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AN ALGEBRAIC STUDY OF LOGICS OF VARIABLE INCLUSION AND ANALYTIC CONTAINMENT

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Contents

Introduction

This thesis investigates a wide family of logics whose common feature is to admit a syntactic definition based on specific variable inclusion principles. This family has been divided into three main components: logics of left variable inclusion, containment logics, and the logic of demodalised analytic implication. The aim is to study such logics, in full generality, within the framework of modern abstract algebraic logic (AAL). Concerning logics of left variable inclusion, a similar task has been reached in [13] for a particular member of the family, namely paraconsistent weak Kleene logic PWK.

Containment logics, for which the recent book [39] is the main reference, have never been approached from an algebraic point of view.

The logic DAI of demodalised analytic implication has been introduced by J.M. Dunn (and independently investigated by R.D. Epstein) as a variation on a time-honoured logical system by C.I. Lewis's student W.T. Parry. DAI has been investigated both proof-theoretically and model-theoretically, but no study so far has focussed on DAI from the viewpoint of abstract algebraic logic. The only investigation into the algebraic semantics of DAI is [35], a paper that — also for historical reasons — does not use the concepts and tools of contemporary abstract algebraic logic.

The key mathematical tool employed in this work is a special class operator called *Płonka sums*, a well known algebraic construction connected with the theory of regular varieties ([4, 51, 52]). Surprisingly enough, it turns out that an appropriate generalization of Płonka sums to logical matrices can be used to model a great amount of logics featuring some variable inclusion requirement in their definitions. More generally, almost all the existing examples of logics defined by a syntactic variable inclusion requirement admit a matrix semantics based on Płonka sums.

The dissertation is structured as follows.

Chapter 1 (subsections 1.1, 1.2, 1.3) provides the necessary preliminaries that will be needed throughout the thesis. The core of the thesis are Chapters 2-5. Chapters 2-3 have a similar structure, and they study in full generality logics of left and right variable inclusion respectively. Firstly, it will be provided a complete matrix semantics for these logics, based on new notion of Płonka sums of matrices. Then, we will produce a method to generate a complete Hilbert style calculus for a wide class of logics of variable inclusion, which contains all the known examples in the literature. The last sections of these chapters are devoted to the investigation of the Leibniz and Suzko reduced models, as well as the classification of logics of variable inclusion within the Leibniz hierarchy.

Chapter 4 studies the algebraization of logics of left varibale inclusion viewed as Gentzen systems and determines the lattice of sublogics of variable inclusion of an arbitrary consequence relation.

The final Chapter 5 focuses on the logic of demodalised analytic implication. Firstly, we present two complete matrix semantics for it, the first one based on Płonka sums, the second one based on a specifically defined notion of "I-product of matrices". Then, we characterize the Leibniz reduced models of DAI and we prove that it is algebraizable.

Logics of variable inclusion in philosophy

One of the most fruitful applications of logics of variable inclusion is, without doubts, the philosophical debate. In this introductory section we briefly review some of the most recent philosophical proposals concerning these logics.

In order to grasp the philosophical appeal of logics of variable inclusion, it is useful to underline their relation with the family of Kleene logics. In his famous book [56], Kleene specifies two different truth-tables with the aim of describing two possible interpretations of classical propositional connectives in presence of partially defined predicates. The first proposal is represented by the Strong Kleene tables (**SK**),

$\wedge$	0	n	1
0	0	0	0
n	0	n	n
1	0	n	1

$\vee$	0	n	1
0	0	n	1
n	n	n	1
1	1	1	1

$\neg$	
1	0
n	n
0	1

while the other one consists on the so-called Weak Kleene tables (**WK**)

$\wedge$	0	n	1
0	0	n	0
n	n	n	n
1	0	n	1

$\vee$	0	n	1
0	0	n	1
n	n	n	n
1	1	n	1

$\neg$	
1	0
n	n
0	1

Each of these tables is the algebraic reduct of two logical matrices, defined as follows:

- $\langle \mathbf{SK}, \{1\} \rangle \Rightarrow \mathsf{K_3}$
- $\langle \mathbf{SK}, \{1, n\} \rangle \Rightarrow$ LP
- $\langle \mathbf{WK}, \{1\} \rangle \Rightarrow \mathsf{B_3}$
- $\langle \mathbf{WK}\{1, n\} \rangle \Rightarrow$ PWK[1]

The logic induced by $\langle \mathbf{SK}, \{1\} \rangle$, usually known as Strong Kleene logic ($\mathsf{K_3}$) found a wide range of applications in philosophy, from non-monotonic reasoning ([95]) to the glorious work on truth by Kripke [59]. $\mathsf{K_3}$ is also the ancestor of a long tradition of logical frameworks for theories of truth, whose milestone is [40], based on the notion of *paracompleteness*, i.e. the idea that sentences of the form $\varphi \vee \neg\varphi$ need not to be always true. The philosophical connection between $\mathsf{K_3}$, paracompleteness and truth is the philosophical interpretation of the element n in $\mathbf{SK}$. Usually, the value n is taken to denote the idea of being "neither true nor false". This, together with the fact that the truth set in $\mathsf{K_3}$ is just $\{1\}$, suffices to show the failure of $\varphi \vee \neg\varphi$. In other words, Strong Kleene logic aims at preserving the validity of the sentences that are true, and true only.

The logic $\langle \mathbf{SK}, \{1, n\} \rangle$, known as the logic of Paradox (LP), has been popularized by Priest [78] as possible solution to paradoxes and to the problem of managing situations with inconsistent information, such as change of time and motion [79]. LP has a similar story to K3, but with respect to the notion of *paraconsistency* that, in this context, amounts to the idea that a sentence of the form $\varphi \wedge \neg\varphi$ needs not to be always false, and false only. In logical terms, the difference between LP and $\mathsf{K_3}$ obviously is the truth set of their characteristic matrices. However, the main

[1] Observe that, on the contrary on what it is stated in [92, p.10], given two arbitrary Kleene logics $\vdash, \vdash'$ it always holds $\vdash \not\leq \vdash'$. The only non trivial cases to show are PWK $\not\leq$ LP and $\mathsf{B_3} \not\leq \mathsf{K_3}$. For the first one, it suffices to notice that $(x \wedge \neg x) \vee (y \wedge \neg y) \vdash_{\mathsf{PWK}} x \wedge y$ while $(x \wedge \neg x) \vee (y \wedge \neg y) \nvdash_{\mathsf{LP}} x \wedge y$. For the second one, just consider that $x \vee y \vdash_{\mathsf{B_3}} x \vee \neg x$ while $x \vee y \nvdash_{\mathsf{K_3}} x \vee \neg x$.

philosophical debate concerns the difference in the intepretation of the element n. In LP, n is intended as being "both true and false" and so, by considering an evaluation that assigns the value n to φ (it always exists), it is easy to observe that $\varphi \wedge \neg\varphi$ can be also true.

The two logics based on **WK**, as noticed, just reflects the structure of their most famous cousins LP and K_3. They coincide with B_3 and PWK respectively, and they are the right and the left variable inclusion companions of Classical Logic (CL).

If the interpretation of the non classical value in the logics based on the Strong Kleene tables is pretty unanimous among philosophers, this is not the case for B_3 and PWK. The first philosophical reason of interest is the "infectious" or "contaminating" behaviour that the element n has over the two classical truth values. It is indeed easy to see that every operation $\delta(n, \vec{c})$ with $\vec{c} \subseteq \{0, n, 1\}$ on **WK** in which n really occurs is such that $\delta(n, \vec{c}) = n$, so, in this sense, n contaminates every sentence in which it occurs. Now, concerning this aspect, the main philosophical problem is how to interpret the value n in a suitable way. The first official proposals originate in [11] and [50], and both suggest that n should be considered as "meaningless". In few words, the idea is that any sentence that contains a meaningless part, is meaningless as well. In particular, Bochvar suggests that the presence of an infectious value in B_3 can be employed for the treatment of Russell's paradox, while Hállden proposes PWK in the context of vagueness and other semantical paradoxes. As an example, Bochvar intrpretation of n is reasonable in order to motivate the fact that $\varphi \nvdash_{\mathsf{B}_3} \varphi \vee \psi$. Indeed, from the fact that φ is true, it is not necessary to conclude that $\varphi \vee \psi$ is true as well, for ψ may contain nonsensical information.

However, recently, J.C. Beall [1] suggested a new interpretation of the Weak Kleene tables. As he says, a common worry about the "meaningless" interpretation of n in **WK** is that a logical proposition is usually assumed to be meaningful by itself, and so it is never the case that a meaningless proposition is (e.g.) conjuncted with a meaningful one. A possible solution relies on the notion of *being off-topic*. Indeed, it may well be that a conjunction of meaningful propositions contains a conjunct that is off-topic with respect to a set of other propositions. Moreover, it is not desirable that a sentence containing some "off-topic" parts should be considered wholly "on-topic". So, it seems that the understanding of n as off-topic fits with its infectious behaviour.

The philosophical interest for logics of variable inclusion is gradually increasing, and it is not confined to PWK and B_3. In [93], the right variable inclusion companion of PWK and the left variable inclusion companion of B_3 are discussed within the framework of theories of truth. As a matter of fact, these logics are defined by a 4 element truth table (see Example 3.2.3) and are both, at the same time, paracomplete and paraconsistent. Such feature is considered as an advantage by the authors of [93], as it is possible to treat both the paradoxes which intuitively should be neither true nor false, and the ones which should be both true and false.

There are more places in the literature where we can find some philosophical insights concerning logics of variable inclusion (see for example [26, 27]), but we conclude here our brief introductory summary.

Chapter 1

Preliminaries

In this section we review the mathematical background that will be used in the dissertation. Any reader which is familiar with standard universal algebra, Płonka sums and basic notions of abstarct algebraic logic can skip the entire part.

1.1 Universal Algebra

We assume the reader is familiar with the notions of *algebra, similarity type, subalgebras, ultraproducts of algebras, homomorphisms, embeddings, homomorphic images of algebras, and congruences on algebras*. Such information can be found in any of the following books [2, 18, 32, 65], that are standard references on universal algebra. We denote algebras by $\mathbf{A}, \mathbf{B}, \mathbf{C} \dots$ respectively with universes $A, B, C \dots$. Given an algebra $\mathbf{A}$, the set of congruences on $\mathbf{A}$ is denoted by $\mathrm{Co}\,\mathbf{A}$. We write $\mathrm{id}^{\mathbf{A}}$ for the smallest congruence on $\mathbf{A}$ and $A \times A$ to denote the total congruence on $\mathbf{A}$. When it will be clear from the context, we will omit the superscript $^{\mathbf{A}}$.

A *closure operator* on a set A is a map $C \colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that for all $X, Y \in \mathcal{P}(A)$ the following conditions hold:

R. $X \subseteq C(X)$.

M. If $X \subseteq Y$, then $C(X) \subseteq C(Y)$.

I. $CC(X) = C(X)$.

A set $X \subseteq A$ is a *closed set* if $C(X) = X$. A *closure system* on A is a collection $\mathscr{C} \subseteq \mathcal{P}A$ closed under arbitrary intersections and such that $A \in \mathscr{C}$. It turns out that the collection of closed sets of a closure operator C on A is a closure system $\mathscr{C}$ on A and that each closure system $\mathscr{C}$ naturally induces a closure operator C whose closed sets are exactly the members of $\mathscr{C}$.

We review the definition of lattice, bounded lattice and Boolean Algebra. As it often happens in algebraic logic, many examples will rely on these classes of algebras. A *lattice* $\mathbf{L}$ is an algebra $\langle L, \wedge, \vee \rangle$ of type $\langle 2, 2 \rangle$, with operations $(\wedge, \vee)$ satisfying the following conditions:

L1. $x \wedge y \approx y \wedge x$; $x \vee y \approx y \vee x$

L2. $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$; $x \vee (y \vee z) \approx (x \vee y) \vee z$

L3. $x \wedge x \approx x$; $x \vee x \approx x$

L4. $x \wedge (x \vee y) \approx x$; $x \vee (x \wedge y) \approx x$.

A lattice is said to be *distributive* iff the following equations hold:

L1. $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$; $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$.

A lattice can be also defined as a partially ordered set $\langle A, \leq \rangle$, in which any two elements a, b in A have both a *supremum* (*join*) denoted by $\sup(a, b)$, and an *infimum* (*meet*) denoted by $\inf(a, b)$.

A *bounded lattice* $\mathbf{L}$ is an algebra $\langle L, \wedge, \vee, 0, 1 \rangle$ of type $\langle 2, 2, 0, 0 \rangle$ satisfying the following conditions:

BL1. $\langle L, \wedge, \vee \rangle$ is a lattice

BL2. $x \wedge 0 \approx 0$; $x \vee 1 \approx 1$.

A *Boolean algebra* $\mathbf{B}$ is an algebra $\langle B, \wedge, \vee, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ satisfying the following conditions:

BA1. $\langle B, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice

BA2. $x \wedge \neg x \approx 0$; $x \vee \neg x \approx 1$

BA3. $\neg(x \wedge y) \approx (\neg x \vee \neg y)$

BA4. $\neg\neg x \approx x$.

A *De Morgan lattice* is a distributive lattice that satisfies **BA3** and **BA4**. Some useful concrete examples of lattices can be obtained from the general theory of closure operators. Indeed, given a closure system $\mathscr{C}$, we have that $\langle \mathscr{C}, \subseteq \rangle$ is a complete lattice under the operations $\bigwedge X_i = \bigcap X_i$

and $\bigvee X_i = C \bigcup X_i$ for any $\{X_i : i \in I\} \subseteq \mathscr{C}$. Given a set A we denote by $\mathrm{CO}(A)$ the set of all closure operators on A and by $\mathrm{CS}(A)$ the set of all closure systems on A. Given $C, C' \in \mathrm{CO}(A)$, by letting

$$C \leq C' \text{ if } C(X) \subseteq C'(X) \text{ for all } X \subseteq A,$$

the usual ordering relation on a power set makes $\langle \mathrm{CO}(A), \leq \rangle$, $\langle \mathrm{CS}(A), \leq \rangle$ ordered sets. The following result underlines the relation between closure operators and closure systems.

Theorem 1.1.1. *[42, Theorem 1.33]. The ordered sets $\langle \mathrm{CO}(A), \leq \rangle$, $\langle \mathrm{CS}(A), \leq \rangle$ are dually isomorphic, i.e.*

$$C \leq C' \iff \mathscr{C} \leq \mathscr{C}'.$$

A fundamental result concerning closure operators and closure systems is the following

Theorem 1.1.2. *[42, Theorem 1.57]. Let A be a set. Then*

(i) *$\langle \mathrm{CO}(A), \leq \rangle$ is a complete lattice, with arbitrary meets and joins defined as*

$$(\bigwedge C_i)(X) := \bigcap C_i X$$

for all $X \subseteq A$, and

$$(\bigvee_{i \in I} C_i)X := \bigcap \{F \in \bigcap_{i \in I} \mathscr{C}_i : F \supseteq X\}$$

(ii) *$\langle \mathrm{CS}(A), \leq \rangle$ is a complete lattice, with arbitrary meets and joins defined as*

$$\bigwedge \mathscr{C}_i := \bigcap \mathscr{C}_i$$

for all $X \subseteq A$, and

$$(\bigvee_{i \in I} \mathscr{C}_i) := \{F \supseteq A : F = \bigcup_{i \in I} F_i \text{ with } F_i \in \mathscr{C}_i \text{ for all } i \in I\}$$

(iii) *These two complete lattices are dually isomorphic.*

The above Theorem will have important applications to logic as well (see Section 1.2 and Chapter 4).

A fundamental topic in universal algebra is the investigation of classes of algebras of the same type closed under one or more constructions.

According to current literature, we will write for an algebra $\mathbf{A}$ and a class of algebras K

$\mathbf{A} \in \mathbb{H}(\mathbf{A})$, iff $\mathbf{A}$ is a homomorphic image of some member of K

$\mathbf{A} \in \mathbb{I}(\mathbf{A})$, iff $\mathbf{A}$ is an isomorphic image of some member of K,

$\mathbf{A} \in \mathbb{S}(\mathbf{A})$, iff $\mathbf{A}$ is a subalgebra of some member of K,

$\mathbf{A} \in \mathbb{P}(\mathsf{K})$, iff $\mathbf{A}$ is a direct product of a nonempty family of members of K,

$\mathbf{A} \in \mathbb{P}_{\mathrm{SD}}(\mathsf{K})$, iff $\mathbf{A}$ is a subdirect product of a nonempty family of members of K,

$\mathbf{A} \in \mathbb{P}_{\mathrm{U}}(\mathsf{K})$ iff $\mathbf{A}$ is an ultraproduct of a nonempty family of members of K.

We say that a class of algebras $\mathbb{K}$ is closed under a class operator $\mathbb{O}$ if $\mathbb{O}(\mathsf{K}) \subseteq \mathbf{K}$, and that $\mathbb{O}$ is idempotent if $\mathbb{O}\mathbb{O}(\mathsf{K}) = \mathbb{O}(\mathsf{K})$. Relevant relations between class operators are: $\mathbb{S}\mathbb{H} \leq \mathbb{H}\mathbb{S}$, $\mathbb{P}\mathbb{S} \leq \mathbb{S}\mathbb{P}$, $\mathbb{P}\mathbb{H} \leq \mathbb{H}\mathbb{P}$. Also $\mathbb{H}$,$\mathbb{S}$ and $\mathbb{I}\mathbb{P}$ are idempotent.

A nonempty class K of algebras of a certain type is called a *variety* if it is closed under homomorphic images, subalgebras and direct products.

If $\mathbb{K}$ is a class of algebras, we will write $\mathbb{V}(\mathsf{K})$ the smallest variety containing K, and we call $\mathbb{V}(\mathsf{K})$ the variety generated by K.

A *quasivariety* is a class of algebras axiomatized by a set of quasi-equations, i.e. implications of the following kind:

$$t_1 \approx u_1 \,\&\ldots\&\, t_n \approx u_n \Rightarrow t \approx u$$

where each t_i with $1 \leq i \leq n$ is a term.

Equivalently, a nonempty class K of algebras of a certain type is a quasivariety if it is closed under subalgebras, products, and ultraproducts. The smallest quasivariety which contains K is the class $\mathbb{S}\mathbb{P}_{\mathrm{U}}\mathsf{K}$.

A *generalised quasi-equation* is a (possibly) infinitary formula

$$\&_{i\in I}\ t_i \approx u_i \Rightarrow t \approx u$$

and a *generalised quasivariety* is a class of algebras axiomatized by generalised quasi-equations.

Given a class of algebras K, a set of equations E and an equation $\epsilon \approx \delta$, if $\&E \Rightarrow \epsilon \approx \delta$ holds in K, then we write $E \vDash_{\mathsf{K}} \epsilon \approx \delta$.

1.2 Abstract Algebraic Logic

For standard textbooks on abstract algebraic logic we refer the reader to [9, 7, 10, 30, 42, 44, 46, 98].

Logics and matrices

Let **Fm** be the absolutely free algebra of a fixed type built up over a countably infinite set Var of variables. Given a formula $\varphi \in Fm$, we denote by $Var(\varphi)$ the set of variables really occurring in φ. Similarly, given $\Gamma \subseteq Fm$, we set

$$Var(\Gamma) = \bigcup\{Var(\gamma) \colon \gamma \in \Gamma\}.$$

A *consequence relation* on a set A is a relation $\vdash \subseteq \mathcal{P}(A) \times A$ s.t. for all $X \cup Y \cup \{x\} \subseteq A$,

R. If $x \in X$, then $X \vdash x$

M. If $X \vdash x$ and $X \subseteq Y$, then $Y \vdash x$

C. If $X \vdash x$ and $Y \vdash y$ for all $y \in X$, then $Y \vdash x$.

Definition 1.2.1. A *logic* is a consequence relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$, which is *substitution-invariant* in the sense that for every substitution (i.e. endomorphism) $\sigma \colon \mathbf{Fm} \to \mathbf{Fm}$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma\Gamma \vdash \sigma\varphi.$$

Given $\varphi, \psi \in Fm$, we write $\varphi \dashv\vdash \psi$ as a shorthand for $\varphi \vdash \psi$ and $\psi \vdash \varphi$. Moreover, we denote by $\mathrm{Cn}_{\vdash} \colon \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ the closure operator associated with $\vdash$. A logic $\vdash$ is *finitary* when the following holds for all $\Gamma \cup \varphi \subseteq Fm$:

$$\Gamma \vdash \varphi \iff \exists \Delta \subseteq \Gamma \text{ s.t. } \Delta \text{ is finite and } \Delta \vdash \varphi.$$

It turns out that there is a bijective correspondence between logics of a fixed type and closure operators on a fixed particular set[1] and, moreover, such correspondence is order preserving. So, in the light of Theorem 1.1.2, the set of logics of a fixed type can be equipped with a complete lattice structure.

A *matrix* is a pair $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is an algebra and $F \subseteq A$. In this case, $\mathbf{A}$ is called the *algebraic reduct* of the matrix $\langle \mathbf{A}, F \rangle$. We denote by $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\text{SD}}$ respectively the class operators of isomorphic copies, substructures, direct products and subdirect products, which apply both to classes of algebras and classes of matrices.

[1] Such sets are known as **M**-sets (see [42, 8, 47]) and they essentially are sets equipped with a monoid action. We do not need the details here.

Every class of matrices M induces a logic as follows:

$$\Gamma \vdash_{\mathsf{M}} \varphi \iff \text{for every } \langle \mathbf{A}, F\rangle \in \mathsf{M} \text{ and hom. } h\colon \mathbf{Fm} \to \mathbf{A}, \\ \text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F.$$

Often, given a matrix $\langle \mathbf{A}, F\rangle$, a homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ is called an *evaluation*. A logic $\vdash$ is *complete* w.r.t. a class of matrices M when it coincides with $\vdash_{\mathsf{M}}$.

A matrix $\langle \mathbf{A}, F\rangle$ is a *model* of a logic $\vdash$ when

$$\text{if } \Gamma \vdash \varphi, \text{ then for every hom. } h\colon \mathbf{Fm} \to \mathbf{A}, \\ \text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F.$$

A set $F \subseteq A$ is a (deductive) *filter* of $\vdash$ on $\mathbf{A}$, or simply a $\vdash$-*filter*, when the matrix $\langle \mathbf{A}, F\rangle$ is a model of $\vdash$. Remarkably, it turns out that given a logic $\vdash$ and an algebra $\mathbf{A}$, the set of all $\vdash$-filters on $\mathbf{A}$, which we denote by $\mathcal{F}i_{\vdash}\mathbf{A}$, is a closure system. So, in the light of the results concerning closure systems presented in section 1.1, we obtain that $\langle \mathcal{F}i_{\vdash}\mathbf{A}, \subseteq\rangle$ is a complete lattice. Moreover, we denote by $\mathrm{Fg}^{\mathbf{A}}_{\vdash}(\cdot)$ its associated closure operator. That is, given a logic $\vdash$, an algebra $\mathbf{A}$ and $X \subseteq A$, $\mathrm{Fg}^{\mathbf{A}}_{\vdash}(X)$ denotes the least $\vdash$-filter on $\mathbf{A}$ containing X. We will say that $\mathrm{Fg}^{\mathbf{A}}_{\vdash}(X)$ is the $\vdash$-filter on $\mathbf{A}$ *generated* by X.

The following theorem provides an inductive characterization of the closure operator $\mathrm{Fg}^{\mathbf{A}}_{\vdash}(\cdot)$

Theorem 1.2.2. *[42, Thm.2.23] Let $\vdash$ be a finitary logic and let $\mathbf{A}$ be an algebra. Then for every $X \subseteq A$, $\mathrm{Fg}^{\mathbf{A}}_{\vdash}(X) = \bigcup_{n\in\omega} X_n$, where the sets X_n are defined inductively as follows:*

$$\begin{aligned} X_0 &:= X \\ X_{n+1} &:= \{a \in A : \textit{ there is } \Gamma \cup \{\varphi\} \subseteq Fm, \Gamma \textit{ finite, such that } \Gamma \vdash \varphi, \\ &\quad \textit{and there is a homomorphism } h : \mathbf{Fm} \to \mathbf{A} \textit{ with } h[\Gamma] \subseteq X_n \textit{ and } h(\varphi) = a\}. \end{aligned}$$

The previous theorem will be particularly useful in Chapter 4. Let $\mathbf{A}$ be an algebra and $F \subseteq A$. A congruence θ of $\mathbf{A}$ is *compatible* with F when for every $a, b \in A$,

$$\text{if } a \in F \text{ and } \langle a, b\rangle \in \theta, \text{ then } b \in F.$$

It turns out that there exists the largest congruence of $\mathbf{A}$ which is compatible with F. This congruence is called the *Leibniz congruence* of F on $\mathbf{A}$, and it is denoted by $\boldsymbol{\Omega}^{\mathbf{A}}F$.

Given an algebra $\mathbf{A}$, $F \subseteq A$ and a logic $\vdash$ the *Suszko congruence* of F on $\mathbf{A}$, is defined as

$$\widetilde{\mathbf{\Omega}}^{\mathbf{A}}_{\vdash} F := \bigcap \{\mathbf{\Omega}^{\mathbf{A}} G : F \subseteq G \text{ and } G \in \mathcal{F}i_{\vdash}\mathbf{A}\}.$$

The *Suszko operator* of $\vdash$ on an algebra $\mathbf{A}$ is the function $\widetilde{\mathbf{\Omega}}^{\mathbf{A}}_{\vdash}$ with domain $\mathcal{F}i_{\vdash}\mathbf{A}$ defined as $F \to \widetilde{\mathbf{\Omega}}^{\mathbf{A}}_{\vdash} F$ for all $F \in \mathcal{F}i_{\vdash}\mathbf{A}$.

Let $\mathbf{A}$ be an algebra. A function $p \colon A^n \to A$ is a *polynomial function* of $\mathbf{A}$ if there are a natural number m, a formula $\varphi(x_1, \dots, x_{n+m})$, and elements $b_1, \dots, b_m \in A$ such that

$$p(a_1, \dots, a_n) = \varphi^{\mathbf{A}}(a_1, \dots, a_n, b_1, \dots, b_m)$$

for every $a_1, \dots, a_n \in A$.

Lemma 1.2.3. *[42, Thm. 4.23] Let $\mathbf{A}$ be an algebra, $F \subseteq A$, and $a, b \in A$.*

$$\begin{aligned} \langle a, b\rangle \in \mathbf{\Omega}^{\mathbf{A}} F \iff & \textit{for every unary pol. function } p \colon \mathbf{A} \to \mathbf{A}, \\ & p(a) \in F \textit{ if and only if } p(b) \in F. \end{aligned}$$

Lemma 1.2.4. *[42, Thm. 5.32] Let $\vdash$ be a logic, $\mathbf{A}$ be an algebra, $F \subseteq A$, and $a, b \in A$.*

$$\begin{aligned} \langle a, b\rangle \in \widetilde{\mathbf{\Omega}}^{\mathbf{A}}_{\vdash} F \iff & \textit{for every unary pol. function } p \colon \mathbf{A} \to \mathbf{A}, \\ & \mathrm{Fg}^{\mathbf{A}}_{\vdash}(F \cup \{p(a)\}) = \mathrm{Fg}^{\mathbf{A}}_{\vdash}(F \cup \{p(b)\}). \end{aligned}$$

The Leibniz and Suszko congruences allow to single out two distinguished classes of models of a logic. More precisely, given a logic $\vdash$, we set

$$\begin{aligned} \mathsf{Mod}(\vdash) &:= \{\langle \mathbf{A}, F\rangle : \langle \mathbf{A}, F\rangle \text{ is a model of } \vdash\} \\ \mathsf{Mod}^{*}(\vdash) &:= \{\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash) : \mathbf{\Omega}^{\mathbf{A}} F = \mathrm{id}\} \\ \mathsf{Mod}^{\mathsf{Su}}(\vdash) &:= \{\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash) : \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F = \mathrm{id}\}. \end{aligned}$$

The above classes of matrices are called, respectively, the classes of models, *Leibniz reduced models*, and *Suszko reduced models* of $\vdash$. It turns out that $\mathsf{Mod}^{\mathsf{Su}}(\vdash) = \mathbb{P}_{\mathsf{SD}}\mathsf{Mod}^{*}(\vdash)$ (see, among others, [42]).

Trivial matrices will play a useful role in the whole thesis. More precisely, a matrix $\langle \mathbf{A}, F\rangle$ is *trivial* if $F = A$. We denote by $\langle \mathbf{1}, \{1\}\rangle$ the trivial matrix, where $\mathbf{1}$ is the trivial algebra. Observe that the latter matrix is a model (resp. Leibniz and Suszko reduced model) of every logic.

Moreover, if $\vdash$ is a logic and $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ is a trivial matrix, then $\langle \mathbf{A}, F\rangle = \langle \mathbf{1}, \{1\}\rangle$.

Given a logic $\vdash$, we set

$$\mathsf{Alg}^*(\vdash) = \{\mathbf{A} : \text{ there is } F \subseteq A \text{ s.t. } \langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\vdash)\}$$

and

$$\mathsf{Alg}(\vdash) = \{\mathbf{A} : \text{ there is } F \subseteq A \text{ s.t. } \langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)\}.$$

In other words, $\mathsf{Alg}(\vdash)(\mathsf{Alg}^*(\vdash))$ is the class of algebraic reducts of matrices in $\mathsf{Mod}^{\mathsf{Su}}(\vdash)(\mathsf{Mod}^*(\vdash))$. The class $\mathsf{Alg}(\vdash)$ is called the *algebraic counterpart* of $\vdash$. For the vast majority of logics $\vdash$, the class $\mathsf{Alg}(\vdash)$ is the class of algebras intuitively associated with $\vdash$.

Lemma 1.2.5. *[42, Lemma 5.78] Let $\vdash$ be a logic defined by a class of matrices* M. *Then* $\mathsf{Alg}(\vdash) \subseteq \mathbb{V}(\mathsf{K})$, *where* K *is the class of algebraic reducts of* M.

Lemma 1.2.6. *Let $\vdash$ be a logic and $\epsilon, \delta \in Fm$. The following are equivalent:*

(i) $\mathsf{Alg}(\vdash) \vDash \epsilon \approx \delta$;

(ii) $\varphi(\epsilon, \vec{z}) \dashv\vdash \varphi(\delta, \vec{z})$, *for every formula* $\varphi(v, \vec{z})$.

Proof. See [42, Lemma 5.74(1)] and [42, Theorem 5.76]. ⊠

Algebraizability and the Leibniz hierarchy

Now, we turn out attention to a fundamental topic in abstract algebraic logic, that is the so-called *Leibniz hierarchy*, see for example [30, 42, 44, 84, 86, 67]. We review only the material which is necessary for the purpose of this thesis. A logic $\vdash$ is *protoalgebraic* if there is a set of formulas $\Delta(x, y)$ such that

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

Remarkably, if $\vdash$ is protoalgebraic, then for every matrix $\langle \mathbf{A}, F\rangle$ it holds $\boldsymbol{\Omega}^{\mathbf{A}} F = \widetilde{\boldsymbol{\Omega}}^{\mathbf{A}} F$, and therefore $\mathsf{Mod}^*(\vdash) = \mathsf{Mod}^{\mathsf{Su}}(\vdash)$.

A logic $\vdash$ is *equivalential* if there is a set of formulas $\Delta(x, y)$ such that for every $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash)$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F \iff \Delta^{\mathbf{A}}(a, b) \subseteq F, \text{ for all } a, b \in A.$$

In this case, $\Delta(x, y)$ is a set of *congruence formulas* for $\vdash$ and we say that it defines the Leibniz congruence of the matrix $\langle \mathbf{A}, F\rangle$. Remarkably, if $\vdash$

is equivalential, then $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{S}$ and $\mathbb{P}$. Moreover, every equivalential logic is protoalgebraic.

The following result is proved in [42], and it simplifies the characterization of equivalentiality.

Lemma 1.2.7. *([42, Corollary 6.56]) A set $\Delta(x, y)$ of formulas defines the Leibniz congruence in a matrix $\langle \mathbf{A}, F\rangle$ if and only if it defines the identity in $\langle \mathbf{A}/\mathbf{\Omega}^{\mathbf{A}}F, F/\mathbf{\Omega}^{\mathbf{A}}F\rangle$.*

A logic $\vdash$ is *truth-equational* if there is a set of equations $\boldsymbol{\tau}(x)$ such that for all $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\vdash)$,

$$a \in F \iff \vDash_{\mathbf{A}} \boldsymbol{\tau}(a), \text{ for all } a \in A.$$

In this case, $\boldsymbol{\tau}(x)$ is a set of *defining equations* for $\vdash$.

Finally, a logic $\vdash$ is *algebraizable* when it is both equivalential and truth-equational. In this case, $\mathsf{Alg}(\vdash)$ is called the *equivalent algebraic semantics* of $\vdash$.

The notion of algebraizability can be characterized independently from matrix semantics. Given a class of algebras K we define the relation $\vDash_{\mathsf{K}}$ between sets of equations and equations as follows:

$$\begin{aligned} \Theta \vDash_{\mathsf{K}} \epsilon \approx \delta \iff & \text{ for every } \mathbf{A} \in \mathsf{K} \text{ and every homomoprhism } h : \mathbf{Fm} \to \mathbf{A} \\ & \text{ if } \vDash_{\mathbf{A}} h(\alpha) \approx h(\beta) \text{ for all } \alpha \approx \beta \in \Theta, \text{ then } \vDash_{\mathbf{A}} h(\epsilon) \approx h(\delta) \end{aligned}$$

A *structural transformer* of formulas into equations is a map

$$\boldsymbol{\tau} \colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$$

which commutes with unions and substitutions, i.e.

$$\boldsymbol{\tau}(\Gamma) = \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma) \text{ and } \boldsymbol{\tau}(\sigma\Gamma) = \sigma\boldsymbol{\tau}\Gamma.$$

Structural transformers $\boldsymbol{\rho} \colon \mathcal{P}(Eq) \to \mathcal{P}(Fm)$ of equations into formulas are defined similarly.

A logic $\vdash$ is algebraizable if there is a generalized quasivariety K and structural transformers $\boldsymbol{\tau}, \boldsymbol{\rho}$ such that conditions (Alg1) and (Alg4) or (Alg2) and (Alg3) below hold:

Alg1. $\Gamma \vdash \varphi$ iff $\boldsymbol{\tau}(\Gamma) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi)$;

Alg2. $E \vDash_{\mathsf{K}} \varphi \approx \psi$ iff $\boldsymbol{\rho}(E) \vdash \boldsymbol{\rho}(\varphi, \psi)$;

Alg3. $\varphi \dashv\vdash \boldsymbol{\rho}(\boldsymbol{\tau}(\varphi))$;

Alg4. $\varphi \approx \psi \dashv\vdash_{\mathsf{K}} \boldsymbol{\tau}(\boldsymbol{\rho}(\varphi, \psi))$.

The following definition originates in [61], but see also [20, 87]:

Definition 1.2.8. A set of formulas Σ is an *antitheorem* of a logic $\vdash$ if $\sigma[\Sigma] \vdash \varphi$ for every substitution σ and formula φ.

Example 1.2.9. For any formula φ, the set $\{\neg(\varphi \to \varphi)\}$ is an antitheorem for all superintuitionistic logics, all axiomatic extensions of MTL-logic [25, 37] including Łukasiewicz logic [24], and all local and global consequences of normal modal logics. ⊠

Remark 1.2.10. Observe that if $\vdash$ has an antitheorem Σ, then $\vdash$ has an antitheorem $\Sigma(x)$ only in variable x. If, moreover, $\vdash$ is finitary, then it has a *finite* antitheorem only in variable x. ⊠

For this reason, we will denote an antitheorem by Σ or by $\Sigma(x)$, in the case it is useful to emphasise that x is the only variable occurring in Σ.

Gentzen systems

By a *sequent*, we understand a finite and nonempty sequence of formulas $\langle \gamma_1, \ldots, \gamma_{n-1}, \gamma_n \rangle$, which we often denote by $\gamma_1, \ldots, \gamma_{n-1} \rhd \gamma_n$.[2] By a *Gentzen system* we understand a substitution invariant consequence relation $\vdash_{\mathcal{G}}$ over the set of sequents. If $\{\Gamma_i \rhd \varphi_i, i \in I\}$ is a set of sequents and $\{\Gamma_i \rhd \varphi_i, i \in I\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi$ we say that $\langle \{\Gamma_i \rhd \varphi_i, i \in I\}, \Gamma \rhd \varphi \rangle$ is a *derivable rule* of $\vdash_{\mathcal{G}}$. A Gentzen system $\vdash_{\mathcal{G}}$ is *adequate* for $\vdash$ when

$$\Gamma \vdash \varphi \iff \vdash_{\mathcal{G}} \Gamma \rhd \varphi.$$

Let A be a set, we denote by $A^* \times A$ the set of all finite and nonempty sequences of elements of A. Let now $\mathbf{A}$ be an algebra and $F \subseteq A^* \times A$. Observe that F can contain sequences of arbitrary finite lengths. The pair $\langle \mathbf{A}, F \rangle$ is a *model* of $\vdash_{\mathcal{G}}$ if and only if for every evaluation $h : \mathbf{Fm} \to \mathbf{A}$ and every $\{\Gamma_i \rhd \varphi_i, i \in I\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi$ if $h[\Gamma_i \rhd \varphi_i] \in F$ for each $i \in I$ then $h(\Gamma \rhd \varphi) \in F$.

In this thesis, and specifically in Chapter 4, we will always consider a fixed kind of Gentzen system. From now on, by $\vdash_{\mathcal{G}}$ we denote the Gentzen calculus defined in the following way.

[2] This definition is a specialization of the one adopted in [85, p.906], where a sequent is a generic pair of finite sequences.

Definition 1.2.11. Let $\vdash$ be a logic. $\vdash_{\mathcal{G}}$ is defined as:

$\{\Gamma_i \rhd \varphi_i,\ i \in I\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi \iff$ for every $\mathbf{A} \in \mathsf{Alg}(\vdash)$ and hom. $h : \mathbf{Fm} \to \mathbf{A}$ $h(\varphi) \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(h(\Gamma))$ whenever $h(\varphi_i) \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(h(\Gamma_i))$ for every $i \in I$.

Given $\mathbf{A} \in \mathsf{Alg}(\vdash)$ we say that $F \subseteq A^* \times A$ *represents* the operation of $\vdash$-generated filter over $\mathbf{A}$ if $F := \{\langle \gamma_1, \ldots, \gamma_{n-1}, \gamma_n\rangle \in A^* \times A : \gamma_n \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\gamma_1, \ldots, \gamma_{n-1})\}$.

The Gentzen system $\vdash_{\mathcal{G}}$ of Definition 1.2.11 is clearly adequate for $\vdash$ and complete w.r.t. the class $\{\langle \mathbf{A}, F\rangle\}$ where $\mathbf{A} \in \mathsf{Alg}(\vdash)$ and F represents the operation of $\vdash$-generated filter over $\mathbf{A}$.

With a slight generalization of Lemma 1.2.3 we can characterize the Leibniz congruence of $F \subseteq A^* \times A$ over $\mathbf{A}$ in the expected way:

Definition 1.2.12. Let $\mathbf{A}$ be an algebra, $F \subseteq A^* \times A$, and $a, b \in A$.

$$\begin{aligned}\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F \iff & \text{for every unary pol. func. } p\colon A \to A^{n-1} \times A,\\ & p(a) \in F \text{ if and only if } p(b) \in F\ (0 \neq n \in \omega).\end{aligned}$$

A model $\langle \mathbf{A}, F\rangle$ of a Gentzen system $\vdash_{\mathcal{G}}$ is *Leibniz reduced* if $\boldsymbol{\Omega}^{\mathbf{A}} F = \mathrm{id}$. The general theory of the algebraization of Gentzen systems originates in [7] and, for the finitary case, is firstly explicitly formulated in [89] and deepened in [83]. The recent [85] contains a general investigation of the topic following the modern concepts of abstract algebraic logic. As a detailed discussion of this subject is outside the scope of this thesis, we only review few notion that will be employed in Chapter 4. Firstly, the definition of structural transformer can be generalised to the level of sequents. More precisely, any function $\boldsymbol{\tau} : \mathcal{P}(Seq) \to \mathcal{P}(Eq)$, ($\boldsymbol{\rho} : \mathcal{P}(Eq) \to \mathcal{P}(Seq)$) that commutes with unions and substitutions is a structural transformer from sequents to equations (from equations to sequents). Moreover, we say that $\vdash_{\mathcal{G}}$ is *equivalential* if it has a set of sequents $\rho(a, b)$ such that for $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{*}(\vdash^{l}_{\mathcal{G}})$

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F \iff \boldsymbol{\rho}(a, b) \in F.$$

A useful alternative framework that allows for a semantic treatment of Gentzen systems is the one of the so-called *generalized matrices*[3] (see [42, 45]).

A *genralized matrix* (*g-matrix* for short), is a pair $\langle \mathbf{A}, \mathscr{C}\rangle$ where $\mathbf{A}$ is an algebra and $\mathscr{C} \subseteq \mathcal{P}(A)$ is a closure system on A. An evaluation

[3] In [45] they are called Abstract Logics.

