \left(\frac{1}{\varepsilon}\frac{1}{\widetilde{s}}\widetilde{\chi}_\varepsilon\right)\frac{\partial \tau_\varepsilon}{\partial x_3}\widetilde{s}\mathrm{d}\widetilde{s}\mathrm{d}x_3$$

Using the strong convergence in $L^2(\Omega_{1,a})$ of $\frac{\partial \tau_\varepsilon}{\partial x_3}$ and $\gamma_4^\varepsilon(v_\varepsilon) \to 0$ in $L^2(\Omega_{1,a})$ we see that

$$\int_{\Omega_{1,a}} \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon}{\partial x_3}\widetilde{\chi}_\varepsilon \mathrm{d}\widetilde{s}\mathrm{d}x_3 \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.63}$$

We use (4.62) and (4.63) in (4.60) to see that

$$\int_{\Omega_{1,a}} \frac{1}{\varepsilon}\tau_\varepsilon \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3}\mathrm{d}\widetilde{s}\mathrm{d}x_3 \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.64}$$

Combining (4.59) and (4.64) and substituting them into (4.58) we get

$$\left|\int_{\Omega_{1,a}} \left(\frac{1}{\varepsilon}\tau_\varepsilon \frac{\partial^2 \widetilde{\chi}_\varepsilon}{\partial \widetilde{s}\partial x_3}\widetilde{s} + \frac{1}{\varepsilon}\tau_\varepsilon \frac{\partial \widetilde{\chi}_\varepsilon}{\partial x_3}\right)\mathrm{d}\widetilde{s}\mathrm{d}x_3\right| \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0,$$

and finally, using this convergence in (4.57) we conclude that

$$\left\|\frac{\partial \widetilde{\chi}}{\partial x_3}\right\|^2_{L^2(\Omega_{1,a})} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus, the $L^2(\Omega_{1,a})$ convergence of $\frac{\partial \widetilde{\chi}}{\partial x_3}$ is strong. To sum up, $\widetilde{\chi}_\varepsilon$, $\frac{\partial \widetilde{\chi}}{\partial \widetilde{s}}$ and $\frac{\partial \widetilde{\chi}}{\partial x_3}$ all converge strongly in $L^2(\Omega_{1,a})$. Therefore, $\widetilde{\chi}_\varepsilon$ converges to a function $\widetilde{\chi}$ strongly in $H^1(\Omega_{1,a})$.

Step 3: $v = 0$

From (4.55) for $\gamma_5^\varepsilon(v_\varepsilon)$ and $\gamma_6^\varepsilon(v_\varepsilon)$ we see that $\frac{\partial \rho}{\partial \widetilde{s}} = \frac{\partial \rho}{\partial x_3} = 0$. Combining this with the Dirichlet boundary conditions $\rho(0) = \rho(l) = 0$ we conclude that $\rho = 0$. From (4.55) for $\gamma_4^\varepsilon(v_\varepsilon)$ we clearly have that $\chi = 0$. Finally, combining (4.55) for $\gamma_1^\varepsilon(v_\varepsilon)$, $\gamma_2^\varepsilon(v_\varepsilon)$ and $\chi = 0$ we conclude that $\tau = 0$. Since $\tau = \chi = \rho = 0$, we see that $v = 0$, but at the same time we have that $\|v\|_{H^1(\Omega_{1,a},\mathbb{R}^3)} = 1$, so we get a contradiction. □

4.7 Generalization to curved rods

In this part we give a conjecture about the stretching and torsional modes on a curved rod with non-uniform cross-section.

We start presenting the domain $\Omega_{\varepsilon,\mathcal{K}}$, where $\varepsilon > 0$ is a small parameter that corresponds to the thickness of the elastic curved rod and $\mathcal{K}$ correspond to the curve defined by the curved rod. We follow similar notations as those in Section 3. Let $l > 0$ and let $B \subseteq \mathbb{R}^2$ be a connected bounded domain such that the boundary is $\mathcal{C}^3$ with $m \in \mathbb{N}$ connected components. We consider the sets

$$S = B \times (0, l), \qquad s_1 = \overline{B} \times \{0, l\}, \qquad s_2 = \partial B \times (0, l).$$

Note that $\partial S = s_1 \cup s_2$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^3-diffeomorphism such that it satisfies the following properties.

i) $F(z) = (F_1(z), F_2(z), z_3) \quad (z = (z_1, z_2, z_3) \in S)$.

ii) $F_i(0, 0, z_3) = 0 \quad (i = 1, 2, \quad 0 \leq z_3 \leq l)$.

iii) The determinant of the Jacobian matrix of F is positive for all $z \in S$.

We define $F^\varepsilon(z) = (\varepsilon F_1(z), \varepsilon F_2(z), z_3)$. With this notation, we consider the following sets in $\mathbb{R}^3$.

$$\mathcal{S}_\varepsilon = F^\varepsilon(S), \quad \mathcal{S}_{1,\varepsilon} = F^\varepsilon(s_1), \quad \mathcal{S}_{2,\varepsilon} = F^\varepsilon(s_2).$$

It is easy to see $\partial \mathcal{S}_\varepsilon = \mathcal{S}_{1,\varepsilon} \cup \mathcal{S}_{2,\varepsilon}$. Moreover, we obtain that $\mathcal{S}_1 = F(S)$. Let $\mathcal{K} : [0, l] \to \mathbb{R}^3$ be a generic space curve with curvature κ and torsion (or bicurvature) τ. Let $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$ be the Frenet reference of the curve $\mathcal{K}$, so that $\mathcal{T}$ is the tangent vector, $\mathcal{N}$ is the normal vector and $\mathcal{B}$ is the binormal vector of the curve $\mathcal{K}$. Define $\mathcal{P}(y) = \mathcal{K} + y_1\mathcal{N} + y_2\mathcal{B}$. Finally, we set

$$\Omega_{\varepsilon,\mathcal{K}} = \mathcal{P}(\mathcal{S}_\varepsilon), \quad \Gamma_{1,\varepsilon,\mathcal{K}} = \mathcal{P}(\mathcal{S}_{1,\varepsilon}), \quad \Gamma_{2,\varepsilon,\mathcal{K}} = \mathcal{P}(\mathcal{S}_{2,\varepsilon}).$$

