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Modified Lee-Carter Methods with LASSO type Smoothing and Adjusting for Lifespan Disparity

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Abstract

Extrapolative methods like Lee-Carter and its later variants are widely accepted for forecasting mortality in industrial countries due to simplicity, both for single population forecasting and coherent forecasting. This model assumes an invariant age component and a linear time component for forecasting. The latter requires a second level estimation to increase forecast accuracy. We propose to apply the Lee-Carter method on smoothed mortality rates obtained by *LASSO* type regularization and hence to partially adjust the time component to match the observed lifespan disparity ($e_0^\dagger$). Smoothing by lasso produces less error during fitting period compared to other spline based smoothing techniques. Also matching with $e_0^\dagger$ – a more informative indicator of longevity than e_0, made the time component more reflective of countries' mortality patterns. We further extend this methodology for coherent forecasting as well. In this setting, choosing the appropriate reference population remains an arbitrary process. We propose to obtain the reference population on the basis of closest $\bar{e}_0^\dagger$. Hence the common factor of coherent model is estimated utilizing only a subset of the available years (the best fitting period), and these same years are considered as country-specific fitting period as well. Both of the proposed methods have been found to be more accurate during out-of-sample evaluation compared to corresponding existing models and provide more optimistic forecasts.

Sommario

I metodi estrapolativi come quello proposto da Lee-Carter e le sue varianti successive sono ampiamente accettate per la previsione della mortalità nei paesi industriali a causa della semplicità, sia per la previsione della popolazione singola che per le previsioni coerenti. Questo modello assume una componente per l'età fissa nel tempo e una componente temporale lineare per la previsione. Quest'ultima richiede una stima di secondo livello per aumentare l'accuratezza della previsione. Si propone di applicare il metodo Lee-Carter sui tassi di mortalità uniformi ottenuti dalla regolarizzazione del tipo *LASSO* e quindi di correggere parzialmente la componente temporale in modo che corrisponda alla disparità di durata della vita osservata ($e_0^\dagger$). Il lisciamento con il lasso produce meno errori durante il periodo di adattamento rispetto ad altre tecniche basate sulle spline. Anche il matching con $e_0^\dagger$ – un indicatore più informativo della longevità rispetto e_0 – permette che la componente temporale rifletta meglio del miglioramento della mortalità degli ultimi anni. Estendiamo ulteriormente questa metodologia anche per le previsioni coerenti. In quest'ambito, la scelta delle popolazioni di riferimento appropriate rimane un processo arbitrario. Proponiamo di ottenere la popolazione di riferimento sulla base del più vicino $\bar{e}_0^\dagger$. Quindi, si stima il fattore comune del modello coerente utilizzando solamente un sottoinsieme degli anni disponibili (il periodo di adattamento migliore), lo stesso periodo viene considerato anche per la stima del modello specifico per il paese. Entrambi i metodi proposti risultano essere più accurati durante la valutazione out-of-sample, rispetto al corrispondente modello esistente ed entrambi forniscono previsioni più ottimistiche.

To my Mom

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Chapter 1

Introduction

Improved mortality has been observed globally during the twentieth century and is always considered a positive change for the socio-economic advancement of a country. In many developed countries, social security systems (including disability and survivorship benefits), medical care for elderly are affected by mortality trends, especially given increased longevity (Brouhns *et al.*, 2002). This improvement have different patterns across countries, however its elementary structure is preserved. To analyze and understand mortality, life table is the main instrument to statistically study the different aspects of mortality. Life table illustrates the distribution of death for a group of individuals. The period life table represents the mortality conditions at a specific moment in time. The core component of a life table is the mortality rates of that particular population and all other components indicating stochasticity of the death process or remaining life expectancy are also estimated based on the observed mortality rates. For a period life table, age-specific mortality rates are defined as

$$m_x = \frac{D_x}{P_x}, \tag{1.1}$$

where D_x are the observed death counts in a calendar year and P_x is the mid year population of that year for age group x. The mortality measures may refer to overall mortality or be decomposed by cause of death as well (Booth and Tickle, 2008). Beside improvement of socio-economical background and technologies, changes in mortality rates are also attributable to changes in population composition arising from subgroup heterogeneity as per equation 1.1. Hence understanding the mortality pattern requires clear definition of underlying model (Booth and Tickle, 2008). Three core components for measuring demographic events, namely age, period (or time) and cohort, are usually employed to classify the underlying model. The simple most model considers mortality rates as a function of age. To illustrate one-to-one relation among age-specific deaths, the post-war Swedish female mortality rates are plotted in Figure 1.1.

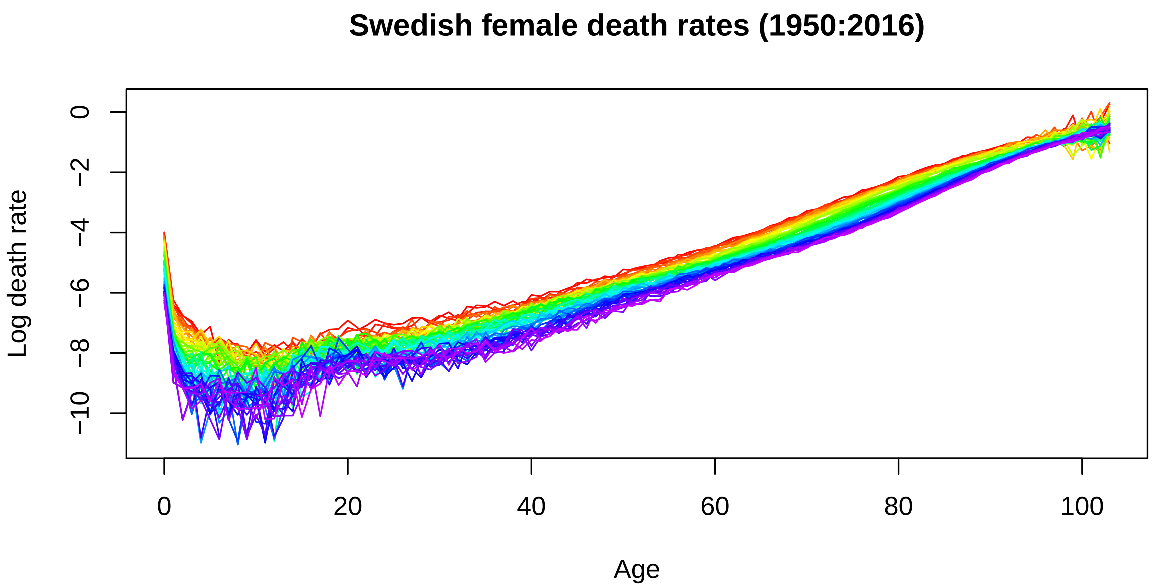


FIGURE 1.1: Swedish female mortality rates (1950-2016). Data are taken from Human Mortality Database (HMD, 2018). Years are plotted using a rainbow palette so the earlier years are shown in red, followed by orange, yellow, green, blue and indigo with the most recent years plotted in violet.

Among all the countries, Sweden has the longest time series data for mortality starting from 1751 (HMD, 2018). Due to demographic transition, changes occurred in the population structure with slow pace of change until the first half of the last century (Canudas-Romo, 2008). Two trends dominated the mortality decline in last century: the first one was a reduction in mortality caused by infectious diseases affecting mainly young ages, particularly in the first half of the century; the second was decreasing mortality from chronic diseases particularly affecting older ages (Booth *et al.*, 2002). Similar to Sweden, mortality is concentrating at older ages in most of the industrialized countries (Canudas-Romo, 2008). This rapid aging of the industrialized countries thus turned into a growing concern for the governments and societies. Policymakers greatly rely on mortality projection to determine appropriate pension benefits and to understand the costing of different economic assumptions and invent regulations regarding the retirement age. As a result, accurate forecasts of mortality and life expectancy became core requirement for decision making in social, health-care and financial sectors. Stochastic modeling of mortality forecasting are particularly gaining rapid recognition in this context, United Nations and several industrialized countries already adapted stochastic forecasting techniques (Booth and Tickle, 2008).

1.1 Overview: Development of mortality forecasting techniques

Although modeling mortality has a very long history, the boost in research on mortality forecasting is more recent (Booth and Tickle, 2008). Mortality forecasting became necessary since aging became common in industrialized countries. Population aging is a common phenomenon of second demographic transition as a consequence of low fertility and low mortality (Sobotka, 2008). This is reflected in increasing trend of life expectancy for most of the industrialized countries, Oeppen and Vaupel (2002) observed liner trend of increase in record high life expectancy for over 160 years. The term population aging was always conceptualized as higher proportion of older people in population structure, however, now it is more rationalized as the members of populations are living longer lives (Lutz *et al.*, 2008). This shifting or postponement of mortality thus brought new requirements for population forecasting, requiring more in depth information regarding improvement of mortality in each ages (Vaupel, 2010).

The preliminary question for making forecast is the mortality measure used to be forecast and it highly depends on the purpose of the forecast and data availability. Mortality forecasting generally involves the specification of an underlying model of the data and a model for forecasting. Age and sex-specific mortality rates (or probabilities) are of primary interest for most of the industrialized countries, along with derived life tables. For countries with limited data resources, the major interest for forecasting is the life expectancy at birth (Booth and Tickle, 2008). Another major challenge for forecasting age-specific mortality is the dimensionality problem. Dimensionality refers to the total number of data 'cells' which are modelled, equal to the product of the numbers of categories for the factors classifying the data (Booth and Tickle, 2008). Models with lower dimensionality problem is hence preferred as they can represent the data more parsimoniously. Parsimonious models for mortality forecasting avoid over-parametrization which may avoid the problems of correlated model parameters and associated complications (Booth and Tickle, 2008). Consequences due to dimensionality problem unable the Gompertz law or other widely used mortality models for mortality forecasting as most of these models have highly correlated parameters.

Several parametric and nonparametric methods have been proposed over the years in order to forecast age-specific mortality rates and life expectancy. The simplest way for parametric forecasting is to parameterize the available series of life table and hence to extrapolate each of the parameters separately for obtaining the forecast from the assumed model (Keyfitz, 1991). In addition to different subjective approaches, the ground breaking approach on probabilistic forecasting was proposed by Lee and Carter (1992). The advantages of Lee-Carter (LC) method include its simplicity as it has fewer parameters with straightforward explanation and robustness in situations where age-specific log mortality rates have linear trends (Booth *et al.*, 2002). To increase the precision of LC method in presence of irregular mortality schedule, later studies restricted the fitting period to post-war years along with other modifications (Lee and Miller, 2001). More generally, Booth *et al.* (2002) noticed the length of fitting period might greatly affect point forecast accuracy.

Application of smoothing technique is also getting popular for improving mortality forecast (Booth and Tickle, 2008). Smoothing can effectively reduce the noises of the observed data to overcome outlier problem and can effectively increase forecast accuracy for LC variants (Girosi and King, 2006). De Jong and Tickle (2006) combined spline smoothing and estimation via the Kalman filter to fit a generalized version of the Lee-Carter model, whereas Hyndman and Ullah (2007) proposed smoothing the mortality curves for each year using constrained regression splines prior to fitting a model using principal components decomposition following the functional data paradigm. Later two dimensional smoothing is proposed by Camarda *et al.* (2012) which considers both the age and period effect on mortality following the work of Currie *et al.* (2006). Hyndman and Ullah (2007) introduced nonparametric approaches of mortality forecasting where they used functional data analysis on smoothed mortality rates obtained from one dimensional smoothing. These nonparametric methods were found to be more robust as they are more efficient than other LC variants even in presence of outliers and provide more accurate forecasts (Shang *et al.*, 2011).

Numerous other methods were proposed later for mortality forecasting, although many of them were somehow extensions of basic LC method. Renshaw and Haberman (2000) considered the concept of Generalized Linear Model to model mortality reduction factors and identified the conditions under which the underlying structure of the proposed model is identical to that of the LC method. A further extension by Renshaw and Haberman (2003) also accounts for the cohort effect. Other stochastic models have been also introduced to integrate the cohort dimension in mortality by Cairns *et al.* (2011). Besides several LC variants we find

- Application of Bayesian framework on LC methodology (see Wiśniowski *et al.*, 2015, for example)
- Different distribution of causes of deaths (see Booth and Tickle, 2008, for example)
- Consideration of risk factors and behavioral impact (see Janssen *et al.*, 2013, for an example on smoking status)
- Different approaches than LC framework. For example, mortality forecasting from distribution of death (De Beer *et al.*, 2017) or using a Relational model both for smoothing and projecting mortlaity De Beer (2012).

Nevertheless, many of these approaches are newer and still LC variants are most applied techniques for mortality forecasting (Bohk-Ewald and Rau, 2017). Most of these studies have focused on mortality forecasting of single population. Lately multi-population forecasting is getting more widespread as it seeks to ensure that the forecasts for related populations hold certain structural relationship based on past pattern of mortality and theoretical understanding. Li and Lee (2005) modified the traditional LC method for forecasting mortality as the sum of a common trend and the population specific rates converging toward that trend in future. Later Hyndman *et al.* (2013) extended nonparametric approach of Hyndman and Ullah (2007) for coherent forecasting; Ahmadi and Li (2014) considered generalized linear modeling and Bergeron-Boucher *et al.* (2017) considered compositional data analysis on distribution of death for coherent forecasting. In another approach, Janssen *et al.* (2013) considered smoking epidemic for coherent forecasting.

Given the existence of so many forecasting methods, it becomes particularly useful to assess which model is the most useful in specific context. This assessment has been done by Shang *et al.* (2011) even though only a specific context has been considered, i.e. industrialized countries characterized by low mortality, high life expectancies, lower adult and early senescence mortality, stable pattern of mortality transition over time and high quality mortality data. Similarly only low mortality, industrialized countries were always considered for coherent forecasting (see Seligman *et al.*, 2016, for example). Lee and Carter (1992) remained the main method for comparison due to its wide accessibility. Several of these studies concluded that none of the methods are uniquely best for all low mortality countries (see Shang *et al.*, 2011; Shang, 2012, for example). As most of the methods are based on Lee and Carter (1992), there is still scope of research to develop it to improve the forecast. For instance, application of different smoothing technique than that used by Hyndman and Ullah (2007) may produce different results which is still subject to analyze. New techniques to estimate the parameters of LC method may also improve the forecast accuracy.

In addition, few forecasting attempts have been made considering high mortality countries from Central and Eastern European (CEE) region or large number of different mortality forecasting methods (see Bohk and Rau, 2015, for example). Even though the mortality pattern is still different from that of Western European countries (see Vallin and Meslé, 2001), there are also some similarities (Bálint and Kovács, 2015) including increasing population aging (see Gavrilova and Gavrilov, 2009). Bohk and Rau (2015) compared and contrast forecasting techniques on countries to evaluate impact of recent financial crisis for some of the high mortality countries and mentioned irregular mortality developments are particularly difficult to forecast due to major changes in long-term trends. Therefore, the Eastern European and Central European countries would also get benefit from an accurate mortality forecasting by comparing the outcomes of different forecasting techniques. Moreover, comparison of mortality forecasting techniques on high mortality countries may guide to understand adequately the further scope of developing mortality forecasting models.

1.2 Main contributions of the thesis

I proposed two new mortality forecasting techniques by modifying the existing Lee-Carter methodology: one for single population and the other one for coherent forecasting. Instead of widely used spline based smoothing techniques, LASSO type regularization have been used to smooth mortality rates prior to model fitting in both of the proposed methods. In addition, I incorporate lifespan disparity during parameter estimation of the models and to our knowledge, the present study is the first attempt to consider lifespan disparity in ground of mortality forecasting. Finally I proposed a way for choosing reference population which is applicable for all existing coherent forecasting techniques as well and considered best fitting period in order to make coherent forecasts ensure the most of the interactions among the populations in reference group.

This thesis consists of 7 chapters. In chapter 2, I mention the source of mortality data used in this study and briefly review some of the existing methods which I consider later for comparison with proposed methods. The evaluation procedure for mortality forecasting methods used in this study are also placed in this chapter. In chapter 3, I assess the performances of the existing methods for some comparatively high-mortality countries. The proposed methodology for single population mortality forecasting are placed in chapter 4. Here I illustrate the application of Lasso for smoothing mortality rates prior to model fitting and adjustment of the time component of LC model according to lifespan disparity. I extend this proposed methodology for coherent forecasting in chapter 5. Besides the proposed estimation technique for coherent setup, I also show a new scheme for choosing best reference population in this chapter. Both for chapter 4 and 5, I consider only low mortality industrialized countries for developing and illustrating the methodologies. Following up the results of chapter 3, I apply the proposed methodology to comparatively high mortality countries in chapter 6. The concluding remarks are placed in chapter 7. Here I discuss the findings and possibility of further extension.

Chapter 2

Data and Methodology

2.1 Data

The data used in this study came from Human Mortality Database (HMD, 2018). The Human Mortality Database (HMD) provides detailed mortality and population data for the available countries with an open access policy. The database is maintained by Department of Demography at the University of California, Berkeley, USA, and the Max Planck Institute for Demographic Research in Rostock, Germany. HMD provides original estimates of death rates and life tables for national populations (countries or areas), as well as the input data used in constructing those tables. The input data consist of death counts from vital statistics, plus census counts, birth counts, and population estimates from country-specific statistics offices or other reliable sources (HMD, 2018). I have considered male and female populations separately for the following 20 low mortality countries to illustrate the new forecasting techniques for single population and coherent setting: Australia (AUS), Austria (AUT), Belgium (BEL), Canada (CAN), Denmark (DNK), Finland (FIN), France (FRA), Germany (DEU), Ireland (IRL), Italy (ITA), Japan (JPN), The Netherlands (NLD), New Zealand (NZL), Norway (NOR), Portugal (PRT), Spain (ESP), Sweden (SWE), Switzerland (CHE), United Kingdom (UK) and USA (USA). The data of Germany is not available before 1990, therefore, I combined the data of East and West Germany together for getting longer time series data. Total populations instead of smaller subpopulations are considered for France, New Zealand and United Kingdom. For single country forecasting, individual available data and age groups are considered. I started fitting the models from 1950 for all the countries except for Germany (started from 1956 due to data unavailability from 1950). HMD covers the period of 1956 to 2011 for all of these countries, which is considered for coherent forecasting.

In context of high mortality regime, nine Central and Eastern European (CEE) countries have been considered: Belarus (BLR), Bulgaria (BGR), Estonia (EST), Hungary (HUN), Latvia (LVA), Lithuania (LTU), Russia (RSU), Slovakia (SVK) and Ukraine (UKR). These countries have comparatively higher mortality regime compare to Western Europe. The data I used for these countries and their last observed life expectancy at birth are given below in Table 2.1. For most of them, the life table started from 1950; only Bulgaria has available data from 1947. However, I did not utilize the whole available data for several countries due to lower data quality, an issue mentioned by the HMD (HMD, 2018).

TABLE 2.1: Fitting periods for the CEE countries and life expectancy at birth (HMD, 2018).

Country	Starting year	End Year	e_0 (Female)	e_0 (Male)
Belarus	1970	2014	78.43	67.81
Bulgaria	1950	2010	77.25	70.31
Estonia	1959	2013	81.33	72.72
Hungary	1960	2014	79.24	72.26
Latvia	1970	2013	78.73	69.26
Lithuania	1959	2013	79.37	68.52
Russia	1970	2014	76.48	65.26
Slovakia	1962	2014	80.32	73.25
Ukraine	1970	2013	76.21	66.31

2.2 Existing mortality forecasting techniques

I considered 7 different variants of the Lee-Carter (LC) method along with a coherent mortality forecasting method and the Bayesian Hierarchical Model used by United Nations (Raftery *et al.*, 2013) for producing probabilistic forecasts. Among them, 4 parametric LC variants including Lee and Carter (1992); Lee and Miller (2001); Booth *et al.* (2002); Brouhns *et al.* (2002) and 3 non-parametric variants of Hyndman-Ullah method, Robust Hyndman-Ullah method and Weighted Hyndman-Ullah method (Hyndman and Ullah, 2007) are considered.

2.2.1 Lee-Carter Method (1992) and variants

Since its development, Lee-Carter (LC) method has been one of the most applicable methods till now. Use of principal components for mortality forecasting came to practice through the work of Lee and Carter (1992). The two-factor LC model is given below,

$$\ln m_{x,t} = a_x + b_x k_t + \epsilon_{x,t}. \tag{2.1}$$

Here, $m_{x,t}$ is the central mortality rate at age x for year t; a_x represents the average of log-mortality at age x over time; b_x is the first principal component capturing relative change in the log mortality rate at each age x; k_t represents the overall level of mortality in year t; and $e_{x,t}$ is the model residual. The constraints of the model are;

$$\sum_{x=0}^{p} b_x = 1 \text{ and, } \sum_{t=1}^{n} k_t = 0$$

Singular Value Decomposition (SVD) is done on $Z_{x,t} = [\ln(m_{x,t}) - \hat{a}_x]$ to obtain the Ordinary Least Square (OLS) estimate of LC model. SVD decomposes the $Z_{x,t}$ into the product of three matrices. Symbolically,

$$\text{SVD}(Z_{x,t}) = ULV' = L_1 U_{x_1} V_{t_1} + \ldots L_n U_{x_n} V_{t_n}.$$

For estimation of the age and time component Lee and Carter (1992) considered the rank-1 approximation only as it explains most of the variance. Then the estimates of model parameters are,

$$\hat{k}_t = L_1 V_{t_1} \text{ and, } \hat{b}_x = U_{x_1}.$$

Lee-Carter method makes a second stage estimate of k_t by finding the value of k_t which, for a given population age distribution and previously estimated a_x and b_x produces exactly the observed number of total deaths for the fitting period of the model (Lee and Carter, 1992). An *ARIMA(0,1,0)* with drift is then fitted for adjusted $\hat{k}_t$, and used to forecast future mortality. Later Lee and Miller (2001) proposed three modifications on the basic LC model; (i) the fitting period is restricted from 1950 and onward to reduce structural shifts, (ii) adjustment of k_t is done by matching life expectancy, and (iii) 'jump-off error' is eliminated by forecasting forward from observed (rather than fitted) rates.

Lee-Carter model can be extended by including higher order terms also instead of rank-1 approximation considered in earlier two approaches. Higher order terms were modeled by Booth *et al.* (2002) and forecasts were later developed by using univariate ARIMA processes (Renshaw and Haberman, 2003). The key modifications of this method are: (i) the fitting period is determined by a statistical 'goodness of fit' criterion, under the assumption that k_t is linear; and (ii) the adjustment of k_t involves fitting to the age distribution of deaths rather than to the total number of deaths in the basic LC model. This model is a significant development in the research of forecasting mortality as it slightly eliminates a shortcoming of LC model, which assumes invariant b_x whereas evidence of substantial age-time interaction is common (Shang, 2012).

Brouhns *et al.* (2002) considered the underlying deaths are distributed in a Poisson regression and assumed to have following log-bilinear form of the mortality rates.

$$D_{x,t} \sim \text{Poisson}\left\{E_{x,t}m_{x,t}\right\} \text{ with } m_{x,t} = \exp(a_x + b_x k_t);$$

where $E_{x,t}$ are the population exposed for death at age x in time t. The constraints of the basic LC model also holds for this method. One of the main advantages of using this approach is that it allows to have maximum likelihood estimation of the model parameters instead of OLS or Gauss-Newton algorithm (Brouhns *et al.*, 2002). This shows some further development scope to utilize Bayesian approach on LC methods.

2.2.2 Nonparametric Approaches: Hyndman and Ullah (2007) methods

To address the problem of lack of across-age smoothness, heterogeneity of deaths over long time period (Girosi and King, 2006) and consideration of only first principal component in LC variants, Hyndman and Ullah (2007) proposed a functional data model that utilizes second and higher order principal components to capture additional variation in mortality rates. This technique uses a penalized regression spline with partial monotonic constraint to smooth the log mortality rates prior to model fitting. The spline based smoothing techniques are discussed in following section prior to model fitting.

2.2.2.1 Spline based smoothing techniques

For the nonparametric approach, Hyndman and Ullah (2007) applied weighted penalized regression splines independently for each year. This one dimensional smoothing involves calculating a vector β which minimizes the following:

$$|w(y) - X\beta|^2 + \lambda^2 \beta^T D\beta. \tag{2.2}$$

Here, y is a vector of observations (mortality rates); X is a matrix representing linear spline bases; $D = diag(0, 0, 1, 1, \ldots 1)$ is a diagonal matrix; w is a vector of weights; and λ is the smoothing parameter. For smoothing mortality rates, observations in year t are given by $y_i = m_{x_{i,t}}$ for age group x_i years old (where $i = 0, \ldots, 110+$). The weights w_i are taken as the inverse of the estimated variances of y_i. Assuming the life table deaths follow a Poisson distribution, Hyndman and Ullah (2007) estimated the variance of y_i as $\sigma^2 \approx (Ex_{i,t}Mx_{i,t})^{-1}$ by Taylor series approximation. Here $Ex_{i,t}$ is the mid-year population of people aged x_i years in year t. Moreover these splines are constrained to ensure that the resulting function $f(x)$ is monotonically increasing for $x > c$ for some c (for example, 60 years). Hyndman and Ullah (2007) proposed to use a modified version of the method described by Wood (1994) to implement this constraint.

Another widely used smoothing technique is the two dimensional splines (Camarda *et al.*, 2012). Although, this method is not utilized for forecasting, I compared findings from one dimensional smoothing and Lasso with two dimensional technique in Chapter 4 for sake of best smoothing technique in terms of accuracy. This method fits a two-dimensional P-spline model with equally-spaced B-splines along X and Y axes (age and calendar year respectively). The response variables must be a matrix of Poisson distributed death counts in this approach. For this splines offset can be provided, else all weights are assumed to be unity. In a Poisson regression setting applied to actual death counts, the offset will be the logarithm of the matrix of exposure population.

To smooth the mortality rates, this method utilizes a smoothing function which is the *Kronecker* product of B-spline basis over the two axes and includes a discrete penalization directly on the differences of the B-splines coefficients. The smoothing parameters λ are mainly used to tune the smoothness/accuracy of the fitted values (Currie *et al.*, 2006). For optimizing the smoothing parameters, both AIC or BIC can be considered.

2.2.2.2 Model fitting

The following continuous smooth function $f_t(x)$ is assumed for discrete ages.

$$\ln m_t(x_i) = f_t(x_i) + \sigma_t(x_i)\epsilon_{t,i}; \quad i = 1, \ldots, p; \quad t = 1, \ldots, n; \tag{2.3}$$

where $m_t(x_i)$ represents mortality rates for each age x_i in time t; $\sigma_t(x_i)$ is the noise component and $\epsilon_{t,i}$ is i.i.d. standard normal variable. Hyndman and Ullah (2007) proposed to use weighted penalized regression splines to estimate $f_t(x)$. This weighting controls for heterogeneity due to $\sigma_t(x)$ and a monotonic constraint for upper ages can lead to better estimates. In this study, equal weights are applied to the approximate inverse variances $w_{x,t} = m_{x,t}E_{x,t}$, and used weighted penalized regression splines to estimate the curve $f_t(x)$ for each year (Hyndman and Ullah, 2007). Weighted penalized regression splines are preferable in terms of computational time and allow monotonicity constraints (Hyndman and Ullah, 2007). Details of estimation procedure of interval forecast are given elsewhere (Hyndman and Ullah, 2007). Functional principal component analysis utilizes a set of continuous functions and is decomposed into functional principal components and their associated scores. Symbolically,

$$f_t(x) = a(x) + \sum_{j=1}^{J} b_j(x)k_{t,j} + e_t(x); \quad t = 1, \dots, n;$$

where $a(x)$ is the mean function $\left(= \frac{1}{n}\sum_{t=1}^{n} f_t(x)\right)$, $b_j(x)$ are set of first J functional principal components, $k_{t,j}$ are set of uncorrelated principal component scores, $e_t(x)$ is the residual function. It should be noted that $J < n$ is considered for optimal number of functional principal components. ARIMA model is suggested to forecast principal component scores as they have minimum AIC of the fitted model, however, almost every suitable time series model can be applied as well (Shang, 2012). Two more version of HU method were also proposed for special situations. The first one is generally referred to as robust Hyndman and Ullah method ($\mathrm{HU_R}$), proposed to forecast in presence of outliers. This approach investigates the integrated squared error for each year by calculating following measures of accuracy for the functional principal component approximation of the functional data.

$$\int_{x-1}^{x_p} \left(f_t(x) - a(x) - \sum_{j=1}^{J} b_j(x)k_{t,j} \right)^2 dx.$$

After assigning zero weight to outliers, the $\mathrm{HU_R}$ fits the mortality rates from which forecasts of age-specific life expectancies can be estimated without affect of prospective outliers. The second one is another weighted version of HU where recent years get more weight during model fitting than years from distant past. The new method can be showed symbolically as follows,

$$f_t(x) = \widehat{a}^*(x) + \sum_{j=1}^{J} b_j^*(x) k_{t,j} + e_t(x); \tag{2.4}$$

where, $a^*(x)$ is the weighted functional mean such as,

$$\widehat{a}^*(x) = \sum_{t=1}^{n} w_t f_t(x), \quad \sum_{t=1}^{n} w_t = 1, \text{ where, } w_t = \kappa(1-\kappa)^{n-t}; \quad t = 1, \dots, n.$$

This w_t is the new weights defined by Hyndman and Ullah (2007) for $0 < \kappa < 1$; a geometrically decaying weight parameter. The optimal value of κ is chosen by minimizing an overall forecast error measure within the validation data set among a set of possible candidates. Details of the methods can be found in (Hyndman and Shang, 2009).

2.2.3 Coherent mortality forecasting

In recent years, coherent or multi-population forecasting methods are getting more popular as these approaches try to capture the influence of global improvement of health, communication, science on a specific population. The standard Lee and Carter (1992) model and its variants are defined for forecasting single population and often used for females and males independently. Li and Lee (2005) modified the standard LC model to forecast mortality for countries by taking into account their membership in a group, rather than forecasting individually. To do that, Li and Lee (2005) first identified the central tendencies within a group of countries, addressing a common factor and hence adopted the historical particularities of each country as their due weight in projecting individual-country trends for forecasting mortality. Thus, in the short term, inter-country mortality differences in trends may be preserved, but ultimately age-specific death rates within the group of countries are constrained to maintain a constant ratio with one another (Li and Lee, 2005). This extended model can be formulated as,

$$\ln m_{x,t,i} = a_{x,i} + B_x K_t + b_{x,i} k_{t,i} + \epsilon_{x,t,i}; \tag{2.5}$$

where i stands for specific country in the group, $a_{x,i}$ is the country specific average log mortality rate. The term B_x and K_t are the relative speed of change in mortality at each age x and a mortality index capturing the main time trend for the reference population respectively. Li and Lee (2005) mentioned the term $B_x K_t$ as common factor since this quantity is common for all the countries of the group. The error term of equation 2.5 is the country specific estimate of error. To obtain the country-specific estimates of Li-Lee model, at first Lee and Miller (2001) model is fitted on reference population, may refereed to as common factor model. The reference group is constructed by adding all the populations of the group from which the common factor is extracted to use in the country level mortality forecasting mentioned in equation 2.5. Both for K_t and $k_{t,i}$, random walk with drift is used for forecasting. Following Lee and Miller (2001), actual data is used for mortality forecasting (rather than fitted data) to avoid jump-off error. Choosing reference population remains one of the biggest problem in coherent forecasting. Several approaches tried different strategies for using particular countries as reference population considering geographic, economic similarities, and other criteria (Kjærgaard *et al.*, 2016). For high-mortality countries, all populations are combined together as reference population for each of these countries. Li and Lee (2005) also considered a group of low-mortality countries as reference for making coherent forecast for high mortality countries with the optimistic assumption that these countries will catch-up the low mortality countries in future.

Besides these above mentioned variants of LC and HU method, there are other approaches for mortality forecasting. Most these techniques are highly based on original LC method. For example, Tuljaparker et al (2000) used the Lee-Carter model without any adjustment on time component and start fitting from 1950; Girosi and King (2006) considered more than one component in Lee-Carter model and later in another study they extended the Lee-Carter model to incorporate age-period-cohort effects. Using the parameter estimation technique of Brouhns *et al.* (2002), later several approaches have been proposed on Bayesian framework. For simplicity of findings and wide applicability, only LC and HU variants are considered for comparison in this study.

2.2.4 Bayesian Probabilistic Projections (UN life expectancy forecast)

Several mortality forecasting technique have been proposed in Bayesian framework to overcome invariant mortality improvement of LC variants (for example, Cairns *et al.*, 2011; Wiśniowski *et al.*, 2015; Bohk-Ewald and Rau, 2017). Moreover, one of the major shortcomings of all the above mentioned variants of LC model is that these methods require age-specific death rates for at least three decades to fit the model; certainly such detailed data are no available for many developing countries (Raftery *et al.*, 2013). Instead of forecasting mortality rates in Bayesian framework, Raftery *et al.* (2013) proposed an alternative approach to forecast life expectancy at birth using Bayesian framework. They applied a Bayesian Hierarchical Model to forecast period life expectancy directly using a random walk model with drift. The newly defined drift term is a non-linear function of current life expectancy and reflects the fact that life expectancy which tends to changes more slowly for the countries with the lowest and highest life expectancies, and more quickly for the countries in the middle. The United Nations (UN) produces estimates of age-specific mortality and period life expectancy at birth for all member countries and updates in every two years in UN World Population Prospects (UN, 2013). The UN projects life expectancy in the next time period deterministically using the equation

$$\ell_{c,t+1} = \ell_{c,t} + g(\ell_{c,t}). \tag{2.6}$$

Forecast of life expectancy is done by a double-logistic function of the current level of life expectancy. Symbolically,

$$\begin{aligned} g(\ell_{c,t}|\theta^c) = & \frac{k^c}{1 + \exp\left(-\frac{A_1}{\Delta_2^c}(\ell_{ct} - \Delta_1^c - A_2\Delta_2^c)\right)} \\ & + \frac{z^c - k^c}{1 + \exp\left(-\frac{A_1}{\Delta_4^c}\left(\ell_{ct} - \sum_{i=1}^{3}\Delta_i^c - A_2\Delta_4^c\right)\right)}. \end{aligned} \tag{2.7}$$

$\Delta_1^c, \Delta_2^c, \Delta_3^c, \Delta_4^c, k^c, z^c$ are the six parameters of the double logistic function for country c at time t. The estimation technique changed since World Population Prospect 2012 (UN, 2013) as the UN Population Division used a probabilistic model for the first time to forecast life expectancy at birth using the methods of Raftery *et al.* (2013). They utilized the following hierarchical model to turned the old deterministic model into a probabilistic one (with uncertainty) and hence adopted a Bayesian approach to estimate the model parameters. Hence the hierarchical model become,

$$\ell_{c,t+1} = \ell_{c,t} + g\left(\ell_{c,t}|\theta^{(c)}\right) + \epsilon_{c,t+1}. \tag{2.8}$$

Raftery *et al.* (2013) defined proper prior for all 13 parameters of the model in such a way that the prior distributions are more diffuse than the posterior distributions. Thus, the above-mentioned hierarchical model turned into a Bayesian Hierarchical Model. This method has one advantage over any other parametric or non-parametric methods: it is flexible on choosing prior to get fast, slow or medium pace for change in life expectancy level. The UN method has been proposed to forecast life expectancy using data of World Population Prospects or similar format. As a consequence, one of the major disadvantages of this method is that it does not forecast considering whole life tables like the previous methods. This makes it complicated to compare the outcomes with LC variants. This method takes single value of life expectancy for each five years and also return the forecast as a median of five (calendar) years period utilizing a Bayesian Hierarchical Model. I considered this method only for high mortality countries to evaluate whether it can provide more optimistic forecast of life expectancy at birth than LC or HU variants or coherent forecasting.

2.3 Assessing the performance of the mortality forecasting techniques

To evaluate a forecast technique, I considered two criteria: how optimistic the forecast is in long run and hence the accuracy level during out-of-sample evaluation period (except for UN method). Pessimistic forecast of life expectancy is a common case for many high mortality countries. With a jagged pattern of mortality improvement over the years, it is possible that forecast could be lower than last observed life expectancy (Booth *et al.*, 2002). I considered it as failure of the model to capture mortality trend for that particular country because a forecast showing future decline in mortality pattern are contrary to the basic assumption of the models regarding future mortality improvement (Lee and Miller, 2001). Moreover, lower estimates of life expectancy may occur due to a seasonal jump of life expectancy during out-of-sample evaluation period or short run which is not the case for long forecast horizon (Shang, 2012). I consider the following two measures for checking the forecast accuracy of mortality rates:

Mean absolute forecast error,

$$\text{MAE} = \frac{1}{(p+1)q} \sum_{j=1}^{q} \sum_{x=0}^{p} \left| y_{x,j} - \widehat{y}_{x,j|j-h} \right| ; \tag{2.9}$$

mean squared forecast error,

$$\text{MSE} = \frac{1}{(p+1)q} \sum_{j=1}^{q} \sum_{x=0}^{p} \left(y_{x,j} - \widehat{y}_{x,j|j-h} \right)^2 ; \tag{2.10}$$

and for life expectancy at birth, I consider the mean error of life expectancy,

$$\text{ME} = \frac{1}{q} \sum_{j=1}^{q} \left(e_{0,j} - \hat{e}_{0,j} \right) . \tag{2.11}$$

Here $y_{x,j}$ represents the observed mortality rate for age x in year j and $\widehat{y}_{x,j}$ represents the forecast; $e_{0,j}$ represents the observed life expectancy at birth in year j and $\hat{e}_{0,j}$ represents the forecast. Unlike Shang *et al.* (2011) or Shang (2012), I choose mean squared forecast error over mean forecast error as measure of forecast accuracy. Mean forecast error could be misleading most of the times as it may conceal forecast inaccuracy due to the offsetting effect of large positive and negative forecast errors or very low error in forecasting (actual). From the available mortality data, I kept the data of last 10 years for out-of-sample evaluation of the forecasting technique. Using the data in the fitting period, I made the one-step-ahead and ten-step-ahead point forecasts, and determine the forecast accuracy by comparing the forecasts with the holdout data in the out-of-sample evaluation period. The analysis performed in this thesis can be implemented by "Demography"package of R for all existing LC and HU variants along with one-dimensional smoothing. Two-dimensional smoothing is done using the R package "MortalitySmooth"; Lasso are performed using "smoothAPC"; Kannisto models are fitted using "MortalityLaws"and the R package "bayesLife"is utilized for Bayesian forecast. The proposed methods are implemented by modifying the existing functions in "Demography"and are attached in appendix.

2.4 Prediction interval of forecast

To compare the interval forecast of e_0, I applied the existing semi-parametric bootstrapping technique proposed by (Hyndman and Booth, 2008). In this technique, the fitted mortality rates from forecasting technique are simulated a large number of times to add disturbance in time component of the model. Life expectancies are then estimated for each set of the simulated log mortality rates. Prediction interval are then constructed by 80% or 95% percentile of the simulated sets of the life expectancies.

Chapter 3

Findings from (Comparatively) High Mortality Countries

The aim of this chapter is to assess the performance of selected forecasting methods for countries characterized by a higher mortality regime vis-à-vis Western countries. I compared and contrasted the performance of the mortality forecasting models for nine CEE countries- Belarus, Bulgaria, Estonia, Hungary, Latvia, Lithuania, Russia, Slovakia and Ukraine. The selected CEE countries differ from Western European and other non-Eastern European countries in five ways: a) higher mortality, b) irregular mortality trends, c) increasing mortality differences between countries over time (the divergence of mortality regime), d) shorter time series data and e) lower quality mortality data. A jagged, inconsistent pattern of mortality improvement over the years are responsible for this irregularity in the mortality pattern of these countries. Age- and sex-specific difference among males and females mortality are higher also in these countries which is highly attributable to different distribution of causes-of-deaths. Following Shang *et al.* (2011), I limited the main interest on LC variants. Moreover, I considered two more approaches for comparison—coherent mortality forecasting and life expectancy forecasting using a Bayesian Hierarchical model adapted by the United Nations (UN). I extended this comparison to a coherent setup because in opposition to the concept of coherent mortality forecasting, these countries have diverging mortality patterns compared to those of low-mortality countries (Li and Lee, 2005). Thus, this extension can provide more insight regarding the assessment of coherent mortality forecasting. UN projections are widely accepted even for countries with limited data. Since these CEE countries have a lower quality of mortality data, therefore, I extended the comparison to include this technique. Overall, this sort of comparison of mortality forecasting for high-mortality countries may help us to better understand the scope of developing mortality forecasting models and may help policymakers in terms of policy implications relating to age- and cause-specific mortality. Unlike the study by Shang (2012), comparing and contrasting on the above mentioned comparatively high mortality countries may give us better insight regarding the impact of recent mortality improvements on future mortality.

3.1 Comparison of mortality projection models for the nine selected CEE countries

Point forecasts of life expectancy at birth in 2050 obtained using all the Lee-Carter (LC) variants along with UN forecasts are given in Table 3.1. The results of the different types of models are discussed later. Here, LC stands for the basic Lee and Carter (1992); LC_P stands for the model with a Poisson regression (Brouhns *et al.*, 2002); LM stands for the modified LC model proposed by Lee and Miller (2001); BMS stands for the modified LC model proposed by Booth *et al.* (2002); HU stands for the non-parametric approach proposed by Hyndman and Ullah (2007); HU_R stands for the robust Hyndman and Ullah (2007); HU_W stands for the weighted Hyndman and Ullah (2007); LL stands for the coherent mortality forecast proposed by Li and Lee (2005) and UN stands for the UN life expectancy forecast using a Bayesian hierarchical model (Raftery *et al.*, 2013).

TABLE 3.1: Forecast of life expectancy at birth in 2050 for selected CEE countries.

Country	e_0	LC	LC_P	LM	BMS	HU	HU_R	HU_W	LL	UN
Belarus	78.43	68.68	75.92	78.43	76.82	75.68	77.50	**80.55**	75.60	79.95
Bulgaria	77.25	82.16	79.77	80.24	79.76	78.49	80.83	**82.61**	77.55	80.63
Estonia	81.33	85.67	85.58	85.75	**90.30**	84.20	82.75	84.95	84.71	85.70
Hungary	79.24	82.77	82.66	83.27	85.40	84.19	82.44	**87.09**	82.60	83.54
Latvia	78.73	81.66	81.54	81.68	**85.83**	81.10	81.34	82.06	80.10	82.43
Lithuania	79.37	82.17	80.92	81.77	**84.82**	80.24	79.63	82.42	81.63	82.76
Russia	76.48	73.62	73.12	76.24	72.93	73.54	78.45	76.43	74.73	**79.08**
Slovakia	80.32	84.52	84.26	84.27	**85.75**	83.59	83.12	83.73	82.37	84.44
Ukraine	76.21	70.90	74.05	76.20	73.67	73.37	**80.03**	78.47	74.60	79.24

Results are shown for females only.
e_0 is the last observed life expectancy during the fitting period from HMD (Table 2.1).

3.1.1 LC and HU variants

BMS produced the most optimistic forecast of life expectancy for three Baltic countries and Slovakia. HU_R produced the highest forecast for the Ukraine; the UN forecast was the highest for Russia and for rest of the countries HU_W produced the most optimistic point forecast of life expectancy. In addition to other exceptional cases, all the models produced lower forecasts than the last observed life expectancy for Belarus, Russia and the Ukraine, the countries with the highest mortality levels. Only HU_W could produce a higher forecast of life expectancy than the last observed e_0 for Belarus; all other methods extrapolated lower or almost equal future life expectancy. For Russia and the Ukraine, the HU_R method was appropriate in this sense (for the Ukraine, HU_W was also appropriate). The results of HU variants show a better fit than these models. This is attributable to smoothing and the implications of more than one principal component to explain the higher variation (Hyndman and Ullah, 2007). HU_R is proposed to give a better fit and optimistic forecast in presence of outliers during fitting period, which is very common for all these countries. For illustration, the observed female mortality rates for Russia and the Ukraine are given below for the fitting period (Figure 3.1).

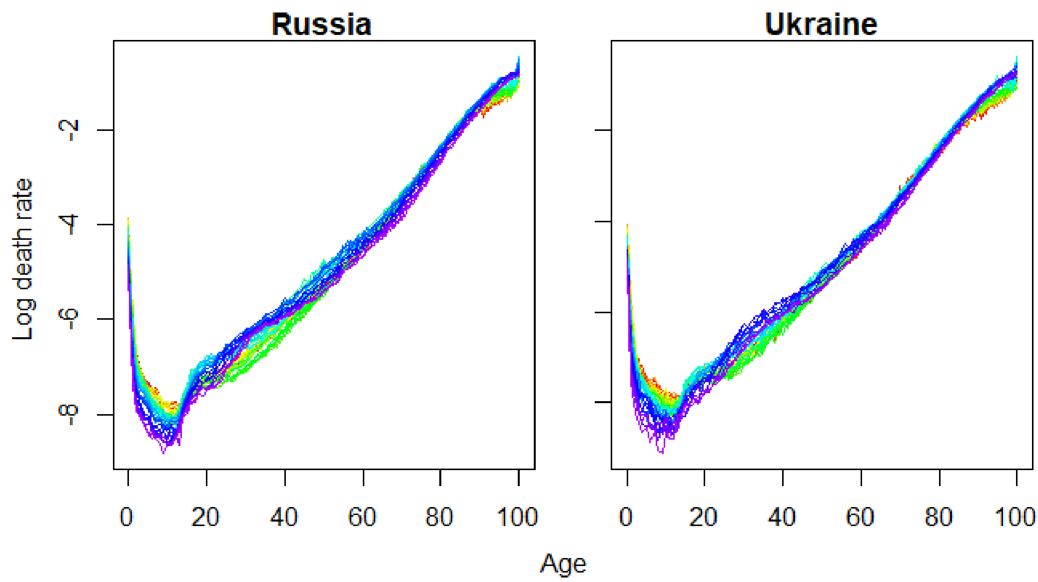


FIGURE 3.1: Observed mortality rates of female Russia (1970-2014) and Ukraine (1970-2013). Years are plotted using a rainbow palette so the earlier years are shown in red, followed by orange, yellow, green, blue and indigo with the most recent years plotted in violet.