$h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ *satisfies* a sequent $\Gamma \rhd \varphi$ in the g-matrix $\langle \mathbf{A}, \mathscr{C} \rangle$ when $h(\varphi) \in C(h(\Gamma))$. A g-matrix $\langle \mathbf{A}, \mathscr{C} \rangle$ is a *g-model* of a sequent $\Gamma \rhd \varphi$ if for all homomorphisms $h : \mathbf{Fm} \to \mathbf{A}$ it holds $h(\varphi) \in C(h(\Gamma))$. The g-matrix $\langle \mathbf{A}, \mathscr{C} \rangle$ is a g-model of the rule $\{\Gamma_i \rhd \varphi_i\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi$ $(i \in I)$ when every evaluation satisfying the premises also satisfies the conclusion. Also logics can be treated within the semantic of g-matrices. Indeed, we say that $\langle \mathbf{A}, \mathscr{C} \rangle$ is a g-model of a logic $\vdash$ if for every evaluation $h : \mathbf{Fm} \to \mathbf{A}$, $\Gamma \vdash \varphi$ implies $h(\varphi) \in C(h(\Gamma))$. The g-matrices we deal with in this work are of the form $\langle \mathbf{A}, \mathcal{F}\mathrm{i}_{\vdash}(\mathbf{A}) \rangle$. Consider the family of g-matrices $\mathsf{M} := \{\langle \mathbf{A}, \mathcal{F}\mathrm{i}_{\vdash}(\mathbf{A}) \rangle : \mathbf{A} \in \mathsf{Alg}(\vdash)\}$ and define a Gentzen system $\vdash^{\mathsf{M}}_{\mathcal{G}}$ as

$$\{\Gamma_i \rhd \varphi_i,\ i \in I\} \vdash^{\mathsf{M}}_{\mathcal{G}} \Gamma \rhd \varphi \iff \text{for every } h : \mathbf{Fm} \to \mathbf{A} \text{ and} \langle \mathbf{A}, \mathscr{C} \rangle \in \mathsf{M}$$
$$h(\varphi) \in C(h(\Gamma)) \text{ whenever } h(\varphi_i) \in C(h(\Gamma_i)) \text{ for every } i \in I.$$

As $\mathcal{F}i_{\vdash}\mathbf{A}$ is a closure system on A whose associated closure operator is $\mathrm{Fg}^{\mathbf{A}}_{\vdash}$, it is immediate to check that $\vdash_{\mathcal{G}} = \vdash^{\mathsf{M}}_{\mathcal{G}}$.

A homomorphism $h : \mathbf{A}_1 \to \mathbf{A}_2$ is a *strict homomorphism* from $\langle \mathbf{A}_1, \mathscr{C}_1 \rangle$ to $\langle \mathbf{A}_2, \mathscr{C}_2 \rangle$ when

$$a \in C_1(X) \iff h(a) \in C_2(h(X)) \text{ for all } X \cup \{a\} \subseteq A_1.$$

The following Lemma describes an important property of strict homomorphisms between g-matrices.

Lemma 1.2.13. *[42, Proposition 5.42] A map $h \in Hom(\mathbf{A}_1, \mathbf{A}_2)$ is a strict homomorphism from $\langle \mathbf{A}_1, \mathscr{C}_1 \rangle$ to $\langle \mathbf{A}_2, \mathscr{C}_2 \rangle$ if and only if $h^{-1}\mathscr{C}_2 = \mathscr{C}_1$.*

The last notion we need to introduce concerning this topic is a special congruence that, roughly, is a generalization of the Leibniz congruence to the case of g-matrices. The *Tarski congruence* of a g-matrix $\langle \mathbf{A}, \mathscr{C} \rangle$ is $\widetilde{\mathbf{\Omega}}^{\mathbf{A}}\mathscr{C}$ and it is the largest congruence which is compatible with every $F \in \mathscr{C}$. In symbols, $\widetilde{\mathbf{\Omega}}^{\mathbf{A}}\mathscr{C} := \bigcap_{F \in \mathscr{C}} \mathbf{\Omega}^{\mathbf{A}} F$. The *reduction* of a g-matrix $\langle \mathbf{A}, \mathscr{C} \rangle$ is the g-matrix $\langle \mathbf{A}_{/\widetilde{\mathbf{\Omega}}^{\mathbf{A}}\mathscr{C}}, \mathscr{C}_{/\widetilde{\mathbf{\Omega}}^{\mathbf{A}}\mathscr{C}} \rangle$. A g-matrix $\langle \mathbf{A}, \mathscr{C} \rangle$ is *Tarski reduced* when $\widetilde{\mathbf{\Omega}}^{\mathbf{A}}\mathscr{C} = \mathrm{id}$. Given a logic $\vdash$, the class $\mathsf{Alg}(\vdash)$ can be equivalently characterized as the class of the algebraic reducts of the reduced g-model of $\vdash$.

1.3 Płonka Sums

For standard references on Płonka sums we refer the reader to [74, 73, 76, 90]. A *semilattice* is an algebra $\mathbf{A} = \langle A, \vee \rangle$, where $\vee$ is a binary commutative, associative and idempotent operation. Given a semilattice $\mathbf{A}$ and $a, b \in A$, we set

$$a \leq b \iff a \vee b = b.$$

It is easy to see that $\leq$ is a partial order on A.

Definition 1.3.1. A *direct system of algebras* consists of

(i) a semilattice $I = \langle I, \vee \rangle$;

(ii) a family of algebras $\{\mathbf{A}_i : i \in I\}$ of the same type with disjoint universes;

(iii) a homomorphism $f_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$, for every $i, j \in I$ such that $i \leq j$;

moreover, f_{ii} is the identity map for every $i \in I$, and if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$.

Let X be a direct system of algebras as above. The *Płonka sum* of X, in symbols $\mathcal{P}_ł(X)$ or $\mathcal{P}_ł(\mathbf{A}_i)_{i \in I}$, is the algebra defined as follows. The universe of $\mathcal{P}_ł(\mathbf{A}_i)_{i \in I}$ is the union $\bigcup_{i \in I} A_i$. Moreover, for every n-ary basic operation f and $a_1, \ldots, a_n \in \bigcup_{i \in I} A_i$, we set

$$f^{\mathcal{P}_ł(\mathbf{A}_i)_{i \in I}}(a_1, \ldots, a_n) := f^{\mathbf{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n))$$

where $a_1 \in A_{i_1}, \ldots, a_1 \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_n$.

Observe that if in the above display we replace f by any complex formula φ in n-variables, we still have that

$$\varphi^{\mathcal{P}_ł(\mathbf{A}_i)_{i \in I}}(a_1, \ldots, a_n) = \varphi^{\mathbf{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n)).$$

Notation: Given $\mathbf{A} \cong \mathcal{P}_ł(\mathbf{A}_i)_{i \in I}$ and $a \in A$, we denote by i_a the fiber a belongs to. Similarly, given formula φ and $h : \mathbf{Fm} \to \mathbf{A}$, we denote by $i_h(\varphi)$ the fiber $h(\varphi)$ belongs to. We will often write $\varphi^{\mathcal{P}_ł}$ instead of $\varphi^{\mathcal{P}_ł(\mathbf{A}_i)_{i \in I}}$ when no confusion shall occur.

Remark 1.3.2. Initially, the notion of Płonka sum was defined only for classes of algebras not containing nullary operations in their type. In [75], Płonka extended the theory to algebras with constants symbols in their type. In such a case, the semilattice I of a direct system $X = \langle I, f_{i,j}, A_i \rangle$ is required to have a bottom element $\perp$, and for every constant symbol c in the type we set $\mathrm{c}^{\mathcal{P}_ł} = \mathrm{c}^{\mathbf{A}_\perp}$. In other words, the constant symbols in the Płonka sums are computed in the algebra $\mathbf{A}_\perp$.

In this thesis, apart from Chapter 5, we generally assume that the algebraic languages do not contain constant symbols, unless specified otherwise.

The theory of Płonka sums is strictly related with a special kind of operation:

Definition 1.3.3. Let $\mathbf{A}$ be an algebra of type ν. A function $\cdot\colon A^2 \to A$ is a *partition function* in $\mathbf{A}$ if the following conditions are satisfied for all $a, b, c \in A$, $a_1, \dots, a_n \in A^n$ and for any operation $g \in \nu$ of arity $n \geqslant 1$.

P1. $a \cdot a \approx a$

P2. $a \cdot (b \cdot c) \approx (a \cdot b) \cdot c$

P3. $a \cdot (b \cdot c) \approx a \cdot (c \cdot b)$

P4. $g(a_1, \dots, a_n) \cdot b \approx g(a_1 \cdot b, \dots, a_n \cdot b)$

P5. $b \cdot g(a_1, \dots, a_n) \approx b \cdot a_1 \cdot_{\dots} \cdot a_n$

This definition of partition function if formulated in [3, p.398], and it not the only one that can be found in the literature (see [73]). However, for our purposes, it is the more compact. The next result makes explicit the relation between Płonka sums and partition functions:

Theorem 1.3.4. *[73, Thm. II] Let $\mathbf{A}$ be an algebra of type ν with a partition funtion $\cdot$. The following conditions hold:*

1. *A can be partitioned into $\{A_i : i \in I\}$ where any two elements $a, b \in A$ belong to the same component A_i exactly when*

$$a = a \cdot b \text{ and } b = b \cdot a.$$

Moreover, every A_i is the universe of a subalgebra $\mathbf{A}_i$ of $\mathbf{A}$.

2. *The relation $\leq$ on I given by the rule*

$$i \leq j \iff \text{there exist } a \in A_i, b \in A_j \text{ s.t. } b \cdot a = b$$

is a partial order and $\langle I, \leq \rangle$ is a semilattice.

3. *For all $i, j \in I$ such that $i \leq j$ and $b \in A_j$, the map $f_{ij}\colon A_i \to A_j$, defined by the rule $f_{ij}(x) = x \cdot b$ is a homomorphism. The definition of f_{ij} is independent from the choice of b, since $a \cdot b = a \cdot c$, for all $a \in A_i$ and $c \in A_j$.*

4. *$X = \langle \langle I, \leq \rangle, \{\mathbf{A}_i\}_{i \in I}, \{f_{ij} \colon i \leq j\} \rangle$ is a direct system of algebras such that $\mathcal{P}_{\nmid}(X) = \mathbf{A}$.*

It is worth remarking that the construction of Płonka sums preserves the validity of the so-called *regular identities*, i.e. identities of the form $\varphi \approx \psi$ such that $Var(\varphi) = Var(\psi)$, while it falsifies any non regular identity (see [73] for details).

An example: Involutive bisemilattices The class of *involutive bisemilattices* has been introduced in [13] as the variety generated by the algebraic counterpart of the logic PWK.

Definition 1.3.5. An *involutive bisemilattice* is an algebra $\mathbf{B} = \langle B, \vee, \wedge, \neg, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ satisfying:

I1. $x \vee x \approx x$;

I2. $x \vee y \approx y \vee x$;

I3. $x \vee (y \vee z) \approx (x \vee y) \vee z$;

I4. $\neg(\neg x) \approx x$;

I5. $x \wedge y \approx \neg(\neg x \vee \neg y)$;

I6. $x \wedge (\neg x \vee y) \approx x \wedge y$;

I7. $0 \vee x \approx x$;

I8. $1 \approx \neg 0$.

Involutive bisemilattices form an equational class denoted by $\mathcal{IBSL}$. Examples of involutive bisemilattices include any Boolean algebra, as well as any semilattice with zero. In the latter case, the two binary operations coincide and the unary operation is the identity.

It is not difficult to note that the variety $\mathcal{IBSL}$ is the *regularization*[4] of the variety $\mathcal{BA}$, of Boolean algebras (see [75, 13]), i.e. $\vDash_{\mathcal{IBSL}} \varepsilon \approx \delta$ if and only if $\vDash_{\mathcal{BA}} \varepsilon \approx \delta$ and $Var(\varepsilon) = Var(\delta)$.
It turns out that Płonka sums provide a useful tool to represent algebras belonging to regular varieties. We recall here the representation theorem for involutive bisemilattices.

Therefore, involutive bisemilattices, as well as bisemilattices admit a representation as Płonka sums over a direct system of Boolean algebras.

Theorem 1.3.6 ([13, Thm. 46])**.**

1) *If X is a direct system of Boolean algebras, then $\mathcal{P}_{\nmid}(X)$ is an involutive bisemilattice.*

2) *If $\mathbf{B}$ is an involutive bisemilattice, then $\mathbf{B} \cong \mathcal{P}_{\nmid}(X)$, where X is a direct system of Boolean algebras.*

It is possible to observe that by setting $a \cdot b = a \wedge (a \vee b)$ we obtain that $\cdot$ is a partition function for any $\mathbf{A} \in \mathcal{IBSL}$.

[4]For the theory of regular varieties and regularizations we refer the reader to [76].

Chapter 2

Left-variable Inclusion Logics

2.1 Introduction

Given a logic $\vdash$, it is always possible to define a new consequence relation $\vdash^l$ by imposing the following syntactic requirement:

$$\Gamma \vdash^l \varphi \iff \text{there is } \Delta \subseteq \Gamma \text{ s.t. } Var(\Delta) \subseteq Var(\varphi) \text{ and } \Delta \vdash \varphi.$$

In this case, we say that logic $\vdash^l$ is the *left variable inclusion companions* of $\vdash$. Intuitively, such logics preserves all the inferences of $\vdash$ in which the variable that occur in the premises also occur in the conclusion. The prototypical example of left variable inclusion logic belongs to the realm of three-valued logics and it is the left variable inclusion companion of classical (propositional) logic, known as *paraconsistent weak Kleene logic* (PWK for short) [56, 50]. The fact that this logic coincide with the variable inclusion companion of Classical Logic was shown in [27, 97]. One of the most notable facts about PWK is the presence of a non-sensical, infectious truth value [92, 28], which made them a valuable tool in modelling reasonings with non-existing objects [80], computer-programs affected by errors [38] as well as recent developments in the theory of truth [93]. Some philosophical applications of PWK are summarised and discussed the Introduction. Recent work [13] linked PWK to the algebraic theory of regular varieties, i.e. equational classes axiomatized by equations $\varphi \approx \psi$ such that $Var(\varphi) = Var(\psi)$. As sketched in the preliminaries, the representation theory of regular varieties is largely due to the pioneering work of Płonka [73], and is tightly related to the special class-operator $\mathcal{P}_{\mathrm{ł}}(\cdot)$, called *Płonka sums* (see Subsection 1.3). Over the years, regular varieties have been studied in depth both from a purely algebraic perspective [74, 54, 51, 52] and in connection to their topological duals [49, 14, 91, 12, 62]. The machinery of Płonka sums has also found useful applications in the study of the constraint satisfaction problem [3], database semantics [63, 81] and algebraic methods in computer science [17].

One of the main results of [13] states that the algebraic counterpart of PWK is the class of Płonka sum of Boolean algebras. This observation led us to investigate the relations between left variable inclusion companions and Płonka sums in full generality.

The chapter is structured as follows. We begin by generalising the construction of Płonka sums from algebras to logical matrices (Section 2.2). This allows us to condense the connection between left variable inclusion principles and Płonka sums in the following slogan: The left variable inclusion companion $\vdash^l$ of a logic $\vdash$ is complete w.r.t. the class of Płonka sums of matrix models of $\vdash$ (Corollary 2.2.8).

As a matter of fact, left variable inclusion companions $\vdash^l$ are especially well-behaved in case the original logic $\vdash$ has an *l-partition function* [90], a feature shared by a great amount of logics. The importance of partition functions is reflected both at a syntactic and at a semantic level. Accordingly, on the one hand we present a general method to transform every Hilbert-style calculus for a finitary logic $\vdash$ with an l-partition function into an Hilbert-style calculus for $\vdash^l$ (Theorem 2.3.9). On the other hand, l-partition functions can be exploited to tame the structure of the matrix semantics $\mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$ of $\vdash^l$, given by the so-called Suszko reduced models of $\vdash^l$. In particular, we obtain a full description of $\mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$ in case $\vdash$ is a finitary equivalential logic with a l-partition function (Theorems 2.5.3 and 2.6.4). We close our investigation by determining the location of $\vdash^l$ in the Leibniz hierarchy (Section 2.7).

2.2 The left variable inclusion companion of a logic

The definition of *direct system* can be extended, as follows, to logical matrices:

Definition 2.2.1. A *l-direct system* of matrices consists in

(i) a semilattice $I = \langle I, \vee \rangle$;

(ii) a family of matrices $\{\langle \mathbf{A}_i, F_i \rangle\}_{i \in I}$ with disjoint universes;

(iii) a homomorphism $f_{ij}\colon \mathbf{A}_i \to \mathbf{A}_j$ such that $f_{ij}[F_i] \subseteq F_j$, for every $i, j \in I$ such that $i \leq j$

such that f_{ii} is the identity map for every $i \in I$, and if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$.

Given directed system of matrices X as above, we set

$$\mathcal{P}_{\nmid}(X) := \langle \mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}, \bigcup_{i\in I} F_i \rangle.$$

The matrix $\mathcal{P}_{\nmid}(X)$ is the *Płonka sum* of the l-direct system of matrices X. Given a class M of matrices, we denote by $\mathcal{P}_{\nmid}(\mathsf{M})$ the class of all Płonka sums of directed systems of matrices in M. The following observation is a routine computation:

Lemma 2.2.2. $\mathbb{S}\mathcal{P}_{\nmid}(\mathsf{M}) \subseteq \mathcal{P}_{\nmid}(\mathbb{S}(\mathsf{M}))$ *and* $\mathbb{P}\mathcal{P}_{\nmid}(\mathsf{M}) \subseteq \mathcal{P}_{\nmid}(\mathbb{P}(\mathsf{M}))$, *for every class of matrices* M.

Definition 2.2.3. Let $\vdash$ be a logic. The *left variable inclusion* companion of $\vdash$ is the relation $\vdash^l \subseteq \mathcal{P}(Fm) \times Fm$ defined for every $\Gamma \cup \{\varphi\} \subseteq Fm$ as

$$\Gamma \vdash^l \varphi \iff \text{there is } \Gamma' \subseteq \Gamma \text{ s.t. } Var(\Gamma') \subseteq Var(\varphi) \text{ and } \Gamma' \vdash \varphi$$

It is immediate to check that $\vdash^l$ is indeed a logic. We will often refer to the left variable inclusion of a logic simply as its variable inclusion companion, as, in this chapter, we will not introduce any different syntactic filters on inferences.

Example 2.2.4. Let $\vdash$ be propositional Classical Logic. Then $\vdash^l$ is the logic known as Paraconsistent Weak Kleene, originally introduced in [56]. This logic is equivalently defined, *syntactically*, by imposing the variable inclusion constrain, as in Definition 2.2.3, to Classical Logic or, *semantically* via the so-called *weak Kleene tables* (**WK**) displayed below

$\wedge$	0	n	1
0	0	n	0
n	n	n	n
1	0	n	1

$\vee$	0	n	1
0	0	n	1
n	n	n	n
1	1	n	1

$\neg$	
1	0
n	n
0	1

by the matrix $\langle \mathbf{WK}, \{1, n\}\rangle$ (see [13, 27]).

It is not difficult to check that the algebra $\mathbf{WK} = \langle \{0, 1, n\}, \wedge, \vee, \neg, 0, 1\rangle$ is the Płonka sum of the two-element Boolean algebra and the trivial (Boolean) algebra $\mathbf{n}$ (the index set is the two element semilattice).

Example 2.2.5. The left variable inclusion companions of Strong Kleene logic[1] and of the logic of Paradox (introduced in [78]) have been introduced and discussed in [92]. They are semantically defined, by adding a non sensical truth value to the (single) matrix inducing Strong Kleene and the logic of Paradox, respectively.

In [13], it is shown that an algebraic semantics for PWK is obtained via the Płonka sum of Boolean algebras. This idea can be generalized to the variable inclusion companion of any logic $\vdash$.

Lemma 2.2.6. *Let $\vdash$ be a logic and X be an l-direct systems of models of $\vdash$. Then $\mathcal{P}_{\nmid}(X)$ is a model of $\vdash^l$.*

Proof. Assume that X is an in Definition 3.2.4. Then suppose that $\Gamma \vdash^l \varphi$ and consider a homomorphism $v\colon \mathbf{Fm} \to \mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}$ such that $v[\Gamma] \subseteq \bigcup_{i\in I} F_i$. By the definition of $\vdash^l$, there exists $\Delta \subseteq \Gamma$ such that $Var(\Delta) \subseteq Var(\varphi)$ and $\Delta \vdash \varphi$. Consider an enumeration $Var(\varphi) = \{x_1, \dots, x_n\}$. There are $i_1, \dots, i_n \in I$ such that $v(x_1) \in A_{i_1}, \dots, v(x_n) \in A_{i_n}$. We set $j := i_1 \vee \dots \vee i_n$.

Now, consider a homomorphism $g\colon \mathbf{Fm} \to \mathbf{A}_j$ such that

$$g(x_m) = f_{i_m j}(v(x_m)), \text{ for every } m \leq n.$$

We claim that $g[\Delta] \subseteq F_j$. To prove this, consider an arbitrary formula $\delta \in \Delta$. Since $Var(\Delta) \subseteq \{x_1, \dots, x_n\}$, we can assume that $Var(\delta) = \{x_{m_1}, \dots, x_{m_k}\} \subseteq \{x_1, \dots, x_n\}$ for some $k \leq n$. Set $l := i_{m_1} \vee \dots \vee i_{m_k}$. From the definition of $\mathcal{P}_{\nmid}(X)$ we have that

$$v(\delta) = \delta^{\mathcal{P}_{\nmid}}(v(x_{m_1}), \dots, v(x_{m_k})) = \delta^{\mathbf{A}_l}(f_{i_{m_1} l}(v(x_{m_1})), \dots, f_{i_{m_k} l}(v(x_{m_k}))).$$

Since $v(\delta) \in \bigcup_{i\in I} F_i$, this implies that

$$\delta^{\mathbf{A}_l}(f_{i_{m_1} l}(v(x_{m_1})), \dots, f_{i_{m_k} l}(v(x_{m_k}))) \in F_l. \tag{2.1}$$

Now observe that $l \leq j$. Therefore there is a homomorphism $f_{lj}\colon \mathbf{A}_l \to \mathbf{A}_j$ such that $f_{lj}[F_l] \subseteq F_j$. Together with (2.1), this implies that

$$\begin{aligned} g(\delta) &= \delta^{\mathbf{A}_j}(f_{i_{m_1} j}(v(x_{m_1})), \dots, f_{i_{m_k} j}(v(x_{m_k}))) \\ &= \delta^{\mathbf{A}_j}(f_{lj} \circ f_{i_{m_1} l}(v(x_{m_1})), \dots, f_{lj} \circ f_{i_{m_k} l}(v(x_{m_k}))) \\ &= f_{lj}\delta^{\mathbf{A}_l}(f_{i_{m_1} l}(v(x_{m_1})), \dots, f_{i_{m_k} l}(v(x_{m_k}))) \\ &\in f_{lj}[F_l] \subseteq F_j. \end{aligned}$$

[1] For details about the three valued logics introduced by Kleene, see [56].

This establishes our claim.

Recall that $\Delta \vdash \varphi$. Since $\langle \mathbf{A}_j, F_j \rangle$ is a model of $\vdash$ and by the claim $g[\Delta] \subseteq F_j$, we conclude that $g(\varphi) \in F_j$. But this means that

$$\begin{aligned} v(\varphi) &= \varphi^{\mathcal{P}_{\nmid}}(v(x_1), \dots, v(x_n)) \\ &= \varphi^{\mathbf{A}_j}(f_{i_1 j}(v(x_1)), \dots, f_{i_n j}(v(x_n))) \\ &= g(\varphi) \in F_j \subseteq \bigcup_{i \in I} F_i. \end{aligned}$$

Hence we conclude that $\mathcal{P}_{\nmid}(X)$ is a model of $\vdash^l$ as desired. ⊠

Recall that $\mathbf{1}$ is the trivial algebra. The following construction, originally designed in [60], will be used throughout the whole chapter. Given an algebra $\mathbf{A}$, there is always a direct system of algebras given by $\mathbf{A}$ and $\mathbf{1}$ equipped with the identity endomophisms and the unique homomorphism $f \colon \mathbf{A} \to \mathbf{1}$. We denote by $\mathbf{A} \oplus \mathbf{1}$ the Płonka sum of this direct system. Observe that $\mathbf{A} \oplus \mathbf{1}$ is the algebra with universe $A \cup \{1\}$ and basic operations f defined as follows:

$$f^{\mathbf{A} \oplus \mathbf{1}}(a_1, \dots, a_n) := \begin{cases} f^{\mathbf{A}}(a_1, \dots, a_n) & \text{if } a_1, \dots, a_n \in A \\ 1 & \text{otherwise.} \end{cases}$$

Observe that the above construction can be lifted to matrices. More precisely, given an arbitrary matrix $\langle \mathbf{A}, F \rangle$, there is always an l-direct system of matrices given by $\langle \mathbf{A}, F \rangle$ and $\langle \mathbf{1}, \{1\} \rangle$ equipped with the identity endomorphisms and the unique homomorphism $f \colon \mathbf{A} \to \mathbf{1}$. The Płonka sum of this system is the matrix $\langle \mathbf{A} \oplus \mathbf{1}, F \cup \{1\} \rangle$.

Theorem 2.2.7. *Let $\vdash$ be a logic and M be a class of matrices containing $\langle \mathbf{1}, \{1\} \rangle$. If $\vdash$ is complete w.r.t. M, then $\vdash^l$ is complete w.r.t. $\mathcal{P}_{\nmid}(\mathsf{M})$.*

Proof. In the light of Lemma 2.2.6 it will be enough to show that if $\Gamma \nvdash^l \varphi$, then $\Gamma \nvdash_{\mathcal{P}_{\nmid}(\mathsf{M})} \varphi$. To this end, suppose that $\Gamma \nvdash^l \varphi$. Define

$$\begin{aligned} \Gamma^+ &:= \{\gamma \in \Gamma : Var(\gamma) \subseteq Var(\varphi)\} \\ \Gamma^- &:= \{\gamma \in \Gamma : Var(\gamma) \nsubseteq Var(\varphi)\}. \end{aligned}$$

Clearly $\Gamma = \Gamma^+ \cup \Gamma^-$. Since $\Gamma \nvdash^l \varphi$, we know that $\Gamma^+ \nvdash \varphi$. Together with the fact that $\vdash$ is complete w.r.t. M, this implies that there exists a matrix $\langle \mathbf{A}, F \rangle \in \mathsf{M}$ and a homomorphism $v \colon \mathbf{Fm} \to \mathbf{A}$ such that $v[\Gamma^+] \subseteq F$ and $v(\varphi) \notin F$.

Since $\langle \mathbf{A}, F\rangle, \langle \mathbf{1}, \{1\}\rangle \in \mathsf{M}$, we have that $\langle \mathbf{A} \oplus \mathbf{1}, F \cup \{1\}\rangle \in \mathcal{P}_{\nmid}(\mathsf{M})$. Now, consider the homomorphism $g \colon \mathbf{Fm} \to \mathbf{A} \oplus \mathbf{1}$ defined for every variable $x \in Var$ as follows:

$$g(x) := \begin{cases} v(x) & \text{if } x \in Var(\varphi) \\ 1 & \text{otherwise.} \end{cases}$$

From the definition of $\mathbf{A} \oplus \mathbf{1}$ it follows that

$$\begin{aligned} g[\Gamma^-] &\subseteq \{1\} \subseteq F \cup \{1\} \\ g(\gamma) &= v(\gamma) \text{ for every } \gamma \in \Gamma^+ \cup \{\varphi\}. \end{aligned}$$

Together with the fact that $v[\Gamma^+] \subseteq F$ and $v(\varphi) \notin F$, this implies that

$$g[\Gamma] = g[\Gamma^+ \cup \Gamma^-] \subseteq F \cup \{1\} \text{ and } g(\varphi) \notin F \cup \{1\}.$$

Hence we conclude that $\Gamma \nvdash_{\mathcal{P}_{\nmid}(\mathsf{M})} \varphi$ as desired. ⊠

Corollary 2.2.8. *Let $\vdash$ be a logic. The variable inclusion companion $\vdash^l$ is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\nmid}(\mathsf{Mod}(\vdash)) \quad \mathcal{P}_{\nmid}(\mathsf{Mod}^*(\vdash)) \quad \mathcal{P}_{\nmid}(\mathsf{Mod}^{\mathsf{Su}}(\vdash)).$$

Proof. Observe that $\vdash$ is complete w.r.t. any of the classes $\mathsf{Mod}(\vdash)$, $\mathsf{Mod}^*(\vdash)$, $\mathsf{Mod}^{\mathsf{Su}}(\vdash)$. Moreover any of these classes contains the (trivial) matrix $\langle \mathbf{1}, \{1\}\rangle$. Thus we can apply Theorem 2.2.7. ⊠

The proof of Theorem 2.2.7 shows also the following result, which significantly simplifies the characterization of a complete matrix semantics for a left variable inclusion logic:

Theorem 2.2.9. *Let $\vdash$ be a logic which is complete w.r.t. the class of matrices* M. *Then $\vdash^l$ is complete w.r.t.the class* $\{\langle \mathbf{A} \oplus \mathbf{1}\rangle, F \cup \{1\} : \langle \mathbf{A}, F\rangle \in \mathsf{M}\}$.

Remark 2.2.10. Observe that, given a logic $\vdash$ which is complete w.r.t. the class of matrices M, the completeness of $\vdash^l$ can be obtained in different ways. Indeed, in general, $\vdash^l$ is complete w.r.t. the class $\{\langle \mathbf{A} \oplus \mathbf{B}, F \cup B : \langle \mathbf{A}, F\rangle \in \mathsf{M}\}$, for any algebra $\mathbf{B}$. This turns out to be particularly significant when the logic $\vdash$ is defined by a single matrix, i.e. when the class M contains exactly one element.

Another aspect which deserves our attention is that, if the logic $\vdash$ does not possesses an antitheorem, the completeness of $\vdash^l$ is completely independent from the presence of trivial matrices. More precisely, Theorem 2.2.7 specializes to the following result:

Corollary 2.2.11. *Let $\vdash$ be a logic without antitheorems and M be a complete class of matrices for $\vdash$ with at least two elements. Then $\vdash^l$ is complete w.r.t. $\mathcal{P}_\mathfrak{t}(\mathsf{M})$.*

Proof. The fact that $\vdash^l \subseteq \vdash^{\mathcal{P}_\mathfrak{t}(\mathsf{M})}$ follows by Lemma 2.2.6. The other direction can be proved by using the very same strategy adopted in the proof of Theorem 2.2.7. The key fact is that the absence of antitheorems entails that, if $\Gamma \nvdash^l \varphi$, then for every matrix $\langle \mathbf{A}, F \rangle \in \mathsf{M}$ there exists a homomorphism $v : \mathbf{Fm} \to \mathbf{A}$ such that $v[\Gamma] \in F$. This directly allows to conclude the argument of the proof without relying on the presence of the trivial matrix $\langle \mathbf{1}, 1 \rangle$. ⊠

2.3 Hilbert-style axiomatizations

Definition 2.3.1. A logic $\vdash$ has an *l-partition function* if there is a formula $x \cdot y$, in which the variables x and y really occur, such that $x \vdash x \cdot y$ and the equations **P1.**, ..., **P5.** in Definition 1.3.3 hold in $\mathsf{Alg}(\vdash)$ for every n-ary connective f.

In this case, $x \cdot y$ is a *l-partition function* for $\vdash$.

Remark 2.3.2. By Lemma 1.2.6, the above Definition can be rephrased in purely logical terms, by requiring that $x \vdash x \cdot y$ and that

$$\varphi(\epsilon, \vec{z}) \dashv\vdash \varphi(\delta, \vec{z}) \text{ for every formula } \varphi(v, \vec{z}),$$

for every identity of the form $\epsilon \approx \delta$ in **P1.**, ..., **P5.** ⊠

Example 2.3.3. Logics with an l-partition function abound in the literature. Indeed, the term $x \cdot y := x \wedge (x \vee y)$ is an l-partition function for every logic $\vdash$ such that $\mathsf{Alg}(\vdash)$ has a lattice reduct. Such examples include all modal and substructural logics. On the other hand, $x \cdot y := (y \to y) \to x$ is an l-partition function for all logics $\vdash$ such that $\mathsf{Alg}(\vdash)$ has a Hilbert algebra reduct [33]. ⊠

Remarkably, the presence of an l-partition function is inherited by the construction of regalurizations:

Lemma 2.3.4. *Let $\vdash$ be a logic. The operation $\cdot$ is an l-partition function for $\vdash$ if and only if it is an l-partition function for $\vdash^l$.*

Proof. From Remark 2.3.2 the fact that $\cdot$ is an l-partition function for $\vdash$ is witnessed by the validity of some inferences $\varphi \vdash \psi$ such that $Var(\varphi) \subseteq Var(\psi)$. Hence these inferences also holds in $\vdash^l$. With another application of Remark 2.3.2 we conclude that $\cdot$ is an l-partition function for $\vdash^l$.

The other direction follows from the inclusion $\vdash^l \subseteq \vdash$. ⊠

The following result is the generalization of Theorem 1.3.4 to the setting of logical matrices.

Theorem 2.3.5. *Let $\vdash$ be a logic with an l-partition function $\cdot$, and $\langle \mathbf{A}, F\rangle$ be a model of $\vdash$ such that $\mathbf{A} \in \mathsf{Alg}(\vdash)$. Conditions (1-4) of Theorem 1.3.4 hold. Moreover, setting $F_i := F \cap A_i$ for every $i \in I$, the triple*

$$X = \langle\langle I, \leq\rangle, \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}\rangle$$

is an l-direct system of matrices such that $\mathcal{P}_ł(X) = \langle \mathbf{A}, F\rangle$.

Proof. In the light of Theorem 1.3.4, it will be enough to show that $f_{ij}[F_i] \subseteq F_j$ for every $i, j \in I$ such that $i \leq j$. To this end, consider $a \in F_i$ and $b \in A_j$ with $i \leq j$. Since $\cdot$ is an l-partition function for $\vdash$, we have $x \vdash x \cdot y$. Together with the fact that $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $a \in F$, this implies that $a \cdot b \in F$. Observe that $a \cdot b \in A_j$ by (ii) and, therefore, that $a \cdot b \in F_j$. Hence, by (iii), we have that $f_{ij}(a) = a \cdot b \in F_j$. ⊠

Definition 2.3.6. Let $\vdash$ be a logic with an l-partition function $\cdot$, and $\langle \mathbf{A}, F\rangle$ be a model of $\vdash$ such that $\mathbf{A} \in \mathsf{Alg}(\vdash)$. The *Płonka fibers* of $\langle \mathbf{A}, F\rangle$ are the matrices $\{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}$ given by condition (iv) of the above result.

Lemma 2.3.7. *Let $\vdash$ be a finitary logic with l-partition function $\cdot$ and $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash^l)$, with $\mathbf{A} \in \mathsf{Alg}(\vdash^l)$. Then, the Płonka fibers of $\langle \mathbf{A}, F\rangle$ are models of $\vdash$.*

Proof. Let $\langle \mathbf{A}_i, F_i\rangle$ be a Płonka fiber of $\langle \mathbf{A}, F\rangle$ and $\Gamma \vdash \varphi$, with Γ a finite set. Then consider a homomorphism $v \colon \mathbf{Fm} \to \mathbf{A}_i$ such that $v[\Gamma] \subseteq F_i$. Then, there are cases: either Γ is empty or not. First, suppose $\Gamma = \emptyset$. Then clearly $\emptyset \vdash^l \varphi$. Since $\mathbf{A}_i$ is a subalgebra of $\mathbf{A}$ and $\langle \mathbf{A}, F\rangle$ is a model of $\vdash^l$, this implies that $v(\varphi) \in F \cap A_i = F_i$. Then consider the case where Γ is non void. Then there are $\gamma_1, \ldots, \gamma_n \in Fm$ such that $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. Since $\cdot$ is an l-partition function, we have $x \vdash x \cdot y$. In particular, this implies that $\varphi \vdash \varphi \cdot (\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots))$. Then $\Gamma \vdash \varphi \cdot (\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots))$. Since the variable inclusion constraint holds for this inference, we obtain that

$$\Gamma \vdash^l \varphi \cdot (\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots)).$$

Since $\mathbf{A}_i$ is a subalgebra of $\mathbf{A}$ and $\langle \mathbf{A}, F\rangle$ is a model of $\vdash^l$, this implies that

$$v(\varphi \cdot (\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots))) \in A_i \cap F = F_i.$$

Since $v(\varphi)$ and $v(\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots)))$ belong to A_i, this implies that

$$\begin{aligned} v(\varphi) &= v(\varphi) \cdot v(\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots))) \\ &= v(\varphi \cdot (\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots))) \end{aligned}$$

and, therefore, that $v(\varphi) \in F_i$, as desired. ⊠

By a *Hilbert-style calculus with finite rules* we understand a (possibly infinite) set of Hilbert-style rules, each of which has finitely many premises.

Definition 2.3.8. Let $\mathcal{H}$ be a Hilbert-style calculus with finite rules, which determines a logic $\vdash$ with an l-partition function $\cdot$. Let $\mathcal{H}^l$ be the Hilbert-style calculus given by the following rules:

$$\emptyset \rhd \psi \tag{H1}$$
$$\gamma_1, \ldots, \gamma_n \rhd \varphi \cdot (\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots)) \tag{H2}$$
$$x \rhd x \cdot y \tag{H3}$$
$$\chi(\delta, \vec{z}) \lhd \rhd \chi(\epsilon, \vec{z}) \tag{H4}$$

for every

(i) $\emptyset \rhd \psi$ rule in $\mathcal{H}$;

(ii) $\gamma_1, \ldots, \gamma_n \rhd \varphi$ rule in $\mathcal{H}$;

(iii) $\epsilon \approx \delta$ equation in the definition of l-partition function, and formula $\chi(v, \vec{z})$.

Theorem 2.3.9. *Let $\vdash$ be a logic with l-partition function $\cdot$ defined by a Hilbert-style calculus with finite rules $\mathcal{H}$. Then $\mathcal{H}^l$ is a complete Hilbert-style calculus for $\vdash^l$.*

Proof. Let $\vdash_{\mathcal{H}^l}$ be the logic determined by $\mathcal{H}^l$. We begin by showing that $\vdash_{\mathcal{H}^l} \leq \vdash^l$. It will be sufficient to show that every rule in $\mathcal{H}^l$ holds in $\vdash^l$. This is clear for (P1). Moreover, the rules (P3, P4) are valid in $\vdash^l$, because $\cdot$ is a partition function for $\vdash^l$ by Lemma 2.3.4. It only remains to prove that (P2) holds in $\vdash^l$. To this end, consider a rule $\gamma_1, \ldots, \gamma_n \rhd \varphi$ in $\mathcal{H}$. Clearly we have that $\gamma_1, \ldots, \gamma_n \vdash \varphi$. Since $\cdot$ is an l-partition function for $\vdash$, we have $x \vdash x \cdot y$. In particular, $\varphi \vdash \varphi \cdot (\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots))$. Hence we conclude that

$$\gamma_1, \ldots, \gamma_n \vdash^l \varphi \cdot (\gamma_1 \cdot (\gamma_2 \cdot \ldots (\gamma_{n-1} \cdot \gamma_n) \ldots)),$$

as desired.

To prove $\vdash^l \leq \vdash_{\mathcal{H}^l}$, we reason as follows. Consider $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash_{\mathcal{H}^l}$). Observe that clearly $\mathbf{A} \in \mathsf{Alg}(\vdash_{\mathcal{H}^l})$. Moreover, $\cdot$ is an l-partition function in $\vdash_{\mathcal{H}^l}$ by Remark 2.3.2 and (P3,P4). Hence we can apply Theorem 2.3.5, obtaining that $\langle \mathbf{A}, F \rangle = \mathcal{P}_{\nmid}(X)$, where X is the l-direct system of matrices $\langle I, \{\langle \mathbf{A}_i, F_i \rangle\}_{i \in I}, \{f_{ij} : i \leq j\} \rangle$ given in the statement of Theorem

2.3.5. Thanks to the rules of $\mathcal{H}^l$ we can replicate the construction in the proof of Lemma 2.3.7 obtaining that each fiber $\langle \mathbf{A}_i, F_i \rangle$ is a model of $\vdash$. This observation, together with the fact that $\langle \mathbf{A}, F \rangle = \mathcal{P}_{\mathfrak{k}}(X)$ and Corollary 2.2.8, implies that $\langle \mathbf{A}, F \rangle$ is a model of $\vdash^l$. Hence we conclude that $\mathsf{Mod}^{\mathsf{Su}}(\vdash_{\mathcal{H}^l}) \subseteq \mathsf{Mod}(\vdash^l)$. This implies that $\vdash^l \leq \vdash_{\mathcal{H}^l}$. $\boxtimes$

The proof of the above result establishes the following:

Corollary 2.3.10. *If $\vdash$ is a finitary logic with an l-partition function, then* $\mathsf{Mod}^{\mathsf{Su}}(\vdash^l) \subseteq \mathcal{P}_{\mathfrak{k}}(\mathsf{Mod}(\vdash))$.

Remark 2.3.11. Observe that the Hilbert calculus $\mathcal{H}^l$ is infinite, as witnessed by condition (H4) in Definition 2.3.8. However, there can be cases in which $\mathcal{H}^l$ can be reduced to a finite calculus. In [42, p.230], it is stated that the Leibniz congruence is *finitizable* in a class of matrices when there is a finite set $\phi \subseteq Fm$ such that for every matrix $\langle \mathbf{A}, F \rangle$ in the class and every $a, b \in A$ it holds

$$\langle a, b \rangle \in \mathbf{\Omega}^{\mathbf{A}} F \text{ if and only if for all } \delta \in \phi \text{ and all } \vec{c} \in \vec{A} \tag{2.2}$$
$$\delta^{\mathbf{A}}(a, \vec{c}) \in F \iff \delta^{\mathbf{A}}(b, \vec{c}) \in F. \tag{2.3}$$

In other words, the quantification employed in Lemma 1.2.3 can be reduced to a finite set. This, together with Lemma 1.2.6, tell us that if the Leibniz congruence is finitizable in $\mathsf{Mod}(\mathcal{H}^l)$, then the infinite set of rules determined by (H4) can be reduced to a finite one, therefore leading to a finite Hilbert calculus for $\vdash^l$. A very interesting investigation would be to determine under which conditions the finitizability of the Leibniz congruence transfers from $\mathcal{H}$ to $\mathcal{H}^l$.

