Doctoral School of General and Quantitative Economics

THESIS BOOK

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Pairwise Comparison Matrices in Multi-Criteria Decision Making

Ph.D. dissertation

Supervisor:

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Department of Operations Research and Actuarial Sciences

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Contents

Chapter 1

Research literature and the justification of the topic

A decision problem arises whenever we have to choose between multiple alternatives. The goal can be to choose the best alternative, or to give a ranking of the possible alternatives. Sometimes the problem can be simplified to a decision with only one criterion, for example a company might want to only consider profit in a particular decision. In these cases we have a single-criterion decision problem, in other words, we have to minimize or maximize an objective function, which can be solved with the traditional tools of operations research. But often even a criterion seemingly as simple as profit may not be simplified to a single criterion, as there can be many influencing factors. If such a simplification is not possible, we are facing a multi-criteria decision problem.

In everyday decisions we do not utilize extensive methodology to make a small decision, because the time and possibly the resource demand would be too high. In these situations we decide swiftly based on established patterns and heuristics. When facing bigger and more important decisions though, it may be worthwhile to use such a well-founded methodology, which lets us analyze and evaluate the problem in smaller parts. For this though, one

needs more powerful tools than everyday heuristics.

The field of Multi-Criteria Decision Making (MCDM) independently of the exact method used is about modeling the preferences of the decision maker. In such complex problems the decision maker is generally not able to accurately take so many criteria into consideration, and directly and precisely determine the importance of criteria, to in the end have a decision that most accurately reflects his own subjective preferences. We are able to aid the decision maker in this by using decision making tools, hence this field is also called Multi-Criteria Decision Aid (MCDA, which sometimes stands for Multi-Criteria Decision Analysis). MCDM is not just about aiding a single decision maker in one decision, it is also about for example group decisions, decisions in a stochastic environment and ranking in other situations. Furthermore, there is a substantial overlapping of themes with the fields of voting theory and social choice.

The evaluations of alternatives or the weights of criteria are often not available as exact numerical values, only the estimates of their ratios can be obtained directly. For example a decision maker can rarely provide accurate information about how much weight does a criterion carry in his decision. The ratios of the weights of criteria can generally be better estimated by the decision maker. In this case, the question the decision maker has to answer is how many times is a criterion more important than an other one. Hence, the comparisons are cardinal, the answers to the question are numerical values. From these ratios in the case of n elements to be compared, an $n \times n$ pairwise comparison matrix (PCM) can be constructed. If cardinal transitivity holds for a PCM, it is called *consistent*.

Thus, the goal is to determine the weights of criteria using the pairwise comparisons of criteria, more precisely to determine the estimates of the weights, which are arranged in a vector, the so called *weight vector*. The weight vector is viewed as the final estimate of the preferences (of criteria) of the decision maker. There are many ways to determine the weight vector.

The Eigenvector Method (EM) is the oldest method to determine the weight vector. It was introduced by Saaty together with PCMs in his paper from 1977 [19]. The Eigenvector Method determines the right principal eigenvector as the weight vector.

Pairwise comparison matrices were first introduced in conjunction with The Analytic Hierarchy Process (AHP), proposed by Saaty [19, 20], and it is still the most frequent field of their application to this day. The popularity of the AHP is thanks to its simplicity, the method of pairwise comparisons, and to its structure and the possibility to divide criteria into subcriteria.

Pareto-efficiency or Pareto-optimality (or simply efficiency) is a core concept of economics. It means a distribution, activity etc. cannot be trivially improved, or in other words, there can be no improvement in anyone's or anything's status without worsening the status of somebody or something else. It is also possible to define the Pareto-efficiency of a weight vector corresponding to a pairwise comparison matrix. A weight vector corresponding to a PCM is efficient, if it is not possible to improve the approximation of a matrix element by changing the elements of the weight vector without worsening the approximation of an other element of the matrix. Pareto-efficiency is a natural requirement. However, Blanquero, Carrizosa and Conde showed, that the principal right eigenvector, which is the weight vector of the Eigenvector Method is not always efficient [5, Section 3]. Bozóki [6] also showed, that efficiency is also not dependent on the extent of inconsistency. It was also showed by Blanquero, Carrizosa and Conde, that Pareto-efficiency is equivalent to the strong connectivity of a directed graph which can be determined from the matrix and the corresponding weight vector.