Although HU_W produced an optimistic forecast for Belarus and Ukraine, still the results are subject to analyze. HU_W is employed for countries with long time series data; which was not the case for these countries and there were severe mortality crisis during mid 1990s in both of the countries (Shkolnikov *et al.*, 1998). Similarly, despite providing more optimistic forecasts and greater forecast accuracy than the other methods, forecasts using HU_R are affected by unusual improvements in age-specific mortality patterns more than forecasts using the other two HU variants (see the country-specific example of Hungary for more details). It should be noted that although I tried to utilize all available data from the HMD; it was not possible for all these countries. Due to caution notes from the HMD, I started the fitting period only from the best available years mentioned in the HMD (Table 2.1), making data quality a restriction for the application of the models. BMS also has some flaws regardless of providing the optimistic forecast for the Baltic counties. BMS considers only the best fitting period instead of taking into account all the observed data. For all nine countries, the data of the last 20 years was found to be more significant for forecasting during model fitting.

To determine the forecast accuracy of the methods (except for the UN forecast), I analyzed the MAE, MSE and ME(e_0) during the out-of-sample evaluation period. The comparison of these measures between the models are given below in Table 3.2, 3.3 and 3.4, respectively. As mentioned previously, the models failing to forecast e_0 higher than the last observed e_0 of corresponding fitting periods (Table 3.1) are omitted. In terms of MAE and MSE, the lowest errors were found for HU_W. For high-mortality contexts, the lowest MAE and MSE were not always obtained using the identical method for a country. Most of the times, all the models overestimated the mortality rates and produced lower life expectancy as a consequence. The basic LC method returned a high MAE for Estonia and Lithuania. Some values of MSE indicate over-fitting of corresponding models. Forecast accuracies were close for the basic LC variants, except for BMS. Contrary to the optimistic forecast obtained for all three Baltic countries and Slovakia; BMS was not the best forecast technique in term of MAE or MSE. The different results of the LC variants imply the influence of different adjusting techniques for the time component of the model. The LC method without any adjustment for the time component produced different results than those of the existing LC variants. During out-of-sample evaluation all the methods severely underestimated the life expectancies except for LC in case of Lithuania. Nevertheless, in terms of forecast accuracy or an optimistic forecast, it was not possible to declare a particular model unquestionably and uniquely best for all of these countries. For the developed countries, almost the same situations were observed, prompting the conclusion that no model performed uniquely well for all countries (Shang *et al.*, 2011; Shang, 2012).

Table 3.2: Comparison of MAE from different methods during out-of-sample evaluation period.

Country	LC	LC_P	LM	BMS	HU	HU_R	HU_W	LL
Belarus	-	-	0.238	-	-	-	**0.123**	-
Bulgaria	0.198	0.178	0.178	0.180	0.173	0.164	**0.150**	0.191
Estonia	0.351	0.347	0.369	0.347	0.365	0.422	**0.346**	0.378
Hungary	0.235	0.231	0.183	0.170	0.180	0.171	**0.156**	0.184
Latvia	0.244	0.244	0.268	0.245	0.252	0.310	0.256	**0.205**
Lithuania	0.390	0.217	0.228	0.225	0.199	0.200	**0.187**	0.210
Russia	-	-	-	-	-	**0.219**	-	-
Slovakia	0.192	0.189	0.230	0.182	0.200	0.189	**0.185**	0.228
Ukraine	-	-	-	-	-	**0.168**	0.171	-

A blank place means the forecast of e_0 was lower than the last observed value of e_0.

Table 3.3: Comparison of MSE from different methods during out-of-sample evaluation period.

Country	LC	LC_P	LM	BMS	HU	HU_R	HU_W	LL
Belarus	-	-	0.109	-	-	-	**0.038**	-
Bulgaria	0.068	0.057	0.067	0.067	0.061	0.059	**0.052**	0.074
Estonia	0.215	**0.206**	0.272	0.210	0.214	0.277	**0.206**	0.275
Hungary	0.117	0.112	0.081	0.077	0.085	0.081	**0.070**	0.076
Latvia	0.143	0.143	0.180	0.142	0.142	0.194	0.162	**0.116**
Lithuania	0.347	0.104	0.135	0.118	**0.090**	0.097	0.092	0.115
Russia	-	-	-	-	-	**0.072**	-	-
Slovakia	0.084	**0.080**	0.121	0.083	0.096	0.096	0.094	0.138
Ukraine	-	-	-	-	-	0.046	0.050	-

A blank place means the forecast of e_0 was lower than the last observed value of e_0.

Table 3.4: Comparison of ME(e_0) from different methods during out-of-sample evaluation period.

Country	LC	LC_P	LM	BMS	HU	HU_R	HU_W	LL
Belarus	-	-	**-1.888**	-	-	-	-1.981	-
Bulgaria	**-0.443**	-0.725	-0.820	-1.143	-1.064	-1.182	-0.464	-1.225
Estonia	-2.017	-2.179	**-2.012**	-2.143	-3.181	-3.865	-2.740	-2.206
Hungary	-0.896	-1.179	-0.514	-0.224	-0.887	-0.710	**-0.582**	-0.845
Latvia	-1.393	-1.400	-1.388	-1.398	-1.712	-2.249	-1.481	**-0.748**
Lithuania	0.680	-0.436	**-0.078**	-0.890	-0.622	-0.465	-0.155	-0.124
Russia	-	-	-	-	-	**-2.587**	-	-
Slovakia	-0.440	-0.638	-0.544	-0.199	-1.074	-0.798	-0.665	**-0.363**
Ukraine	-	-	-	-	-	**-1.085**	-1.450	-

A blank place means the forecast of e_0 was lower than the last observed value of e_0.

3.1.2 Coherent forecasting

I consider the reference group for coherent forecast only from the countries of interest, so only the high-mortality CEE countries instead of combining with countries from low-mortality regimes (Li and Lee, 2005). The models are fitted from 1970 to 2010 for coherent forecasting due to data problems in earlier years and ended with Bulgarian mortality data for 2010. The coherent mortality forecast could not produce more optimistic results than the LC and HU variants for any of the countries. However, LL was the most accurate method for Latvia in terms of MAE and MSE and for Slovakia in terms of MSE. The scope of coherent mortality forecasting for these comparatively high-mortality countries became restricted due to the lack of long time series data. The results of coherent mortality forecasting for theses countries are shown in Table 3.1. LL produced the most pessimistic forecast among all methods for Bulgaria, Hungary, Latvia and Slovakia. For the sake of comparability, the same fitting period (1970-2010) is used for different LC and HU variants, which results are shown in Table 3.5.

TABLE 3.5: Forecast of life expectancy in 2050 by different methods considering harmonized fitting period (1970:2010).

Country	e_0	LC	LC_P	LM	BMS	HU	HU_R	HU_W	LL
Belarus	76.49	75.12	74.20	77.41	75.20	73.18	73.35	**78.58**	75.60
Bulgaria	77.25	79.92	**80.09**	79.95	**80.09**	76.92	79.17	77.43	77.55
Estonia	80.55	84.73	84.66	84.82	**88.23**	79.91	79.93	80.57	84.71
Hungary	78.34	82.30	82.42	82.78	84.79	85.71	86.42	**86.44**	82.60
Latvia	77.39	**80.28**	79.60	79.70	79.75	78.62	78.35	79.83	80.10
Lithuania	78.73	80.96	79.76	80.52	79.66	79.41	78.96	81.45	**81.63**
Russia	74.86	71.75	72.39	75.94	71.14	72.97	73.14	**74.89**	74.73
Slovakia	79.15	83.50	83.45	83.49	**84.67**	82.69	82.67	83.45	82.37
Ukraine	75.19	74.45	73.36	75.70	73.00	72.52	74.46	**78.90**	74.60

Results are shown for females only.
e_0 is life expectancy at birth of 2010 from the HMD (Table 2.1).

The performance of the coherent mortality forecast was greatly affected by three aspects: (i) combining mortality rates of a population with large exposure to mortality rates of a population with comparatively smaller exposure size; (ii) high adult male mortality for several of these countries and (iii) irregular trend of life expectancy of joint mortality data because of (i) and (ii). The observed joint mortality rates and fitted parameters of first stage LC models are illustrated in Figure 3.2 and 3.3 for more details.

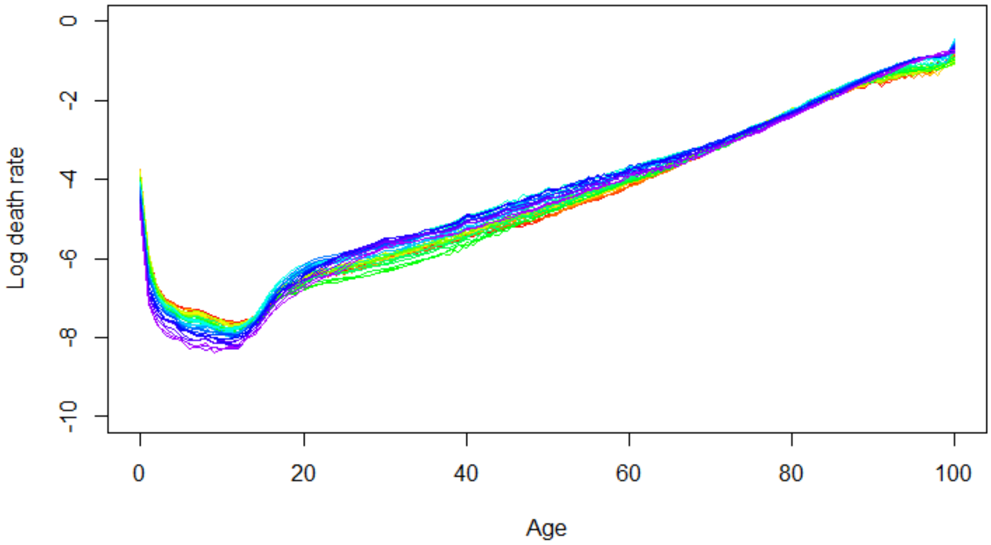


FIGURE 3.2: Observed log mortality rates for combined mortality data of (comparatively) high mortality countries (1970:2010). Years are plotted using a rainbow palette so the earlier years are shown in red, followed by orange, yellow, green, blue and indigo with the most recent years plotted in violet. Mortality crisis in recent period is more visible than previous trend specially for adult and later senescence age groups.

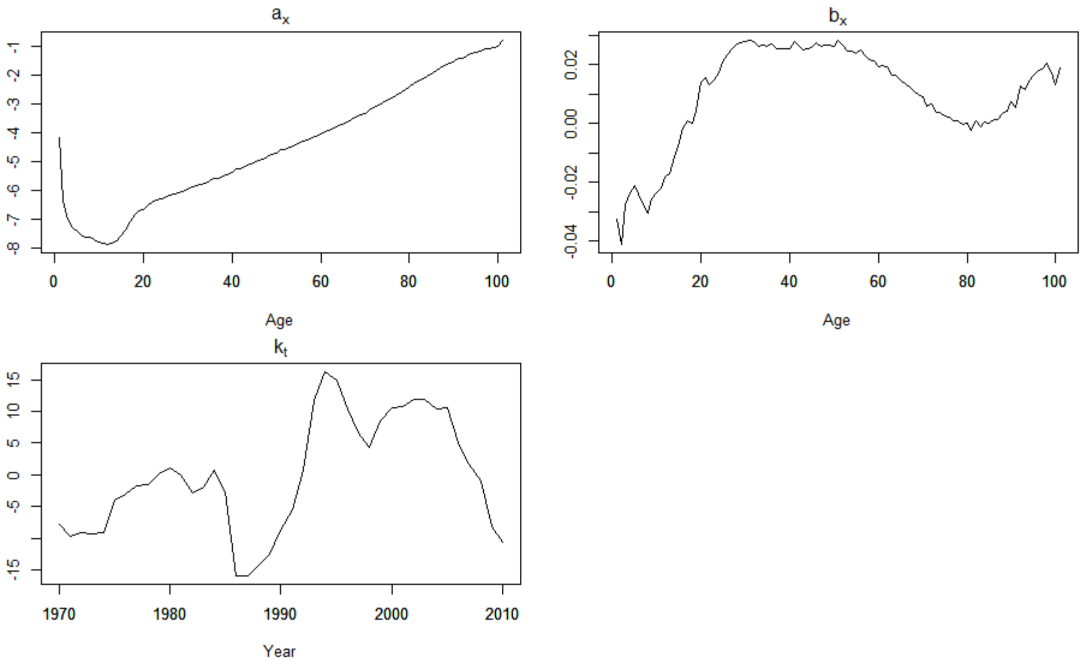


FIGURE 3.3: Estimated parameters of first stage LC modeling on joint mortality data (1970:2010). Irregular trend of e_0 in joint mortality data is reflected on k_t.

Although LL performed reasonably well for most of these high-mortality countries, it failed to do so for Belarus, Russia and the Ukraine. During estimation of the common factor, countries with high mortality dominated the comparatively low-mortality countries. This might be due to mixing large exposure with smaller exposure or combining mortality rates of populations with different age- and cause-specific mortality patterns. Because country-level mortality has two parts, common factor from the reference population and country-specific estimate of $b_{x,i}k_{t,i}$, the country-specific forecast is affected because the common factor of the reference population greatly affects completely different mortality patterns. This is another consequence of using coherent forecasting for population groups with increasing mortality differences over time (mortality divergence). LL adjusts the time component of the common factor according to the estimated life expectancy of combined mortality data. The estimated life expectancy for these high-mortality countries has irregular trend that is different than the country-specific individual trend of life expectancy in most of the cases.

Coherent forecasting of current study also indicates necessity of a rigid assumption for choosing a similar group of countries (Kjærgaard *et al.*, 2016). The six countries for which the coherent mortality forecast of life expectancies is higher than the last observed life expectancies are all currently member states of the European Union. In the current methodology of coherent forecasting, the fitting period of combined mortality data might be shorter for individual countries with longer time series data; this may affect the forecast as well.

The problem of a shorter fitting period for all LC and HU variants are re-discovered during the analysis of the forecasts with a harmonized, shortened fitting period. In addition to the coherent mortality forecasting, previous methods also suffered for shortened fitting periods. Using the harmonized fitting period for Bulgaria and Estonia, the HU model failed to produce a higher forecast than the last observed life expectancy; $\mathrm{HU_R}$ failed to do so for Estonia and Russia. It was mentioned in earlier studies that long time series is preferable for the fitting of these models, however, that condition could not be held for many of these countries due to data problems (Booth *et al.*, 2002). As the HMD mentioned, there was lower data quality for some of the years for a few of these countries due to the data source; I tried to fit the models by both including and excluding those years. The forecasts obtained by considering those years during the fitting period were misleading. However, it was not possible to fit the models for Estonia and Lithuania by excluding the problematic years, as they are almost in the middle of the fitting period for Estonia, and exclusion could make the fitting period too short for the forecasts in the case of Lithuania. In several cases, particular models failed to produce higher estimates of life expectancies than the last observed one; estimates obtained considering all available data for model fitting differed.

3.1.3 UN forecasting

I extend the comparison of the mortality forecasting models with probabilistic forecasting obtained through a Bayesian framework (UN forecast) only for the CEE countries as LC variants could not perform well for several of them. It should be noted that the UN life expectancy forecasts shown in Table 3.1 refer to years 2050-55, as this technique uses 5 calendar years. For the nine countries considered in this study, I utilized the life expectancy at birth in five-year intervals from the HMD instead of using the UN data. I project the life expectancy at birth up to year 2100; to compare with the LC and HU variants the results are shown up to 2050. In addition to the data from HMD, I also used the data from the World Population Prospects (UN, 2013), the comparison between these two different data sources is given in the appendix. The simulation was done with 160,000 iterations (10,000 burn-in), the thinning interval was 10, and the number of chains was three. It was already mentioned in the previous section that an irregular trend in mortality is visible in the case of several of these countries. The trend of life expectancies computed on a five-year basis was also similar to that; fluctuations in the trend for life expectancies remained for Belarus, Russia and the Ukraine during the fitting period. Like HMD, a similar pattern was also observed in case of data from World Population Prospects. Nevertheless, UN forecasts showed an increasing trend for life expectancy at birth for all these countries (both for HMD and World Population Prospects). Unlike several LC variants and coherent forecasting, the forecast of life expectancies was higher than the last observed one for Belarus, Russia and the Ukraine. For all other countries, the forecast produced using the UN forecast technique fell between the other forecast methods. Of all the forecast methods, the UN forecast were lowest for Hungary and Latvia among all the methods. The forecast of life expectancies (with prediction intervals) are attached in appendix.

One shortcoming of the UN forecasting technique is that it is not based on life table like the previous LC or HU variants. This prevents age-specific mortality forecasting and comparing forecast accuracy in out-of-sample evaluation period, as done before for the LC and HU variants or for the coherent forecasting.

3.2 Country-specific illustration: Hungary

3.2.1 Past mortality trends

I assessed all the models in the previous section without focusing on a specific country to get an overall view of mortality forecasting in a high-mortality context. To illustrate the performance of different mortality forecasting models in a more detailed way, Hungary is considered in this section as a representative of these countries. Hungary is chosen because it has a high-mortality regime similar to many other countries in Eastern Europe. The common features of the mortality scenario of these countries can be characterized by the presence of a high level of mortality from Cardiovascular diseases and several external causes of deaths (Bálint and Kovács, 2015). In the beginning of the 1990s, life expectancy of the Hungarian population was among the lowest in Europe. The recent gains in longevity are related to specific causes of death. Decomposition of life expectancy at birth showed that the seven-year gain in male life expectancy between 1990 and 2013 was mainly attributable to a decline in cardiovascular mortality, which corresponds to 40% of the total gain in longevity (Bálint and Kovács, 2015). The decline in mortality due to external causes of deaths has resulted in an increase in life expectancy of 1.7 years. The trend of life expectancy at birth of Hungary is plotted below in Figure 3.4.

FIGURE 3.4: Observed life expectancy at birth of Hungary (1960-2014).

The main source of gain in female life expectancy was the result of the decline in adult and senescence mortality for some particular causes of deaths. Previous studies revealed the contribution of the female population aged 64 years and over was more substantial on increasing life expectancy than that of middle-aged females those were also affected by the economic crisis in past (Bálint and Kovács, 2015; Bohk and Rau, 2015). The log mortality rates for Hungarian Males and Females from the HMD are given below (Figure 3.5). The irregular patterns in mortality are visible in the young and senescence age groups along with gender gap in different age groups.

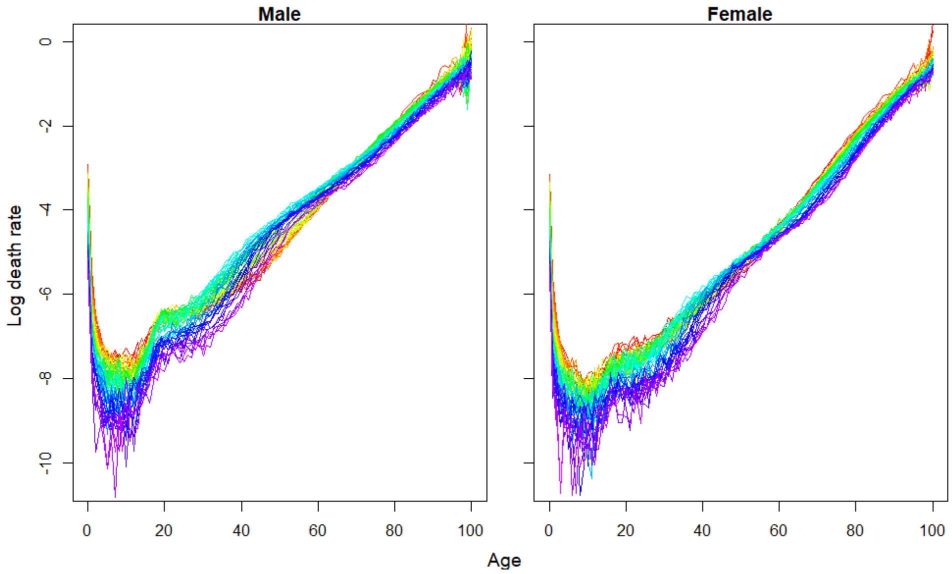


FIGURE 3.5: Observed log mortality rates of Hungary (1960-2014). Years are plotted using a rainbow palette as Figure 3.1.

3.2.2 Forecast of life expectancies

To compare the different methods more precisely in a high-mortality context, I compared the forecasts for all the LC and HU variants and coherent forecasting for males and females separately for Hungary. The results are given below in Table 3.6. I extrapolated the results for 20, 30 and 40 years ahead (24, 34, 44 years for LL) to see any possible convergence of the forecasts using different methods. Life expectancy at birth was 72.26 and 79.24 years, respectively, for males and females in 2014; while for coherent forecasting, it was 70.59 and 78.34 years, respectively, in 2010. Except for LC, $\mathrm{LC_P}$ and BMS, all the models produced a higher forecast of life expectancy than the last observed e_0 of Hungarian males. For females, all the models produced an optimistic forecast. I also forecast life expectancy at age 65 to see the performance of the life expectancy forecasting in later ages and extrapolated for 20, 30 and 40 years ahead (Table 3.7). Life expectancy at age 65 was 14.56 years and 18.40 years, respectively, for males and female in 2014, while it was 13.94 years and 17.88 years, respectively, in 2010 for coherent forecasting. Although LC and BMS failed to produce optimistic forecasts of life expectancies at birth, the forecast of e_{65} was higher than the last observed e_{65} for both of the methods; which was not the case for LC_P. This better performance in later ages are rather acceptable compare to the pessimistic forecast of life expectancy at birth which is more sensitive on change of mortality in different part of life span. In the other cases, all the models performed well, although higher forecasts are observed from HU_W for females compared to the other methods. Moderate improvements in the mortality of people aged 65 years or over were observed in previous studies as well, unlike the pattern observed for the middle-aged population (Bálint and Kovács, 2015).

TABLE 3.6: Forecast of life expectancies at birth for Hungarian males and females.

Forecast period	LC	LC_P	LM	BMS	HU	HU_R	HU_W	LL
Male, 2034	70.56	70.14	73.93	70.14	77.35	77.53	**77.96**	73.19
Female, 2034	81.17	81.06	81.58	82.90	82.22	80.85	**83.86**	80.96
Male, 2044	70.31	70.34	74.42	70.34	79.33	79.70	**80.42**	74.23
Female, 2044	82.18	82.07	82.65	84.50	83.47	81.94	**85.91**	82.00
Male, 2054	69.99	70.50	74.82	70.50	81.03	81.63	**82.97**	75.24
Female, 2054	83.16	83.05	83.68	85.98	84.64	82.99	**87.81**	83.01

TABLE 3.7: Forecast of remaining life expectancies at age 65 for Hungarian males and females.

Forecast period	LC	LC_P	LM	BMS	HU	HU_R	HU_W	LL
Male, 2034	**21.04**	14.75	17.14	14.77	16.65	17.29	17.42	15.21
Female, 2034	20.29	20.19	20.26	20.93	20.59	19.35	**21.72**	19.49
Male, 2044	**22.84**	15.24	18.61	15.26	17.71	18.49	19.10	15.77
Female, 2044	21.15	21.06	21.18	22.12	21.52	20.07	**23.35**	20.17
Male, 2054	**25.02**	15.73	20.27	15.76	18.73	19.67	21.10	16.35
Female, 2054	22.00	21.91	22.08	23.27	22.41	20.82	**24.90**	20.87

3.2.3 Forecast of mortality rates

Besides optimistic forecast of life expectancies and accuracy during out-of-sample evaluation period, I examined more deeply the ability of the methods to capture the mortality improvement over the life span in this section. It should be noted that some methods have different performances for different stages of lifespan (Table-3.6 & 3.7). The fitted parameters of the LC model with a forecast of parameter k_t and the observed and fitted log mortality rates for females are presented below (Figures 3.6 and 3.7). As discussed before, the parameter k_t is used for the forecast and the blue spread of k_t in Figure 3.6

may be implemented for interval forecasting as well (Hyndman and Ullah, 2007).

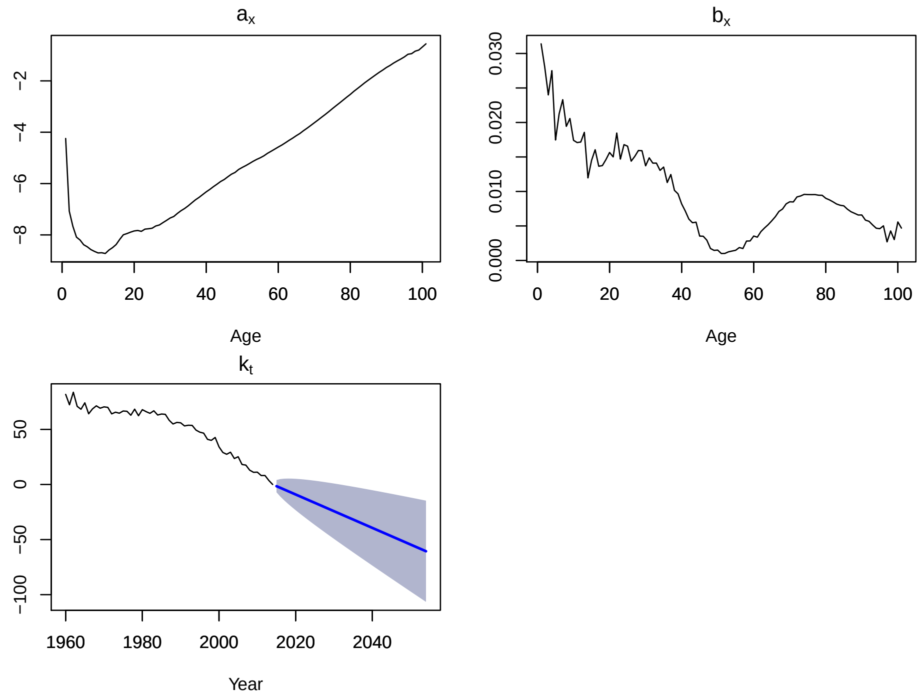


FIGURE 3.6: Fitted components of basic LC method for females of Hungary (1960:2014). The blue area of the parameter k_t presents spread of the parameter under a random walk with drift which is used to make forecast.

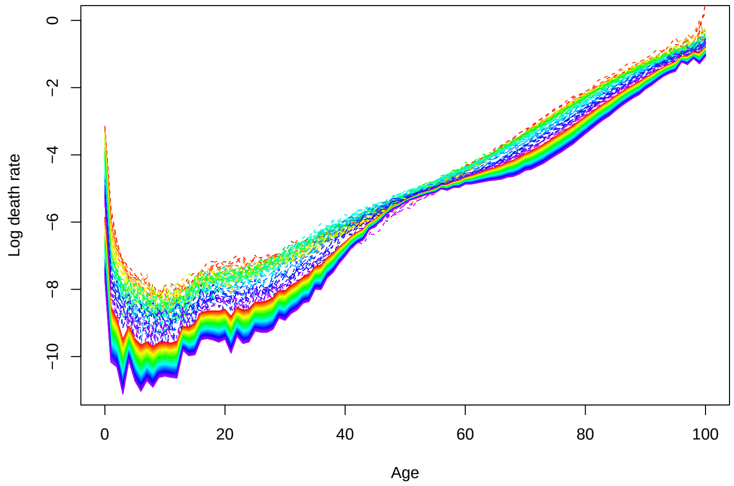


FIGURE 3.7: Observed (1960:2014) and forecast (2015:2050) of log mortality rates for females of Hungary by basic LC method. Years are plotted using a rainbow palette as before. Observed mortality rates are plotted using dotted line whereas forecast are plotted with regular line.

Figure 3.8 shows the forecast of log mortality rates till 2050 for Hungarian Females using different methods.

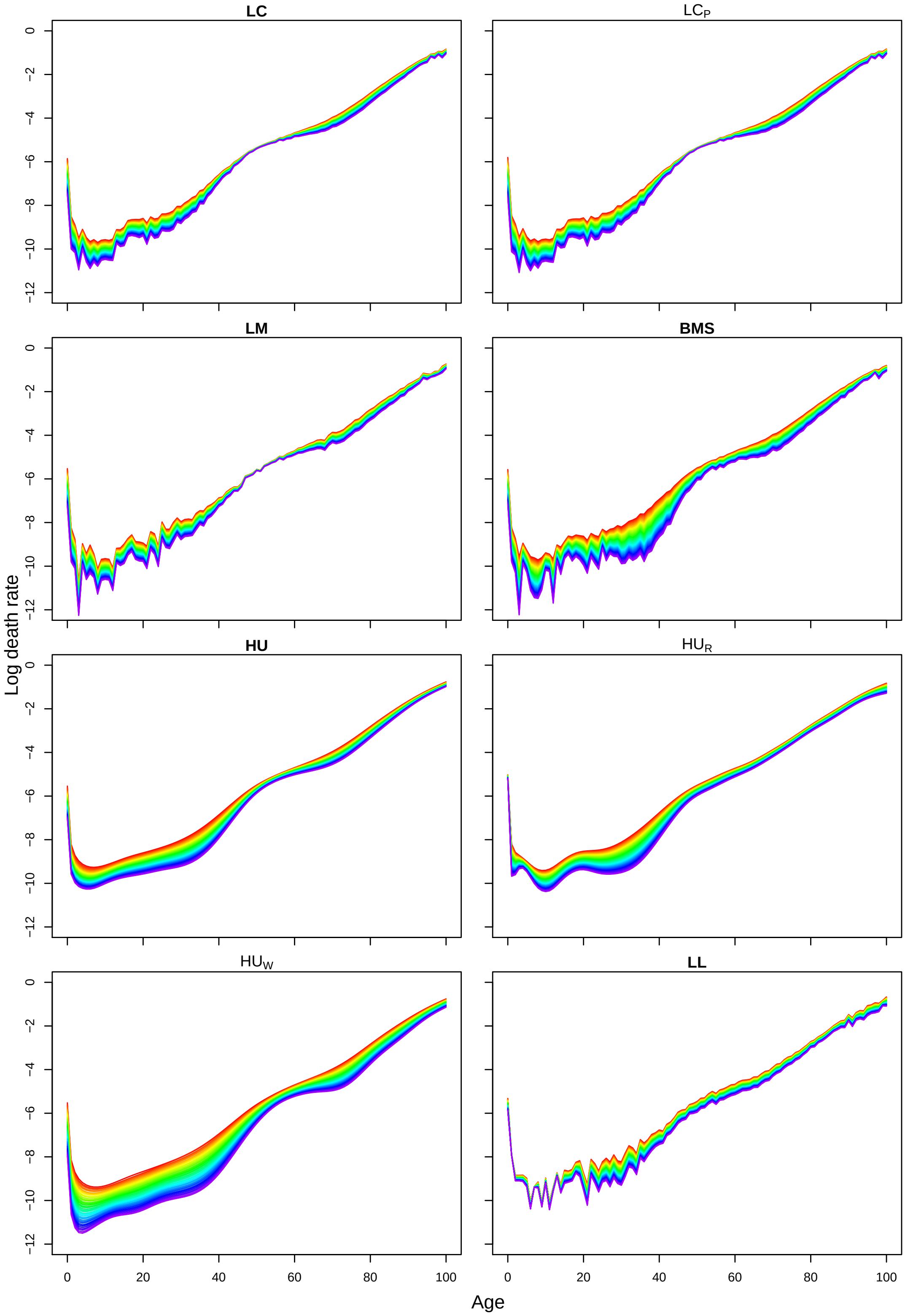


FIGURE 3.8: Comparison of forecast of log mortality rates till 2050 for females of Hungary. Years are plotted using a rainbow palette as before.

Except for HU and HU_W, all other methods are affected by the recent short-run improvement of early-aged mortality around age 5 to 15. For Hungarian female mortality, this improvement occurred only for a few of the very recent years (Figure 3.5) which highly affects the forecasts of BMS, HU_R and LL. All LC variants suffered from reduction in variability over years in senescence and old ages. This is a clear consequence of the estimated b_x in current LC methodology. However, this feature is mitigated with the coherent forecasting since part of the information about rate-of-change in LL is shared among different countries with diverging mortality patterns. Besides, combined mortality rates of the reference population were higher across the life span compares to observed female mortality rates for Hungary. As a consequence, the coherent forecasts for the Hungarian Females are more pessimistic than those of the other methods (Table 3.6 and 3.7). This has been already mentioned as a drawback of current settings for choosing reference population in coherent forecasting. Clearly, the increasing mortality differences over time between the countries of the reference group have high impact on individual country-specific forecasts. Forecasts of senescence mortality are virtually similar for all methods except for coherent forecasting due to this reason.

Chapter 4

A Modified Lee-Carter Method with LASSO type Smoothing and Adjusting for Lifespan Disparity

Here it is proposed to fit the Lee-Carter model over smoothed mortality rates and to partially adjust the fitted time component of Lee-Carter method according to observed lifespan disparity, $e_0^\dagger$ (Vaupel and Canudas-Romo, 2003; Zhang and Vaupel, 2009). Instead of spline based smoothing techniques I smoothed the observed mortality rates using *LASSO* type regularization (Dokumentov *et al.*, 2018). The rational behind these modifications and details of the estimation procedure are explained in the following sections.

4.1 Application of smoothing technique

Application of smoothing technique is common to improve mortality forecast (Booth and Tickle, 2008; Hyndman and Booth, 2008). Smoothing techniques are widely used for extrapolation and reducing noises of the observed data to overcome outliers problem which is quit common for mortality data (see Hyndman and Ullah, 2007, for example). Smoothing is particularly useful in Lee-Carter framework for another particular reason. From equation 2.1, Lee-Carter (LC) type model estimates the product of time component (k_t) and age component (b_x) obtained from singular value decomposition of mortality matrix and then add it with observed and invariant a_x. Although LC method assumes linear trend of k_t, still this is not completely linear even after all proposed adjustment policies. Similarly, the estimated b_x have irregular trend in different parts of the life span. The product of these two parameters with irregular trends become more jagged over time which affects the estimated log mortality rates obtained using equation 2.1 (Girosi and King, 2006). To illustrate this, the product of b_x and k_t for Swedish Females are plotted below. Typical estimate of a_x for LC variants can be seen in Figure 3.6 or Figure 4.7.

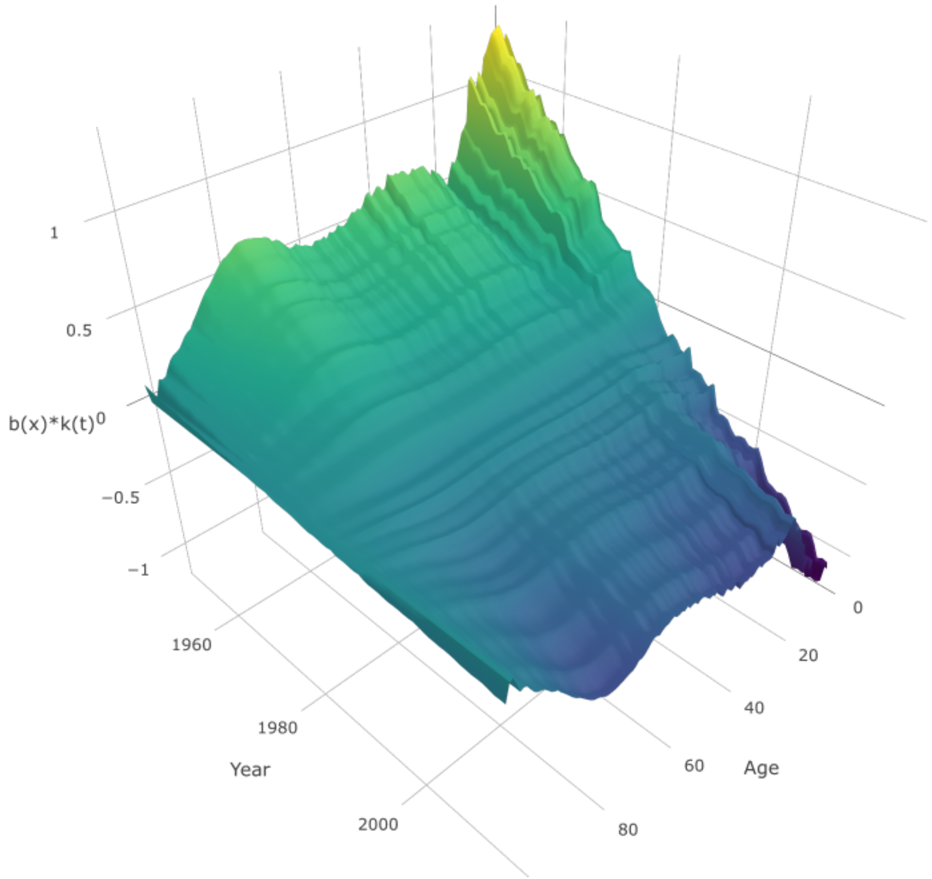


FIGURE 4.1: Product of time component and age component for Swedish Females (1950:2016) using LC (Lee and Carter, 1992). Irregular trends are visible in earlier and later part of lifespan.

To overcome this problem, different smoothing techniques can be applied in three different ways: (a) smoothing the observed mortality rates first and then fitting the model for forecasting; (b) smoothing over the fitted parameters and to make the forecast by standard time series techniques or, (c) applying smoothing on fitted/forecasted rates by the model. Following Hyndman and Ullah (2007), I also applied the smoothing prior to model fitting. Although spline based smoothing techniques are widely used for mortality, I used LASSO type regularization to smooth the mortality curves. One of the shortcomings of these spline-based smoothing techniques is, they might over-smooth the mortality curves which thus reduces the accuracy of smoothed data to be used for model fitting. This is observed when we compared the smoothing techniques on different countries, having countries of high-mortality regime suffered more than low-mortality ones. To address this problem, a smoothing technique need to find an optimal balance between reducing noise and keeping accuracy of the observed data, particularly for mortality forecasting. Spline based smoothing techniques are already discussed in Chapter 2. The methodology of Lasso is discussed in following section.

4.1.1 Smoothing by *LASSO*

For the proposed modifications in LC methodology, first I smooth the mortality rates using *LASSO* type regularization. Dokumentov *et al.* (2018) defined the Lasso derived from a two-dimensional thin plate spline which is used to smooth the mortality rates considering age and period effects. For observed mortality rates y, age x and year t, the two dimensional thin plate spline is defined as a function $f(x,t)$ which minimizes the following:

$$\mathcal{J}\left(\{y_i\}_{i=i}^{n}, f\right) = \sum_{i=1}^{n}\left(y_i - f(x_i,t_i)\right)^2 + \lambda\int\left[\frac{\delta^2 f}{\delta x^2} + 2\frac{\delta^2 f}{\delta x \delta t} + \frac{\delta^2 f}{\delta t^2}\right] dxdt. \tag{4.1}$$

Here, $\lambda > 0$ is the smoothing parameter with $(x_i,t_i)_{i=1}^{n}$ knots (following, Wood, 2006). The expression in equation 4.1 can be approximated by a sum if the knots form a regular grid. In that case, the second partial derivatives can be approximated also as linear combinations of function values at nearby knots. Denoting the mortality rates as a vector y as before (which is a two dimensional data packed as vector) and letting $\{f(x_i,t_i)\}_{i=1}^{n}$ as vector z, the expression of 4.1 can be re-written as,

$$\mathcal{J}(y,z) \approx ||y-z||_{L_2}^2 + \frac{\lambda}{n}\left(||D_{xx}z||_{L_2}^2 + ||D_{xt}z||_{L_2}^2 + ||D_{tt}z||_{L_2}^2\right), \tag{4.2}$$

where,

$$D_{xx} = \left\{\frac{\delta^2}{\delta x^2} f(x_i,t_i)\right\}_{i=1}^{n}, \; D_{xt} = \left\{\frac{\delta^2}{\delta x \delta t} f(x_i,t_i)\right\}_{i=1}^{n} \text{ and } D_{tt} = \left\{\frac{\delta^2}{\delta t^2} f(x_i,t_i)\right\}_{i=1}^{n}.$$

This expression of equation 4.2 can be approximated with thin plate spline computed in its knots taking the points of z for which right side of 4.2 is minimized. Following Schuette (1978), Dokumentov *et al.* (2018) replaced the L_2 norm by a L_1 norm to smooth with quintile Lasso. Schuette (1978) showed that L_1 norm is more robust than L_2 norm in presence of outliers, which occurs very often in case of mortality data. (Schuette, 1978) proposed to use different λ coefficients before every derivative to adjust the influence of each derivative distinctly on smoothing. Hence the smoothing can be defined as,

$$Q(y) = \arg\min_{z} \{K(y,z)\}, \tag{4.3}$$

$$\text{where, } K(y,z) = ||y - Mz||_{L_1} + \lambda_{xx} ||D_{xx}z||_{L_1} + \lambda_{xt} ||D_{xt}z||_{L_1} + \lambda_{tt} ||D_{tt}z||_{L_1}.$$

Here, M will be an identity matrix as same number of knots and data points are considered which are positioned at same places (Dokumentov *et al.*, 2018). Following Wood (2006), Dokumentov *et al.* (2018) stacked matrices M, $\lambda_{xx}D_{xx}$, $\lambda_{xt}D_{xt}$, $\lambda_{tt}D_{tt}$ on top of each other to give

$$R = [M', \lambda_{xx}D'_{xx'}, \lambda_{xt}D'_{xt'}, \lambda_{tt}D'_{tt}]' = \begin{bmatrix} I & I & I \\ \lambda_{xx}D_{xx} & 0 & 0 \\ \lambda_{xt}D_{xt} & 0 & 0 \\ \lambda_{tt}D_{tt} & 0 & 0 \end{bmatrix}. \tag{4.4}$$

In next step, the data vector y is extended by zeros until its length is equal to the number of rows in R. So, the extended y became: $y_{ext} = [y', 0']'$. The right hand side of 4.3 is then replaced with the equivalent expression,

$$K(y,z) = ||y_{ext} - Rz||_{L_1}. \tag{4.5}$$

Obtaining $Q(y)$ is then a quantile regression problem. Like spline based smoothing techniques, the smoothing from Lasso also depends on values of λ_{xx} (controls the flexibility of the smooth surface in age direction), λ_{xt} (controls the flexibility of the smooth surface in age and time direction) and λ_{tt} (controls the flexibility of the smooth surface in time direction). A subsequent problem to utilize quantile Lasso is to optimize $\lambda_{xx}, \lambda_{xt}$ and λ_{tt}, because the smoothness of the obtained mortality curves depend on these three parameters (also accuracy of Lasso). It is possible to have many local minima while dealing with mortality data. To overcome this problem, Dokumentov *et al.* (2018) used "Nelder–Mead"optimization to get the global minima of the function.

4.2 Definition of lifespan disparity

The second proposed modification on traditional LC method is to partially adjust the fitted time component of LC model according to the observed lifespan disparity instead of total number of deaths or life expectancy at birth or age distribution of deaths. Lifespan disparity illustrates the variation in the lifespan distribution which is the differences in the length of life across members of a population. In this study, I utilize the definition of Vaupel and Canudas-Romo (2003) and Zhang and Vaupel (2009) where lifespan disparity is defined as average number of life years lost at birth. Symbolically,

$$e_0^{\dagger} = \frac{\displaystyle\int_0^{\omega} e_x d_x \, dx}{l_0} \approx \frac{\displaystyle\sum_0^{\omega} e_x l_x m_x}{l_0} \tag{4.6}$$

Here, ω is the maximum attainable age, d_x is the distribution of death and l_x is the number of people alive at age x (l_0 is the life table radix). Thus estimation of $e_0^{\dagger}$ is simple and straightforward. It can be easily implemented by the above expression with the assumption that deaths are Poisson distributed.

There are several benefits of considering $e_0^{\dagger}$ as an alternative measure of longevity. First, it is a robust indicator of mortality improvement as it considers distribution of deaths along with remaining life expectancies (equation 4.6). Unlike life expectancy at birth, $e_0^{\dagger}$ provides more information about shrinking/expansion of mortality and it can be utilized also to get information on mortality shifting (Zhang and Vaupel, 2009). An extension of $e_0^{\dagger}$ can be also used to derive threshold age between premature and senescence mortality for a population (Zhang and Vaupel, 2009). Like life expectancy, the lifespan disparity measure has the property that it can be additively decomposed at any age such that the components before and after this age sum to the total life disparity (Vaupel and Canudas-Romo, 2003). Vaupel *et al.* (2011) analyzed the trend of $e_0^{\dagger}$ for 40 different countries from different regions considering long time period data. They observed high correlation between $e_0^{\dagger}$ and e_0 among countries and concluded that progress in reducing premature deaths reduces variation in lifespan, whereas progress in reducing deaths at older ages increases variation in lifespan. Due to its stability over time, previous study considered $e_0^{\dagger}$ to evaluate forecast performances of different forecasting techniques rather than just considering the fitted mortality rates or life expectancy (Bohk-Ewald *et al.*, 2017). It should be noted that, there are several other definition of lifespan disparity including classic statistical variability measures (standard deviation or the interquartile range) or common equality measures (for example, Gini coefficient) and others (Bohk-Ewald *et al.*, 2017). As all these measures are highly correlated, it is expected that their impact on the results would be minor (Vaupel *et al.*, 2011; Bohk-Ewald *et al.*, 2017).

4.3 Model fitting and forecasting

After obtaining the smoothed mortality rates by Lasso, I followed the standard Lee-carter methodology to obtain the initial estimate of the age and time component from the smoothed mortality data. The Lee-Carter model mentioned before is given below for recall,

$$\ln(m_{x,t}) = a_x + b_x k_t + \epsilon_{x,t}, \tag{4.7}$$

with same constraints mentioned by Lee and Carter (1992). To imply the proposed methodology, modification on time component begins with estimation of the observed $e_0^\dagger$ (obtained from observed, non-smoothed mortality rates). It should be noted that $e_0^\dagger$ is less sensitive to smoothing performed by Lasso.