Example 2.3.12. Consider the following axiomatization of classical logic:

$$\emptyset \rhd \neg(\varphi \lor \varphi) \lor \varphi \tag{A1}$$
$$\emptyset \rhd (\varphi \lor \psi) \lor \neg\varphi \tag{A2}$$
$$\emptyset \rhd \neg(\varphi \lor \psi) \lor (\varphi \lor \psi) \tag{A3}$$
$$\emptyset \rhd \neg(\neg\psi \lor \zeta) \lor (\neg(\varphi \lor \psi) \lor (\varphi \lor \zeta)) \tag{A4}$$
$$\varphi, \neg\varphi \lor \psi \rhd \psi \tag{R1}$$

A Hilbert-style calculus for PWK is axiomatized, following Definition 2.3.8, as follows ($\varphi \to \psi$ is a shorthand for $\neg\varphi \lor \psi$):

$$\emptyset \rhd \neg(\varphi \vee \varphi) \vee \varphi \tag{A*1}$$
$$\emptyset \rhd (\varphi \vee \psi) \vee \neg\varphi \tag{A*2}$$
$$\emptyset \rhd \neg(\varphi \vee \psi) \vee (\varphi \vee \psi) \tag{A*3}$$
$$\emptyset \rhd \neg(\neg\psi \vee \zeta) \vee (\neg(\varphi \vee \psi) \vee (\varphi \vee \zeta)) \tag{A*4}$$
$$\varphi, \neg\varphi \vee \psi \rhd \psi \wedge (\psi \vee (\varphi \wedge (\varphi \vee (\neg\varphi \vee \psi)))) \tag{R*1}$$
$$\varphi \rhd \varphi \wedge (\varphi \vee \psi) \tag{R*2}$$
$$\chi(\epsilon, \vec{z}) \lhd\rhd \chi(\delta, \vec{z}) \tag{R*}$$

(R*1) and (R*2) are obtained by setting $x \cdot y := x \wedge (x \vee y)$ as l-partition function for Classical Logic.

⊠

Remark 2.3.13. It is also worth remarking that, in general, $\mathcal{H}^l$ does not feature any linguistic clause on rules, on the contrary of the usual deductive systems for logics of left variable inclusion in the literature. Indeed, taking into account the logic PWK, [13] contain a Hilbert calculus and [29] feature a sequent calculus, both with explicit syntactic restrictions on the applicability of rules.

2.4 Suszko reduced models of $\vdash^l$

In this section we investigate the structure of the Suszko reduced models $\mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$ of the variable inclusion companion $\vdash^l$ of a logic $\vdash$. To this end, we rely on the following technical observation:

Lemma 2.4.1. *Let $\vdash$ be a logic with an l-partition function $\cdot$, and $X = \langle\langle I, \leq\rangle, \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}\rangle$ an l-direct system of models of $\vdash$. Given an upset $J \subseteq I$, we define for every $i \in I$,*

$$G_i := \begin{cases} A_i & \text{if } i \in J \\ F_i & \text{otherwise.} \end{cases}$$

Then $\bigcup_{i\in I} G_i$ is a $\vdash^l$-filter on $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$.

Proof. It is clear that the matrices $\{\langle \mathbf{A}_i, G_i\rangle : i \in I\}$ naturally give rise to an l-direct system of matrices, when equipped with the homomorphisms in X. Moreover, by assumption each $\langle \mathbf{A}_i, G_i\rangle$ is a model of $\vdash$. Thus $\bigcup_{i\in I} G_i$ is a $\vdash^l$-filter on $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ by Lemma 2.2.6. ⊠

The following result identifies the Płonka sums of matrices in $\mathsf{Mod}^{\mathsf{Su}}(\vdash)$ that belong to $\mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$.

Theorem 2.4.2. *Let $\vdash$ be a logic with an l-partition function $\cdot$, and let $X = \langle\langle I, \leq\rangle, \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}\rangle$ be an l-direct system of matrices in $\mathsf{Mod}^{\mathsf{Su}}(\vdash)$. The following conditions are equivalent:*

(i) $\mathcal{P}_{\text{ł}}(X) \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$.

(ii) *For every $n, i \in I$ such that $\langle \mathbf{A}_n, F_n\rangle$ is trivial and $n < i$, there exists $j \in I$ s.t. $n \leq j, i \nleq j$ and $\mathbf{A}_j$ is non trivial.*

Proof. (i)⇒(ii): Suppose that $\mathcal{P}_{\text{ł}}(X) \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$, and consider $n, i \in I$ such that $\langle \mathbf{A}_n, F_n\rangle$ is trivial and $n < i$. The fact that $\langle \mathbf{A}_n, F_n\rangle$ is both trivial and belongs to $\mathsf{Mod}^{\mathsf{Su}}(\vdash)$ implies that $\mathbf{A}_n$ is the trivial algebra. Then $\langle \mathbf{A}_n, F_n\rangle = \langle \mathbf{1}, \{1\}\rangle$. Moreover, set $a := f_{ni}(1)$. Since $n < i$, we know that $a \neq 1$. Together with the fact that $\mathcal{P}_{\text{ł}}(X) \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$, this implies that there is a $\vdash^l$-filter G of $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ such that $\bigcup_{i\in I} F_i \subseteq G$ and $\langle a, 1\rangle \notin \mathbf{\Omega}^{\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}} G$. Thus, by Lemma 1.2.4, there is a formula $\varphi(x, \vec{z})$ and elements $\vec{c} \in \bigcup_{i\in I} A_i$ such that

$$\varphi^{\mathcal{P}_{\text{ł}}}(a, \vec{c}) \in G \Longleftrightarrow \varphi^{\mathcal{P}_{\text{ł}}}(1, \vec{c}) \notin G. \tag{2.4}$$

We can assume w.l.o.g. that all the elements in the sequence $\vec{c}$ belong to the same component A_k of the Płonka sum $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$.[2]

We claim that indeed $\varphi(1, \vec{c}) \notin G$. Suppose the contrary towards a contradiction. Then $\varphi(1, \vec{c}) \in G$. First observe that

$$\begin{aligned}
\varphi^{\mathcal{P}_{\text{ł}}}(a, \vec{c}) &= \varphi^{\mathbf{A}_{i\vee k}}(f_{i,i\vee k}(a), f_{k,i\vee k}(\vec{c})) &(2.5)\\
&= \varphi^{\mathbf{A}_{i\vee k}}(f_{i,i\vee k} \circ f_{n,i}(1), f_{k,i\vee k}(\vec{c})) &(2.6)\\
&= \varphi^{\mathbf{A}_{i\vee k}}(f_{n\vee k,i\vee k} \circ f_{n,n\vee k}(1), f_{n\vee k,i\vee k} \circ f_{k,n\vee k}(\vec{c})) &(2.7)\\
&= f_{n\vee k,i\vee k}\, \varphi^{\mathbf{A}_{n\vee k}}(f_{n,n\vee k}(1), f_{k,n\vee k}(\vec{c})) &(2.8)\\
&= f_{n\vee k,i\vee k}\, \varphi^{\mathcal{P}_{\text{ł}}}(1, \vec{c}) &(2.9)\\
&= f_{n\vee k,i\vee k}(\varphi^{\mathcal{P}_{\text{ł}}}(1, \vec{c})) \cdot^{\mathbf{A}_{i\vee k}} f_{i,i\vee k}(a) &(2.10)\\
&= \varphi^{\mathcal{P}_{\text{ł}}}(1, \vec{c}) \cdot^{\mathcal{P}_{\text{ł}}} a &(2.11)\\
&\in G. &(2.12)
\end{aligned}$$

[2]More precisely, if $\vec{c} = c_1, \ldots, c_m$ and $c_1 \in A_{p_1}, \ldots, c_m \in A_{p_m}$, then we set $k := p_1 \vee \cdots \vee p_m$ and replace c_i by $f_{p_i k}(c_i)$.

The above equalities are justified as follows: (2.7) is a consequence of the fact that X is an l-direct system of matrices and that $n \vee k \leq i \vee k$ (since $n \leq i$), (2.10) follows from the fact that $x \cdot y$ is the projection on the first component on the algebra $\mathbf{A}_{i \vee k}$. Condition (2.12) follows from the fact that $\varphi(1, \vec{c}) \in G$, G is a $\vdash^l$-filter and, by Lemma 2.3.4 $\cdot$ is an l-partition function for $\vdash^l$, hence $x \vdash^l x \cdot y$.

Hence we have that $\varphi(a, \vec{c}), \varphi(1, \vec{c}) \in G$, which contradicts (2.4), establishing the claim.

From the claim and (2.4) we get that $\varphi(a, \vec{c}) \in G$ and $\varphi(1, \vec{c}) \notin G$. Set $j := n \vee k$ and $m := k \vee i$. We claim that j is such that: (A) $n \leq j$, (B) $\mathbf{A}_j$ is non trivial and (C) $i \nleq j$. We proceed to prove (A, B, C).

(A): Since $j = n \vee k$, we have that $n \leq j$.

(B): Observe that

$$\varphi^{\mathcal{P}_{\natural}}(1, \vec{c}) = \varphi^{\mathbf{A}_j}(f_{nj}(1), f_{kj}(\vec{c})) \in A_j.$$

Together with $\varphi(1, \vec{c}) \notin G$, this implies that $\varphi^{\mathcal{P}_{\natural}}(1, \vec{c}) \in A_j \smallsetminus G$.

On the other hand, since $F_n = A_n$, we have that

$$f_{nj}(1) \in f_{nj}[F_n] \subseteq F_j \subseteq A_j \cap G.$$

Thus both $A_j \cap G$ and $A_j \smallsetminus G$ are non-empty. We conclude that $\mathbf{A}_j$ is non trivial.

(C): Suppose, by contradiction, that $i \leq j$. In particular, this implies that $m = j$ (indeed, $i \leq j = n \vee k$, thus $i \vee k \leq n \vee k$, i.e. $m \leq j$; on the other hand, since $n < i$ then $n \vee k \leq i \vee j$, i.e. $j \leq m$). Therefore we have that

$$\begin{aligned}
\varphi^{\mathcal{P}_{\natural}}(1, \vec{c}) &= \varphi^{\mathbf{A}_j}(f_{nj}(1), f_{kj}(\vec{c})) && (2.13)\\
&= \varphi^{\mathbf{A}_j}(f_{ij} \circ f_{ni}(1), f_{kj}(\vec{c})) && (2.14)\\
&= \varphi^{\mathbf{A}_j}(f_{ij}(a), f_{kj}(\vec{c})) && (2.15)\\
&= \varphi^{\mathbf{A}_m}(f_{im}(a), f_{km}(\vec{c})) && (2.16)\\
&= \varphi^{\mathcal{P}_{\natural}}(a, \vec{c}) \in G. && (2.17)
\end{aligned}$$

The above equalities are justified as follows. (2.14) follows from the fact that $i \leq m = j$. (2.15) is a consequence of $a = f_{ni}(1)$. (2.16) from $j = m$ and (2.17) from $m = i \vee k$. This establishes the above equalities, yielding that $\varphi^{\mathcal{P}_{\natural}}(1, \vec{c}) \in G$. But this contradicts the fact that $\varphi(1, \vec{c}) \notin G$.

Hence (A), (B) and (C) hold establishing our claim. In particular, this implies that $j \in I$ satisfies the condition in the statement.

(ii)⇒(i): By Lemma 2.2.6 we know that $\mathcal{P}_{\nmid}(X)$ is a model of $\vdash^{l}$. It only remains to prove that it is Suszko reduced. To this end, let θ be the Suszko congruence of $\mathcal{P}_{\nmid}(X)$.

Observe that, in order to prove that θ is the identity, it will be enough to show that it does not identify distinct elements in components of the Płonka sum which are comparable with respect to the order $\leq$. To prove this, suppose indeed that θ does not identify different elements in components of the Płonka sum which are comparable. Then consider two different elements $a, b \in A = \bigcup_{i \in I} A_i$. There exist $i, j \in I$ such that $a \in A_i$ and $b \in A_j$. If i and j are comparable, then by assumption $\langle a, b \rangle \notin \theta$. Then consider the case where i and j are incomparable. Set $k := i \vee j$. Clearly we have that $i, j < k$. In particular, we have that $b \cdot b = b \in A_j$ and $a \cdot b \in A_k$ and, therefore, $b \cdot b \neq a \cdot b$. Since j and k are comparable, this implies that $\langle b \cdot b, a \cdot b \rangle \notin \theta$. In particular, this means that $\langle a, b \rangle \notin \theta$ as well. As a consequence we conclude that θ is the identity.

By the above observation, to prove that θ is the identity, it will be enough to show that it does not identify elements in components of the Płonka sum $\mathcal{P}_{\nmid}(X)$ which are comparable with respect to $\leq$. To this end, consider two different elements $a, b \in A$ such that $a \in A_i$ and $b \in A_j$ with $i \leq j$. We have two cases: either $i = j$ or $i < j$.

First consider the case where $i = j$, that is $a, b \in A_i$. By assumption, we have that $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$. Therefore we can assume w.l.o.g. that there is a $\vdash$-filter G_i on $\mathbf{A}_i$ such that $F_i \subseteq G_i$, some elements $\vec{c} \in A_i$, and a formula $\varphi(x, \vec{z})$ such that $\varphi^{\mathbf{A}_i}(a, \vec{c}) \in G_i$ and $\varphi^{\mathbf{A}_i}(b, \vec{c}) \notin G_i$. For every $l \neq i$, define

$$G_l := \begin{cases} A_l & \text{if } i \leq l \\ F_l & \text{otherwise.} \end{cases}$$

An analogous argument to the one described in the proof Lemma 2.4.1 shows that $G := \bigcup_{i \in I} G_i$ is a $\vdash^{l}$-filter on $\mathcal{P}_{\nmid}(\mathbf{A}_i)_{i \in I}$. Moreover, observe that

$$\begin{aligned} \varphi^{\mathcal{P}_{\nmid}}(a, \vec{c}) &= \varphi^{\mathbf{A}_i}(a, \vec{c}) \in G \\ \varphi^{\mathcal{P}_{\nmid}}(b, \vec{c}) &= \varphi^{\mathbf{A}_i}(b, \vec{c}) \notin G. \end{aligned}$$

We conclude that $\langle a, b \rangle \notin \theta$.

Then we consider the case where $i < j$. We have cases: either $\mathbf{A}_i$ is trivial or not. If $\mathbf{A}_i$ is non trivial, then $F_i \neq A_i$ as $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$. Then for every $l \in I$, we define

$$G_l := \begin{cases} A_l & \text{if } i < l \\ F_l & \text{otherwise.} \end{cases}$$

By Lemma 2.4.1 we know that $G := \bigcup_{i \in I} G_i$ is a $\vdash^l$-filter on $\mathcal{P}_{\nmid}(\mathbf{A}_i)_{i \in I}$. Then choose an element $c \in A_i \smallsetminus F_i$. We have that

$$c \cdot^{\mathcal{P}_{\nmid}} a = c \cdot^{\mathbf{A}_i} a = c \in A_i \smallsetminus F_i = A_i \smallsetminus G_i$$

and $c \cdot^{\mathcal{P}_{\nmid}} b \in A_j = G_j$. Therefore, $c \cdot a \notin G$ and $c \cdot b \in G$. Hence we conclude that $\langle a, b \rangle \notin \theta$, as desired.

Then we consider the case where $\mathbf{A}_i$ is trivial. We have cases: either $F_i = \emptyset$ or $F_i = A_i$. First suppose that $F_i = \emptyset$. Iterating the argument in the previous paragraph (taking $c := a$) we obtain that $\langle a, b \rangle \notin \theta$. Then consider the case where $F_i = A_i$. Observe that in this case $\langle \mathbf{A}_i, F_i \rangle$ is a trivial matrix. Therefore we can apply the assumption, obtaining an element $k \in I$ such that $\mathbf{A}_k$ is non trivial, $i < k$ and $j \nleq k$. Then for every $l \in I$ we define

$$G_l := \begin{cases} A_l & \text{if } k \vee j \leq l \\ F_l & \text{otherwise.} \end{cases}$$

By Lemma 2.4.1 we know that $G := \bigcup_{i \in I} G_i$ is a $\vdash^l$-filter on $\mathcal{P}_{\nmid}(\mathbf{A}_i)_{i \in I}$. Since $\mathbf{A}_k$ is non trivial and $\langle \mathbf{A}_k, F_k \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$, there is $c \in A_k \smallsetminus F_k$. Since $k < k \vee j$, we have that

$$\begin{aligned} c \cdot^{\mathcal{P}_{\nmid}} a &= c \cdot^{\mathbf{A}_k} f_{ik}(a) = c \in A_k \smallsetminus F_k = A_k \smallsetminus G_k \\ c \cdot^{\mathcal{P}_{\nmid}} b &\in A_{j \vee k} = G_{j \vee k}. \end{aligned}$$

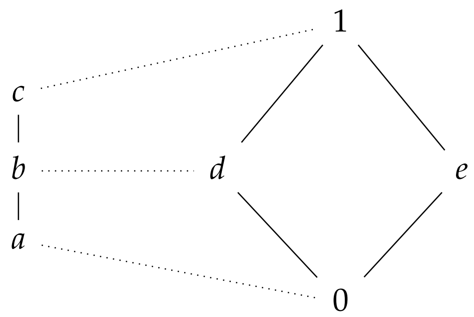
Hence we conclude that $c \cdot^{\mathcal{P}_{\nmid}} a \notin G$ and $c \cdot^{\mathcal{P}_{\nmid}} b \in G$. But this means that $\langle a, b \rangle \notin \theta$. ⊠

Theorem 2.4.2 identifies the Suszko reduced models of $\vdash^l$, which can be expressed in terms of Płonka sums of Suszko reduced models of $\vdash$. It is natural to wonder whether it is true that *all* Suszko models of $\vdash^l$ are of this kind. The following shows that this does not hold in general:

Example 2.4.3. Consider the logic $\vdash$ determined by the following class of matrices:

$$\mathsf{M} := \{\langle \mathbf{A}, F \rangle : \mathbf{A} \text{ is a distributive lattice and } F \text{ is an upset}\}.$$

Let $\mathbf{A}_1$ be the three element lattice $a < b < c$ and let $F_1 = \{b, c\}$. Moreover, let $\mathbf{A}_2$ be the four-element Boolean lattice (with universe $\{0, d, e, 1\}$ with 0 as bottom element), and let $F_2 = A_2 \smallsetminus \{0\}$. Clearly both $\langle \mathbf{A}_1, F_1 \rangle$ and $\langle \mathbf{A}_2, F_2 \rangle$ are models of $\vdash$ (as they belong to M). However, it is easy to see that $\langle \mathbf{A}_1, F_1 \rangle \notin \mathsf{Mod}^{\mathsf{Su}}(\vdash)$. Now, let $f \colon \mathbf{A}_1 \to \mathbf{A}_2$ be any of the two embeddings of $\mathbf{A}_1$ into $\mathbf{A}_2$. Clearly these two matrices plus f give rise to an l-direct system X of matrices (of course one should pedantically add the identity endomorphisms) depicted in the following figure. We denote by $\langle \mathbf{B}, G \rangle$ the Płonka sum $\mathcal{P}_{\text{ł}}(X)$.



Since $\langle \mathbf{A}_1, F_1 \rangle$ and $\langle \mathbf{A}_2, F_2 \rangle$ are models of $\vdash$, by Lemma 2.2.6 $\langle \mathbf{B}, G \rangle$ is a model of $\vdash^l$. Moreover, it is possible to see that it is indeed Suszko reduced. Elements belonging to the algebra $\mathbf{A}_1$, as for example b and c (any other pair of elements in $\mathbf{A}_1$ is distinguished by the identity function), can be distinguished by means of the function $\wedge^{\mathbf{B}}$, the filter G and the element e, as follows:

$$b \wedge^{\mathbf{B}} e = d \wedge^{\mathbf{A}_2} e = 0 \notin G$$

$$c \wedge^{\mathbf{B}} e = 1 \wedge^{\mathbf{A}_2} e = e \in G.$$

One can reason similarly (using G as filter) for pairs of elements belonging to $\mathbf{A}_2$ (we illustrate the only interesting case):

$$d \wedge^{\mathbf{B}} b = d \wedge^{\mathbf{A}_2} d = d \in G$$

$$e \wedge^{\mathbf{B}} b = e \wedge^{\mathbf{A}_2} d = 0 \notin G.$$

On the other hand, pairs of elements belonging to different algebras are distinguished by considering the filter $H := F_1 \cup A_2$ on $\mathbf{B}$ (the fact that it is a filter is guaranteed by Lemma 2.4.1) , the function $\wedge^{\mathbf{B}}$ and the element a. Consider, for instance, the elements b and d:

$$b \wedge^{\mathbf{B}} a = a \notin H;$$

$$d \wedge^{\mathbf{B}} a = d \wedge^{\mathbf{A}_2} 0 = 0 \in H.$$

This is enough to show that $\langle \mathbf{B}, G \rangle$ is Suszko reduced.

To conclude the example we need to disprove that $\langle \mathbf{B}, G\rangle$ is a Płonka sum of any Suszko reduced model of $\vdash$. Suppose that $\langle \mathbf{B}, G\rangle$ is the Płonka sum of an l-direct system Y of Suszko reduced models $\langle \mathbf{B}_1, G_1\rangle, \dots, \langle \mathbf{B}_n, G_n\rangle$ of $\vdash$. First observe that $n \leq 2$. Suppose the contrary towards a contradiction. Then $n \geqslant 3$. We choose three elements $b_1 \in B_1, b_2 \in B_2$ and $b_3 \in B_3$. Clearly b_1, b_2 and b_3 are different. Moreover, for every $1 \leq i < j \leq 3$ we have that either $b_i \cdot b_j \neq b_i$ or $b_j \cdot b_i \neq b_j$. It is easy to see that no such three elements exist in $\mathbf{B}$, which is a contradiction. Hence $n \leq 2$. We have cases. If $n = 1$, then $\langle \mathbf{B}_1, G_1\rangle = \langle \mathbf{B}, G\rangle$. In particular, this implies that $\langle \mathbf{B}, G\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ and, therefore, $\mathbf{B} \in \mathsf{Alg}(\vdash)$. By Lemma 1.2.5 this implies that $\mathbf{B}$ is a lattice, which is false. Thus we obtain that $n = 2$. Now, by Lemma 1.2.5 we know that $\mathbf{B}_1$ and $\mathbf{B}_2$ are lattices. Since the only way of partitioning $\mathbf{B}$ into two subalgebras that are distributive lattice is $\{A_1, A_2\}$, we conclude that w.l.o.g. $\mathbf{B}_1 = \mathbf{A}_1$ and $\mathbf{B}_2 = \mathbf{A}_2$. ⊠

2.5 Equivalential logics

It turns out that, in the setting of finitary equivalential logics $\vdash$, the class of matrices $\mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$ has a very transparent description in terms of Płonka sums, as we proceed to prove (see Theorem 2.5.3).

Lemma 2.5.1. *Let $\vdash$ be an equivalential finitary logic with an l-partition function. Then*

$$\mathsf{Mod}^*(\vdash^l) \subseteq \mathbb{I}\mathcal{P}_{\text{ł}}(\mathsf{Mod}^*(\vdash)).$$

Proof. Recall from Lemma 2.3.4 that also $\vdash^l$ has an l-partition function. Then consider $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\vdash^l)$ and let

$$X = \langle \langle I, \leq \rangle, \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}\rangle$$

be the l-direct system of matrices given in Theorem 2.3.5. We know that $\mathcal{P}_{\text{ł}}(X) = \langle \mathbf{A}, F\rangle$. Moreover, by Lemma 2.3.7, we know that each fiber of X is a model of $\vdash$. It only remains to prove that the fibers of X are Leibniz reduced.

We claim that $\bigcup_{i\in I} \boldsymbol{\Omega}^{\mathbf{A}_i} F_i$ is a congruence of $\mathbf{A}$. To show this, let $\Delta(x, y)$ be a set of congruence formulas for $\vdash$. Then consider an n-ary basic operation λ and elements $a_1, \dots, a_n, b_1, \dots, b_n \in A$ such that $\langle a_j, b_j\rangle \in \bigcup_{i\in I} \boldsymbol{\Omega}^{\mathbf{A}_i} F_i$, for all $1 \leq j \leq n$. This implies that are indexes $m_1, \dots, m_n \in I$ such that $a_j, b_j \in A_{m_j}$, for all $j \leq n$. The fact that Δ is a set of congruence formulas for $\vdash$ implies that

$$\Delta^{\mathcal{P}_{\text{ł}}}(a_j, b_j) = \Delta^{\mathbf{A}_j}(a_j, b_j) \subseteq F_j.$$

Set $k := m_1 \vee \cdots \vee m_n$. We have that

$$\bigcup_{j \leq n} \Delta^{\mathbf{A}_k}(f_{m_j k}(a_j), f_{m_j k}(b_j)) \subseteq F_k \tag{2.18}$$

From the fact that Δ is a set of congruence formulas for $\vdash$ it follows that

$$\bigcup_{j \leq n} \Delta(x_j, y_j) \vdash \Delta(\lambda(\vec{x}), \lambda(\vec{y})). \tag{2.19}$$

Together with (2.18) and (2.19), the fact that $\langle \mathbf{A}_k, F_k \rangle$ is a model of $\vdash$ implies that

$$\begin{aligned} &\Delta^{\mathbf{A}_k}(\lambda^{\mathcal{P}_{\nmid}}(a_1, \dots, a_n), \lambda^{\mathcal{P}_{\nmid}}(b_1, \dots, b_n)) \\ &= \Delta^{\mathbf{A}_k}(\lambda(f_{m_1 k}(a_1), \dots, f_{m_n k}(a_n)), \lambda(f_{m_1 k}(b_1), \dots, f_{m_n k}(b_n))) \\ &\subseteq F_k. \end{aligned}$$

Together with the fact that Δ is a set of congruence formulas for $\vdash$, this implies that

$$\langle \lambda^{\mathcal{P}_{\nmid}}(\vec{a}), \lambda^{\mathcal{P}_{\nmid}}(\vec{b}) \rangle \in \mathbf{\Omega}^{\mathbf{A}_k} F_k \subseteq \bigcup_{i \in I} \mathbf{\Omega}^{\mathbf{A}_i} F_i.$$

This establishes the claim.

Since each $\mathbf{\Omega}^{\mathbf{A}_i} F_i$ is compatible with F_i, we know that the congruence $\bigcup_{i \in I} \mathbf{\Omega}^{\mathbf{A}_i} F_i$ is compatible with F. In particular, this implies that $\bigcup_{i \in I} \mathbf{\Omega}^{\mathbf{A}_i} F_i \subseteq \mathbf{\Omega}^{\mathbf{A}} F$. Since $\mathbf{\Omega}^{\mathbf{A}} F$ is the identity relation, we conclude that so is each $\mathbf{\Omega}^{\mathbf{A}_i} F_i$. Hence we obtain that $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^*(\vdash)$ for every $i \in I$ and, therefore, that

$$\langle \mathbf{A}, F \rangle = \mathcal{P}_{\nmid}(X) \subseteq \mathcal{P}_{\nmid}(\mathsf{Mod}^*(\vdash)).$$

We conclude that $\mathsf{Mod}^*(\vdash^l) \subseteq \mathcal{P}_{\nmid}(\mathsf{Mod}^*(\vdash))$, as desired. $\boxtimes$

Corollary 2.5.2. *If $\vdash$ is an equivalential finitary logic with an l-partition function, then*

$$\mathsf{Mod}^{\mathsf{Su}}(\vdash^l) \subseteq \mathbb{IP}_{\nmid}(\mathsf{Mod}^*(\vdash)) = \mathbb{IP}_{\nmid}(\mathsf{Mod}^{\mathsf{Su}}(\vdash)).$$

Proof. First recall that $\mathsf{Mod}^{\mathsf{Su}}(\vdash) = \mathsf{Mod}^*(\vdash)$, since $\vdash$ is equivalential. Thus it will be enough to prove that $\mathsf{Mod}^{\mathsf{Su}}(\vdash^l) \subseteq \mathcal{P}_{\nmid}(\mathsf{Mod}^*(\vdash))$. We have that

$$\begin{aligned} \mathsf{Mod}^{\mathsf{Su}}(\vdash^l) &= \mathbb{P}_{\mathsf{SD}}\mathsf{Mod}^*(\vdash^l) && (2.20) \\ &\subseteq \mathbb{SP}\mathsf{Mod}^*(\vdash^l) && (2.21) \\ &\subseteq \mathbb{SP}\mathcal{P}_{\nmid}(\mathsf{Mod}^*(\vdash)) && (2.22) \\ &\subseteq \mathcal{P}_{\nmid}(\mathbb{SP}\mathsf{Mod}^*(\vdash)) && (2.23) \\ &= \mathcal{P}_{\nmid}(\mathsf{Mod}^*(\vdash)). && (2.24) \end{aligned}$$

The non trivial inclusions above are justified as follows: (2.22) is a consequence of Lemma 2.5.1, (2.23) follows from Lemma 2.2.2, and (2.24) from the fact that $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{S}$ and $\mathbb{P}$, since $\vdash$ is equivalential. Hence we conclude that $\mathsf{Mod}^{\mathsf{Su}}(\vdash^l) \subseteq \mathcal{P}_{\mathfrak{k}}(\mathsf{Mod}^*)$. ⊠

We are now ready to provide a full characterization of the Suszko reduced models of the variable inclusion companion of a finitary equivalential logic (with l-partition function).

Theorem 2.5.3. *Let $\vdash$ be an equivalential and finitary logic with an l-partition function, and $\langle \mathbf{A}, F\rangle$ be a matrix. The following conditions are equivalent:*

(i) $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$.

(ii) *There exists an l-direct system of matrices $X \subseteq \mathsf{Mod}^*(\vdash)$ indexed by a semilattice I such that $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_{\mathfrak{l}}(X)$ and for every $n, i \in I$ such that $\langle \mathbf{A}_n, F_n\rangle$ is trivial and $n < i$, there exists $j \in I$ s.t. $n \leq j, i \nleq j$ and $\mathbf{A}_j$ is non trivial.*

Proof. This is a consequence of Theorem 2.4.2 and Corollary 2.5.2. ⊠

Example 2.5.4. Observe that all substructural logics [48, 68] are finitary, equivalential, and have an l-partition function. The same holds for all local and global consequences of normal modal logics [6]. As a consequence, the above result provides a description of the Suszko reduced models of the regularizations of all substructural and modal logics (when the latter are understood as local and global consequences of normal modal logics [5, 23, 58]). ⊠

2.6 Logics with antitheorems

The goal of this section is to show that if $\vdash$ is a logic with antitheorems (see Definition 1.2.8), then the description of the Suszko reduced models of its variables inclusion companion can be sustantially improved (see Theorems 2.4.2 and 2.5.3), as we show in this section.

The next result discloses the semantic meaning of inconsitency terms. It should be observed that algebraic versions of it first appeared in [57] and [21] in the setting of varieties and quasi-varieties of algebras respectively.

Lemma 2.6.1. *Let $\vdash$ be a logic. The following are equivalent:*

(i) $\vdash$ *has an antitheorem* Σ.

(ii) *If* $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash)$ *is non trivial, then it has no trivial submatrix.*

Proof. (i)⇒(ii): Suppose that $\vdash$ has an antitheorem Σ. We can assume w.l.o.g. that Σ is in variable x only. Suppose, in view of a contradiction, that there is a non trivial matrix $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash)$ with a trivial submatrix $\langle \mathbf{B}, B\rangle$. Since $\langle \mathbf{A}, F\rangle$ is non trivial, there exists an element $a \in A \smallsetminus F$. Consider any homomorphism $v\colon \mathbf{Fm} \to \mathbf{A}$ such that $v(x) = b$ and $v(y) = a$, where b is any element of B. Since $\Sigma = \Sigma(x)$ and $\langle \mathbf{B}, B\rangle$ is a submatrix of $\langle \mathbf{A}, F\rangle$, we have that $v[\Sigma] \subseteq B \subseteq F$. Together with the fact that $\Sigma \vdash y$, this implies that $a = v(y) \in F$, which is a contradiction.

(ii)⇒(i): Let $Fm(x)$ be the set of formulas in variable x only. We show that $Fm(x)$ is an antitheorem for $\vdash$. To this end, consider a substitution σ and a formula ψ. It is enough to show that $\sigma[Fm(x)] \vdash \psi$. Let $\varphi := \sigma(x)$. Observe that $\sigma[Fm(x)]$ coincides with the universe of the subalgebra $\mathrm{Sg}^{\mathbf{Fm}}(\varphi)$ of $\mathbf{Fm}$ generated by φ. Consider the matrices

$$\begin{aligned} \mathsf{M}_1 &:= \langle \mathbf{Fm}, \mathrm{Cn}_{\vdash}(\mathrm{Sg}^{\mathbf{Fm}}(\varphi))\rangle \\ \mathsf{M}_2 &:= \langle \mathrm{Sg}^{\mathbf{Fm}}(\varphi), \mathrm{Sg}^{\mathbf{Fm}}(\varphi)\rangle. \end{aligned}$$

Clearly, M_1 is a model of $\vdash$ and M_2 a trivial submatrix of M_1. By the assumption, we get that M_1 is a trivial matrix, i.e. $Fm = \mathrm{Cn}_{\vdash}(\mathrm{Sg}^{\mathbf{Fm}}(\varphi))$. Hence we conclude that

$$\psi \in Fm = \mathrm{Cn}(\mathrm{Sg}^{\mathbf{Fm}}(\varphi)) = \mathrm{Cn}_{\vdash}(\sigma[Fm(x)]).$$

Clearly this implies that $\sigma[Fm(x)] \vdash \psi$, as desired. ⊠

Remarkably, Theorem 2.4.2 can be substantially improved for logics possessing an antitheorem:

Theorem 2.6.2. *Let* $\vdash$ *be a logic with an l-partition function and an antitheorem. For every l-direct system* X *of matrices in* $\mathsf{Mod}^{\mathsf{Su}}(\vdash)$*, the following conditions are equivalent:*

(i) $\mathcal{P}_l(X) \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$.

(ii) *X contains at most one trivial component.*

Proof. For the sake of simplicity, throughout the proof we set

$$X = \langle \langle I, \leq\rangle, \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}\rangle.$$

First we claim that if a component $\langle \mathbf{A}_n, F_n \rangle$ of X is trivial, then so is $\langle \mathbf{A}_k, F_k \rangle$, for every $k \geqslant n$. To prove this, consider a trivial component $\langle \mathbf{A}_n, F_n \rangle$ of X and $k \geqslant n$. Observe that

$$f_{nk}[A_n] = f_{nk}[F_n] \subseteq F_k.$$

Then $\langle f_{nk}[A_n], f_{nk}[F_n] \rangle$ is a trivial submatrix of $\langle \mathbf{A}_k, F_k \rangle$. Since $\vdash$ has an antitheorem, we can apply Lemma 2.6.1 obtaining that $\langle \mathbf{A}_k, F_k \rangle$ is trivial. This establishes the claim.

(i)⇒(ii): Suppose, in view of a contradiction, that $\mathcal{P}_{\nmid}(X) \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{l})$ and that X contains two distinct trivial components $\langle \mathbf{1}_n, \{1_n\} \rangle$ and $\langle \mathbf{1}_k, \{1_k\} \rangle$ (their algebraic reducts are trivial, as the components of X belong to $\mathsf{Mod}^{\mathsf{Su}}(\vdash)$). Set $\langle \mathbf{A}, F \rangle := \mathcal{P}_{\nmid}(X)$. Observe that, for every formula $\varphi(x, \vec{z})$ in which x really occurs, and every tuple $\vec{c} \in A$, we have that

$$\varphi^{\mathbf{A}}(1_n, \vec{c}), \varphi^{\mathbf{A}}(1_k, \vec{c}) \in F.$$

To prove this, observe that the element $\varphi(1_n, \vec{c})$ belongs to a component $\langle \mathbf{A}_l, F_l \rangle$ of X with $n \leq l$. By the previous claim, we know that $\langle \mathbf{A}_l, F_l \rangle$ is trivial and, therefore, that $\varphi(1_n, \vec{c}) \in F_l \subseteq F$, as desired. A similar argument shows that $\varphi(1_k, \vec{c}) \in F$ as well. Hence for every unary polynomial function p of $\mathbf{A}$ we have that

$$\mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(F \cup \{p(1_n)\}) = \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(F \cup \{p(1_k)\}).$$

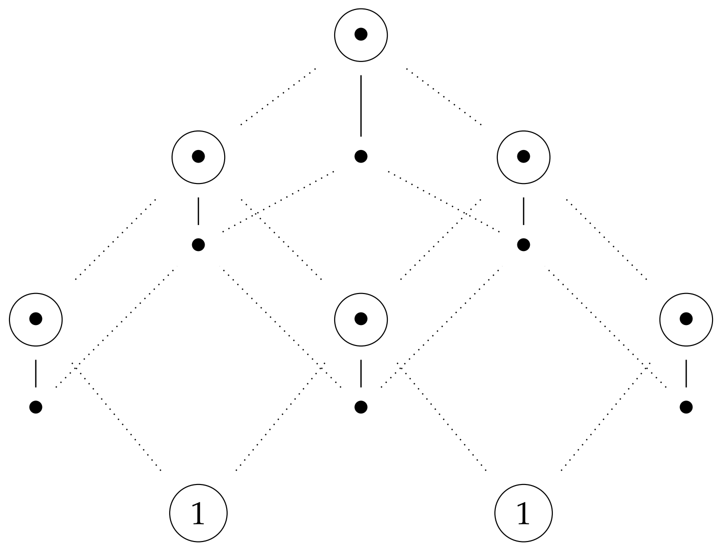
By Lemma 1.2.4 this implies that $\langle 1_n, 1_k \rangle \in \widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\vdash} F$. Since $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{l})$, this implies that $1_n = 1_k$, which is a contradiction.

(ii)⇒(i): Suppose that X contains at most one trivial matrix. If X contain no trivial component. Then by Theorem 2.4.2 we obtain that $\mathcal{P}_{\nmid}(X) \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{l})$. Then consider the case where X contains exactly one trivial component. By the claim we obtain that this component is the maximum of $\langle I, \leq \rangle$. Again, with an application of Theorem 2.4.2, we conclude that $\mathcal{P}_{\nmid}(X) \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{l})$. ⊠

The assumption that $\vdash$ has an antitheorem in the above theorem is essential, as shown in the following

Example 2.6.3. The statement of Theorem 2.6.2 is in general false for logics without an antitheorem, as witnessed by the following example based of $\mathrm{CL}^{\wedge\vee}$. In particular, it happens to have a Suszko reduced model of $\vdash^{l}$, which is the Płonka sum of Suszko of reduced models of $\vdash$ containing two trivial matrices.

Let $\vdash$ be the $\{\wedge, \vee\}$-fragment of classical propositional logic. Moreover, let $\mathbf{1}$ be the trivial lattice and $\mathbf{L}_2 = \langle\{\bot, 1\}, \wedge, \vee\rangle$ the 2-element distributive lattice (with $\bot < 1$). Consider the l-direct system X of matrices formed by 6 copies of the matrix $\langle \mathbf{L}_2, \{1\}\rangle$ and two trivial matrices $\langle \mathbf{1}, \{1\}\rangle$ sketched in the following figure (lines represent lattice order in the Płonka fibers, dotted lines the homomorphisms, and circles, filters in any fiber).



Clearly each matrix in X, which contains two trivial matrices, is a Suszko reduced model of $\vdash$. Moreover, by applying Theorem 2.4.2, one immediately checks that $\mathcal{P}_{\mathrm{ł}}(X) \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{l})$. ⊠

Drawing consequences from Theorem 2.6.2, we obtain a very transparent description of the Suszko reduced models of the variable inclusion logic companion of a finitary equivalential logic with an l-partition function and antitheorems:

Theorem 2.6.4. *Let $\vdash$ be an equivalential and finitary logic with an l-partition function and antitheorems, and $\langle \mathbf{A}, F\rangle$ be a matrix. The following conditions are equivalent:*

(i) $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{l})$.

(ii) *There exists an l-direct system of matrices $X \subseteq \mathsf{Mod}^{*}(\vdash)$ with at most one trivial component such that $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_{\mathrm{ł}}(X)$.*

Proof. This is a combination of Theorems 2.6.2 and 2.5.3. ⊠

Example 2.6.5. It is worth to observe that the above result provides a full description of the Suszko reduced models of the regularizations of most well-known logics, including all logics mentioned in Example 1.2.9. ⊠

2.7 Classification in the Leibniz hierarchy

We conclude this chapter by investigating the location of logics of variable inclusions in the Leibniz hierarchy. To this end, recall that a logic $\vdash$ is inconsistent if $\Gamma \vdash \varphi$ for every $\Gamma \cup \{\varphi\} \subseteq Fm$. Equivalently, $\vdash$ is inconsistent if $\emptyset \vdash x$ for some variable x. A logic is *consistent* when it is not inconsistent.

Theorem 2.7.1. *Let $\vdash$ be a logic.*

(i) *If $\vdash$ is consistent, then $\vdash^l$ is not protoalgebraic.*

(ii) *If $\vdash$ is finitary, algebraizable and has an l-partition function, then $\vdash^l$ is truth-equational.*

Proof. (i): We reason by contraposition. Suppose that $\vdash^l$ is protoalgebraic. Then there is a set of formulas $\Delta(x,y)$ such that $\emptyset \vdash^l \Delta(x,x)$ and $x, \Delta(x,y) \vdash^l y$. Together with $x, \Delta(x,y) \vdash^l y$, the definition of $\vdash^l$ implies that there is a subset $\Sigma(y) \subseteq \Delta(x,y)$ such that $\Sigma(y) \vdash y$. Since $\emptyset \vdash^l \Delta(x,x)$, we have that $\emptyset \vdash^l \Sigma(y)$. From $\Sigma(y) \vdash y$ and $\emptyset \vdash^l \Sigma(y)$ it follows that $\emptyset \vdash^l y$. By the definition of $\vdash^l$ we conclude that $\emptyset \vdash y$ and, therefore, that $\vdash$ is inconsistent.