Consider a consistent pairwise comparison matrix, which is modified in one element and its reciprocal. Thus we get a pairwise comparison matrix which differs from a consistent one in only one element (and its reciprocal), which is called a simple perturbed PCM. Farkas [12] investigated simple perturbed PCMs, but not from an efficiency point of view.

There are instances when not all pairwise comparisons are available, only a subset of them. In other instances it is not possible or desired to ask all $\binom{n}{2}$ comparisons of the decision maker. In these cases only some of the elemets of a PCM will be filled in, the others will be missing. In this case we have an incomplete pairwise comparison matrix (IPCM) [14]. The extension of the Eigenvector Method proposed by Shiraishi, Obata and Daigo [21] to incomplete pairwise comparison matrices is one of the most important methods to determine the weight vector, which also gives a completion which is optimal according to the CR inconsistency index.

Bozóki, Fülöp and Rónyai [7] have proved that the connectedness of a graph corresponding to the IPCM is a necessary and sufficient condition for the unique existence of the (optimal) completion according to the Eigenvector Method mentioned above. It was also them who proposed a method using the method of cyclic coordinates [17][page 253–254] to determine the optimal completion.

It is true in the case of complete PCMs that determining the dominant eigenvalue and eigenvector (in other words the weight vector according to the Eigenvector Method) is a slow process. Regarding this problem, Fülöp [13] proposed a fast algorithm. This method has a similar base to the algorithm introduced in Chapter 5.2. in the dissertation, but it does not use cyclic coordinates.

Chapter 2

Applied methods

A more detailed introduction of the methodology of pairwise comparison matrices is found in the 2nd Chapter of the dissertation. Here only the definitions, theorems, notations and methods necessary to the new results are introduced. The motivation of the introduction of pairwise comparison matrices and weight vectors is detailed above.

2.1 Pairwise comparison matrices

Definition 1. The $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in \mathbb{R}_+^{n \times n}$ matrix is a pairwise comparison matrix (PCM) if

1. $a_{ij} > 0$ and

2. $a_{ij} = 1/a_{ji}$,

for all $i, j = 1, \ldots, n$ pair of indices.

From the second property it follows, that $a_{ii} = 1$. The set of $n \times n$ PCMs is denoted $\mathcal{PCM}_n$.

An $\mathbf{A} \in \mathcal{PCM}_n$ PCM can thus be written in the following general form:

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & 1 & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & 1 & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & 1 \end{pmatrix}. \tag{2.1}$$

If cardinal transitivity holds for a PCM, it is called *consistent*.

Definition 2. The $\mathbf{A} \in \mathcal{PCM}_n$ PCM is consistent if

$$a_{ik}a_{kj} = a_{ij} \tag{2.2}$$

for all $i, j, k = 1, \ldots, n$ indices.

Thus, in the case of a consistent PCM, if for example criterion A is twice more important than criterion B, and B is 3 times more important than C, then A is 6 times more important than C. The set of consistent $n \times n$ PCMs will be denoted $\mathcal{PCM}_n^*$. A PCM is called *inconsistent* if it is not consistent.

Like mentioned before, there are instances when not all pairwise comparisons are available, only a subset of them. In other instances it is not possible or desired to ask all $\binom{n}{2}$ comparisons of the decision maker. In these cases only some of the elemets of a PCM will be filled in, the others will be missing. In this case we have an incomplete pairwise comparison matrix (IPCM) [14]:

Definition 3. $\mathbf{A}$ is an *incomplete pairwise comparison matrix*, if it has the following form:

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & - & \ldots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \ldots & - \\ - & 1/a_{23} & 1 & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & - & 1/a_{3n} & \ldots & 1 \end{pmatrix},$$

where in the positions with dashes are the missing elements, and $a_{ij} > 0$.