The procedure to estimate the model parameters are same as mentioned before in 2.2.1 except for everything is done on smoothed mortality rates. For estimation of the age and time component rank-1 approximation is considered only as it explains most of the variance (Lee and Carter, 1992; Booth *et al.*, 2002). The initial estimates of model parameters are same as mentioned in 2.2.1. For recall,

$$\hat{k}_t = L_1 V_{t_1} \text{ and, } \hat{b}_x = U_{x_1}.$$

Lee and Carter (1992) proposed to conduct a second stage estimate of k_t by finding the value of k_t which, for a given population age distribution and previously estimated a_x and b_x produces exactly the observed number of total deaths for the fitting period of the model. I propose the adjustment of the estimated k_t by partially matching with observed $e_0^\dagger$. This is done by solving the following equation:

$$e^\dagger_{0\ observed} = \sum_{0}^{\omega} \exp(\hat{a}_x + \hat{b}_x.k_{t\ adj}) e_x l_x / l_0. \tag{4.8}$$

The e_x and l_x of equation 4.8 are obtained from life table estimated from fitted data. An *ARIMA(0,1,0)* with drift is then fitted for adjusted $\hat{k}_t$, from which forecast are done.

$$\hat{k}_t = c + \hat{k}_{t-1} + \xi_t \tag{4.9}$$

Here c is the drift term and ξ_t is the model residual. Previous studies applied different econometric models for forecasting but the best output was obtained for Random Walk with Drift (Hyndman and Ullah, 2007). Following Lee and Miller (2001), I used the actual data for forecasting to avoid jump-off error; which is found to be more accurate during out-of-sample evaluation than that of fitted data. Also the forecast are more optimistic for actual data than that of fitted data.

4.3.1 Errors in the mortality forecast

The mortality curves obtained from Lasso are the closest to observed ones compared to spline based techniques for all countries considered in this study. Still, it is necessary to incorporate the smoothing error while estimating the forecast variance. Following Lee and Carter (1992), the forecast error for h year ahead forecast from base period t will be,

$$E_{x,t+h} = \alpha_x + (\hat{b}_x + \beta_x)u_{t+h} + \beta_x \hat{k}_{t+h} + \epsilon_{x,t+h} + \epsilon_s \tag{4.10}$$

Lee and Carter (1992) defined α_x and β_x as errors in estimating the model parameters a_x and b_x respectively whereas u contains the errors due to innovations and errors in estimating the drift. For the proposed model; α_x, β_x and u will contain the individual level of error due to smoothing as well. I estimated the model parameters without smoothing the data and observed the slight effect of smoothing on trend of estimated parameters (with newly adjusted $\hat{k}_t$). Lee and Carter (1992) mentioned that the elements in 4.10 are uncorrelated from the results obtained by bootstrapping and informal experiments. Same concept is applicable for smoothing because the smoothing errors are very low and it is independently done before model fitting. For the error unexplained from the estimated parameters are attributable to smoothing effect, that is why I added the last term as error from smoothing which is independent from other sources of errors to estimate the model. This error term is different than that of nonparametric approaches as those methods considered the smoothed mortality rates as functional form of age implied through smoothing (Hyndman and Ullah, 2007). Consequently Lasso and the proposed adjustment on k_t reduce the overall variance of the forecast. The variance of the forecast for modified Lee-Carter will be,

$$\sigma^2_{E_{x,t+h}} = \sigma^2_{\alpha_x} + \hat{b}^2_x \sigma^2_{u_{t+h}} + \sigma^2_{\beta_x}\left(\hat{k}^2_{t+h} + \sigma^2_{u_{t+h}}\right) + \sigma^2_{E_{x,t+h}} + \sigma^2_s \tag{4.11}$$

During the estimation, the obtained variance for adjusted $\hat{k}_t$ found to be lower than previous model and smoothing also produced lower error.

4.4 Results

4.4.1 Smoothing techniques

For model validation, I considered only the low mortality countries in this chapter. Since I smoothed the mortality rates first before model fitting, so I compared the smoothing techniques before checking the model fitting. I compared the smoothed mortality rates by Lasso (Dokumentov *et al.*, 2018) with one dimensional spline (Hyndman and Ullah, 2007) and two dimensional spline (Camarda *et al.*, 2012). Accuracy of smoothing techniques has been evaluated in terms of lowest mean absolute error (MAE) and mean squared error (MSE). I compared the results for all the low-mortality countries mentioned in chapter 2 and for all of these Lasso provided most accurate mortality curves (with lowest errors). For illustration, the smoothed mortality rates from all three techniques for US Females are given below (Figure 4.2).

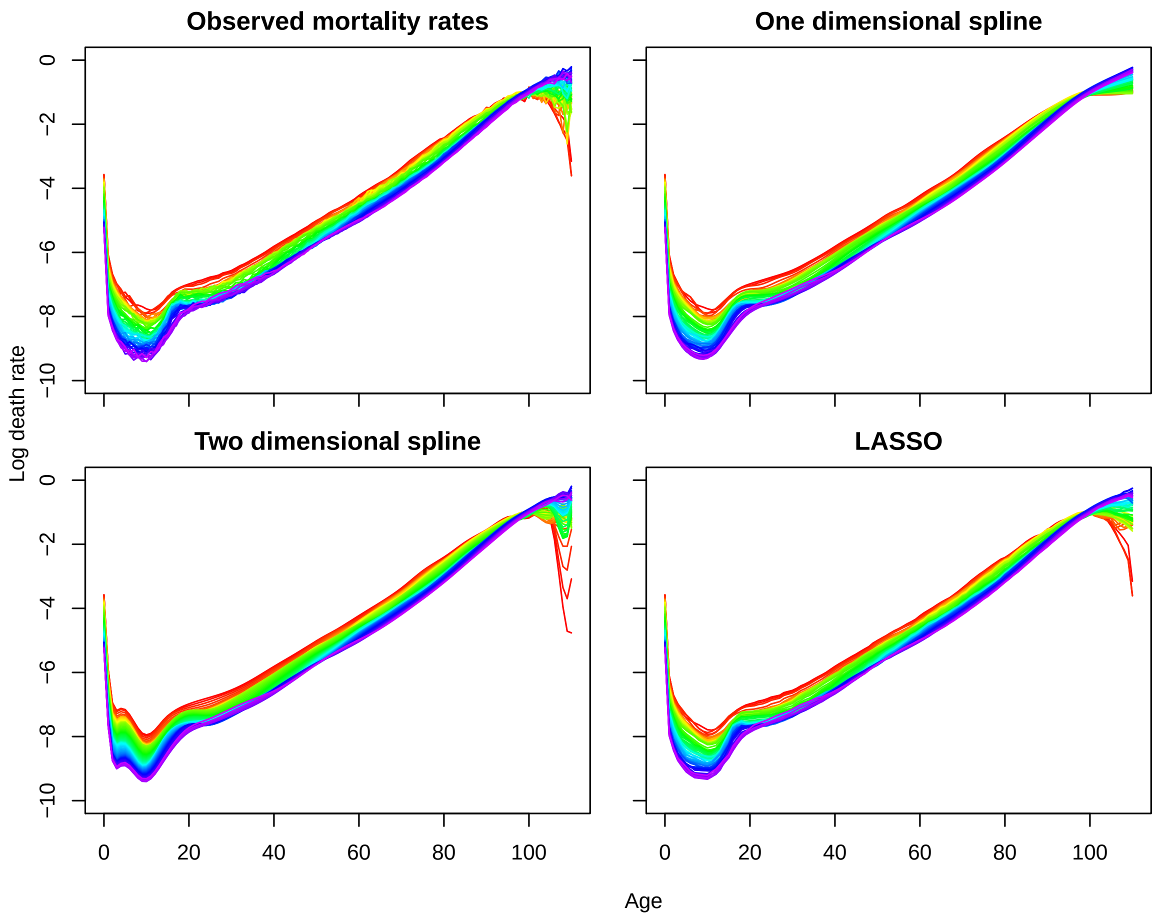


FIGURE 4.2: Comparison of smoothing techniques for US Female mortality (1950:2016). Years are plotted using a rainbow palette so the earlier years are shown in red, followed by orange, yellow, green, blue and indigo with the most recent years plotted in violet.

The mortality trends are regular for almost all the low-mortality countries, so the smoothing techniques also performed well for those countries. USA is considered here because the trend of mortality is slightly different than other countries. US data is characterized by presence of distinct accidental hump and high centenarian mortality rates in earlier era are. To date, USA is one of the most populous countries having the lowest observed life expectancy at birth among the G-7 countries (HMD, 2018), which is highly attributable to particular cause-specific deaths (Tuljapurkar *et al.*, 2000). Compared to spline based smoothing techniques, Lasso provided less smoothed mortality surface for US Females; same pattern is observed for other lower mortality countries as well. For US Females (and many other countries), an unexplainable second drop of mortality is visible in earlier life span in case of two dimensional spline. Previous study mentioned that smoothing with two dimensional technique could be misleading for earlier life as it might produce biased result for earlier part of life (Dokumentov *et al.*, 2018). This bias for infant and child mortality can be avoided by starting the smoothing from age 10 or later but this is not so convenient for mortality forecasting as most jagged pattern of b_x is observed in earlier and later part of life. The transformed error for all three smoothing techniques are illustrated for US Females in Figure 4.3.

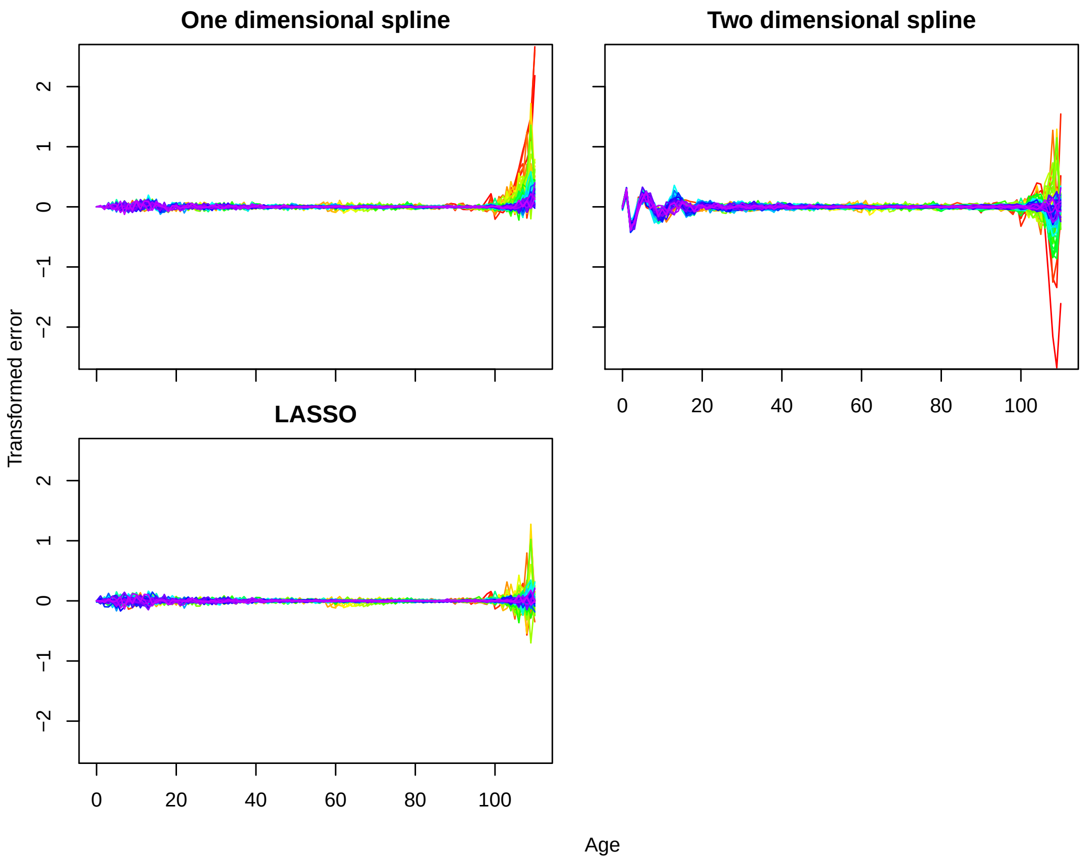


FIGURE 4.3: Errors of smoothing techniques for female mortality of USA(1950:2016). Years are plotted using a rainbow palette as before.

The surfaces of the smoothed death rates along with errors are plotted in Figure 4.4 for the smoothing techniques. The standardized residuals are presented in two dimensional plots in Figure 4.5.

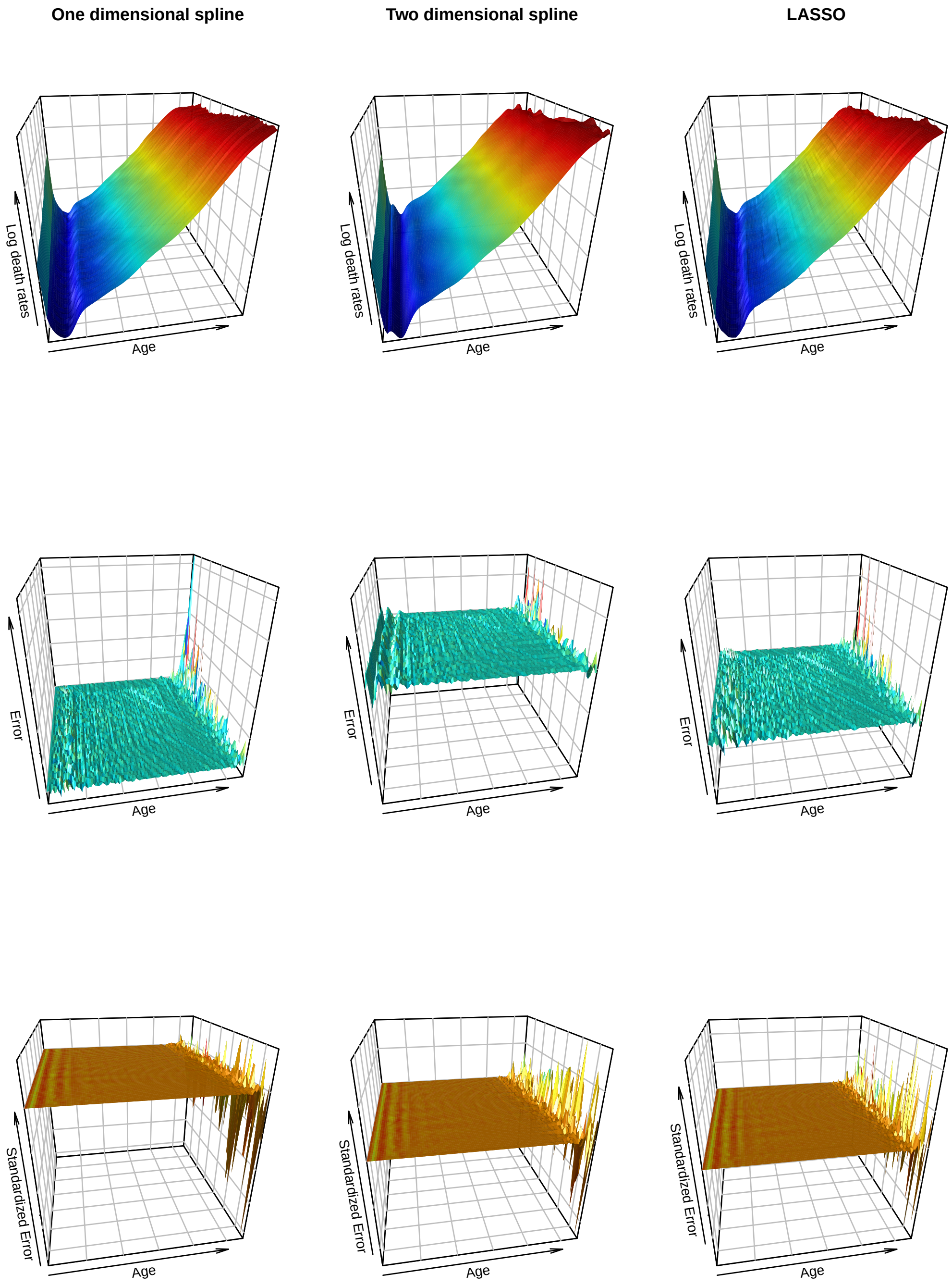


FIGURE 4.4: Mortality surfaces and corresponding errors from difference smoothing techniques for US Females (1950:2016).

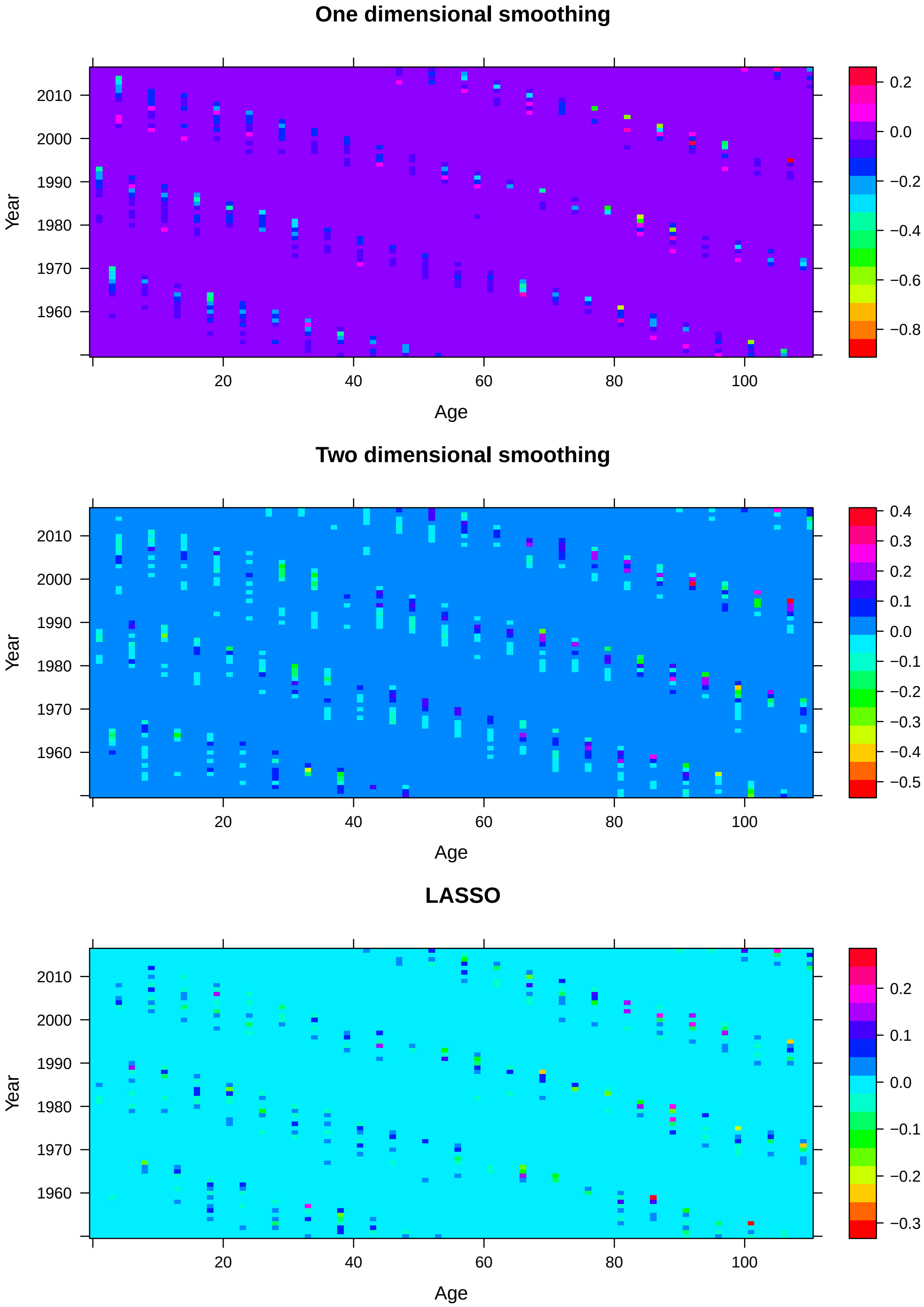


FIGURE 4.5: Standardized residuals of the smoothing techniques for US Females (1950:2016). Difference scale of colors are used due to different distribution of errors.

Standardized residuals in Figure 4.4 and 4.5 have different patterns than that observed before in Figure 4.3. Due to different distributions of errors over the ages and time, it is quite difficult to compare the residuals among these three techniques from Figure 4.5. To illustrate the aspects of Figure 4.5 more clearly, the standardized residuals for some specific years are plotted in Figure 4.6.

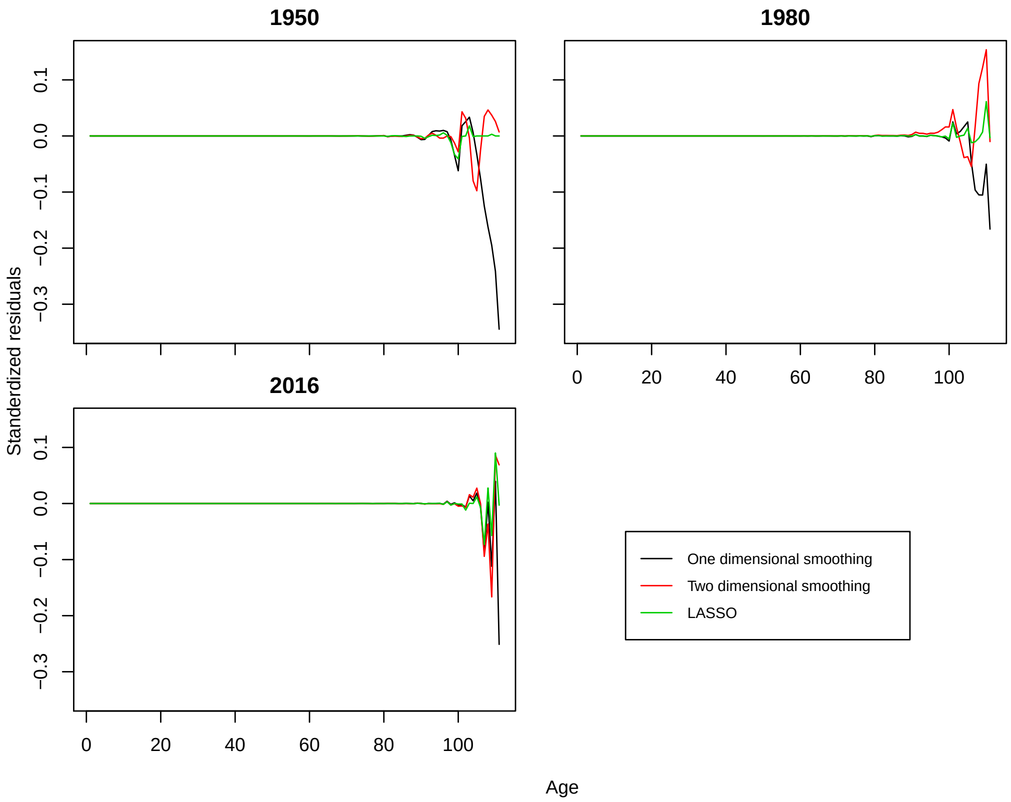


FIGURE 4.6: Standardized residuals of the smoothing techniques for US Females in the years 1950, 1980 and 2016.

Unlike two dimensional smoothing, Lasso and one dimensional smoothing technique performed well for earlier part of life. Presence of accidental hump was well-captured by Lasso compared to spline based techniques. Spline based smoothing are less affective for older ages. One dimensional smoothing could not capture the high centenarian mortality in beginning of 1950s due to monotonic constraint, whereas two dimensional smoothing over-fitted that part. The accuracy of smoothing for US female mortality by these three techniques are summarized in following table (Table 4.1).

TABLE 4.1: Accuracy of smoothing techniques for US Females (1950:2016).

Measure of accuracy	1 dim Smoothing	2 dim Smoothing	*LASSO*
MAE(m_x)×100	3.170	4.817	1.909
MSE(m_x)×100	0.953	1.162	0.215
ME(e_0)	-0.154	0.139	0.950
MAE(e_0)	0.002	0.087	0.016

The errors for mortality rates are magnified by 100 times to show the comparison more precisely. For both absolute and squared errors, Lasso is more accurate than other two smoothing techniques. Following Dokumentov *et al.* (2018), I also compared the accuracy of smoothing from MAE and MSE. I did not consider BIC or other measure of goodness of fit as BIC requires a clearly defined likelihood function. I further reconstructed the life tables from smoothed mortality rates and from that I compared the fitted life expectancy at birth with observed one. Although two dimensional smoothing was most accurate in terms of ME(e_0), still it is questionable due to unexplainable drop of mortality curves. Considering the sign of the errors, Lasso overestimated the life expectancies for most of the years whereas one dimensional smoothing underestimated for most of the years. Due to different pattern of outcome obtained from ME(e_0), I also examined the mean absolute error of life expectancy for more insight. The results showed highest accuracy for one dimensional smoothing followed by lasso and two dimensional smoothing. I did not consider MAE(e_0) to evaluate forecasting in later sections as ME(e_0) is more robust indicator for forecast accuracy which also provides information regarding underestimation or overestimation of life expectancies. Better accuracy for Lasso can be explained from the context of smoothing. Lasso is more a fitting technique with higher emphasize to keeping the fitted values close to observed one. As a result, although the surface is less smoothed for Lasso, still it provides less error for fitting the mortality curves; which is not the case for spline based smoothing techniques.

4.4.2 Model fitting and forecast accuracy

The proposed methodology performs better than all 4 LC variants and 3 HU variants for several countries during out-of-sample evaluation. I denoted the proposed method by $\mathrm{LC}_{e_0^\dagger}$ all over this section. For existing models I use the same notation of previous chapter: LC stands for the basic Lee and Carter (1992); $\mathrm{LC_P}$ stands for Lee-Carter model with Poisson regression (Brouhns *et al.*, 2002); LM stands for modified Lee-Carter model proposed by Lee and Miller (2001); BMS stands for modified Lee-Carter model proposed by Booth *et al.* (2002); HU stands for the non-parametric approach proposed by Hyndman and Ullah (2007); $\mathrm{HU_R}$ stands for robust Hyndman and Ullah (2007); $\mathrm{HU_W}$ stands for weighted Hyndman and Ullah (2007). As non-parametric approaches are different than that of LC variants and the proposed methodology is more close to those, I compared the individual parameters of $\mathrm{LC}_{e_0^\dagger}$ with LC variants only. For illustration of the characteristics of the proposed methodology, I consider the Swedish female mortality in this section. Following the illustration of Figure 4.1, I considered

Sweden again for its consistent and stable trend of mortality improvement since long. In addition, Sweden has the longest time series data for mortality starting from 1751 and well known for mortality data with highest quality (HMD, 2018). Following previous chapters, models are fitted from 1950 and afterward to avoid war and epidemic during demographic transition. The estimated a_x and b_x from LC variants and $\mathrm{LC}_{e_0^\dagger}$ are plotted in Figure 4.7. For same estimation technique, these two parameters are same for all four LC variants.

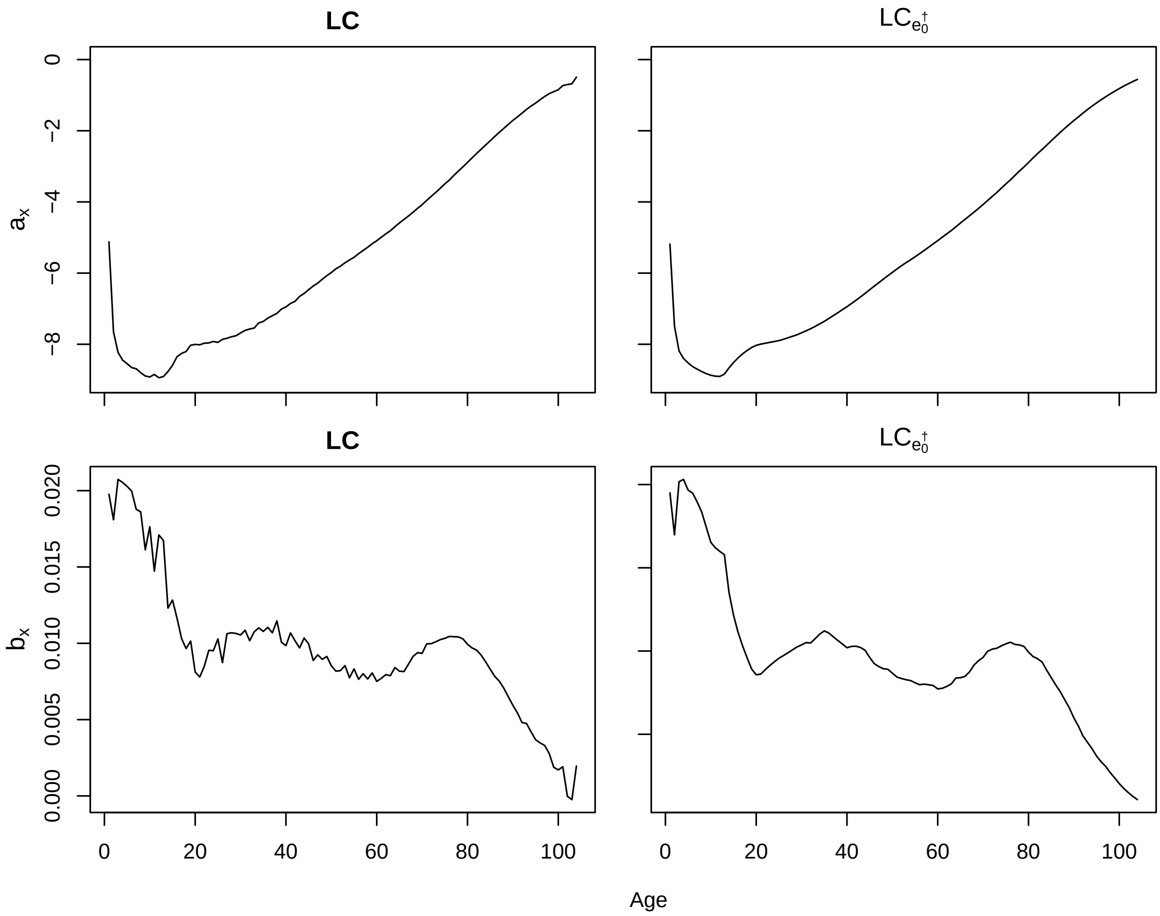


FIGURE 4.7: Estimated a_x and b_x for Swedish Females (1950:2016) from LC variants and $\mathrm{LC}_{e_0^\dagger}$. The estimated parameters from $\mathrm{LC}_{e_0^\dagger}$ slightly different than that of LC variants due to application of smoothing technique prior to model fitting.

It can be noticed that the estimates and forecasts from k_t are different for all of these variants due to different adjustment methods and jump-off policy. Sweden has long and steady trend of mortality improvement which is also reflected after adjusting for lifespan disparity. The newly adjusted $\hat{k}_t$ has more regular and linear trend than estimates from earlier Lee-Carter methods. The trend of adjusted $\hat{k}_t$ along with forecast till 2050 by an *ARIMA (0,1,0)* for Swedish female is presented below in Figure 4.8. The regularity of new $\hat{k}_t$ is attributable to both smoothing technique and new adjustment policy for k_t.

Compare to the previous LC variants, the proposed $\hat{k}_t$ have less standard error which also consequently creates narrower confidence bound for forecast intervals.

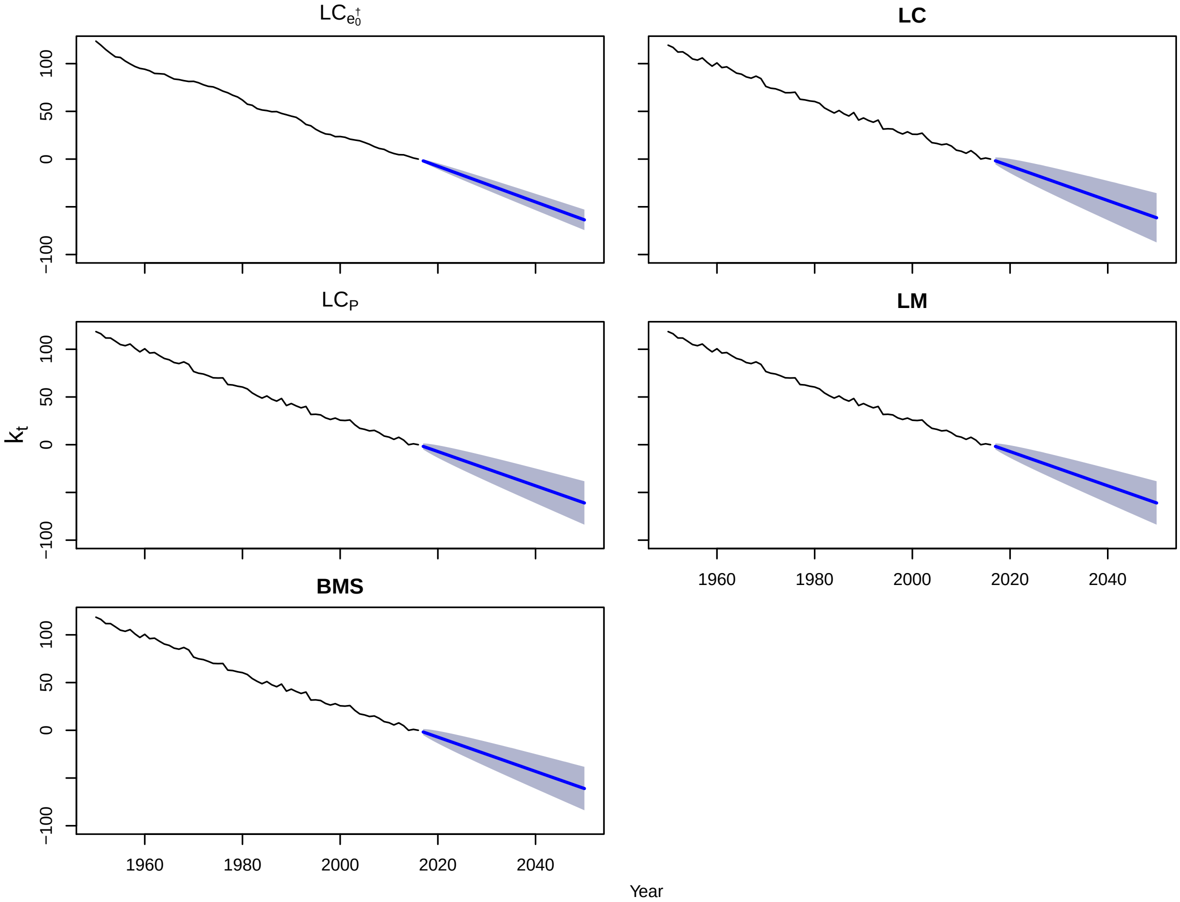
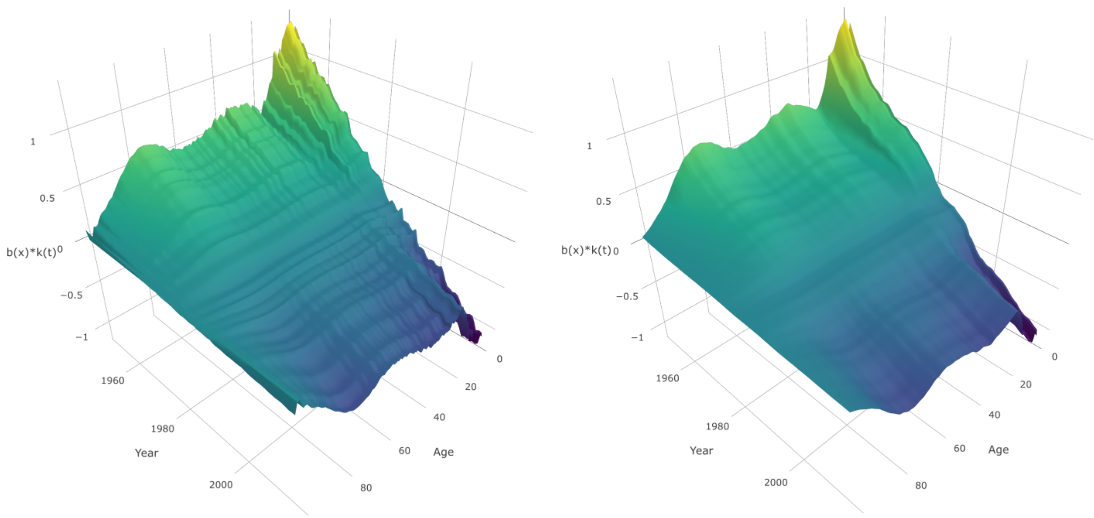


FIGURE 4.8: Random walk with drift on estimated k_t of Swedish Females (1950:2016) for LC variants and $\mathrm{LC}_{e_0^\dagger}$.

For Swedish female mortality I used the life tables constructed up to age 103 years due to missing values at older ages. For the trend of k_t shown in Figure 4.8, the model was fitted in the years 1950 to 2016 (the last available year during this analysis). Although I plotted the trend for 1950 to 2016, the best fitting period obtained for BMS was 1978 to 2016. The product of $\hat{b}_x$ (obtained from smoothed mortality rates) with more regular $\hat{k}_t$ ultimately produce more smoothed surface than that observed for LC (Figure 4.1). The other LC variants also shown almost similar surface like basic LC (Lee and Carter, 1992). The new smooth surface of the product of $\hat{b}_x$ and $\hat{k}_t$ are given below in Figure 4.9, which can explain more variation as well. For Swedish Females, LC can explain 77.4% of observed variation whereas both $\mathrm{LC}_{e_0^\dagger}$ and HU variants can explain 97.7% of that. Variance explained by the models for all the countries are attached in appendix. It should be noted that unlike rank-1 approximation in LC variants and $\mathrm{LC}_{e_0^\dagger}$, the HU variants are estimated considering 3 components.



(a) LC (b) $\mathrm{LC}_{e_0^\dagger}$

FIGURE 4.9: Product of time component and age component for Swedish Females (1950:2016) using (a) LC (Lee and Carter, 1992) and (b) proposed $\mathrm{LC}_{e_0^\dagger}$. (a) is illustrated with larger interface in Figure 4.1.

Next, I compared the forecast accuracy for all different versions of LC method. For out-of-sample evaluation of the methods, I fitted the model for all the available time period except the last 10 years to compare with the forecast obtained by fitted model with respect to observed mortality rates. The forecast accuracy of all 8 models ($\mathrm{LC}_{e_0^\dagger}$, 4 LC variants and 3 HU variants) for the females of 20 low-mortality countries are given below in Table 4.2, 4.3 and 4.4. Table 4.2 showed the accuracy in terms of mean absolute errors (MAE) for mortality rates, Table 4.3 for mean squared errors (MSE) for mortality rates and mean error of obtained life expectancy at birth ($\mathrm{ME}(e_0)$) are summarized in Table 4.4. Country-specific highest forecast accuracy (lowest error) are marked using bold texts in all of these tables. The proposed modified version of Lee-Carter method ($\mathrm{LC}_{e_0^\dagger}$) generated less forecast error for many of these low-mortality countries, both in terms of absolute and squared errors. For MAE and MSE, the models with lowest errors can be identified by bold texts in values. For $\mathrm{ME}(e_0)$, a value close to zero are shown in bold text to indicate lowest error. I skipped the results of $\mathrm{HU_W}$ method for case of US Females both for out-of-sample evaluation as it showed decreasing trend of life expectancy forecast for US Females.

Among all of these 20 low-mortality female populations, $LC_{e_0^\dagger}$ produced lowest forecast errors in terms of MAE for 10 of them. LM also produced same value of MAE as $LC_{e_0^\dagger}$ for US Females. In terms of MSE, $LC_{e_0^\dagger}$ produced lowest forecast errors for 11 countries. To obtain ME(e_0), I reconstructed the life tables from the forecast of mortality rates during out-of-sample evaluation period and compared the mean error of obtained life expectancy at birth. As for mean errors of obtained life expectancy at birth $LC_{e_0^\dagger}$ was most accurate for 5 countries followed by LC for 4 countries. LM was most accurate for Austria only while HU was most accurate for Germany only. Previous studies also mentioned that none of the methods were uniquely best for all countries (Shang, 2012). Comparatively better performance of $LC_{e_0^\dagger}$ are attributable to both of the proposed modifications. Beside application of Lasso as a more accurate smoothing technique, adjustment of time component according to lifespan disparity reflects more insight of the mortality trend of a population rather than total number of deaths or life expectancy at birth. From the definition of $e_0^\dagger$, it contains information of both remaining life expectancies and corresponding distribution of death, which is found to be more affective to obtain accurate forecast.

Table 4.2: MAE of the forecast methods during out-of-sample evaluation period.

Country	LC	LC_P	LM	BMS	HU	HU_R	HU_W	$LC_{e_0^\dagger}$
Australia	0.132	0.132	0.117	0.124	0.093	0.130	0.114	**0.109**
Austria	0.193	0.189	0.195	0.170	0.181	0.200	0.172	**0.166**
Belgium	0.152	0.152	0.169	0.150	0.154	0.147	**0.141**	0.152
Canada	0.091	0.090	0.089	0.091	0.093	0.103	0.087	**0.079**
Denmark	0.255	0.260	0.249	0.245	0.233	0.252	0.211	**0.208**
Finland	0.247	0.247	0.251	0.236	0.227	0.239	0.213	**0.185**
France	0.117	0.117	0.092	0.106	0.087	0.091	**0.076**	0.081
Germany	0.097	0.097	0.093	0.098	0.089	0.110	**0.085**	**0.085**
Ireland	0.351	0.319	0.242	0.227	0.225	0.243	**0.195**	0.230
Italy	0.140	0.126	0.107	0.107	0.087	0.092	**0.080**	0.091
Japan	0.428	0.391	0.133	0.310	**0.085**	0.100	0.087	0.119
Netherlands	0.134	**0.132**	0.143	0.144	0.132	0.135	**0.132**	0.137
New Zealand	0.209	0.209	0.236	0.203	0.204	0.202	0.184	**0.180**
Norway	**0.192**	**0.192**	0.236	0.196	0.217	0.224	0.211	0.206
Portugal	**0.177**	0.178	0.192	0.193	0.178	0.181	**0.160**	0.185
Spain	0.204	0.164	0.118	0.151	0.133	0.156	**0.102**	0.113
Sweden	0.170	0.170	0.193	0.174	0.160	0.171	**0.148**	0.152
Switzerland	0.212	0.211	0.246	0.206	0.215	0.218	0.215	**0.199**
United Kingdom	0.123	0.122	0.082	0.097	0.095	0.121	0.094	**0.071**
USA	0.100	0.100	**0.070**	0.117	0.100	0.082	-	**0.070**

Table 4.3: MSE of the forecast methods during out-of-sample evaluation period.

Country	LC	LC_P	LM	BMS	HU	HU_R	HU_W	$LC_{e_0^\dagger}$
Australia	0.033	0.033	0.033	0.032	0.030	0.035	**0.023**	0.030
Austria	0.093	0.089	0.106	**0.072**	**0.072**	0.080	0.087	0.079
Belgium	0.055	**0.054**	0.076	0.055	0.062	0.058	0.057	0.060
Canada	0.020	0.020	0.019	0.019	0.019	0.022	0.019	**0.015**
Denmark	0.131	0.137	0.144	0.130	0.119	0.134	0.107	**0.104**
Finland	0.130	0.130	0.172	0.120	0.128	0.135	0.122	**0.094**
France	0.026	0.027	0.018	0.020	0.014	0.015	**0.012**	0.013
Germany	0.018	**0.017**	0.020	0.019	**0.017**	0.022	**0.017**	**0.017**
Ireland	0.248	0.200	0.140	0.116	0.112	0.128	**0.102**	0.131
Italy	0.046	0.040	0.029	0.022	**0.019**	0.020	**0.019**	**0.019**
Japan	0.293	0.249	0.110	0.146	**0.016**	0.022	**0.016**	0.027
Netherlands	0.048	0.047	0.055	0.053	0.045	**0.043**	0.044	0.050
New Zealand	0.094	0.095	0.138	0.094	0.111	0.103	0.093	**0.077**
Norway	0.095	**0.094**	0.142	0.099	0.129	0.124	0.129	0.127
Portugal	0.071	0.069	0.083	0.093	0.074	0.077	**0.067**	0.074
Spain	0.073	0.047	0.033	0.040	0.031	0.039	**0.022**	0.031
Sweden	0.077	0.077	0.113	0.084	0.072	0.075	0.068	**0.067**
Switzerland	0.121	0.122	0.166	**0.116**	0.135	0.134	0.135	**0.116**
United Kingdom	0.027	0.026	0.016	0.017	0.016	0.024	0.016	**0.011**
USA	0.019	0.019	0.012	0.023	0.015	0.013	-	**0.011**

Table 4.4: ME(e_0) of the forecast methods during out-of-sample evaluation period.

Country	LC	LC_P	LM	BMS	HU	HU_R	HU_W	$LC_{e_0^\dagger}$
Australia	0.007	0.038	0.010	0.207	-0.083	-0.124	0.001	0.054
Austria	0.288	0.195	**0.003**	0.235	-0.585	-1.148	-0.018	-0.033
Belgium	**0.050**	0.110	0.062	0.144	-0.189	-0.194	0.155	0.132
Canada	-0.109	-0.040	-0.014	-0.041	-0.042	-0.116	-0.236	**0.039**
Denmark	-0.955	-1.119	-0.608	-1.067	-0.808	-0.819	-0.727	**-0.575**
Finland	0.456	0.448	-0.016	0.421	-0.581	-0.757	-0.452	**-0.013**
France	0.165	0.157	0.101	0.161	0.055	0.073	**0.048**	0.171
Germany	0.135	0.193	0.183	0.310	**0.003**	-0.414	0.159	0.215
Ireland	**-0.115**	-0.586	-0.452	-0.414	-1.054	-1.307	-0.400	-0.551
Italy	0.657	0.542	0.413	0.551	0.164	**-0.142**	0.379	0.420
Japan	1.232	1.006	0.768	0.956	0.306	**0.115**	0.469	0.674
Netherlands	-0.329	-0.294	-0.220	-0.389	-0.437	-0.625	-0.437	**-0.191**
New Zealand	-0.429	-0.335	-0.293	**-0.111**	-0.309	-0.576	-0.283	-0.249
Norway	0.064	**0.027**	-0.111	-0.103	-0.487	-0.855	-0.569	-0.155
Portugal	**-0.580**	-0.764	-0.795	-0.697	-0.981	-0.836	-0.561	-0.775
Spain	0.347	**-0.046**	-0.053	-0.074	-0.473	-0.824	-0.472	-0.088
Sweden	**0.062**	0.111	0.105	0.063	-0.151	-0.089	-0.114	0.167
Switzerland	0.300	0.233	0.197	0.167	0.095	**0.073**	0.094	0.308
United Kingdom	-0.120	-0.150	-0.158	**-0.082**	-0.669	-0.961	-0.669	-0.118
USA	-0.072	-0.054	-0.037	-0.233	-0.258	-0.174	-	**-0.001**

To explain the accuracy more precisely, the mortality surface of the fitted models for

Swedish Females are plotted in Figure 4.10 during the fitted period. The more smoothed surface for $\mathrm{LC}_{e_0^\dagger}$ are attributed to both the smoothing and new adjustment policy.