(ii): Suppose that $\vdash$ is finitary, algebraizable and has an l-partition function. In particular, $\vdash$ is truth-equational with set of defining equations $\boldsymbol{\tau}(x)$. We will show that $\boldsymbol{\tau}(x)$ is a set of defining equations for $\vdash^l$ as well. To this end, consider $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^*(\vdash^l)$. Since $\vdash$ is finitary, equivalential and with an l-partition function, we can apply Lemma 2.5.1 obtaining that there exists an l-direct system of matrices $X \subseteq \mathsf{Mod}^*(\vdash)$ such that $\langle \mathbf{A}, F \rangle \cong \mathcal{P}_{\natural}(X)$. For the sake of simplicity, we set

$$X = \langle \langle I, \leq \rangle, \{\langle \mathbf{A}_i, F_i \rangle\}_{i \in I}, \{f_{ij} : i \leq j\} \rangle$$

and assume w.l.o.g. that $\langle \mathbf{A}, F \rangle = \mathcal{P}_{\natural}(X)$. Consider an element $a \in A$. There is $i \in I$ such that $a \in A_i$. We have that

$$\mathbf{A} \vDash \boldsymbol{\tau}(a) \iff \mathbf{A}_i \vDash \boldsymbol{\tau}(a) \iff a \in F_i \iff a \in F. \tag{2.25}$$

The above equivalences are justified as follows. The first one follows from the fact that $\mathbf{A} \cong \mathcal{P}_{\natural}(\mathbf{A}_i)_{i \in I}$. The second one follows from the fact that $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^*(\vdash)$ and that $\boldsymbol{\tau}(x)$ is a set of defining equations for $\vdash$. The last one follows from the observation that $\langle \mathbf{A}, F \rangle \cong \mathcal{P}_{\nmid}(X)$.

By (2.25) we obtain that for every $a \in A$,

$$\mathbf{A} \vDash \boldsymbol{\tau}(a) \iff a \in F.$$

Hence we conclude that $\boldsymbol{\tau}(x)$ is a set of defining equations for $\vdash^l$ and, therefore, $\vdash^l$ is truth-equational. ⊠

In [13, Theorem 48] it is proved that the variety of involutive bisemilattices, i.e. the closure under Płonka sums of the variety of Boolean algebras [71], is not the equivalent algebraic semantics of *any* algebraizable logic. This result can be strengthened as follows:

Theorem 2.7.2. *Let* K *be a class of algebras containing trivial algebras and closed under Płonka sums. There is no protoalgebraic logic* $\vdash$ *such that* $\mathsf{Alg}(\vdash) = \mathsf{K}$.

Proof. Suppose, in view of a contradiction, that there are a class of algebras K containing trivial algebras and closed under Płonka sums, and a protoalgebraic logic $\vdash$ such that $\mathsf{Alg}(\vdash) = \mathsf{K}$. Observe that K contains algebras of arbitrarily large cardinality, since it contains all Płonka sums of arbitrarily large direct systems of *trivial* algebras. In particular, this implies that K contains non trivial algebras. Since $\mathsf{Alg}(\vdash) = \mathsf{K}$, the same holds for $\mathsf{Alg}(\vdash)$. It is not difficult to see that this implies $x \nvdash y$.

Since $\vdash$ is protoalgebraic, there is a set of formulas $\Delta(x, y)$ such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Since $x \nvdash y$ and $x, \Delta(x, y) \vdash y$, we conclude that $\Delta(x, y) \neq \emptyset$. Then consider $\varphi(x, y) \in \Delta(x, y)$. Since $\emptyset \vdash \Delta(x, x)$, we conclude that $\emptyset \vdash \varphi(x, x)$.

Then consider the direct system given by two trivial algebras $\mathbf{1}_a$ and $\mathbf{1}_b$ with a homomorphism $f_{ab} \colon \mathbf{1}_a \to \mathbf{1}_b$. Let $\mathbf{A}$ be the Płonka sum of this direct system. Clearly $\mathbf{A} \in \mathsf{K}$, since K contains trivial algebras and is closed under Płonka sums. In particular, this implies that $\mathbf{A} \in \mathsf{K} = \mathsf{Alg}(\vdash)$. Therefore there is $F \subseteq A$ such that $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$.

Now, observe that the variable x really occurs in $\varphi(x, x)$, since we do not allow the presence of constant symbols in this chapter. Hence we obtain that

$$\varphi^{\mathbf{A}}(1_a, 1_a) = 1_a \text{ and } \varphi^{\mathbf{A}}(1_b, 1_b) = 1_b.$$

Together with the fact that $\emptyset \vdash \varphi(x, x)$, this implies that A is the smallest $\vdash$-filter on $\mathbf{A}$. In particular, this implies that A is the *unique* $\vdash$-filter on $\mathbf{A}$, Since F is a $\vdash$-filter on $\mathbf{A}$, we conclude that $A = F$. Hence $\langle \mathbf{A}, A \rangle$ is a Suszko reduced model of $\vdash$. This implies that $\mathbf{A}$ is trivial, which is false. ⊠

Chapter 3

Right-variable Inclusion Logics

3.1 Introduction

In the previous Chapter 2 we saw that every logic $\vdash$ admits a sublogic $\vdash^l$. Now, we turn our attention to another sublogic of $\vdash$, which we denote by $\vdash^r$ and whose syntactic characterization is

$$\Gamma \vdash^r \varphi \iff \begin{cases} \Gamma \vdash \varphi \text{ and } Var(\varphi) \subseteq Var(\Gamma) & \text{or} \\ \Sigma \subseteq \Gamma, \end{cases}$$

where Σ is an antitheorem of $\vdash$ (see Definition 1.2.8).

We will refer to the two companions $\vdash^l$ and $\vdash^r$ of the logic $\vdash$ as *left variable inclusion logic* and *right variable inclusion logic*, respectively. The logic $\vdash^r$ is often called *containment logic*.

As in the case of $\vdash^l$, examples of right variable inclusion logics are well known when $\vdash$ is Classical Logic. In such a case, the role is played by a logic in the family of Kleene three-valued logics [56], known as *Bochvar logic* B_3 [11].

A common feature shared by B_3 and PWK is the peculiar infectious behaviour of the non-classic truth-value. This aspect motivates a wide philosophical literature whose focus is the comparison between PWK, B_3 and other logics of variable inclusion with respect to paradoxes and other topics related with theories of truth.

It is worth remarking that the technical work concerning containment logics is less developed than what happens for left variable inclusion logics. Indeed, as noticed in Chapter 2, [13] contains a detailed algebraic investigation of PWK, while no similar work has been produced neither for its cousin B_3, nor for any other containment logic. The present chapter, following the aim of the previous one, faces this task.

The chapter is divided as follows.

In Section 3.2, logics of right variable inclusion are formally introduced. Moreover, by providing the correct notion of Płonka sum of logical matrices, we obtain soundness and completeness for arbitrary, finitary, logics $\vdash^r$ with respect to Płonka sums of matrix models of $\vdash$.

In Section 3.3, we focus on a specific class of logics, namely those possessing a binary term called r-partition function (see Definition 3.3.1). We provide a method for obtaining a Hilbert style axiomatization for a finitary logic $\vdash^r$ (Theorem 3.3.9) out of one axiomatization for the logic $\vdash$. It is worthwhile mentioning that almost all examples of containment logics, including B_3, belong to this class and that the obtained calculi are free of syntactic restrictions on rules.

Sections 3.4 and 3.5 study the structure of the Leibniz and Suszko reduced models of $\vdash^r$ (Theorems 3.4.1 and 3.5.4). It turns out that the property of a model to be (Leibniz or Suszko) reduced is actually rendered by some conditions on the semilattice structure of the system of the matrix models involved in the construction of the Płonka sums. In case, $\vdash$ is truth-equational, the description of the Suszko reduced models can be considerably refined (see Theorem 3.5.7). The chapter is closed by a the brief Section 3.6, where containment logics are classified into the Leibniz hierarchy: we show that $\vdash^r$ is neither truth-equational, nor protoalgebraic.

3.2 Logics of right variable inclusion

Logics of *right variable inclusion*, more often called *containment logics*, see for e.g. [39, 69], are defined according to the following:

Definition 3.2.1. Let $\vdash$ be a logic. $\vdash^r$ is the logic defined as

$$\Gamma \vdash^r \varphi \iff \begin{cases} \Gamma \vdash \varphi \text{ and } Var(\varphi) \subseteq Var(\Gamma) & \text{or} \\ \Sigma(x) \subseteq \Gamma, \end{cases}$$

where $\Sigma(x)$ is an antitheorem of $\vdash$.

Example 3.2.2. The most famous example of right variable inclusion logic is Bochvar logic [11]. This can also, equivalently, be defined by the so-called weak Kleene tables[1] (displayed below) with $\{1\}$ as the unique designated value.

[1]It is worthwhile noticing that the logic PWK is induced by the same algebra with $\{1, n\}$ as filter (see [13] for details).

$\wedge$	0	n	1
0	0	n	0
n	n	n	n
1	0	n	1

$\vee$	0	n	1
0	0	n	1
n	n	n	n
1	1	n	1

$\neg$	
1	0
n	n
0	1

As already noticed in Example 2.2.4, it is not difficult to check that the algebra $\mathbf{WK} = \langle\{0, 1, n\}, \wedge, \vee, \neg\rangle$ is the Płonka sum of the two-element Boolean algebra and the trivial (Boolean) algebra $\mathbf{n}$ (the index set is the two element semilattice).

The fact that the logic induced by the above algebra and the filter $\{1\}$ is the right variable inclusion companion of (propositional) Classical Logic has been stated in [97] (it is also a consequence of Theorem 3.2.10).

$\boxtimes$

Example 3.2.3. The logic $\mathsf{K}^{\mathsf{w}}_{4\mathsf{n}}$, one among the four-valued regular logics counted by Tomova (see [94, 72]), is another example of containment logic. In particular, as a consequence of our analysis (see Remark 3.2.11), $\mathsf{K}^{\mathsf{w}}_{4\mathsf{n}}$ is the containment companion of PWK. It is defined as the logic induced by the matrix given by the algebra displayed in the following table and the filter $\{1, b\}$.

$\wedge$	0	b	n	1
0	0	b	n	0
b	b	b	n	b
n	n	n	n	n
1	0	b	n	1

$\vee$	0	b	n	1
0	0	b	n	1
b	b	b	n	b
n	n	n	n	n
1	1	b	n	1

$\neg$	
1	0
b	b
n	n
0	1

$\boxtimes$

We generalise the definition of direct system of algebras (see Definition 1.3.1) to logical matrices as follows

Definition 3.2.4. A *r-direct system* of matrices consists in

(i) A semilattice $I = \langle I, \vee\rangle$.

(ii) A family of matrices $\{\langle \mathbf{A}_i, F_i\rangle : i \in I\}$ such that $I^+ := \{i \in I : F_i \neq \emptyset\}$ is a sub-semilattice of I.

(iii) a homomorphism $f_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$, for every $i, j \in I$ such that $i \leq j$, satisfying also that:

- f_{ii} is the identity map, for every $i \in I$;
- if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$;
- if $F_j \neq \emptyset$ then $f_{ij}^{-1}[F_j] = F_i$, for any $i \leq j$.

Remark 3.2.5. The notion of direct system of matrices in the above definition is essentially different, as highlighted by the nomenclature, from the one introduced in [15]. The differences mainly concern the interplay between homomorphisms of the system and matrices' filters.

Given an r-directed system of matrices X, we define a new matrix as

$$\mathcal{P}_{\nmid}(X) \coloneqq \langle \mathcal{P}_{\nmid}(\mathbf{A}_i)_{i \in I}, \bigcup_{i \in I} F_i \rangle.$$

We will refer to the matrix $\mathcal{P}_{\nmid}(X)$ as the *Płonka sum* over the r-direct system of matrices X. Given a class M of matrices, in this chapter $\mathcal{P}_{\nmid}(\mathsf{M})$ will denote the class of all Płonka sums of r-directed systems of matrices in M.

Remark 3.2.6. Recall that, in general, the index $i_h(\Gamma)$ is defined provided that the set $Var(\Gamma)$ is finite. In order to assure the existence of $i_h(\Gamma)$, we assume, throughout the whole chapter, that the logic $\vdash^r$ is finitary. Moreover, observe that, for every homomorphism $h \colon \mathbf{Fm} \to \mathcal{P}_{\nmid}(X)$ from the formula algebra into a generic Płonka sum over an r-direct system of matrices X, and every $\Gamma \cup \{\varphi\} \subseteq Fm$, it is immediate to check that $Var(\varphi) \subseteq Var(\Gamma)$ implies $i_h(\varphi) \leq i_h(\Gamma)$.

We remind that a matrix of the form $\langle \mathbf{A}, A \rangle$ is called *trivial*. A set of models of a logic $\vdash$ is said to be non trivial, if it does not contain trivial matrices. We indicate by $\mathsf{Mod}_{+}(\vdash)$ the set of non trivial models of a logic $\vdash$. Moreover, we denote by $\mathbf{1}$ the one-element algebra.

Lemma 3.2.7. *Let X be an r-direct system of non trivial models of a logic $\vdash$. Then $\mathcal{P}_{\nmid}(X)$ is a model of $\vdash^r$.*

Proof. Let X be an r-direct system of non trivial models of $\vdash$. Assume $\Gamma \vdash^r \varphi$. Since $\vdash^r$ is finitary (see Remark 3.2.6), there exists a finite subset $\Delta \subseteq \Gamma$, such that $\Delta \vdash^r \varphi$. We distinguish the following cases:

(a) $\Sigma(x) \subseteq \Delta$, where $\Sigma(x)$ is an antitheorem of $\vdash$;

(b) $\Delta \vdash \varphi$ with $Var(\varphi) \subseteq Var(\Delta)$.

Suppose (a) is the case. Then $\Sigma(x) \vdash \psi$, for any $\psi \in Fm$. Let $\langle \mathbf{A}_i, F_i \rangle \in X$. Preliminarily, observe that, for any homomorphism $v \colon \mathbf{Fm} \to \mathbf{A}_i$, we have $v[\Sigma(x)] \nsubseteq F_i$ (as, otherwise we would have $v(\psi) \in F_i$, for any formula ψ, implying that $F_i = A_i$, in contradiction with the fact that $\langle \mathbf{A}_i, F_i \rangle$ is non trivial). From this fact, it easily follows that, for any homomorphism $h \colon \mathbf{Fm} \to \mathcal{P}_{\nmid}(\mathbf{A}_i)$, $h[\Sigma(x)] \nsubseteq F$.
Therefore $\Sigma(x) \vdash_{\mathcal{P}_{\nmid}(X)} \varphi$, hence also $\Delta \vdash_{\mathcal{P}_{\nmid}(X)} \varphi$.

Suppose (b) is the case, i.e. $\Delta \vdash \varphi$ with $Var(\varphi) \subseteq Var(\Delta)$. Let $h \colon \mathbf{Fm} \to \mathcal{P}_{\nmid}(\mathbf{A}_i)$ be a homomorphism such that $h[\Delta] \subseteq \bigcup_{i \in I} F_i$. Since Δ is a finite set, then we can fix $j := i_h(\Delta)$ and, for any formula $\delta \in \Delta$, we have $h(\delta) \in F_{i_h(\delta)}$. This implies that each $i_h(\delta) \in I^+$ and, as I^+ forms a sub-semilattice of I, we have that $j \in I^+$. Now, define $g \colon \mathbf{Fm} \to \mathbf{A}_j$ as

$$g(x) := f_{i_h(x)j} \circ h(x),$$

for every $x \in Var(\Delta)$. For any $\delta \in \Delta$, we have $g(\delta) = f_{i_h(\delta)j} \circ h(\delta)$, hence $g[\Delta] \subseteq F_j$. From the fact that $\Delta \vdash \varphi$ and $\langle \mathbf{A}_j, F_j \rangle \in \mathsf{Mod}(\vdash)$, it follows that $g(\varphi) \in F_j$. Setting $k := i_h(\varphi)$, by Remark 3.2.6, we have $k \leq j$ and this, together with the observation that $F_j \neq \emptyset$, implies $f_{kj}^{-1}[F_j] = F_k$. Moreover, we claim that $F_k \neq \emptyset$. Suppose, by contradiction, that $F_k = \emptyset$. Then, by definition of r-direct system of matrices, we have that $f_{kj}^{-1}[F_j] = \emptyset$, that is: there exists no $a \in A_k$ such that $f_{kj}(a) \in F_j$. On the other hand, since $Var(\varphi) \subseteq Var(\Delta)$, then $g(\varphi) = f_{kj} \circ h(\varphi) \in F_j$, a contradiction.

From the fact that $g(\varphi) \in F_j$ together with $f_{kj}^{-1}[F_j] = F_k$, we conclude $h(\varphi) \in F_k$. This proves that $h(\varphi) \in F_k \subseteq \bigcup_{i \in I} F_i$. ⊠

Remark 3.2.8. Observe that in Lemma 3.2.7, the assumption on the non triviality of models of the logic $\vdash$ is crucial, as witnessed by the following example. Let $\vdash$ be a theoremless logic possessing an antitheorem $\Sigma(x)$ (an example is the almost inconsistent logic). Set $X = \langle \mathbf{A} \oplus \mathbf{1}, A \rangle$ to be the r-direct system of models of $\vdash$, consisting of the two algebras $\mathbf{A}$ and $\mathbf{1}$ with the unique homomorphism $f \colon \mathbf{A} \to \mathbf{1}$ (plus the identity homomorphisms). Then $\Sigma(x) \vdash y$, for an arbitrary variable y, and therefore $\Sigma(x) \vdash^r y$. However, $\mathcal{P}_{\nmid}(X)$ is not a model of the latter inference (consider, for instance, an evaluation $v \colon \mathbf{Fm} \to \mathcal{P}_{\nmid}(\mathbf{A} \oplus \mathbf{1})$ such that $v(x) = a \in A$ and $v(y) = 1$).

Observe that, if the logic $\vdash$ does not possess an antitheorem, then the following holds:

Corollary 3.2.9. *Let X be an r-direct system of models of a logic $\vdash$ possessing no antitheorems. Then $\mathcal{P}_{\nmid}(X)$ is a model of $\vdash^r$.*

Given a logic $\vdash$ which is complete with respect to a class M of matrices, we set $\mathsf{M}^{\emptyset} := \mathsf{M} \cup \langle \mathbf{A}, \emptyset \rangle$, for any arbitrary $\mathbf{A} \in \mathsf{Alg}(\vdash)$.

Theorem 3.2.10. *Let $\vdash$ be a logic which is complete w.r.t. a class of non trivial matrices M. Then $\vdash^r$ is complete w.r.t. $\mathcal{P}_{\nmid}(\mathsf{M}^{\emptyset})$.*

Proof. We aim at showing that $\vdash^r = \vdash_{\mathcal{P}_{\nmid}(\mathsf{M}^{\emptyset})}$.
($\vdash^r \leq \vdash_{\mathcal{P}_{\nmid}(\mathsf{M}^{\emptyset})}$). Firstly, observe that, using the same argument applied in Lemma 3.2.7, if $\Sigma(x)$ is an antitheorem of $\vdash$, then $\mathcal{P}_{\nmid}(\mathsf{M}^{\emptyset})$ is a model of the rule $\Sigma(x) \vdash^r \varphi$, for any $\varphi \in Fm$. Moreover, if the matrix $\langle \mathbf{A}, \emptyset \rangle$ is a model of $\vdash$, then the claim follows from Lemma 3.2.7.

We are left with treating the case where $\langle \mathbf{A}, \emptyset \rangle$ is not a model of $\vdash$. Consider a Płonka sum $\langle \mathbf{A}, \bigcup_{i \in I} F_i \rangle$ of matrices in $\mathsf{M}^{\emptyset}$ and suppose that $\Gamma \vdash^r \varphi$, with $Var(\varphi) \subseteq Var(\Gamma)$. W.l.o.g. we can assume Γ to be finite (since, in virtue of Remark 3.2.6, $\vdash^r$ is finitary). Let $h\colon Fm \to \mathbf{A}$ a homomorphism such that $h[\Gamma] \subseteq \bigcup_{i \in I} F_i$. Suppose, in view of a contradiction, that $h(\varphi) \notin \bigcup_{i \in I} F_i$. Set $i_h(\varphi) = j$ and $i_h(\Gamma) = k$; since $Var(\varphi) \subseteq Var(\Gamma)$ then $j \leq k$, by Remark 3.2.6. We define a homomorphism $v\colon \mathbf{Fm} \to \mathbf{A}_k$, as follows

$$v(x) := f_{lk} \circ h(x),$$

where $l = i_h(x)$. Clearly, $v[\Gamma] = f_{kk} \circ h[\Gamma] = h[\Gamma] \subseteq F_k$ and $v(\varphi) = f_{jk} \circ h(\varphi) \in A_k \smallsetminus F_k$, since $h(\varphi) \in A_j \smallsetminus F_j$ and $F_j = f_{jk}^{-1}[F_k]$ (as we know that $F_k \neq \emptyset$). Therefore, we have $\Gamma \not\vdash \varphi$, which is a contradiction.

($\vdash_{\mathcal{P}_{\nmid}(\mathsf{M}^{\emptyset})} \leq \vdash^r$). By contraposition, we prove that $\Gamma \nvdash^r \varphi$ implies $\Gamma \nvdash_{\mathcal{P}_{\nmid}(\mathsf{M}^{\emptyset})} \varphi$. To this end, assume $\Gamma \nvdash^r \varphi$. Firstly, consider the case where $Var(\varphi) \subseteq Var(\Gamma)$. It follows that $\Gamma \nvdash \varphi$. Since M is a class of matrices complete for $\vdash$, then there exists a matrix $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{M}$ and a homomorphism $h\colon \mathbf{Fm} \to \mathbf{A}_i$ such that $h[\Gamma] \subseteq F_i$ and $h(\varphi) \notin F_i$. Upon considering the r-direct system $X = \langle \langle \mathbf{A}_i, F_i \rangle, \{i\}, id \rangle$, h is a homomorphism from $\mathbf{Fm}$ to $\mathcal{P}_{\nmid}(X)$ such that $h[\Gamma] \subseteq \bigcup_{i \in I} F_i$ and $h(\varphi) \notin \bigcup_{i \in I} F_i$, proving that $\Gamma \nvdash_{\mathcal{P}_{\nmid}(\mathsf{M}^{\emptyset})} \varphi$.

The only other case to consider is $Var(\varphi) \nsubseteq Var(\Gamma)$. Preliminarily, observe that the assumption $\Gamma \nvdash^r \varphi$ implies that Γ contains no antitheorem $\Sigma(x)$ for $\vdash$. Therefore, since M is a class of models complete with respect to $\vdash$, there exist a matrix $\langle \mathbf{B}, G\rangle \in \mathsf{M}$ and a homomorphism $v\colon \mathbf{Fm} \to \mathbf{B}$ such that $v[\Gamma] \subseteq G$ and $v(\varphi) \notin G$. Consider the r-direct system formed by the matrices $\langle \mathbf{B}, G\rangle$ and $\langle \mathbf{A}, \emptyset\rangle$ for an appropriate $\mathbf{A} \in \mathsf{Alg}(\vdash)$ (observe that the choice $\mathbf{A} = \mathbf{1}$ is always appropriate), indexed over the two element chain $\{1,2\}$ with f_{12} any homomorphism from $\mathbf{B}$ to $\mathbf{A}$ (plus the identity homomorphisms f_{11} and f_{22}). Denote by $\mathbf{B} \oplus \mathbf{A}^{\emptyset}$ the Płonka sum over the r-direct system just described.

For an arbitrary $a \in A$, we define the homomorphism $g\colon \mathbf{Fm} \to \mathbf{B} \oplus \mathbf{A}^{\emptyset}$ as follows

$$g(x) := \begin{cases} v(x) \text{ if } x \in Var(\Gamma), \\ a \text{ otherwise.} \end{cases}$$

Clearly, $g[\Gamma] = v[\Gamma] \subseteq G$. On the other hand, since $Var(\varphi) \nsubseteq Var(\Gamma)$, there exists $y \in Var(\varphi)$ such that $y \notin Var(\Gamma)$. Therefore $g(y) = a$ and, by the construction of $\mathbf{B} \oplus \mathbf{A}^{\emptyset}$, we have $g(\varphi) \in A$ and $A \cap G = \emptyset$. This shows that $\Gamma \nvdash_{\mathcal{P}_{\natural}(\mathsf{M}^{\emptyset})} \varphi$, as desired. ⊠

Remark 3.2.11. As a consequence of Theorem 3.2.10, we have that $\mathsf{K}^{\mathsf{w}}_{4\mathsf{n}} = \vdash^{r}_{\mathsf{PWK}}$, i.e. the logic $\mathsf{K}^{\mathsf{w}}_{4\mathsf{n}}$, introduced in Example 3.2.3, is the right variable inclusion companion of PWK (for a general discussion, see Chapter 4). This follows by simply observing that PWK is complete with respect to the matrix $\langle \mathbf{WK}, \{1, n\}\rangle$ and the matrix defining $\mathsf{K}^{\mathsf{w}}_{4\mathsf{n}}$ is the Płonka sum of $\langle \mathbf{WK}, \{1, n\}\rangle$ (n is simply replaced by b) and the matrix $\langle \mathbf{n}, \emptyset\rangle$.

Remark 3.2.12. Observe that that Theorem 2.2.9 can be adapted to the case of $\vdash^{r}$. Indeed, the proof of Theorem 5.2.12 shows that, given a complete class of matrices M for $\vdash$, we the classs $\{\langle \mathbf{A} \oplus \mathbf{1}, F\rangle : \langle \mathbf{A}, F\rangle \in \mathsf{M}\}$ is a complete matrix semantics for $\vdash^{r}$.

Theorem 3.2.10 provides a complete class of matrices for an arbitrary logic of right variable inclusion. This class is obtained performing Płonka sums over r-direct systems of models of $\vdash$ together with the matrices $\langle \mathbf{A}, \emptyset\rangle$ for any $\mathbf{A} \in \mathsf{Alg}(\vdash)$. Obviously, it is not generally the case that the matrix $\langle \mathbf{A}, \emptyset\rangle$ is a model of a logic $\vdash$. For this reason, it is not always true that Płonka sums over a r-direct systems of models of $\vdash$ provide a complete matrix semantics for $\vdash^{r}$. In this sense, the right variable inclusion companion of a logic is a logic of "Płonka sums" (of matrices) in weaker sense compared to the case of the left variable inclusion companion, fully described in [15]. Nonetheless, if $\langle \mathbf{1}, \emptyset\rangle \in \mathsf{Mod}(\vdash)$, the correspondence between $\vdash^{r}$ and Płonka sums is fully recovered. This is actually the case of every theoremless logic, such as Strong Kleene Logic, $\vdash^{\wedge,\vee}_{\mathrm{CL}}$.

Example 3.2.13. A very simple example of right companion $\vdash^r$ which is not complete w.r.t. a complete class of models of $\vdash$ is Bochvar logic B_3. It is indeed easy to check that all the matrices in the class $\mathcal{P}_{\nmid}(\mathsf{Mod}^{*}(\mathrm{CL}))$ are models of the inference $x \vdash x \vee y$, while it is the case that $x \nvdash_{\mathsf{B}_3} x \vee y$.

Corollary 3.2.14. *A containment logic $\vdash^r$ is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\nmid}(\mathsf{Mod}_{+}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),\ \mathcal{P}_{\nmid}(\mathsf{Mod}^{*}_{+}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),\ \mathcal{P}_{\nmid}(\mathsf{Mod}^{\mathsf{Su}}_{+}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),$$

for $\mathbf{A} \in \mathsf{Alg}(\vdash)$.

Moreover, observing that if $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}(\vdash)$ then $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}^{*}(\vdash)$, the following hold

Corollary 3.2.15. *Let $\vdash$ a logic such that $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}(\vdash)$, then a finitary logic $\vdash^r$ is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\nmid}(\mathsf{Mod}_{+}(\vdash)),\ \mathcal{P}_{\nmid}(\mathsf{Mod}^{*}_{+}(\vdash)),\ \mathcal{P}_{\nmid}(\mathsf{Mod}^{\mathsf{Su}}_{+}(\vdash)).$$

In case $\vdash$ does not possess antitheorems, then the above corollaries can be restated as follows

Corollary 3.2.16. *Let $\vdash$ a logic without antitheorems. Then $\vdash^r$ is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\nmid}(\mathsf{Mod}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),\ \mathcal{P}_{\nmid}(\mathsf{Mod}^{*}(\vdash) \langle \mathbf{A}, \emptyset \rangle),\ \mathcal{P}_{\nmid}(\mathsf{Mod}^{\mathsf{Su}}(\vdash \langle \mathbf{A}, \emptyset \rangle),$$

for any $\mathbf{A} \in \mathsf{Alg}(\vdash)$.

Corollary 3.2.17. *Let $\vdash$ a logic without antitheorems such that $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}(\vdash)$, then $\vdash^r$ is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\nmid}(\mathsf{Mod}(\vdash)),\ \mathcal{P}_{\nmid}(\mathsf{Mod}^{*}(\vdash)),\ \mathcal{P}_{\nmid}(\mathsf{Mod}^{\mathsf{Su}}(\vdash)).$$

3.3 Hilbert style calculi for right variable inclusion logics with an r-partition function

Definition 3.3.1. A logic $\vdash$ has an *r-partition function* if there is a formula $x * y$, in which the variables x and y really occur, such that

(i) $x, y \vdash x * y$,

(ii) $x * y \vdash x$,

and the term operation $*$ is a partition function in every $\mathbf{A} \in \mathsf{Alg}(\vdash)$.

Remark 3.3.2. By Lemma 1.2.6, the above Definition can be rephrased in purely logical terms, by requiring that $x, y \vdash x * y$, $x * y \vdash x$ and that

$$\varphi(\varepsilon, \vec{z}) \dashv\vdash \varphi(\delta, \vec{z}) \text{ for every formula } \varphi(v, \vec{z}),$$

for every identity of the form $\varepsilon \approx \delta$ in Definition 1.3.3. ⊠

From now on, we will use both the formula $x * y$ and the term operation $*$ to denote an r-partition function with respect to a logic $\vdash$.

Notice that the above definition is essentially different from the definition of logic with a l-partition function introduced in the previous Chapter 2. However, in most cases (for instance, all substructural logics, classical and modal logics) the very same formula plays both the role of a r-partition function and of a l-partition function.

Example 3.3.3. Logics with an r-partition function abound in the literature. Indeed, the term $x * y := x \wedge (x \vee y)$ is a partition function for every logic $\vdash$ such that $\mathsf{Alg}(\vdash)$ has a lattice reduct. Such examples include all modal and substructural logics [48]. On the other hand, the term $x * y := (y \to y) \to x$ plays the role of an r-partition function for all the logics $\vdash$ whose class $\mathsf{Alg}(\vdash)$ possesses a Hilbert algebra (see [33]) or a BCK algebra (see [53]) reduct.

Remark 3.3.4. It is easily checked that a logic $\vdash$ has r-partition function $*$ if and only if $\vdash^r$ has an r-partition function $*$.

In the following, we extend Płonka representation theorem to r-direct systems of logical matrices.

Theorem 3.3.5. *Let $\vdash$ be a logic with r-partition function $*$, and $\langle \mathbf{A}, F\rangle$ be a model of $\vdash$ such that $\mathbf{A} \in \mathsf{Alg}(\vdash)$. Then Theorem 1.3.4 holds for $\mathbf{A}$. Moreover, by setting $F_i := F \cap A_i$ for every $i \in I$, the triple*

$$X = \langle\langle I, \leq\rangle, \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} \colon i \leq j\}\rangle$$

is an r-direct system of matrices such that $\mathcal{P}_{\nmid}(X) = \langle \mathbf{A}, F\rangle$.

Proof. Theorem 1.3.4 holds for $\mathbf{A}$, by simply observing that $*$ is a partition function for $\mathbf{A}$.
For the remaining part, it will be enough to show:

(a) for every $i, j \in I$ such that $i \leq j$, if $F_j \neq \emptyset$ then $f_{ij}^{-1}[F_j] = F_i$;

(b) I^+ is a sub-semilattice of I.

In order to prove (a), consider $i, j \in I$ such that $i \leq j$ and let F_j be non-empty. Assume, in view of a contradiction, that $f_{ij}^{-1}[F_j] \neq F_i$. This implies that $F_i \nsubseteq f_{ij}^{-1}[F_j]$ or that $f_{ij}^{-1}[F_j] \nsubseteq F_i$. In the first case, let $a \in F_i$ such that $f_{ij}(a) = c \in A_j \smallsetminus F_j$. As $F_j \neq \emptyset$, then there exists an element $b \in F_j$. Since $*$ is an r-partition function for $\vdash$, then $x, y \vdash x * y$ holds. However, we have that $a, b \in F$ while $a *^{\mathbf{A}} b = f_{ij}(a) *^{\mathbf{A}_j} b = c *^{\mathbf{A}_j} b = c \notin F$. A contradiction. In the second case, let $a \in A_i \smallsetminus F_i$ be such that $f_{ij}(a) \in F_j$. Fix $f_{ij}(a) = c$. Again, as $*$ is an r-partition function for $\vdash$ it holds $x * y \vdash x$. This, however, is in contradiction with the fact that $a *^{\mathbf{A}} c = f_{ij}(a) *^{\mathbf{A}_j} c = c *^{\mathbf{A}_j} c = c \in F$ while $a \notin F$. This proves (a).

In order to prove (b), consider $i, j \in I^+$ and let $k = i \vee j$, with $i, j, k \in I$. As $*$ is an r-partition function for $\vdash$, $x, y \vdash x * y$. Since $i, j \in I^+$, then F_i and F_j are non-empty, therefore there exist two elements $a \in F_i$, $b \in F_j$. We have $a *^{\mathbf{A}} b = f_{ik}(a) *^{\mathbf{A}_k} f_{jk}(b) \in A_k$. This, together with the fact that $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash)$ implies $a * b \in F_k$, i.e. $F_k \neq \emptyset$. So $k \in I^+$ and this proves (b). ⊠

Given a logic $\vdash$ with a r-partition function $*$ and a model $\langle \mathbf{A}, F\rangle$ of $\vdash$ such that $\mathbf{A} \in \mathsf{Alg}(\vdash)$, we call *Płonka fibers* of $\langle \mathbf{A}, F\rangle$ the matrices $\{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}$ given by the decomposition in Theorem 3.3.5, in the same vein of Definiton 2.3.6 in the previous chapter. From now on, when considering a model $\langle \mathbf{A}, F\rangle$ of a logic $\vdash$ with r-partition function, we will assume that $\langle \mathbf{A}, F\rangle = \mathcal{P}_{\text{ł}}(X)$, for a given r-direct system $X = \langle \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}, \langle I, \leq\rangle\rangle$, without explicitly mentioning the r-direct system X.

Lemma 3.3.6. *Let $\vdash^r$ be a logic with r-partition function $*$, and $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash^r)$, with $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$. Then, the Płonka fibers $\langle \mathbf{A}_i, F_i\rangle$, such that $i \in I^+$, are models of $\vdash$.*

Proof. Let $\Gamma \vdash \varphi$ and suppose, by contradiction, that there exist a matrix $\langle \mathbf{A}_j, F_j\rangle$, with $j \in I^+$, and a homomorphism $h\colon \mathbf{Fm} \to \mathbf{A}_j$ such that $h[\Gamma] \subseteq F_j$ and $h(\varphi) \notin F_j$. Preliminarily, observe that $Var(\varphi) \nsubseteq Var(\Gamma)$ and, moreover, if $\vdash$ has an antitheorem $\Sigma(x)$, then $\Sigma(x) \nsubseteq \Gamma$, for otherwise $\Gamma \vdash^r \varphi$, which is in contradiction with our assumption that $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash^r)$. Denote by X the (non-empty) set of variables occurring in φ but not in Γ and, for $\gamma \in \Gamma$, let $X_\gamma := \{\gamma * x : x \in X\}$ and $\Gamma^-_\gamma := \Gamma \smallsetminus \{\gamma\}$. Since $*$ is a r-partition function for $\vdash^r$, we have $\gamma * x \vdash^r \gamma$. Therefore $\gamma * x \vdash \gamma$

and $X_\gamma \vdash \gamma$, which implies $X_\gamma, \Gamma^-_\gamma \vdash \varphi$, for any $\gamma \in \Gamma$. Observe that $Var(\varphi) \subseteq Var(X_\gamma) \cup Var(\Gamma^-)$, hence $X_\gamma, \Gamma^-_\gamma \vdash^r \varphi$.

Since $h(\gamma), h(\varphi) \in A_j$ and $x \in Var(\varphi)$, for every $x \in X$, we have that $h(\gamma * x) = h(\gamma)$, whence $h[X_\gamma] = h(\gamma)$. Now, for an arbitrary $a \in A$, we define a homomorphism $g \colon \mathbf{Fm} \to \mathbf{A}$, as follows

$$g(x) := \begin{cases} h(x) \text{ if } \ x \in Var(\Gamma) \cup Var(\varphi) \\ a \text{ otherwise.} \end{cases}$$

We have $g[X_\gamma] = h[X_\gamma] = h(\gamma) \in F_j$, $g[\Gamma^-_\gamma] = h[\Gamma^-_\gamma] \in F_j$ and $g(\varphi) = h(\varphi) \notin F_j$. A contradiction. ⊠

In this section we show how to provide a sound and complete Hilbert style calculus for a logic of right variable inclusion possessing an r-partition function. Interestingly enough, the calculi we present do not present syntactic limitations on their rules.

Throughout this section, we implicitly assume that the logic $\vdash$ possesses an antitheorem. Our analysis can be easily adapted to the case where $\vdash$ does not have antitheorems (see Remark 3.3.10).

Recall from the previous chapter that by a *Hilbert-style calculus with finite rules* we understand a (possibly infinite) set of Hilbert-style rules, each of which has finitely many premises.

Definition 3.3.7. Let $\mathcal{H}$ be a Hilbert-style calculus with finite rules, which determines a logic $\vdash$ with an r-partition function $*$ and an antitheorem $\Sigma(x)$. Let $\mathcal{H}^r$ be the Hilbert-style calculus given by the following rules:

$$x * \varphi \rhd \varphi \tag{P0}$$
$$x, y \rhd x * y \tag{P1}$$
$$x * y \rhd x \tag{P2}$$
$$\{\gamma_1, \dots, \gamma_n\} \smallsetminus \{\gamma_i\}, \gamma_i * \varphi \rhd \varphi \tag{P3}$$
$$\Sigma(x) \rhd \varphi \tag{P4}$$
$$\chi(\delta, \vec{z}) \lhd\rhd \chi(\varepsilon, \vec{z}) \tag{P5}$$

for every

(i) $\rhd\varphi$ axiom in $\mathcal{H}$

(ii) $\gamma_1, \dots, \gamma_n \rhd \varphi$ rule in $\mathcal{H}$ with γ_i such that $i \in \{i, \dots, n\}$;

(iii) $\varepsilon \approx \delta$ equation in the definition of partition function, and formula $\chi(v, \vec{z})$.

Lemma 3.3.8. *Let $\vdash$ be a logic with an r-partition function $*$, an antitheorem $\Sigma(x)$ and let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash_{\mathcal{H}^r})$. Then:*

(i) $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_{\nmid}(X)$, *where* $X = \langle \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}, \langle I, \leq\rangle\rangle$ *is an r-direct system of matrices;*

(ii) *if X contains a trivial matrix then $\mathbf{A} = \mathbf{1}$.*

Proof. (i) Since $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash_{\mathcal{H}^r})$, $\mathbf{A} \in \mathsf{Alg}(\vdash_{\mathcal{H}^r})$. Moreover, observe that $*$ is an r-partition function for $\vdash_{\mathcal{H}^r}$ (thanks to conditions (P1), (P2), (P5)). These facts, together with Theorem 3.3.5, implies that $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_{\nmid}(X)$, where $X = \langle \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}, \langle I, \leq\rangle\rangle$ is an r-direct system of matrices.
(ii) Suppose that, for some $j \in I$, $\langle \mathbf{A}_j, F_j\rangle$ is a trivial fiber of $\langle \mathbf{A}, F\rangle$, i.e. $F_j = A_j$. Since $\Sigma(x)$ is an antitheorem (for $\vdash$) and (P4) is a rule of $\mathcal{H}^r$, then, for every $i \in I$, we have $A_i = F_i$, i.e. each fiber is trivial. Indeed, if there exists a non trivial fiber $\langle \mathbf{A}_k, F_k\rangle$ and an element $c \in A_k \smallsetminus F_k$, then the evaluation $h\colon \mathbf{Fm} \to \mathbf{A}$, defined as $h(x) = a$, $h(y) = c$ (for an arbitrary $a \in \mathbf{A}_j$) is such that $h[\Sigma(x)] \subseteq F$ while $h(y) \notin F$, against the fact that $\Sigma(x) \vdash_{\mathcal{H}^r} y$. Moreover, the fact that each fiber is trivial, together with $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}} F = id$ immediatly implies $\mathbf{A} = \mathbf{1}$. ⊠

Theorem 3.3.9. *Let $\vdash$ be a logic with an r-partition function $*$ and an antitheorem $\Sigma(x)$. Moreover, let $\mathcal{H}$ be a Hilbert style calculus with finite rules. If $\mathcal{H}$ is complete for $\vdash$, then $\mathcal{H}^r$ is complete for $\vdash^r$.*

Proof. Let us denote with $\vdash_{\mathcal{H}^r}$ the logic defined by $\mathcal{H}^r$. We show that $\vdash_{\mathcal{H}^r} = \vdash^r$.

($\leq$). In order to verify the desired inequality, it is enough to prove that every rule of $\mathcal{H}^r$ holds in $\vdash^r$. This is immediate for (P0), (P1), (P2), (P4) and (P5), as, by Remark 3.3.4, $*$ is the r-partition function for $\vdash^r$. By condition (ii) in Definition 3.3.1, it holds $\gamma_1 * \varphi \vdash \gamma_1$, hence (by monotonicity) $\gamma_1 * \varphi, \gamma_2, \ldots, \gamma_n \vdash \gamma_1$. Moreover, as $\mathcal{H}$ is complete for $\vdash$ and $\gamma_1, \ldots, \gamma_n \rhd \varphi$ is a rule in $\mathcal{H}$, we have that $\gamma_1, \ldots, \gamma_n \vdash \varphi$ and therefore, by transitivity, we obtain $\gamma_1 * \varphi, \gamma_2, \ldots, \gamma_n \vdash \varphi$. Since $Var(\varphi) \subseteq Var(\gamma_1 * \varphi, \gamma_2, \ldots, \gamma_n)$ we conclude $\gamma_1 * \varphi, \gamma_2, \ldots, \gamma_n \vdash^r \varphi$, and this proves that (P3) holds in $\vdash^r$.