In this subsection we discuss high-frequency eigenvalues of a thin elastic curved rod with non-uniform cross-section that varies along the rod. We study the following eigenvalue problem.

$$\begin{cases} L[u] + \mu u = \mathbf{0} & \text{in } \Omega_{\varepsilon,\mathcal{K}}, \\ u = \mathbf{0} & \text{on } \Gamma_{1,\varepsilon,\mathcal{K}}, \\ \sigma(u)\,\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{2,\varepsilon,\mathcal{K}}. \end{cases} \tag{4.65}$$

Here $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega_{\varepsilon,\mathcal{K}}$. We denote by $\{\mathfrak{m}_k(\varepsilon)\}_{k=1}^{+\infty}$ the set of eigenvalues of problem (4.65) and we recall that for any $\varepsilon > 0$ there is an infinite discrete sequence of positive eigenvalues

$$0 < \mathfrak{m}_k(\varepsilon) \leq \mathfrak{m}_2(\varepsilon) \leq \cdots \leq \mathfrak{m}_k(\varepsilon) \leq \mathfrak{m}_{k+1}(\varepsilon) \leq \cdots \text{ with } \lim_{k\to+\infty} \mathfrak{m}_k(\varepsilon) = +\infty$$

which are arranged in increasing order, counting multiplicities. Tambača [31] gives a result on the convergence of the eigenvalues $\mathfrak{m}_k(\varepsilon)$ in the case of a thin curved rod with simply connected, constant cross-section and such that its barycenter or "center of mass" is also constant. We combine the main result in Tambača [31] with the tools and the know-how we give in Section 3, to conclude that for each $k \in \mathbb{N}$ we have $\mathfrak{m}_k(\varepsilon) = O(\varepsilon^2)$. Thus, we know

$$\lim_{k\to+\infty} \mathfrak{m}_k(\varepsilon) = +\infty \text{ for each } \varepsilon > 0,$$
$$\lim_{\varepsilon\to 0} \mathfrak{m}_k(\varepsilon) = 0 \text{ for each } k \in \mathbb{N}.$$

These eigenvalues are low-frequency eigenvalues corresponding to the bending mode. However, we conjecture that there exist high-frequency eigenvalues corresponding to stretching and torsional vibrations, which do not tend to zero as the thinness gets smaller. These high-frequency eigenvalues cannot be analyzed when the subindex k is fixed.

In order to state the main conjecture we first introduce several notations. Denote $\mathrm{d}y' = \mathrm{d}y_1\mathrm{d}y_2$ and define the set $\widehat{\mathcal{S}}(y_3)$ to be the cross-section of $\mathcal{S}_1$ at $y_3 \in [0,l]$. Furthermore, for $1 \leq i,j \leq 2$, we define the functions

$$H(y_3) = \int_{\widehat{\mathcal{S}}(y_3)} 1\,\mathrm{d}y', \quad K_i(y_3) = \int_{\widehat{\mathcal{S}}(y_3)} y_i\,\mathrm{d}y', \quad A_{ij}(y_3) = \int_{\widehat{\mathcal{S}}(y_3)} y_iy_j\,\mathrm{d}y' \quad (y_3 \in [0,l])$$

and write $H = H(y_3)$, $K_i = K_i(y_3)$, $A_{ij} = A_{ij}(y_3)$, and $Y = \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}$. For every $y_3 \in [0,l]$, let $p \in H^1(\widehat{\mathcal{S}}(y_3))$ be the unique solution of the following problem.

$$\int_{\widehat{\mathcal{S}}(y_3)} \left(\left(\tfrac{\partial p}{\partial y_1} - y_2\right)\tfrac{\partial r}{\partial y_1} + \left(\tfrac{\partial p}{\partial y_2} + y_1\right)\tfrac{\partial r}{\partial y_2}\right)\mathrm{d}y' = 0 \text{ for every } r \in H^1(\widehat{\mathcal{S}}(y_3)), \quad \int_{\widehat{\mathcal{S}}(y_3)} p\,\mathrm{d}y' = 0. \tag{4.66}$$

With abuse of notation, we write $p = p(y_1,y_2,y_3)$ as a function of (y_1,y_2,y_3), $y = (y_1,y_2,y_3) \in \mathcal{S}_1$. We then define

$$J = J(y_3) = \int_{\widehat{\mathcal{S}}(y_3)} \left(\left(\frac{\partial p}{\partial y_1} - y_2\right)^2 + \left(\frac{\partial p}{\partial y_2} + y_1\right)^2\right)\mathrm{d}y'.$$

Remark 4.8. Assume that we have $K_1 = K_2 = 0$ and $A_{12} = A_{21} = 0$. In this case $p \equiv 0$, so that

$$J = \int_{\widehat{\mathcal{S}}(y_3)} (y_1^2 + y_2^2)\,\mathrm{d}y' = A_{11} + A_{22}.$$

With this notation, we state the following conjecture.

Conjecture 4.9. *Let $\{\mathfrak{m}_k(\varepsilon)\}_{k\in\mathbb{N}}$ be the set of eigenvalues of the eigenvalue problem (4.65). Then, for every $k \in \mathbb{N}$ there exists a sequence $(q(k,\varepsilon))_{\varepsilon>0}$ with $q(k,\varepsilon) \in \mathbb{N}$, $q(k,\varepsilon) < q(k+1,\varepsilon)$ and $q(k,\varepsilon) \to +\infty$ as $\varepsilon \to 0$ and a constant $\Pi_k > 0$ such that*

$$\lim_{\varepsilon\to 0} \mathfrak{m}_{q(k,\varepsilon)}(\varepsilon) = \Pi_k.$$