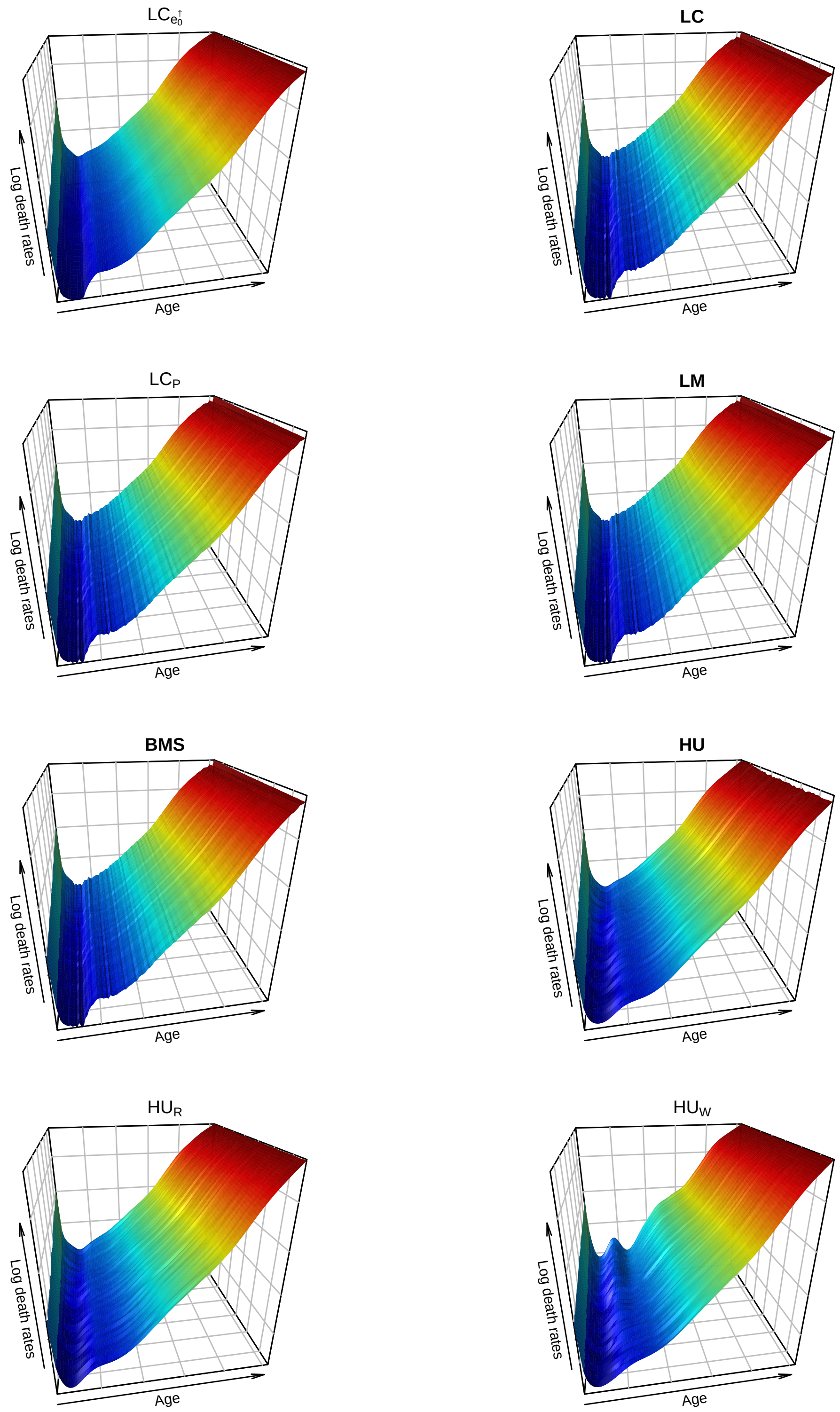


FIGURE 4.10: Mortality surface of the fitted models for Swedish Females (1950:2016).

4.4.3 Forecast of life expectancy

I compared and contrasted the forecast of life expectancy at birth in 2050 by all these forecasting techniques for the low-mortality countries. The forecast of life expectancy at birth till 2050 for these low-mortality countries are given below in Table 4.5. Country-specific highest forecast are marked using bold texts.

TABLE 4.5: Forecast of female life expectancy at birth in 2050 for 20 low mortality countries.

Country	LC	LC_P	LM	BMS	HU	HU_R	HU_W	$LC_{e_0^\dagger}$
Australia	89.24	89.26	89.24	**89.65**	89.28	88.66	89.42	89.43
Austria	88.95	88.88	88.92	**89.65**	88.56	89.07	88.51	88.95
Belgium	87.83	87.90	88.02	87.94	**88.42**	88.30	88.31	88.25
Canada	88.86	88.79	88.78	88.62	88.39	88.47	**89.19**	88.82
Denmark	86.81	86.78	**86.95**	86.51	85.91	86.93	86.21	86.75
Finland	89.25	89.36	89.51	88.96	88.01	88.43	89.22	**89.99**
France	90.26	90.34	90.45	90.54	90.61	90.55	90.13	**90.74**
Germany	87.59	87.76	87.74	**88.37**	87.87	87.28	87.53	87.82
Ireland	88.23	87.88	88.36	88.79	89.75	88.61	**89.82**	88.05
Italy	90.26	90.28	90.30	**90.80**	90.40	90.12	90.30	90.42
Japan	**93.72**	93.71	93.67	93.60	91.29	90.57	90.07	93.63
Netherlands	86.87	86.86	86.93	86.86	86.95	86.90	86.96	**87.22**
New Zealand	87.96	87.92	88.05	**88.65**	87.93	87.95	87.93	88.15
Norway	88.04	87.98	88.14	88.11	88.42	88.50	88.42	**88.55**
Portugal	88.41	88.50	88.43	89.40	86.24	**90.22**	86.24	88.60
Spain	90.66	90.27	90.57	90.22	89.93	89.81	**91.47**	90.53
Sweden	88.19	88.15	88.23	87.79	87.95	87.87	88.09	**88.38**
Switzerland	89.83	89.73	89.87	89.56	89.87	88.85	89.21	**90.06**
United Kingdom	87.12	87.16	87.16	87.54	**88.06**	87.13	87.67	87.36
USA	**85.68**	85.42	85.30	84.66	84.08	84.15	80.88	85.12

$LC_{e_0^\dagger}$ was the forecasting technique producing the most optimistic forecasts of life expectancy at birth for 6 countries followed by BMS for 5 countries. The difference between most optimistic forecast and forecast produced by $LC_{e_0^\dagger}$ were very low for Canada and Japan. Despite of remarkable performance during out-of-sample evaluation for several countries, HU_W produced lower forecast of life expectancy at 2050 for US Females (which is indicated by blanc space in Table 4.5). To illustrate the forecast of life expectancy at birth, the forecast of mortality rates and prediction interval of e_0 till 2050 are plotted below for Swedish Females in Figure 4.11 and 4.12 respectively.

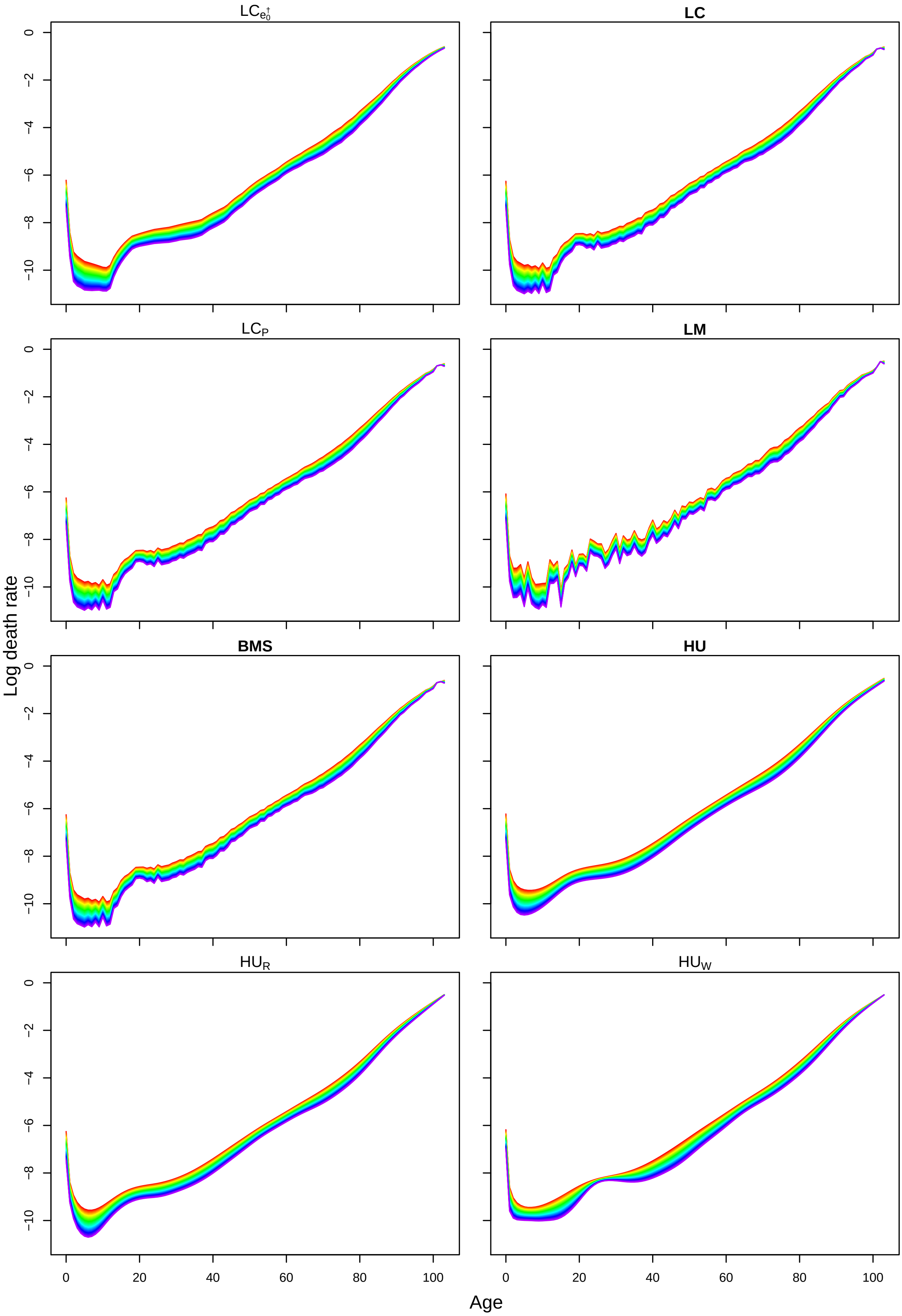


FIGURE 4.11: Forecast of Swedish female mortality rates till 2050. Years are plotted using a rainbow palette so the earlier years are shown in red, followed by orange, yellow, green, blue and indigo with the most recent years plotted in violet.

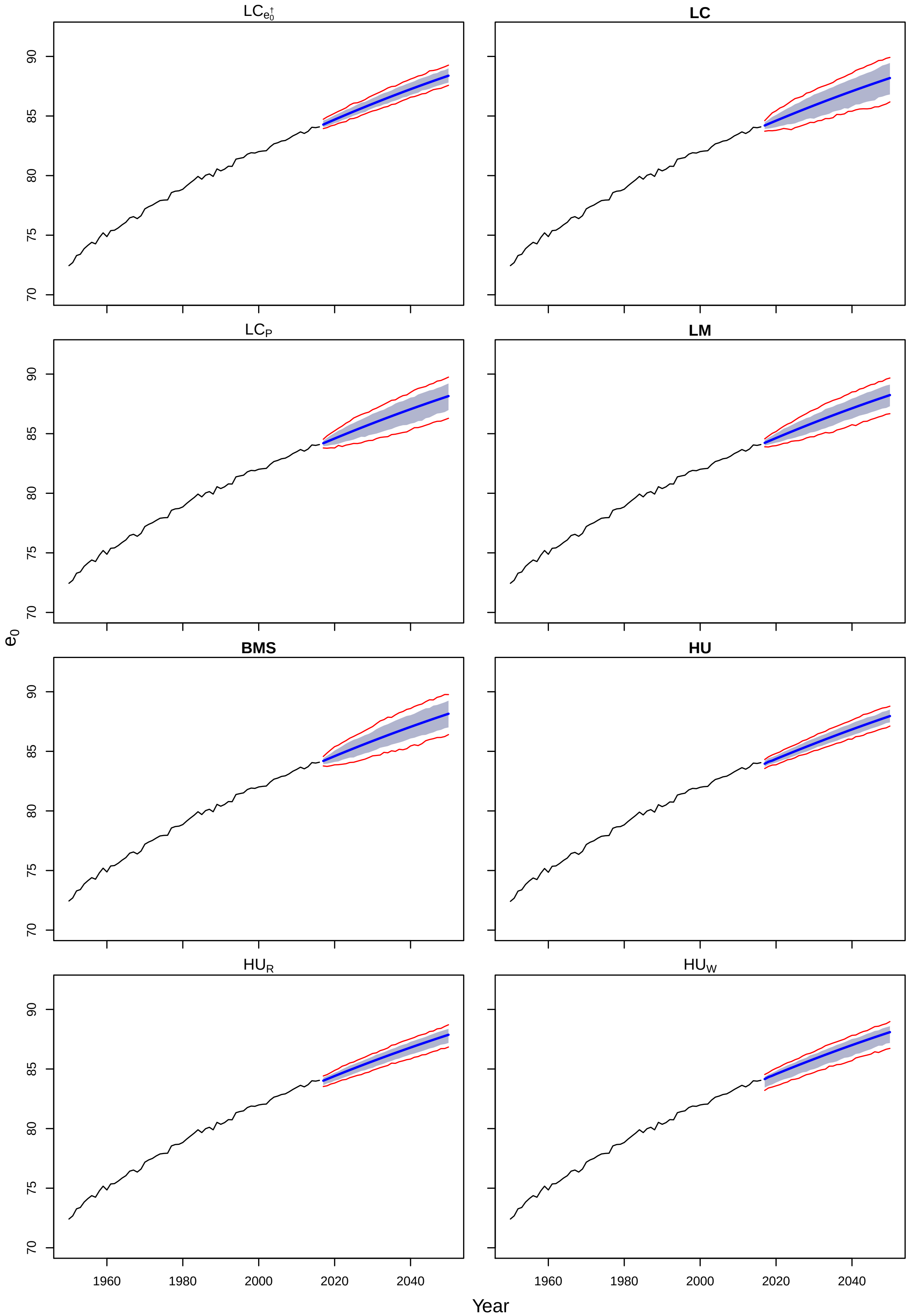


FIGURE 4.12: Prediction interval of Swedish female e_0 by different methods till 2050. The blue area represents 80% prediction interval and red lines are for 95% prediction interval.

The forecasts of $\mathrm{LC}_{e_0^\dagger}$ are smoother than other LC variants but less smoothed than HU variants. I already showed that Lasso smoothed mortality rates more accurately than one dimensional smoothing (Figure 4.2 and 4.3). Although, $\mathrm{HU_W}$ was more accurate than $\mathrm{LC}_{e_0^\dagger}$ for Swedish Females during out-of-sample evaluation, the mortality forecast for Swedish Females were more optimistic from $\mathrm{LC}_{e_0^\dagger}$ than $\mathrm{HU_W}$ (Table 4.5). Due to different techniques used in HU variants, the forecasts are also different for each of them. For mortality forecasting, $\mathrm{LC}_{e_0^\dagger}$ and LM consider actual data, whereas other methods utilize the fitted data. Due to lack of smoothing and different adjusting of time component, the forecast of LM have more jagged pattern than any other methods.

To compare the interval forecast of life expectancy at birth, I applied the existing semi-parametric bootstrapping technique proposed by Hyndman and Booth (2008). This technique was mainly designed for non-parametric forecasting obtained from functional data analysis (Hyndman and Ullah, 2007). Following Hyndman and Booth (2008), simulated forecasts of log mortality rates are obtained by adding disturbances to the forecast of $\hat{k}_t$ which are then multiplied by the fixed age component $\hat{b}_x$. Hence the life expectancies are estimated for each set of simulated log mortality rates. Prediction intervals are then constructed from the percentiles of the simulated life expectancies. Due to lower standard error obtained for newly adjusted $\hat{k}_t$, the interval forecast obtained from $\mathrm{LC}_{e_0^\dagger}$ are narrower than previous LC variants. Previous studies mentioned this problem of interval forecasting that the prediction intervals of the Lee-Carter type models are too narrow and may lead to underestimate the coverage probability (see Lee and Carter, 1992; Shang, 2012, for example). Interval forecasting thus remained as a common issue for Lee-Carter type models that has room for improvement. The prediction intervals obtained from $\mathrm{LC}_{e_0^\dagger}$ for female populations of low mortality countries are attached in appendix.

Chapter 5

Extending the Proposed Lee-Carter Method for Coherent Mortality Forecasting

In this chapter I extend the proposed methodology ($\text{LC}_{e_0^\dagger}$) for multi-population forecasting along with other modifications. Multi-population or coherent forecasting is getting more focus in recent era as it seeks to ensure that the forecasts for related population holds certain structural relationship based on past pattern of mortality and theoretical understanding. The concept of coherent mortality forecasting was introduced by Li and Lee (2005) as an extended hierarchical interface of the Lee-Carter (LC) method. One of the core problems for coherent forecasting is to choose appropriate reference population for a particular population. Adding a low-mortality population with comparatively high-mortality regime might show better accuracy during out-of-sample evaluation or most optimistic forecast for population with higher mortality but question remained about validity of choosing these populations. Another issue concerns the gap between male and female mortality throughout the lifespan. This issue also involves the choice of reference populations (see Li and Lee, 2005, for example). Nevertheless, many of the coherent approaches used males and females of the same country together for coherent forecasting with assumption of convergence in future mortality (for example, Li and Lee, 2005). Some studies gave priority to choose the reference population on the basis of environmental or geo-social issues, however these approaches could not explain the pattern of coherence in all cases (Kjærgaard *et al.*, 2016; Ahcan *et al.*, 2014). Kjærgaard *et al.* (2016) find that a reference population made of small number of countries tend to perform better than that of larger group in term of forecast accuracy, whereas choosing countries with closest life expectancy found to be better strategy for several countries.

Instead of life expectancy or geographical or socio-cultural aspects, choosing reference populations according to closest mortality pattern over time is more preferable in this sense. Since the second half of the last century, each of the populations have different pattern of mortality improvement over time which has great impact on country-specific mortality forecasting as the common factor obtained from reference group depends on it. To illustrate this, the parameter b_x obtained from individual LC for all 20 low-mortality countries is plotted in Figure 5.1. Recall from 2.1 that b_x is the first principal component capturing relative change in the log mortality rate at each age x.

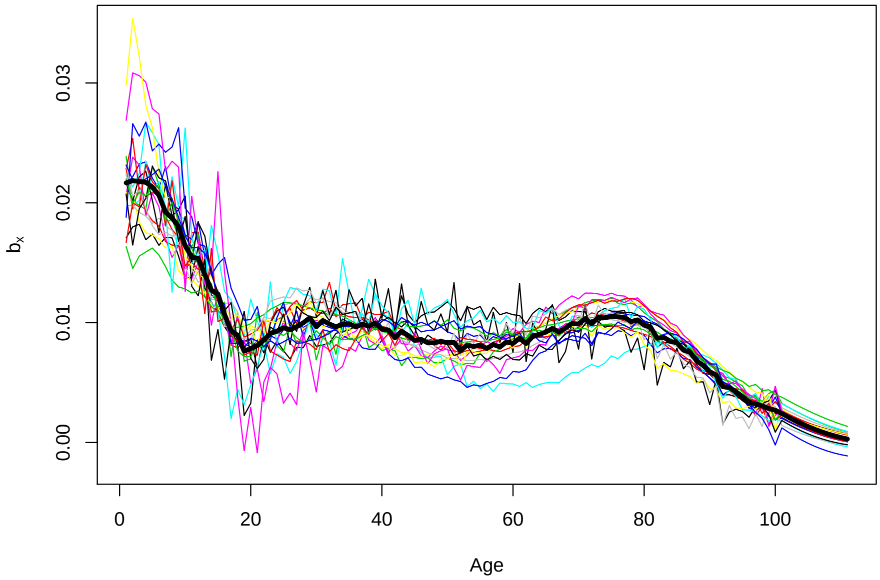


FIGURE 5.1: Estimated b_x from fitted Lee-Carter model for females of 20 low mortality countries (1956:2011). The bold black line is the mean trend of b_x for comparing the countries.

From Figure 5.1, it is clearly visible that each of the countries have their own distinct characteristics of mortality trend translated from different pattern of age of death distribution. Thus, the best reference populations characterized by closest age of death distribution or equivalent measure may perform well to explain the group's mean trend of mortality improvement. Another issue is that none of the previous approaches consider period effect on coherent forecasting. Period effect may be reflected by a deviation of usual mortality pattern since a particular calendar year which may be results of new intervention or a change in international relations. For example, old-age mortality was considerably higher in East Germany than West Germany prior to reunification, but it converged quickly afterward (Vogt and Missov, 2017). Before reunification, the diffusion of longevity process were different in East Germany, but after that, adaptation of new health care policies make the convergence faster. Similarly, rapid increase in life expectancy may be observed for Portugal after joining European Union. Considering full fitting period in case of coherent forecasting will be thus misleading because clearly the relation between the populations will not have high impact on mortality before a certain threshold point of time (period effect).

5.1 Extended method

5.1.1 Choice of reference population

The reference group of a particular country is chosen on the basis of nearest lifespan disparity ($e_0^\dagger$). The notations and practical implementation are the same as equation 4.6. For a particular population I consider only those populations for which the difference between observed $e_0^\dagger$ is lowest. For a particular population **i**, another population **j** will be in reference group if:

$$\left|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right| = min, \tag{5.1}$$

compared to other available populations. To implement this, I estimate the $e_0^\dagger$ for all populations under consideration over the common fitting period and from that I estimate the population specific $\bar{e}_0^\dagger$. For m years of available mortality rates and p populations, the estimates can be presented in the following matrix notation:

$$\begin{array}{c} Year \\ \\ t_1 \\ t_2 \\ \vdots \\ \vdots \\ t_{m-1} \\ t_m \\ \\ \\ \end{array}
\begin{array}{c} \begin{array}{cccc} Pop_1 & Pop_2 & \dots & Pop_p \end{array} \\
\begin{pmatrix} e^\dagger_{0(t_1,1)} & e^\dagger_{0(t_1,2)} & \dots & e^\dagger_{0(t_1,p)} \\ e^\dagger_{0(t_2,1)} & e^\dagger_{0(t_2,2)} & \dots & e^\dagger_{0(t_2,p)} \\ \vdots & \vdots & & \vdots \\ & & & \\ e^\dagger_{0(t_{m-1},1)} & e^\dagger_{0(t_{m-1},2)} & \dots & e^\dagger_{0(t_{m-1},p)} \\ e^\dagger_{0(t_m,1)} & e^\dagger_{0(t_m,2)} & \dots & e^\dagger_{0(t_m,p)} \end{pmatrix} \\
\begin{array}{cccc} \bar{e}^\dagger_{0(.,1)} & \bar{e}^\dagger_{0(.,2)} & \dots & \bar{e}^\dagger_{0(.,p)}. \end{array} \end{array}$$

For a particular population i, I sort all $\bar{e}^{\dagger}_{0(t,j)}$ in ascending order. Populations j for which $\bar{e}^{\dagger}_{0(t,j)}$ is closest to $\bar{e}^{\dagger}_{0(t,i)}$ will be considered in reference group for population i. The trend of sorted $\bar{e}^{\dagger}_{0(i,j)}$ obtained over time may show inconsistent pattern for full fitting period. For low-mortality populations considered in this study, I found the consistent pattern of sorted $\bar{e}^{\dagger}_{0(i,j)}$ starting from the period of 1982 as the sorted $\bar{e}^{\dagger}_{0(i,j)}$ have almost same pattern since then. It should be noted that the number of populations and required gap between $\bar{e}^{\dagger}_{0(t,i)}$ and $\bar{e}^{\dagger}_{0(t,j)}$ may very in reference group as not all the populations have symmetric distance of $\bar{e}^{\dagger}_{0}$ between each other.

5.1.2 Coherent forecast of mortality rates

Let us recall the Li and Lee (2005) again from equation 2.5,

$$\ln m_{x,t,i} = a_{x,i} + B_x K_t + b_{x,i} k_{t,i} + \epsilon_{x,t,i}, \tag{5.2}$$

where i stands for specific country in the group, $a_{x,i}$ is the country specific average log mortality rate. The term B_x is relative speed of change in mortality at each age x and mortality index K_t captures the main time trend for the reference group respectively. The term $B_x K_t$ is known as common factor as this quantity is common for all the countries of the group.

To extend the proposed methodology ($\mathrm{LC}_{e_0^{\dagger}}$), first I obtain the smoothed mortality rates by using Lasso for each population. I added the populations in reference group according to closest $\bar{e}^{\dagger}_{0}$ mentioned before. The standard Lee-carter methodology is applied to obtain the initial estimate of the age and time component from the combined smoothed mortality data. From the experience of high-mortality countries (Section 3.1.2), in this chapter I applied same weight on all populations to overcome problem of combining larger exposures with smaller ones.

$$m_{x,t} = \frac{1}{p} \sum_{i=1}^{p} m_{x,i,t}. \tag{5.3}$$

As these $m_{x,i,t}$ are observed and comes from life table, it has almost null influence of population size. I used same weight for fitting Li and Lee (2005) also in this chapter. In this stage, fitted two-factor Lee-Carter model over combined mortality rates will be,

$$\ln(\hat{m}_{x,t}) = \hat{a}_x + \hat{B}_x \hat{K}_t. \tag{5.4}$$

Following Lee and Miller (2001), Li and Lee (2005) made a second stage estimate of K_t by finding the value of K_t which, for a given population age distribution and previously estimated a_x and b_x produces exactly the observed life expectancy for the fitting period of the model. Following the proposed methodology ($\mathrm{LC}_{e_0^\dagger}$), I adjusted the estimated K_t by partially matching with observed $e_0^\dagger$ of the combined smoothed mortality rates. It is done by solving the following equation:

$$e^{\dagger}_{0\ observed} = \sum_{0}^{\omega} \exp(\hat{a}_x + \hat{B}_x.K_{t\ adj})e_x l_x/l_0. \tag{5.5}$$

The e_x and l_x of equation 5.5 are obtained from life table estimated from fitted data of combined mortality rates. After obtaining the adjusted $\hat{K}_t$, I identified the most appropriate period for which this common factor should be estimated using equation 5.4.

The concept of best fitting period comes from the assumption of linear trend of $\hat{k}_t$ in Lee-Carter type models (Booth *et al.*, 2002). For departure of $\hat{k}_t$ from linear trend, the linear fit of $\hat{k}_t$ can be improved by proper restriction of the fitting period. Booth *et al.* (2002) defined the starting of the fitting period by means of statistical measure of relative lack of fit of the model. For getting the best fitting period, first I obtain the close approximation of deviance(t) which is equal to $\chi^2(t)$ statistic of the lack of fit in observed distribution of death $D_{x,t}$,

$$\chi^2(t) = \sum_{x} \frac{\left[D_{x,t} - D'_{x,t}\right]^2}{D'_{x,t}}. \tag{5.6}$$

Here $D'_{x,t}$ are fitted deaths which can be obtained from observed exposure $N_{x,t}$ as follow:

$$D'_{x,t} = N_{x,t}\left[\exp(\hat{a}_x + \hat{B}_x.K_{t\ adj})\right]. \tag{5.7}$$

I denote the starting year of the best fitting period by S, whereas the end will be the last year of the available data for model fitting. Following Booth *et al.* (2002), the total lack of fit to the log-linear model comes from two different sources: (i) the base lack of fit from log-additive model and the adjustment of $\hat{K}_t$ and the additional lack of fit from the imposition of the ARIMA model on $\hat{K}_{t\ adj}$. The base lack of fit for the period S years prior to last observed data is measured by

$$\chi^2_{\mathrm{logadd}}(S) = \sum_{t} \chi^2_{\mathrm{logadd}}(t).$$

Here the $D'_{x,t}$ are derived from $\hat{K}_{t\ adj}$. The total lack of fit to the log-linear model is

$$\chi^2_{\text{loglin}}(S) = \sum_t \chi^2_{\text{loglin}}(t).$$

Here the $D'_{x,t}$ are derived from the linear fit of $\hat{K}_{t\ adj}$. This total lack of fit will be greater than or equal to base lack of fit. In order to compare $\chi^2_{\text{loglin}}(S)$ with $\chi^2_{\text{logadd}}(S)$ they are divided by corresponding degrees of freedom (df) to produce mean-χ^2 statistic. For n age groups and m years in the fitting period, the df for $\chi^2_{\text{loglin}}(S)$ is $n(m-2)$ and df for $\chi^2_{\text{logadd}}(S)$ is $(n-1)(m-1)$. The choice of best fitting period S is determined by the extent of the additional lack of fit relative to the total lack of fit. The additional lack of fit will be small in case of good fit of the ARIMA model. Additional lack of fit is defined as ratio of total to base lack of fit,

$$R(S) = \frac{\chi^2_{\text{loglin}}(S)/[n(m-2)]}{\chi^2_{\text{logadd}}(S)/[(n-1)(m-2)]}. \tag{5.8}$$

Marginal effect of including each additional years in S can be obtained from ratio of the differences in total and base mean-χ^2 statistics for S and $S+1$,

$$RD(S) = \frac{\left[\chi^2_{\text{loglin}}(S) - \chi^2_{\text{loglin}}(S+1)\right]/n}{\left[\chi^2_{\text{logadd}}(S) - \chi^2_{\text{logadd}}(S+1)\right]/(n-1)}. \tag{5.9}$$

Booth *et al.* (2002) mentioned small values of $R(S)$ and $RD(S)$ indicate that the additional lack of fit is relatively small. To choose best fitting period, a value of S for which $R(S)$ and $RD(S)$ are substantially smaller than corresponding statistics for preceding values of S. Thus, the smallest value obtained for a particular S indicates that the inclusion of year $S-1$ (and preceding years) in the fitting period results in a relatively large reduction in goodness of fit of the ARIMA model (Booth et al, 2002).

For getting country level coherent forecast, basic LC model is then fitted on country-specific mortality rates without the common factor. To obtain the country-specific ordinary least square estimates of $b_{x,i}$ and $k_{t,i}$, SVD is performed on the following expression

$$Z_{x,i,t} = \ln(m_{x,i,t}) - \hat{a}_{x,i} - \hat{B}_x\hat{k}_{t\ adj} \tag{5.10}$$

The estimation procedure are as before. However, in this stage the LC is fitted without any adjustment for country-specific $k_{t,i}$ and it is fitted for the best fitting period obtained during estimation of common factor. A random walk with drift is then fitted for both $\hat{K}_{t\ adj}$ and $\hat{k}_{t,i}$. To eliminate jump-off error, I use the actual data to get the forecast.

5.1.3 Data

I have considered male and female populations for the previously used 20 low-mortality countries in this chapter. Unlike high-mortality countries or $\mathrm{LC}_{e_0^\dagger}$ in previous chapter, I considered life table constructed up to age 110 for all the populations in case of coherent forecasting. This inclusion is important for mortality forecasting of aging societies in coherent settings as centenarians supposed to be benefited from other populations of reference group. Due to presence of zero deaths and also missing death rates after age 100 for several years, I replaced all the mortality rates for age 100 to 110+ by extrapolated values obtained from Kannisto model (Thatcher *et al.*, 1998). Details of Kannisto model are attached in Appendix.

5.2 Results

5.2.1 Best reference population

The sorted $\bar{e}_0^\dagger$ for all the 40 populations from 20 low-mortality countries are given below in Table 5.1. I estimated $\bar{e}_0^\dagger$ considering different starting year, the pattern obtained in Table 5.1 remained steady since 1982. Any starting year after 1982 produces different values of population-specific $\bar{e}_0^\dagger$, but the pattern among populations remained unchanged.

TABLE 5.1: Sorted $\bar{e}_0^\dagger$ over the period 1982:2011 for the 20 low-mortality countries.

Population	$\mathrm{ESP_F}$	$\mathrm{CHE_F}$	$\mathrm{JPN_F}$	$\mathrm{SWE_F}$	$\mathrm{FIN_F}$	$\mathrm{ITA_F}$	$\mathrm{AUT_F}$	$\mathrm{NOR_F}$
$\bar{e}_0^\dagger$	9.888	9.922	9.933	9.969	10.011	10.043	10.101	10.150
Population	$\mathrm{DEU_F}$	$\mathrm{NLD_F}$	$\mathrm{FRA_F}$	$\mathrm{BEL_F}$	$\mathrm{AUS_F}$	$\mathrm{IRL_F}$	$\mathrm{PRT_F}$	$\mathrm{UK_F}$
$\bar{e}_0^\dagger$	10.246	10.326	10.378	10.454	10.467	10.493	10.618	10.791
Population	$\mathrm{CAN_F}$	$\mathrm{SWE_M}$	$\mathrm{NLD_M}$	$\mathrm{NZL_F}$	$\mathrm{DNK_F}$	$\mathrm{JPN_M}$	$\mathrm{NOR_M}$	$\mathrm{IRL_M}$
$\bar{e}_0^\dagger$	10.881	10.911	11.039	11.100	11.103	11.310	11.340	11.341
Population	$\mathrm{CHE_M}$	$\mathrm{UK_M}$	$\mathrm{ITA_M}$	$\mathrm{AUS_M}$	$\mathrm{DEU_M}$	$\mathrm{BEL_M}$	$\mathrm{USA_F}$	$\mathrm{DNK_M}$
$\bar{e}_0^\dagger$	11.464	11.495	11.547	11.648	11.717	11.800	11.802	11.831
Population	$\mathrm{CAN_M}$	$\mathrm{AUT_M}$	$\mathrm{ESP_M}$	$\mathrm{NZL_M}$	$\mathrm{FIN_M}$	$\mathrm{FRA_M}$	$\mathrm{PRT_M}$	$\mathrm{USA_M}$
$\bar{e}_0^\dagger$	11.857	11.949	11.990	11.999	12.154	12.516	12.808	13.127

Country codes are same as mentioned in section 2.1. M and F stands for males and females respectively.

Although future convergence for male-female mortality is assumed, the sorted $\bar{e}_0^\dagger$ are ranked naturally with lowest values for females followed by males except for some overlapping. This steady pattern from beginning of 1980s can be explained by the impact of changes in risk factors during end of 1970s for many industrialized countries which lead to significant decline in cardiovascular diseases (Ouellette *et al.*, 2014). Previous research suggests that decline in mortality from diseases of the cardiovascular system in high-income countries is attributable to changes in major risk factors (including smoking), and to innovation of specific treatments (see Capewell *et al.*, 2010, for an example). A more modest turning point in all-cause mortality was observed in beginning of 1980s due to change in cancer mortality trends in the 1980s for many of these countries (Ouellette *et al.*, 2014).

The closest populations for a particular one can be obtained using the equation 5.1 on Table 5.1. To illustrate, I applied the equation 5.1 for French Females and sorted the results. The closest populations for French Females to be use in reference group are given below in Table 5.2.

TABLE 5.2: Reference population for French Females based on closest difference in $\bar{e}_0^\dagger$ during 1982:2011.

Population	$\mathrm{FRA_F}$	$\mathrm{NLD_F}$	$\mathrm{BEL_F}$	$\mathrm{AUS_F}$	$\mathrm{IRL_F}$	$\mathrm{DEU_F}$	$\mathrm{NOR_F}$	$\mathrm{PRT_F}$
$\left\lvert\bar{e}^\dagger_{0,\mathrm{FRA_F}} - \bar{e}^\dagger_{0j}\right\rvert$	-	0.051	0.076	0.088	0.115	0.131	0.227	0.240
Population	$\mathrm{AUT_F}$	$\mathrm{ITA_F}$	$\mathrm{FIN_F}$	$\mathrm{SWE_F}$	$\mathrm{UK_F}$	$\mathrm{JPN_F}$	$\mathrm{CHE_F}$	$\mathrm{ESP_F}$
$\left\lvert\bar{e}^\dagger_{0,\mathrm{FRA_F}} - \bar{e}^\dagger_{0j}\right\rvert$	0.276	0.334	0.367	0.408	0.412	0.444	0.455	0.489
Population	$\mathrm{CAN_F}$	$\mathrm{SWE_M}$	$\mathrm{NLD_M}$	$\mathrm{NZL_F}$	$\mathrm{DNK_F}$	$\mathrm{JPN_M}$	$\mathrm{NOR_M}$	$\mathrm{IRL_M}$
$\left\lvert\bar{e}^\dagger_{0,\mathrm{FRA_F}} - \bar{e}^\dagger_{0j}\right\rvert$	0.502	0.533	0.660	0.721	0.725	0.932	0.962	0.963
Population	$\mathrm{CHE_M}$	$\mathrm{UK_M}$	$\mathrm{ITA_M}$	$\mathrm{AUS_M}$	$\mathrm{DEU_M}$	$\mathrm{BEL_M}$	$\mathrm{USA_F}$	$\mathrm{DNK_M}$
$\left\lvert\bar{e}^\dagger_{0,\mathrm{FRA_F}} - \bar{e}^\dagger_{0j}\right\rvert$	1.086	1.117	1.169	1.270	1.339	1.421	1.424	1.453
Population	$\mathrm{CAN_M}$	$\mathrm{AUT_M}$	$\mathrm{ESP_M}$	$\mathrm{NZL_M}$	$\mathrm{FIN_M}$	$\mathrm{FRA_M}$	$\mathrm{PRT_M}$	$\mathrm{USA_M}$
$\left\lvert\bar{e}^\dagger_{0,\mathrm{FRA_F}} - \bar{e}^\dagger_{0j}\right\rvert$	1.478	1.571	1.612	1.620	1.776	2.137	2.430	2.749

Country codes are same as mentioned in section 2.1. *M* and *F* stands for males and females respectively.

Unlike previous studies, I did not fix the number of populations in best reference group (Kjærgaard *et al.*, 2016). Based on existing research, two assumptions are considered for choosing best reference group: 1) smaller number of countries are preferable for a persimmons model (for computational purpose) and 2) future convergence in male-female mortality pattern. For both Li and Lee (2005) and the proposed method, the best reference group is hence determined from the analysis of forecast accuracy during out-of-sample evaluation period. As reference group, a combination of countries producing lowest forecast error during out-of-sample evaluation is chosen. The proposed method of coherent forecasting is denoted by $\text{LL}_{e_0^\dagger}$ and previous one of Li and Lee (2005) by LL as before. Details of forecast accuracies for proposed methods are discussed in next section, here I discuss the findings for French Females only for illustration. I added populations one at a time in reference group with previous combination for both $\text{LL}_{e_0^\dagger}$ and LL and used same weight for both methods as mentioned in equation 5.3. The forecast accuracy for French Female during out-of-sample evaluation is plotted below in Figure 5.2.

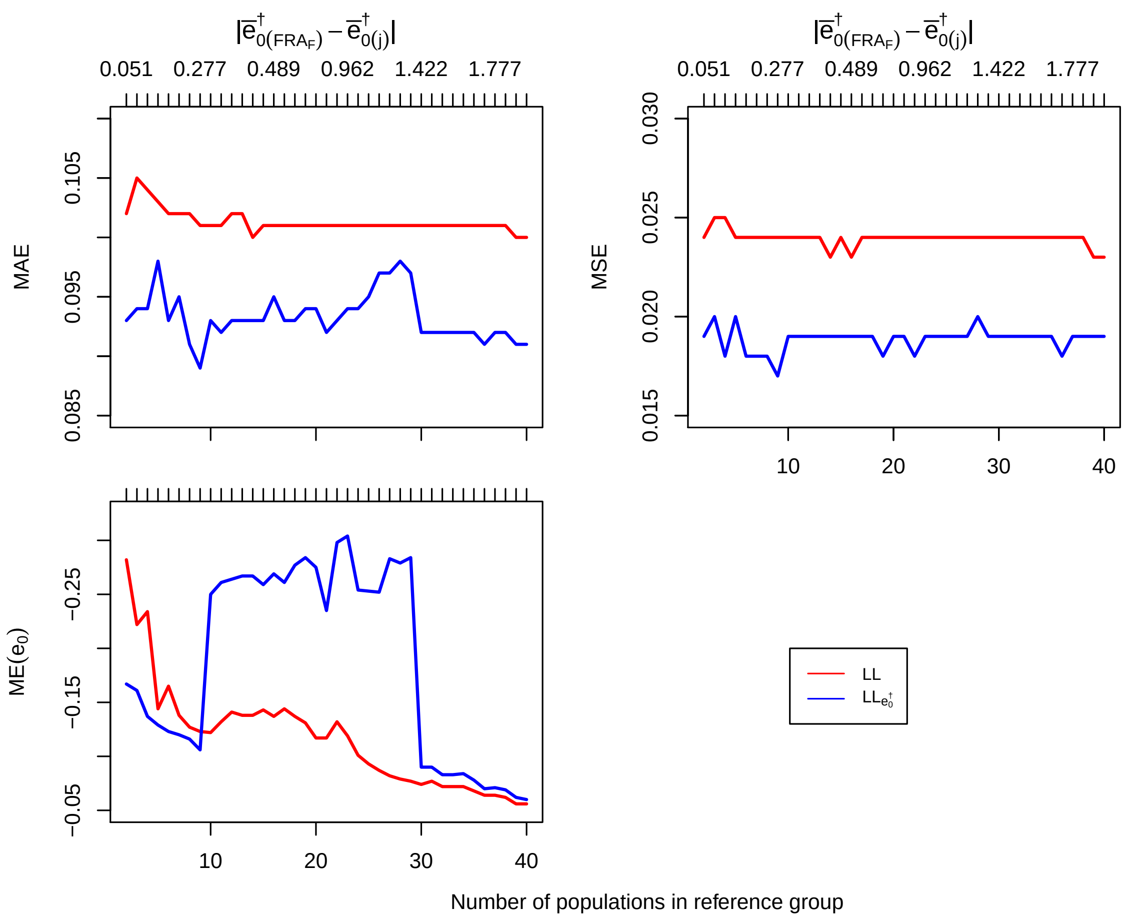


FIGURE 5.2: Forecast accuracies for French Females during out-of-sample evaluation period (2002:2011) considering different sizes of reference group.

For all three measures of forecast accuracy there is a distinct fall (rise) on accuracy (error) level after adding some certain countries using $\text{LL}_{e_0^\dagger}$. LL did not show sharp threshold like $\text{LL}_{e_0^\dagger}$ for mean error of e_0. All three measures indicated same best reference population for French female in case of $\text{LL}_{e_0^\dagger}$. For LL, MAE and MSE indicates same best reference group, whereas best reference populations indicated by $\text{ME}(e_0)$ were different. Different combination of populations as best reference group from different measures of forecast accuracies are observed for many other populations as well. However, this threshold in level of forecast accuracy is important, because it clearly shows which is the best reference group. For several other populations, I observed that mean error of e_0 in coherent forecast diminishes after adding lots of populations even though those populations are not so close to that of interest in terms of observed mortality level. Adding these populations with huge gap in level of population-specific $\bar{e}_0^\dagger$ does not have high impact on MAE or MSE during out-of-sample evaluation (Figure 5.2). The best reference populations obtained for French Females according to distance of $\bar{e}_0^\dagger$ with other population is plotted below in Figure 5.3.

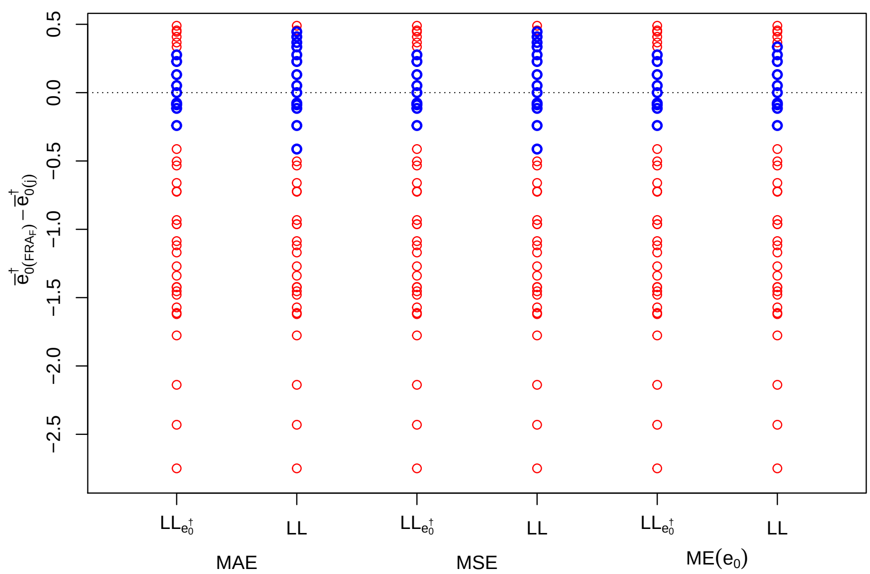


FIGURE 5.3: The best reference populations for French Females according to different measures of forecast accuracy. The blue points are for those populations which are in best reference group. The black line represents French Females.