($\geqslant$). For the desired inequality, it is enough to show that $\mathsf{Mod}^{\mathsf{Su}}(\vdash_{\mathcal{H}^r}) \subseteq \mathsf{Mod}(\vdash^r)$. So let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash_{\mathcal{H}^r})$. By Lemma 3.3.8-(i), we know that $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_{\nmid}(X)$, where $X = \langle \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}, \langle I, \leq\rangle\rangle$ is an r-direct system of matrices.

In order to show that $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}(\vdash_{\mathcal{H}})$, for each $i \in I^+$ we adapt the proof strategy of Lemma 3.3.6 to the calculus $\mathcal{H}^r$ as follows. Suppose

$\Gamma \rhd \varphi$ is a rule of $\mathcal{H}$, and assume towards a contradiction that for $\langle \mathbf{A}_i, F_i \rangle$ ($i \in I^+$) there exists $h\colon \mathbf{Fm} \to \mathbf{A}_i$ such that $h[\Gamma] \subseteq F_i$, while $h(\varphi) \in A_i \smallsetminus F_i$. We distinguish the cases where (a): $\Gamma = \emptyset$, (b): $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. In the case of (a) then by condition (P0), $x * \varphi \rhd \varphi$ holds in $\mathcal{H}^r$. So consider $v\colon \mathbf{Fm} \to \mathbf{A}$ defined as $v(x) = a \in F_i$ (w.l.o.g. we choose $x \notin Var(\varphi)$) and $v(y) = h(y)$, for every $y \in Var(\varphi)$. As

$$v(x * \varphi) = v(x) * v(\varphi) = v(x) * h(\varphi) = a \in F_i$$

and $v(\varphi) = h(\varphi) \in A_i \smallsetminus F_i$, we obtain that v falsifies a rule of $\mathcal{H}^r$, which is a contradiction.
The strategy for proving the remaining case (b) can be carried out in a very similar way by using condition (P3).
Therefore, recalling that $\mathcal{H}$ is complete for $\vdash$ we have proved that $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}(\vdash)$, for each $i \in I^+$. By Lemma 3.3.8-(ii), we know that if X contains a trivial matrix $\langle \mathbf{A}_j, F_j \rangle$, then $\mathbf{A} = \mathbf{1}$.

So, two cases may arise: (1) $\mathbf{A} = \mathbf{1}$, (2) X contains no trivial fibers. If (1) then clearly $\langle \mathbf{A}, F \rangle \in \{\langle \mathbf{1}, \emptyset \rangle, \langle \mathbf{1}, \{1\} \rangle\}$. As $\vdash^r$ is a theoremless logic $\{\langle \mathbf{1}, \emptyset \rangle, \langle \mathbf{1}, 1\} \subseteq \mathsf{Mod}(\vdash^r)$. If (2), then we can apply Lemma 3.2.7, so $\langle \mathbf{A}, F \rangle = \mathcal{P}_{\text{I}}(X) \in \mathsf{Mod}(\vdash^r)$. ⊠

Remark 3.3.10. It is easy to check that, if the logic $\vdash$ does not possess antitheorems, then a Hilbert-style calculus for $\vdash^r$ can be defined by simply dropping condition (P4) from Definition 3.3.7. The completeness of $\vdash^r$ with respect to such calculus can be proven by adapting the strategy in the proof of Theorem 3.3.9.

Example 3.3.11. Recall from Example 2.3.12 the following Hilbert style calculus for Classical Logic:

$$\emptyset \rhd \neg(\varphi \vee \varphi) \vee \varphi \tag{A1}$$
$$\emptyset \rhd (\varphi \vee \psi) \vee \neg \varphi \tag{A2}$$
$$\emptyset \rhd \neg(\varphi \vee \psi) \vee (\varphi \vee \psi) \tag{A3}$$
$$\emptyset \rhd \neg(\neg \psi \vee \zeta) \vee (\neg(\varphi \vee \psi) \vee (\varphi \vee \zeta)) \tag{A4}$$
$$\varphi, \neg \varphi \vee \psi \rhd \psi \tag{R1}$$

Theorem 3.3.9 allows to provide the following complete Hilbert style calculus for Bochvar logic B3. Here $\varphi * \psi$ is an abbreviation for $\varphi \wedge (\varphi \vee \psi)$.

$$x * \neg(\varphi \vee \varphi) \vee \varphi \rhd \neg(\varphi \vee \varphi) \vee \varphi \tag{A'1}$$
$$x * (\varphi \vee \psi) \vee \neg\varphi \rhd (\varphi \vee \psi) \vee \neg\varphi \tag{A'2}$$
$$x * \neg(\varphi \vee \psi) \vee (\varphi \vee \psi) \rhd \neg(\varphi \vee \psi) \tag{A'3}$$
$$x * \neg(\neg\psi \vee \zeta) \vee (\neg(\varphi \vee \psi) \vee (\varphi \vee \zeta)) \rhd \neg(\neg\psi \vee \zeta) \vee (\neg(\varphi \vee \psi) \vee (\varphi \vee \zeta)) \tag{A'4}$$
$$\varphi * \psi, \neg\varphi \vee \psi \rhd \psi \tag{R'1}$$
$$x, \neg x \rhd \varphi \tag{R'2}$$
$$x, y \rhd x \wedge (x \vee y) \tag{R'3}$$
$$\chi(\delta, \vec{z}) \lhd \rhd \chi(\varepsilon, \vec{z}) \tag{R'}$$

for every formula $\chi(x, \vec{z})$ and equation $\delta \approx \varepsilon$ in Definition 1.3.3.

Remark 3.3.12. The observations made in Remark 2.3.11 clearly can be adapted to the calculus $\mathcal{H}^r$. Indeed, also $\mathcal{H}^r$ is infinite.

3.4 Leibniz reduced models of a containment logic

In this section we provide a description of both the Leibniz reduced models (see Theorem 3.4.1) and the Suszko reduced models (see Theorem 3.5.4) of containment logics, possessing an r-partition function.

Theorem 3.4.1. *Let $\vdash^r$ a logic with an r-partition function $*$, $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash^r)$ with $\mathbf{A} \neq \mathbf{1}$ and $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$. TFAE:*

(i) *$\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\vdash^r)$;*

(ii) *$I^+ = \{i\}$ and, either $\mathbf{A} = \mathbf{A}_i$ or $\mathbf{A} = \mathbf{A}_i \oplus \mathbf{1}$, with $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}^*(\vdash)$.*

Proof. $(i) \Rightarrow (ii)$. Let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\vdash^r)$. Since $\mathsf{Alg}^*(\vdash^r) \subseteq \mathsf{Alg}(\vdash^r)$, then, by applying Theorem 3.3.5, the matrix $\langle \mathbf{A}, F\rangle$ is a Płonka sum over an r-direct system X of matrices.

We first prove $I^+ = \{i\}$. Suppose, in view of a contradiction, that $I^+ \neq \{i\}$. Clearly $I^+ \neq \emptyset$; differently, $\langle \mathbf{A}, F\rangle$ is a Płonka sum of matrices with empty filters, any of which cannot be Leibniz reduced as we have assumed that $\mathbf{A} \neq \mathbf{1}$.

We can consider, w.l.o.g. two elements $i, j \in I^+$ such that $i \leq j$ (this is justified by the fact that I^+ is a semilattice). Since $F_i \neq \emptyset$, let $a \in F_i$ and $f_{ij}(a) = b \in F_j$. We claim that $\langle a, b\rangle \in \mathbf{\Omega}^{\mathbf{A}} F$.

In order to show this, we use the characterization provided in Lemma 1.2.3. Let $\varphi(v, \vec{z})$ be an arbitrary unary polynomial function and assume $\varphi^{\mathbf{A}}(a, \vec{c}) \in F$, with $\vec{c} \in A_s$, for some $s \in I$. Clearly, $\varphi^{\mathbf{A}}(a, \vec{c}) \in F_k$, where $k = i \vee s$. Observe that $j, k \in I^+$, hence also $k \vee j = p \in I^+$ (as I^+ is a sub-semilattice of I). In particular:

$$\varphi^{\mathbf{A}}(b, \vec{c}) = \tag{3.1}$$
$$\varphi^{\mathbf{A}}(f_{ij}(a), \vec{c}) = \tag{3.2}$$
$$\varphi^{\mathbf{A}_p}(f_{jp}(f_{ij}(a)), f_{sp}(\vec{c})) = \tag{3.3}$$
$$\varphi^{\mathbf{A}_p}(f_{kp}(f_{ik}(a)), f_{kp}(f_{sk}(\vec{c}))) = \tag{3.4}$$
$$f_{kp}(\varphi^{\mathbf{A}_k}(f_{ik}(a), f_{sk}(\vec{c})) = \tag{3.5}$$
$$f_{kp}(\varphi^{\mathbf{A}}(a, \vec{c})) \in F_p. \tag{3.6}$$

In particular, (3.4) holds as $s \vee j = p$; (3.5) by observing that $f_{ip} = f_{jp} \circ f_{ij} = f_{kp} \circ f_{ik}$ and $s \leq k \leq p$; (3.6) since $\varphi^{\mathbf{A}}(a, \vec{c}) \in F_k$ implies that $f_{kp}(\varphi^{\mathbf{A}}(a, \vec{c})) \in F_p$.

Similarly, assume $\varphi(b, \vec{c}) \in F$, that is $\varphi(b, \vec{c}) \in F_p$. Suppose, towards a contradiction that $\varphi(a, \vec{c}) \notin F$, which means $\varphi^{\mathbf{A}}(a, \vec{c}) = \varphi^{\mathbf{A}_k}(f_{ik}(a), f_{sk}(\vec{c})) \notin F_k$, whence $f_{kp}(\varphi^{\mathbf{A}_k}(f_{ik}(a), f_{sk}(\vec{c}))) \notin F_p$. However,

$$\begin{aligned}
f_{kp}(\varphi^{\mathbf{A}_k}(f_{ik}(a), f_{sk}(\vec{c}))) &= \\
\varphi^{\mathbf{A}_p}(f_{kp}(f_{ik}(a)), f_{kp}(f_{sk}(\vec{c}))) &= \\
\varphi^{\mathbf{A}_p}(f_{jp}(f_{ij}(a)), f_{kp}(f_{sk}(\vec{c}))) &= \\
\varphi^{\mathbf{A}_p}(f_{jp}(b), f_{sp}(\vec{c})) &= \\
\varphi^{\mathbf{A}}(b, \vec{c}) &\in F_p.
\end{aligned}$$

This is a contradiction, so $\varphi(a, \vec{c}) \in F_k \subseteq F$. This established our claim that $\langle a, b \rangle \in \mathbf{\Omega}^{\mathbf{A}} F$. Therefore $a = b$, which implies that $i = j$, i.e. I^+ does not possess two different comparable elements. Then I^+ is a singleton.

We now prove $\mathbf{A} \in \{\mathbf{A}_i, \mathbf{A}_i \oplus \mathbf{1}\}$. Consider $I^- := I \smallsetminus I^+$ and suppose there exists $j \in I^-$ such that $\mathbf{A}_j \neq \mathbf{1}$. Firstly observe it must be $i < j$. Otherwise, it is easy to show that for $a \in A_j$ and $b = f_{jq}(a) \in A_q$ with $q = i \vee j$, $\langle a, b \rangle \in \mathbf{\Omega}^{\mathbf{A}} F$. To this end, consider a unary polynomial function $\varphi(x, \vec{z})$, and assume that $\varphi^{\mathbf{A}}(a, \vec{c}) \in F = F_i$. Observe that from this and $I^+ = \{i\}$, it follows that $j \leq i$ and also that $\vec{c} \in A_k$ with $i = j \vee k$, as otherwise $\varphi^{\mathbf{A}}(a, \vec{c}) \in A_p$ (for some $p \neq j$) and $F_p = \emptyset$. Now we have

$\varphi^{\mathbf{A}}(a,\vec{c}) = \varphi^{\mathbf{A}_i}(f_{ji}(a), f_{ki}(\vec{c})) = \varphi^{\mathbf{A}}(b,\vec{c})$. That is $\varphi^{\mathbf{A}}(a,\vec{c}) \in F$ if and only if $\varphi^{\mathbf{A}}(b,\vec{c}) \in F$. This shows that: if $j \in I^-$, then $i < j$.
We claim that $\mid I^- \mid \leq 1$. To this end, suppose by contradiction, that there exist $j,k \in I^-$. By the above argument, $i < j,k$. Since $I^+ = \{i\}$, this implies that for every $q \in I$, with $q \neq i$, $q \in I^-$. Let $a \in A_j$ and $b \in A_k$ and let, moreover, $\varphi(x,\vec{z})$ be a unary polynomial function and consider the elements $\vec{c} \in A_s$ (for some $s \in I$). Clearly $\varphi^{\mathbf{A}}(a,\vec{c}) \in A_{j\vee s}$ and $\varphi^{\mathbf{A}}(b,\vec{c}) \in A_{k\vee s}$. As $i < s \vee p, s \vee k$, we have $F_{j\vee s} = F_{k\vee s} = \emptyset$, therefore $\langle a,b\rangle \in \mathbf{\Omega}^{\mathbf{A}}F$. Therefore, $a = b$, which implies that $j = k$, a contradiction. This proves our claim.
Moreover, observe that, if $I^- = \{j\}$, i.e. I^- is a singleton, then $\mathbf{A}_j = \mathbf{1}$ (a proof of this fact is analogous to the above claim, namely if $\mathbf{A}_j$ contains two distinct elements a,b, then $\langle a,b\rangle \in \mathbf{\Omega}^{\mathbf{A}}F$). This is enough to show that either $\mathbf{A} = \mathbf{A}_i$ or $\mathbf{A} = \mathbf{A}_i \oplus \mathbf{1}$.
It only remains to show $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}^*(\vdash)$. Firstly, observe that, by Lemma 3.3.6, $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}(\vdash)$.

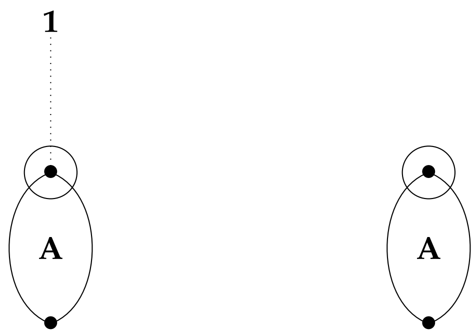
There are two cases: either $\mathbf{A} = \mathbf{A}_i$ or $\mathbf{A} = \mathbf{A}_i \oplus \mathbf{1}$. In the former, $\langle \mathbf{A}_i, F_i\rangle = \langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\vdash)$, hence there is nothing to prove. In the latter, if $\mathbf{A} = \mathbf{A}_i \oplus \mathbf{1}$, this means that for $a,b \in A_i$, with $a \neq b$, the exists a unary polynomial function $\varphi(x,\vec{z})$ such that it holds $\varphi^{\mathbf{A}}(a,\vec{c}) \in F$ if and only if $\varphi^{\mathbf{A}}(b,\vec{c}) \notin F$, for some $\vec{c} \in A$. Observe that $\vec{c} \in A_i$ is the only interesting choice, for otherwise $\varphi(a,\vec{c}), \varphi(b,\vec{c}) \in 1$, i.e. $\varphi(a,\vec{c}), \varphi(b,\vec{c}) \notin F$. Then, $\varphi^{\mathbf{A}}(a,\vec{c}) = \varphi^{\mathbf{A}_i}(a,\vec{c}) \in F_i$ and $\varphi^{\mathbf{A}}(b,\vec{c}) = \varphi^{\mathbf{A}_i}(b,\vec{c}) \notin F_i$. This shows that $\mathbf{\Omega}^{\mathbf{A}}F_i = \mathrm{id}$, hence $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}^*(\vdash)$.

$(ii) \Rightarrow (i)$. Let $\mathcal{P}_{\nmid}(X) = \langle \mathbf{A}, F\rangle$ satisfying (ii). By Lemma 3.2.7, $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash)$. Moreover, since $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}^*(\vdash)$, for any pair of elements $a,b \in A_i$, there exists a unary polynomial function $\varphi(x,\vec{z})$ such that, for some $\vec{c} \in A_i$,

$$\varphi^{\mathbf{A}}(a,\vec{c}) \in F_i \text{ if and only if } \varphi^{\mathbf{A}}(b,\vec{c}) \notin F_i.$$

In order to prove (i), we just need to show that $\langle d,1\rangle \notin \mathbf{\Omega}^{\mathbf{A}}F$, for an arbitrary $d \in A_i$. To this end, let $e \in F_i$. Clearly $e *^{\mathbf{A}} d = e \in F$, while $e *^{\mathbf{A}} 1 = 1 \notin F$. That is, the function $*$ is a unary polynomial function witnessing that $\langle d,1\rangle \notin \mathbf{\Omega}^{\mathbf{A}}F$. This concludes our proof. ⊠

Example 3.4.2. Theorem 3.4.1 allows to provide a full description of the Leibniz reduced models of Bochvar logic B_3. They may assume the two possible structures outlined in the following drawing (where $\mathbf{A}$ is a Boolean algebra, with the singleton of the top element as filter and dotted lines represent Płonka homomorphisms)



$\boxtimes$

3.5 Suszko reduced models of a containment logic

We recall a result that will be used in the description of the Suszko reduced models of $\vdash^r$.

Lemma 3.5.1. *[42, Prop. 2.24] Let $\langle \mathbf{A}, F\rangle, \langle \mathbf{B}, G\rangle$ be two matrices and let $h\colon \mathbf{A} \to \mathbf{B}$ a homomorphism from the algebra $\mathbf{A}$ into the algebra $\mathbf{B}$ such that $F = h^{-1}[G]$. If G is a $\vdash$-filter on $\mathbf{B}$ then F is a $\vdash$-filter on $\mathbf{A}$.*

In what follows, given a logic $\vdash$ and an algebra $\mathbf{A} \in \mathsf{Alg}(\vdash)$, we say that, if $\langle \mathbf{A}, G\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ then G is a *Suszko filter* of $\vdash$ over $\mathbf{A}$.

Lemma 3.5.2. *Let $\vdash^r$ be a logic with an r-partition function $*$. If $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^r)$ then $\mid I^+ \mid \leq 1$.*

Proof. Assume that $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^r)$, so by Thereom 3.3.5 $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_{\sharp}(X)$. Suppose, by contradiction, that there exists $i, j \in I^+$ with $i \neq j$. Let $k = i \vee j$. Since I^+ is a semilattice, then $k \in I^+$. Let $a \in F_i$ and $b = f_{ik}(a) \in F_k$, we claim that $\langle a, b\rangle \in \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F$, giving raise to a contradiction. To show the claim, suppose, again by contradiction, that $\langle a, b\rangle \notin \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F$. Then, there exists a $\vdash^r$-filter $G \supseteq F$, a unary polynomial function $\varphi(x, \vec{v})$ and elements $\vec{c} \in A$ such that, it holds

$$\varphi(a, \vec{c}) \in G \iff \varphi(b, \vec{c}) \notin G.$$

Observe that $a \in G_i, b \in G_k$ (as $G \supseteq F$). Suppose that $\varphi(a, \vec{c}) \in G$. W.l.o.g. consider $\vec{c} \in A_q$, hence $\varphi(a, \vec{c}) \in G_s$, with $s = i \vee q$. On the other hand,

$\varphi(b,\vec{c}) \in A_p$ with $p = k \vee q$. Clearly, as $i \leq k$, we have $s \leq p$ and so $p = s \vee k$. This, together with the fact that $s, k \in I^+$ implies that $p \in I^+$ (as, otherwise, we would have $\varphi(a,\vec{c}), b \in G$, while $\varphi(a,\vec{c}) * b \notin G$, against the fact that G is a $\vdash^r$ filter). In particular, we obtain $f_{sp}(\varphi(a,\vec{c})) \in G_p$. Now, recalling that $f_{ip} = f_{sp} \circ f_{is} = f_{ks} \circ f_{ik}$ we have

$$\begin{aligned} f_{sp}(\varphi(a,\vec{c})) &= \\ f_{sp}(\varphi(f_{is}(a), f_{qs}(\vec{c}))) &= \\ \varphi(f_{sp}(f_{is}(a)), f_{sp}(f_{qs}(\vec{c}))) &= \\ \varphi(f_{kp}(f_{ik}(a)), f_{qp}(\vec{c})) &= \\ \varphi(f_{kp}(b), f_{qp}(\vec{c})) &= \\ \varphi(b,\vec{c}) \in G_p, & \end{aligned}$$

a contradiction. ⊠

Lemma 3.5.3. *Let $\vdash^r$ be a logic possessing an r-partition function $*$ and $\mathbf{A} \cong \mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I} \in \mathsf{Alg}(\vdash^r)$. If $G_i \neq A_i$ is a non-empty $\vdash$-filter, then*

$$\langle \mathbf{A}, \bigcup_{k\leq i}(f_{ki}^{-1}(G_i))\rangle \in \mathsf{Mod}(\vdash^r),$$

Proof. Firstly observe that, by Lemma 3.5.1, $\langle \mathbf{A}_k, f_{ki}^{-1}(G_i)\rangle \in \mathsf{Mod}(\vdash)$, for each $k \leq i$. In general, there are two possibilities, (1) $\vdash$ does not have an antitheorem, (2) $\Sigma(x)$ is an antitheorem of $\vdash$.

Since $\mathbf{A} \cong \mathcal{P}_{\nmid}(\mathbf{A}_i)$, by construction it is immediate to check that $\langle \mathbf{A}, \bigcup_{k\leq i} f_{ki}^{-1}(G_i)\rangle$ is isomorphic to a Płonka sum over an r-direct system of matrices and so, by Corollary 3.2.16 $\langle \mathbf{A}, \bigcup_{k\leq i} f_{ki}^{-1}(G_i)\rangle \in \mathsf{Mod}(\vdash^r)$.

If (2) we need to verify that for each $k \leq i$ $f_{ki}^{-1}(G_i) \neq A_k$. Suppose the contrary towards a contradiction. Consider an arbitrary homomorphism $h\colon \mathbf{Fm} \to \mathbf{A}_k$. Clearly $h(\Sigma(x)) \in A_k = f_{ki}^{-1}(G_i)$ and so $f_{ki}(h(\Sigma(x))) \in G_i$. This entails $f_{ki} \circ h$ is an evaluation that maps $\Sigma(x)$ into a subset of G_i. Consider now $d \in A_i \smallsetminus G_i$ and an evaluation $v\colon \mathbf{Fm} \to \mathbf{A}_i$ such that $v(x) = f_{ki} \circ h(x)$ and $v(y) = d$ for all the variables $y \neq x$. Clearly we have $v(\Sigma(x)) \in G_i$ and $v(y) \notin G_i$ against the assumption that $\langle \mathbf{A}_i, G_i\rangle$ is a model of $\vdash$. This, by same argument used in case (1), proves that $\langle \mathbf{A}, \bigcup_{k\leq i} f_{ki}^{-1}(G_i))\rangle$ is a $\vdash^r$ model. ⊠

Theorem 3.5.4. *Let $\vdash^r$ be a logic with an r-partition function $*$ and $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash^r)$ such that $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$, $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ for every $i \in I^+$. Assume, moreover, that, for each $j \in I$, $\mathbf{A}_j \in \mathsf{Alg}(\vdash)$ and there exists a Suszko filter G_j over $\mathbf{A}_j$ such that $F_i \subseteq f_{ij}^{-1}(G_j)$. The following are equivalent:*

(i) $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{r})$;

(ii) *(a)* $I^{+} = \emptyset$ *or*

(b) $I^{+} = \{i\}$ *is the bottom of* I.

Proof. (i)⇒(ii). By Lemma 3.5.2, we have that $\mid I^{+} \mid \leq 1$, that is, either $I^{+} = \emptyset$, namely $F = \emptyset$, or $I^{+} = \{i\}$, i.e. $F = F_i$. In order to prove (ii) we only need to show that if $I^{+} = \{i\}$ then i is the bottom element of I. We reason by absurdum, so assume that i is not the bottom element of I, i.e. there exists $j \in I$ such that $i \nleq j$.
Let $a \in A_j$ and $s = i \vee j$; consider an element $b = f_{js}(a) \in A_s$. We know $F_j = \emptyset$ so, by Definition 3.2.4, $b \notin F_s$. Moreover, as $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{r})$ there exists a $\vdash^{r}$-filter $G \supseteq F$ and a unary polynomial function $\varphi(v, \vec{z})$ such that for $\vec{c} \in A_k$, it holds

$$\varphi(a, \vec{c}) \in G \iff \varphi(b, \vec{c}) \notin G.$$

W.l.o.g. assume $\varphi(a, \vec{c}) \in G_q \subseteq A_q$ (with $q = j \vee k$). Now, as $G_i \neq \emptyset$ and $G_q \neq \emptyset$, by Theorem 3.3.5, we have $G_p \neq \emptyset$ (with $p = s \vee k$). Observe also that this implies $f_{qp}(\varphi(a, \vec{c})) \in G_p$. Moreover, by applying the same strategy used in the proof of Lemma 3.5.2

$$f_{qp}(\varphi(a, \vec{c})) = \varphi(b, \vec{c}) \in G_p$$

which is a contradiction. The same argument can be applied to the case $\varphi(b, \vec{c}) \in G$. This proves (ii).

(ii)⇒(i). We have to show that each of the conditions (a) and (b) implies (i).
(a)⇒(i). Assume the Płonka decomposition of $\langle \mathbf{A}, F\rangle$ is such that $I^{+} = \emptyset$. Consider $a, b \in A$, with $a \neq b$. We aim at showing $\langle a, b\rangle \notin \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F$. Consider first the case when $a \in A_i, b \in A_j$ for arbitrary $i \neq j$. We assume w.l.o.g. that if i, j are comparable then $i < j$. Now, as $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$ consider a non-empty $\vdash$filter $G_i \neq A_i$. By Lemma 3.5.3, $\langle \mathbf{A}, \bigcup_{k \leq i}(f_{ki}^{-1}(G_i))\rangle$ is a model of $\vdash^{r}$. In particular, as $F = \emptyset$, $\bigcup_{k \leq i} f_{ki}^{-1}(G_i))$ is a $\vdash^{r}$-filter extending F.

Now fix $c \in G_i$. We have that $c * a = c \in \bigcup_{k \leq i} f_{ki}^{-1}(G_i))$, while $c * b \notin \bigcup_{k \leq i} f_{ki}^{-1}(G_i))$, proving $\langle a, b\rangle \notin \mathbf{\Omega}^{\mathbf{A}} \bigcup_{k \leq i} f_{ki}^{-1}(G_i))$, i.e. $\langle a, b\rangle \notin \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F$. This proves $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{r})$, as desired.

The only case left is $a, b \in A_i$. As $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$ there exists $\langle \mathbf{A}_i, G_i\rangle \in \mathsf{Mod}(\vdash)$ such that $\langle a, b\rangle \notin \mathbf{\Omega}^{\mathbf{A}_i} G_i$ i.e. there exist $\vec{c} \in A_i$ and a unary polynomial function $\varphi(v, \vec{z})$ satisfying $\varphi(a, \vec{c}) \in G_i$ if and only if $\varphi(b, \vec{c}) \notin G_i$. Observe this implies $G_i \neq A_i$, for otherwise $\mathbf{\Omega}^{\mathbf{A}_i} G_i = A_i \times A_i$ and, by

Lemma 3.5.3, this entails $\langle \mathbf{A}, \bigcup_{k \leq i} f_{ki}^{-1}(G_i) \rangle \in \mathsf{Mod}(\vdash^r)$. So, we obtain $\varphi(a, \vec{c}) \in \bigcup_{k \leq i} f_{ki}^{-1}(G_i)$ if and only if $\varphi(b, \vec{c}) \notin \bigcup_{k \leq i} f_{ki}^{-1}(G_i)$, proving $\langle a, b \rangle \notin \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F$.

(b)⇒(i). Assume that $I^+ = \{i\}$ is the bottom of I and consider arbitrary $a, b \in A$. Again, we aim at showing $\langle a, b \rangle \notin \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F$. The case $a, b \in A_i$ is immediate, as $F = F_i$ and $\widetilde{\mathbf{\Omega}}^{\mathbf{A}_i} F_i = id$, for $\langle A_i, F_i \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$. So let $a \in A_j, b \in A_k$ assuming w.l.o.g. that if j, k are comparable then $j < k$. The argument of Lemma 3.5.3, together with the fact that there exists a Suszko filter G_j such that $F_i \subseteq f_{ij}^{-1}(G_j)$ for each $j \geqslant i$, imply that $\langle \mathbf{A}, \bigcup_{s \leq j} f_{sj}^{-1}(G_j) \rangle$ is a model of $\vdash^r$ and $F \subseteq \bigcup_{s \leq j} f_{sj}^{-1}(G_j)$. Moreover, as $G_j \neq \emptyset$, we can fix $c \in G_j$. Clearly $c * a \in \bigcup_{s \leq j} f_{sj}^{-1}(G_j)$ and $c * b \notin \bigcup_{s \leq j} f_{sj}^{-1}(G_j)$, so $\langle a, b \rangle \notin \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F$, as desired.

The only case left is $a, b \in A_j$. Again consider $\langle \mathbf{A}_j, G_j \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ such that $F_i \subseteq f_{ij}^{-1}(G_j)$ and let $H_j \supseteq G_j$ be the $\vdash$-filter on $\mathbf{A}_j$ such that $\langle a, b \rangle \notin \mathbf{\Omega}^{\mathbf{A}_j} H_j$. This is to say that there exist a unary polynomial function $\varphi(v, \vec{z})$ and $\vec{c} \in A_j$ such that $\varphi(a, \vec{c}) \in H_j$ if and only if $\varphi(b, \vec{c}) \notin H_j$. As $H_j \supseteq G_j$ and $F_i \subseteq f_{ij}^{-1}(G_j)$ we have $F_i \subseteq f_{ij}^{-1}(H_j)$. This, as before, implies $\langle \mathbf{A}, \bigcup_{k \leq j} f_{kj}^{-1}(H_j) \rangle$ is a model of $\vdash^r$ and therefore we obtain $\varphi(a, \vec{c}) \in \bigcup_{k \leq j} f_{kj}^{-1}(H_j)$ if and only if $\varphi(b, \vec{c}) \notin \bigcup_{k \leq j} f_{kj}^{-1}(H_j)$. This proves $\langle a, b \rangle \notin \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F$ and it concludes the proof. ⊠

3.5.1 Truth equational logics

If the logic $\vdash$ is truth equational, the characterization of the Suszko reduced models can be significantly simplified. The reason relies on the fact that if a logic is truth-equational, then the Leibniz operator and the Suszko operators behaves in a suitable way, as witnessed by the following

Theorem 3.5.5 ([42],Theorem 6.106). *A logic $\vdash$ is truth-equational if and only if the Suszko operator is injective over the set of its filters, for any algebra.*

Lemma 3.5.6. *Let $\vdash$ be a truth equational logic with an r-partition function $*$. Consider $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\vdash^r)$ with $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$ for each $i \in I$. If $k \leq j$ and $\langle \mathbf{A}_j, G_j \rangle, \langle \mathbf{A}_k, G_k \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ then $G_k \cap f_{kj}^{-1}(G_j) = G_k$.*

Proof. Let $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\vdash^r)$ as in the statement. Consider $k \leq j$ and let $\langle \mathbf{A}_j, G_j \rangle, \langle \mathbf{A}_k, G_k \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$. Preliminary observe that $\vdash$ is truth-equational, therefore $G_j, G_k \neq \emptyset$. By Lemma 3.5.1, $f_{kj}^{-1}(G_j)$ is a $\vdash$-filter on $\mathbf{A}_k$ with $f_{kj}^{-1}(G_j) \neq \emptyset$.

Consider now $G_k \cap f_{kj}^{-1}(G_j)$, which is again a non-empty $\vdash$-filter on $\mathbf{A}_k$. Clearly $G_k \cap f_{kj}^{-1}(G_j) \subseteq G_k$ so, as the Suszko operator is monotone (see [42, Lemma 5.37]), $\widetilde{\mathbf{\Omega}}^{\mathbf{A}_k} G_k \cap f_{kj}^{-1}(G_j) \subseteq \widetilde{\mathbf{\Omega}}^{\mathbf{A}_k} G_k = id$, which entails $\widetilde{\mathbf{\Omega}}^{\mathbf{A}_k} G_k \cap f_{kj}^{-1}(G_j) = id$. By Theorem 3.5.5, the Suszko operator in injective and this, together with $\widetilde{\mathbf{\Omega}}^{\mathbf{A}_k} G_k \cap f_{kj}^{-1}(G_j) = \widetilde{\mathbf{\Omega}}^{\mathbf{A}_k} G_k$, implies $G_k \cap f_{kj}^{-1}(G_j) = G_k$. ⊠

The next Theorem is a refinement of Theorem 3.5.4 that characterizes the Suszko reduced models of $\vdash^r$ when $\vdash$ is a truth-equational logic.

Theorem 3.5.7. *Let $\vdash$ be a truth equational logic with an r-partition function $*$. Consider $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\vdash^r)$ such that $\mathbf{A}_j \in \mathsf{Alg}(\vdash)$ for each $j \in I$, $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ for every $i \in I^+$.The following are equivalent:*

(i) $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^r)$;

(ii) *(a)* $I^+ = \emptyset$ *either*

(b) $I^+ = \{i\}$ *is the bottom of* I.

Proof. (i)⇒(ii). This direction follows from Theorem 3.5.4 (it is enough to verify that the additional assumption in Theorem 3.5.4 is not used in the proof of this direction).

(ii)⇒(i). We need to show that both (a) and (b) implies (i).

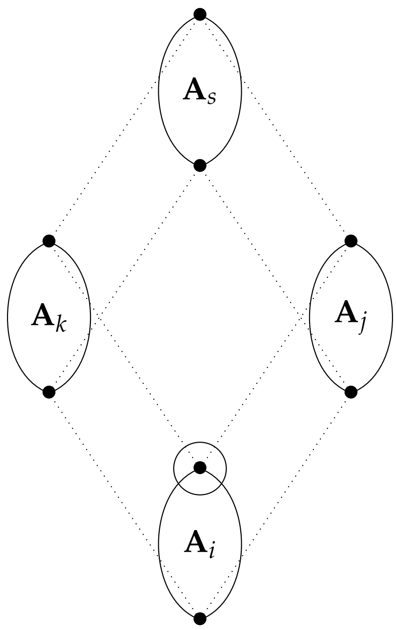
(a)⇒(i). This implication can be proved using the very same argument of Theorem 3.5.4.

(b)⇒(i). Let $I^+ = \{i\}$ be the bottom of I and consider $a, b \in A$. The case $a, b \in A_i$ is immediate, as $\langle a, b \rangle \notin \widetilde{\mathbf{\Omega}}^{\mathbf{A}} F_i$. If $a \in A_j, b \in A_k$ (if j and k are comparable assume $j \leq k$) we can consider $\langle \mathbf{A}_j, G_j \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ and, by applying Lemma 3.5.3 and Lemma 3.5.6, we obtain that $\langle \mathbf{A}, \bigcup_{s \leq j} f_{sj}^{-1}(G_j) \rangle$ is a model of $\vdash^r$ and that $F_i = F \subseteq \bigcup_{s \leq j} f_{sj}^{-1}(G_j)$. If $j \neq k$ then, as before, we can fix $c \in G_j$ and observe that $c * a \in \bigcup_{s \leq j} f_{sj}^{-1}(G_j)$ while $c * b \notin \bigcup_{s \leq j} f_{sj}^{-1}(G_j)$. If $j = k$, then, from the fact that $\langle \mathbf{A}_j, G_j \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash)$, we deduce that there exists a $\vdash$-filter $H_j \supseteq G_j$ such that $\langle a, b \rangle \notin \mathbf{\Omega}^{\mathbf{A}_j} H_j$. This is equivalent to the fact that $\varphi(a, \vec{c}) \in H_j$ if and only if $\varphi(b, \vec{c}) \notin H_j$, for a unary polynomial function $\varphi(v, \vec{z})$ and $\vec{c} \in A_j$. Clearly $H_j \supseteq G_j$ implies $\bigcup_{s \leq j} f_{sj}^{-1}(H_j) \supseteq \bigcup_{s \leq j} f_{sj}^{-1}(G_j)$, so by Lemma 3.5.3, $\bigcup_{s \leq j} f_{sj}^{-1}(H_j)$ is a $\vdash^r$-filter extending $\bigcup_{s \leq j} f_{sj}^{-1}(G_j)$. As $\bigcup_{s \leq j} f_{sj}^{-1}(H_j) \cap A_j = H_j$ we have that $\varphi(a, \vec{c}) \in \bigcup_{s \leq j} f_{sj}^{-1}(H_j)$ if and only if $\varphi(b, \vec{c}) \notin \bigcup_{s \leq j} f_{sj}^{-1}(H_j)$.

This proves that in all the considered cases $\langle a, b\rangle \notin \widetilde{\boldsymbol{\Omega}}^{\mathbf{A}} F$, i.e. $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^r)$. $\boxtimes$

Corollary 3.5.8. *Let $\vdash^r$ be a logic with an r-partition function. Then* $\mathsf{Mod}^{*}(\vdash^r) \subsetneq \mathsf{Mod}^{\mathsf{Su}}(\vdash^r)$.

Example 3.5.9. The following is an example of Suszko reduced model (which is not Leibniz reduced) of Bochvar logic B_3 ($\mathbf{A}_i$, $\mathbf{A}_j\mathbf{A}_k$, $\mathbf{A}_s$ are Boolean algebras, circles indicate filters and dotted lines represent Płonka homomorphisms).



3.6 Classification in the Leibniz hierarchy

In this short final section, we turn our attention to a fundamental topic in abstract algebraic logic, that is the so-called *Leibniz hierarchy*. Intuitively, the hierarchy provides a taxonomy, where logics are classified and every class in it witnesses how deep is the link between a logic and its algebraic counterpart (for a detailed discussion, see [42, 67]). We will see that containment logics occupy very low levels in the Leibniz hierarchy, showing that their relation with the respective algebraic counterpart is quite weak.

We recall from the preliminaries the material which is necessary for the purpose of the present subsection.

A logic $\vdash$ is *protoalgebraic* if there is a set of formulas $\Delta(x, y)$ such that

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

Remarkably, if a logic $\vdash$ is protoalgebraic, then $\mathsf{Mod}^*(\vdash) = \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ (see [42, Corollary 6.3]).

A logic $\vdash$ is *truth-equational* if there is a set of equations $\boldsymbol{\tau}(x)$ such that for all $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^*(\vdash)$,

$$a \in F \iff \mathbf{A} \vDash \boldsymbol{\tau}(a), \text{ for all } a \in A.$$

In this case, $\boldsymbol{\tau}(x)$ is a set of *defining equations* for $\vdash$.

A well-known result concerning truth-equationality is the the following:

Lemma 3.6.1. *[42, Corollary 6.92] If a logic is truth-equational logic, then it has theorems.*

Theorem 3.6.2. *Let $\vdash$ be a logic. Then*

(i) *$\vdash^r$ is not protoalgebraic;*

(ii) *$\vdash^r$ is not truth-equational.*

Proof. (i) is obtained by observing that $\vdash^r$ is theoremless, hence disproves condition $\vdash^r \Delta(x, x)$ of the characterization of protoalgebraicity.
(ii) $\vdash^r$ does not have theorems and this, together with Lemma 3.6.1, entails that it is not truth equational. $\boxtimes$

Chapter 4

Sublogics of Variable Inclusion and Gentzen System

4.1 Introduction

The present chapter faces two main topics. Firstly, it studies the properties of left variable inclusion logics viewed as Gentzen systems, while Section 4.3 investigates the structure of the lattice of sublogics of variable inclusion of a given logic.

We saw that Theorems 2.7.1, 3.6.2 state that no logic of variable inclusion is, in general, protoalgebraic. More precisely, a part from truth equationality (under particular assumptions), no other property of algbraizability transfer from a logic to its variable inclusion companions. Section 4.2 investigates how the algebraizability properties transmit from the calculus $\vdash_{\mathcal{G}}$ to the calculus $\vdash^{l}_{\mathcal{G}}$. It turns out that, in terms of algebraization, some logics of left variable inclusion possess stronger properties when considered as Gentzen systems[1]. Indeed, the main result of Section 4.2 (Theorem 4.2.7) is that, under reasonable assumptions, the property of equivalentiality transfers from $\vdash_{\mathcal{G}}$ to $\vdash^{l}_{\mathcal{G}}$.

4.2 Equivalentiality of Gentzen systems for $\vdash^{l}$

In the preset section, unless stated otherwise, we always assume $\vdash$ to be a finitary logic with an l-partition function $\cdot$ and an antitheorem $\Sigma = \{\epsilon_1, \dots, \epsilon_n\}$. Moreover, given a logic $\vdash$ we let $\vdash_{\mathcal{G}}$ be the calculus repre-

[1] At the time this thesis is written, a full investigation of logics of varibale inclusion, including $\vdash^{r}$, as Gentzen systems, is a work in progress.

senting the operation of generated filter, as described in the preliminaries (1.2). In the next lemmas, given a logic $\vdash$ an algebra $\mathbf{A}$ and $X \subseteq A$, by $\mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X) = \bigcup_{n\in\omega} F_n$ we denote the inductive characterization of the $\vdash$-filter generated by X over $\mathbf{A}$, as stated in Theorem 1.2.2. The next results, mainly consisting of a bunch of technical lemmas, provide the basis for the main theorem of the section.

The following lemma characterizes a wide class of models of the calculus $\vdash^{\mathcal{G}}$.