Moreover, Π_k is the k-th eigenvalue of the spectral problem

$$(\Pi,\theta,\zeta,w_1,w_2) \in (0,+\infty) \times H_0^1(0,l)^4, \quad (\theta,\zeta,w_1,w_2) \neq (0,0,0,0),$$

$$\begin{aligned}\int_0^l \Bigg\{ Y\Bigg[H\left(\frac{\mathrm{d}\zeta}{\mathrm{d}y_3} - \kappa w_1\right)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}y_3} - \kappa\omega_1\right) \\ + K_2\kappa\left(\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}y_3} - \kappa\omega_1\right) + \chi\left(\frac{\mathrm{d}\zeta}{\mathrm{d}y_3} - \kappa w_1\right)\right) + A_{22}\kappa^2\chi\theta\Bigg] + \lambda_2 J\frac{\mathrm{d}\theta}{\mathrm{d}y_3}\frac{\mathrm{d}\chi}{\mathrm{d}y_3}\Bigg\}\mathrm{d}y_3 \\ = \Pi\int_0^l \{H(\zeta\varphi + w_1\omega_1 + w_2\omega_2) - K_2(\theta\omega_1 + \chi w_1) + K_1(\theta\omega_2 + \chi w_2) + (A_{11}+A_{22})\theta\chi\}\,\mathrm{d}y_3 \\ \textit{for all } (\chi,\varphi,\omega_1,\omega_2) \in H_0^1(0,l)^4.\end{aligned}$$

Remark 4.10. Note that the spectral problem does not depend on the torsion τ.

The ideas of the proof of Theorem 4.1, should also be valid for the current case. We remark where the main differences lie.

First, we perform a change of variable to transform $\Omega_{\varepsilon,\mathcal{K}}$ into $\mathcal{S}_1$. After the change of variable, the energy becomes

$$\int_{\Omega_{\varepsilon,\mathcal{K}}} (\lambda_1(\operatorname{tr} u)^2 + 2\lambda_2 \sum_{i,j=1}^{3} e_{ij}(u)^2)\,\mathrm{d}x = \int_{\mathcal{S}_1} \left(\lambda_1(\Theta_1^\varepsilon(v) + \Theta_2^\varepsilon(v) + \Theta_3^\varepsilon(v))^2 + \sum_{i=1}^{6} \Theta_i^\varepsilon(v)^2\right) \mathrm{d}y,$$

where $v = (v_1, v_2, v_3) = v(y) = u(\mathcal{P}(y))$ $(y \in \mathcal{S}_1)$ and

$$\begin{aligned}
\Theta_1^\varepsilon(v) =& \frac{1}{(1-\varepsilon\kappa y_2)^2}\Bigg[\frac{\partial v_1}{\partial y_1} + \tau y_3\left(\frac{\partial v_1}{\partial y_2} + \varepsilon\frac{\partial v_2}{\partial y_1}\right) - \tau y_2\left(\frac{\partial v_1}{\partial y_3} + \varepsilon\frac{\partial v_3}{\partial y_1}\right) - \varepsilon\tau^2 y_2 y_3\left(\frac{\partial v_2}{\partial y_3} + \frac{\partial v_3}{\partial y_2}\right)\\
&+ \varepsilon\tau^2 y_3^2\frac{\partial v_2}{\partial y_2} + \varepsilon\tau^2 y_2\frac{\partial v_3}{\partial y_3} + \varepsilon\frac{\tau\kappa y_3 + \kappa' y_2}{1-\varepsilon\kappa y_2}v_1\\
&+ \frac{1}{1-\varepsilon\kappa y_2}\left(-(1-\varepsilon\kappa y_2)^2\kappa + \varepsilon(\tau' y_3 - \tau^2 y_2) + \varepsilon^2((\kappa'\tau - \kappa\tau')y_2y_3 + \varepsilon^2\kappa\tau^2(y_2^2 + y_3^2)\right)v_2\\
&+ \frac{\varepsilon}{1-\varepsilon\kappa y_2}\left(\varepsilon(\kappa\tau' - \kappa'\tau)y_2^2 - \tau' y_2 - \tau^2 y_3\right)v_3\Bigg],\\
\Theta_2^\varepsilon(v) =& \frac{1}{\varepsilon}\frac{\partial v_2}{\partial y_2},\\
\Theta_3^\varepsilon(v) =& \frac{1}{\varepsilon}\frac{\partial v_3}{\partial y_3},\\
\Theta_4^\varepsilon(v) =& \frac{1}{1-\varepsilon\kappa y_2}\Bigg[\frac{1}{2}\left(\frac{1}{\varepsilon}\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\right) - \frac{1}{2}\tau y_2\left(\frac{\partial v_2}{\partial y_3} + \varepsilon\frac{\partial v_3}{\partial y_2}\right) + \frac{1}{1-\varepsilon\kappa y_2}\kappa v_1 + \varepsilon\frac{\tau\kappa y_3}{1-\varepsilon\kappa y_2}v_2\\
&- \frac{1}{1-\varepsilon\kappa y_2}\tau v_3 + \tau y_3\frac{\partial v_2}{\partial y_2}\Bigg],\\
\Theta_5^\varepsilon(v) =& \frac{1}{1-\varepsilon\kappa y_2}\Bigg[\frac{1}{2}\left(\frac{1}{\varepsilon}\frac{\partial v_1}{\partial y_3} + \frac{\partial v_3}{\partial y_1}\right) + \frac{1}{2}\tau y_3\left(\frac{\partial v_2}{\partial y_3} + \frac{\partial v_3}{\partial y_2}\right) - \tau y_2\frac{\partial v_3}{\partial y_3} + \tau v_2\Bigg],\\
\Theta_6^\varepsilon(v) =& \frac{1}{\varepsilon}\left(\frac{\partial v_2}{\partial y_3} + \frac{\partial v_3}{\partial y_2}\right).
\end{aligned}$$

As in Section 4.2, we need a candidate to torsional or stretching eigenfunction. In this case, since we do not have axial symmetry and we have influence from the curvature κ and the torsion τ, it is difficult to construct a candidate on $\Omega_{\varepsilon,\mathcal{K}}$ before taking the limit, even though we can predict the behavior as $\varepsilon \to 0$.

Due to the complexity of the weak form, the manipulation of Korn's inequality will also be complicated.