For French Females, the ME(e_0) showed a sharp rise after adding Italian Females in reference group. The $\bar{e}_0^\dagger$ for French Females were closer to populations added in reference group prior to Italian Females (Table 5.2). The observed e_0 of French and Italian Females along with forecast of e_0 from both LL and $LL_{e_0^\dagger}$ are plotted below in Figure 5.4. Both for LL and $LL_{e_0^\dagger}$, the reference group consists of 10 populations which includes up to Austrian Females (Table 5.2). The observed life expectancies for both of the populations were close to each other till 2004. Irregular divergence is visible afterwards and French e_0 was more close to forecast of LL than that obtained from proposed $LL_{e_0^\dagger}$. Clearly, the adjustment policy considered in $LL_{e_0^\dagger}$ made this difference for the populations considered to obtain the best reference group. To get more insight of the reference populations for French females, the trend of the $e_0^\dagger$ is plotted in Figure 5.5 for some of the populations of the possible best reference group.

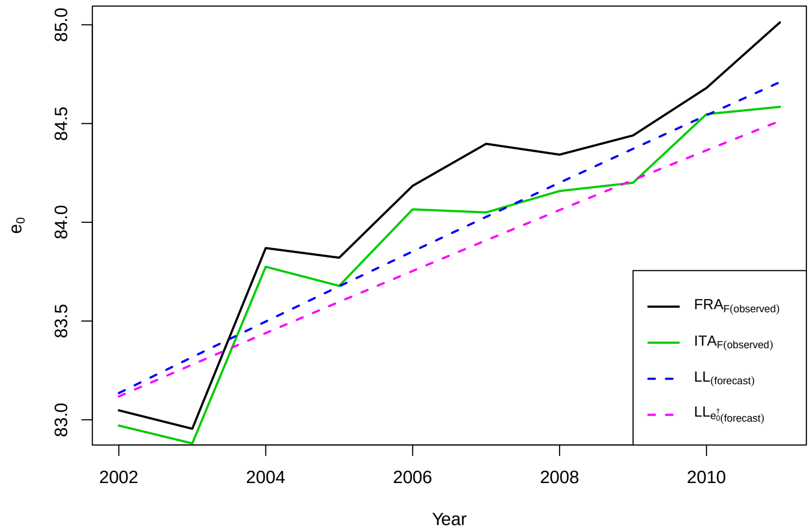


FIGURE 5.4: Observed and forecast of e_0 for French Females (2002:2011). Observed Italian Females e_0 are also added for comparison.

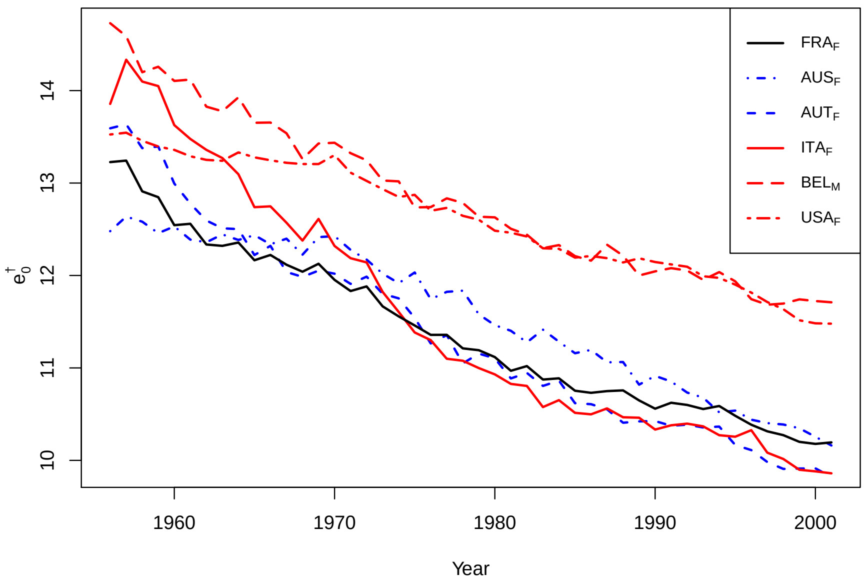


FIGURE 5.5: Trend of $e_0^\dagger$ for French Females and some other populations (1956:2001). The populations in blue lines are in the best reference group for French Females and red are for those populations which are not in the best reference group.

For Italian Females, recent trend of $e_0^\dagger$ is close to other populations in best reference group seen for French Females. The past trend was quite different compare to those which are in the best reference group. This trend also explain the rise in forecast accuracy after adding many populations. For instance, Belgian Males are ranked as 29 th closest population for French Females and the level of mortality is much higher for them compare to French Females. Merging lots of these divergent populations together in reference group finally increase accuracy level. Although adding these huge number of countries increase level of accuracy for life expectancy, but it does not decrease MAE or MSE than that observed for smaller number of populations in reference group.

5.2.2 Optimal size of best reference group

Unlike previous approaches, I did not restrict the number of populations in best reference group (Kjærgaard *et al.*, 2016). From empirical analysis, I find different results for LL and $\mathrm{LL}_{e_0^\dagger}$. The best reference groups obtained from LL and $\mathrm{LL}_{e_0^\dagger}$ for all 40 populations are attached in appendix. The distribution of number of countries in best reference group according to difference in $\left|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right|$ is plotted below in Figure 5.6.

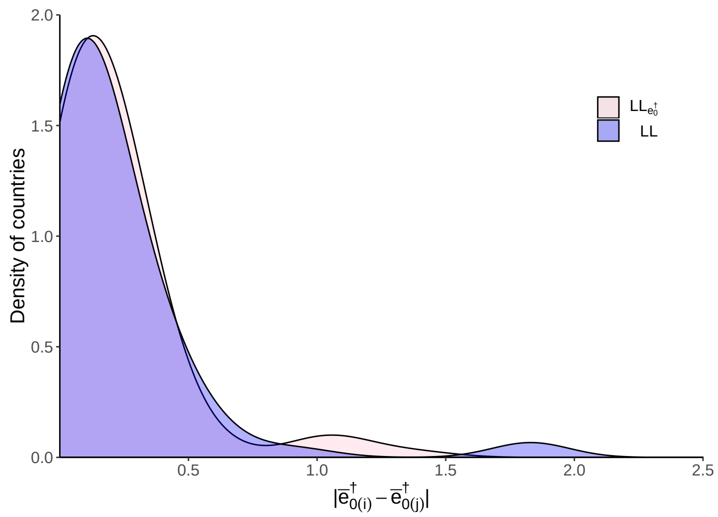


FIGURE 5.6: Distribution of countries obtaining reference population according to difference in $\left|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right|$.

Since different measures of forecast accuracy provides different combinations of populations as best reference group, I considered all three measures of forecast accuracy together for the distribution obtained in Figure 5.6. In case of same accuracy level for two or more combinations, the combination having smaller number of countries is chosen as best reference group for a parsimonious model. The optimal number of populations in reference group and corresponding differences in $\left|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right|$ considering all forecast errors are summarized below in Table 5.3. The errors are considered separately for males, females and both sexes together.

TABLE 5.3: Summary statistics for best reference group.

	Summary statistics	$LL_{e_0^\dagger}$ Male	Female	All	LL Male	Female	All
Number of populations	Mean	5	5	5	5	5	5
	Median	5	5	5	4	4	4
	IQR	6	4	5	4	4	4
Difference in $\left\|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right\|$	Mean	0.25	0.16	0.21	0.28	0.19	0.24
	Median	0.15	0.14	0.15	0.15	0.09	0.12
	IQR	0.26	0.20	0.24	0.26	0.20	0.25

5.2.3 Forecast accuracy

$LL_{e_0^\dagger}$ returns lower MAE and MSE than LL during out-of-sample evaluation for all combinations of reference population (except for coherent forecast of US Females). The forecast accuracy for French Females are given below in Table 5.4.

TABLE 5.4: Comparison of forecast accuracy for French Females during out of sample evaluation period (2002-2011).

Reference population	MAE LL	MAE $LL_{e_0^\dagger}$	MSE LL	MSE $LL_{e_0^\dagger}$	ME(e_0) LL	ME(e_0) $LL_{e_0^\dagger}$	Best fitting period ($LL_{e_0^\dagger}$)
FRA_F+NLD_F	0.103	0.093	0.024	0.019	-0.282	-0.167	1956:2001
Above+BEL_F	0.105	0.094	0.025	0.020	-0.222	-0.161	1956:2001
Above+AUS_F	0.105	0.094	0.025	0.018	-0.235	-0.137	1979:2001
Above+IRL_F	0.103	0.098	0.024	0.020	-0.144	-0.129	1979:2001
Above+DEU_F	0.102	0.093	0.024	0.018	-0.165	-0.123	1977:2001
Above+NOR_F	0.102	0.095	0.024	0.018	-0.138	-0.120	1979:2001
Above+PRT_F	0.101	0.091	0.024	0.018	-0.127	-0.116	1974:2001
Above+AUT_F	0.101	0.089	0.023	0.017	-0.124	-0.105	1974:2001
Above+ITA_F	0.101	0.093	0.024	0.019	-0.123	-0.250	1964:2001
Above+FIN_F	0.101	0.092	0.024	0.019	-0.133	-0.261	1960:2001
Above+SWE_F	0.102	0.093	0.024	0.019	-0.141	-0.264	1960:2001
Above+UK_F	0.102	0.093	0.024	0.019	-0.139	-0.267	1960:2001
Above+JPN_F	0.100	0.093	0.023	0.019	-0.138	-0.212	1957:2001
Above+CHE_F	0.101	0.093	0.024	0.019	-0.144	-0.208	1957:2001
Above+ESP_F	0.101	0.095	0.024	0.019	-0.138	-0.201	1957:2001
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Blue texts are used for showing the lowest errors obtained by $LL_{e_0^\dagger}$, while red texts are used for showing lowest error obtained by LL.

It has been already mentioned before that different measures of forecast accuracy often leads to different best reference groups. For all combinations of reference group for French Females, the lowest forecast errors were obtained from $LL_{e_0^\dagger}$. The mortality surface of the fitted model for French Females are illustrated in Figure 5.7 for explain the results more closely. The surface obtained from $LL_{e_0^\dagger}$ is slightly more smoothed than LL. Due to use of same weight (Equation 5.3), the effect of smoothing diminishes slowly with increase in number of populations in reference group.

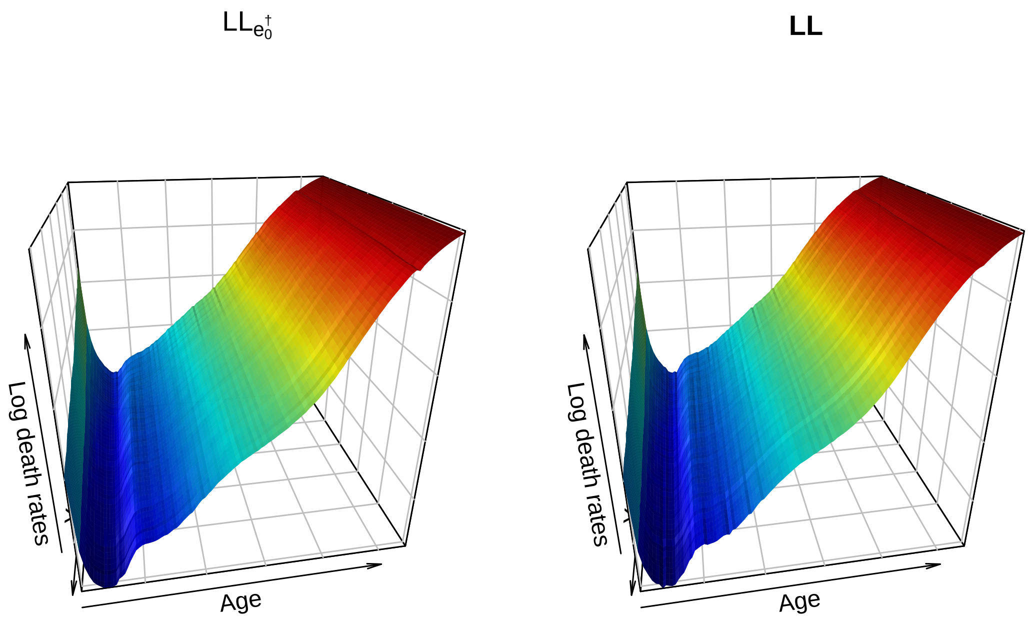


FIGURE 5.7: Mortality surface of fitted coherent models for French Females (1956:2011).

For LL, the lowest MAE and MSE were obtained from same combination of reference group while the lowest ME(e_0) were obtained from a combination of less number of populations in the reference group (Table 5.4). One important feature of the proposed $LL_{e_0^\dagger}$ is the concept of best fitting period. I did not find best fitting period for all of the combinations (Table 5.4). The best reference group obtained for French Females consists of 9 populations and it has the best fitting period for 1974 to 2001. However, this best fitting period slowly shifts to full fitting period eventually after adding more countries. A reference group consists of all 40 low-mortality populations consider full observed period (1956:2001) as best fitting period with 97.3% explained variation. The comparison of different measures of forecast accuracy during out-of-sample evaluation period is given below in Table 5.5.

Table 5.5: Comparison of minimum values of different measures of forecast accuracy for female populations of 20 countries during out-of-sample evaluation period (2002-2011).

Country	MAE		MSE		ME(e_0)	
	LL	$\text{LL}_{e_0^\dagger}$	LL	$\text{LL}_{e_0^\dagger}$	LL	$\text{LL}_{e_0^\dagger}$
Australia	0.121	**0.096**	0.038	**0.022**	-0.011	**-0.004**
Austria	0.193	**0.156**	0.100	**0.066**	0.007	**0.003**
Belgium	0.153	**0.135**	0.070	**0.053**	**0.002**	-0.009
Canada	0.080	**0.068**	0.016	**0.013**	-0.042	**-0.041**
Denmark	0.231	**0.200**	0.129	**0.102**	-0.672	**-0.542**
Finland	0.223	**0.199**	0.125	**0.080**	**-0.017**	-0.072
France	0.100	**0.089**	0.023	**0.017**	-0.123	**-0.106**
Germany	0.087	**0.082**	0.017	**0.014**	**0.049**	0.082
Ireland	0.251	**0.242**	0.143	**0.130**	**-0.998**	-1.063
Italy	0.085	**0.078**	0.019	**0.015**	**-0.005**	-0.032
Japan	0.120	**0.116**	0.034	**0.031**	**0.431**	0.465
The Netherlands	0.149	**0.138**	0.068	**0.056**	-0.516	**0.514**
New Zealand	0.224	**0.184**	0.118	**0.838**	**-0.298**	-0.301
Norway	0.230	**0.188**	0.150	**0.100**	**-0.036**	-0.217
Portugal	0.174	**0.140**	0.074	**0.047**	-0.597	**-0.281**
Spain	0.100	**0.096**	0.023	**0.021**	**0.008**	0.011
Sweden	0.169	**0.149**	0.079	**0.063**	-0.037	**-0.012**
Switzerland	0.222	**0.165**	0.161	**0.074**	0.105	0.105
United Kingdom	0.079	**0.069**	0.014	**0.010**	-0.343	**-0.331**
USA	0.054	0.054	0.005	0.005	**-0.156**	-0.221

Since $\text{LL}_{e_0^\dagger}$ produces lower error than LL for all combinations, the population-specific lowest forecast error for both $\text{LL}_{e_0^\dagger}$ and LL are shown in Table 5.5. For several countries, forecast errors decrease or return to same level of lowest observed error after adding 30 or more populations in reference group. Nevertheless, as mentioned before, I considered the reference group consists of lower number of populations. $\text{LL}_{e_0^\dagger}$ and LL produced same level of MAE and MSE for US Females. For mean error of life expectancy, LL performed better than that of $\text{LL}_{e_0^\dagger}$; LL produced lower error for 11 populations. Unusual rise in ME(e_0) is observed in case of $\text{LL}_{e_0^\dagger}$ for several other countries, for all of them I noticed same pattern as discussed before in the section of best reference group. Best fitting period observed for corresponding best reference group are attached in appendix and it is given for all three measures of forecast accuracies. The variance explained by the fitted coherent models are also attached in appendix.

5.2.4 Forecast of life expectancy

The forecast of female life expectancy at birth for all 20 low-mortality countries are plotted below in Figure 5.8 and forecast in 2050 are presented in Table 5.6. For both of the methods, I checked forecast accuracy during out-of-sample evaluation and from three possible combinations of best reference group I considered the one having lower number of populations to make coherent forecast.

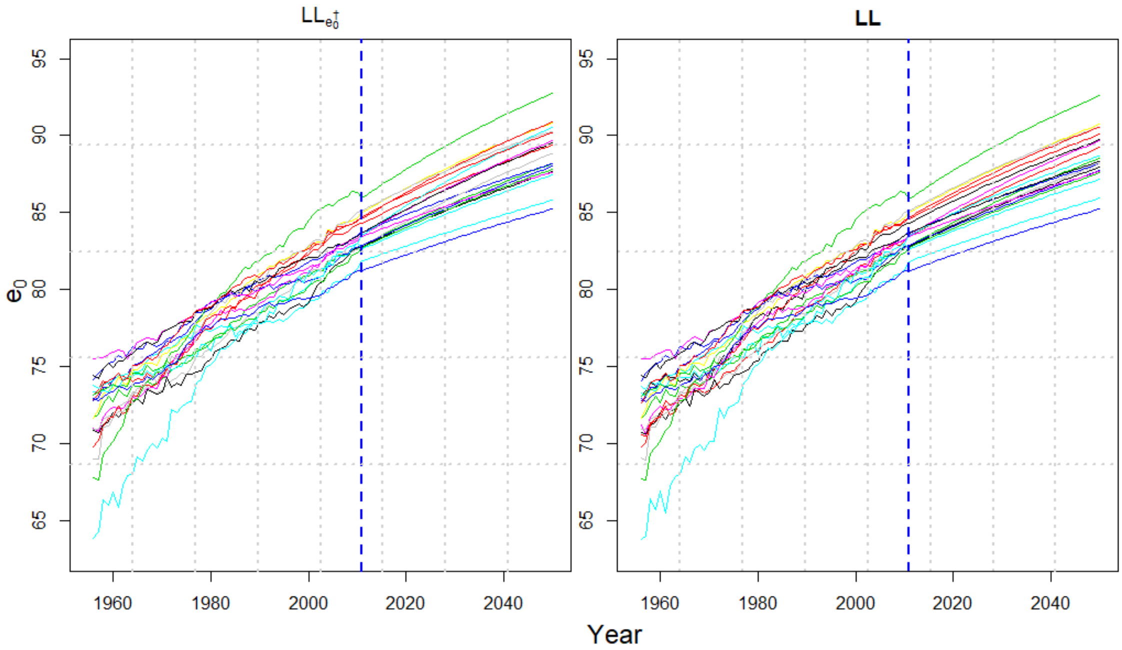


FIGURE 5.8: Observed (1956:2011) and forecast (2012:2050) of female life expectancy at birth for 20 low-mortality countries.

TABLE 5.6: Comparison of coherent forecast of life expectancy at birth in 2050 for female populations of 20 low-mortality countries.

Country	LL	LL$_{e_0^\dagger}$	Country	LL	LL$_{e_0^\dagger}$
Australia	**89.719**	89.382	Japan	92.636	**92.777**
Austria	89.216	**89.382**	The Netherlands	87.721	**88.009**
Belgium	**88.538**	88.184	New Zealand	87.159	**87.432**
Canada	88.166	88.166	Norway	**87.662**	87.636
Denmark	**85.943**	85.813	Portugal	88.692	**90.537**
Finland	89.665	**89.668**	Spain	**90.567**	90.278
France	90.730	**90.824**	Sweden	88.327	**89.535**
Germany	88.154	**88.851**	Switzerland	90.557	**90.194**
Ireland	**87.934**	87.649	United Kingdom	87.571	**87.802**
Italy	90.082	**90.887**	USA	85.235	**85.237**

Except for Australia, Belgium, Denmark, Ireland, Norway, Spain and Switzerland, the forecast of life expectancy were higher for $\text{LL}_{e_0^\dagger}$ than that of LL. To get more insight of the obtained results I identified the populations for which $\text{LL}_{e_0^\dagger}$ was most optimistic and least optimistic than that of LL. For Portugal and Sweden the forecast obtained from $\text{LL}_{e_0^\dagger}$ have highest positive difference with that of LL whereas the highest negative difference were observed for Belgium and Switzerland. The forecast of these four countries are plotted below in Figure 5.9.

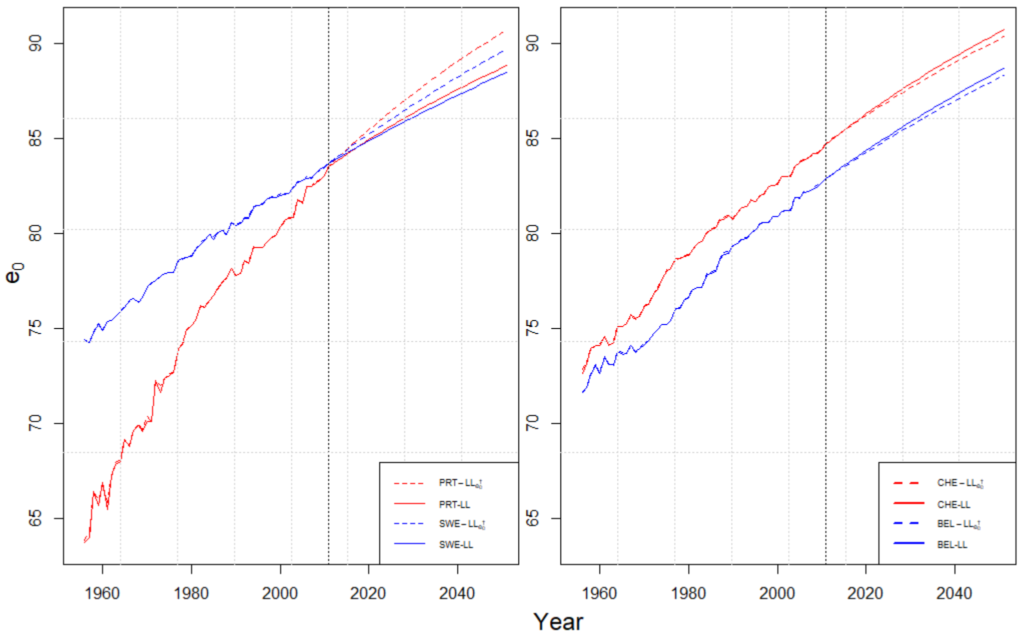


FIGURE 5.9: Observed (1956:2011) and forecast (2012:2050) of female life expectancy at birth for Portugal, Sweden, Switzerland and Belgium. For Portugal and Sweden the positive difference of forecast were highest in 2050 while highest negative difference of forecast were observed for Switzerland and Belgium.

Among these four countries, Portugal is mentioned before for remarkable improvement in health status and rapid increase of life expectancy (Van Oyen *et al.*, 2013). Between 2000 and 2015 the female life expectancy increased by almost four years for Portugal, almost 5 years for males (HMD, 2018). However, these improvements have not been followed at the same pace for different income groups and disparities exist for other important dimensions of health. Cardiovascular diseases and cancer are the largest contributors to mortality(Van Oyen *et al.*, 2013). This asymmetry in distribution of death is reflected also in trend of $e_0^\dagger$ (Table 5.1). Portuguese Females ranked 15th in terms of sorted $\bar{e}_0^\dagger$ (Table 5.1). Unlike Portugal, Sweden and Switzerland have almost steady pattern of mortality improvement since beginning. However, this scenario is different for Danish Females. During the past decades, the life expectancy of Danish women has lagged behind that of women in neighboring Western European countries (Jacobsen *et al.*, 2002). Among various causes-of-deaths, ischaemic heart diseases followed by lung cancer are responsible for lower life expectancy of Danish Females. Danish female mortality is mentioned before for having distinct pattern of remarkable middle-aged mortality (Juel *et al.*, 2000).

5.2.5 Prediction interval of forecast

I followed the same methodology for prediction interval as mentioned before in section 2.4. The practical implication is slightly different for coherent forecasting for hierarchical structure of the model than that done before for single population forecasting. Also, for the proposed method it became more complicated due to best fitting period. From equation 5.2, it is clear the possible sources of variation are the common factor and the country-specific product of $b_{x,i}$ and $k_{t,i}$. Assuming independence for both common factor and $b_{x,i}k_{t,i}$ during estimation procedure, large number of future deaths are simulated from both of the estimated model for reference group and country-specific model separately and finally the simulated common factor is added with country-specific model to obtain simulated future death rates using equation 5.2. To avoid computational complexity and to keeping country-specific $b_{x,i}k_{t,i}$ independent from common factor obtained from the reference group, I considered full fitting period instead of best fitting period for constructing the prediction intervals. Following the results of forecast of life expectancy till 2050, the prediction interval of e_0 is plotted below in Figure 5.10 for Portugal, Sweden, Switzerland and Belgian Females. For Sweden and Belgium, the prediction interval of the $\mathrm{LL}_{e_0^\dagger}$ is slightly wider than that of LL, whereas prediction interval from LL are slightly wider for Portugal and Switzerland. The prediction intervals using $\mathrm{LL}_{e_0^\dagger}$ for all female populations of the low mortality countries are attached in appendix.

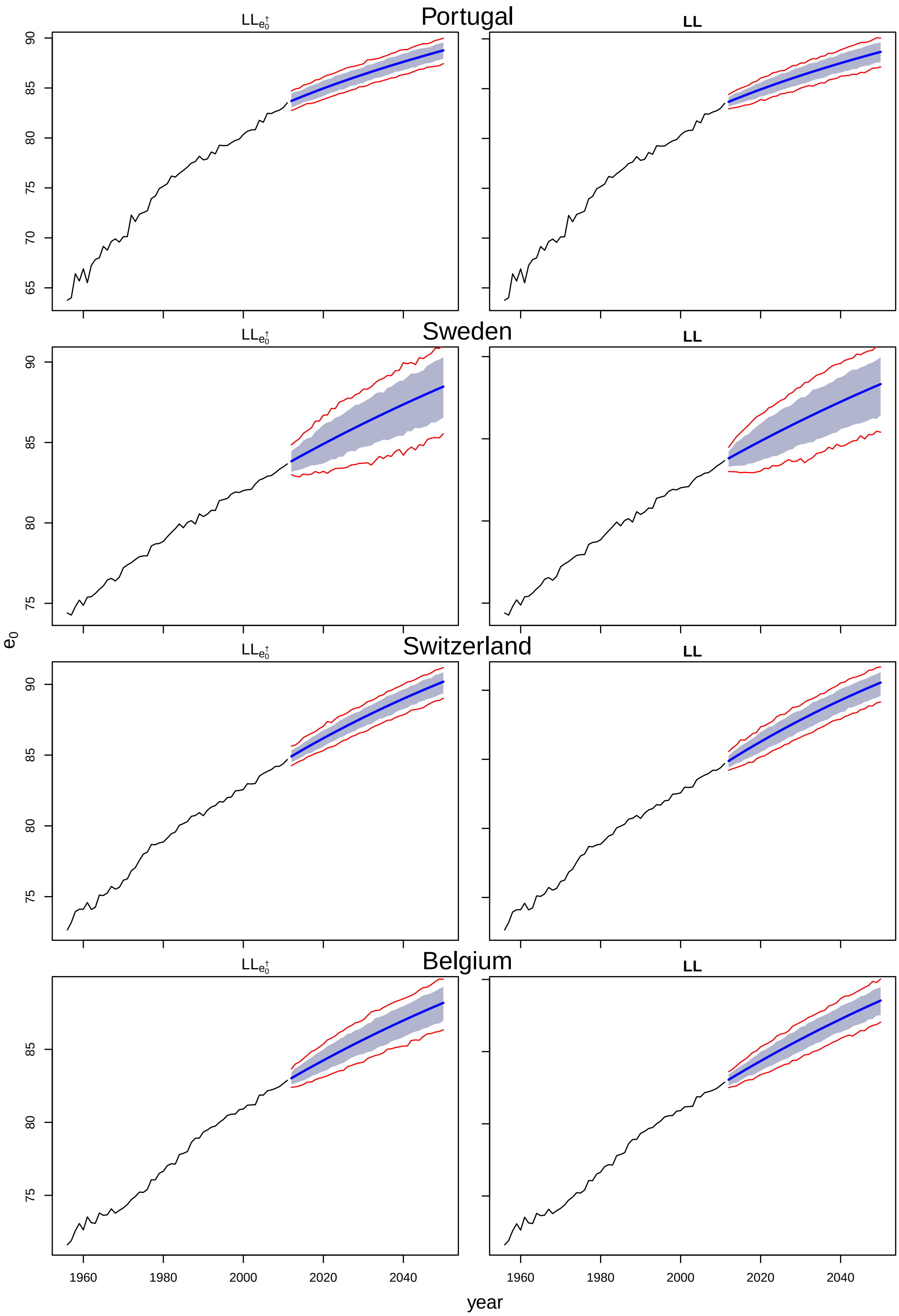


FIGURE 5.10: Prediction interval of Portuguese, Swedish, Swiss and Belgian female e_0 till 2050. The blue are represents 80% prediction interval and red lines are for 95% prediction interval.

Chapter 6

Application of Proposed Methods on (Comparatively) High Mortality Countries

In this chapter the proposed methods ($\mathrm{LC}_{e_0^\dagger}$ and $\mathrm{LL}_{e_0^\dagger}$) are applied on nine CEE countries- Belarus, Bulgaria, Estonia, Hungary, Latvia, Lithuania, Russia, Slovakia and Ukraine. The findings of the existing models on these countries are already illustrated in Chapter 3. Like low-mortality countries, none of the existing models performed uniquely well for these countries and performances of some models were also subject to further analysis. This chapter consists of two parts. In first part I compared the $\mathrm{LC}_{e_0^\dagger}$ with the existing single population forecasting methods (LC variants) and in second part I compared the findings of $\mathrm{LL}_{e_0^\dagger}$ with LL only.

6.1 Findings from $\mathrm{LC}_{e_0^\dagger}$

6.1.1 Application of *LASSO*

I compared Lasso with spline based techniques for all of these CEE populations and Lasso found to be most accurate one for all of these countries as well in terms of mean absolute error (MAE) and mean squared error (MSE) of the mortality rates. The smoothed mortality rate for Hungarian Females are plotted below in Figure 6.1. Similar to low-mortality countries, smoothing by Lasso seems less smoothed than that of other two methods. The surfaces of the smoothed mortality rates are plotted in Figure 6.2.

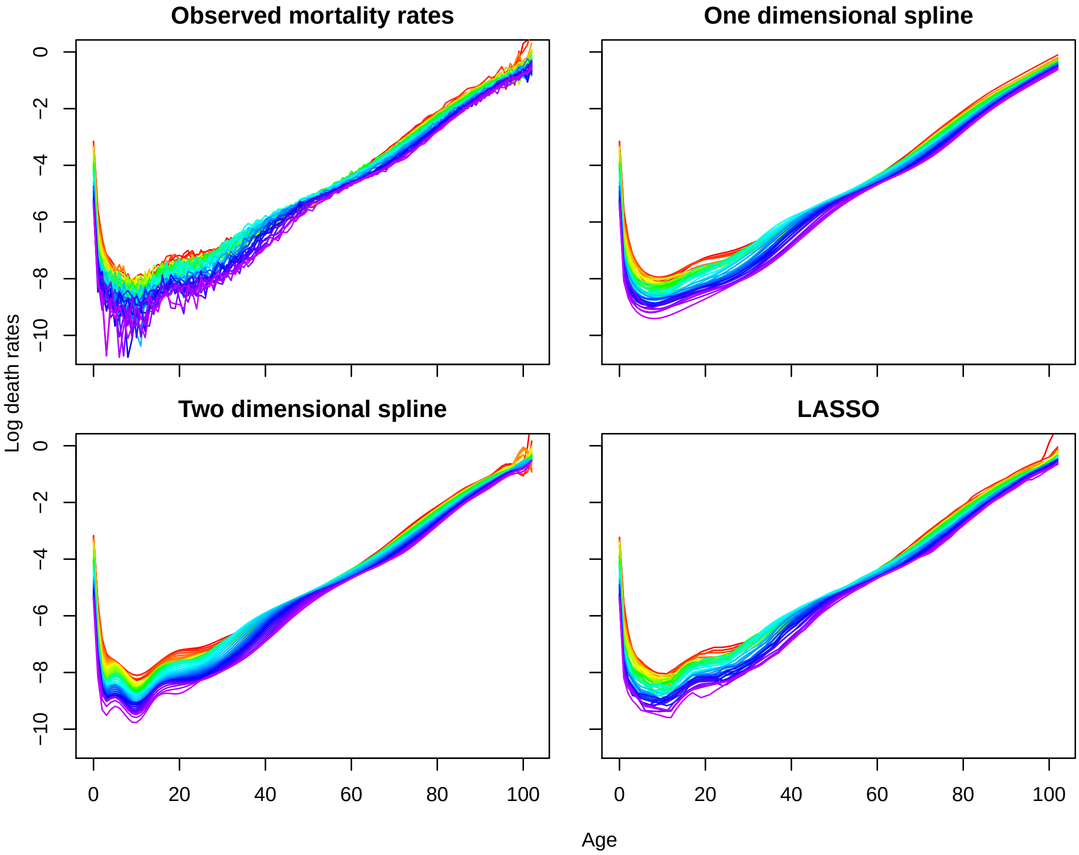


FIGURE 6.1: Comparison of smoothing techniques for Hungarian Female mortality (1960:2014). Years are plotted using a rainbow palette so the earlier years are shown in red, followed by orange, yellow, green, blue and indigo with the most recent years plotted in violet.

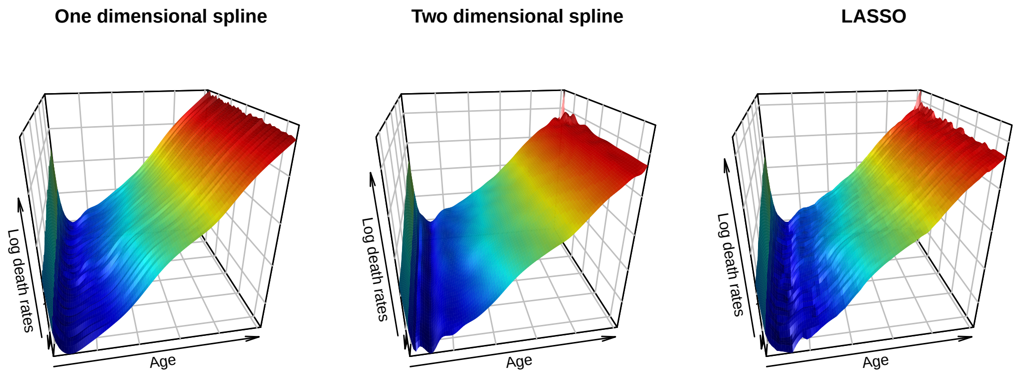


FIGURE 6.2: Mortality surfaces from difference smoothing techniques for Hungarian Females (1960:2014).

The corresponding errors of all three smoothing techniques are plotted below in Figure 6.3. All three methods produce errors in earlier part of life span and it was highest for two dimensional smoothing (also for senescence mortality). The accuracy measures of all these smoothing techniques are given below in Table 6.1. Lasso smooths the mortality rates with highest level of accuracy. In terms of mean error of life expectancy at birth, two dimensional smoothing seems more accurate. However, it is also the least accurate one in terms of MAE and MSE of mortality rates, followed by one dimensional smoothing.

TABLE 6.1: Accuracy of smoothing techniques for Hungarian Females (1960:2014).

Measure of accuracy	1 dim Smoothing	2 dim Smoothing	**LASSO**
MAE(m_x)×100	8.721	9.705	7.393
MSE(m_x)×100	2.503	2.772	1.977
ME(e_0)	-0.032	-0.019	0.036
MAE(e_0)	0.032	0.157	0.052

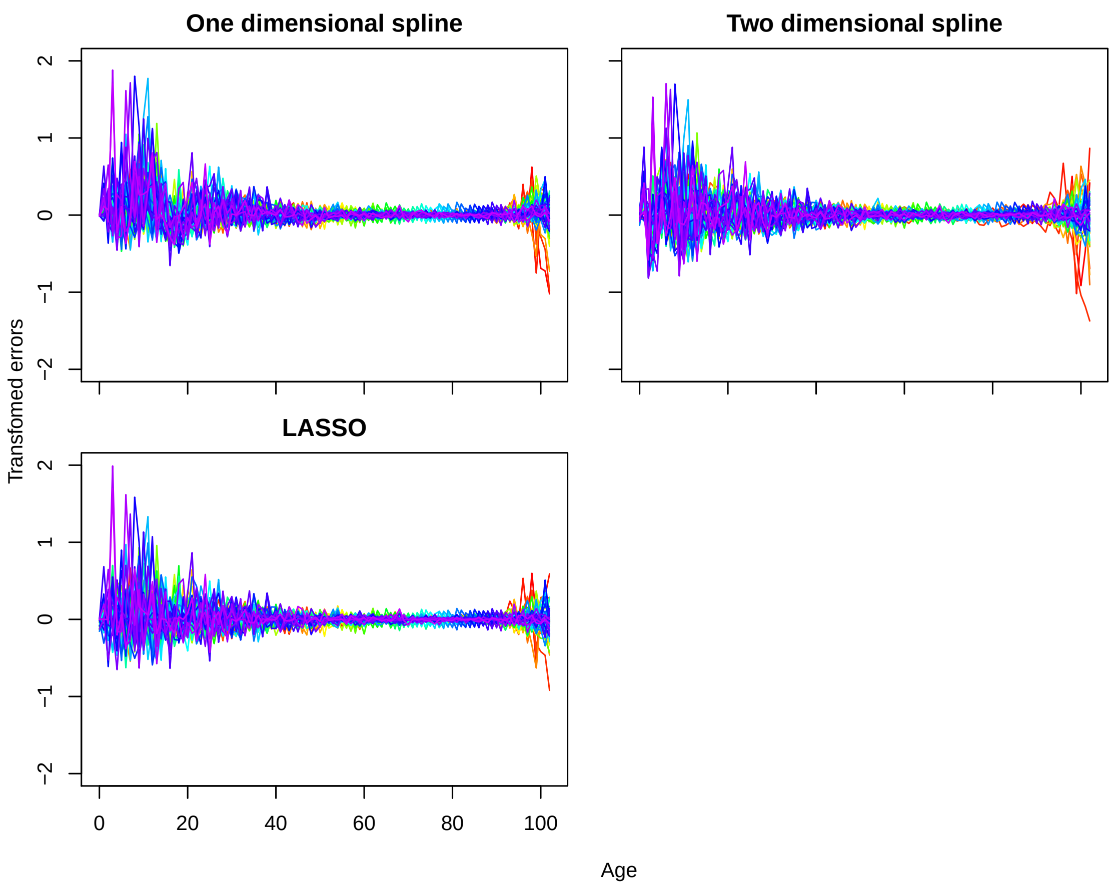


FIGURE 6.3: Errors of smoothing techniques for female mortality of Hungary(1960:2014). Years are plotted using a rainbow palette as before.

6.1.2 Forecast accuracy

Forecast accuracy in terms of MAE, MSE and ME(e_0) for all of these methods during out-of-sample evaluation are summarized below in Table 6.2, 6.3 and 6.4 respectively.

TABLE 6.2: Comparison of MAE during out-of-sample evaluation period.

Country	LC	$\mathrm{LC_P}$	LM	BMS	HU	$\mathrm{HU_R}$	$\mathrm{HU_W}$	$\mathrm{LC}_{e_0^\dagger}$
Belarus	-	-	0.238	-	-	-	**0.123**	-
Bulgaria	0.198	0.178	0.178	0.180	0.173	**0.164**	0.150	0.168
Estonia	0.351	0.347	0.369	0.347	0.365	0.422	0.346	**0.314**
Hungary	0.235	0.231	0.183	0.170	0.180	0.171	**0.156**	0.170
Latvia	**0.244**	**0.244**	0.268	0.245	0.252	0.310	0.256	0.249
Lithuania	0.390	0.217	0.228	0.225	0.199	0.200	**0.187**	0.193
Russia	-	-	-	-	-	**0.219**	-	-
Slovakia	0.192	0.189	0.230	0.182	0.200	0.189	0.185	**0.169**
Ukraine	-	-	-	-	-	**0.168**	0.171	-

A blank place means the forecast of e_0 was lower than the last observed value of e_0.

TABLE 6.3: Comparison of MSE during out-of-sample evaluation period.

Country	LC	$\mathrm{LC_P}$	LM	BMS	HU	$\mathrm{HU_R}$	$\mathrm{HU_W}$	$\mathrm{LC}_{e_0^\dagger}$
Belarus	-	-	0.109	-	-	-	**0.038**	-
Bulgaria	0.068	0.057	0.067	0.067	0.061	0.059	**0.052**	0.057
Estonia	0.215	0.206	0.272	0.210	0.214	0.277	0.206	**0.177**
Hungary	0.117	0.112	0.081	0.077	0.085	0.081	**0.070**	0.072
Latvia	0.143	0.143	0.180	**0.142**	**0.142**	0.194	0.162	0.153
Lithuania	0.347	0.104	0.135	0.118	**0.090**	0.097	0.092	0.092
Russia	-	-	-	-	-	**0.072**	-	-
Slovakia	0.084	0.080	0.121	0.083	0.096	0.096	0.094	**0.071**
Ukraine	-	-	-	-	-	**0.046**	0.050	-

A blank place means the forecast of e_0 was lower than the last observed value of e_0.

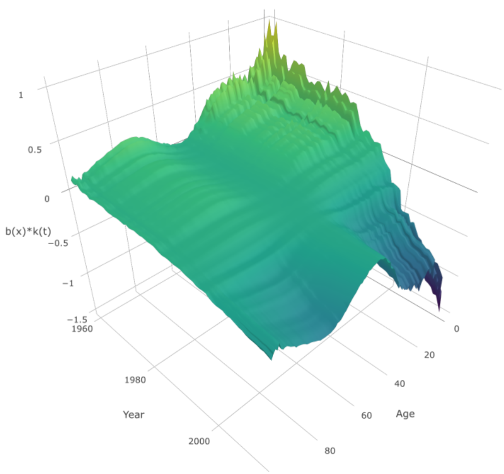
The country-specific highest forecast accuracy are marked with bold texts. The methods failed to make optimistic forecast of life expectancy in 2050 are omitted for comparison of forecast accuracy during out-of-sample evaluation. In terms of MAE and MSE, $\mathrm{LC}_{e_0^\dagger}$ found to be the most accurate method to forecast mortality rates for Estonia and Slovakia. Both of these countries have better mortality scenario than other CEE countries. For mean error of life expectancy, $\mathrm{LC}_{e_0^\dagger}$ found to be the most accurate method for Estonia, Hungary and Latvia. Although $\mathrm{LC}_{e_0^\dagger}$ was not the most accurate method for majority of these countries, still it produced low ME(e_0) for several of them.

TABLE 6.4: Comparison of ME(e_0) during out-of-sample evaluation period.

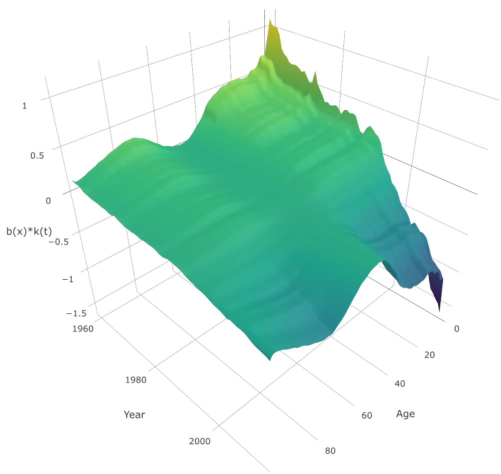
Country	LC	LC$_\text{P}$	LM	BMS	HU	HU$_\text{R}$	HU$_\text{W}$	LC$_{e_0^\dagger}$
Belarus	-	-	**-1.888**	-	-	-	-1.981	-
Bulgaria	**-0.443**	-0.725	-0.820	-1.143	-1.064	-1.182	-0.464	-0.786
Estonia	-2.017	-2.179	-2.012	-2.143	-3.181	-3.865	-2.740	**-1.977**
Hungary	-0.896	-1.179	-0.514	-0.224	-0.887	-0.710	-0.582	**-0.536**
Latvia	-1.393	-1.400	-1.388	-1.398	-1.712	-2.249	-1.481	**-1.381**
Lithuania	0.680	-0.436	**-0.078**	-0.890	-0.622	-0.465	-0.155	-0.179
Russia	-	-	-	-	-	**-2.587**	-	-
Slovakia	-0.440	-0.638	-0.544	**-0.199**	-1.074	-0.798	-0.665	-0.595
Ukraine	-	-	-	-	-	**-1.085**	-1.450	-

A blank place means the forecast of e_0 was lower than the last observed value of e_0.

The benefit of using Lasso and new adjustment technique for $\hat{k}_t$ is already illustrated before for low-mortality countries (see Figure 4.9 for example). The product of b_x (obtained in LC$_{e_0^\dagger}$) and k_t for Hungarian Females are plotted below in Figure 6.4. The surface obtained for LC$_{e_0^\dagger}$ is smoother than ordinary LC. This also increases the goodness of fit, the ordinary LC explained 70.4% variation during fitting period whereas LC$_{e_0^\dagger}$ explained 85.9% for Hungarian Females.



(a) LC



(b) LC$_{e_0^\dagger}$

FIGURE 6.4: Product of time component and age component for Hungarian Females (1960:2014) using LC (Lee and Carter, 1992) and proposed LC$_{e_0^\dagger}$.

The variance explained by the fitted models for the high-mortality countries are attached in appendix. The mortality surface of the fitted models for Hungarian Females are illustrated in Figure 6.5.