Lemma 4.2.1. *Let $\mathbf{A}$ be an algebra in the type of $\vdash$ and $F \subseteq A^* \times A$ a set of sequences representing the operation of $\vdash$-generated filter, then $\langle \mathbf{A}, F\rangle$ is a model of $\vdash_{\mathcal{G}}$.*

Proof. Let $\mathbf{A}$ be an algebra. By applying the Tarski process to the g-matrix $\langle \mathbf{A}, \mathcal{F}\mathrm{i}_{\vdash}(\mathbf{A})\rangle$ we obtain the g-matrix $\langle A_{/\cap\Omega F}, \mathcal{F}\mathrm{i}(\mathbf{A}_{/\cap\Omega F})\rangle$. Consider $\Gamma_i \rhd \alpha_i \vdash_{\mathcal{G}} \Gamma \rhd \alpha$, and let $h : \mathbf{Fm} \to \mathbf{A}$ be a homomorphism such that $h(\Gamma_i \rhd \alpha_i) \in F$, which means $h(\alpha_i) \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(h(\Gamma_i))$. Let also $\pi : \mathbf{A} \to \mathbf{A}_{/\cap\Omega F}$ be the canonical map onto the quotient. Since π is a strict homomorphism, we have $\pi(h(\alpha_i)) \in \mathrm{Fg}^{\mathbf{A}/\cap\Omega F}_{\vdash}(\pi(h(\Gamma_i)))$ and this, together with the fact that $A_{/\cap\Omega F} \in \mathsf{Alg}(\vdash)$, implies $\pi(h(\alpha)) \in \mathrm{Fg}^{\mathbf{A}/\cap\Omega F}_{\vdash}(\pi(h(\Gamma)))$. By Lemma 1.2.13 we conclude $h(\alpha) \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(h(\Gamma))$. This proves that $\langle \mathbf{A}, F\rangle$ is a model of $\vdash_{\mathcal{G}}$. ⊠

The following observation mirrors the one stated in Remark 3.2.6 for $\vdash^r$.

Remark 4.2.2. Observe that, if $Var(\Gamma) \subseteq Var(\alpha)$, then for any $h : \mathbf{Fm} \to \mathcal{P}_{\nmid}(\mathbf{A}_i)$ and all $\mathcal{P}_{\nmid}(\mathbf{A}_i)$, $I_h(\Gamma) \leq I_h(\alpha)$. Indeed, let $Var(\Gamma) = x_1, \dots, x_m$, $Var(\Gamma) \subseteq Var(\alpha)$ and $k = I_h(\Gamma)$, that is $k = i_h(x_1) \vee \dots \vee i_h(x_m)$. So, for each $i_h(x_n) (1 \leq n \leq m)$ there exist $\beta \in Var(\alpha)$ s.t. $i_h(\beta) = i_h(x_n)$. Therefore $k \leq I_h(\alpha)$.

Lemma 4.2.3 explains how the inductive construction of the filter $\mathrm{Fg}^{\mathbf{A}}_{\vdash}(X)$ behaves with respect to the homomorphisms of a Płonka sum.

Lemma 4.2.3. *Consider $\mathbf{A} = \mathcal{P}_l(B_i)_{i\in I}$, let $l, i \in I$ be such that $l \leq i$ and $X \subseteq B_i$. Consider $\mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X) = \bigcup_{n\in\omega} F_n$. If $\gamma(c_1, \dots, c_n) \in F_n \cap B_l$ then $\gamma(c_1, \dots, c_n) \cdot a \in F_n \cap B_i$, for any $a \in B_i$, $\vec{c} \in A$.*

Proof. We reason, by induction, on the construction of $\mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X) = \bigcup_{n\in\omega} F_n$. *Base*. For $n = 0$, we have that $\mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X) = X$. Suppose $\gamma(\vec{c}) \in F_0 \cap B_l = X \cap B_l$, where $\vec{c} = c_1, \dots, c_n$. Necessarily, $l = i$ (as $X \cap B_l = \emptyset$, for $l \neq i$)

and $\gamma(\vec{c}) \in X \subseteq B_i$, which implies $\gamma(\vec{c}) \cdot a = \gamma(\vec{c}) \in F_0 \cap B_i$, for any $a \in B_i$.
Inductive step. Let $\gamma(\vec{c}) \in (F_{n+1} \cap B_l) \smallsetminus F_n$. This means that there there is a rule $\delta_1(\vec{x}), \ldots, \delta_m(\vec{x}) \vdash^l \alpha(\vec{x})$ and $\vec{d} \in A$ such that $\vec{\delta}(\vec{d}) \in F_n$ and $\alpha(\vec{d}) = \gamma(\vec{c}) \in B_l$. Observe that $Var(\vec{\delta}) \subseteq Var(\alpha)$, hence, by Remark 4.2.2, any term (among $\vec{\delta}$) $\delta_k(\vec{d}) \in B_k$ (here k is ranging over $1, \ldots, m$), with $k \leq l \leq i$. That is, $\delta_k(\vec{d}) \in F_n \cap B_k$. By inductive hypothesis, we have $\delta_1(\vec{d}) \cdot a, \ldots, \delta_n(\vec{d}) \cdot a \in F_n \cap B_i$. Now, observe that the following rule holds

$$\delta_1(\vec{x}) \cdot y, \ldots, \delta_m(\vec{x}) \cdot y \vdash^l \alpha(\vec{x}) \cdot y.$$

Therefore, by interpreting $\vec{x} \mapsto \vec{d}$ and $y \mapsto a$, we obtain $\delta_1(\vec{d}) \cdot a, \ldots, \delta_m(\vec{d}) \cdot a \in F_n \cap B_i$ and, therefore $\alpha(\vec{d}) \cdot a \in F_{n+1} \cap B_i$. Since, by assumption, $\alpha(\vec{d}) = \gamma(\vec{c})$, then $\alpha(\vec{d}) \cdot a = \gamma(\vec{c}) \cdot a) \in F_{n+1} \cap B_i$.

⊠

The next lemma state a crucial connection between the two closure operators $\mathrm{Fg}^{\mathbf{A}}_{\vdash}$ and $\mathrm{Fg}^{\mathbf{A}}_{\vdash^l}$. In particular, given a specific Płonka sum, the operations of $\vdash$-filter generation and of $\vdash^l$-filter generation relate as follows:

Lemma 4.2.4. *Let* $\vdash^l$ *be a logic with a partition function,* $\mathbf{A} \in \mathsf{Alg}(\vdash^l)$ *such that* $\mathbf{A} \cong \mathcal{P}_{\prime}(\mathbf{B}_i)_{i \in I}$ *and* $X \subseteq B_i$*. Then* $\mathrm{Fg}^{\mathbf{B}_i}_{\vdash}(X) = \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X) \cap B_i$*, for each* $i \in I$*.*

Proof. ($\subseteq$) We prove the claim by induction on the construction of $\mathrm{Fg}^{\mathbf{B}_i}_{\vdash}(X) = \bigcup_{n \in \omega} F_n$.
Base. Immediate, as $F_0 = X \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X) \cap B_i$.
Inductive step. Let $a \in F_{n+1} \smallsetminus F_n$, then there exists a set of formulas $\gamma_1, \ldots, \gamma_n, \alpha$ such that $\gamma_1, \ldots, \gamma_n \vdash \alpha$ and elements $\vec{c} \in B_i$ such that $\gamma_1(\vec{c}), \ldots, \gamma_n(\vec{c}) \in F_n$ and $a = \alpha(\vec{c})$. By induction hypothesis,
$\gamma_1(\vec{c}), \ldots, \gamma_n(\vec{c}) \in \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X) \cap B_i$. Moreover, since $\cdot$ is an l-partition function for $\vdash^l$, we have $\gamma_1, \ldots, \gamma_n \vdash^l \alpha \cdot \gamma_1 \cdot_{\ldots} \cdot \gamma_n$, which implies $\vdash^l_{\mathcal{G}} \gamma_1, \ldots \gamma_n \rhd \alpha \cdot \gamma_1 \cdot_{\ldots} \cdot \gamma_n$ and, therefore, $a \cdot \gamma_1(\vec{c}) \cdot_{\ldots} \cdot \gamma_n(\vec{c}) \in \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X)$. Since $\alpha(\vec{c}) = a \in B_i$ then $a \cdot \gamma_1(\vec{c}) \cdot_{\ldots} \cdot \gamma_n(\vec{c}) = a$, whence $a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X) \cap B_i$.

($\supseteq$) By induction on the construction of $\mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(X) = \bigcup_{n \in \omega} F_n$.
Base. $F_0 \cap B_i = X \cap B_i = X \subseteq \mathrm{Fg}^{\mathbf{B}_i}_{\vdash}(X)$.
Inductive step. Let $a \in F_{n+1} \cap B_i \smallsetminus F_n$. So there are formulas $\gamma_1, \ldots, \gamma_n, \alpha$ such that $\gamma_1, \ldots, \gamma_n \vdash^l \alpha$ and $\vec{c} \in A$ such that $\gamma_1(\vec{c}), \ldots, \gamma_n(\vec{c}) \in F_n$, $\alpha(\vec{c}) = a$. Since $Var(\Gamma) \subseteq Var(\alpha)$, then by Remark 4.2.2 $\gamma_1(\vec{c}), \ldots, \gamma_n(\vec{c}) \in B_l$ with $l \leq i$. Two case may arise: either (1) $l = i$ or (2) $l < i$.

1. In this case, $\gamma_1(\vec{c}), \ldots, \gamma_n(\vec{c}) \in F_n \cap B_i$ hence, by inductive hypothesis, $\gamma_1(\vec{c}), \ldots, \gamma_n(\vec{c}) \in \mathrm{Fg}_{\vdash}^{\mathbf{B}_i}(X)$. Moreover, since $\gamma_1, \ldots, \gamma_n \vdash \alpha$ then $\vdash_{\mathcal{G}} \gamma_1, \ldots, \gamma_n \rhd \alpha$ and, by Definition of $\vdash_{\mathcal{G}}$, $\alpha(\vec{c}) = a \in \mathrm{Fg}_{\vdash}^{\mathbf{B}_i}(X)$.

2. In this case, $\gamma_1(\vec{c}), \ldots, \gamma_n(\vec{c}) \in F_n \cap B_l$; therefore, by Lemma 4.2.3, $\gamma_1(\vec{c}) \cdot a, \ldots, \gamma_n(\vec{c}) \cdot a \in F_n \cap B_i$. By induction hypothesis, $\gamma_1(\vec{c}) \cdot a, \ldots, \gamma_n(\vec{c}) \cdot a \in \mathrm{Fg}_{\vdash}^{\mathbf{B}_i}(X)$. Moreover, since $\gamma_1, \ldots, \gamma_n \vdash^l \alpha$, then $\gamma_1 \cdot y, \ldots, \gamma_n \cdot y \vdash^l \alpha \cdot y$, so the following rule holds:

$$\gamma_1(\vec{x}) \cdot y, \ldots, \gamma_n(\vec{x}) \cdot y \vdash \alpha(\vec{x}) \cdot y.$$

This implies that $\alpha(\vec{c}) \cdot a = a \cdot a = a \in \mathrm{Fg}_{\vdash}^{\mathbf{B}_i}(X)$, as desired.[2]

⊠

The previous Lemma has the following immediate consequence.

Corollary 4.2.5. *Let $\mathbf{A} \cong \mathcal{P}_{\nmid}(\mathbf{B}_j)$, $H \subseteq A^* \times A$ representing the operation of generated $\vdash^l$-filter and $G \subseteq B_i^* \times B_i$ representing the operation of generated $\vdash$-filter. Then $G = H \cap (B_i^* \times B_i)$.*

The last needed technical lemma describes another aspect of the closure operator $\mathrm{Fg}_{\vdash^l}^{\mathbf{A}}$ in a Płonka sum.

Lemma 4.2.6. *Let $\mathbf{A} \cong \mathcal{P}_{\nmid}(\mathbf{B}_i)$, $X \subseteq B_i$, for some $i \in I$. If $i \nleq j \in I$ then $\mathrm{Fg}_{\vdash^l}^{\mathbf{A}}(X) \cap B_j = \mathrm{Fg}_{\vdash^l}^{\mathbf{A}}(\emptyset) \cap B_j$.*

Proof. We reason by induction on the construction of $\mathrm{Fg}_{\vdash^l}^{\mathbf{A}}(X) = \bigcup_{n \in \omega} F_n$.

Base. Let $\mathrm{Fg}_{\vdash^l}^{\mathbf{A}}(X) = F_0 = X$ and $\mathrm{Fg}_{\vdash^l}^{\mathbf{A}}(\emptyset) = \emptyset$. By definition of direct system, $B_i \cap B_j = \emptyset$; moreover $X \subseteq B_i$, therefore $\mathrm{Fg}_{\vdash^l}^{\mathbf{A}}(X) \cap B_j = F_0 \cap B_j = X \cap B_j = \emptyset = \mathrm{Fg}_{\vdash^l}^{\mathbf{A}}(\emptyset) \cap B_j$.
Inductive step. Let $a \in (F_{n+1} \cap B_j) \smallsetminus F_n$, i.e. the element a has been added at the $n+1$ step of the construction, so there exists formulas $\gamma_1, \ldots, \gamma_n, \alpha$ such that $\gamma_1, \ldots, \gamma_n \vdash^l \alpha$ and elements $\vec{c} \in A$ such that $\gamma_1(\vec{c}), \ldots, \gamma_n(\vec{c}) \in F_n$ and $\alpha(\vec{c}) = a \in B_j$. Since $\gamma_1, \ldots, \gamma_n \vdash^l \alpha$, we have that $Var(\gamma_1, \ldots, \gamma_n) \subseteq Var(\alpha)$, and this implies, by Remark 4.2.2, that $\vec{c} \in B_k$ with $k \leq j$. Notice that, for every formula $\gamma_1, \ldots, \gamma_n$ we have that $\gamma_l(\vec{c}) \in B_m$ (with l ranging over $1, \ldots, n$) and $i \nleq m$ (as $m \leq k \leq j$ and

[2] Observe that the eveluation into the component B_i: $x_1 \mapsto c_1 \cdot a, \ldots, x_n \mapsto c_n \cdot a, y \mapsto a$ is well defined.

if $i \leq m$ would lead to a contradiction with the assumption $i \not\leq j$). Therefore, by induction hypothesis, $\gamma_l(\vec{c}) \in \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\emptyset)$ for every $l = 1, \ldots, n$, hence $\alpha(\vec{c}) = a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\emptyset)$.

$\boxtimes$

The previous lemmas allow to prove the main result of this section, namely that the left variable inclusion companion of a logic may possess an equivalential Gentzen system.

Theorem 4.2.7. *Let* $\vdash_{\mathcal{G}}$ *be an equivalential Genzen system for* $\vdash$ *under the transformer* $\boldsymbol{\rho}$*. Then the Gentzen system* $\vdash^l_{\mathcal{G}}$ *is equivalential under the transformer* $\boldsymbol{\rho}^l : Eq \to \mathcal{P}(Seq)$ *mapping* $x \approx y \longmapsto \{\boldsymbol{\rho}(x,y), \epsilon_i(x) \lhd \rhd \epsilon_i(y)\}$ *for every* $1 \leq i \leq n$*.*

Proof. Our goal is to prove that $\boldsymbol{\rho}^l(x,y)$ is a set of congruence formulas. By Lemma 1.2.7, it will be enough to show that, for $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\vdash^l_{\mathcal{G}})$

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F \iff \boldsymbol{\rho}^l(a,b) \in F.$$

($\Rightarrow$). Let $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F = \mathrm{id}$, i.e. $a = b$. This means $\boldsymbol{\rho}^l(a,b) = \boldsymbol{\rho}^l(a,a)$.

By equivalentiality of $\vdash_{\mathcal{G}}$ we have $\vdash_{\mathcal{G}} \boldsymbol{\rho}(x,x)$, i.e. for $\vec{\gamma}(x) \rhd \varphi(x) \in \boldsymbol{\rho}(x,x)$ it holds $\vdash_{\mathcal{G}} \vec{\gamma}(x) \rhd \varphi(x)$ which by the adequacy of $\vdash_{\mathcal{G}}$ implies $\vec{\gamma}(x) \vdash \varphi(x)$. Clearly, since $Var(\vec{\gamma}(x)) = \{x\} \subseteq Var(\varphi(x))$ then $\vec{\gamma}(x) \vdash^l \varphi(x)$. This implies $\vdash^l_{\mathcal{G}} \vec{\gamma}(x) \rhd \varphi(x)$, therefore we conclude $\vec{\gamma}(a) \rhd \varphi(a) \in F$ and this proves $\boldsymbol{\rho}(a,a) = \boldsymbol{\rho}(a,b) \in F$. It only remains to show that $\epsilon_i(a) \lhd \rhd \epsilon_i(b) \in F$. As $a = b$ we have $\epsilon_i(a) = \epsilon_i(b)$ and hence $\epsilon_i(a) \lhd \rhd \epsilon_i(b) = \epsilon_i(a) \lhd \rhd \epsilon_i(a) \in F$. This shows $\boldsymbol{\rho}^l(a,b) \in F$.

($\Leftarrow$). We prove this direction by contraposition.
Let $\langle \mathbf{A}', F'\rangle \in \mathsf{Mod}^*(\vdash^l_{\mathcal{G}}), a', b' \in A'$ such that $\langle a', b'\rangle \notin \boldsymbol{\Omega}^{\mathbf{A}'} F' = id$, that is $a' \neq b'$. Suppose, by contradiction, that $\boldsymbol{\rho}^l(a', b') \in F'$.

Observe that the assumption that $\boldsymbol{\rho}^l(a', b') \in F'$ together with the definition of $\vdash^l_{\mathcal{G}}$ and Theorem 2.3.5 implies that there is $\mathbf{A} = \mathcal{P}_{\text{ł}}(\mathbf{B}_i) \in \mathsf{Alg}(\vdash^l)$ and elements $a, b \in A$ such that if $\vec{\gamma}(a,b) \rhd \delta(a,b) \in \boldsymbol{\rho}^l(a,b)$ then $\delta(a,b) \in Fg^{\mathbf{A}}_{\vdash^l}(\vec{\gamma}(a,b))$. We distinguish two cases.

(A) $a \in B_i, b \in B_j$ with $i \neq j$, for $i, j \in I$;

(B) $a, b \in B_i$, for $i \in I$.

(A) Let $a \in B_i$, $b \in B_j$ with $i \neq j$. By the fact that $\{\epsilon_j(x) \lhd \rhd \epsilon_j(y)\} \subseteq \boldsymbol{\rho}^l(x,y)$, for every $1 \leq j \leq n$ (the index j is ranging over the number of members of Σ), we have $\mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\Sigma(a)) = \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\Sigma(b))$ and $\Sigma(b) \subseteq B_j$, as Σ

depends on one variable only. Hence $\Sigma(b) \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\Sigma(a)) \cap B_j$ and, by the same reason, $\Sigma(a) \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\Sigma(b)) \cap B_i$. Clearly, either $i < j$ or $i \not\leq j$ (the case $j < i$ is analogous to $i < j$). Firstly, suppose $i < j$. Since $\Sigma(a) \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\Sigma(b)) \cap B_i$, by Lemma 4.2.6, it holds $\Sigma(a) \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\emptyset)$. Let $l \in I$ any index such that $i \leq l$. We claim that $B_l \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\emptyset)$. Indeed, since $\Sigma(x)$ is an antitheorem for $\vdash$, we have $\Sigma(x) \vdash y \cdot x$ and, as $Var(\Sigma(x)) = \{x\} \subseteq Var(y \cdot x)$, it follows $\Sigma(x) \vdash^l y \cdot x$. Now, let $d \in B_l$, then $d = d \cdot a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\Sigma(a)) = \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\emptyset)$, where the first equality is justified by Theorem 2.3.5. This proves our claim.
From the fact that $\mathbf{A} \in \mathsf{Alg}(\vdash^l)$ there exists a filter F such that $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$. Clearly, as $a \neq b$, $(a,b) \notin \widetilde{\mathbf{\Omega}}^{\mathbf{A}}(F) = id$, i.e. there exists a filter G extending F and a unary polynomial function $\psi(x,\vec{c})$ (for some $\vec{c} \in A$) such that $\psi(a,\vec{c}) \in G$ if and only if $\psi(b,\vec{c}) \notin G$ (x appearing in ψ the only interesting case). It is always the case that $\psi(a,\vec{c}) \in B_l$, $\psi(b,\vec{c}) \in B_{l'}$ with $i \leq l, l'$, therefore $\psi(a,\vec{c}) \in G$ and $\psi(b,\vec{c}) \in G$ (as $B_l, B'_l \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\emptyset)$). A contradiction.

In the case whether $i \not\leq j$, the argument is analogous and relies on Lemma 4.2.6, reasoning on the fact that for every unary polynomial function $\psi(x,\vec{c})$ we have $\psi(a,\vec{c}) \in B_l$, $\psi(b,\vec{c}) \in B_{l'}$ with $i \leq l$, $j \leq l'$ and $B_l, B_{l'} \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\emptyset)$.

(B) In this case, we will show that $\langle a', b'\rangle \in \mathbf{\Omega}^{\mathbf{A}}F' = id$, getting a contradiction. This is equivalent to prove that for every unary polynomial function $q(x) = \langle \psi_1(x,\vec{c}), \ldots, \psi_{n-1}(x,\vec{c}), \varphi(x,\vec{c})\rangle$ it holds $q(a') \in F'$ iff $q(b') \in F'$. Our strategy is to show the derivability of the rule $\boldsymbol{\rho}^l(x,y), q(x) \vdash^l_{\mathcal{G}} q(y)$. Let now be $H \subseteq A^* \times A$ representing the operation of $\vdash^l$ generated filter over $\mathbf{A}$. So, the goal is to prove $q(b) \in H$ under the assumption that $\boldsymbol{\rho}^l(a,b), q(a) \in H$. Observe that w.l.o.g. we can take parameters $\vec{c} \in B_k$, with $k \in I$. Set $j = i \vee k$.
We claim that $\boldsymbol{\rho}(a \cdot \vec{c}, b \cdot \vec{c}) \in H \cap B^*_j \times B_j$. Indeed, let $\delta_1(a,b), \ldots, \delta_n(a,b) \rhd \gamma(a,b) \in \boldsymbol{\rho}(a,b) \subseteq H \cap B^*_i \times B_i$. Then $\gamma(a,b) \in \mathrm{Fg}^{\mathbf{A}}_{\vdash^l}(\vec{\delta}(a,b))$, which implies that there are formulas such that $\vec{\delta}(x,y) \vdash^l \alpha(x,y)$ where $\alpha(a,b) = \gamma(a,b)$. Observe that the rule

$$\delta_1(x,y) \cdot \vec{z}, \ldots, \delta_n(x,y) \cdot \vec{z} \vdash^l \alpha(x,y) \cdot \vec{z}$$

is derivable. By the adequacy of $\vdash^l_{\mathcal{G}}$ it holds

$$\vdash^l_{\mathcal{G}} \delta_1(x,y) \cdot \vec{z}, \ldots, \delta_n(x,y) \cdot \vec{z} \rhd \alpha(x,y) \cdot \vec{z}.$$

Therefore

$$\delta_1(a,b) \cdot \vec{c}, \ldots, \delta_n(a,b) \cdot \vec{c} \rhd \alpha(a,b) \cdot \vec{c} \in H,$$

where we have just substituted $a, b, \vec{c}$ for x, y, and $\vec{z}$, respectively. By condition (v) in Definition 2.3.1 the above is equivalent to

$$\delta_1(a \cdot \vec{c}, b \cdot \vec{c}), \ldots, \delta_n(a \cdot \vec{c}, b \cdot \vec{c}) \rhd \gamma(a \cdot \vec{c}, b \cdot \vec{c}) \in H,$$

which established our claim. Let $G \subseteq B_j^* \times B_j$ be a set representing the operation of $\vdash$-generated filter over $\mathbf{B}_j$. By Corollary 4.2.5, $G = H \cap B_j^* \times B_j$. Since, by assumption, $q(a) \in H$ and, by construction, $q(a) \in B_j^* \times B_j$, we have $q(a) \in G$. Moreover, in virtue of the above claim $\boldsymbol{\rho}(a \cdot \vec{c}, b \cdot \vec{c}) \in G$. The equivalentiality of $\vdash_{\mathcal{G}}$ guarantees that $\langle a \cdot \vec{c}, b \cdot \vec{c} \rangle \in \boldsymbol{\Omega}^{\mathbf{B}_j} G$. Therefore $q(b) \in G$, whence $q(b) \in H$. We have shown the admissibility of $\boldsymbol{\rho}^l(x, y), q(x) \vdash_{\mathcal{G}}^l q(y)$, as desired. This implies $\langle a', b' \rangle \in \boldsymbol{\Omega}(F')$, a contradiction.

⊠

4.2.1 A non-equivalential case: the logic LP^n

The assumption made in Theorem 4.2.7 about the presence of an antitheorem is fundamental. Indeed, in case $\vdash$ lacks such a set, it may turn out that $\vdash_{\mathcal{G}}^l$ (for $\vdash^l$) is not equivalential, although $\vdash_{\mathcal{G}}$ is.

The next part of the section is conceived to illustrate this case. In order to present a concrete example, we need to introduce the following class of algebras:

Definition 4.2.8. A *Kleene lattice* is a distributive lattice with an additional unary operation $\neg$ satisfying the following conditions:

(i) $x \vee y \approx \neg(\neg x \wedge \neg y)$

(ii) $x \wedge y \approx \neg(\neg x \vee \neg y)$

(iii) $\neg\neg x \approx x$

(iv) $x \wedge \neg x \leq y \vee \neg y$.

We need to introduce here a linguistic expansion of the variety of Kleene lattices (obtained by adding a constant), called DMF in [66]. An algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, n \rangle$ of type $\langle 2, 2, 1, 0 \rangle$ belongs to the variety DMF if it is a Kleene lattice with a constant n satisfying $\neg n \approx n$. We will refer to n as the fixed point (with respect to $\neg$).

We denote by $\mathbf{KL}_3 = \langle \{a, n, b\}, \wedge, \vee, \neg, n \rangle$ the 3-element Kleene lattice with a fixed point n (whose Hasse diagram is) depicted in the following figure (arrows represent negation)

$$\begin{tikzcd} b \arrow[dd, bend right, <->] \arrow[d, dash] \\ n \arrow[loop right] \arrow[d, dash] \\ a \end{tikzcd}$$

Some basic properties of DMF are summarized in the following Lemma from [66].

Lemma 4.2.9. *The following hold in* DMF*:*

(i) $x \wedge \neg x \leq n \leq y \vee \neg y$

(ii) $x \leq y \Rightarrow \neg y \leq \neg x$

(iii) $x \approx \neg x \Rightarrow x \approx n$

(iv) $(x \vee n \approx y \vee n \text{ and } \neg x \vee n \approx \neg y \vee n) \Rightarrow x \approx y$.

Definition 4.2.10. $\vdash_{\mathrm{LP}^n}$ is the logic defined by the matrix $\langle \mathbf{KL_3}, \{n, b\} \rangle$.

The $\{n\}$-free reduct of $\vdash_{\mathrm{LP}^n}$ is known as "the logic of Paradox" [78] (see Introduction). In [82], it is shown that it consists of the logic obtained by adding the axiom $\varphi \vee \neg\varphi$ to the Hilbert calculus for Belnap-Dunn four-valued logic $\mathcal{B}$. The logic $\vdash_{\mathrm{LP}^n}$ is obtained simply by adding n as an additional axiom. It is immediate to observe that $\vdash_{\mathrm{LP}^n}$ does not posses antitheorems.

Lemma 4.2.11. *Let* $\mathbf{A} \in \mathsf{DMF}$, $P := \{a \in A : n \leq a\}$*. Then the following hold:*

(i) $\langle \mathbf{A}, P \rangle \in \mathsf{Mod}(\vdash_{\mathrm{LP}^n})$*;*

(ii) P *is the smallest* $\vdash_{\mathrm{LP}^n}$*-filter over* $\mathbf{A}$*;*

(iii) *any lattice filter* F *extending* P *is a* $\vdash_{\mathrm{LP}^n}$*-filter over* $\mathbf{A}$.

Proof. (i) Notice that the $\{n\}$-free reduct of $\mathbf{A}$ is a De Morgan Lattice; therefore, whenever $F \subseteq A$ is a lattice filter over $\mathbf{A}$, then $\langle \mathbf{A}, F \rangle$ is a model of $\mathcal{B}$ (see [43] for details). It is immediate to check that P is a lattice filter on $\mathbf{A}$, hence $\langle \mathbf{A}, P \rangle \in \mathsf{Mod}(\mathcal{B})$. Moreover, $n \in P$ and, for every $a \in A$, $n = n \wedge n = n \wedge \neg n \leq a \vee \neg a$, whence $a \vee \neg a \in P$. This shows that $\langle \mathbf{A}, P \rangle \in \mathsf{Mod}(\vdash_{\mathrm{LP}^n})$.

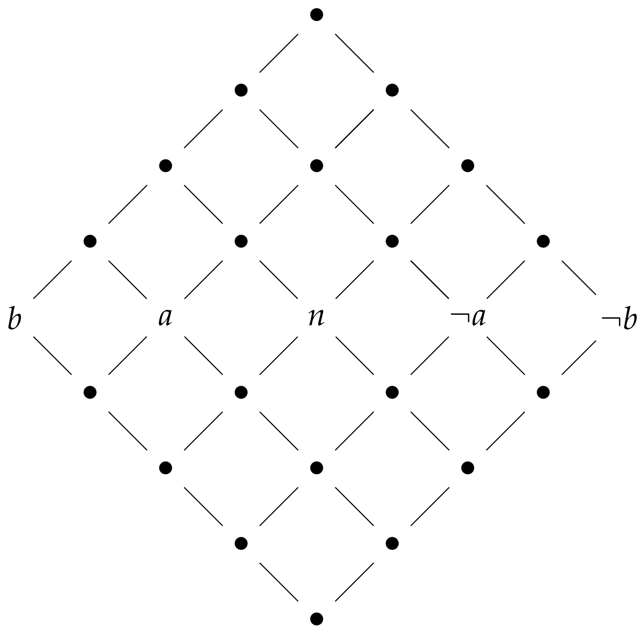
(ii) follows from the fact that n is contained in every $\vdash_{\mathrm{LP}^n}$-filter and that every $\vdash_{\mathrm{LP}^n}$-filter is a lattice filter.

(iii) Let $P \subseteq F$, with F a lattice filter. Then $n \in F$, therefore $a \vee \neg a \in F$, for every $a \in A$. $\boxtimes$

Lemma 4.2.12. *Let* $\mathbf{A} \in \mathsf{DMF}$ *and* $a \in A$. *Then* $Fg^{\mathbf{A}}_{\vdash_{\mathrm{LP}^n}}(a) = \uparrow(a \wedge n) = \{b \in A \colon a \wedge n \leq b\}$.

Proof. It is easily checked that the set $\uparrow(a \wedge n)$ is a lattice filter. Moreover, for any $b \in A$, $a \wedge n \leq n = n \wedge \neg n \leq b \vee \neg b$, therefore $b \vee \neg b \in \uparrow(a \wedge n)$ which shows that it is a $\vdash_{\mathrm{LP}^n}$-filter. To show that $\uparrow(a \wedge n)$ is the smallest filter containing a, assume that $F \subseteq A$ is any $\vdash_{\mathrm{LP}^n}$-filter over $\mathbf{A}$ containing a. Since F is a filter, necessarily $n \in F$, whence $a \wedge n \in F$. Since F is a lattice filter, then $\uparrow(a \wedge n) \subseteq F$. ⊠

An example of DMF



We are now ready to show that the Gentzen calculus $\vdash^{\mathrm{LP}^n}_{\mathcal{G}}$ for $\vdash_{\mathrm{LP}^n}$ is equivalential.

Theorem 4.2.13. $\vdash^{\mathrm{LP}^n}_{\mathcal{G}}$ *is an equivalential Gentzen calculus for* $\vdash_{\mathrm{LP}^n}$ *under the transformer* $\boldsymbol{\rho}(x \approx y) = \{x \rhd\lhd y, \neg x \rhd\lhd \neg y\}$.

Proof. We apply the same strategy of Theorem 4.2.7, showing that $\boldsymbol{\rho}(x, y)$ is a set of congruence formulas, meaning that for $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^*(\vdash^{\mathrm{LP}^n}_{\mathcal{G}})$

$$\langle a, b \rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F \iff \boldsymbol{\rho}(a, b) \in F.$$

($\Rightarrow$). The fact that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F$ implies $a = b$ and so $\boldsymbol{\rho}(a,b) = \boldsymbol{\rho}(a,a) = \{a \lhd \rhd a, \neg a \lhd \rhd \neg a\}$. It is immediate to check that $\vdash_{\mathrm{LP}^n} \boldsymbol{\rho}(x,x)$ so $\boldsymbol{\rho}(a,a) = \boldsymbol{\rho}(a,b) \in F$.

($\Leftarrow$). We reason by contraposition. So let $\langle \mathbf{A}', F'\rangle \in \mathsf{Mod}^*(\vdash_{\mathcal{G}}^{\mathcal{LP}^n})$, $a', b' \in A'$ and $a' \neq b'$. Suppose by contradiction $\boldsymbol{\rho}(a',b') \in F'$. So, by definition of $\vdash_{\mathcal{G}}^{\mathrm{LP}^n}$ there exists $\mathbf{A} \in \mathsf{Alg}(\vdash_{\mathrm{LP}^n}) \subseteq \mathsf{DMF}$, $a, b \in A$ with $a \neq b$ such that $b \in \mathrm{Fg}^{\mathbf{A}}_{\vdash_{\mathrm{LP}^n}}(a)$, $a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash_{\mathrm{LP}^n}}(b)$ and $\neg b \in \mathrm{Fg}^{\mathbf{A}}_{\vdash_{\mathrm{LP}^n}}(\neg a)$, $\neg a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash_{\mathrm{LP}^n}}(\neg b)$.

Applying Lemma 4.2.12, we get that $a \wedge n \leq b$, $b \wedge n \leq a$, $\neg a \wedge n \leq \neg b$ and $\neg b \wedge n \leq \neg a$. Therefore $a \wedge n = b \wedge n$ and $\neg a \wedge n = \neg b \wedge n$, whence

$$a \vee n = a \vee \neg n = \neg(\neg a \wedge n) = \neg(\neg b \wedge n) = b \vee \neg n = b \vee n,$$

and

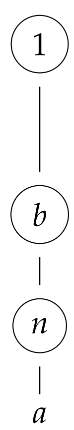
$$\neg a \vee n = \neg a \vee \neg n = \neg(a \wedge n) = \neg(b \wedge n) = \neg b \vee \neg n = \neg b \vee n.$$

Therefore, by Lemma 4.2.9-(iv), $a = b$, a contradiction.

$\boxtimes$

We can now focus on our main example, which highlights the importance of the assumption on an antitheorem in Theorem 4.2.7.

Example 4.2.14. In order to simplify notation, set $\vdash\ = \ \vdash_{\mathrm{LP}^n}$. Consider the two matrices $\langle \mathbf{KL_3}, \{n, b\}\rangle$ and $\langle \mathbf{1}, \{1\}\rangle$, which are Leibniz reduced models of $\vdash$. We apply the same construction used in Section 2.2 to obtain the Płonka sum $\mathbf{A} = \langle \mathbf{KL_3} \oplus \mathbf{1}, \{n, b\} \cup \{1\}\rangle$ (visualized in the following drawing).



$\vdash$ is a logic with partition function (as it has a lattice reduct), hence, by Theorem 2.4.2, $\langle \mathbf{A}, \{n, b, 1\}\rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{l})$, whence $\mathbf{A} \in \mathsf{Alg}(\vdash^{l})$.

We claim that $\vdash^{l}_{\mathcal{G}}$ is not equivalential.

Suppose, towards a contradiction, that there exists a transformer $\boldsymbol{\rho}^{l}(x, y)$ witnessing the equivalentiality of $\vdash^{l}_{\mathcal{G}}$. Let $F \subseteq A^{*} \times A$ be a set representing the operation of generated filter of $\vdash^{l}_{\mathrm{LP}^{n}}$ over $\mathbf{A}$. Consider the 2-valued unary polynomial function $\varphi(x) = \langle x, x \wedge a\rangle$. Observe that $1 \wedge a = 1 \in \mathrm{Fg}^{\mathbf{A}}_{\vdash^{l}}(1)$, and $n \wedge a = a \notin \mathrm{Fg}^{\mathbf{A}}_{\vdash^{l}}(n)$; this last fact is justified as, by Lemma 4.2.12, $a \notin Fg^{\mathbf{KL_3}}_{\vdash}(n)$ and, by Lemma 4.2.4, $Fg^{\mathbf{KL_3}}_{\vdash}(n) = \mathrm{Fg}^{\mathbf{A}}_{\vdash^{l}}(n) \cap KL_3$.

By the previous considerations, we have $\varphi(1) \in F$ and $\varphi(n) \notin F$. By the polynomial characterization of the Leibniz congruence it follows that $\langle n, 1\rangle \notin \boldsymbol{\Omega}^{\mathbf{A}}F$. Since $\boldsymbol{\rho}^{l}(x, y)$ witnesses the equivalentiality of $\vdash^{l}_{\mathcal{G}}$, there exists at least one sequent $\vec{\gamma}(x, y) \rhd \delta(x, y) \in \boldsymbol{\rho}^{l}(x, y)$ such that $\vec{\gamma}(1, n) \rhd \delta(1, n) \notin F$, which means $\delta(1, n) \notin \mathrm{Fg}^{\mathbf{A}}_{\vdash^{l}}(\vec{\gamma}(1, n))$.

It can be checked that, for every term $t(x, y)$, $t^{\mathbf{A}}(1, n) \in \{c, n\}$. Indeed, if the term depends on x then $t^{\mathbf{A}}(1, n) = 1$. Otherwise (if x does not appear in t), $t(x, y)$ depends on y only, and $t^{\mathbf{A}}(1, n) = n$. Moreover we have that $\mathrm{Fg}^{\mathbf{A}}_{\vdash^{l}}(n) = \mathrm{Fg}^{\mathbf{A}}_{\vdash^{l}}(\neg n) = \mathrm{Fg}^{\mathbf{A}}_{\vdash^{l}}(1)$. Remembering that the transformer $\boldsymbol{\rho}^{l}$ is a set of sequents in two variables, we have $\vec{\gamma}(1, n) \in \{1, n\}$, and $\delta(1, n) \in \{1, n\}$ for every sequent in $\boldsymbol{\rho}^{l}(1, n)$. Therefore $\delta(1, n) \in \mathrm{Fg}^{\mathbf{A}}_{\vdash^{l}}(\vec{\gamma}(c, n))$, a contradiction.

⊠

4.3 Sublogics of variable inclusion

In this section we consider how many different sublogics of a given logic $\vdash$ can be obtained by applying the definitions of right and left variable inclusion. In the whole section, unless stated otherwise, we assume that $\vdash$ is a finitary logic, and that it possesses a binary term $\pi(x, y)$ that behaves as an r-partition function for $\vdash^{r}$ and as an l-parition function for $\vdash^{l}$. As already noticed in Chapters 2, 3, this is a very natural assumption, for a huge amount of non ad-hoc examples do have such term. We distinguish the case in which the initial logic $\vdash$ possesses or not an antitheorem $\Sigma(x)$. This condition will deeply affect the results.

Remark 4.3.1. Observe that, according with Theorem 3.2.10 and Corollary 2.2.8, given $\mathsf{M}^{\vdash}$ a complete class of matrices for $\vdash$ containing $\langle \mathbf{1}, 1\rangle$ as only trivial matrix (e.g. $\mathsf{Mod}^{*}(\vdash)$), it is always possible to obtain a complete class of non trivial matrices M for $\vdash^{l}$, and a complete class of matrices $\mathsf{M}^\star$ for $\vdash^r$ containing $\langle \mathbf{1}, 1\rangle$ as only trivial matrix. Moreover, by applying again Theorem 3.2.10 and Corollary 2.2.8 to M and $\mathsf{M}^\star$ we have that $\mathcal{P}_{\nmid}(\mathsf{M} \cup \langle \mathbf{1}, \emptyset\rangle)$ is complete for $\vdash^{lr}$ while $\mathcal{P}_{\nmid}(\mathsf{M}^\star)$ is complete for $\vdash^{rl}$.

In what follows, we write $\bullet$ to denote any (possibly empty) sequence of elements among $\{l, r\}$. So, $\vdash^\bullet$ will denote an arbitrary logic obtained by replacing $\bullet$ with a sequence of elements among $\{l, r\}$. We denote the length of a sequence $\bullet$ as $L(\bullet)$.

The reading of a sequence $\bullet$ is from left to right. So, if $\bullet = u_1 \ldots u_n$ with ($u_i \in \{l, r\}$ for $1 \leq i \leq n$) the logic $\vdash^\bullet$ is the logic obtained by applying the definition of u_m to the logic $\vdash^{u_1 \ldots u_{m-1}}$ for every $1 \leq m \leq n$.

Example 4.3.2. Let $\vdash$ be Classical Logic. Then

(i) $\vdash^l = \mathsf{PWK}$

(ii) $\vdash^r = \mathsf{B}_3$

(iii) $\vdash^{lr} = \mathsf{K}^{\mathsf{w}}_{4\mathsf{n}}$ (see Example 3.2.3)

An immediate consequence of Remark 4.3.1 is that $\vdash^{\bullet l} \geqslant \vdash^{\bullet lr}$ and $\vdash^{\bullet r} \geqslant \vdash^{\bullet rl}$. This fact will be useful for the next theorems.

4.3.1 Logics without antitheorems

We start our investigation assuming that $\vdash$ does not have antitheorems.

Lemma 4.3.3. *Let $\vdash$ be a logic without antitheorems. If $\Gamma \vdash^{\bullet rl \bullet'} \varphi$ then there exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$ and $Var(\Delta) = Var(\varphi)$.*

Proof. By induction on the length of $\bullet'$.
(B). If $L(\bullet') = 0$ the proof is immediate, so it remains to consider $L(\bullet') = 1$. There are cases: (a) $\bullet' = l$ or (b): $\bullet' = r$. if (a) then $\Gamma \vdash^{\bullet rll} \varphi$ implies $\Gamma \vdash^{\bullet rl} \varphi$, so there exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash^{\bullet r} \varphi$ and $Var(\Delta) \subseteq Var(\varphi)$. This implies $\Delta \vdash^{\bullet} \varphi$ and $Var(\varphi) \subseteq Var(\Delta)$. Now, suppose $\Delta \nvdash \varphi$. This implies $\Delta \nvdash^l \varphi$ and $\Delta \nvdash^r \varphi$, which is in contradiction with the fact that $\Delta \vdash^\bullet \varphi$. So $\Delta \vdash \varphi$.The case of (b) is analogous.