Finally, when searching for an upper bound as in Section 3.5 and Section 4.4, we have to minimize the operator

$$\mathcal{M}_{\mathcal{K}}(\phi)=\int_{\mathcal{S}_1}\Bigg\{\lambda_1\left(\frac{\partial\phi_1}{\partial y_1}+\frac{\partial\phi_2}{\partial y_2}+\frac{\mathrm{d}\zeta}{\mathrm{d}y_3}-(-\theta y_2+w_1)\kappa\right)^2+2\lambda_2\Bigg[\sum_{i,j=1}^{2}E_{ij}(\phi)^2$$
$$+2\left(\frac{1}{2}\frac{\partial\phi_3}{\partial y_1}+\frac{1}{2}\left(-\frac{\mathrm{d}\theta}{\mathrm{d}y_3}y_2+\frac{\mathrm{d}w_1}{\mathrm{d}y_3}\right)+\zeta\kappa-(\theta y_1+w_2)\tau\right)^2$$
$$+2\left(\frac{1}{2}\frac{\partial\phi_3}{\partial y_2}+\frac{1}{2}\left(-\frac{\mathrm{d}\theta}{\mathrm{d}y_3}y_1+\frac{\mathrm{d}w_2}{\mathrm{d}y_3}\right)+(-\theta y_2+w_1)\tau\right)^2+\left(\frac{\mathrm{d}\zeta}{\mathrm{d}y_3}-(-\theta y_2+w_1)\kappa\right)^2\Bigg]\Bigg\}\mathrm{d}y, \tag{4.67}$$

where $\phi=(\phi_1,\phi_2,\phi_3)$ is a test function. In order to minimize $\mathcal{M}_{\mathcal{K}}$, we have to be careful. Note that in (4.67) there are no cross-terms involving (ϕ_1,ϕ_2) and ϕ_3 so we can deal with them separately. We put

$$\phi_i(y)=\sum_{p,q=1}^{2}\alpha_{pq}^{(i)}(y_3)y_py_q+\sum_{p=1}^{2}\beta_p^{(i)}(y_3)y_p.$$

Here $\alpha_{12}^{(i)}=\alpha_{21}^{(i)}$ for $i=1,2$. Analogously, but following similar steps as in Section 3.5, we deduce

$$\alpha_{11}^{(1)}=0,\qquad \alpha_{12}^{(1)}=-\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\theta,\qquad \alpha_{22}^{(1)}=0,$$
$$\alpha_{11}^{(2)}=\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\theta,\qquad \alpha_{12}^{(2)}=0,\qquad \alpha_{22}^{(2)}=-\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\theta,$$
$$\beta_1^{(1)}=-\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}y_3}-w_1\kappa\right),\quad \beta_2^{(2)}=-\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}y_3}-w_1\kappa\right),\quad \beta_1^{(2)}=\beta_2^{(1)}=0.$$

We define

$$\widetilde{\mathcal{M}}_{\mathcal{K}}(\phi_3)=\int_{\mathcal{S}_1}2\lambda_2\Bigg[2\left(\frac{1}{2}\frac{\partial\phi_3}{\partial y_1}+\frac{1}{2}\left(-\frac{\mathrm{d}\theta}{\mathrm{d}y_3}y_2+\frac{\mathrm{d}w_1}{\mathrm{d}y_3}\right)+\zeta\kappa-(\theta y_1+w_2)\tau\right)^2$$
$$+2\left(\frac{1}{2}\frac{\partial\phi_3}{\partial y_2}+\frac{1}{2}\left(-\frac{\mathrm{d}\theta}{\mathrm{d}y_3}y_1+\frac{\mathrm{d}w_2}{\mathrm{d}y_3}\right)+(-\theta y_2+w_1)\tau\right)^2\Bigg]\mathrm{d}y. \tag{4.68}$$

Note that in contrast with (3.53) and (4.33), $\phi_3=0$ does not minimize $\widetilde{\mathcal{M}}_{\mathcal{K}}$. In this case, let ϕ_3 be a solution of

$$\frac{\partial\phi_3}{\partial y_1}=2\left(-\frac{1}{2}\frac{\mathrm{d}w_1}{\mathrm{d}y_3}-\zeta\kappa+w_2\tau+\theta\tau y_1+\frac{1}{2}\frac{\mathrm{d}\theta}{\mathrm{d}y_3}\frac{\partial p}{\partial y_1}\right),$$
$$\frac{\partial\phi_3}{\partial y_2}=2\left(-\frac{1}{2}\frac{\mathrm{d}w_2}{\mathrm{d}y_3}-w_1\tau+\theta\tau y_2+\frac{1}{2}\frac{\mathrm{d}\theta}{\mathrm{d}y_3}\frac{\partial p}{\partial y_1}\right),$$

where p is defined as in (4.66), so that

$$\widetilde{\mathcal{M}}_{\mathcal{K}}(\phi_3) = \int_{\mathcal{S}_1} \left(\left(\frac{1}{2} \frac{\mathrm{d}\theta}{\mathrm{d}y_3} \left(\frac{\partial p}{\partial y_1} - y_2 \right) \right)^2 + \left(\frac{1}{2} \frac{\mathrm{d}\theta}{\mathrm{d}y_3} \left(\frac{\partial p}{\partial y_2} + y_1 \right) \right)^2 \right) \mathrm{d}y. \tag{4.69}$$