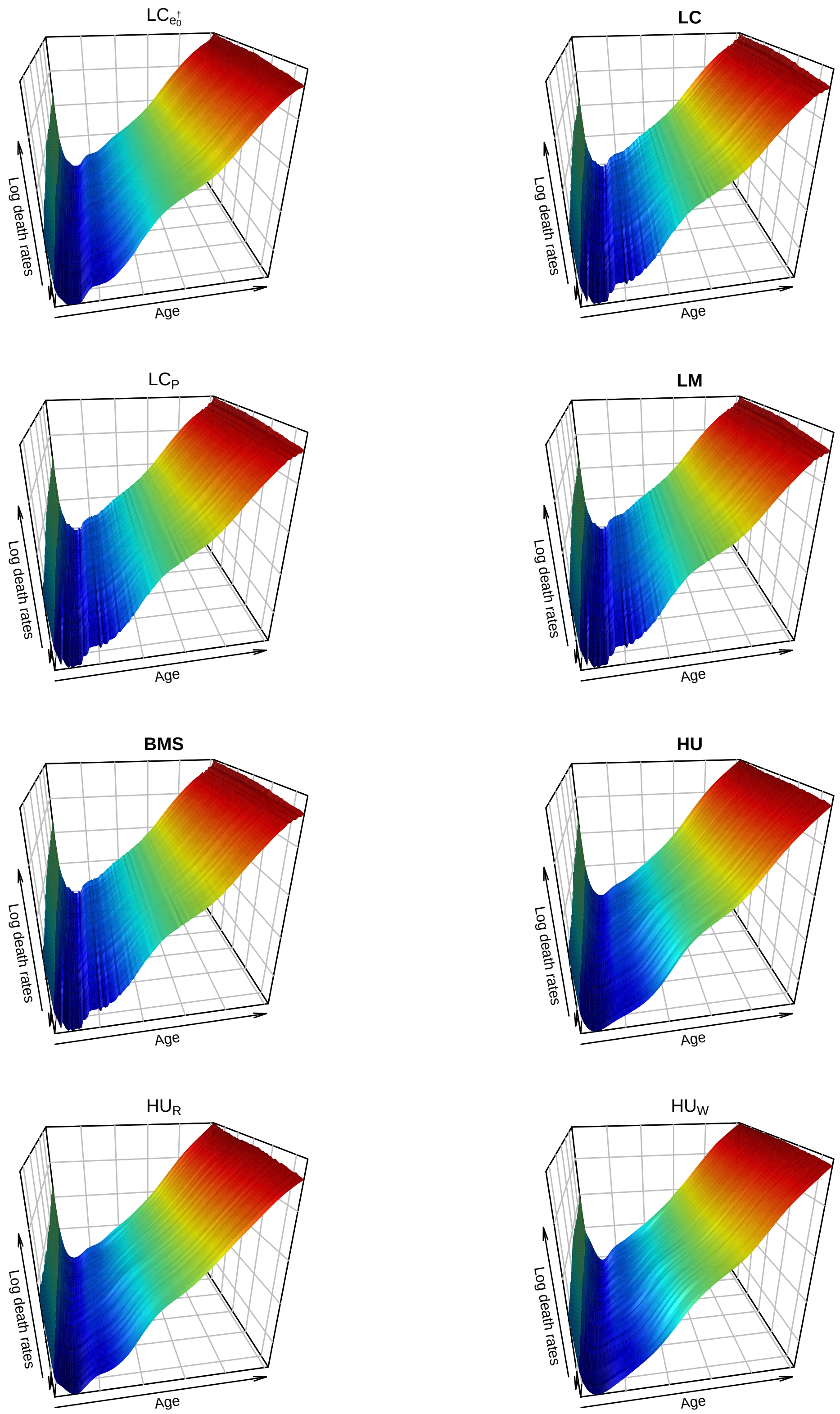


FIGURE 6.5: Mortality surface of the fitted models for Hungarian Females (1960:2014).

6.1.3 Forecast of life expectancy at birth

Forecast of female life expectancy at birth from different models are given below in Table 6.5 for all these nine CEE countries. Similar to most of the LC variants, $LC_{e_0^\dagger}$ also failed to produce optimistic forecast for Belarus, Russia and the Ukraine. Unlike low-mortality countries, $LC_{e_0^\dagger}$ did not produce most optimistic forecast of life expectancy for any of these CEE countries.

TABLE 6.5: Forecast of female life expectancy at birth in 2050 for selected CEE countries.

Country	e_0	LC	LC_P	LM	BMS	HU	HU_R	HU_W	$LC_{e_0^\dagger}$
Belarus	78.43	68.68	75.92	78.43	76.82	75.68	77.50	**80.55**	78.16
Bulgaria	77.25	82.16	79.77	80.24	79.76	78.49	80.83	**82.61**	79.90
Estonia	81.33	85.67	85.58	85.75	**90.30**	84.20	82.75	84.95	86.15
Hungary	79.24	82.77	82.66	83.27	85.40	84.19	82.44	**87.09**	83.40
Latvia	78.73	81.66	81.54	81.68	**85.83**	81.10	81.34	82.06	81.69
Lithuania	79.37	82.17	80.92	81.77	**84.82**	80.24	79.63	82.42	81.00
Russia	76.48	73.62	73.12	76.24	72.93	73.54	**78.45**	76.43	75.95
Slovakia	80.32	84.52	84.26	84.27	**85.75**	83.59	83.12	83.73	83.62
Ukraine	76.21	70.90	74.05	76.20	73.67	73.37	**80.03**	78.47	75.76

e_0 is the last observed life expectancy during the fitting period from HMD (Table 2.1).

The forecast from LC and HU variants in Table 6.5 are same as presented earlier in chapter 3 (Table 3.1). The proposed $LC_{e_0^\dagger}$ could not overcome the problem of a lower and pessimistic estimate of future life expectancies for Belarus, Russia and Ukraine. The interval forecast of life expectancy at birth for all these methods are illustrated below in Figure 6.6 for Hungarian Females. LC and HU_W produced unusually wider prediction interval compare to other method.

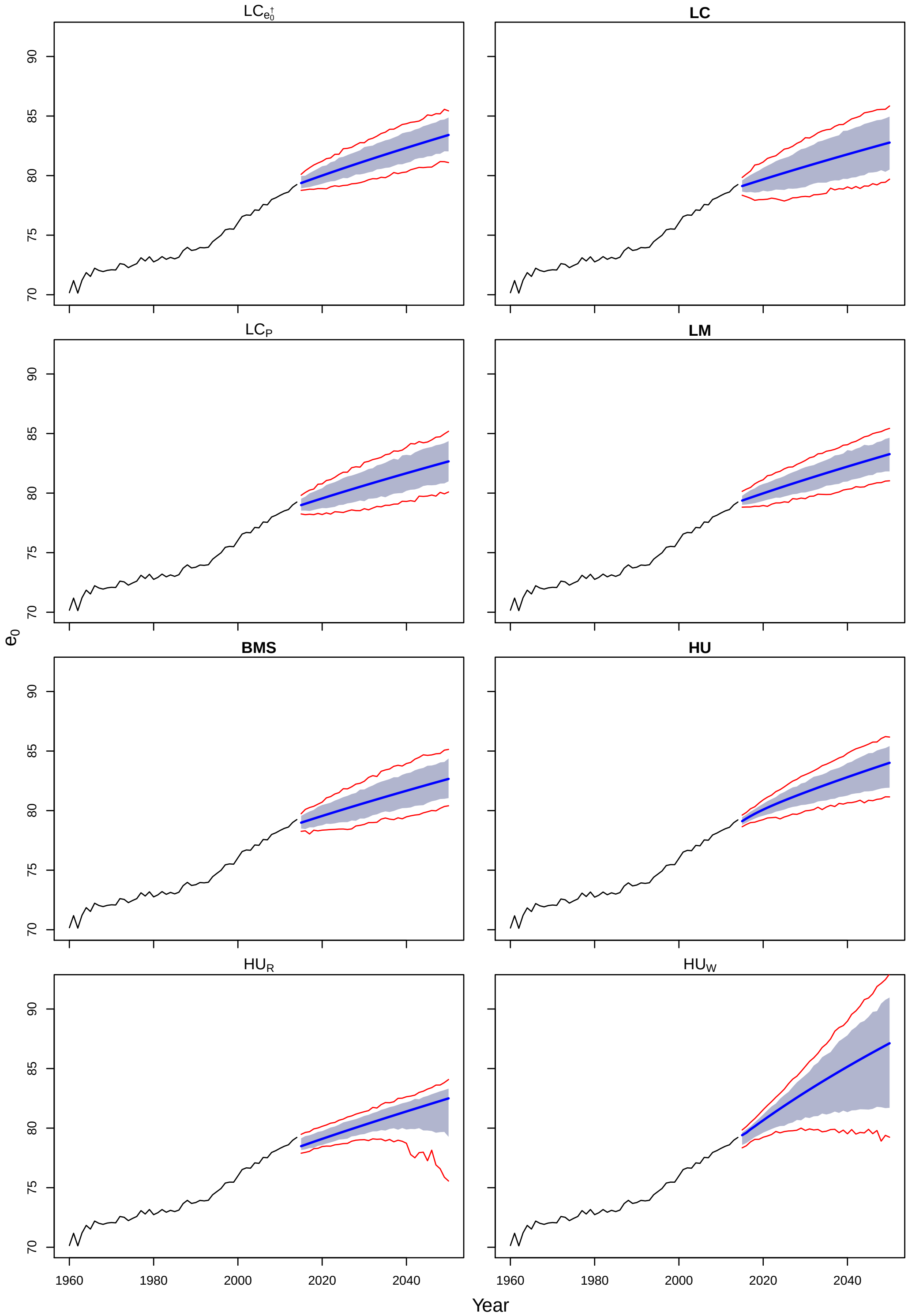


FIGURE 6.6: Prediction interval of Hungarian female e_0 by different methods till 2050. The blue area represents 80% prediction interval and red lines are for 95% prediction interval.

6.2 Findings from $\text{LL}_{e_0^\dagger}$

6.2.1 Best reference population

The common time frame for the CEE countries is shorter than the low-mortality countries. Another complexity is due to severe mortality crisis during mid 1990s, some of the populations suffered from it till early 2000s. Thus, in absence of any stable combination, the pattern starting from 2001 is considered. The sorted $\bar{e}_0^\dagger$ of the CEE populations over the period 2001:2010 are given below in Table 6.6. Except for some populations this is the most stable pattern observed for these countries. The combination of sorted closest populations are different for each different starting year. Similar to the low-mortality countries, life table constructed up to age 110 year are considered also for these countries. All the mortality rates for age 100 to 110+ are replaced/extrapolated using Kannisto model (Thatcher *et al.*, 1998). For country-specific illustration, I applied the equation 5.1 for Hungarian Females. The closest populations for Hungarian Females to be use in reference group are given below in Table 6.7.

TABLE 6.6: Sorted $\bar{e}_0^\dagger$ over the period 2001:2010 for the high-mortality CEE countries.

Population	SVK_F	BGR_F	EST_F	HUN_F	LTU_F	BLR_F
$\bar{e}_0^\dagger$	10.373	10.887	11.116	11.193	11.487	11.598
Population	LVA_F	UKR_F	SVK_M	RUS_F	BGR_M	HUN_M
$\bar{e}_0^\dagger$	11.717	12.101	12.596	12.722	13.062	13.192
Population	EST_M	BLR_M	LVA_M	LTU_M	UKR_M	RUS_M
$\bar{e}_0^\dagger$	14.125	14.485	14.536	15.051	15.116	15.799

Country codes are same as mentioned in section 2.1.
M and *F* stands for males and females respectively.

TABLE 6.7: Reference population for Hungarian Females based on closest difference in $\bar{e}_0^\dagger$ during 2001:2010.

Population	HUN_F	EST_F	LTU_F	BGR_F	BLR_F	LVA_F
$\left\lvert \bar{e}_{0,\text{FRA}_\text{F}}^\dagger - \bar{e}_{0j}^\dagger \right\rvert$	-	0.078	0.294	0.307	0.405	0.524
Population	SVK_F	UKR_F	SVK_M	RUS_F	BGR_M	HUN_M
$\left\lvert \bar{e}_{0,\text{FRA}_\text{F}}^\dagger - \bar{e}_{0j}^\dagger \right\rvert$	0.820	0.908	1.403	1.528	1.869	1.998
Population	EST_M	BLR_M	LVA_M	LTU_M	UKR_M	RUS_M
$\left\lvert \bar{e}_{0,\text{FRA}_\text{F}}^\dagger - \bar{e}_{0j}^\dagger \right\rvert$	2.931	3.291	3.343	3.858	3.923	4.605

Country codes are same as mentioned in section 2.1.
M and *F* stands for males and females respectively.

Although these populations are mentioned for divergent mortality pattern, still the sorted $\bar{e}_0^\dagger$ are ranked naturally with lowest values for females followed by males except for some overlapping.

6.2.2 Optimal size of best reference group

To obtain the best reference group for a particular population, I checked the forecast accuracy during out-of-sample evaluation. Similar to the low-mortality countries, for all three measures of forecast accuracy there is a clear fall (rise) on accuracy (error) level after adding some certain countries using $\mathrm{LL}_{e_0^\dagger}$; LL also have the same criterion. Different combination of populations are obtained as best reference group from different measures of forecast accuracies for the CEE countries also. The distribution of number of CEE countries in best reference group according to difference in $\left|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right|$ is plotted below in Figure 6.7.

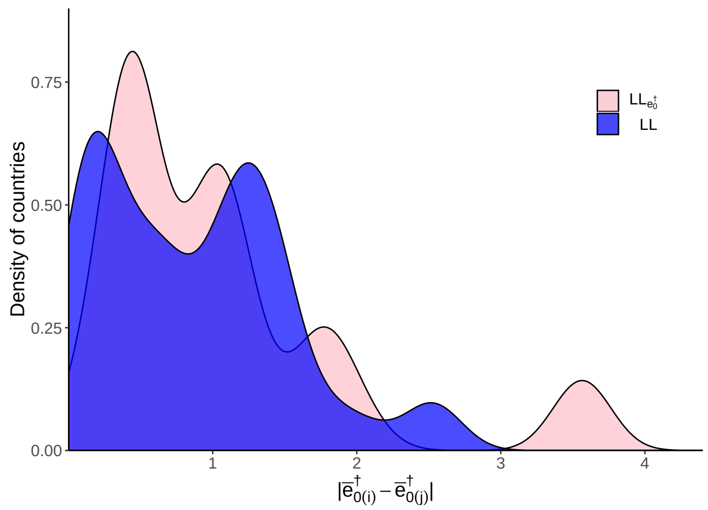


FIGURE 6.7: Distribution of CEE countries obtaining reference population according to difference in $\left|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right|$.

For some CEE populations, the reference group was larger and this made the distributions more widespread than that observed before for low-mortality countries (Figure 5.6). Diverging mortality pattern between the countries are responsible for this wide distribution of $\left|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right|$ in case of CEE countries. For instance, best reference group of Lithuanian Males obtained from $\mathrm{LL}_{e_0^\dagger}$ contains 14 populations (including Lithuanian Males) with difference in $\bar{e}_0^\dagger$ of 3.564. The optimal number of populations in reference group and corresponding differences in $\left|\bar{e}_{0i}^{\dagger} - \bar{e}_{0j}^{\dagger}\right|$ considering all forecast errors are summarized below in Table 6.8. The errors are considered separately for males, females and both sexes together.

TABLE 6.8: Summary statistics for best reference group.

	Summary statistics	$LL_{e_0^\dagger}$ Male	$LL_{e_0^\dagger}$ Female	$LL_{e_0^\dagger}$ All	LL Male	LL Female	LL All
Number of populations	Mean	5	4	5	2	4	3
	Median	6	3	4	3	3	2
	IQR	9	3	6	2	3	4
Difference in $\left|\bar{e}_{0i}^{\dagger} - \bar{e}_{0j}^{\dagger}\right|$	Mean	1.60	0.64	1.05	0.84	0.88	0.87
	Median	1.51	0.51	0.88	0.13	0.96	0.74
	IQR	0.82	0.55	0.66	1.34	0.57	1.10

I could not obtain best reference group for several of these CEE populations as the corresponding models failed to produce optimistic forecast in 2050. More discussion about the forecast accuracy will be given in next section. Mortality divergence of these populations are strongly reflected in the summary statistics obtained for the best reference group. The trend of $e_0^\dagger$ for all of these populations are plotted below in Figure 6.8.

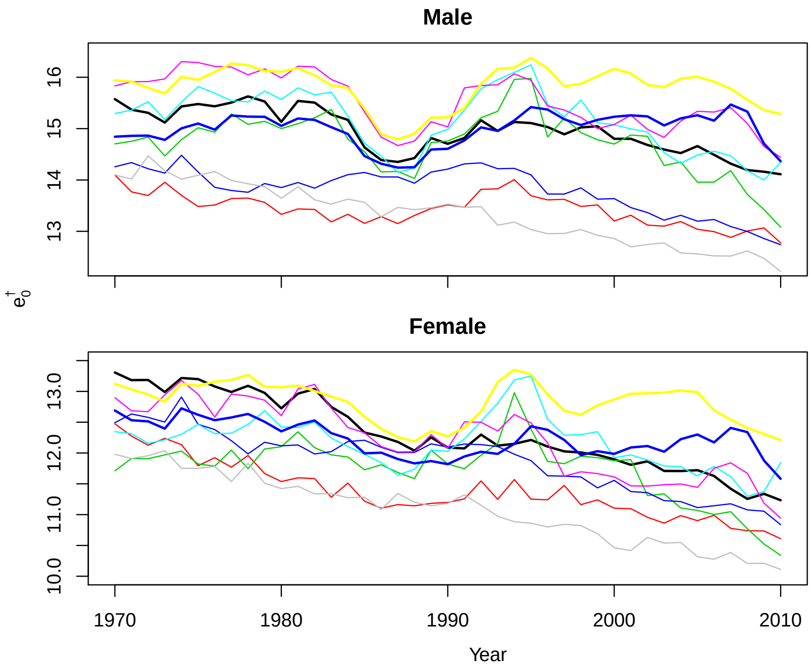


FIGURE 6.8: Trend of $e_0^\dagger$ for the high-mortality CEE countries (1970:2010). The bold black, blue and yellow line represents Belarus, Russia and the Ukraine.

In Figure 6.8, the bold black, blue and yellow lines represent Belarus, Russia and the Ukraine respectively. These countries are mentioned before for highest mortality levels. Except for Hungary and Slovakia (represented by thin blue and gray line respectively in Figure 6.8) all of these populations suffered from mortality crisis during mid 1990s and it was severe for Belarus, Russia and the Ukraine; specially for males (Shkolnikov *et al.*, 1998). To illustrate the divergence of mortality for these CEE countries, the fitted b_x from individual LC model is plotted below in Figure 6.9 for all the females populations of the CEE countries. In addition to high-mortality regime, the mortality improvement over the years are too divergent for obtaining any coherent forecast from these populations compare to that observed for low-mortality countries (Figure 5.1).

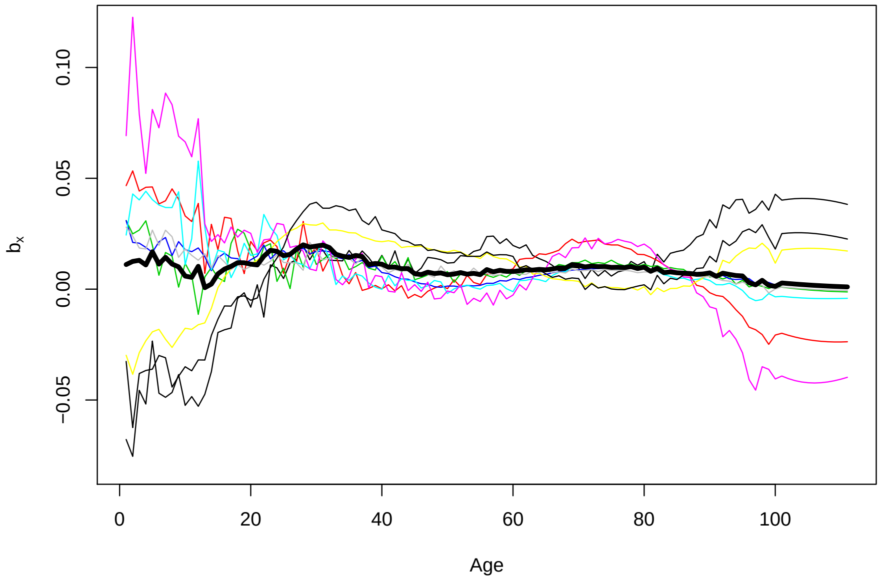


FIGURE 6.9: Estimated b_x from fitted Lee-Carter model for females of nine CEE countries (1970:2010). The bold black line is the mean trend of b_x for comparing the countries.

6.2.3 Forecast accuracy

The Comparison of minimum values of different measures of forecast accuracy for female populations of CEE countries during out-of-sample evaluation period are given below in Table 6.9. $\text{LL}_{e_0^\dagger}$ is omitted from this comparison for Belarus as $\text{LL}_{e_0^\dagger}$ failed to produce optimistic forecast of life expectancy for Belarus. Except for Russia and the Ukraine, $\text{LL}_{e_0^\dagger}$ was most accurate method during out-of-sample evaluation in terms of MAE and MSE for mortality rates. It was not the case for mean error of life expectancy; $\text{LL}_{e_0^\dagger}$

found to be most accurate for Hungary only. Moreover, best fitting period for all of these populations are observed in case of $\text{LL}_{e_0^\dagger}$. Details about the reference populations according to forecast accuracy and best fitting period are attached in Appendix.

TABLE 6.9: Comparison of minimum values of different measures of forecast accuracy for female populations of CEE countries during out-of-sample evaluation period (2001-2010).

Country	MAE		MSE		ME(e_0)	
	LL	$\text{LL}_{e_0^\dagger}$	LL	$\text{LL}_{e_0^\dagger}$	LL	$\text{LL}_{e_0^\dagger}$
Belarus	0.159	-	0.058	-	-0.772	-
Bulgaria	0.187	**0.173**	0.070	**0.057**	**-0.971**	-1.066
Estonia	0.335	**0.292**	0.236	**0.180**	**-1.819**	-1.940
Hungary	0.175	**0.137**	0.070	**0.050**	-0.909	**-0.663**
Latvia	0.180	**0.174**	0.100	**0.090**	**0.394**	-0.577
Lithuania	0.204	**0.170**	0.100	**0.072**	**-0.117**	-0.192
Russia	**0.122**	0.135	**0.025**	0.031	**-0.866**	-1.264
Slovakia	0.211	**0.157**	0.128	**0.066**	**-0.225**	-0.352
Ukraine	**0.111**	0.113	0.023	0.023	**-0.343**	-0.667

A blank place means forecast of e_0 in 2050 was lower than the observed e_0 of 2010.

The mortality surface of the fitted model for Hungarian Females are illustrated in Figure 6.10 to illustrate the findings more closely. The surface obtained from $\text{LL}_{e_0^\dagger}$ is more smoothed than LL.

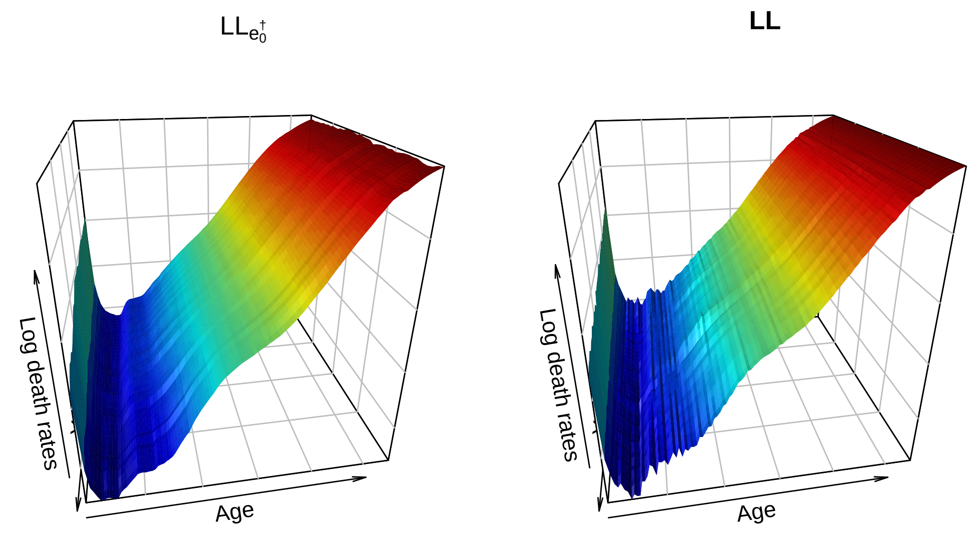


FIGURE 6.10: Mortality surfaces of the fitted coherent models for Hungarian Females (1970:2010).

As of Figure 6.10, the effect of smoothing is highly visible for all of these high-mortality countries unlike that observed for low-mortality countries. The variance explained by the fitted models for females of high-mortality countries are attached in appendix.

6.2.4 Forecast of life expectancy

The forecast of female life expectancy at birth in 2050 for all nine CEE countries are given below in Table 6.10.

TABLE 6.10: Comparison of coherent forecast of life expectancy at birth in 2050 for female populations of selected CEE countries.

Country	e_0	LL	$\mathrm{LL}_{e_0^\dagger}$
Belarus	76.49	**77.01**	76.37
Bulgaria	77.25	**80.15**	79.96
Estonia	80.55	84.82	**85.43**
Hungary	78.34	**82.96**	82.59
Latvia	77.39	80.78	**81.40**
Lithuania	78.73	80.51	**82.35**
Russia	74.86	**75.70**	75.05
Slovakia	79.15	83.16	**83.21**
Ukraine	75.19	75.90	**76.83**

e_0 is life expectancy at birth of 2010 from the HMD.

For both of the methods, I checked forecast accuracy during out-of-sample evaluation and from three possible combinations of best reference group I considered the one having lower number of populations to make coherent forecast. $\mathrm{LL}_{e_0^\dagger}$ produced the most optimistic forecast in 2050 for three Baltic countries, Slovakia and the Ukraine. However, it failed to produce optimistic forecast for Belarus. The methodology of Hyndman and Booth (2008) is followed to construct the prediction interval as before. The prediction interval obtained from $\mathrm{LL}_{e_0^\dagger}$ found to be wider than that of LL in case of Hungarian Females. The interval forecast of life expectancy at birth for the coherent forecasts are illustrated below in Figure 6.11 for Hungarian Females.

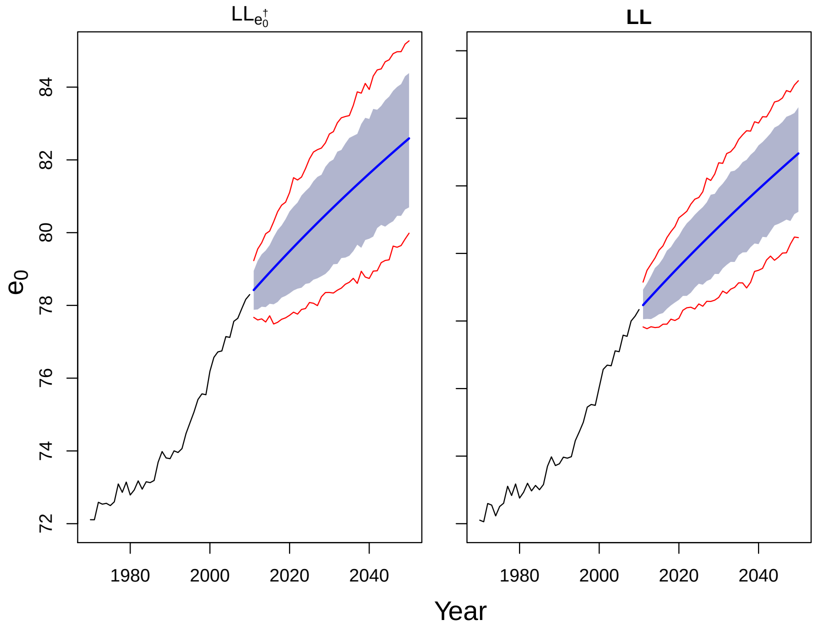


FIGURE 6.11: Prediction interval of Hungarian female e_0 by coherent forecast till 2050. The blue area represents 80% prediction interval and red lines are for 95% prediction interval.

Chapter 7

Conclusions

7.1 Discussion

The Lee-Carter method and its later variants are widely accepted probabilistic approaches for mortality and life expectancy forecast in many industrial countries. In this line of research on Lee-Carter framework, I introduced two new mortality forecasting techniques in this thesis: one for single population and another one for coherent forecasting. Application of Lasso type smoothing prior to fitting the model overcome the problem of a jagged trend of age-component over the lifespan. I incorporate lifespan disparity during parameter estimation of the existing Lee-Carter model and to our knowledge, the present study is the first attempt to consider lifespan disparity in ground of mortality forecasting. Both of these modifications increase the precision of the forecasts. Moreover the modifications are further applied for coherent forecasting, along with introducing a scheme for choosing the reference population is provided. Choosing the appropriate reference group is an old puzzle for coherent forecasting and different reference populations bring about quite different results. I addressed this problem by proposing a robust definition of reference population based on closest trend of lifespan disparity. This definition is found to be applicable for existing coherent forecasting techniques as well. Last but not least, consideration of best fitting period also found to be an adequate addition for conducting coherent forecast; both in terms of forecast accuracy during out-of-sample evaluation and more optimistic forecast.

Although the results are promising for both of the methods, there are still many open questions that deserve further investigation. I outline some of them, which are closely related to the work presented in the thesis.

1. Interval forecast of life expectancy at birth is narrow for several of the populations,

this narrow prediction interval is more notable in case of $\mathrm{LC}_{e_0^\dagger}$ than $\mathrm{LL}_{e_0^\dagger}$. This is an old criticism regarding Lee-Carter variants (including Li and Lee, 2005). In the proposed model it happened due to application of smoothing and new adjustment technique which made the time component more linear. As a consequence it reduced the variance of the ARIMA model. In addition, variance of the model is lower in the proposed method, which also affects the interval forecast.

2. I considered only rank-1 approximation for model fitting. This proposed modifications generated almost similar level of explained variation as of higher order approximation. Therefore, I did not consider higher order term to avoid computational complexity. It should be noted that, the higher order terms are also not free from invariance problem and thus those model also contain constraints for parameter estimation (see Hyndman and Ullah, 2007, for example).

3. In the proposed definition of reference population, I considered both genders together. Although several approaches considered males and females together for coherent forecasting, still, consideration of males and females together in reference group is a topic of debate due to different pattern of mortality over the lifespan. There are two responses regarding this issue. First, during sorting out the closest populations, males and females were separated naturally from the value of $\bar{e}_0^\dagger$. Getting best reference population from opposite gender did not happen very often except for few populations. Second, most of these countries consider same policy for both genders regarding old health care system, as aging is common issue for both genders. Thus, practical implementation is not a big problem.

4. For coherent forecasting, I applied equal weight on mortality rates of all populations to construct joint mortality matrix (both for $\mathrm{LL}_{e_0^\dagger}$ and LL in chapter 5 and 6). This step solves the problem of mixing population with large exposure with smaller one (for example, Figure 3.6). However, consideration of equal weight has a consequence. Different mortality patterns from different populations are result of both the exposure size and distinct distribution of causes-of-deaths. Application of same weight underestimate these patterns in joint mortality matrix. Defining an appropriate population-specific weight to adjust the problem of different distribution of causes-of-deaths will be complicated because different data sources will be needed to obtain harmonized data for causes-of-death. Although the low-mortality countries are converging in terms of aging, still, each of these populations are distinct in terms of distribution of deaths (see Figure 5.1 and Figure 6.9 for example).

5. Consideration of life tables with longer lifespan are preferable in coherent settings. As most of the populations do not have data after age 100 for several years, I replaced/extrapolated the mortality rates of age 100:110+ using Kannisto model. In this way, the missing mortality rates might be imputed (keeping the fitted rates closest to the observed data), but on the other hand it reduces the variability for centenarian mortality (see Figure 5.1 for an example).

6. Clearly, smoothing can significantly improve the forecast accuracy. For coherent forecasting, I smoothed each of the populations separately using Lasso and then combined them in common factor model. Instead of smoothing prior to combine the mortality rates, applying Lasso after combining populations may produce different results. However, smoothing after combining populations will create computational complexity because in this way smoothing will be needed every time a new population is added in the reference group.

7. One important auxiliary outcome of this thesis is to verify the performances of the existing Lee-Carter variants on high-mortality regime. I applied the proposed methods on these countries also and the performance of the proposed methods were promising for several of these countries. Nevertheless, most of the models (including the proposed methods) failed to capture mortality pattern for some particular populations. Irregular mortality patterns followed by a severe mortality crisis in later part of fitting period are responsible for pessimistic forecasts of life expectancies. Unlike the low-mortality countries, the performance of $\text{LC}_{e_0^\dagger}$ was not very optimistic in case of CEE countries. It happened because $e_0^\dagger$ explains the mortality trend of a population better than other contemporary longevity indicators and thus it suffered from the severe mortality crisis of several of these CEE countries. Therefore, the newly adjusted time component contains the effect of irregular mortality trend and produces pessimistic forecasts of life expectancies.

In addition to all of these issues, I did not compare the forecast-accuracy or forecast by bringing two classes of models together (single and coherent). I am leaving this issue for personal subjective opinion of getting the forecast independently for a particular population or from a coherent point of view. It should be noted that I considered few types of time series models for forecasting and finally considered only RWD in both of the models. Hyndman and Ullah (2007) also tried different time series model for forecasting and obtained the best results using RWD.

7.2 Future directions of research

Based on the results and limitations of the current research, I might point out some future scope of research on mortality forecasting. The first issue will be to overcome to problem of invariant b_x in Lee-carter framework. One possible solution to do that is to adopt Bayesian approaches on parameter estimation. Secondly, adaptation of cohort effect might get more insight of the mortality scenario of a population. Thirdly, I introduced a new systematic approach to obtain best reference group for a population. Although $e_0^\dagger$ better reflects the distribution of death for a population, further research on this field may produce better results than proposed method. Fourthly, although Lasso is found to be more effective smoothing technique for our data, it is a slightly time consuming method. Faster algorithm for getting optimal results for Lasso will be helpful. Fifthly, all these techniques are based on series life tables. In new age of big data revolution, there will be plenty more opportunity than current life table based mortality forecasting. Some countries already started register based database, so it will be a complete new ground of research for mortality forecasting.

There are two more possible methodological issues which are closely related to mortality forecasting. For sake of coherent forecasting, it is wise to consider life table of longer time series and longer life span. New method on the ground of Mathematical Demography for defining the life span will be helpful for future research on mortality forecasting. Another innovation in ground of Formal Demography will be helpful for any research using life expectancy at birth. Despite of being widely accepted measure of longevity, life expectancy at birth is not free from age-specific bias. Revision of its definition may change the current limitation.

As final statement of this thesis, like many other approaches on mortality forecast, the proposed methods are also in same framework of Lee and Carter (1992). Further researches may consider a fresh start leaving this circle behind.

Appendix A

Old Age Mortality

Kannisto model

Mortality rates are not available for some of the later age-groups for the whole fitting period. Considering shifting mortality of almost all the populations, I fitted Kannisto's model at later age group for coherent forecasting. For ages $x = 80, 81, \ldots 110+$, let the observed death counts are noted as D_x and exposure as E_x. Mortality rates for later age groups are then extrapolated by fitting the Kannisto's model (Thatcher *et al.*, 1998) of old age mortality on observed death rates M_x to estimate the underlying hazards function μ_x as,

$$\mu_{x,(a,b)} = \frac{ae^{b(x-80)}}{1 + ae^{b(x-80)}}; \qquad a, b \geq 0.$$

Sensitivity of $e_0^\dagger$ due to Kannisto fitted mortality rates

I tried different combinations of fitting period and then added different combination of smoothed data with observed data. In this analysis I used the data obtained from fitting period at age 80:100 and adding the smoothed data of age 100:100+ with observed data till age 99. Among various combinations I tried, this combination was the closest to real data and the difference of estimated and observed $e_0^\dagger$ during the fitting period (1956-2011) were lowest for this combination.

Appendix B

Variance Explained by the Fitted Models

TABLE B.1: Variance explained by the fitted single population forecasting models for the low mortality countries.

Country	LC	LC_P	LM	BMS	HU	HU_R	HU_W	$\text{LC}_{e_0^\dagger}$
Australia	0.890	0.890	0.890	0.795	0.986	0.880	0.994	0.956
Austria	0.854	0.854	0.854	0.542	0.983	0.836	0.993	0.975
Belgium	0.831	0.831	0.831	0.549	0.983	0.813	0.992	0.922
Canada	0.901	0.901	0.901	0.888	0.988	0.889	0.994	0.980
Denmark	0.681	0.681	0.681	0.644	0.971	0.640	0.989	0.942
Finland	0.774	0.774	0.774	0.355	0.977	0.748	0.990	0.958
France	0.939	0.939	0.939	0.938	0.992	0.956	0.996	0.960
Germany	0.957	0.957	0.957	0.934	0.983	0.952	0.997	0.972
Ireland	0.735	0.735	0.735	0.560	0.984	0.703	0.993	0.909
Italy	0.955	0.955	0.955	0.925	0.994	0.957	0.996	0.977
Japan	0.955	0.955	0.955	0.955	0.997	0.976	0.996	0.965
Netherlands	0.870	0.870	0.870	0.870	0.983	0.874	0.983	0.929
New Zealand	0.639	0.639	0.639	0.447	0.959	0.548	0.959	0.942
Norway	0.654	0.654	0.654	0.349	0.943	0.600	0.943	0.906
Portugal	0.905	0.905	0.905	0.852	0.992	0.906	0.992	0.946
Spain	0.915	0.915	0.915	0.910	0.991	0.886	0.996	0.945
Sweden	0.774	0.774	0.774	0.774	0.977	0.712	0.990	0.977
Switzerland	0.791	0.791	0.791	0.613	0.985	0.778	0.994	0.942
United Kingdom	0.920	0.920	0.920	0.888	0.987	0.915	0.995	0.965
USA	0.916	0.916	0.916	0.863	0.985	0.862	0.993	0.941

Note: Country-specific available years and age-groups are considered for model fitting.

Table B.2: Variance explained by the fitted single population forecasting models for the high mortality CEE countries.

Country	LC	LC_P	LM	BMS	HU	HU_R	HU_W	$LC_{e_0^\dagger}$
Belarus	0.465	0.465	0.465	0.423	0.904	0.915	0.944	0.592
Bulgaria	0.712	0.712	0.712	0.300	0.970	0.783	0.982	0.861
Estonia	0.312	0.312	0.312	0.349	0.899	0.814	0.955	0.862
Hungary	0.703	0.703	0.703	0.679	0.975	0.693	0.989	0.859
Latvia	0.420	0.420	0.420	0.419	0.877	0.405	0.934	0.659
Lithuania	0.483	0.483	0.483	0.414	0.898	0.429	0.952	0.877
Russia	0.529	0.529	0.529	0.506	0.964	0.773	0.978	0.513
Slovakia	0.567	0.567	0.567	0.413	0.964	0.478	0.984	0.959
Ukraine	0.583	0.583	0.583	0.518	0.965	0.750	0.980	0.647

Note: Country-specific available years and age-groups are considered for model fitting.

Table B.3: Variance explained by the fitted coherent forecasting models for the low mortality countries (1956:2011).

Country	LL	$LL_{e_0^\dagger}$	Country	LL	$LL_{e_0^\dagger}$
Australia	0.888	0.886	Japan	0.964	0.961
Austria	0.820	0.810	The Netherlands	0.832	0.834
Belgium	0.806	0.801	New Zealand	0.607	0.600
Canada	0.912	0.917	Norway	0.594	0.583
Denmark	0.594	0.570	Portugal	0.906	0.902
Finland	0.722	0.706	Spain	0.945	0.961
France	0.940	0.957	Sweden	0.723	0.713
Germany	0.949	0.978	Switzerland	0.749	0.742
Ireland	0.688	0.678	United Kingdom	0.925	0.926
Italy	0.960	0.952	USA	0.931	0.931

Table B.4: Variance explained by the fitted coherent forecasting models for the high mortality CEE countries (1970:2010).

Country	LL	$LL_{e_0^\dagger}$	Country	LL	$LL_{e_0^\dagger}$	Country	LL	$LL_{e_0^\dagger}$
Belarus	0.511	0.485	Hungary	0.648	0.827	Russia	0.341	0.824
Bulgaria	0.406	0.687	Latvia	0.357	0.620	Slovakia	0.450	0.819
Estonia	0.779	0.745	Lithuania	0.429	0.663	Ukraine	0.637	0.705

Appendix C

Prediction Interval for Forecast of Life Expectancy at Birth

The interval forecast of the female e_0 till 2050 are plotted in this section for low-mortality countries using the proposed methods.

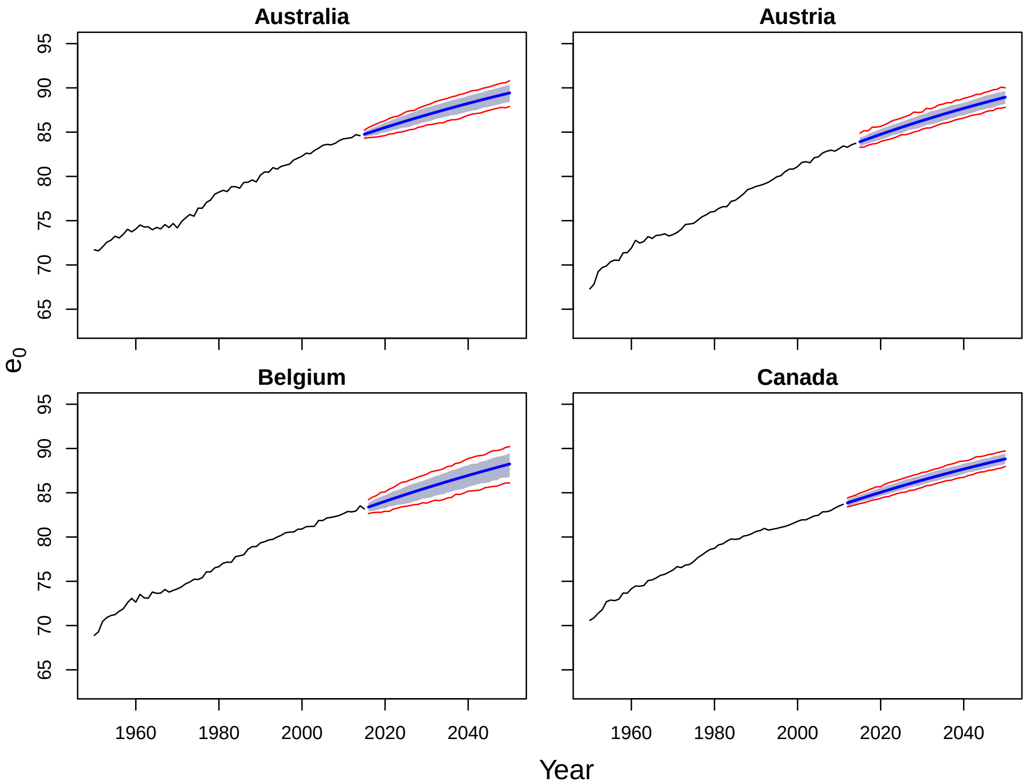


FIGURE C.1: Prediction interval of female e_0 for Australia, Austria, Belgium and Canada till 2050 using $\mathrm{LC}_{e_0^\dagger}$. The blue are represents 80% prediction interval and red lines are for 95% prediction interval.

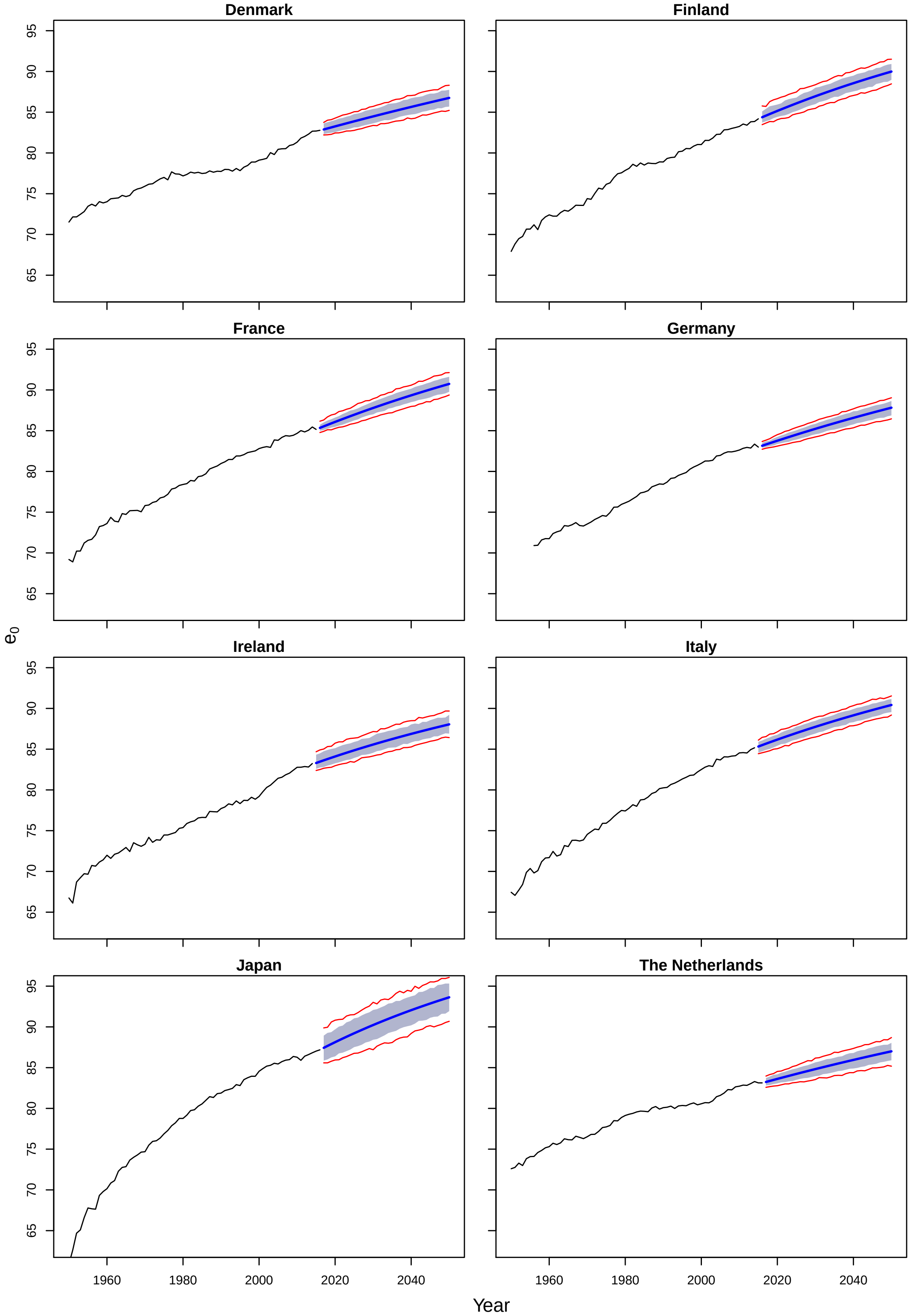


FIGURE C.2: Prediction interval of female e_0 for Denmark, Finland, France, Germany, Ireland, Italy, Japan and the Netherlands till 2050 using $LC_{e_0^\dagger}$. The blue are represents 80% prediction interval and red lines are for 95% prediction interval.