(IND). Suppose the statement holds for $L(\bullet') = n$ and consider $L(\bullet') = n + 1$. That is, $\bullet'$ can be of the following forms: (a) $\bullet' = s \cup \{l\}$ with $L(s) = n$, or (b) $\bullet' = s \cup \{r\}$ with $L(s) = n$. In the case of (a), as $\Gamma \vdash^{\bullet rl \bullet'} \varphi$ we have that there exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash^{\bullet rls} \varphi$ and $Var(\Delta) \subseteq Var(\varphi)$. As $L(s) = n$, by inductive hypothesis there exists $\Sigma \subseteq \Delta$ such that $\Sigma \vdash \varphi$

and $Var(\Sigma) = Var(\varphi)$. Observing that $\Sigma \subseteq \Delta \subseteq \Gamma$ we obtain our conclusion. The case for (b) can be proved with the same strategy.

$\boxtimes$

Corollary 4.3.4. *Let $\vdash$ be a logic without antitheorems. Then $\vdash^{rl} \leq \vdash^{\bullet}$.*

Remark 4.3.5. Observe that every logic $\vdash^{\bullet}$ such that $l \in \bullet$ does not have antitheorems. Indeed, let $\vdash$ be a logic and suppose $\Sigma(x)$ is an antitheorem for $\vdash^{l}$. Let X be a direct system of matrices such that

(i) $I = \{i, j\}$ with $i \leq j$

(ii) $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}(\vdash)$ be non trivial

(iii) $\langle \mathbf{A}_j, F_j \rangle$ such that $\mathbf{A}_j = \mathbf{1}, F_j = 1$

(iv) $f_{ij} : \mathbf{A}_i \to \mathbf{A}_j$ be the unique homomorphism

Then by Corollary 2.2.8 $\mathcal{P}_{\nmid}(X) = \langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\vdash^{l})$. The fact that $\Sigma(x)$ is an antitheorem for $\vdash^{l}$ implies $\Sigma(x) \vdash^{l} y$ for $y \in Var$. Let now $h : \mathbf{Fm} \to \mathcal{P}_{\nmid}(\mathbf{A}_i)_{i \in I}$ be such that $h(x) = 1, h(y) = c$ with $c \in A_i \smallsetminus F_i$ (note that such c exists as $A_i \neq F_i$). Then clearly $h(\Sigma(x)) \subseteq F$, while $h(y) \notin F$, a contradiction.

The following theorem characterizes the relation among the sublogics of varibale inclusion of an antitheorem-free logic $\vdash$.

Theorem 4.3.6. *Let $\vdash \neq \vdash^{r}, \vdash^{l}$ be a logic without antitheorems. The following relations hold:*

(i) $\vdash^{l} \not\leq \vdash^{r}$ *and* $\vdash^{r} \not\leq \vdash^{l}$

(ii) $\vdash^{l} \cap \vdash^{r} = \vdash^{lr} \lneq \vdash^{l}, \vdash^{r}$

(iii) $\vdash^{rl} = \vdash^{rl\bullet} = \vdash^{lrl\bullet}$

Proof. (i) it immediately follows by noticing that $\pi(x, y) \vdash^{r} x$ while $\pi(x, y) \nvdash^{l} x$ and $x \vdash^{l} \pi(x, y)$ while $x \nvdash^{r} \pi(x, y)$.

(ii) As a direct consequence of Remark 4.3.1 we have $\vdash^{lr} \leq \vdash^{l}$. We now prove using contraposition that $\vdash^{lr} \leq \vdash^{r}$. So assume $\Gamma \nvdash^{r} \varphi$. There are cases, namely (1) $\Gamma \nvdash \varphi$ or (2) $Var(\varphi) \nsubseteq Var(\Gamma)$. (1) immediately implies $\Gamma \nvdash^{l} \varphi$, so $\Gamma \nvdash^{lr} \varphi$. If it is case of (2), assume towards a contradiction that $\Gamma \vdash^{lr} \varphi$. This entails that $\Gamma \vdash^{l} \varphi$ and that $Var(\varphi) \subseteq Var(\Gamma)$, which is a contradiction. So $\Gamma \nvdash^{lr} \varphi$.

Now, $\vdash^l \cap \vdash^r \leq \vdash^{lr}$ follows by noticing that in the lattice of sublogics of $\vdash$ it holds $\vdash^l \wedge \vdash^r = \vdash^l \cap \vdash^r$, and so, as $\vdash^{lr} \leq \vdash^r, \vdash^l$ it follows $\vdash^l \cap \vdash^r \leq \vdash^{lr}$. For the other direction, assume $\Gamma \vdash^{lr} \varphi$. This entails $\Gamma \vdash^l \varphi$ with $Var(\varphi) \subseteq Var(\Gamma)$. Furthermore, as $\vdash^l \leq \vdash$, we have $\Gamma \vdash \varphi$ which finally entails $\Gamma \vdash^r \varphi$.

Moreover, the fact that $\pi(x,y) \vdash^r x$ while $\pi(x,y) \nvdash^{lr} x$ and $x \vdash^l \pi(x,y)$ while $x \nvdash^{lr} \pi(x,y)$ proves the desired proper inequality.

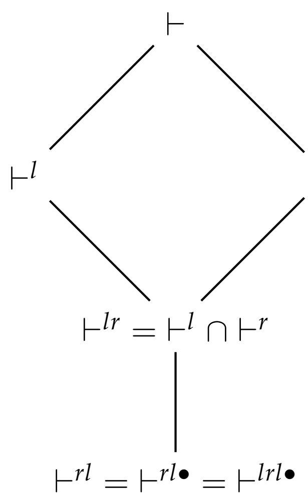
(iii) That $\vdash^{rl\bullet} \leq \vdash^{rl}$ follows again by remark 4.3.1. That $\vdash^{rl} \leq \vdash^{rl\bullet}$ follows immediately from Corollary 4.3.4. Now we prove $\vdash^{lrl\bullet} \leq \vdash^{rl}$. To this end, assume $\Gamma \vdash^{lrl\bullet} \varphi$. By Lemma 4.3.3 we have that there exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$ and $Var(\Delta) = Var(\varphi)$. So, it follows $\Delta \vdash^{rl} \varphi$ and, by monotonicity, we obtain $\Gamma \vdash^{rl} \varphi$. The fact that $\vdash^{rl} \leq \vdash^{lrl\bullet}$ is a consequence of Corollary 4.3.4. ⊠

Remark 4.3.7. Observe that if a logic $\vdash$ has a theorem φ, then $\vdash^{rl} \lneq \vdash^{lr}$. Indeed it is immediate to verify that $\pi(x,y) \vdash^{lr} \varphi(x)$ while $\pi(x,y) \nvdash^{rl} \varphi(x)$.

Corollary 4.3.8. *Let $\vdash$ be a logic with a partition function and without antitheorems. Then, the lattice of sublogics of variable inclusion of $\vdash$ has at most 5 elements.*

As result of the previous Theorem 4.3.6, the following Figure 4.3.1 represents the lattice of sublogics of variable inclusion of an arbitrary logic $\vdash$ that does not have antitheorems:

Figure 4.3.1



4.3.2 Logics with antitheorems

We now turn to the case in which the logic $\vdash$ does posses an antitheorem $\Sigma(x)$. In the next Theorem 4.3.9 we assume w.l.o.g. $\Sigma(x) = \{\epsilon_1(x), \ldots, \epsilon_n(x)\}$.

Theorem 4.3.9. *Let $\vdash$ be a logic with antitheorems. Then the following relations hold*

(i) $\vdash^{rl} \nleq \vdash^{lr}$ *and* $\vdash^{lr} \nleq \vdash^{rl}$

(ii) $\vdash^{l} \cap \vdash^{r} \gneq \vdash^{lr}, \vdash^{rl}$

(iii) $\vdash^{rlr} \lneq \vdash^{rl}$ *and* $\vdash^{lrl} \lneq \vdash^{lr}$

(iv) $\vdash^{rlr} \lneq \vdash^{lr} \cap \vdash^{rl}$

(v) $\vdash^{lrl} = \vdash^{lrlr} = \vdash^{rlrl} \lneq \vdash^{rlr}$.

(vi) $\vdash^{rlrl\bullet} = \vdash^{lrl\bullet}$

where $\bullet$ denotes any (possibly empty) sequence of elements among $\{l, r\}$.

Proof. (i). Firstly we show $\vdash^{rl} \nleq \vdash^{lr}$. To this end it is immediate to verify that $\Sigma(x) \vdash^{rl} \pi(x,y)$ while $\Sigma(x) \nvdash^{lr} \pi(x,y)$.

For the other inequality, first observe that

$$Var(\pi(y,z)) \subseteq Var(y, \pi(\epsilon_1(x), z), \ldots, \pi(\epsilon_n(x), z))$$

and, moreover

$$y, \pi(\epsilon_1(x), z), \ldots, \pi(\epsilon_n(x), z) \vdash^{l} \pi(y,z),$$

as $y \vdash^{l} \pi(y,z)$ and $\{y\} \subseteq \{y, \pi(\epsilon_1(x), z), \ldots, \pi(\epsilon_n(x), z)\}$. So, this proves

$$y, \pi(\epsilon_1(x), z), \ldots, \pi(\epsilon_n(x), z) \vdash^{lr} \pi(y,z).$$

This, together with the fact that for no $\Delta \subseteq \{y, \pi(\epsilon_1(x), z), \ldots, \pi(\epsilon_n(x), z)\}$ it holds $\Delta \vdash^{r} \pi(y,z)$ and $Var(\Delta) \subseteq \{y, z\}$ shows

$$y, \pi(\epsilon_1(x), z), \ldots, \pi(\epsilon_n(x), z) \nvdash^{rl} \pi(y,z),$$

as desired.

(ii). We first prove $\vdash^{l} \cap \vdash^{r} \geqslant \vdash^{lr}, \vdash^{rl}$. Let $\Gamma \vdash^{lr} \varphi$, then, as $\vdash^{l}$ does not have antitheorems, it must be that $\Gamma \vdash^{l} \varphi$ and $Var(\varphi) \subseteq Var(\Gamma)$. This, together with $\vdash^{l} \leq \vdash$ entails $\Gamma \vdash \varphi$, so $\Gamma \vdash^{r} \varphi$. So, $\Gamma \vdash^{l} \cap \vdash^{r} \varphi$. That $\vdash^{l} \cap \vdash^{r} \geqslant \vdash^{rl}$ is proved in the same way.

As the inferences described in point (i) hold both in $\vdash^l$ and $\vdash^r$, we obtain $\vdash^{lr}, \vdash^{rl} \lneq \vdash^l \cap \vdash^r$.

(iii). The fact that $\vdash^{rlr} \leq \vdash^{rl}$ and $\vdash^{lrl} \leq \vdash^{lr}$ is a direct consequence of Remark 4.3.1.

This, together with the fact that

$$y, \pi(\epsilon_1(x), z), \ldots, \pi(\epsilon_n(x), z) \nvdash^{lrl} \pi(y, z)$$

and $\Sigma(x) \nvdash^{rlr} \pi(x, y)$ proves the desired proper inequalities.

(iv). We first prove $\vdash^{rlr} \leq \vdash^{lr} \cap \vdash^{rl}$. That $\vdash^{rlr} \leq \vdash^{rl}$ follows, again by Remark 4.3.1. Consider $\Gamma \vdash^{rlr} \varphi$, so, as $\vdash^{rl}$ does not have antitheorems, $\Gamma \vdash^{rl} \varphi$ with $Var(\varphi) \subseteq Var(\Gamma)$. This entail that there exists $\Delta \subseteq \Gamma$, $\Delta \vdash^r \varphi$ and $Var(\Delta) \subseteq Var(\varphi)$. As, $\vdash^r \leq \vdash$ we obtain $\Delta \vdash \varphi$, so $\Delta \vdash^l \varphi$ which, by monotonicity entails $\Gamma \vdash^l \varphi$. Recalling that $Var(\varphi) \subseteq Var(\Gamma)$ we conclude $\Gamma \vdash^{lr} \varphi$.

The proper inclusion is proved by noticing that $\Sigma(x) \vdash^{lr} \pi(x, y)$, $\Sigma(x) \vdash^{rl} \pi(x, y)$ while $\Sigma(x) \nvdash^{rlr} \pi(x, y)$.

(v). As by remark 4.3.5 $\vdash^l, \vdash^{lr}, \vdash^{rl}$ are logics without antitheorems, then by Lemma 4.3.3 we know that $\Gamma \vdash^{lrl} \varphi$ entails that there exists $\Delta \subseteq \Gamma$, $\Delta \vdash \varphi$ and $Var(\varphi) = Var(\Delta)$ (the same holds for $\vdash^{lrlr}$ and $\vdash^{rlrl}$). As this immediately implies $\Delta \vdash^{lrlr} \varphi$ and $\Delta \vdash^{rlrl} \varphi$, by monotonicity we conclude $\Gamma \vdash^{lrlr} \varphi$ and $\Gamma \vdash^{rlrl} \varphi$, so $\vdash^{lrlr} = \vdash^{rlrl} = \vdash^{lrl}$.

It only remains to prove that $\vdash^{rlrl} \lneq \vdash^{rlr}$. To this end, it suffices to note that $\pi(y, z), \Sigma(x) \vdash^{rlr} \pi(y, x)$ while $\pi(y, z), \Sigma(x) \nvdash^{rlrl} \pi(y, x)$.

(vi). The equality $\vdash^{rlrl\bullet} = \vdash^{lrl\bullet}$ is a straightforward application of Lemma 4.3.3, using the same strategy of point (v). ⊠

Corollary 4.3.10. *Let $\vdash$ be a logic with a partition function and antitheorems. Then, the lattice of sublogics of variable inclusion of $\vdash$ has exactly 8 elements.*

Observe that we do not consider the logic $\vdash^{rl} \vee \vdash^{lr}$ among the family of logics of variable inclusion. As already noticed, a logic without antitheorems has a lattice of only 4 proper sublogics.

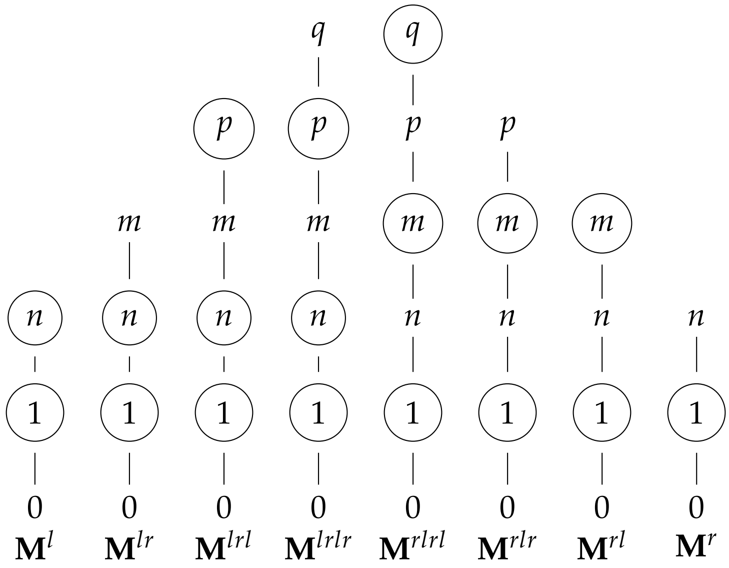
The next Figure 4.3.2 describes the lattice of sublogics of variable inclusion of a logic $\vdash$ with antitheorems.

Figure 4.3.2

$$
\begin{tikzcd}
 & \vdash \arrow[dl, dash] \arrow[dr, dash] & \\
\vdash^{l} \arrow[dr, dash] & & \vdash^{r} \arrow[dl, dash] \\
 & \vdash^{r} \cap \vdash^{l} \arrow[d, dash] & \\
 & \vdash^{rl} \vee \vdash^{lr} \arrow[dl, dash] \arrow[dr, dash] & \\
\vdash^{lr} \arrow[dr, dash] & & \vdash^{rl} \arrow[dl, dash] \\
 & \vdash^{rl} \cap \vdash^{lr} \arrow[d, dash] & \\
 & \vdash^{rlr} \arrow[d, dash] & \\
 & \vdash^{rlrl\bullet} = \vdash^{lrl} = \vdash^{lrl\bullet} &
\end{tikzcd}
$$

We now apply the results. The following example shows how the situation works for our favorite logic, namely CL.

Example 4.3.11. Let $\vdash =$ CL, defined by the matrix $\langle \mathbf{B}_2, 1 \rangle$ where $\mathbf{B}_2$ is the two-elements Boolean algebra. We now draw the diagram of the characteristic matrices for $\vdash^{l}, \vdash^{r}, \vdash^{lr}, \vdash^{rl}, \vdash^{lrl}, \vdash^{rlr}$.



Letting $\pi(x, y) = x \wedge (x \vee y)$ it is not difficult to apply Theorem 4.3.9, and to observe that there are 8 (proper) sublogics of variabale inclusion of CL. In particular, the 8 different logics are $\vdash^{l} = \mathsf{PWK}, \vdash^{r} = \mathsf{B}_3, \vdash^{l \cap r}, \vdash^{lr}, \vdash^{rl}, \vdash^{rl \cap lr}, \vdash^{rlr}, \vdash^{lrl}$. Notice also that the matrices $\mathbf{M}^{lrl}, \mathbf{M}^{rlrlr}, \mathbf{M}^{lrlr}$ define the same logic.

Chapter 5

The Logic of Demodalised Analytic Implication

5.1 Introduction

Containment logics (see Chapter 3) are a family of logics based on the idea that a necessary condition for an argument to be valid is that its conclusion be "analytically contained" in its premisses. This idea can be specified to a single operation, by saying that a necessary condition for an implication to be valid is that its consequent be "analytically contained" in its antecedent. What determines the right notion of analytic containment depends on the particular logic at issue. For propositional logics, it usually amounts to an inclusion constraint among the sets of propositional variables occurring in the sentences under consideration. In this chapter, we deal with a logic $\vdash$ with a binary connective $\rightarrow$ in their language that satisfy what Ferguson [38] calls the *proscriptive principle for theorems* ($\mathbf{PP}^{\rightarrow}$). $\mathbf{PP}^{\rightarrow}$ requires that if $\vdash \varphi \rightarrow \psi$, then $Var(\psi) \subseteq Var(\varphi)$. There can be many ways $\mathbf{PP}^{\rightarrow}$ can be justified, for instance as a relevance constraint — indeed, a tighter one than the usual variable-sharing requirement of relevance logics.

This idea slightly differs from the motivation behind logics of variable inclusion *tout court*. Here, indeed, the requirement of variable inclusion does not directly involve the consequence relation $\vdash$, but only a specific operation in the expanded language.

Historically, the first logical system obeying $\mathbf{PP}^{\rightarrow}$ was PAI, the logic of analytic implication introduced in the early 1930's [69], and modified in later writings [70], by C.I. Lewis' student W.T. Parry. Although Parry's approach never became mainstream within relevance logics, it

drew the attention of notable scholars over the following decades (see e.g. [41, 55, 96, 34]). In particular, Kit Fine's analysis determined that PAI is ultimately obtained by imposing a "linguistic strainer" on the modal logic S5. Observe, indeed, that although the language of PAI is the same as for classical logic, a necessity connective can be introduced as usual in relevance logics via $\Box\varphi = (\varphi \to \varphi) \to \varphi$. For further information on PAI and its history, see the comprehensive monograph [38].
In 1972, J. Michael Dunn [34] explored what happens if we "demodalise" PAI by adding to it the axiom of collapse of modality $\varphi \to \Box\varphi$. The resulting logic DAI of *demodalised analytic implication* turns out to be much smoother, and to have more interesting formal properties, than Parry's original system. As a relevance logic, DAI is in the same ballpark as Lewis' systems of strict implication, to the extent that all these logics are proper expansions of Classical Logic.

Later on, DAI was independently rediscovered by R.D. Epstein under the heading of *dependence logic* [36]. From the proof-theoretic viewpoint, DAI has been endowed with Hilbert-style calculi [34, 36], tableaux calculi [22], and sequent calculi [31]. Model-theoretically, it has been analysed both with the standard methods of possible-world semantics [34] and via more unusual semantics especially tailored by Epstein for the needs of containment logics and their neighbours (see [36]).

The chapter is structured as follows. Section 5.2 presents two complete matrix semantics for DAI, based on Płonka sums and particular "twist products" of models of Classical Logic respectively. Section 5.3 characterizes the Leibniz reduced models of DAI that can be represented in terms of Płonka sums, while Section 5.4 classifies DAI in the Leibniz hierarchy. On the contrary on variable inclusion logics, DAI is an algebraizable logic, and the last part of the chapter is devoted to the presentation of the quasivariety that plays the role of equivalent algebraic semantics.

5.2 Semantics for DAI

5.2.1 Demodalised analytic implication

Demodalised analytic implication DAI is semantically defined in [35] by means of the so called *dependence models*.

Definition 5.2.1. (Essentially [35, p.20]). A *dependence model* is a triple $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S} \rangle$ composed by a non-empty set $\mathfrak{S}$, and maps $\mathfrak{v} : Var \to \{0, 1\}$, $\mathfrak{s} : Var \cup \{0, 1\} \to \mathcal{P}(\mathfrak{S})$ extended to formulas according to the following:

(D0) $\mathfrak{s}(0) = \mathfrak{s}(1) = \emptyset$

(D1) $\mathfrak{s}(\varphi) = \bigcup\{\mathfrak{s}(x) : x \in Var(\varphi)\}$

(D2) $\mathfrak{v}(\neg\varphi) = 1$ if and only if $\mathfrak{v}(\varphi) = 0$

(D3) $\mathfrak{v}(\varphi \wedge \psi) = 1$ if and only if $\mathfrak{v}(\varphi) = \mathfrak{v}(\psi) = 1$

(D4) $\mathfrak{v}(\varphi \rightarrow \psi) = 1$ if and only if $\mathfrak{s}(\psi) \subseteq \mathfrak{s}(\varphi)$ and not both $\mathfrak{v}(\varphi) = 1$ and $\mathfrak{v}(\psi) = 0$

(D5) $\mathfrak{v}(0) = 0$ and $\mathfrak{v}(1) = 1$.

We define $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S} \rangle \vDash \varphi$ to mean $\mathfrak{v}(\varphi) = 1$.

Definition 5.2.2. ([35, p.20]) The logic DAI is defined as follows. $\Gamma \vdash_{\mathsf{DAI}} \varphi$ iff for every model$\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S} \rangle$

$$\text{if} \langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S} \rangle \vDash \gamma \text{ for every } \gamma \in \Gamma, \text{ then } \langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S} \rangle \vDash \varphi.$$

In the whole chapter, two distinguished similarity types will play a crucial role in what follows. The former one is the $\langle 2, 2, 1, 0, 0 \rangle$-type of Classical Logic and Boolean algebras ($\mathcal{BA}$), with primitive operation symbols $\wedge, \vee, \neg, 0$, and 1; this type will be denoted by $\mathcal{L}_0$. The latter one, hereafter referred to as $\mathcal{L}_1$, is the expansion of $\mathcal{L}_0$ by an additional binary operation symbol $\rightarrow$.

Definition 5.2.3. An *implicative involutive bisemilattice* is an algebra

$$\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \neg, 0, 1 \rangle$$

of type $\mathcal{L}_1$ that satisfies the following conditions:

1. The reduct $\mathbf{A}^- = \langle A, \wedge, \vee, \neg, 0, 1 \rangle$ is an involutive bisemilattice (see Definition 1.3.5).
2. $\mathbf{A}$ satisfies the following quasiequations:

   (I1) $x \approx \neg x \Rightarrow y \approx z$;
   (I2) $x \wedge (x \vee y) \approx x \Rightarrow x \rightarrow y \approx \neg x \vee y$;
   (I3) $x \rightarrow y \approx (x \rightarrow y) \vee 1 \Rightarrow x \wedge (x \vee y) \approx x$.

The quasivariety of implicative involutive bisemilattices will be denoted by $\mathcal{IIBSL}$.

Given $\mathbf{A} \in \mathcal{IIBSL}$ we denote by $\mathbf{A}^-$ its $\to$-free reduct.

Definition 5.2.4. We define the class $\mathcal{PIBSL}$ as the algebras of type $\mathbf{A} = \langle A, \wedge, \vee, \neg, \to 0, 1\rangle$ such that $\mathbf{A}^- \simeq \mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$, and the binary operation $\to$ is defined for every $x, y \in A$ as

$$x \to y := \begin{cases} \neg x \vee y \text{ if } x \wedge (x \vee y) \approx x \\ (x \wedge \neg x) \wedge (y \wedge \neg y) \text{ otherwise.} \end{cases}$$

Given $\mathbf{A} \in \mathcal{IBSL}$, we write $\mathbf{A}^{\to}$ for the $\mathcal{PIBSL}$ obtained by adding the operation $\to$ of Definition 5.2.4 to $\mathbf{A}$.

Remark 5.2.5. Observe that, whenever $\mathbf{A} \in \mathcal{IIBSL}$, the arrow-free reduct $\mathbf{A}^-$ of $\mathbf{A}$ is an involutive bisemilattice whose Płonka sum representation either contains no trivial Boolean fibres, or, by (I1), is the trivial algebra. In particular, neither **WK** nor any nontrivial semilattice with zero is the $\mathcal{L}_0$-reduct of any member of $\mathcal{IIBSL}$.

5.2.2 Matrix semantics based on Płonka sums

We now extend the notion of direct system and of Płonka sum to logical matrices.

Definition 5.2.6. A *d-direct system* of matrices consists in

(i) A semilattice with bottom $I = \langle I, \vee\rangle$

(ii) A family of matrices $\{\langle \mathbf{A}_i, F_i\rangle_{i\in I} : \mathbf{A}_i \in \mathcal{BA}$ and F_i is a lattice filter$\}$

(iii) a homomorphism $f_{ij}\colon \mathbf{A}_i \to \mathbf{A}_j$, for every $i, j \in I$ such that $i \leq j$, satisfying also that:

- f_{ii} is the identity map for every $i \in I$;
- if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$;
- $f_{ij}^{-1}[F_j] = F_i$.

Given a d-directed system of matrices X, we define a new matrix of type $\mathcal{L}_0$ as

$$\mathcal{P}_{\text{ł}}(X) := \langle \mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}, \bigcup_{i\in I} F_i\rangle.$$

We will refer to the matrix $\mathcal{P}_{\text{ł}}(X)$ as the *Płonka sum* over the d-direct system of matrices X. Given a class M of matrices (i.e. matrices formed by a Boolean algebra and a lattice filter), $\mathcal{P}_ł(\mathsf{M})$ will denote the class of all Płonka sums of d-directed systems of matrices in M.

Given a Płonka sum $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_ł(X)$ over a d-direct system of matrixes X, we set $\mathcal{P}_ł^{\rightarrow}(X) = \langle \mathbf{A}^{\rightarrow}, F\rangle$. Similarly, we denote by $\mathcal{P}_ł(\mathbf{A}_i)^{\rightarrow}_{i\in I}$ the $\mathcal{PIBSL}$ whose $\rightarrow$-free reduct is $\mathcal{P}_ł(\mathbf{A}_i)_{i\in I}$. It is worth noticing that, if $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_ł^{\rightarrow}(X)$, then $\mathbf{A} \in \mathcal{PIBSL}$. The interpretation of the constant symbols satisfies the condition stated in Remark 1.3.2. That is, given a Płonka sum $\mathcal{P}_ł(\mathbf{A}_i)_{i\in I}$, we have $0^{\mathcal{P}_ł} = 0^{\mathbf{A}_\perp}, 1^{\mathcal{P}_ł} = 1^{\perp}$, where $\perp$ is bottom element of the semilattice I.

Remark 5.2.7. Observe that, in general, $a \rightarrow^{\mathcal{P}_ł^{\rightarrow}} b \neq f_{i_a k}(a) \rightarrow^{\mathbf{A}_k} f_{i_b k}(b)$ where $k = i_a \vee i_b$.

Lemma 5.2.8. *Let $\mathcal{P}_ł(X) = \langle \mathbf{A}, F\rangle$ be a Płonka sum over a d-direct system of matrices. Then $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\mathrm{CL})$.*

Proof. Consider $\mathcal{P}_ł(X) = \langle \mathbf{A}, F\rangle$ and let $\Gamma \vdash_{\mathrm{CL}} \varphi$. Let also $h : \mathbf{Fm} \to \mathbf{A}$ be an evaluation such that $h(\Gamma) \subseteq F$. Suppose, towards a contradiction that $h(\varphi) \notin F$. As CL is finitary, w.l.o.g. we can take $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ and compute $k = i_h(\gamma_1) \vee \dots \vee i_h(\gamma_n)$ and $i = i_h(\varphi)$. Observe that, as X is a d-direct system,

$$h(\gamma_1) \in F_{i_h(\gamma_1)}, \dots, h(\gamma_n) \in F_{i_h(\gamma_n)}$$

implies

$$f_{i_h(\gamma_1)k}(h(\gamma_1)) \in F_k, \dots, f_{i_h(\gamma_n)k}(h(\gamma_n)) \in F_k.$$

Now, fixing $j = i \vee k$, clearly we have $f_{kj}(h(\Gamma)) \subseteq (F_j)$. Moreover, as $h(\varphi) \notin F_i$, $f_{ij}(h(\varphi)) \in A_j \smallsetminus F_j$. Define now a valuation $v : \mathbf{Fm} \to \mathbf{A}_i$ as

$$v(x) := \begin{cases} f_{i_h(x)i} \circ h(x) \text{ if } x \in Var(\Gamma \cup \varphi) \\ a \in A_i \text{ otherwise.} \end{cases}$$

Then clearly $v(\Gamma) \subseteq F_i$ while $v(\varphi) \in A_i \smallsetminus F_i$, against the fact that $\Gamma \vdash^{\mathrm{CL}} \varphi$ and $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}(\mathrm{CL})$. This is a contradiction, so $h(\varphi) \in F$. ⊠

Lemma 5.2.9. *If $\mathcal{P}_ł(X) = \langle \mathbf{A}, F\rangle$ is non trivial, then so is $\langle \mathbf{A}_i, F_i\rangle$ for each $i \in I$.*

Proof. We reason by contraposition. So, suppose there is a trivial fiber $\langle \mathbf{A}_i, F_i\rangle$. We show that any arbitrary fiber $\langle \mathbf{A}_k, F_k\rangle$ is trivial as well, i.e. $\langle \mathbf{A}, F\rangle$ is trivial. Fix $j = i \vee k$. Since $A_i = F_i$ for any $v : \mathbf{Fm}\,(\mathcal{L}_0) \to \mathbf{A}_i$ we have that $v(x \wedge \neg x) \in F_i$ and so $f_{ij}(v(x \wedge \neg x)) \in F_j$. The fact that

$x \wedge \neg x \vdash_{CL} y$ for every $y \in Var(\mathcal{L}_1)$, together with $\langle \mathbf{A}_j, F_j \rangle \in \mathsf{Mod}(\mathrm{CL})$, entails that $A_j = F_j$. Indeed, if there were $c \in A_j \smallsetminus F_j$, we would define $h : \mathbf{Fm}(\mathcal{L}_0) \to \mathbf{A}_j$ such that $h(x) = f_{ij} \circ v(x)$ and $h(y) = c$, whence $h(x \wedge \neg x) \in F_j, h(y) \notin F_j$. This proves that if $A_i = F_i$ then $A_j = F_j$ for each $i \leq j$. Moreover, as by Definition 5.2.6 $f_{kj}^{-1}[F_j] = F_k$ we obtain $A_k = F_k$. This proves that each fiber is trivial. ⊠

We now define a particular way of obtaining a Płonka sum over a d-direct system matrices out of an arbitrary dependence model $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S} \rangle$.

Lemma 5.2.10. *Let $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S} \rangle$ be a dependence model. Then there exists a d-direct system of matrices $X = \langle \mathcal{P}(\mathfrak{S}), \{\langle \mathbf{A}_i, F_i \rangle\}_{i \in \mathcal{P}(\mathfrak{S})}, f_{ij}(i \leq j)\}$, a matrix $\mathcal{P}_{\nmid}^{\to}(X)$ and a homomorphisms $h : \mathbf{Fm} \to \mathcal{P}_{\nmid}^{\to}(\mathbf{A}_i)_{i \in \mathcal{P}(\mathfrak{S})}$ such that*

$$h(x) := \begin{cases} 1_{\mathfrak{s}(x)} \text{ if } \mathfrak{v}(x) = 1 \\ 0_{\mathfrak{s}(x)} \text{ if } \mathfrak{v}(x) = 0 \end{cases}$$

Proof. Let $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S} \rangle$ be a dependence model. Clearly, given a set $\mathfrak{S}$, its powerset $\mathcal{P}(\mathfrak{S})$ can naturally be equipped with a 0-semilattice structure $\langle \mathcal{P}(\mathfrak{S}), \cup, \emptyset \rangle$. Now, for every element $i, j \in \mathcal{P}(\mathfrak{S})$ with $i \leq j$ it is possible to associate two matrices $\langle \mathbf{A}_i, F_i \rangle, \langle \mathbf{A}_j, F_j \rangle \in \mathsf{Mod}(\mathrm{CL})$ with $\mathbf{A}_i, \mathbf{A}_j \in \mathcal{BA}$ (w.l.o.g for each $i \in I$ assume $A_i \neq F_i$), and a homomorphism $f_{ij} : \mathbf{A}_i \to \mathbf{A}_j$ satisfying the conditions in Definition 5.2.6 (a possible choice is to set $F_i = 1_i$ for every $i \in \mathcal{P}(\mathfrak{S})$). The fact that h defined as

$$h(x) := \begin{cases} 1_{\mathfrak{s}(x)} \text{ if } \mathfrak{v}(x) = 1 \\ 0_{\mathfrak{s}(x)} \text{ if } \mathfrak{v}(x) = 0 \end{cases}$$

is a well-defined homomorphism from $\mathbf{Fm}$ to $\mathcal{P}_{\nmid}^{\to}(\mathbf{A}_i)_{i \in \mathcal{P}(\mathfrak{S})}$ is immediate. ⊠

Lemma 5.2.11. *Let X be a d-direct system, $\langle \mathbf{A}, F \rangle \cong \mathcal{P}_{\nmid}^{\to}(X)$, and $h : \mathbf{Fm} \to \mathcal{P}_{\nmid}^{\to}(\mathbf{A}_i)_{i \in I}$ a homomorphism. Then, setting:*

(i) $I = \mathfrak{S}$

(ii) $\mathfrak{s}(x) = \downarrow i_h(x)$

(iii) $\mathfrak{v} : Fm \to \{0, 1\}$ *defined as*

$$\mathfrak{v}(x) := \begin{cases} 1 \text{ if } h(x) \in F \\ 0 \text{ otherwise} \end{cases}$$

$\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S} \rangle$ is a dependence model.

Proof. Let $\langle \mathbf{A}, F\rangle$ as in the statement. We verify that $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S}\rangle$ satisfies the conditions of Definition 5.2.1. This is immediate for (i), while that (ii) respects condition (D0)-(D1) follows from the fact that h is a homomorphism. We now prove that $\mathfrak{v} : Fm \to \{0,1\}$ respects conditions (D2)-(D5). We do the cases of (D3) and of (D4), as (D2) is similar and (D5) is immediate.

(D3). Observe first that, as $\mathcal{P}_{\natural}(X) \in \mathsf{Mod}(\mathrm{CL})$ and h is a homomorphism, $h(x) \wedge h(y) \in F$ if and only $h(x), h(y) \in F$. Therefore the following equivalences hold:

$$\begin{aligned} \mathfrak{v}(x) \wedge \mathfrak{v}(y) = 0 &\iff \\ \text{w.l.o.g. } \mathfrak{v}(x) = 0 &\iff \\ h(x) \notin F &\iff \\ h(x \wedge y) \notin F &\iff \\ \mathfrak{v}(x \wedge y) = 0. \end{aligned}$$

(D4). Firstly, we claim that for $\varphi, \psi \in Fm$,

$$i_h\varphi \leq i_h\psi \iff \mathfrak{s}(\varphi) \subseteq \mathfrak{s}(\psi).$$

($\Rightarrow$) Suppose $i_h\varphi \leq i_h\psi$ and let $x_1, \ldots, x_n$ be the variables occurring in φ, $y_1, \ldots, y_m$ be the variables occurring on ψ. By Definition 5.2.6 $i_h\varphi = \bigvee_{1\leq i\leq n} i_h x_i$ and $i_h\psi = \bigvee_{1\leq j\leq m} i_h y_j$. As $\downarrow i_h\varphi = \bigcup_{1\leq i\leq n} \downarrow i_h(x_i) = \mathfrak{s}(\varphi)$ and $\downarrow i_h\psi = \bigcup_{1\leq j\leq m} \downarrow i_h(y_i) = \mathfrak{s}(\psi)$ we obtain $\mathfrak{s}(\varphi) \subseteq \mathfrak{s}(\psi)$ as desired.

($\Leftarrow$) The fact $\mathfrak{s}(\varphi) \subseteq \mathfrak{s}(\psi)$ implies $\downarrow i_h\varphi \subseteq \downarrow i_h\psi$ and this, together with the fact that $i_h\varphi = \bigvee_{1\leq i\leq n} i_h x_i$ and $i_h\psi = \bigvee_{1\leq j\leq m} i_h y_j$ entails $i_h\varphi \subseteq i_h\psi$. This proves our claim.

Now, observe that $\mathfrak{v}(x) \to \mathfrak{v}(y) = 1$ if and only if $\mathfrak{s}(y) \subseteq \mathfrak{s}(x)$ and (a). $\mathfrak{v}(x) = 0$ or (b).$\mathfrak{v}(y) = 1$.

Consider the case of (a), so $\mathfrak{v}(x) = 0$. Notice that, using the previous claim the following equivalences hold:

$$\begin{aligned} \mathfrak{v}(x) \to \mathfrak{v}(y) = 1 \text{ and } \mathfrak{v}(x) = 0 &\iff \\ h(x) \notin F \text{ and } h(x) \cdot h(y) = h(x) &\iff \\ h(x \to y) = h(\neg x \vee y) \in F &\iff \\ \mathfrak{v}(x \to y) = 1 \end{aligned}$$

The case (b) is similar, and we omit it. ☒

In the following Theorem, we denote by $\vdash_{\mathcal{P}_{\nmid}^{\rightarrow}}$ the logic defined by the class of matrices of the form $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_{\nmid}^{\rightarrow}(X)$, where X is a d-direct system of matrices. We aim at showing a completeness theorem for DAI with respect to such class of matrices that originates from a Płonka sum of models for Classical Logic.

Theorem 5.2.12. $\vdash_{\mathcal{P}_{\nmid}^{\rightarrow}} =$DAI.

Proof. ($\leq$). Let $\Gamma \vdash_{\mathcal{P}_{\nmid}^{\rightarrow}} \varphi$ and assume, towards a contradiction that $\Gamma \nvdash_{\mathsf{DAI}} \varphi$. Therefore, there exists a dependence model $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S}\rangle$ such that $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S}\rangle \vDash \Gamma$ and $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S}\rangle \nvDash \varphi$. By Lemma 5.2.10 there exists a (w.l.o.g. non trivial) matrix $\mathcal{P}_{\nmid}^{\rightarrow}(X)$ such that $I \cong \mathcal{P}(\mathfrak{S})$ and a valuation $h : \mathbf{Fm} \to \mathcal{P}_{\nmid}^{\rightarrow}(\mathbf{A}_i)_{i\in I}$ defined as

$$h(x) := \begin{cases} 1_{\mathfrak{s}(x)} \text{ if } \mathfrak{v}(x) = 1 \\ 0_{\mathfrak{s}(x)} \text{ if } \mathfrak{v}(x) = 0 \end{cases}$$

It is immediate to verify that $\mathfrak{v}(\gamma) = 1$ implies $h(\Gamma) \in F$ and that $\mathfrak{v}(\varphi) = 0$ implies $h(\varphi) \notin F$. So, h is a valuation that proves $\Gamma \nvdash_{\mathcal{P}_{\nmid}^{\rightarrow}} \varphi$, which is a contradiction.

($\geqslant$). Let $\Gamma \vdash_{\mathsf{DAI}} \varphi$ and assume by contradiction that $\Gamma \nvdash_{\mathcal{P}_{\nmid}^{\rightarrow}} \varphi$, i.e. there exists $\mathcal{P}_{\nmid}^{\rightarrow}(X) = \langle \mathbf{A}, F\rangle$ and a valuation $h : \mathbf{Fm} \to \mathbf{A}$ that maps $h(\Gamma) \subseteq F$, $h(\varphi) \notin F$. By Lemma 5.2.11 there is a dependence model $\langle \mathfrak{v}, \mathfrak{s}, \mathfrak{S}\rangle$ such that

(i) $I = \mathfrak{S}$

(ii) $\mathfrak{s}(x) = \downarrow i_h(x)$

(iii) $\mathfrak{v} : Fm \to \{0, 1\}$ is defined as

$$\mathfrak{v}(x) := \begin{cases} 1 \text{ if } h(x) \in F \\ 0 \text{ otherwise.} \end{cases}$$

Clearly $h(\Gamma) \subseteq F$ implies $\mathfrak{v}(\Gamma) = 1$ and $h(\varphi) \notin F$ implies $\mathfrak{v}(\varphi) = 0$, leading to the contradiction that $\Gamma \nvdash_{\mathsf{DAI}} \varphi$.

$\boxtimes$

5.2.3 Matrix semantics based on "twist products"

We now introduce a construction on involutive bisemilattices. In analogy with other constructions in the broad family of twist products (see e.g. [19]), it yields an algebra in an expanded type, whose universe is the

Cartesian product of the arguments, and whose operations are partly *internal* (meaning operations in the original type, defined componentwise) and partly *external* (operations not in the original type, involving an interplay between the components).