Then ϕ_3 minimizes $\widetilde{\mathcal{M}}_{\mathcal{K}}$. Indeed, let ψ^* be another function so that we have $\psi^* = \phi_3 + \psi$ (with $\psi = -\phi_3 + \psi^*$). Using the properties of p in (4.66) and (4.69), we compute

$$\begin{aligned}\widetilde{\mathcal{M}}_{\mathcal{K}}(\psi^*) &= \int_0^l \int_{\widehat{\mathcal{S}}(y_3)} \left(\left(\frac{1}{2} \frac{\mathrm{d}\theta}{\mathrm{d}y_3} \left(\frac{\partial p}{\partial y_1} - y_2 \right) + \frac{\partial \psi}{\partial y_1} \right)^2 + \left(\frac{1}{2} \frac{\mathrm{d}\theta}{\mathrm{d}y_3} \left(\frac{\partial p}{\partial y_2} + y_1 \right) + \frac{\partial \psi}{\partial y_2} \right)^2 \right) \mathrm{d}y \\ &= \widetilde{\mathcal{M}}_{\mathcal{K}}(\phi_3) + \int_0^l \int_{\widehat{\mathcal{S}}(y_3)} \left(\left(\frac{\partial \psi}{\partial y_1} \right)^2 + \left(\frac{\partial \psi}{\partial y_2} \right)^2 \right) \mathrm{d}y \geq \widetilde{\mathcal{M}}_{\mathcal{K}}(\phi_3).\end{aligned}$$

Therefore, ϕ minimizes the operator $\mathcal{M}_{\mathcal{K}}$.

A Appendix

In this appendix we give the proofs of Lemma 3.5 and Lemma 3.6 and some additional facts which we used before in the proof of the main results.

Proof of Lemma 3.5. a) Let $\phi, \psi \in \mathcal{C}_0^{+\infty}(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(t)\mathrm{d}t = 1$ and $\int_{\mathbb{R}} \phi(t)\mathrm{d}t = 1$. For any $\Phi \in \mathcal{C}_0^{+\infty}(\mathbb{R}^3)$ with $\mathrm{supp}(\Phi) \subseteq F(S)$, we construct h_1 such that

$$\langle h_1, \Phi\rangle = \left(\alpha_2, \widehat{\Phi}\right)_{L^2(F(S))} - \left(\alpha_1, \int_{\mathbb{R}} \widehat{\widehat{\Phi}}(s, y_2, y_3)\mathrm{d}s\, \phi(y_1)\right)_{L^2(F(S))}$$

where

$$\widehat{\Phi}(y) = \int_{-\infty}^{y_1} \left(\Phi(t, y_2, y_3) - \left(\int_{\mathbb{R}} \Phi(s, y_2, y_3)\mathrm{d}s\right)\phi(t)\right)\mathrm{d}t,$$
$$\widehat{\widehat{\Phi}}(y) = \int_{-\infty}^{y_2} \left(\Phi(y_1, \tau, y_3) - \left(\int_{\mathbb{R}} \Phi(y_1, t, y_3)\mathrm{d}t\right)\psi(\tau)\right)\mathrm{d}\tau.$$

Note $\langle h_1, \cdot\rangle$ denotes the linear functional on $\mathcal{C}_0^{+\infty}(F(S))$. With these definitions, the following holds.

$$\widehat{\frac{\partial\Phi}{\partial y_1}} = \Phi(y), \quad \widehat{\widehat{\frac{\partial\Phi}{\partial y_1}}} = 0, \quad \widehat{\widehat{\frac{\partial\Phi}{\partial y_2}}} = \Phi(y).$$

Using these facts and combining it with property (3.26), we can see after some computations that

$$\left\langle h_1, \frac{\partial\Phi}{\partial y_1}\right\rangle = (\alpha_2, \Phi)_{L^2(F(S))} \quad \text{and} \quad \left\langle h_1, \frac{\partial\Phi}{\partial y_2}\right\rangle = -(\alpha_1, \Phi)_{L^2(F(S))}$$

which proves $\frac{\partial h_1}{\partial y_2} = \alpha_1$ and $\frac{\partial h_1}{\partial y_1} = -\alpha_2$ in the distribution sense. Moreover, it can also be shown that $|\langle h_1, \Phi\rangle| \leq C\|\Phi\|_{L^2(F(S))}$ for some constant $C > 0$. Using that $\mathcal{C}_0^{+\infty}(F(S))$ is dense in $L^2(F(S))$ and Riesz's Theorem we deduce that $h_1 \in L^2(F(S))$. Furthermore, since $\frac{\partial h_1}{\partial y_1}$, $\frac{\partial h_1}{\partial y_2}$ belong to $L^2(F(S))$, we can take values on the boundary and $h_1|_{\partial F(S)} \in L^2(\partial F(S))$. Similar arguments can be done for h_2. This proves item a) of the lemma.

b) We change variables according to (3.1) and work with z in S. Before beginning with the proof of this item we introduce some notation. Recall that B was an arbitrary connected bounded domain in $\mathbb{R}^2$ and that $s_2 = \partial B \times (0, l)$. Write $\partial B = b_1 \cup \cdots \cup b_m$ where b_i are its connected components. With this notation, for $i = 1, \ldots, m$ we define $\varsigma_i = b_i \times (0, l)$ so that $s_2 = \varsigma_1 \cup \cdots \cup \varsigma_m$. We parametrize the boundary ∂B by the arclength θ and, accordingly, each b_i by θ_i. Through this notes, $n = (n_1, n_2, n_3)$ will denote the unit outward normal vector on s_2.

Let $\widetilde{h}_1(z) = h_1(F(z))$ and let $\widetilde{\phi} = \widetilde{\phi}(z) \in \mathcal{C}^{+\infty}(\overline{S})$ be a smooth test function such that

$\widetilde{\phi}(z_1, z_2, 0) = \widetilde{\phi}(z_1, z_2, l) = 0$, namely, $\widetilde{\phi}|_{s_1^{(+)} \cup s_1^{(-)}} = 0$. We compute

$$\begin{aligned}
\int_{s_2} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta} \mathrm{d}A &= \int_{s_2} \widetilde{h}_1 \left(\frac{\partial \widetilde{\phi}}{\partial z_1} \frac{\partial z_1}{\partial \theta} + \frac{\partial \widetilde{\phi}}{\partial z_2} \frac{\partial z_2}{\partial \theta} \right) \mathrm{d}A = \int_{s_2} \widetilde{h}_1 \left(-n_2 \frac{\partial \widetilde{\phi}}{\partial z_1} + n_1 \frac{\partial \widetilde{\phi}}{\partial z_2} \right) \mathrm{d}A \\
&= \int_{s_2} \left(n_1 \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial z_2} - n_2 \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial z_1} \right) \mathrm{d}A \\
&= \int_S \left(\frac{\partial}{\partial z_1} \left(\widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial z_2} \right) - \frac{\partial}{\partial z_2} \left(\widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial z_1} \right) \right) \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 = \int_S \left(\frac{\partial \widetilde{h}_1}{\partial z_1} \frac{\partial \widetilde{\phi}}{\partial z_2} - \frac{\partial \widetilde{h}_1}{\partial z_2} \frac{\partial \widetilde{\phi}}{\partial z_1} \right) \mathrm{d}z.
\end{aligned}$$