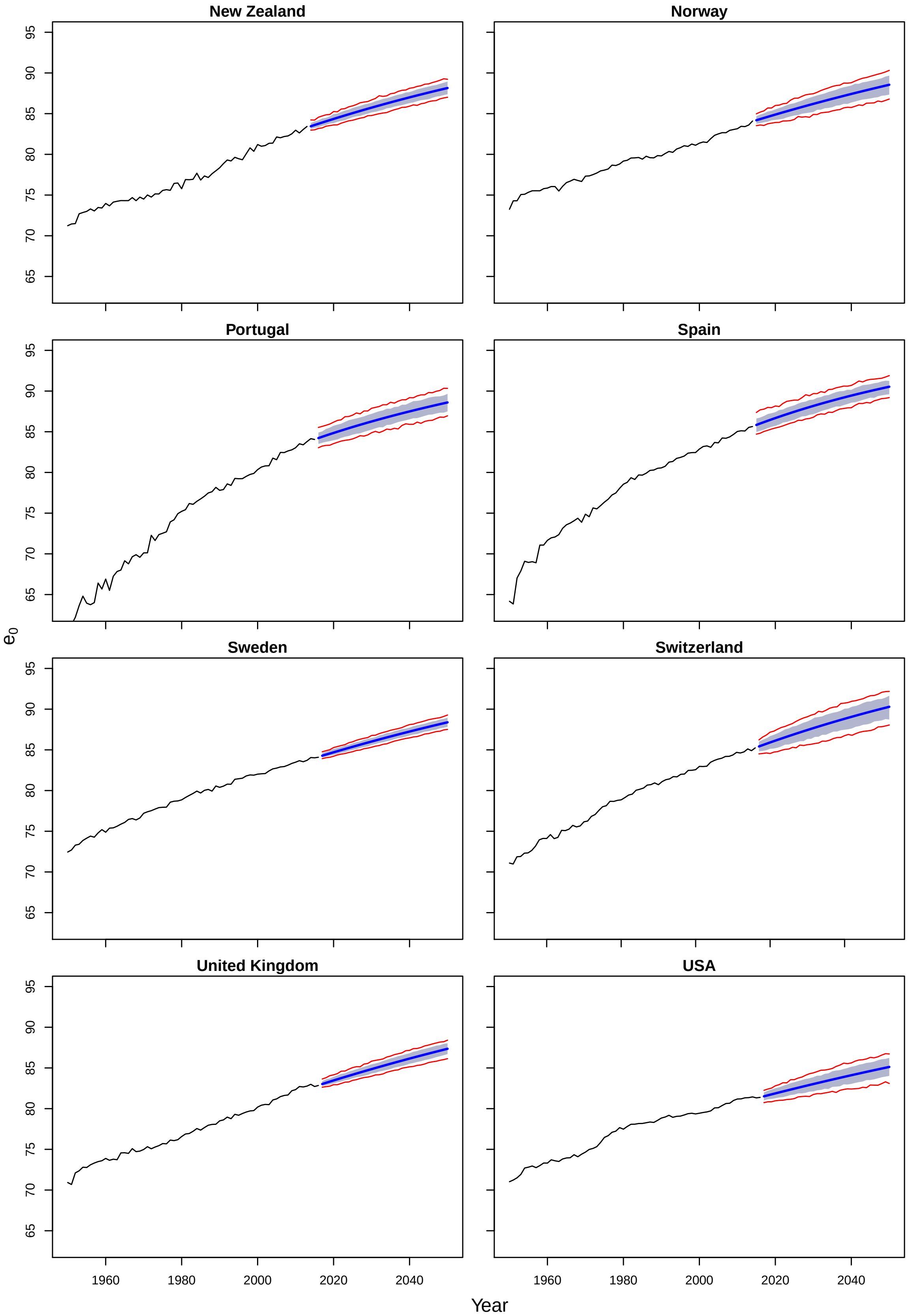


FIGURE C.3: Prediction interval of female e_0 for New Zealand, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom and USA till 2050 using LC$_{e_0^\dagger}$. The blue are represents 80% prediction interval and red lines are for 95% prediction interval.

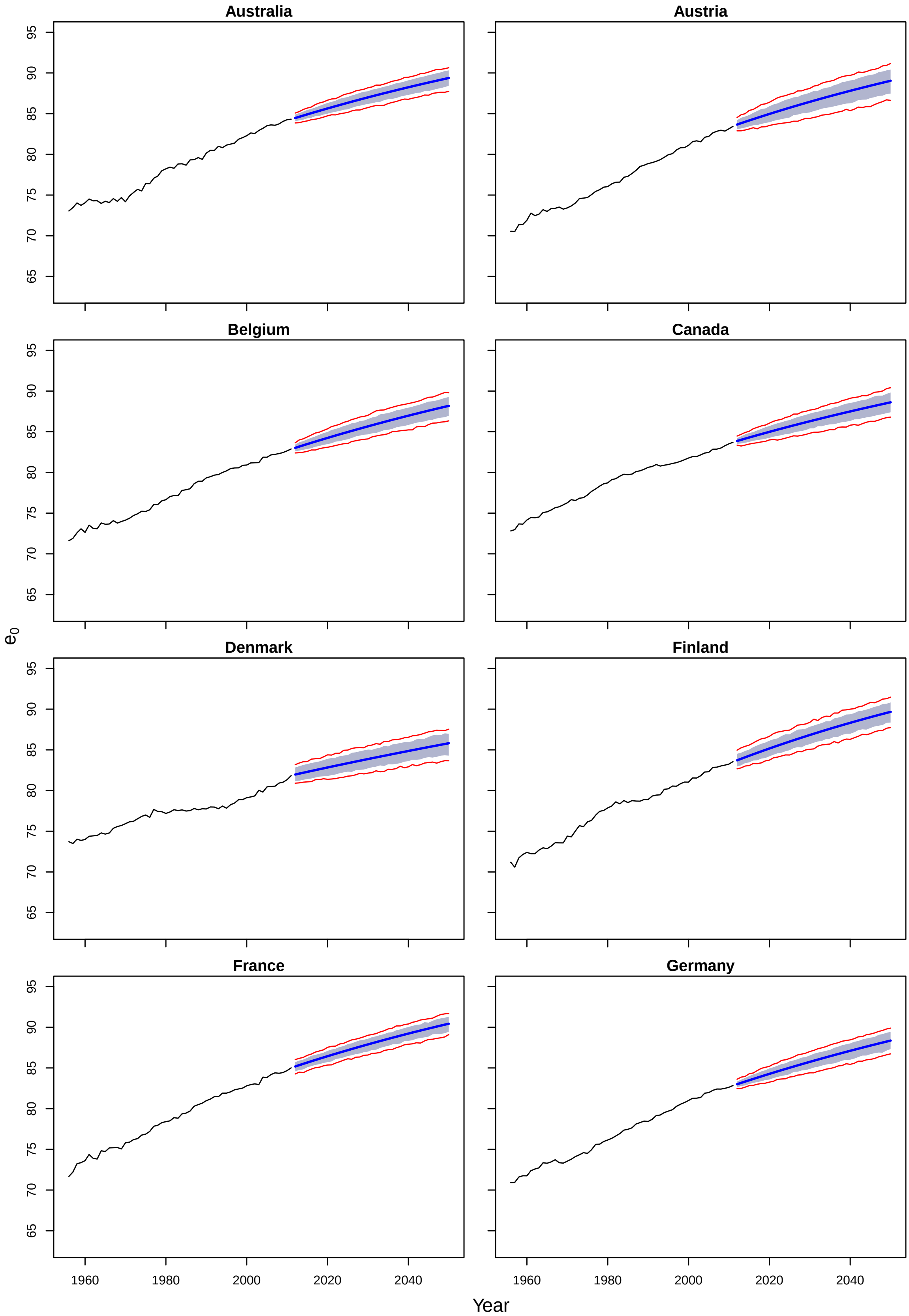


FIGURE C.4: Prediction interval of female e_0 for Australia, Austria, Belgium, Canada, Denmark, Finland, France and Germany till 2050 using $\mathrm{LL}_{e_0^\dagger}$. The blue are represents 80% prediction interval and red lines are for 95% prediction interval.

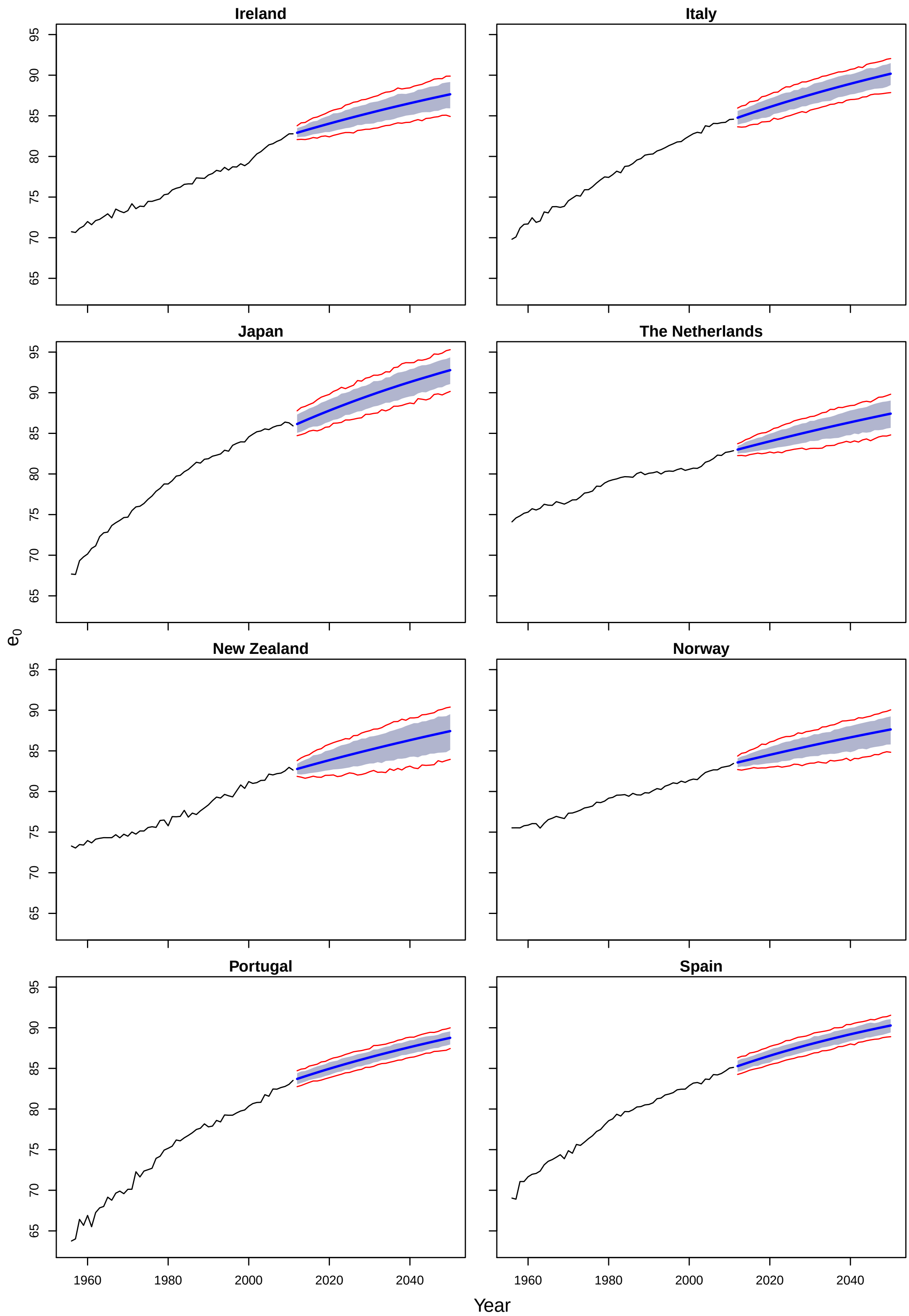


FIGURE C.5: Prediction interval of female e_0 for Ireland, Italy, Japan, the Netherlands, New Zealand, Norway, Portugal and Spain till 2050 using $LL_{e_0^\dagger}$. The blue are represents 80% prediction interval and red lines are for 95% prediction interval.

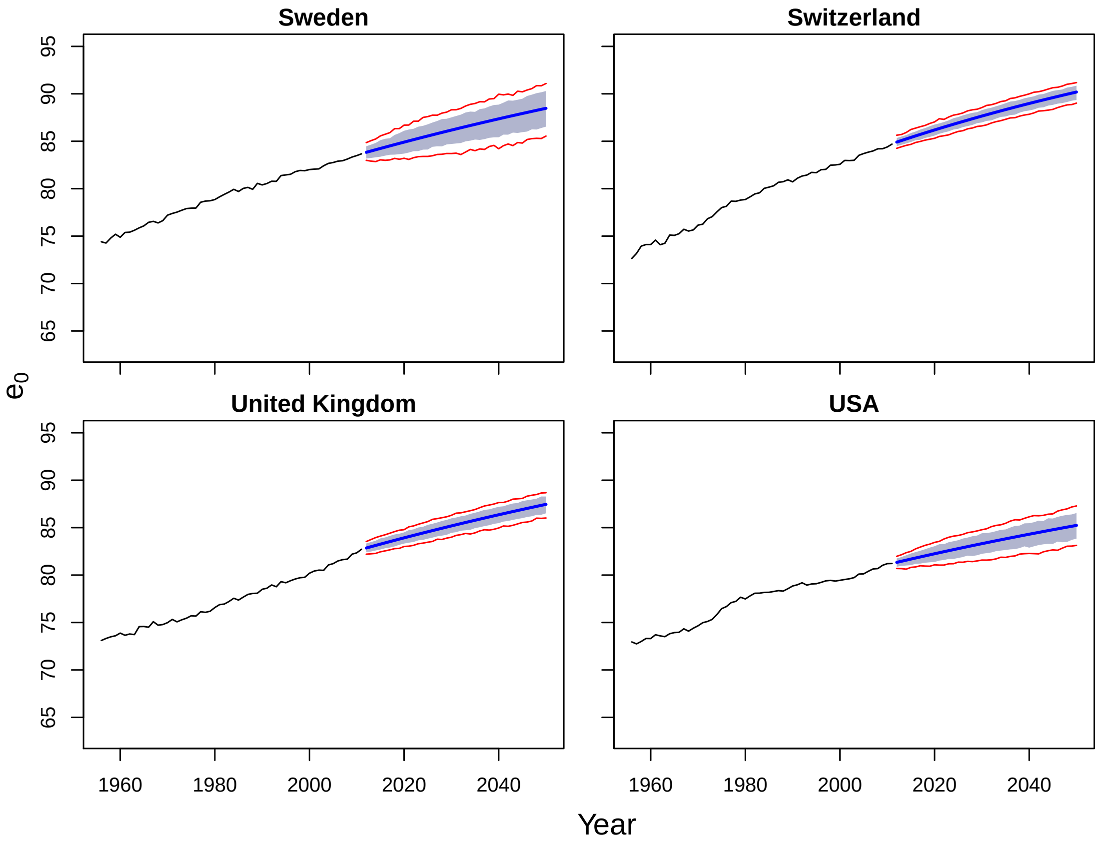


FIGURE C.6: Prediction interval of female e_0 for Sweden, Switzerland, United Kingdom and USA till 2050 using $\mathrm{LL}_{e_0^\dagger}$. The blue are represents 80% prediction interval and red lines are for 95% prediction interval.

Appendix D

Best Reference Group for Coherent Forecasting

The best reference groups obtained from LL and $\mathrm{LL}_{e_0^\dagger}$ according to different measures of forecast accuracy during out-of-sample evaluations are summarized in this section. The combinations without mentioning best fitting period utilized the full available time frame in case of $\mathrm{LL}_{e_0^\dagger}$. German male forecast accuracies were increasing indefinitely, so it is omitted for LL. For CEE countries, models failed to produce optimistic forecasts from any combination of reference group are omitted in this section.

Table D.1: Best reference group for males of low-mortality countries according to lowest MAE in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}^\dagger_{0i} - \bar{e}^\dagger_{0j} \right\rvert$
Australia	$\mathrm{DEU_M}$	0.06
Austria	$\mathrm{ESP_M}$	0.04
Belgium	$\mathrm{USA_F}, \mathrm{DNK_M}, \mathrm{CAN_M}, \mathrm{DEU_M}, \mathrm{AUT_M}, \mathrm{AUS_M}$	0.15
Canada	10 populations	0.30
Denmark	$\mathrm{CAN_M}$	0.025
Finland	$\mathrm{NZL_M}, \mathrm{ESP_M}, \mathrm{AUT_M}$	0.204
France	$\mathrm{PRT_M}, \mathrm{FIN_M}, \mathrm{NZL_M}, \mathrm{ESP_M}, \mathrm{AUT_M}$	0.56
Germany	-	-
Ireland	$\mathrm{NOR_M}$	0.00048
Italy	$\mathrm{UK_M}, \mathrm{CHE_M}, \mathrm{AUS_M}$	0.1
Japan	$\mathrm{NOR_M}$	0.03
Netherlands	$\mathrm{NZL_M}, \mathrm{DNK_F}, \mathrm{SWE_M}$	0.12
New Zealand	$\mathrm{ESP_M}, \mathrm{AUT_M}$	0.049
Norway	$\mathrm{IRL_M}$	0.0004
Portugal	$\mathrm{FRA_M}, \mathrm{USA_M}, \mathrm{FIN_M}, \mathrm{NZL_M}, \mathrm{ESP_M}, \mathrm{AUT_M}$	0.858
Spain	$\mathrm{NZL_M}, \mathrm{AUT_M}, \mathrm{CAN_M}$	0.13
Sweden	$\mathrm{CAN_M}$	0.03
Switzerland	$\mathrm{UK_M}, \mathrm{ITA_M}, \mathrm{IRL_M}, \mathrm{NOR_M}, \mathrm{JPN_M}, \mathrm{AUS_M}$	0.184
UK	$\mathrm{CHE_M}, \mathrm{ITA_M}, \mathrm{AUS_M}, \mathrm{IRL_M}, \mathrm{NOR_M}, \mathrm{JPN_M}, \mathrm{DEU_M}, \mathrm{BEL_M}$	0.335
USA	$\mathrm{PRT_M}$	0.31

Table D.2: Best reference group for males of low-mortality countries according to lowest MSE in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}^{\dagger}_{0i} - \bar{e}^{\dagger}_{0j} \right\rvert$
Australia	DEU_M	0.06
Austria	ESP_M	0.04
Belgium	$USA_F, DNK_M, CAN_M, DEU_M, AUT_M, AUS_M$	0.15
Canada	$DNK_M, USA_F, BEL_M, AUT_M, ESP_M, DEU_M, NZL_F, AUS_M, FIN_M$	0.29
Denmark	CAN_M	0.029
Finland	$NZL_M, ESP_M, AUT_M, CAN_M$	0.29
France	$PRT_M, FIN_M, NZL_M, ESP_M, AUT_M$	0.56
Germany	-	-
Ireland	$NOR_M, JPN_M, CHE_M, UK_M$	0.15
Italy	$UK_M, CHE_M, AUS_M, DEU_M, IRL_M$	0.20
Japan	$NOR_M, IRL_M, CHE_M, UK_M, DNK_M$	0.206
Netherlands	$NZL_M, DNK_F, SWE_M, CAN_F$	0.15
New Zealand	$ESP_M, AUT_M, CAN_M, FIN_M, DNK_M, USA_F, BEL_M$	0.199
Norway	IRL_M	0.0004
Portugal	$FRA_M, USA_M, FIN_M, NZL_M, ESP_M, AUT_M, CAN_M$	0.95
Spain	13 populations	0.52
Sweden	CAN_M	0.03
Switzerland	$UK_M, ITA_M, IRL_M, NOR_M, JPN_M, AUS_M, DEU_M$	0.25
UK	$CHE_M, ITA_M, AUS_M, IRL_M, NOR_M, JPN_M, DEU_M$	0.304
USA	PRT_M	0.31

Table D.3: Best reference group for males of low-mortality countries according to lowest ME(e_0) in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}^{\dagger}_{0i} - \bar{e}^{\dagger}_{0j} \right\rvert$
Australia	DEU_M, ITA_M	0.10
Austria	ESP_M	0.04
Belgium	$USA_F, DNK_M, CAN_M, DEU_M, AUT_M, AUS_M, ESP_M$	0.19
Canada	10 populations	0.30
Denmark	CAN_M, USA_F	0.029
Finland	NZL_M, ESP_M, AUT_M	0.204
France	$PRT_M, FIN_M, NZL_M, ESP_M, AUT_M$	0.56
Germany	-	-
Ireland	$NOR_M, JPN_M, CHE_M, UK_M$	0.15
Italy	UK_M, CHE_M, AUS_M	0.1
Japan	NOR_M	0.03
Netherlands	NZL_M, DNK_F, SWE_M	0.12
New Zealand	ESP_M	0.008
Norway	IRL_M	0.004
Portugal	FRA_M	0.29
Spain	NZL_M, AUT_M	0.04
Sweden	CAN_M, UK_M, NLD_M	0.127
Switzerland	$UK_M, ITA_M, IRL_M, NOR_M, JPN_M, AUS_M$	0.184
UK	CHE_M	0.03
USA	PRT_M, FRA_M	0.61

TABLE D.4: Best reference group for females of low-mortality countries according to lowest MAE in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}^{\dagger}_{0i} - \bar{e}^{\dagger}_{0j} \right\rvert$
Australia	$\mathrm{BEL_F, IRL_F, FRA_F}$	0.088
Austria	13 populations	0.39
Belgium	$\mathrm{AUS_F, IRL_F, NLD_F}$	0.12
Canada	$\mathrm{SWE_M, UK_F, NLD_F}$	0.15
Denmark	$\mathrm{NZL_F, NLD_M}$	0.06
Finland	$\mathrm{ITA_F, SWE_F, JPN_F, CHE_F, AUT_F, ESP_F, NOR_F, DEU_F, NLD_F, FRA_F}$	0.36
France	13 populations	0.44
Germany	-	-
Ireland	$\mathrm{AUS_F, BEL_F}$	0.03
Italy	$\mathrm{FIN_F, AUT_F, SWE_F, NOR_F, JPN_F, CHE_F, ESP_F}$	0.15
Japan	$\mathrm{CHE_F, SWE_F}$	0.01
Netherlands	$\mathrm{FRA_F}$	0.05
New Zealand	$\mathrm{DNK_F, NLD_M, SWE_M}$	0.06
Norway	$\mathrm{AUT_F}$	0.048
Portugal	$\mathrm{IRL_F, AUS_F}$	0.151
Spain	$\mathrm{CHE_F, JPN_F, SWE_F}$	0.08
Sweden	$\mathrm{JPN_F, FIN_F, CHE_F}$	0.046
Switzerland	$\mathrm{JPN_F, ESP_F, SWE_F, FIN_F, ITA_F, AUT_F, NOR_F, DEU_F, NLD_F}$	0.403
UK	$\mathrm{CAN_F, SWE_M, PRT_F, NLD_M, IRL_F}$	0.297
USA	$\mathrm{BEL_M}$	0.002

TABLE D.5: Best reference group for females of low-mortality countries according to lowest MSE in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}^{\dagger}_{0i} - \bar{e}^{\dagger}_{0j} \right\rvert$
Australia	$\mathrm{BEL_F, IRL_F, FRA_F}$	0.088
Austria	13 populations	0.39
Belgium	$\mathrm{AUS_F, IRL_F, NLD_F}$	0.12
Canada	$\mathrm{SWE_M, UK_F, NLD_F, NZL_F, DNK_F}$	0.22
Denmark	$\mathrm{NZL_F, NLD_M}$	0.06
Finland	13 populations	0.44
France	13 populations	0.44
Germany	11 populations	0.31
Ireland	$\mathrm{AUS_F, BEL_F}$	0.03
Italy	$\mathrm{FIN_F, AUT_F, SWE_F, NOR_F, JPN_F}$	0.11
Japan	$\mathrm{CHE_F, SWE_F}$	0.01
Netherlands	$\mathrm{FRA_F}$	0.05
New Zealand	$\mathrm{DNK_F}$	0.003
Norway	$\mathrm{AUT_F, DEU_F}$	0.09
Portugal	$\mathrm{IRL_F, AUS_F}$	0.151
Spain	$\mathrm{CHE_F, JPN_F, SWE_F}$	0.08
Sweden	$\mathrm{JPN_F, FIN_F, CHE_F, ITA_F}$	0.07
Switzerland	$\mathrm{JPN_F, ESP_F, SWE_F, FIN_F}$	0.088
UK	$\mathrm{CAN_F, SWE_M, PRT_F, NLD_M}$	0.248
USA	$\mathrm{BEL_M}$	0.002

TABLE D.6: Best reference group for females of low-mortality countries according to lowest ME(e_0) in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}^{\dagger}_{0i} - \bar{e}^{\dagger}_{0j} \right\rvert$
Australia	$\mathrm{BEL_F, IRL_F, FRA_F, NLD_F, PRT_F, DEU_F}$	0.22
Austria	$\mathrm{NOR_F, ITA_F, FIN_F, SWE_F, DEU_F, JPN_F, CHE_F}$	0.17
Belgium	$\mathrm{AUS_F, IRL_F, NLD_F, PRT_F}$	0.16
Canada	$\mathrm{SWE_M}$	0.03
Denmark	$\mathrm{NZL_F, NLD_M}$	0.06
Finland	$\mathrm{ITA_F, SWE_F, JPN_F, CHE_F}$	0.04
France	$\mathrm{NLD_F, BEL_F, AUS_F, IRL_F, DEU_F, NOR_F, PRT_F, AUT_F, ITA_F}$	0.33
Germany	$\mathrm{NLD_F}$	0.079
Ireland	$\mathrm{AUS_F}$	0.02
Italy	$\mathrm{FIN_F}$	0.03
Japan	$\mathrm{CHE_F, SWE_F}$	0.01
Netherlands	$\mathrm{FRA_F}$	0.05
New Zealand	$\mathrm{DNK_F, NLD_F, SWE_M, JPN_M, CAN_F, NOR_M, IRL_M}$	0.241
Norway	$\mathrm{AUT_F}$	0.048
Portugal	$\mathrm{IRL_F, AUS_F, BEL_F, UK_F, FRA_F}$	0.24
Spain	$\mathrm{CHE_F, JPN_F, SWE_F, FIN_F, ITA_F, AUT_F, NOR_F, DEU_F, NLD_F}$	0.43
Sweden	$\mathrm{JPN_F, FIN_F}$	0.041
Switzerland	$\mathrm{JPN_F, ESP_F, SWE_F, FIN_F, ITA_F, AUT_F, NOR_F, DEU_F, NLD_F}$	0.403
UK	$\mathrm{CAN_F, SWE_M}$	0.12
USA	$\mathrm{BEL_M, DNK_M, CAN_F, DEU_M}$	0.084

TABLE D.7: Best reference group for males of low-mortality countries according to lowest MAE in $\mathrm{LL}_{e_0^{\dagger}}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\lvert \bar{e}^{\dagger}_{0i} - \bar{e}^{\dagger}_{0j} \right\rvert$
Australia	$\mathrm{DEU_M}$	0.06
Austria	$\mathrm{ESP_M}$	0.04
Belgium	9 populations	0.199
Canada	10 populations	0.30
Denmark	$\mathrm{CAN_M, USA_F, BEL_M}$	0.03
Finland	$\mathrm{NZL_M, ESP_M, AUT_M}$	0.204
France	$\mathrm{PRT_M}$	0.29
Germany	12 populations	0.376
Ireland	$\mathrm{NOR_M, JPN_M, CHE_M, UK_M}$	0.15
Italy	$\mathrm{UK_M, CHE_M, AUS_M}$	0.1
Japan	$\mathrm{NOR_M}$	0.03
Netherlands	$\mathrm{NZL_M, DNK_F, SWE_M, CAN_F}$ (1974:)	0.15
New Zealand	$\mathrm{ESP_M}$	0.008
Norway	$\mathrm{IRL_M}$	0.0004
Portugal	9 populations (1965:)	1.00
Spain	$\mathrm{NZL_M, AUT_M, CAN_M, CAN_M}$	0.13
Sweden	$\mathrm{CAN_M}$	0.03
Switzerland	$\mathrm{UK_M, ITA_M}$	0.08
UK	10 populations	0.33
USA	$\mathrm{PRT_M, FRA_M, FIN_M, NZL_M, ESP_M, AUT_M}$ (1965:)	1.17

Note: Combinations without mentioning the best fitting period have best fit for 1956 to 2001.

Table D.8: Best reference group for males of low-mortality countries according to lowest MSE in $LL_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right\|$
Australia	DEU_M	0.06
Austria	ESP_M	0.04
Belgium	$USA_F, DNK_M, CAN_M, DEU_M, AUT_M, AUS_M, ESP_M$	0.19
Canada	$DNK_M, USA_F, BEL_M, AUT_M, ESP_M, DEU_M, NZL_F, AUS_M, FIN_M$	0.29
Denmark	$CAN_M, USA_F, BEL_M, DEU_M$	0.11
Finland	$NZL_M, ESP_M, AUT_M, CAN_M$	0.29
France	PRT_M	0.29
Germany	13 populations	0.377
Ireland	$NOR_M, JPN_M, CHE_M, UK_M$	0.15
Italy	$UK_M, CHE_M, AUS_M, DEU_M$	0.16
Japan	NOR_M	0.03
Netherlands	$NZL_M, DNK_F, SWE_M, CAN_F$ (1974:)	0.15
New Zealand	ESP_M	0.008
Norway	$IRL_M, JPN_M, CHE_M, UK_M$	0.15
Portugal	9 populations (1965:)	1.00
Spain	13 populations	0.52
Sweden	CAN_M	0.03
Switzerland	UK_M, ITA_M	0.08
UK	10 populations	0.33
USA	$PRT_M, FRA_M, FIN_M, NZL_M, ESP_M, AUT_M, DEU_M$ (1965:)	1.40

Note: Combinations without mentioning the best fitting period have best fit for 1956 to 2001.

Table D.9: Best reference group for males of low-mortality countries according to lowest $ME(e_0)$ in $LL_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right\|$
Australia	DEU_M, ITA_M	0.10
Austria	ESP_M	0.04
Belgium	9 populations	0.25
Canada	10 populations	0.30
Denmark	CAN_M, USA_F, BEL_M	0.03
Finland	$NZL_M, ESP_M, AUT_M, DNK_M, USA_F$ (1964:)	0.35
France	PRT_M	0.29
Germany	12 populations	0.376
Ireland	$NOR_M, JPN_M, CHE_M, UK_M$	0.15
Italy	UK_M, CHE_M, AUS_M	0.1
Japan	NOR_M	0.03
Netherlands	$NZL_M, DNK_F, SWE_M, CAN_F$ (1974:)	0.15
New Zealand	ESP_M	0.008
Norway	IRL_M	0.004
Portugal	$FRA_M, USA_M, FIN_M, NZL_M$ (1965:)	1.00
Spain	NZL_M, AUT_M	0.04
Sweden	CAN_M	0.03
Switzerland	UK_M, ITA_M	0.08
UK	10 populations	0.33
USA	$PRT_M, FRA_M, FIN_M, NZL_M$ (1965:)	1.12

Note: Combinations without mentioning the best fitting period have best fit for 1956 to 2001.

TABLE D.10: Best reference group for females of low-mortality countries according to lowest MAE in $LL_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right\|$
Australia	BEL_F, IRL_F, FRA_F	0.088
Austria	$NOR_F, ITA_F, FIN_F, SWE_F, DEU_F, JPN_F, CHE_F, ESP_F, NLD_F$ (1957:)	0.22
Belgium	$AUS_F, IRL_F, NLD_F, PRT_F, DEU_F, NOR_F$ (1974:)	0.30
Canada	$SWE_M, UK_F, NLD_F, NZL_F$	0.21
Denmark	NZL_F, NLD_M, SWE_M	0.19
Finland	$ITA_F, SWE_F, JPN_F, CHE_F$	0.08
France	$NLD_F, BEL_F, AUS_F, IRL_F, DEU_F, NOR_F, PRT_F, AUT_F$ (1974:)	0.33
Germany	$NLD_F, NOR_F, FRA_F, AUT_F, ITA_F$	0.20
Ireland	AUS_F, BEL_F	0.03
Italy	$FIN_F, AUT_F, SWE_F, NOR_F, JPN_F, CHE_F, ESP_F, DEU_F$ (1957:)	0.20
Japan	$CHE_F, SWE_F, ESP_F, FIN_F$	0.07
Netherlands	$FRA_F, DEU_F, BEL_F, AUS_F$ (1977:)	0.14
New Zealand	$DNK_F, NLD_M, SWE_M, JPN_M, CAN_F, NOR_M$ (1976:)	0.24
Norway	AUT_F	0.048
Portugal	$IRL_F, AUS_F, BEL_F, UK_F, FRA_F$ (1966:)	0.24
Spain	$CHE_F, JPN_F, SWE_F, FIN_F, ITA_F, AUT_F, NOR_F$ (1957:)	0.26
Sweden	JPN_F, FIN_F, CHE_F	0.04
Switzerland	13 populations (1958:)	0.57
UK	CAN_F (1972:)	0.08
USA	BEL_M	0.002

Note: Combinations without mentioning the best fitting period have best fit for 1956 to 2001.

TABLE D.11: Best reference group for females of low-mortality countries according to lowest MSE in $LL_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right\|$
Australia	BEL_F	0.012
Austria	$NOR_F, ITA_F, FIN_F, SWE_F$	0.13
Belgium	$AUS_F, IRL_F, NLD_F, PRT_F, DEU_F, NOR_F$ (1974:)	0.30
Canada	$SWE_M, UK_F, NLD_F, NZL_F$	0.21
Denmark	NZL_F, NLD_M, SWE_M	0.19
Finland	$ITA_F, SWE_F, JPN_F, CHE_F$	0.08
France	$NLD_F, NEL_F, AUS_F, IRL_F, DEU_F, NOR_F, PRT_F, AUT_F$ (1974:)	0.33
Germany	$NLD_F, NOR_F, FRA_F, AUT_F, ITA_F, BEL_F, AUS_F$ (1977:)	0.22
Ireland	AUS_F, BEL_F	0.03
Italy	$FIN_F, AUT_F, SWE_F, NOR_F, JPN_F, CHE_F, ESP_F, DEU_F, NLD_F$	0.28
Japan	$CHE_F, SWE_F, ESP_F, FIN_F$	0.07
Netherlands	$FRA_F, DEU_F, BEL_F, AUS_F$ (1977:)	0.14
New Zealand	$DNK_F, NLD_M, , SWE_M, JPN_M, CAN_F, NOR_M$ (1976:)	0.24
Norway	AUT_F	0.04
Portugal	$IRL_F, AUS_F, BEL_F, UK_F, FRA_F$ (1966:)	0.24
Spain	$CHE_F, JPN_F, SWE_F, FIN_F, ITA_F, AUT_F, NOR_F$ (1957:)	0.26
Sweden	JPN_F, FIN_F, CHE_F	0.04
Switzerland	10 populations (1957:)	0.45
UK	CAN_F (1972:)	0.08
USA	BEL_M	0.002

Note: Combinations without mentioning the best fitting period have best fit for 1956 to 2001.

TABLE D.12: Best reference group for females of low-mortality countries according to lowest ME(e_0) in $\mathrm{LL}_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\lvert \bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger \right\rvert$
Australia	$\mathrm{BEL_F}, \mathrm{IRL_F}, \mathrm{FRA_F}$	0.088
Austria	$\mathrm{NOR_F}$	0.04
Belgium	$\mathrm{AUS_F}$	0.012
Canada	$\mathrm{SWE_M}, \mathrm{UK_F}, \mathrm{NLD_F}, \mathrm{NZL_F}, \mathrm{DNK_F}$	0.22
Denmark	$\mathrm{NZL_F}, \mathrm{NLD_M}, \mathrm{SWE_M}$	0.19
Finland	$\mathrm{ITA_F}, \mathrm{SWE_F}, \mathrm{JPN_F}, \mathrm{CHE_F}$	0.08
France	$\mathrm{NLD_F}, \mathrm{BEL_F}, \mathrm{AUS_F}, \mathrm{IRL_F}, \mathrm{DEU_F}, \mathrm{NOR_F}, \mathrm{PRT_F}, \mathrm{AUT_F}$	0.33
Germany	$\mathrm{NLD_F}, \mathrm{NOR_F}, \mathrm{FRA_F}, \mathrm{AUT_F}, \mathrm{ITA_F}$	0.20
Ireland	$\mathrm{AUS_F}, \mathrm{BEL_F}, \mathrm{FRA_F}$	0.11
Italy	$\mathrm{FIN_F}$	0.03
Japan	$\mathrm{CHE_F}, \mathrm{SWE_F}, \mathrm{ESP_F}, \mathrm{FIN_F}$	0.07
Netherlands	$\mathrm{FRA_F}$	0.05
New Zealand	$\mathrm{DNK_F}$	0.003
Norway	$\mathrm{AUT_F}$	0.04
Portugal	$\mathrm{IRL_F}, \mathrm{AUS_F}, \mathrm{BEL_F}, \mathrm{UK_F}, \mathrm{FRA_F}, \mathrm{CAN_F}, \mathrm{NLD_F}, \mathrm{SWE_M}, \mathrm{DEU_F}$ (1966:)	0.37
Spain	$\mathrm{CHE_F}, \mathrm{JPN_F}, \mathrm{SWE_F}, \mathrm{FIN_F}, \mathrm{ITA_F}, \mathrm{AUT_F}, \mathrm{NOR_F}$ (1957:)	0.26
Sweden	$\mathrm{JPN_F}, \mathrm{FIN_F}, \mathrm{CHE_F}$	0.04
Switzerland	13 populations	0.57
UK	$\mathrm{CAN_F}$ (1972:)	0.08
USA	$\mathrm{BEL_M}$	0.002

Note: Combinations without mentioning the best fitting period have best fit for 1956 to 2001.

TABLE D.13: Best reference group for males of CEE countries according to lowest MAE in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger \right\rvert$
Belarus	-	-
Bulgaria	$\mathrm{HUN_M}$	0.129
Estonia	12 populations	2.526
Hungary	$\mathrm{BGR_M}$	0.129
Latvia	$\mathrm{BLR_M}, \mathrm{EST_M}, \mathrm{LTU_M}, \mathrm{UKR_M}, \mathrm{RUS_M}, \mathrm{HUN_M}, \mathrm{BGR_M}$	1.473
Lithuania	-	-
Russia	-	-
Slovakia	$\mathrm{RUS_F}$	0.125
Ukraine	-	-

Table D.14: Best reference group for males of CEE countries according to lowest MSE in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}_{0i}^{\dagger} - \bar{e}_{0j}^{\dagger} \right\rvert$
Belarus	-	-
Bulgaria	HUN_M	0.129
Estonia	10 populations	2.023
Hungary	BGR_M	0.129
Latvia	$BLR_M, EST_M, LTU_M, UKR_M, RUS_M, HUN_M, BGR_M$	1.473
Lithuania	-	-
Russia	-	-
Slovakia	RUS_F	0.125
Ukraine	-	-

Table D.15: Best reference group for males of CEE countries according to lowest ME(e_0) in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}_{0i}^{\dagger} - \bar{e}_{0j}^{\dagger} \right\rvert$
Belarus	-	-
Bulgaria	HUN_M	0.129
Estonia	12 populations	2.526
Hungary	BGR_M	0.129
Latvia	$BLR_M, EST_M, LTU_M, UKR_M, RUS_M, HUN_M, BGR_M$	1.473
Lithuania	-	-
Russia	-	-
Slovakia	RUS_F	0.125
Ukraine	-	-

Table D.16: Best reference group for females of CEE countries according to lowest MAE in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}_{0i}^{\dagger} - \bar{e}_{0j}^{\dagger} \right\rvert$
Belarus	$LTU_F, LVA_F, HUN_F, EST_F, UKR_F, BGR_F, SVK_M, RUS_F$	1.123
Bulgaria	$EST_F, HUN_F, SVK_F, LTU_F$	0.600
Estonia	HUN_F, BGR_F, LTU_F	0.372
Hungary	$EST_F, LTU_F, BGR_F, BLR_F$	0.405
Latvia	$BLR_F, LTU_F, UKR_F, HUN_F, EST_F, BGR_F, SVK_M, RUS_F, SVK_F$	1.344
Lithuania	$BLR_F, LVA_F, HUN_F, EST_F, BGR_F, UKR_F, SVK_M$	1.109
Russia	11 populations	1.763
Slovakia	BGR_F, EST_F	0.743
Ukraine	$LVA_F, SVK_M, BLR_F, LTU_F, RUS_F, HUN_F, BGR_M$	0.961

TABLE D.17: Best reference group for females of CEE countries according to lowest MSE in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}^{\dagger}_{0i} - \bar{e}^{\dagger}_{0j} \right\rvert$
Belarus	$\mathrm{LTU_F, HUN_F, EST_F, UKR_F, BGR_F, SVK_M, RUS_F}$	1.123
Bulgaria	$\mathrm{EST_F, HUN_F, SVK_F, LTU_F, BLR_F, LVA_F, UKR_F}$	1.214
Estonia	$\mathrm{HUN_F, BGR_F, LTU_F, BLR_F, LVA_F}$	0.601
Hungary	$\mathrm{EST_F, LTU_F, BGR_F, BLR_F, LVA_F, SVK_F, UKR_F, SVK_M}$	1.403
Latvia	$\mathrm{BLR_F, LTU_F, UKR_F, HUN_F, EST_F, BGR_F, SVK_M, RUS_F, SVK_F}$	1.344
Lithuania	$\mathrm{BLR_F, LVA_F, HUN_F}$	0.294
Russia	$\mathrm{SVK_M, BGR_M, HUN_M, UKR_F, LVA_F, BLR_F}$	1.123
Slovakia	$\mathrm{BGR_F, EST_F}$	0.743
Ukraine	$\mathrm{LVA_F}$	0.384

TABLE D.18: Best reference group for females of CEE countries according to lowest ME(e_0) in LL.

Country	Other populations in best reference group	$\left\lvert \bar{e}^{\dagger}_{0i} - \bar{e}^{\dagger}_{0j} \right\rvert$
Belarus	$\mathrm{LTU_F, HUN_F, EST_F, UKR_F, BGR_F, SVK_M, RUS_F}$	1.123
Bulgaria	$\mathrm{EST_F, HUN_F, SVK_F, LTU_F}$	0.600
Estonia	$\mathrm{HUN_F, BGR_F, LTU_F, SVK_F}$	0.743
Hungary	$\mathrm{EST_F}$	0.078
Latvia	$\mathrm{BLR_F, LTU_F, UKR_F, HUN_F, EST_F, BGR_F, SVK_M, RUS_F, SVK_F}$	1.344
Lithuania	$\mathrm{BLR_F, LVA_F}$	0.230
Russia	$\mathrm{SVK_M, BGR_M, HUN_M, UKR_F, LVA_F, BLR_F}$	1.123
Slovakia	$\mathrm{BGR_F, EST_F, HUN_F, LTU_F, BLR_F, LVA_F}$	1.344
Ukraine	$\mathrm{LVA_F, SVK_M, BLR_F, LTU_F, RUS_F}$	0.621

TABLE D.19: Best reference group for males of CEE countries according to lowest MAE in $\mathrm{LL}_{e_0^{\dagger}}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\lvert \bar{e}^{\dagger}_{0i} - \bar{e}^{\dagger}_{0j} \right\rvert$
Belarus	-	-
Bulgaria	$\mathrm{HUN_M, RUS_F, SVK_M, UKR_F, EST_M, LVA_F}$ (1974:)	1.345
Estonia	9 populations (1975:)	1.674
Hungary	$\mathrm{BGR_M, RUS_F}$ (1974:)	0.470
Latvia	$\mathrm{BLR_M, EST_M, LTU_M, UKR_M, RUS_M, HUN_M, BGR_M, RUS_F}$ (1975:)	1.814
Lithuania	13 populations (1975:)	3.564
Russia	-	-
Slovakia	$\mathrm{RUS_F, BGR_M, UKR_F, HUN_M, LVA_F, BLR_F}$ (1974:)	0.998
Ukraine	-	-

Note: Combinations without mentioning the best fitting period have best fit for 1970 to 2000.

TABLE D.20: Best reference group for males of CEE countries according to lowest MSE in $\mathrm{LL}_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\lvert \bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger \right\rvert$
Belarus	-	-
Bulgaria	$\mathrm{HUN_M, RUS_F, SVK_M, UKR_F, EST_M}$ (1974:)	1.062
Estonia	9 populations (1975:)	1.674
Hungary	$\mathrm{BGR_M, RUS_F}$ (1974:)	0.470
Latvia	$\mathrm{BLR_M, EST_M, LTU_M, UKR_M, RUS_M, HUN_M, BGR_M, RUS_F}$ (1975:)	1.814
Lithuania	13 populations (1975:)	3.564
Russia	-	-
Slovakia	$\mathrm{RUS_F, BGR_M, UKR_F, HUN_M, LVA_F, BLR_F}$(1974:)	0.998
Ukraine	-	-

Note: Combinations without mentioning the best fitting period have best fit for 1970 to 2000.