The object in Definition 3 is not a matrix, thus it cannot be formally handled that way either. This can be circumvented by writing variables in place of the missing elements, while taking reciprocal symmetry into account. This way, the number of variables will be equal to the number off missing elements in the upper triangle.

There is a simple graph representation of IPCMs that will be important in calculating weight vectors. The nodes of the graph correspond to the criteria, and the edges correspond to the non-missing elements of the matrix, which are the pairs of criteria which have been compared. Formally:

Definition 4. The $G_{\mathbf{A}}(V, E)$ undirected graph corresponding to the $n \times n$ IPCM $\mathbf{A}$ is defined as follows:

$$V = \{1, \ldots, n\},$$

$$E = \{e(i, j) | a_{ij} \text{ (and } a_{ji}) \text{ are known, } i \neq j\}.$$

As a special case, the graph corresponding to a complete PCM is the complete graph K_n. For an example of the graph representation see the full length thesis.

Two of the new results is about simple and double perturbed pairwise comparison matrices, thus being familiar with the following definitions is necessary.

Definition 5. A PCM $\mathbf{A}_\delta \in \mathcal{PCM}_n$ is simple perturbed if it can be rearraged by row and column swaps to the following form:

$$\mathbf{A}_\delta = \begin{pmatrix} 1 & \boldsymbol{x_1\delta} & x_2 & \ldots & x_{n-1} \\ \frac{\boldsymbol{1}}{\boldsymbol{x_1\delta}} & 1 & \frac{x_2}{x_1} & \ldots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \ldots & \frac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \ldots & 1 \end{pmatrix}, \tag{2.3}$$

where $0 < \delta \neq 1$ and $x_1, x_2, \ldots, x_{n-1} > 0$.

Definition 6. A PCM $\mathbf{A}_\delta \in \mathcal{PCM}_n$ is double perturbed if it can be rearraged by row and column swaps to one of the following forms:

Case 1 ($n \geq 4$):

$$
\mathbf{P}_{\gamma,\delta} =
\begin{pmatrix}
1 & \boldsymbol{\delta x_1} & \boldsymbol{\gamma x_2} & x_3 & \ldots & x_{n-1} \\
1/(\boldsymbol{\delta x_1}) & 1 & x_2/x_1 & x_3/x_1 & \ldots & x_{n-1}/x_1 \\
1/(\boldsymbol{\gamma x_2}) & x_1/x_2 & 1 & x_3/x_2 & \ldots & x_{n-1}/x_2 \\
1/x_3 & x_1/x_3 & x_2/x_3 & 1 & \ldots & x_{n-1}/x_3 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1/x_{n-1} & x_1/x_{n-1} & x_2/x_{n-1} & x_3/x_{n-1} & \ldots & 1
\end{pmatrix}, \quad (2.4)
$$

Case 2 ($n \geq 4$):

$$
\mathbf{R}_{\gamma,\delta} =
\begin{pmatrix}
1 & \boldsymbol{\delta x_1} & x_2 & x_3 & x_4 & \ldots & x_{n-1} \\
1/(\boldsymbol{\delta x_1}) & 1 & x_2/x_1 & x_3/x_1 & x_4/x_1 & \ldots & x_{n-1}/x_1 \\
1/x_2 & x_1/x_2 & 1 & \boldsymbol{\gamma x_3/x_2} & x_4/x_2 & \ldots & x_{n-1}/x_2 \\
1/x_3 & x_1/x_3 & \boldsymbol{x_2/(\gamma x_3)} & 1 & x_4/x_3 & \ldots & x_{n-1}/x_3 \\
1/x_4 & x_1/x_4 & x_2/x_4 & x_3/x_4 & 1 & \ldots & x_{n-1}/x_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1/x_{n-1} & x_1/x_{n-1} & x_2/x_{n-1} & x_3/x_{n-1} & x_4/x_{n-1} & \ldots & 1
\end{pmatrix},
$$

$$(2.5)$$

Here $x_1, \ldots, x_{n-1} > 0$ and $0 < \delta, \gamma \neq 1$.

Case 2 had to be separated into two separate subcases 2.A ($n = 4$) and 2.B ($n \geq 5$) due to algebraic issues.