With the change of variables $(y_1, y_2, y_3) = (F_1(z), F_2(z), z_3)$ and (3.26), with some computations it can be seen that

$$\int_S \left(\frac{\partial \widetilde{h}_1}{\partial z_1} \frac{\partial \widetilde{\phi}}{\partial z_2} - \frac{\partial \widetilde{h}_1}{\partial z_2} \frac{\partial \widetilde{\phi}}{\partial z_1} \right) \mathrm{d}z = - \int_{F(S)} \left(\alpha_2 \frac{\partial \phi}{\partial y_2} + \alpha_1 \frac{\partial \phi}{\partial y_1} \right) \mathrm{d}y$$

where $\phi \in \mathcal{C}^{+\infty}(\overline{F(S)})$. Due to (3.27), we conclude

$$\int_{s_2} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta} \mathrm{d}A = \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta_j} \mathrm{d}A = 0. \tag{A.1}$$

For any $i = 1, \ldots, m$, choose a test function $\widetilde{\phi}$ such that $\widetilde{\phi}|_{\varsigma_j} \equiv 0$ for $j \neq i$. Then (A.1) becomes

$$\sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta_j} \mathrm{d}A = \int_{\varsigma_i} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta_i} \mathrm{d}A = 0.$$

We will now show that $\widetilde{h}_1|_{\varsigma_i}$ does not depend on (z_1, z_2) over ς_i for $i = 1, \ldots, m$. Let $\phi = \phi(\theta, z_3) \in \mathcal{C}^{+\infty}(s_2)$ be a test function such that $\phi(\theta, 0) = \phi(\theta, l) = 0$. We define $\widehat{\phi}$ and χ such that for $i = 1, \ldots, m$

$$\widehat{\phi}|_{\varsigma_i} = \phi|_{\varsigma_i} - \int_{b_i} \phi(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta}, \qquad \chi|_{\varsigma_i} = \int_0^{\theta_i} \widehat{\phi}(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta}.$$

We compute

$$\begin{aligned}
\int_{s_2} \widetilde{h}_1 \phi(\theta, z_3) \mathrm{d}A &= \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \phi(\theta_j, z_3) \mathrm{d}A \\
&= \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \left(\phi(\theta, z_3) - \int_{b_j} \phi(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta} + \int_{b_j} \phi(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta} \right) \mathrm{d}A \\
&= \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \left(\frac{\partial \chi}{\partial \theta_j}(\theta_j, z_3) + \int_{b_j} \phi(\widetilde{\theta}_j, z_3) \mathrm{d}\widetilde{\theta} \right) \mathrm{d}A. \qquad \text{(A.2)}
\end{aligned}$$

From (A.1), we can easily see that for any $j = 1, \dots, m$

$$\int_{\varsigma_j} \widetilde{h}_1 \frac{\partial \chi}{\partial \theta_j}(\theta_j, z_3)\mathrm{d}A = 0.$$

Therefore, we continue the computations in (A.2) and we obtain

$$\begin{aligned}\sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \phi(\theta_j, z_3)\mathrm{d}A &= \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1(\theta_j, z_3) \left(\int_{b_j} \phi(\widetilde{\theta}_j, z_3)\mathrm{d}\widetilde{\theta} \right) \mathrm{d}\theta \mathrm{d}z_3 \\ &= \sum_{j=1}^{m} \int_{\varsigma_j} \phi(\widetilde{\theta}, z_3) \left(\int_{b_j} \widetilde{h}_1(\theta_j, z_3)\mathrm{d}\theta_j \right) \mathrm{d}\widetilde{\theta}\mathrm{d}z_3 \\ &= \sum_{j=1}^{m} \int_{\varsigma_j} \phi(\theta_j, z_3) \left(\int_{b_j} \widetilde{h}_1(\widetilde{\theta}, z_3)\mathrm{d}\widetilde{\theta} \right) \mathrm{d}\theta \mathrm{d}z_3,\end{aligned}$$

where we used Fubini's Theorem and we renamed the variables θ_j and $\widetilde{\theta}$. Sending it all to the left-hand side we see

$$\sum_{j=1}^{m} \int_{\varsigma_j} \left(\widetilde{h}_1(\theta_j, z_3) - \int_{b_j} \widetilde{h}_1(\widetilde{\theta}, z_3)\mathrm{d}\widetilde{\theta} \right) \phi(\theta_j, z_3)\mathrm{d}\theta_j \mathrm{d}z_3 = 0.$$

For any $i = 1, \dots, m$, we choose a test function ϕ such that $\phi|_{\varsigma_j} \equiv 0$ for $j \neq i$ so that the previous equation becomes

$$\int_{\varsigma_i} \left(\widetilde{h}_1(\theta_i, z_3) - \int_{b_i} \widetilde{h}_1(\widetilde{\theta}, z_3)\mathrm{d}\widetilde{\theta} \right) \phi(\theta_i, z_3)\mathrm{d}\theta_i \mathrm{d}z_3 = 0.$$

Since $\phi|_{\varsigma_i}$ is arbitrary, we conclude that

$$\widetilde{h}_1|_{\varsigma_i} = \int_{b_i} \widetilde{h}_1(\widetilde{\theta}, z_3)\mathrm{d}\widetilde{\theta},$$

hence $\widetilde{h}_1|_{\varsigma_i}$ does not depend on θ_i, that is, it does not depend on (z_1, z_2) along ς_i. Therefore, using the regularity of F, we conclude that $h_1|_{g_i}$ does not depend on (y_1, y_2) along g_i. All of the above calculations can be made similarly to prove that $h_2|_{g_i}$ does not depend on (y_1, y_2) along g_i. □