TABLE D.21: Best reference group for males of CEE countries according to lowest $\mathrm{ME}(e_0)$ in $\mathrm{LL}_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\lvert \bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger \right\rvert$
Belarus	-	-
Bulgaria	$\mathrm{HUN_M, RUS_F, SVK_M}$ (1976:)	0.466
Estonia	10 populations (1975:)	2.024
Hungary	$\mathrm{BGR_M, RUS_F}$ (1974:)	0.470
Latvia	$\mathrm{BLR_M, EST_M, LTU_M, UKR_M, RUS_M, HUN_M, BGR_M, RUS_F}$ (1975:)	1.814
Lithuania	13 populations (1975:)	3.564
Russia	-	-
Slovakia	$\mathrm{RUS_F, BGR_M, UKR_F, HUN_M, LVA_F, BLR_F}$(1974:)	0.998
Ukraine	-	-

Note: Combinations without mentioning the best fitting period have best fit for 1970 to 2000.

TABLE D.22: Best reference group for females of CEE countries according to lowest MAE in $\mathrm{LL}_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\lvert \bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger \right\rvert$
Belarus	-	-
Bulgaria	$\mathrm{EST_F, HUN_F, SVK_F}$	0.513
Estonia	$\mathrm{HUN_F, BGR_F}$	0.229
Hungary	$\mathrm{EST_F, LTU_F, BGR_F}$	0.307
Latvia	$\mathrm{BLR_F, LTU_F, UKR_F, HUN_F, EST_F, BGR_F, SVK_M}$ (1971:)	0.879
Lithuania	$\mathrm{BLR_F, LVA_F, HUN_F, EST_F, BGR_F, UKR_F, SVK_M}$ (1971:)	1.109
Russia	$\mathrm{SVK_M}$ (1973:)	0.125
Slovakia	$\mathrm{BGR_F}$	0.514
Ukraine	$\mathrm{LVA_F}$ (1975:)	0.384

Note: Combinations without mentioning the best fitting period have best fit for 1970 to 2000.

Table D.23: Best reference group for females of CEE countries according to lowest MSE in $\mathrm{LL}_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right\|$
Belarus	-	-
Bulgaria	$\mathrm{EST_F, HUN_F, SVK_F}$	0.513
Estonia	$\mathrm{HUN_F, BGR_F, LTU_F}$	0.372
Hungary	$\mathrm{EST_F, LTU_F, BGR_F}$	0.307
Latvia	$\mathrm{BLR_F, LTU_F, UKR_F, HUN_F, EST_F, BGR_F, SVK_M}$ (1971:)	0.879
Lithuania	$\mathrm{BLR_F, LVA_F, HUN_F, EST_F, BGR_F, UKR_F, SVK_M}$ (1971:)	1.109
Russia	$\mathrm{SVK_M}$ (1973:)	0.125
Slovakia	$\mathrm{BGR_F}$	0.514
Ukraine	$\mathrm{LVA_F}$ (1975:)	0.384

Note: Combinations without mentioning the best fitting period have best fit for 1970 to 2000.

Table D.24: Best reference group for females of CEE countries according to lowest $\mathrm{ME}(e_0)$ in $\mathrm{LL}_{e_0^\dagger}$.

Country	Other populations in best reference group (best fitting period from:)	$\left\|\bar{e}_{0i}^\dagger - \bar{e}_{0j}^\dagger\right\|$
Belarus	-	-
Bulgaria	$\mathrm{EST_F, HUN_F, SVK_F}$	0.513
Estonia	$\mathrm{HUN_F}$	0.078
Hungary	$\mathrm{EST_F, LTU_F, BGR_F, BLR_F}$ (1971:)	0.405
Latvia	$\mathrm{BLR_F, LTU_F, UKR_F, HUN_F, EST_F, BGR_F, SVK_M}$ (1971:)	0.879
Lithuania	$\mathrm{BLR_F, LVA_F, HUN_F, EST_F, BGR_F, UKR_F, SVK_M, SVK_F}$ (1971:)	1.114
Russia	$\mathrm{SVK_M}$ (1973:)	0.125
Slovakia	$\mathrm{BGR_F}$	0.514
Ukraine	$\mathrm{LVA_F}$ (1975:)	0.384

Note: Combinations without mentioning the best fitting period have best fit for 1970 to 2000.

Appendix E

UN Forecasting for CEE Countries

Table E.1: Forecast of life expectancies at birth for high-mortality CEE countries using UN forecast for HMD-2018 and WPP-2012†.

Data & Forecast year	Belarus	Bulgaria	Estonia	Hungary	Latvia	Lithuania	Russia	Slovakia	Ukraine
HMD, 2034	78.672	78.901	83.381	81.165	80.361	80.938	77.463	82.109	77.507
WPP, 2034	77.225	78.914	81.395	80.600	78.960	79.807	76.223	81.071	75.621
HMD, 2044	79.370	79.740	84.528	82.311	81.384	81.851	78.265	83.273	78.371
WPP, 2044	78.022	79.858	82.371	81.592	79.724	80.645	77.204	82.057	76.219
HMD, 2054	79.952	80.633	85.703	83.539	82.432	82.763	79.079	84.436	79.243
WPP, 2054	78.812	80.729	83.315	82.707	80.564	81.428	77.991	83.053	76.906

†For WPP-2012 data, the life expectancies were available from 1873 to 2015. The results are shown for females only.

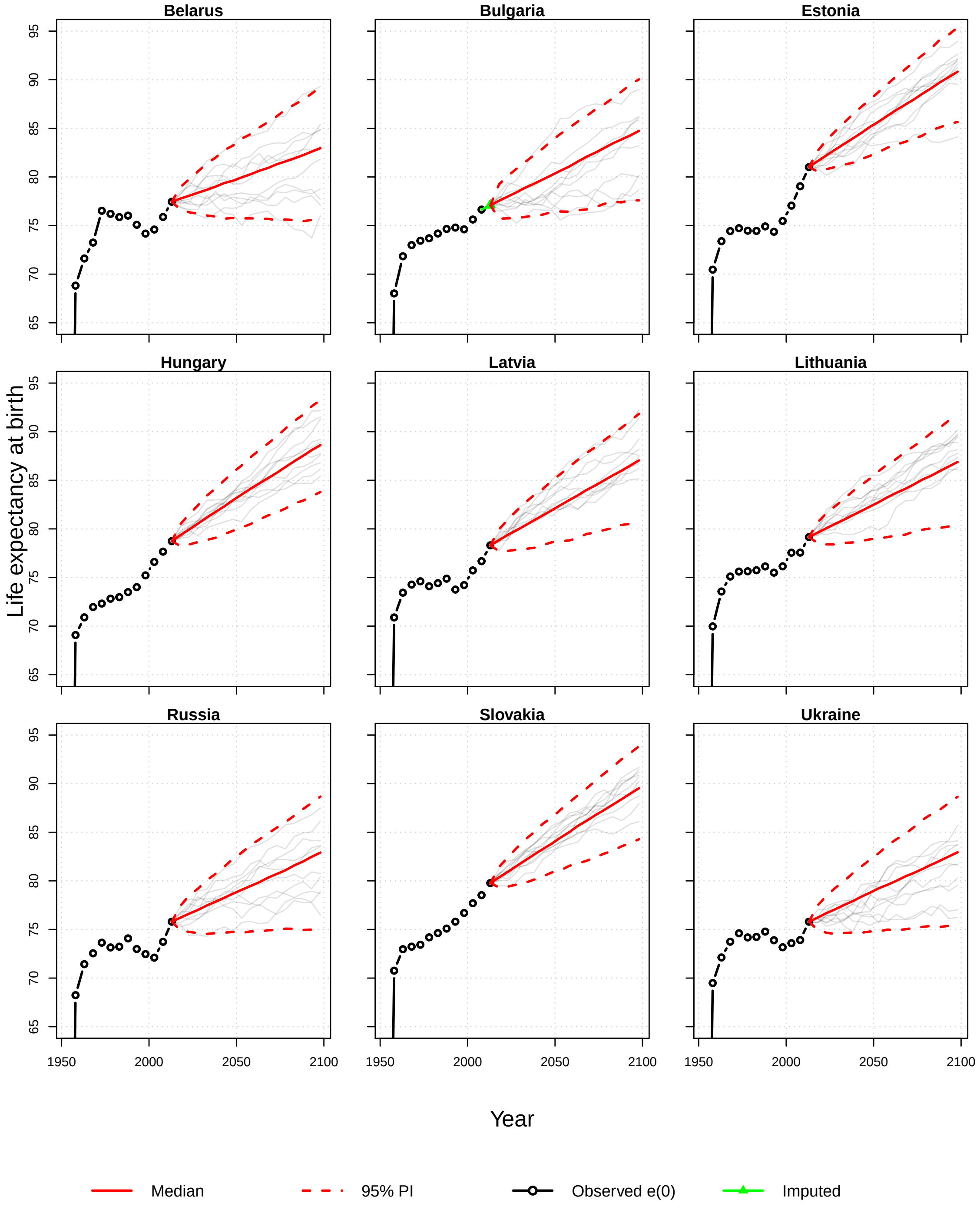


FIGURE E.1: Forecast of life expectancy at birth for (comparatively) high mortality countries by UN forecast with 95% prediction interval (HMD-2018). Results are showed for females only.

Appendix F

R codes for the proposed methods

```
library(forecast)
library(ftsa)
library(demography)
library(smoothAPC)
library(MortalitySmooth)
library(MortalityLaws)


#loading the pre-smoothed mortality rates (by lasso)#
load("forecast_40years_ll_edag.RData")

#adding exposure population one by one for creating demogdata object#
# pop1 is for population of interest#
Ext2 <- pop1+pop2+....+pop40

#adding mortality rates one by one for creating demogdata object#
# rate1 is for population of interest#
Dxxt2 <- rate1+rate2+....+rate40


ages <- 0:110
years <- 1956:2011
Dx2 <- matrix(Dxxt2/n, #n is the total number of population added in Dxxt2#
ncol = length(years),
nrow = length(ages))
```

```
Nx2 <- matrix(Ext2,
ncol = length(years),
nrow = length(ages))

#jmdata2 is the demogdata object containing joint mortality rates#
jmdata2 <- demogdata(data = Dx2, pop = Nx2, ages = ages, years = years,
type = "mortality", label = "JMD", name = "Combined Mortality data")

plot(jmdata2)

#Estimating life tables for obtaining e-dagger#
ltjd<-print(lifetable(jmdata2, max.age = 110))
eddgg<-sapply(ltjd, FUN = function(x) sum(x$ex*x$lx*x$mx))


overall_edgr <- cbind.data.frame("years"=years,"edagger"=eddgg)


#estimation of observed e-dagger#

#basic LC model in demography without no adjustment for k(t)#
ln.female <- lca(jmdata2, max.age= max(jmdata2$age), adjust = "none",
interpolate=TRUE)
a1j<-ln.female$ax
b1j<-ln.female$bx
k1j<-ln.female$kt

length(a1j)
length(b1j)
length(k1j)

mx1j<-exp(b1j*k1j[1]+a1j)

# for getting fitted mortality rates from estimated parameters of LC model#
fitmx <- function (kt,ax,bx,transform=FALSE)
{
```

```
clogratesfit <- outer(kt, bx)
logratesfit <- sweep(clogratesfit,2,ax,"+")
if(transform)
return(logratesfit)
else
return(exp(logratesfit))
}


wwmxj<-fitmx(k1j,a1j,b1j)
wmxj<-t(wwmxj)
plot(wmxj)


ages1 <- jmdata2$age
years1 <- jmdata2$year
fDx1 <- matrix(wmxj,
ncol = length(years1),
nrow = length(ages1))
fNx1 <- Nx2

lcfdata1 <- demogdata(data = fDx1, pop = fNx1, ages = ages1, years = years1,
type = "mortality", label = "JMLC", name = "joint")

plot(lcfdata1)


#extracting component of e-dagger from fitted Lee-carter to use in later codes#
fitswed <- print(lifetable(lcfdata1,years = jmdata2$year,
ages = ages1,max.age = max(jmdata2$age)),type = c("period"))

years<-1956:2011

fitted_edgr <- sapply(fitswed, FUN = function(x) sum(x$ex*x$lx*x$mx))

#extracting lx, ex from life table obtained from fitted LC #
```

```
llxx <- matrix(unlist(lapply(fitswed, FUN = function(x) x[, "lx"])),
nrow = length(years), ncol = length(ages1), byrow = TRUE)


eexx <- matrix(unlist(lapply(fitswed, FUN = function(x) x[, "ex"])),
nrow = length(years), ncol = length(ages1), byrow = TRUE)

mmxx <- matrix(unlist(lapply(fitswed, FUN = function(x) x[, "mx"])),
nrow = length(years), ncol = length(ages1), byrow = TRUE)


#the function for matching k_t with observed e-dagger#
# for lc-edaggger it is performed to match with e-dagger obtained from observed,#
#unsmoothed data#
#same function is used for first order modeling of ll-edagger, where matching is#
#done with e-dagger obtained from observed (smoothed) data#


lcadagger<-function (data, series = names(data$rate)[1], years = data$year,
ages = data$age, max.age = max(data$age), adjust = c("edagger", "none"),
chooseperiod = FALSE, minperiod = 20, breakmethod = c("bai", "bms"),
scale = FALSE, restype = c("logrates", "rates", "deaths"),
interpolate = TRUE)
{
if (class(data) != "demogdata") {
stop("Not demography data")
}
if (!any(data$type == c("mortality", "fertility"))) {
stop("Neither mortality nor fertility data")
}
is.el <- function(el,set)
{
is.element(toupper(el),toupper(set))
}

# Compute expected age from single year mortality rates
```

```
get.e0 <- function(x,agegroup,sex,startage=0)
{
lt(x, startage, agegroup, sex)$ex[1]
}


# Replace zeros with interpolated values
fill.zero <- function(x,method="constant")
{
tt <- 1:length(x)
zeros <- abs(x) < 1e-9
xx <- x[!zeros]
tt <- tt[!zeros]
x <- stats::approx(tt,xx,1:length(x),method=method,f=0.5,rule=2)
return(x$y)
}



adjust <- match.arg(adjust)
restype <- match.arg(restype)
breakmethod <- match.arg(breakmethod)
data <- extract.ages(data, ages, combine.upper = FALSE)
if (max.age < max(ages))
data <- extract.ages(data, min(ages):max.age, combine.upper = TRUE)
startage <- min(data$age)
get.series <- function(data,series)
{
if(!is.el(series,names(data)))
stop(paste("Series",series,"not found"))
i <- match(toupper(series),toupper(names(data)))
return(as.matrix(data[[i]]))
}
mx <- get.series(data$rate, series)
pop <- get.series(data$pop, series)
startyear <- min(years)
```

```
stopyear <- max(years)
if (startyear > max(data$year) | stopyear < min(data$year))
stop("Year not found")
startyear <- max(startyear, min(data$year))
if (!is.null(stopyear))
stopyear <- min(stopyear, max(data$year))
else stopyear <- max(data$year)
id2 <- stats::na.omit(match(startyear:stopyear, data$year))
mx <- mx[, id2]
pop <- pop[, id2]
year <- data$year[id2]
deltat <- year[2] - year[1]
ages <- data$age
n <- length(ages)
m <- sum(id2 > 0)
edgr<-overall_edgr$edagger
mx <- matrix(mx, nrow = n, ncol = m)
if (interpolate) {
mx[is.na(mx)] <- 0
if (sum(abs(mx) < 1e-09, na.rm = TRUE) > 0) {
warning("Replacing zero values with estimates")
for (i in 1:n) mx[i, ] <- fill.zero(mx[i, ])
}
}
mx <- t(mx)
mx[mx == 0] <- NA
lograTES <- log(mx)
pop <- t(pop)
deaths <- pop * mx
ax <- apply(lograTES, 2, mean, na.rm = TRUE)
if (sum(ax < -1e+09) > 0)
stop(sprintf("Some %s rates are zero.\n Try reducing the maximum age
or setting interpolate=TRUE.",
data$type))
clograTES <- sweep(lograTES, 2, ax)
svd.mx <- svd(clograTES)
```

```
sumv <- sum(svd.mx$v[, 1])
bx <- svd.mx$v[, 1]/sumv
kt <- svd.mx$d[1] * svd.mx$u[, 1] * sumv
ktadj <- rep(0, m)
logdeathsadj <- matrix(NA, n, m)
z <- log(t(pop)) + ax
x <- 1:m
ktse <- stats::predict(stats::lm(kt ~ x), se.fit = TRUE)$se.fit
ktse[is.na(ktse)] <- 1
agegroup = ages[4] - ages[3]

edgr<-overall_edgr$edagger

fitmx <- function (kt,ax,bx,transform=FALSE)
{
# Derives mortality rates from kt mortality index,
# per Lee-Carter method
clogratesfit <- outer(kt, bx)
logratesfit <- sweep(clogratesfit,2,ax,"+")
if(transform)
return(logratesfit)
else
return(exp(logratesfit))
}
#the following function is taken from demography package#
findroot <- function(FUN,guess,margin,try=1,...)
{
# First try in successively larger intervals around best guess
for(i in 1:5)
{
rooti <- try(stats::uniroot(FUN,interval=guess+i*margin/3*c(-1,1),...),
silent=TRUE)
if(class(rooti) != "try-error")
return(rooti$root)
}
# No luck. Try really big intervals
```

```
rooti <- try(stats::uniroot(FUN,interval=guess+10*margin*c(-1,1),...),
silent=TRUE)
if(class(rooti) != "try-error")
return(rooti$root)

# Still no luck. Try guessing root using quadratic approximation
if(try<3)
{
root <- try(quadroot(FUN,guess,10*margin,...),silent=TRUE)
if(class(root)!="try-error")
return(findroot(FUN,root,margin,try+1,...))
root <- try(quadroot(FUN,guess,20*margin,...),silent=TRUE)
if(class(root)!="try-error")
return(findroot(FUN,root,margin,try+1,...))
}

# Finally try optimization
root <- try(newroot(FUN,guess,...),silent=TRUE)
if(class(root)!="try-error")
return(root)
else
root <- try(newroot(FUN,guess-margin,...),silent=TRUE)
if(class(root)!="try-error")
return(root)
else
root <- try(newroot(FUN,guess+margin,...),silent=TRUE)
if(class(root)!="try-error")
return(root)
else
stop("Unable to find root")
}

quadroot <- function(FUN,guess,margin,...)
{
x1 <- guess-margin
x2 <- guess+margin
```

```
y1 <- FUN(x1,...)
y2 <- FUN(x2,...)
y0 <- FUN(guess,...)
if(is.na(y1) | is.na(y2) | is.na(y0))
stop("Function not defined on interval")
b <- 0.5*(y2-y1)/margin
a <- (0.5*(y1+y2)-y0)/(margin^2)
tmp <- b^2 - 4*a*y0
if(tmp < 0)
stop("No real root")
tmp <- sqrt(tmp)
r1 <- 0.5*(tmp-b)/a
r2 <- 0.5*(-tmp-b)/a
if(abs(r1) < abs(r2))
root <- guess+r1
else
root <- guess+r2
return(root)
}

# Try finding root using minimization
newroot <- function(FUN,guess,...)
{
fred <- function(x,...){(FUN(x,...)^2)}
junk <- stats::nlm(fred,guess,...)
if(abs(junk$minimum)/fred(guess,...) > 1e-6)
warning("No root exists. Returning closest")
return(junk$estimate)
}
if (adjust == "edagger") {


Fundg<-function(p,bx,ax,edgr,llxxi,eexxi){
edgr - sum(exp(ax + bx*p)*llxxi*eexxi)
}
for (i in 1:m) {
```

```
if (i == 1)
guess <- kt[1]
else guess <- mean(c(ktadj[i - 1], kt[i]))
ktadj[i] <- findroot(Fundg, guess = guess, margin = 3 *
ktse[i], edgr=edgr[i] , llxxi=llxx[i,],
eexxi=eexx[i,],ax = ax, bx = bx)
logdeathsadj[,i]<-z[,i]+bx*ktadj[i]
}
}
else if (adjust == "none")
ktadj <- kt
else stop("Unknown adjustment method")
kt <- ktadj
if (chooseperiod) {
if (breakmethod == "bai") {
x <- 1:m
bp <- strucchange::breakpoints(kt ~ x)$breakpoints
bp <- bp[bp <= (m - minperiod)]
bestbreak <- max(bp)
return(lca(data, series, year[(bestbreak + 1):m],
ages = ages, max.age = max.age, adjust = adjust,
interpolate = interpolate, chooseperiod = FALSE,
scale = scale))
}
else {
RS <- devlin <- devadd <- numeric(m - 2)
for (i in 1:(m - 2)) {
tmp <- lcadagger(data, series, year[i:m], ages = ages,
max.age = max.age, adjust = adjust,
chooseperiod = FALSE,
interpolate = interpolate, scale = scale)
devlin[i] <- tmp$mdev[2]
devadd[i] <- tmp$mdev[1]
RS[i] <- (tmp$mdev[2]/tmp$mdev[1])
}
bestbreak <- order(RS[1:(m - minperiod)])[1] - 1
```

```
out <- lcadagger(data, series, year[(bestbreak + 1):m],
ages = ages, max.age = max.age, adjust = adjust,
chooseperiod = FALSE, interpolate = interpolate,
scale = scale)
out$mdevs <- ts(cbind(devlin, devadd, RS), start = startyear,
deltat = deltat)
dimnames(out$mdevs)[[2]] <- c("Mean deviance total",
"Mean deviance base", "Mean deviance ratio")
return(out)
}
}
logfit <- fitmx(kt, ax, bx, transform = TRUE)
if (restype == "logrates") {
fit <- logfit
res <- logrates - fit
}
else if (restype == "rates") {
fit <- exp(logfit)
res <- exp(logrates) - fit
}
else if (restype == "deaths") {
fit <- exp(logfit) * pop
res <- deaths - fit
}
residuals <- fts(ages, t(res), frequency = 1/deltat, start = years[1],
xname = "Age", yname = paste("Residuals", data$type,
"rate"))
fitted <- fts(ages, t(fit), frequency = 1/deltat, start = years[1],
xname = "Age", yname = paste("Fitted", data$type, "rate"))
names(ax) <- names(bx) <- ages
if (scale) {
avdiffk <- -mean(diff(kt))
bx <- bx * avdiffk
kt <- kt/avdiffk
}
deathsadjfit <- exp(logfit) * pop
```

```
drift <- mean(diff(kt))
ktlinfit <- mean(kt) + drift * (1:m - (m + 1)/2)
deathslinfit <- fitmx(ktlinfit, ax, bx, transform = FALSE) *
pop
dflogadd <- (m - 2) * (n - 1)
mdevlogadd <- 2/dflogadd * sum(deaths * log(deaths/deathsadjfit) -
(deaths - deathsadjfit))
dfloglin <- (m - 2) * n
mdevloglin <- 2/dfloglin * sum(deaths * log(deaths/deathslinfit) -
(deaths - deathslinfit))
mdev <- c(mdevlogadd, mdevloglin)
output <- list(label = data$label, age = ages, year = year,
mx = t(mx), ax = ax, bx = bx, kt = ts(kt, start = startyear,
deltat = deltat), residuals = residuals, fitted = fitted,
varprop = svd.mx$d[1]^2/sum(svd.mx$d^2), y = fts(ages,
t(mx), start = years[1], frequency = 1/deltat, xname = "Age",
yname = ifelse(data$type == "mortality", "Mortality",
"Fertility")), mdev = mdev)
names(output)[4] <- series
output$call <- match.call()
names(output$mdev) <- c("Mean deviance base", "Mean deviance total")
output$adjust <- adjust
output$type <- data$type
return(structure(output, class = "lca"))
}
```

```
#fisrt stage modeling of Li-Lee; LC with adjustment for K_t according#
#to observed (estimated) e(0)#
# the minimum period for model fitting is 20 or more
funmod2 <- lcadagger(jmdata2, ages=0:110,max.age = max(jmdata2$age),
adjust = "edagger", interpolate=TRUE,
breakmethod = "bms", minperiod = k, chooseperiod = TRUE)
```

```
funmod2
plot(funmod2)

plot(funmod2$fitted, main="fitted mortality rates from
first stage model (1956:2011)")
dim(funmod2$fitted$y)

ax12<-funmod2$ax
bx12<-funmod2$bx
Kt12<-funmod2$kt

mx12<-exp(bx12*Kt12[1]+ax12)
fitmx <- function (kt,ax,bx,transform=FALSE)
{
clogratesfit <- outer(kt, bx)
logratesfit <- sweep(clogratesfit,2,ax,"+")
if(transform)
return(logratesfit)
else
return(exp(logratesfit))
}

wwmx2<-fitmx(Kt12,ax12,bx12)
wmx2<-t(wwmx2)
Dxll12 <- matrix(wmx2,
ncol = length(funmod2$year),
nrow = length(ages))

jmdatall12 <- demogdata(data = Dxll12, pop = Nx2, ages = ages,
years = funmod2$year, type = "mortality", label = "JMD",
name = "Joint Mortality Data")

plot(jmdatall12)


thyr<-outer(Kt12, bx12) # extracting the common factor#
```

```
dim(thyr)

#second level modeling for coherent forecast#
#Lee-Carter (1992) model on country-specific data (mortality data without "THYR")#

liLee<-function(data, series = names(data$rate)[1], years = data$year,
ages = data$age, max.age = max(data$age), adjust = c("dt", "dxt",
"e0", "none"), chooseperiod = FALSE, minperiod = 20,
breakmethod = c("bai", "bms"), scale = FALSE,
restype = c("lograte", "rates", "deaths"), interpolate = FALSE)
{
if (class(data) != "demogdata") {
stop("Not demography data")
}
if (!any(data$type == c("mortality", "fertility"))) {
stop("Neither mortality nor fertility data")
}

is.el <- function(el,set)
{
is.element(toupper(el),toupper(set))
}

get.series <- function(data,series)
{
if(!is.el(series,names(data)))
stop(paste("Series",series,"not found"))
i <- match(toupper(series),toupper(names(data)))
return(as.matrix(data[[i]]))
}

fitmx <- function (kt,ax,bx,transform=FALSE)
{
clogratesfit <- outer(kt, bx)
logratesfit <- sweep(clogratesfit,2,ax,"+" )
if(transform)
```

```
return(logratesfit)
else
return(exp(logratesfit))
}

adjust <- match.arg(adjust)
restype <- match.arg(restype)
breakmethod <- match.arg(breakmethod)
data <- extract.ages(data, ages, combine.upper = FALSE)
if (max.age < max(ages))
data <- extract.ages(data, min(ages):max.age, combine.upper = TRUE)
startage <- min(data$age)
mx <- get.series(data$rate, series)
pop <- get.series(data$pop, series)
startyear <- min(years)
stopyear <- max(years)
if (startyear > max(data$year) | stopyear < min(data$year))
stop("Year not found")
startyear <- max(startyear, min(data$year))
if (!is.null(stopyear))
stopyear <- min(stopyear, max(data$year))
else stopyear <- max(data$year)
id2 <- stats::na.omit(match(startyear:stopyear, data$year))
mx <- mx[, id2]
pop <- pop[, id2]
year <- data$year[id2]
deltat <- year[2] - year[1]
ages <- data$age
n <- length(ages)
m <- sum(id2 > 0)
mx <- matrix(mx, nrow = n, ncol = m)
if (interpolate) {
mx[is.na(mx)] <- 0
if (sum(abs(mx) < 1e-09, na.rm = TRUE) > 0) {
warning("Replacing zero values with estimates")
for (i in 1:n) mx[i, ] <- fill.zero(mx[i, ])
```

```
}
}
mx <- t(mx)
mx[mx == 0] <- NA
lograte s <- log(mx)
pop <- t(pop)
deaths <- pop * mx
ax <- apply(lograte s, 2, mean, na.rm = TRUE)
tgy<-ax+thyr
if (sum(ax < -1e+09) > 0)
stop(sprintf("Some %s rates are zero.\n Try reducing the maximum age
or setting interpolate=TRUE.",
data$type))
corates<-sweep(lograte s,2,ax)
colgrates<-corates-thyr
svd.mx <- svd(colgrates)
sumv <- sum(svd.mx$v[, 1])
bx <- svd.mx$v[, 1]/sumv
kt <- svd.mx$d[1] * svd.mx$u[, 1] * sumv
ktadj <- rep(0, m)
logdeathsadj <- matrix(NA, n, m)
z <- log(t(pop)) + ax
x <- 1:m
ktse <- stats::predict(stats::lm(kt ~ x), se.fit = TRUE)$se.fit
ktse[is.na(ktse)] <- 1
agegroup = ages[4] - ages[3]
if (adjust == "dxt") {
options(warn = -1)
for (i in 1:m) {
y <- as.numeric(deaths[i, ])
zi <- as.numeric(z[, i])
weight <- as.numeric(zi > -1e-08)
yearglm <- stats::glm(y ~ offset(zi) - 1 + bx, family = stats::poisson,
weights = weight)
ktadj[i] <- yearglm$coef[1]
logdeathsadj[, i] <- z[, i] + bx * ktadj[i]
```

```
}
options(warn = 0)
}
else if (adjust == "dt") {
FUN <- function(p, Dt, bx, ax, popi) {
Dt - sum(exp(p * bx + ax) * popi)
}
for (i in 1:m) {
if (i == 1)
guess <- kt[1]
else guess <- mean(c(ktadj[i - 1], kt[i]))
ktadj[i] <- findroot(FUN, guess = guess, margin = 3 *
ktse[i], ax = ax, bx = bx, popi = pop[i, ],
Dt = sum(as.numeric(deaths[i,])))
logdeathsadj[, i] <- z[, i] + bx * ktadj[i]
}
}
else if (adjust == "e0") {
e0 <- apply(mx, 1, get.e0, agegroup = agegroup, sex = series,
startage = startage)
FUN2 <- function(p, e0i, ax, bx, agegroup, series, startage) {
e0i - estimate.e0(p, ax, bx, agegroup, series, startage)
}
for (i in 1:m) {
if (i == 1)
guess <- kt[1]
else guess <- mean(c(ktadj[i - 1], kt[i]))
ktadj[i] <- findroot(FUN2, guess = guess, margin = 3 *
ktse[i], e0i = e0[i], ax = ax, bx = bx,
agegroup = agegroup,
series = series, startage = startage)
}
}
else if (adjust == "none")
ktadj <- kt
else stop("Unknown adjustment method")
```

```
kt <- ktadj
if (chooseperiod) {
if (breakmethod == "bai") {
x <- 1:m
bp <- strucchange::breakpoints(kt ~ x)$breakpoints
bp <- bp[bp <= (m - minperiod)]
bestbreak <- max(bp)
return(lca(data, series, year[(bestbreak + 1):m],
ages = ages, max.age = max.age, adjust = adjust,
interpolate = interpolate, chooseperiod = FALSE,
scale = scale))
}
else {
RS <- devlin <- devadd <- numeric(m - 2)
for (i in 1:(m - 2)) {
tmp <- lca(data, series, year[i:m], ages = ages,
max.age = max.age, adjust = adjust, chooseperiod = FALSE,
interpolate = interpolate, scale = scale)
devlin[i] <- tmp$mdev[2]
devadd[i] <- tmp$mdev[1]
RS[i] <- (tmp$mdev[2]/tmp$mdev[1])
}
bestbreak <- order(RS[1:(m - minperiod)])[1] - 1
out <- lca(data, series, year[(bestbreak + 1):m],
ages = ages, max.age = max.age, adjust = adjust,
chooseperiod = FALSE, interpolate = interpolate,
scale = scale)
out$mdevs <- ts(cbind(devlin, devadd, RS), start = startyear,
deltat = deltat)
dimnames(out$mdevs)[[2]] <- c("Mean deviance total",
"Mean deviance base", "Mean deviance ratio")
return(out)
}
}
logfit <- fitmx(kt, ax, bx, transform = TRUE)
if (restype == "logrates") {
```

```
fit <- logfit
res <- lograte̲s - fit
}
else if (restype == "rates") {
fit <- exp(logfit)
res <- exp(lograte̲s) - fit
}
else if (restype == "deaths") {
fit <- exp(logfit) * pop
res <- deaths - fit
}
residuals <- fts(ages, t(res), frequency = 1/deltat, start = years[1],
xname = "Age", yname = paste("Residuals", data$type,
"rate"))
fitted <- fts(ages, t(fit), frequency = 1/deltat, start = years[1],
xname = "Age", yname = paste("Fitted", data$type, "rate"))
names(ax) <- names(bx) <- ages
if (scale) {
avdiffk <- -mean(diff(kt))
bx <- bx * avdiffk
kt <- kt/avdiffk
}
deathsadjfit <- exp(logfit) * pop
drift <- mean(diff(kt))
ktlinfit <- mean(kt) + drift * (1:m - (m + 1)/2)
deathslinfit <- fitmx(ktlinfit, ax, bx, transform = FALSE) *
pop
dflogadd <- (m - 2) * (n - 1)
mdevlogadd <- 2/dflogadd * sum(deaths * log(deaths/deathsadjfit) -
(deaths - deathsadjfit))
dfloglin <- (m - 2) * n
mdevloglin <- 2/dfloglin * sum(deaths * log(deaths/deathslinfit) -
(deaths - deathslinfit))
mdev <- c(mdevlogadd, mdevloglin)
output <- list(label = data$label, age = ages, year = year,
mx = t(mx), ax = ax, bx = bx, kt = ts(kt, start = startyear,
```

```
deltat = deltat), residuals = residuals, fitted = fitted,
varprop = svd.mx$d[1]^2/sum(svd.mx$d^2), y = fts(ages, t(mx),
start = years[1], frequency = 1/deltat, xname = "Age",
yname = ifelse(data$type == "mortality", "Mortality",
"Fertility")), mdev = mdev)
names(output)[4] <- series
output$call <- match.call()
names(output$mdev) <- c("Mean deviance base", "Mean deviance total")
output$adjust <- adjust
output$type <- data$type
return(structure(output, class = "lca"))
}

#country specific model fitting with demogdata object "country"#
llcountry <- liLee(country, years = 1956:2011, max.age = 110, adjust = "none",
interpolate = TRUE)



axicountry<-llcountry$ax
bxicountry<-llcountry$bx
Kticountry<-llcountry$kt



#We need to add the common factor now with the fitted rate from #
#country-specific model#

fitttmx <- function (kt,ax,bx,thyr,transform=FALSE)
{
clogratesfit <- outer(kt, bx)
logratesfitt <- sweep(clogratesfit,2,ax,"+")
logratesfit <- logratesfitt+thyr
if(transform)
return(logratesfit)
else
return(exp(logratesfit))
```

```
}

augfitllcountry<-fitttmx(Kticountry,axicountry,bxicountry, thyr)
augfittllcountry<-t(augfitllcountry)


#creating demogdata object of the fitted rate#
Dxfitllcountry <- matrix(augfittllcountry,
ncol = length(years),
nrow = length(ages))
Nxllcountry <- matrix(countryfor$pop$Female,
ncol = length(years),
nrow = length(ages))

countryfit <- demogdata(data = Dxfitllcountry, pop = Nxllcountry, ages = ages,
years = years, type = "mortality", label = "LABEL", name = "GENDER")

plot(countryfit)


# forecasting of common factor (39 years) to add with country-specific model#

forrthyrcountry<-forecast(funmod2,h=39,jumpchoice = "actual")
dim(forrthyrcountry)
life.expectancy(forecast(funmod2,h=39))

#forcast of common factor#
lioncountry<-outer(forrthyrcountry$kt.f$mean,funmod2$bx)
dim(lioncountry)



#adding the common factor with forecast#
althundercountry<-log(thundercountry$rate$Female)
llthundercountry<-exp(althundercountry+t(lioncountry))
```

```
#extracting previous population as dummy for constructing demogdata object for forecast#
countrycom<-extract.years(country, years = 1973:2011)

#creating demogdata for getting forecast#
#ages <- belarus$age
yearsth <- 2012:2050
Dxthcountry <- matrix(llthundercountry,
ncol = length(yearsth),
nrow = length(ages))
Nxforcountry <- matrix(countrycom$pop$Female,
ncol = length(yearsth),
nrow = length(ages))

countryllthunder <- demogdata(data = Dxthcountry, pop = Nxforcountry,
ages = ages, years = yearsth, type = "mortality", label = "LABEL",
name = "GENDER")

plot(countryllthunder)
compare.demogdata(countrycom,countryllthunder, series = "Female")



##  prediction intervavl ##

part1<-e0(forrthyrcountry, PI=TRUE)

part2<-e0(thundercountry, PI=TRUE)

#simulation of future life expectancy for augmented model#
llflife.expectancy<-function (data, series = NULL, e0level, years = data$year,
type = c("period", "cohort"), age, max.age = NULL, PI = FALSE, ...)
{
type <- match.arg(type)
if (is.element("fmforecast", class(data))) {
if (data$type != "mortality")
stop("data not a mortality object")
```

```
hdata <- list(year = data$model$year, age = data$model$age,
type = data$type, label = data$model$label,
lambda = data$lambda)
hdata$rate <- list(data$model[[4]])
if (min(hdata$rate[[1]], na.rm = TRUE) < 0 | !is.null(data$model$ratio))
hdata$rate <- list(InvBoxCox(hdata$rate[[1]], data$lambda))
if (type == "cohort") {
hdata$year <- c(hdata$year, data$year)
hdata$rate <- list(cbind(hdata$rate[[1]], data$rate[[1]]))
}
names(hdata$rate) <- names(data$model)[4]
if (!is.null(data$model$pop)) {
hdata$pop = list(data$model$pop)
names(hdata$pop) <- names(hdata$rate)
if (type == "cohort") {
n <- ncol(hdata$pop[[1]])
h <- length(hdata$year) - n
hdata$pop[[1]] <- cbind(hdata$pop[[1]], matrix(rep(hdata$pop[[1]][,
n], h), nrow = nrow(hdata$pop[[1]]), ncol = h))
}
}
class(hdata) <- "demogdata"
hdata$rate[[1]][is.na(hdata$rate[[1]])] <- 1 - 1e-05
if (is.null(max.age))
max.age <- max(data$age)
if (missing(age))
age <- min(hdata$age)
x <- stats::window(life.expectancy(hdata, type = type,
age = age, max.age = max.age),
end = max(data$model$year))
xf <- life.expectancy(data, years = years, type = type,
age = age, max.age = max.age)
if (type == "cohort") {
xf <- ts(c(stats::window(x, start = max(data$model$year) -
max.age + age + 1, extend = TRUE), xf),
end = max(time(xf)))
```

```
if (sum(!is.na(xf)) > 0)
xf <- stats::na.omit(xf)
else xf <- stop("Not enough data to continue")
if (min(time(x)) > max(data$model$year) - max.age +
age)
x <- NULL
else x <- stats::window(x, end = max(data$model$year) -
max.age + age)
}
out <- structure(list(x = x, mean = xf, method = "FDM model"),
class = "forecast")
if (is.element("lca", class(data$model)))
out$method = "Coherent LC model"
else if (!is.null(data$product))
out$method = "Coherent FDM model"
if (PI) {
e0calc <- (!is.element("product", names(data$rate)) &
!is.element("ratio", names(data$rate)) & is.null(data$model$ratio))
if (is.null(data$product) & is.null(data$var) & is.null(data$kt.f))
warning("Incomplete information. Possibly this is from a coherent\n
model and you need to pass the entire object.")
else {
sim <- newsimdata
if (e0calc) {
e0sim <- matrix(NA, dim(sim)[2], dim(sim)[3])
simdata <- data
if (type == "cohort")
simdata$year <- min(time(out$mean)) - 1 +
1:dim(sim)[2]
for (i in 1:dim(sim)[3]) {
simdata$rate <- list(as.matrix(sim[, , i]))
names(simdata$rate) <- names(data$rate)[1]
e0sim[, i] <- life.expectancy(simdata, type = type,
age = age, max.age = max.age)
}
e0sim <- e0sim[1:length(xf), , drop = FALSE]
```

```
if (is.element("lca", class(data$model)))
out$level <- e0level
out$lower <- ts(apply(e0sim, 1, quantile, prob = 0.5 -
out$level/200, na.rm = TRUE))
out$upper <- ts(apply(e0sim, 1, quantile, prob = 0.5 +
out$level/200, na.rm = TRUE))
stats::tsp(out$lower) <- stats::tsp(out$upper) <- stats::tsp(out$mean)
}
out$sim <- sim
}
}
return(out)
}
else {
if (!is.element("demogdata", class(data)))
stop("data must be a demogdata object")
if (data$type != "mortality")
stop("data must be a mortality object")
if (is.null(series))
series <- names(data$rate)[1]
if (missing(age))
age <- min(data$age)
return(life.expectancy(data, series = series, years = years,
type = type, age = age, max.age = max.age))
}
}


#for 80% prediciton interval#
Fe0dag80<-llflife.expectancy(thundercountry, e0level=80, age=0, PI=TRUE)
#for 95% prediciton interval#
Fe0dag95<-llflife.expectancy(thundercountry, e0level=95, age=0, PI=TRUE)


## variance explained ##
```

```
# for lc-edagger#
funmod2$varprop

# for coherent method#

numataus<-(log(country$rate$Female) - log(countryfit$rate$Female))^2

denomataus<-(sweep(log(country$rate$Female), 1, llcountry$ax))^2

varexpl <- 1-sum(numataus)/sum(denomataus)
```

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Current Position

Since October 2015; (expected completion: March 2019)
PhD Student in Statistical Sciences, University of Padua.
Thesis title: Modified Lee-Carter Methods with LASSO type Smoothing and Adjusting for Lifespan Disparity
Supervisor: Prof. Stefano Mazzuco

Research interests

- Mortality and Longevity
- Multivariate Techniques
- Fertility

Education

September 2014 – July 2015
European Doctoral School of Demography (2014-15).
Max Planck Institute for Demographic Research (Rostock, Germany) and Warsaw School of Economics (Warsaw, Poland)
Funded by: Max Planck Institute for Demographic Research
Supervisor: Prof. Trifon Missov and Prof. Magdalena Muszyńska
Final grade: B (on average in all courses)

November 2008 – April 2010
Master degree in Statistics, Biostatistics and Informatics.
University of Dhaka, Department of Statistics, Biostatistics and Informatics
Title of dissertation: “Birth Spacing and Fertility in Bangladesh”
Supervisor: Prof. Shahadat Ali Mallick and Prof. Shamal Chandra Karmaker
Final mark: 1st Class (Exam held in 2009, Result published at 2010)

July 2003 – October 2008
Bachelor degree in Statistics.
University of Dhaka, Department of Statistics
Final mark: 1st Class (Exam held in 2008, Result published at 2008).

Visiting period

March 2018 – June 2018
School of Demography, Australian National University
Canberra, Australia.
Supervisor: Prof. Heather Booth and Prof. Vladimir Canudas-Romo

Work experience

March 2012 – August 2014
Department of Mathematics and Statistics, Bangladesh University of Textiles.
Lecturer of Statistics (on study leave since August 25, 2014).
Course Taken: Business Statistics, Bangladesh Studies (Part A-Demography).

February 2011 – March 2012
Department of Mathematics and Statistics, Bangladesh University of Business and Technology.
Lecturer of Statistics.
Course Taken: Business Statistics, Stochastic Process.
Course Designed: Statistical Data Analysis

August 2010 – February 2011
Department of Quantitative Sciences, IUBAT-International University of Business, Agriculture and Technology.
Faculty of Statistics.
Course Taken: Business Statistics.

Awards and Scholarship

- Scholarship from University of Padua for PhD in Cycle XXXI (2015-18)
- Awarded scholarship from UNFPA for presenting paper in 3nd APA Conference in Kuala Lumpur, Malaysia-2015
- InGRID transnational visiting grant for IECM at CED, Barcelona, Spain-2015
- Scholarship from Max Planck Institute for Demographic Research for European Doctoral School of Demography (2014-15)
- Partial scholarship for the 2nd APA Conference for proceeding in Bangkok, Thailand-2012

Computer skills

- **Programming languages:** R (Good), Python (Basic)
- **Statistical Softwares**
  - SPSS (Trained user)
  - Stata (Basic)
- **Operating system:** Windows, Ubuntu (Linux)
- **Document preparation:** LaTeX, MS Office

Language skills

Bangla: native ; English: fluent.

Publications (Since October, 2015)

Articles in journals

Fazle Rabbi, A.M., Mazzuco, S. (2018). Mortality and Life Expectancy Forecast for (Comparatively) High Mortality Countries. *Genus* **74**, 18.

Fazle Rabbi, A.M., Kabir, M., Kabir, R. (2018). What Went Wrong with the Achievement of Replacement Fertility in Bangladesh and Its Consequences on the Demographic Dividend: The Role of Proximate Determinants? *Romanian Journal of Population Studies* **XII(1)**, 103-126.

Islam, M.S., Tareque, M.I., Mondal, M.N.I., Fazle Rabbi, A.M., Khan, H.T.A., Begum, S. (2017). Urban-rural differences in disability-free life expectancy in Bangladesh using the 2010 HIES data. *PLOS ONE* **12(7)**, e0179987.

Fazle Rabbi, A.M., Mazzuco, S., Booth, H., Canudas-Romo, V. Coherent Mortality Forecasting with Modified Lee-Carter Method: Adjusting for Smoothing and Lifespan Disparity (In preparation).

Conference presentations

Fazle Rabbi, A.M., Mazzuco, S. (2018). A Modified Lee-Carter Method Adjusting for Smoothing and Lifespan Disparity. (oral) *European Population Conference*, Brussels, Belgium, June 2018.

Fazle Rabbi, A.M., Kabir, M. (2017). What went wrong with the Achievement of Replacement Fertility in Bangladesh and Its Consequences on the Demographic Dividend: The Role of Proximate Determinants? (poster) *28th International Population Conference of IUSSP*, Capetown, South Africa, October-November 2017.

Fazle Rabbi, A.M., Mazzuco, S. (2017). Mortality and Life Expectancy Forecast for (Comparatively) High Mortality Countries. (poster) *28th International Population Conference of IUSSP*, Capetown, South Africa, October-November 2017.

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Fazle Rabbi, A.M., Mazzuco, S. (2017). Mortality Decline in Bangladesh: Age-Sex Specific Differences. (poster) *12th Population Days, AISP*, Florence, Italy, February 2017.

Fazle Rabbi, A.M. (2016). Mortality transition in Bangladesh. (oral) *European Population Conference*, Mainz, Germany, August-September 2016.

Other teaching experience

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Invited lecture on "Proximate Determinants of Fertility"
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