Definition 5.2.13. Let $\mathbf{A}, \mathbf{B} \in \mathcal{IBSL}$. The *I-product* of $\mathbf{A}$ and $\mathbf{B}$ is the algebra $\mathbf{A} \circledcirc \mathbf{B} = \langle A \times B, \wedge, \vee, \rightarrow, \neg, 0, 1 \rangle$, of type $\mathcal{L}_1 = \langle 2, 2, 2, 1, 0, 0 \rangle$, such that:

1. its $\langle \wedge, \vee, \neg, 0, 1 \rangle$-reduct is the direct product $\mathbf{A} \times \mathbf{B}$;
2. For all $a_1, a_2 \in A$ and all $b_1, b_2 \in B$,

$$\langle a_1, b_1 \rangle \rightarrow \langle a_2, b_2 \rangle = \begin{cases} \left\langle \neg^{\mathbf{A}} a_1 \vee^{\mathbf{A}} a_2, \neg^{\mathbf{B}} b_1 \vee^{\mathbf{B}} b_2 \right\rangle, \text{if } b_1 \leq^{\mathbf{B}}_{\wedge} b_2 \\ \left\langle 0^{\mathbf{A}}, \neg^{\mathbf{B}} b_1 \vee^{\mathbf{B}} b_2 \right\rangle, \text{otherwise.} \end{cases}$$

Hereafter, for $\mathcal{K}, \mathcal{K}' \subseteq \mathcal{IBSL}$, we denote by $\mathcal{K} \circledcirc \mathcal{K}'$ the class

$$I \left(\left\{ \mathbf{A} \circledcirc \mathbf{B} : \mathbf{A} \in \mathcal{K}, \mathbf{B} \in \mathcal{K}' \right\} \right).$$

The aim of this part is to provide a algebraic semantics for DAI that uses the product construction. The completeness of this semantics will be proved by showing its equivalence to the more customary modellings of DAI, either in terms of dependence models or in terms of the models discussed by Dunn in [34, Section 6] (although this is not the main approach employed in that paper). It will be observed that such models are a special case of models based on products of involutive bisemilattices.

We first need a technical lemma.

Lemma 5.2.14. *1. Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{v}, \mathfrak{s} \rangle$ be a dependence model for $\mathcal{L}_1$. Then there exists a countable set X such that the map v^*, defined by*

$$v^* (x) = \langle \mathfrak{v} (x), \mathfrak{s} (x) \rangle,$$

belongs to $Hom\left(\mathbf{Fm}\left(\mathcal{L}_1\right), \mathbf{B}_2 \circledcirc \mathbf{S}_X\right)$.

2. If X is a countable set and $v \in Hom\left(\mathbf{Fm}\left(\mathcal{L}_1\right), \mathbf{B}_2 \circledcirc \mathbf{S}_X\right)$, then $g(M) = \langle X, \pi_1 \circ v, \pi_2 \circ v \rangle$, where π_1 (resp. π_2) denote the operation of left (resp. right) projection, is a dependence model for $\mathcal{L}_1$.

Proof. (1). We only check that v^* respects conjunction and implication.

$$\begin{aligned} v^*(\varphi\wedge\psi) &= \langle \mathfrak{v}(\varphi\wedge\psi), \mathfrak{s}(\varphi\wedge\psi)\rangle \\ &= \left\langle \mathfrak{v}(\varphi)\wedge^{\mathbf{B}_2}\mathfrak{v}(\psi), \mathfrak{s}(\varphi)\vee^{\mathbf{S}_{\mathfrak{S}}}\mathfrak{s}(\psi)\right\rangle \\ &= \langle \mathfrak{v}(\varphi), \mathfrak{s}(\varphi)\rangle \wedge^{\mathbf{B}_2\circledcirc\mathbf{S}_{\mathfrak{S}}} \langle \mathfrak{v}(\psi), \mathfrak{s}(\psi)\rangle \\ &= v^*(\varphi)\wedge^{\mathbf{B}_2\circledcirc\mathbf{S}_{\mathfrak{S}}} v^*(\psi); \end{aligned}$$

moreover, if $\mathfrak{s}(\psi)\subseteq\mathfrak{s}(\varphi)$ and $\mathfrak{v}(\neg\varphi\vee\psi) = \neg^{\mathbf{B}_2}\mathfrak{v}(\varphi)\vee^{\mathbf{B}_2}\mathfrak{v}(\psi) = 1$, then

$$\begin{aligned} v^*(\varphi\to\psi) &= \langle \mathfrak{v}(\varphi\to\psi), \mathfrak{s}(\varphi\to\psi)\rangle = \left\langle 1^{\mathbf{B}_2}, \mathfrak{s}(\varphi)\vee^{\mathbf{S}_{\mathfrak{S}}}\mathfrak{s}(\psi)\right\rangle \\ &= \langle \mathfrak{v}(\varphi), \mathfrak{s}(\varphi)\rangle \to^{\mathbf{B}_2\circledcirc\mathbf{S}_{\mathfrak{S}}} \langle \mathfrak{v}(\psi), \mathfrak{s}(\psi)\rangle = v^*(\varphi)\to^{\mathbf{B}_2\circledcirc\mathbf{S}_{\mathfrak{S}}} v^*(\psi), \end{aligned}$$

while otherwise we have that $v^*(\varphi\to\psi) = \langle 0^{\mathbf{B}_2}, \mathfrak{s}(\varphi)\vee^{\mathbf{S}_{\mathfrak{S}}}\mathfrak{s}(\psi)\rangle = v^*(\varphi)\to^{\mathbf{B}_2\circledcirc\mathbf{S}_{\mathfrak{S}}} v^*(\psi)$.

(2). Again, it suffices to show that $\pi_1\circ v$ and $\pi_2\circ v$ obey the clauses in Definition 5.2.13. $\pi_2\circ v : Fm(\mathcal{L}_1)\to\wp(X)$ is such that

$$\pi_2\circ v(\varphi) = \bigcup\{\pi_2\circ v(p) : p\in var(\varphi)\}.$$

As regards $\pi_1\circ v$, we confine ourselves to showing that $\pi_1\circ v(\varphi\to\psi) = 1$ iff $\pi_1\circ v(\neg\varphi\vee\psi) = 1$ and $\pi_2\circ v(\psi)\subseteq\pi_2\circ v(\varphi)$. However, if the right-hand side of the biconditional is true,

$$\begin{aligned} v(\varphi\to\psi) &= v(\varphi)\to^{\mathbf{B}_2\circledcirc\mathbf{S}_X} v(\psi) \\ &= \langle \pi_1\circ v(\varphi), \pi_2\circ v(\varphi)\rangle \to^{\mathbf{B}_2\circledcirc\mathbf{S}_X} \langle \pi_1\circ v(\psi), \pi_2\circ v(\psi)\rangle \\ &= \left\langle \neg^{\mathbf{B}_2}(\pi_1\circ v(\varphi))\vee^{\mathbf{B}_2}\pi_1\circ v(\psi), \pi_2\circ v(\varphi)\vee\pi_2\circ v(\psi)\right\rangle \\ &= \left\langle 1^{\mathbf{B}_2}, \pi_2\circ v(\varphi)\vee\pi_2\circ v(\psi)\right\rangle, \end{aligned}$$

so $\pi_1\circ v(\varphi\to\psi) = 1$. Similarly, it is easy to check that if the right-hand side of the biconditional is false, then $\pi_1\circ v(\varphi\to\psi) = 0$. ⊠

We now semantically introduce two logics that we want to prove coincident with DAI. With an eye to doing so, we provide a recipe for associating logics to classes of implicative involutive bisemilattices.

Definition 5.2.15. If $\mathcal{K}\subseteq\mathcal{IIBSL}$, let $\mathrm{DAI}_{\mathcal{K}}$ be the logic $\left\langle \mathbf{Fm}(\mathcal{L}_1), \vdash_{\mathrm{DAI}_{\mathcal{K}}}\right\rangle$, where, for all $\Gamma\cup\{\varphi\}\subseteq Fm(\mathcal{L}_1)$,

$$\begin{aligned} &\Gamma\vdash_{\mathrm{DAI}_{\mathcal{K}}}\varphi \text{ iff for every } \mathbf{A}\in\mathcal{K} \text{ and every } h\in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{A}), \\ &\qquad \text{if } 1^{\mathbf{A}}\leq^{\mathbf{A}}_{\vee} h(\gamma) \text{ for all } \gamma\in\Gamma, \text{ then } 1^{\mathbf{A}}\leq^{\mathbf{A}}_{\vee} h(\varphi). \end{aligned}$$

The subclasses of $\mathcal{IIBSL}$ of immediate concern, for us, are $\mathcal{P}r = (\mathcal{BA} \backslash \{\mathbf{B}_1\}) \circledcirc \mathcal{SL}$ and $\mathcal{P}r^x = \{\mathbf{B}_2 \circledcirc \mathbf{S}_X : X \text{ a countable set}\}$. The corresponding logics $\mathrm{DAI}_{\mathcal{P}r}$ and $\mathrm{DAI}_{\mathcal{P}r^x}$ correspond to the logics respectively determined by certain I-products and by the Dunn models referred to in the introduction to this subsection. Observe that $\mathrm{DAI}_{\mathcal{P}r}$ can also be viewed as the logic determined by the class of all matrices of the form $\langle \mathbf{B} \circledcirc \mathbf{S}, F \rangle$, where $\mathbf{B}$ is a nontrivial Boolean algebra, $\mathbf{S}$ is a semilattice with zero, and $F = \left\{ \langle a, b \rangle \in B \times S : 1^{\mathbf{B}} = a \text{ and } 1^{\mathbf{S}} = 0^{\mathbf{S}} \leq^{\mathbf{S}} b \right\}$. Since the latter condition is always satisfied, such $\mathrm{DAI}_{\mathcal{P}r}$-filters have the form $\left\{ \langle a, b \rangle \in B \times S : 1^{\mathbf{B}} = a \right\}$. Analogously, $\mathrm{DAI}_{\mathcal{P}r^x}$ is the logic determined by the class of all matrices of the form $\langle \mathbf{B}_2 \circledcirc \mathbf{S}_X, F \rangle$, where $F = \left\{ \langle a, b \rangle \in B_2 \times S_X : 1^{\mathbf{B}_2} = a \right\}$.

Theorem 5.2.16. $\mathrm{DAI} = \mathrm{DAI}_{\mathcal{P}r} = \mathrm{DAI}_{\mathcal{P}r^x}$.

Proof. We first show the equivalence between DAI and $\mathrm{DAI}_{\mathcal{P}r^x}$. Suppose that $\Gamma \vdash_{\mathrm{DAI}} \varphi$. Then, according to Definition 5.2.2, for every dependence model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{v}, \mathfrak{s} \rangle$, if $\mathfrak{v}(\gamma) = 1$ for all $\gamma \in \Gamma$, then $\mathfrak{v}(\varphi) = 1$. With an eye to establishing that $\Gamma \vdash_{\mathrm{DAI}_{\mathcal{P}r^x}} \varphi$, consider an arbitrary countable set X and fix $v \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{B}_2 \circledcirc \mathbf{S}_X)$ such that $1^{\mathbf{B}_2 \circledcirc \mathbf{S}_X} \leq_{\vee}^{\mathbf{B}_2 \circledcirc \mathbf{S}_X} v(\gamma)$, for all $\gamma \in \Gamma$. By our previous remarks, this holds iff $1^{\mathbf{B}_2} = \pi_1 \circ v(\gamma)$, for all $\gamma \in \Gamma$. By Lemma 5.2.14, $\langle X, \pi_1 \circ v, \pi_2 \circ v \rangle$ is a dependence model, and since $1^{\mathbf{B}_2} = \pi_1 \circ v(\gamma)$ for all $\gamma \in \Gamma$, our assumption implies that $1^{\mathbf{B}_2} = \pi_1 \circ v(\varphi)$. This ensures that $1^{\mathbf{B}_2 \circledcirc \mathbf{S}_X} \leq_{\vee}^{\mathbf{B}_2 \circledcirc \mathbf{S}_X} v(\varphi)$.

Conversely, suppose that $\Gamma \nvdash_{\mathrm{DAI}} \varphi$. Thus, there exists a dependence model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{v}, \mathfrak{s} \rangle$ such that $\mathfrak{v}(\gamma) = 1$ for all $\gamma \in \Gamma$, yet $\mathfrak{v}(\varphi) = 0$. By Lemma 5.2.14.(1), $v^*(p) = \langle \mathfrak{v}(p), \mathfrak{s}(p) \rangle$ belongs to $Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{B}_2 \circledcirc \mathbf{S}_{\mathfrak{S}})$. Since $1^{\mathbf{B}_2} = \mathfrak{v}(\gamma) = \pi_1 \circ v^*(\gamma)$ for all $\gamma \in \Gamma$, then $1^{\mathbf{B}_2 \circledcirc \mathbf{S}_{\mathfrak{S}}} \leq_{\vee}^{\mathbf{B}_2 \circledcirc \mathbf{S}_{\mathfrak{S}}} v^*(\gamma)$, for all $\gamma \in \Gamma$. However, it is not the case that $1^{\mathbf{B}_2 \circledcirc \mathbf{S}_{\mathfrak{S}}} \leq_{\vee}^{\mathbf{B}_2 \circledcirc \mathbf{S}_{\mathfrak{S}}} v^*(\varphi)$. Therefore $\Gamma \nvdash_{\mathrm{DAI}_{\mathcal{P}r^x}} \varphi$.

Since it is trivially the case that $\vdash_{\mathrm{DAI}_{\mathcal{P}r}} \subseteq \vdash_{\mathrm{DAI}_{\mathcal{P}r^x}}$, we prove the converse inclusion. Assume that $\Gamma \vdash_{\mathrm{DAI}_{\mathcal{P}r^x}} \varphi$; since $\mathrm{DAI} = \mathrm{DAI}_{\mathcal{P}r^x}$, Γ can be assumed to be a finite set $\{\gamma_1, \ldots, \gamma_n\}$. Take a nontrivial Boolean algebra $\mathbf{B}$ and a semilattice with zero $\mathbf{S}$, together with $v \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{B} \circledcirc \mathbf{S})$, such that $1^{\mathbf{B} \circledcirc \mathbf{S}} \leq_{\vee}^{\mathbf{B} \circledcirc \mathbf{S}} v(\gamma)$, for all $\gamma \in \Gamma$, but it is not the case that $1^{\mathbf{B} \circledcirc \mathbf{S}} \leq_{\vee}^{\mathbf{B} \circledcirc \mathbf{S}} v(\varphi)$. By the structure of I-products, this means that either $1^{\mathbf{B}} \neq \pi_1 \circ v(\varphi)$ or $1^{\mathbf{S}} \nleq^{\mathbf{S}} \pi_2 \circ v(\varphi)$. If the former, then $\mathbf{B}_2$ falsifies the quasiequation $\gamma_1 \approx 1 \& \ldots \& \gamma_n \approx 1 \Rightarrow \varphi \approx 1$, since $\mathbf{B}_2$ generates $\mathcal{BA}$ as a quasivariety. Call v_1 the falsifying valuation; then for any countable set X and for any $v^+ \in Hom(\mathbf{Fm}(\mathcal{L}_1), \mathbf{B}_2 \circledcirc \mathbf{S}_X)$ such that $\pi_1 \circ v^+ = v_1$, the algebra $\mathbf{B}_2 \circledcirc \mathbf{S}_X \in \mathcal{P}r^x$ will be such that $1^{\mathbf{B}_2 \circledcirc \mathbf{S}_X} \leq_{\vee}^{\mathbf{B}_2 \circledcirc \mathbf{S}_X} v^+(\gamma)$ for all $\gamma \in \Gamma$, but $1^{\mathbf{B}_2 \circledcirc \mathbf{S}_X} \not\leq_{\vee}^{\mathbf{B}_2 \circledcirc \mathbf{S}_X} v^{+}(\varphi)$, contradicting the assumption that $\Gamma \vdash_{\mathrm{DAI}_{\mathcal{P}r^x}} \varphi$. If the latter, then some countable power $\mathbf{S}_2^X$ of the 2-element semilattice with zero $\mathbf{S}_2$ falsifies the same quasiequation, because $\mathbf{S}_2$ generates $\mathcal{SL}$ as a quasivariety. Calling v_2 the falsifying valuation, we get another family of algebras of the form $\mathbf{B}_2 \circledcirc \mathbf{S}_X$, and of homomorphisms v^{+*}, where ${}_2\circ v^{+*} = v_2$, that contradict once more the assumption $\Gamma \vdash_{\mathrm{DAI}_{\mathcal{P}r^x}} \varphi$. ⊠

Theorem 5.2.16 can be viewed as showing that each of the classes $\mathcal{P}r$ and $\mathcal{P}r^x$ is an algebraic semantics for DAI, under the set of defining equations $\boldsymbol{\tau}(x) = \{1 \vee x \approx x\}$.

5.3 Leibniz Reduced models of DAI

In this section we characterize a wide class of Leibniz reduced models of the logic DAI, namely those ones that originates from a d-direct system of matrices.

Recall the following

Lemma 5.3.1 ([42]). *Let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\mathrm{CL})$, and $a, b \in A$. Then*

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F \iff \{\neg a \vee b, \neg b \vee a\} \subseteq F.$$

In other words, the previous Lemma 5.3.1 states that $\{\neg a \vee b, \neg b \vee a\}$ is a set of congruence formulas for Classical Logic.

Theorem 5.3.2. *Let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\mathsf{DAI})$.*

$$\{a \to b, b \to a\} \subseteq F \iff \langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F.$$

Proof. ($\Leftarrow$).This direction follows directly by Lemma 1.2.3.

($\Rightarrow$). Let $a \to, b, b \to a \in F$. Suppose, towards a contradiction $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\mathbf{A}} F$, i.e. there exists a unary polynomial function $\varphi(x, \vec{v})$ such that for parameters $\vec{c} \in A$ it holds

$$\varphi(a, \vec{c}) \in F \iff \varphi(b, \vec{c}) \notin F.$$

If we show the admissibility the rule

$$x \to y, y \to x, \varphi(x, \vec{v}) \vdash_{\mathsf{DAI}} \varphi(y, \vec{v}) \quad \text{(R)}$$

we obtain the desired contradiction. To this end let $\mathcal{P}_{\mathrm{I}}^{\to}(X) = \langle \mathbf{B}, G\rangle \in \mathsf{Mod}(\mathsf{DAI})$ (w.l.o.g. consider it to be non trivial), and $h : \mathbf{Fm} \to \mathbf{B}$ s.t. $h(x \to y), h(y \to x), h(\varphi(x, \vec{v})) \in G$.

Firstly observe that $h(x \to y), h(y \to x) \in G$ implies (i): $h(x), h(y) \in A_i$ and that (ii): $h(x), h(y) \in F_i$ or $h(x), h(y) \in A_i \setminus F_i$. (i) follows from the fact that otherwise $h(x \to y) \in F$ implies $h(y \to x) \notin F$, (ii) is justified by noticing that if $h(x) \in F$ and $h(y) \notin F$ we have that $h(\neg x \vee y) \in F$ if and only if $h(\neg y \vee x) \notin F$.

By induction on the complexity of $\varphi(x, \vec{v})$ we show $h(\varphi(y, \vec{v})) \in G$. We assume that the variable x actually occurs in $\varphi(x, \vec{v})$, for otherwise the rule (R) is trivially admissible.

(B). $\varphi(x, \vec{v}) = x$. This is immediate, as $h(x \to y), h(y \to x) \in G$ together with $h(x) \in G$ implies $h(y) = h(\varphi(y, \vec{v})) \in G$.

The cases for $\varphi(x, \vec{v}) = \neg x$ and $\varphi(x, \vec{v}) = x * z$ for $* \in \{\wedge, \vee\}$ are immediate (here z is an arbitrary variable in $\vec{v}$).

So let $\varphi(x, \vec{v}) = x \to z$. As $h(\varphi(x, \vec{v})) = h(x) \to h(z) \in G$ we have $i_h x \leq i_h z$ and $h(\neg x \vee z) \in G$, so $i_h x = i_h y \leq i_h z$. Moreover $h(\neg x) \vee h(z) \in G$ implies $h(x) \notin G$ or $h(z) \in G$. Recalling that if $h(x) \notin G$ then $h(y) \notin G$ we conclude $h(y \to z) \in G$.

The only case left is $\varphi(x, \vec{v}) = z \to x$, which can be treated in a similar way.

(IND). Now, if $\varphi(x, \vec{v}) = \sigma(x, \vec{v}) * \epsilon(x, \vec{v})$ by induction hypothesis we can assume $h(\sigma(x, \vec{v})) \in G$ if and only if $h(\sigma(y, \vec{v})) \in G$ and $h(\epsilon(x, \vec{v})) \in G$ if and only if $h(\epsilon(y, \vec{v})) \in G$.

This, together with the fact that $h(x), h(y)$ belongs to the same fiber, entails $h(\varphi(y, \vec{v})) \in G$, proving the admissibility the rule

$$x \to y, y \to x, \varphi(x, \vec{v}) \vdash_{\mathsf{DAI}} \varphi(y, \vec{v}).$$

This shows $\varphi(b, \vec{c}) \in F$, and leads us to the desired contradiction. So $\langle a, b \rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F$.

$\boxtimes$

So, Theorem 5.3.2 that $\{x \to y, y \to x\}$ is a set of congruence formulas for DAI. The next Theorem 5.3.3 identifies the Leibniz reduced models of DAI that are obtained as Płonka sum over a d-direct system.

Theorem 5.3.3. *Let* $\langle \mathbf{A}, F \rangle \cong \mathcal{P}_{\mathfrak{t}}^{\to}(X) \in \mathsf{Mod}(\mathsf{DAI})$. *The following are equivalent*

(i) $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^*(\mathsf{DAI})$

(ii) *for each* $i \in I$, $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^*(\mathrm{CL})$ *and* $\mathbf{A}_i \neq \mathbf{1}$ *or* $\mathbf{A} = \mathbf{1}$.

Proof. (i)⇒(ii). Consider $\langle \mathbf{A}, F \rangle \cong \mathcal{P}_{\mathfrak{t}}^{\to}(X) \in \mathsf{Mod}^*(\mathsf{DAI})$. We show that for each $i \in I$ and arbitrary $a, b \in A_i$, $\langle a, b \rangle \notin \boldsymbol{\Omega}^{\mathbf{A}_i} F_i$. Fix $a, b \in A_i$, the fact that $\boldsymbol{\Omega}^{\mathbf{A}}F = id$ and Theorem 5.3.2 imply $\{a \to b, b \to a\} \nsubseteq F$. Moreover, as $a \to b = \neg a \vee b$ and $b \to a = \neg b \vee a$ we obtain $\{\neg a \vee b, \neg b \vee a\} \nsubseteq F_i$ which, by Lemma 5.3.1 entails $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\mathbf{A}_i}F_i$, as desired. It remains to show $\mathbf{A}_i \neq \mathbf{1}$ or $\mathbf{A} = \mathbf{1}$. Now suppose towards a contradiction $\mathbf{A}_i = \mathbf{1}$ and $\mathbf{A} \neq \mathbf{1}$. This implies there exists $j \in I, j \neq i$ and, by Lemma 5.2.9 for every $k \in I$ it holds $A_k = F_k$. Therefore $\boldsymbol{\Omega}^{\mathbf{A}}F = A \times A$, a contradiction.

(ii)⇒(i). for each $i \in I$ let $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}^*(\mathsf{CL})$ with $\mathbf{A}_i \neq \mathbf{1}$ or $\mathbf{A} = \mathbf{1}$.

Observe that if $\mathbf{A} = \mathbf{1}$ then trivially $\boldsymbol{\Omega}^{\mathbf{A}}F = id$. So consider $\mathbf{A} \neq \mathbf{1}$ which entails $\mathbf{A}_i \neq \mathbf{1}$. Fix arbitrary $a, b \in A$. w.l.o.g. consider $a \in A_i$ and $b \in A_j$. Observe first that if $i = j$ then the assumption $\boldsymbol{\Omega}^{\mathbf{A}_i}F_i = id$ directly entails $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\mathbf{A}}F$. Now, if $i \neq j$ consider $i \vee j = k$ (in the case i, j are comparable assume w.l.o.g. $i < j$). Cconsider the unary polynomial function $x \to b$. Clearly $b \to b = 1_{A_j} \in F_j \subseteq F$ while $a \to b = 0_{A_k}$. The fact that $A_i \neq F_i$ and Lemma 5.2.9 imply $A_k \neq F_k$ and therefore $a \to b = 0_{A_k} \notin F_k$. By Lemma 1.2.3 we obtain $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\mathbf{A}}F$.

⊠

Corollary 5.3.4. *Let $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_{\nmid}^{\to}(X)$. Then $\boldsymbol{\Omega}^{\mathbf{A}}F = \bigcup_{i\in i} \boldsymbol{\Omega}^{\mathbf{A}_i}F_i$.*

Proof. ($\subseteq$). Assume $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}F$. By Theorem 5.3.2 $\{a \to b, b \to a\} \subseteq F$ which entails $a, b \in A_i$. So $\{\neg a \vee b, \neg b \vee a\} \in F_i$, i.e. $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}_i}F_i \subseteq \bigcup_{i\in i} \boldsymbol{\Omega}^{\mathbf{A}_i}F_i$

($\supseteq$). This follows from direction (i)⇒(ii) of Theorem 5.3.3. ⊠

5.4 DAI in the Leibniz hierarchy

In the following theorem by "equivalent algebraic semantics" is intended the quasi-variety generated by any class $\mathcal{K}$ w.r.t. the logic DAI is algebraizable.

Lemma 5.4.1. *Let $\mathbf{A} \in \mathcal{IIBSL}$, $F = \bigcup_{i\in I} 1_i$ and $\varphi, \psi \in Fm$. Consider an evaluation $h : \mathbf{Fm} \to \mathbf{A}$. T.F.A.E.*

(i) $h(\varphi \to \psi) = h(\varphi) \to^{\mathbf{A}} h(\psi) \in F$

(ii) *for every $v : \mathbf{Fm} \to (\mathbf{A}^-)^{\to}$ such that $v(x) = h(x)$ ($x \in Var$) $v(\varphi \to \psi) = v(\varphi) \to^{(\mathbf{A}^-)^{\to}} v(\psi) \in F$*

Proof. w.l.o.g. we assume φ, ψ do not contain occurrences of $\to$.

(i)⇒(ii). By Remark 5.2.5 let $\mathbf{A}^- \cong \mathcal{P}_{\nmid}(\mathbf{A}_i) \in \mathcal{IBSL}$. Moreover, recall that $\mathbf{A}$ and $(\mathbf{A}^-)^{\to}$ do have the same $\to$-free reduct and therefore $v(\chi) =$

$h(\chi)$ for every $\to$-free formula χ and for every $v : \mathbf{Fm} \to (\mathbf{A}^-)^{\to}$ that coincides with h on the set Var.

Assume now $h(\varphi \to \psi) \in F$, which, as $F = \bigcup_{i \in I} 1_i$, implies $1 \leq h(\varphi) \to^{\mathbf{A}} h(\psi)$. By (I3) in Definition 5.2.3 we have $h(\varphi) \wedge (h(\varphi) \vee h(\psi)) \approx h(\varphi)$.

So, by (I2) we have $h(\varphi \to \psi) = h(\neg\varphi \vee \psi) \geqslant 1$. Therefore, fixed an arbitrary $v : \mathbf{Fm} \to \mathbf{A}(^-)^{\to}$ as in the statement, we obtain $1 \leq v(\neg\varphi \vee \psi)$ and $v(\varphi) \wedge (v(\varphi) \vee v(\psi)) \approx v(\varphi)$ which, by Definition 5.2.4 entails $v(\varphi \to \psi) = v(\varphi) \to^{(\mathbf{A}^-)^{\to}} v(\psi) \in F$.

(ii)$\Rightarrow$(i). Consider an arbitrary $v : \mathbf{Fm} \to (\mathbf{A}^-)^{\to}$ such that $v(x) = h(x)$ ($x \in Var$) mapping $v(\varphi \to \psi) = v(\varphi) \to^{(\mathbf{A}^-)^{\to}} v(\psi) \in F$. By Definition 5.2.4 we have $i_v(\psi) \leq i_v(\varphi)$ and $v(\neg\varphi \vee \psi) \in F$. As $i_v(\psi) \leq i_v(\varphi)$ implies $v(\varphi \wedge (\varphi \vee \psi)) = h(\varphi \wedge (\varphi \vee \psi)) = h(\varphi) = v(\varphi)$, by (I3) in Definition 5.2.3 we obtain $h(\varphi \to \psi) = h(\neg\varphi \vee \psi) = v(\neg\varphi \vee \psi) \in F$, as desired.

$\boxtimes$

Corollary 5.4.2. *Let $\mathbf{A} \in \mathcal{IIBSL}$, $\varphi \in Fm$. The following are equivalent*

(i) $\models_{\mathbf{A}} \varphi \geqslant 1$

(ii) $\models_{(\mathbf{A}^-)^{\to}} \varphi \geqslant 1$

Proof. The statement is immediate if φ does not contain occurrences of $\to$. Otherwise, the desired equivalence follows by an application of Lemma 5.4.1. $\boxtimes$

Theorem 5.4.3. *The class $\mathcal{IIBSL}$ is the equivalent algebraic semantics of the logic* DAI *via the transformers $\boldsymbol{\tau}(\alpha) = \{1 \vee \alpha \approx \alpha\}$ and $\boldsymbol{\rho}(\alpha \approx \beta) = \{\alpha \to \beta, \beta \to \alpha\}$.*

Proof. We prove that conditions (ALG1) and (ALG4) defined in Section 1.2 holds.

(ALG1).

($\Rightarrow$). Assume $\Gamma \vdash_{\mathsf{DAI}} \varphi$ with $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. Suppose towards a contradiction that there exists $\mathbf{A} \in \mathcal{IIBSL}$ such that $\gamma_1^{\mathbf{A}} \geqslant 1, \ldots, \gamma_n^{\mathbf{A}} \geqslant 1$ and $\varphi^{\mathbf{A}} \not\geqslant 1$. By Corollary 5.4.2 it holds $\gamma_1^{(\mathbf{A}^-)^{\to}} \geqslant 1, \ldots, \gamma_n^{(\mathbf{A}^-)^{\to}} \geqslant 1$ and $\varphi^{\mathbf{A}^{\to}} \not\geqslant 1$ (here by $\psi^{\mathbf{A}^{\to}}$ we understand the computation of ψ on $(\mathbf{A}^-)^{\to}$ obtained by interpreting the variables of ψ in the same way as it is done for $\psi^{\mathbf{A}}$).

This is equivalent to the fact that there exists a valuation $h : \mathbf{Fm} \to (\mathbf{A}^-)^{\to}$ such that $h(\gamma_1) \geqslant 1, \ldots, h(\gamma_n) \geqslant 1$ and $h(\varphi) \not\geqslant 1$. Now, as $\langle \mathbf{A}^{\to}, \bigcup_{i \in I} 1_i \rangle \cong \mathcal{P}_{\mathfrak{l}}^{\to}(X)$ for some d-direct system X, by Theorem 5.2.12

we obtain $\Gamma \nvdash_{\mathsf{DAI}} \varphi$, which contradicts our assumption. So we conclude $\varphi^{\mathbf{A}} \geqslant 1$.

($\Leftarrow$). We reason by contraposition so assume $\Gamma \nvdash_{\mathsf{DAI}} \varphi$. So there is a dependence model that falsifies the inference. By Lemma 5.2.10 we can obtain a d-direct system of models $\langle \mathbf{A}, \bigcup_{i\in I} 1_i \rangle$ and a valuation $h : \mathbf{Fm} \to \mathbf{A}$ such that $h(\Gamma) \subseteq \bigcup_{i\in I} 1_i$ and $h(\varphi) \notin \bigcup_{i\in I} 1_i$. Clealry this means that, under the valuation h, it holds $\gamma_1^{\mathbf{A}} \geqslant 1, \dots, \gamma_n^{\mathbf{A}} \geqslant 1$ and $\varphi^{\mathbf{A}} \ngeqslant 1$. This, together with the fact that $\mathbf{A} \in \mathcal{IIBSL}$ entails $\boldsymbol{\tau}(\Gamma) \nvDash_{\mathcal{IIBSL}} \boldsymbol{\tau}(\varphi)$.
(ALG4). We must prove that

$$\varphi \approx \psi \vDash\!\dashv_{\mathcal{IIBSL}} \{1 \leq_{\vee} \varphi \to \psi, 1 \leq_{\vee} \psi \to \varphi\}.$$

Thus, let $\mathbf{A} \in \mathcal{IIBSL}$ and let $a \in A$. Recall that every $\mathcal{IBSL}$ (whence also $\mathbf{A}^{-}$) satisfies the identities $1 \leq_{\vee} 1 \vee x \approx x \vee \neg x$. Since $a = a \wedge (a \vee a)$, by I2 in Definition 5.2.3 we have that $1 \leq_{\vee} a \to a = a \vee \neg a$. Let now $a, b \in A$, and let $1 \leq_{\vee}^{\mathbf{A}} a \to b, 1 \leq_{\vee}^{\mathbf{A}} b \to a$. By I3, a and b belong to the same fiber $\mathbf{A}_i$ in the Płonka sum representation of $\mathbf{A}^{-}$. Thus, by I2, $1 \leq_{\vee}^{\mathbf{A}_i} b \vee \neg a, a \vee \neg b$, which implies (since $\mathbf{A}_i$ is Boolean) $a \leq_{\vee}^{\mathbf{A}_i} b \leq_{\vee}^{\mathbf{A}_i} a$, hence $a = b$. ⊠

Theorem 5.4.4 ([42]). *Let $\vdash$ be an algebraizable logic with defining equations $E(x)$ and equivalent algebraic semantics the quasivariety $\mathcal{K}$. Then $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{*}(\vdash)$ if and only if $\mathbf{A} \in \mathcal{K}$ and $F = \{a \in A : \vDash_{\mathbf{A}} E(x)[a]\}$.*

So we have

Corollary 5.4.5. $\mathsf{Alg}^{*}(\mathsf{DAI}) = \mathsf{Alg}(\mathsf{DAI}) = \mathcal{IIBSL}$.

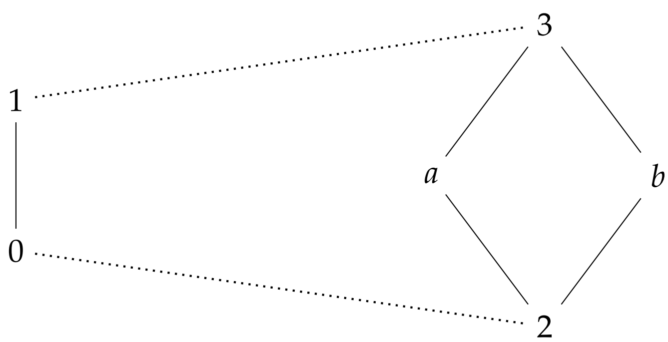
The previous result, together with Theorem 5.3.2, fully characterize the Leibniz reduced models of the logic DAI, as underlined by the following

Corollary 5.4.6. $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{*}(\mathsf{DAI})$ *if and only if* $\mathbf{A} \in \mathcal{IIBSL}$ *and* $F = \bigcup_{i\in I} 1_{\mathbf{A}_i}$.

We obtained that the class $\mathcal{IIBSL}$ coincides with the class $\mathsf{Alg}(\mathsf{DAI})$. Moreover, Theorem 5.3.3 identifies the Leibniz reduced models $\langle \mathbf{A}, F \rangle \cong \mathcal{P}_{\mathbf{I}}^{\to}(X)$. It is natural to wonder whether all the Leibniz reduced models can be represented as Płonka sums arising from d-direct systems. The next example answers negatively.

Example 5.4.7. Consider the following algebra $\mathbf{A}$ where the operations in the language $\{0, 1, \neg, \wedge, \vee\}$ are computed according to the usual Płonka definition, and $\to$ is defined as:

$$x \to y := \begin{cases} \neg x \vee y \text{ if } x \cdot y = x \\ 0 \text{ otherwise.} \end{cases}$$



Clearly $\mathbf{A} \in \mathcal{IIBSL}$, but $\mathbf{A} \notin \mathcal{PIBSL}$, as

$$0 \to^{\mathbf{A}} a = 0 \text{ while } 0 \to^{\mathbf{A}^{\to}} a = 2.$$

So $\langle \mathbf{A}, \{1, 3\} \rangle$ is a Leibniz reduced model of DAI whose algebraic reduct is not in $\mathcal{PIBSL}$.

Conclusions

We conclude by underlining some general outcomes of the investigation.

The algebraic analysis of logics of variable inclusion and of demodalised analytic implication lead to a, perhaps surprising, conclusion: what syntactically amounts to a linguistic requirement of variable inclusion on a consequence relation, semantically "corresponds" to a specific way of summing up logical matrices. More precisely, a generic dependence relation between sets of variables turns out to be representable by an appropriate Płonka sum.

More precisely, it is possible to generalize the machinery of Płonka sums to the level of logical matrices in, at least, three different ways.

- In Chapter 2, we presented the notion of l-direct system of matrices. Its distinctive feature consists in preserving filter membership via homomorphisms: the image of a filter under a Płonka homomorphism is still contained a filter. Such feature turned out to play the role of semantic counterpart of left variable inclusion logics, and it permitted a systematic investigation of such family of logics. The analysis led to the formulation of a complete matrix semantics for logics of left variable inclusion, and, under natural assumptions, it produced a Hilbert-style axiomatization. Such axiomatizations, though infinite, do not contain syntactic restrictions on their rules. Moreover, the general approach just described is suitable for an extensive study of second-order AAL properties.

- A second declination of the notion of direct system of matrices is the one adopted in Chapter 3. There, the formulation of r-direct system of matrices allowed for an extensive study of the other variable inclusion companion of a logic, namely its right one. This second generalization of direct systems essentially differs from the first. Indeed, it figures a condition on the structure of the fibers with non empty filters, which are required to form a sub-semilattice of the index set. This, together with a new interplay between filters and homomorphisms, gave the basis to the analysis of logics of right variable inclusion, mirroring the analysis carried on in Chapter 2. It must be remarked that, however, the results on the characterization of the Leibniz and Suszko reduced models essentially differs in the cases of left and right variable inclusion logics. Moreover, right variable inclusion logics turned out to join less properties, in terms of algebraizability, then their left cousins, failing also to be truth equational.

- The last account of direct system of matrices is the one employed in Chapter 5, under the name of d-direct system of matrices. This third version has a less general appeal than the other two, as it embraces only matrices whose algebraic reduct is Boolean. However, it revealed a new and fascinating aspect about Płonka sums, namely their strict connection with modal operators. Indeed, the investigation of the logic of demodalised analytic implication shows that Płonka sums permits to semantically model a special kind of implication, based on a dependence relation between antecedent and succedent. A possible-world semantics for this connective has been provided in existing literature: such connection between Płonka sums and modality sheds new light on their application to logic.
  The analysis of DAI also revealed that logics admitting a Płonka sums-based semantics are not necessarily relegated to the bottom of the Leibniz hierarchy. Indeed, on the contrary of logics of variable inclusion, DAI turned out to be algebraizable.

In Chapter 4 we faced two distinct topics. Firstly, we focused on the so-called Gentzen algebraizability of logics of left variable inclusion. This revealed that logics of left varibale inclusion, when viewed as Gentzen systems, actually join much stronger properties. Indeed, under precise assumptions, such Gentzen systems are equivalential: this never happens at the level of consequence relations over formulas.
In the remaining section of the Chapter we have considered how logics of variable fit in the lattice of consequence relation over a fixed signature. The outcomes underline that a logic has can have up to eight sublogics of variable inclusion.

Other applications of Płonka sums

The machinery of Płonka sums finds fruitful applications also in the field of algebraic methods in computer science. In [17], Płonka sums have been employed in order to solve the *finite spectrum* problem for the class of finite, linearly ordered involutive bisemilattices. This class, as oberved in Chapter 2, is contained in the variety generated by the "algebraic counterpart"of the logic PWK. The construction allowed to produce and implement an algorithm that, given a natural number n as input, it produces and counts all the non-isomorphic linearly ordered involutive bisemilattices with exactly n elements as outputs.

A remarkable fact is that such algorithm turns out to be up to three times more efficient than the usual Prover9/Mace4 tools.

There are many ways this research can be further developed. On the one hand, the solution of the finite spectrum problem for the entire class of finite $\mathcal{IBSL}$ seems a reachable task, and it currently is work-in-progress. On the other hand, we believe that the techniques employed there can be generalized to a good number of regular varieties.

Future work

Each chapter of this thesis contains many open problems and suggests possible future investigations. In this concluding part, we summarize some of them.

- The Hilbert-style axiomatizations presented in Theorems 3.3.9 and 2.3.9 are infinite. Clearly, this is far from being ideal, and a finite refinement of such calculi would be an interesting achievement to reach. A possible direction in order to approach the problem is sketched in Remark 2.3.11.

- The analysis of logics of left variable inclusion as Gentzen systems suggests that, when such perspective is assumed, a deeper understanding of such logics is available. There are at least two obvious directions in order to develop the topic:

  (i) in this dissertation, the only kind on Gentzen system we took into account is $\vdash^{\mathcal{G}}$. The axiomatization of such calculus, at least in some specific examples, would provide a Gentzen calculus without explicit syntactic restrictions on logical rules: in the best of our knowledge, no calculus of this kind is still known for logics of left variable inclusion;

  (ii) the extension of the investigation also to logics of right variable inclusion constitutes a very natural goal. Moreover, the preservation of equivalentiality is surely not the only feature that deserves our attention. We believe that and a more comprehensive analysis can disclose new interesting results.

- The tools employed in the analysis of DAI do not rely, in an essential way, on the specific properties of Classical Logic and Boolean algebras. So, we believe that a generalization of the results may be a possible future goal. A further motivation consists in observing that the expansion of a language via a dependence-based operation can be motivated in many other cases, different from classical logic.

- Another aspect that emerges from our investigation concerns the role of antitheorems. It turned out that the presence of antitheorems influences in a sensible way the characterization of the Suszko reduced models of logics of variable inclusion and, above all, it is a necessary condition in order for equivalentiality to be preserved in the framework of Gentzen systems. An extensive investigation of the role of antitheorems with respect to Płonka sums may constitute a promising task to reach.

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