Proof of Lemma 3.6. Let $n = (n_1, n_2)$ be the unit outward normal vector on $\partial\widehat{\Omega}(y_3)$ and write $\partial\widehat{\Omega}(y_3) = \widehat{g}_1(y_3) \cup \cdots \cup \widehat{g}_m(y_m)$, where $\widehat{g}_j(y_3)$ are the connected components of $\partial\widehat{\Omega}(y_3)$ ($j = 1, \dots, m$). We use the divergence theorem for the 2-dimensional bounded domain enclosed by

$\widehat{g}_j(y_3)$ to see that for every $y_3 \in [0,l]$ and $j = 1,\dots,m$ we have

$$\int_{\widehat{g}_j(y_3)} n_i \mathrm{d}L = 0, \qquad (i = 1,2) \tag{A.3}$$

$$\int_{\widehat{g}_j(y_3)} y_2 n_1 \mathrm{d}L = 0, \qquad \int_{\widehat{g}_j(y_3)} y_1 n_2 \mathrm{d}L = 0, \tag{A.4}$$

$$\int_{\widehat{g}_j(y_3)} (y_2 n_2 - y_1 n_1) \mathrm{d}L = 0. \tag{A.5}$$

Throughout the next computations, we will use the fact that for $j = 1,\dots,m$ we have that $h_1|_{\widehat{g}_i(y_3)}$, $h_2|_{\widehat{g}_i(y_3)}$ do not depend on $y' = (y_1, y_2)$ along $\widehat{g}_i(y_3)$ (see Lemma 3.5-b)), so we can write $h_p|_{\widehat{g}_j(y_3)} = h_p|_{\widehat{g}_j(y_3)}(y_3)$ for $p = 1,2$. Using the divergence theorem we first calculate

$$\begin{aligned}
\int_{\widehat{\Omega}(y_3)} Q \mathrm{d}y' &= \int_{\widehat{\Omega}(y_3)} \left(\frac{\partial h_1}{\partial y_2} - \frac{\partial h_2}{\partial y_1} \right) \mathrm{d}y' = \int_{\partial\widehat{\Omega}(y_3)} (h_1 n_2 - h_2 n_1)\, \mathrm{d}L \\
&= \sum_{j=1}^{m} \int_{\widehat{g}_j(y_3)} (h_1 n_2 - h_2 n_1)\, \mathrm{d}L \\
&= \sum_{j=1}^{m} \left(h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} n_2 \mathrm{d}L - h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} n_1 \mathrm{d}L \right) = 0.
\end{aligned}$$

The last equality is due to (A.3). We have seen that

$$\int_{\widehat{\Omega}(y_3)} Q \mathrm{d}y' = 0.$$

We now proceed to prove that $\int_{\widehat{\Omega}(y_3)} Q y_i \mathrm{d}y' = 0$ for $i = 1,2$. For that purpose, from (3.28) and (3.29), we see that

$$\begin{aligned}
&\int_{\widehat{\Omega}(y_3)} \left(\frac{\partial h_1}{\partial y_1} + \frac{\partial h_2}{\partial y_2} \right) y_1 \mathrm{d}y' = 0 \\
&\int_{\partial\widehat{\Omega}(y_3)} (y_1 h_2 n_2 + y_1 h_1 n_1)\, \mathrm{d}L - \int_{\widehat{\Omega}(y_3)} h_1 \mathrm{d}y' = 0 \\
&\sum_{j=1}^{m} \left(h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_2 \mathrm{d}L + h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) - \int_{\widehat{\Omega}(y_3)} h_1 \mathrm{d}y' = 0 \\
&\sum_{j=1}^{m} \left(h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) - \int_{\widehat{\Omega}(y_3)} h_1 \mathrm{d}y' = 0
\end{aligned}$$

where we used (A.4) in the last step. Therefore

$$\sum_{j=1}^{m} \left(h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) = \int_{\widehat{\Omega}(y_3)} h_1 \mathrm{d}y'. \tag{A.6}$$

Similarly, again from (3.29), we see

$$\int_{\widehat{\Omega}(y_3)} \left(\frac{\partial h_1}{\partial y_1} + \frac{\partial h_2}{\partial y_2} \right) y_2 \mathrm{d}y' = 0$$

and we get

$$\sum_{j=1}^{m} \left(h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_2 n_2 \mathrm{d}L \right) = \int_{\widehat{\Omega}(y_3)} h_2 \mathrm{d}y'. \tag{A.7}$$

Using integration by parts and (A.4) again we compute

$$\begin{aligned}
\int_{\widehat{\Omega}(y_3)} Q y_1 \mathrm{d}y' &= \int_{\widehat{\Omega}(y_3)} \left(\frac{\partial h_1}{\partial z_2} - \frac{\partial h_2}{\partial z_1} \right) y_1 \mathrm{d}y' \qquad\qquad (\mathrm{A.8}) \\
&= \int_{\partial\widehat{\Omega}(y_3)} \left(y_1 h_1 n_2 - y_1 h_2 n_1 \right) \mathrm{d}L - \int_{\widehat{\Omega}(y_3)} -h_2 \mathrm{d}y' \\
&= \sum_{j=1}^{m} \left(h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_2 \mathrm{d}L - h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) + \int_{\widehat{\Omega}(y_3)} h_2 \mathrm{d}y' \\
&= \sum_{j=1}^{m} \left(-h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) + \int_{\widehat{\Omega}(y_3)} h_2 \mathrm{d}y'.
\end{aligned}$$

Using the relation found in (A.7) and property (A.5), the equation (A.8) becomes

$$\begin{aligned}
&\sum_{j=1}^{m} \left(-h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) + \int_{\widehat{\Omega}(y_3)} h_2 \mathrm{d}y' \\
&\qquad = \sum_{j=1}^{m} \left(-h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) + \sum_{j=1}^{m} \left(h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_2 n_2 \mathrm{d}L \right) \\
&\qquad = \sum_{j=1}^{m} \left(h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} (y_2 n_2 - y_1 n_1) \mathrm{d}L \right) = 0
\end{aligned}$$

and we see that $\int_{\widehat{\Omega}(y_3)} Q y_1 \mathrm{d}y' = 0$. In a similar way, using (A.4), (A.5) and (A.6), we can prove that $\int_{\widehat{\Omega}(y_3)} Q y_2 \mathrm{d}y' = 0$. □

Proposition A.1. *Let $\widetilde{\Omega}$ be a domain in $\mathbb{R}^2$ and let $V_1(y_1, y_2), V_2(y_1, y_2) \in \mathcal{D}'(\widetilde{\Omega})$. If*

$$\frac{\partial V_i}{\partial y_j} + \frac{\partial V_j}{\partial y_i} = 0 \quad \text{for } 1 \le i, j \le 2$$

in the distribution sense, then there exist constants $C_1, C_2, C_3 \in \mathbb{R}$ such that

$$V_1(y_1, y_2) = -C_3 y_2 + C_1, \quad V_2(y_1, y_2) = C_3 y_1 + C_2.$$

Proof. The idea of the proof is to use a 2-dimensional version of the fact that if for $V = (V_1, V_2, V_3)$ and $1 \leq i, j \leq 3$ we have $E_{ij}(V) = \frac{1}{2}(\frac{\partial V_i}{\partial y_j} + \frac{\partial V_j}{\partial y_i}) = 0$, then $V = \mathcal{O}y + C$, where $\mathcal{O} \in M_{3\times 3}(\mathbb{R})$ is an anti-symmetric matrix and $C \in \mathbb{R}^3$ is a constant vector. In addition, this can be shown using that

$$\frac{\partial^2 V_i}{\partial y_j \partial y_k} = \frac{\partial E_{ik}(V)}{\partial y_j} + \frac{\partial E_{ij}(V)}{\partial y_k} - \frac{\partial E_{jk}(V)}{\partial y_i} \quad (1 \leq i, j, k \leq 3).$$

Further details can be seen in Duvaut-Lion [11] and Schwartz [30]. □

Remark A.2. We present here the explicit forms of the matrix $\mathcal{X}$ and the vector $\mathcal{Y}$ used in Section 7 in order to find a minimum.

$$\mathcal{X} = \begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ \mathcal{X}_2^T & \mathcal{X}_3 \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{pmatrix},$$

where

$$\mathcal{X}_1 = \begin{pmatrix} (4\lambda_1+8\lambda_2)A_{11} & (4\lambda_1+8\lambda_2)A_{12} & 0 & 0 & 4\lambda_1 A_{11} & 4\lambda_1 A_{12} \\ (4\lambda_1+8\lambda_2)A_{12} & 4\lambda_2 A_{11}+(4\lambda_1+8\lambda_2)A_{22} & 4\lambda_2 A_{12} & 4\lambda_2 A_{11} & (4\lambda_1+4\lambda_2)A_{12} & 4\lambda_1 A_{22} \\ 0 & 4\lambda_2 A_{12} & 4\lambda_2 A_{22} & 4\lambda_2 A_{12} & 4\lambda_2 A_{22} & 0 \\ 0 & 4\lambda_2 A_{11} & 4\lambda_2 A_{12} & 4\lambda_2 A_{11} & 4\lambda_2 A_{12} & 0 \\ 4\lambda_1 A_{11} & (4\lambda_1+4\lambda_2)A_{12} & 4\lambda_2 A_{22} & 4\lambda_2 A_{12} & (4\lambda_1+8\lambda_2)A_{11}+4\lambda_2 A_{22} & (4\lambda_1+8\lambda_2)A_{12} \\ 4\lambda_1 A_{12} & 4\lambda_1 A_{22} & 0 & 0 & (4\lambda_1+8\lambda_2)A_{12} & (4\lambda_1+8\lambda_2)A_{22} \end{pmatrix},$$

$$\mathcal{X}_2 = \begin{pmatrix} (2\lambda_1+4\lambda_2)K_1 & 0 & 0 & 2\lambda_1 K_1 \\ (2\lambda_1+4\lambda_2)K_2 & 2\lambda_2 K_1 & 2\lambda_2 K_1 & 2\lambda_1 K_2 \\ 0 & 2\lambda_2 K_2 & 2\lambda_2 K_2 & 0 \\ 0 & 2\lambda_2 K_1 & 2\lambda_2 K_1 & 0 \\ 2\lambda_1 K_1 & 2\lambda_2 K_2 & 2\lambda_2 K_2 & (2\lambda_1+4\lambda_2)K_1 \\ 2\lambda_1 K_2 & 0 & 0 & (2\lambda_1+4\lambda_2)K_2 \end{pmatrix}, \quad \mathcal{X}_3 = \begin{pmatrix} (\lambda_1+2\lambda_2)H & 0 & 0 & \lambda_1 H \\ 0 & \lambda_2 H & \lambda_2 H & 0 \\ 0 & \lambda_2 H & \lambda_2 H & 0 \\ \lambda_1 H & 0 & 0 & (\lambda_1+2\lambda_2)H \end{pmatrix},$$

$$\mathcal{Y}_1 = \begin{pmatrix} 4\lambda_1\gamma_1 \\ 4\lambda_1\gamma_2 \\ 0 \\ 0 \\ 4\lambda_1\gamma_1 \\ 4\lambda_1\gamma_2 \end{pmatrix}, \quad \mathcal{Y}_2 = \begin{pmatrix} 2\lambda_1\gamma_0 \\ 0 \\ 0 \\ 2\lambda_1\gamma_0 \end{pmatrix} \quad \text{with} \quad \begin{cases} \gamma_0 = H\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3} - K_1\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2} - K_2\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}, \\ \gamma_1 = K_1\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3} - A_{11}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2} - A_{12}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}, \\ \gamma_2 = K_2\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3} - A_{12}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2} - A_{22}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}. \end{cases}$$

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