2.2 Methods for determining the weight vector

The weight vector can be determined from a pairwise comparison matrix by several methods, of which the dissertation details several. Understanding the new results requires familiarity with two of them. The first one is the Eigenvector Method [19], which is the oldest, and one of the most popular

methods to determine the weight vector. The second one is the Logarithmic Least Squares Method [9, 10, 11, 18], which is also popular and has many useful properties.

2.2.1   Eigenvector Method

Because a PCM is positive, following from the Perron–Frobenius theorem it has a unique largest real eigenvalue (called the principal eigenvalue), and the elements of the corresponding eigenvector can be chosen as positive values. From now on, the principal eigenvalue will be denoted λ_{max}. The Eigenvector Method determines the right principal eigenvector (corresponding to λ_{max}) as the weight vector, which will be denoted $\mathbf{w}^{EM}$. Thus, for $\mathbf{w}^{EM}$, the following holds:

$$\mathbf{A}\mathbf{w}^{EM} = \lambda_{max}\mathbf{w}^{EM}. \tag{2.6}$$

The degree of inconsistency of a PCM can be measured by inconsistency indices. Many such indices can be found in the literature. However, the oldest one, and at the same time one of the most popular ones is the CR (Consistency Ratio) inconsistency index, which is also introduced by Saaty [19], and is as old as the AHP, and it is also closely connected to the Eigenvector Method.

Because for the principal eigenvalue of a PCM $\lambda_{max} \geq n$ holds, and this relation is true with equality if and only if the PCM is consistent, the value of λ_{max} can be used to define an inconsistency index. However, the value of λ_{max} is dependent on the dimension of the PCM (which is the number of criteria), namely n. Because of this, some kind of normalization is needed. This is the reason for the form of the first value to be calculated, the CI (Consistency Index):

$$CI = \frac{\lambda_{max} - n}{n - 1}.$$

However, CI is still not adequate to compare PCMs of different size, because the average CI value for larger (randomly generated) matrices is

larger. Therefore, even this value has to be further normalized to get the CR index. For this, CR is calculated for a multitude of $n \times n$ PCMs. The average of these is called RI-nek (Random Index), which is dependent on n. Thus, RI will be a real number for each n, which can be recorded in a table (see for example [22, Table 1]).

To extend the Eigenvector Method to incomplete pairwise comparison matrices, based on the idea of Shiraishi, Obata and Daigo [21], the completion which gives the lowest CR inconsistency is used to calculate the weight vector by the eigenvector method. Because we consider the weight vector corresponding to an exact completion, this method has the good property of not only providing a weight vector, but a completion as well.

Because the CR index is a linear transformation of the principal eigenvalue, their minimization is equivalent. Formally, in case of the IPCM $\mathbf{A}(\mathbf{x})$ the following problem is to be solved:

$$\min_{\mathbf{x} > \mathbf{0}} \lambda_{\max}(\mathbf{A}(\mathbf{x})). \tag{2.7}$$

The eigenvector corresponding to the minimal $\lambda_{\max}$ in (2.7) will be the weight vector provided by the eigenvector method for the incomplete case. Vector $\mathbf{x}$, where the minimum is attained, will provide the optimal completion.

However, problem (2.7) does not always have a unique solution, as the connectedness of the graph is a necessary condition. Bozóki, Fülöp and Rónyai [7] have proved, that it is also a sufficient condition.

2.2.2 Logarithmic Least Squares Method

The so called Least Squares Method (LSM) [8] minimizes the distance of the ratios of the weight vector from the matrix elements in quadratic norm, which is intuitive, but suffers from several problems. The modification of the LSM is the ogarithmic Least Squares Method (LLSM), which compares the logarithms of the ratios of the matrix elements with the logarithms of the ratios of the weight vector [9, 10, 11, 18]. Formally the LLSM gives that

$\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ vector as weight vector which is the solution to the following optimization problem:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\log a_{ij} - \log \frac{w_i}{w_j} \right)^2 \qquad (2.8)$$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i > 0, \quad i = 1, \ldots, n.$$

Unlike the LSM, in the LLSM there is always a unique solution: the solution of problem (2.8) is the vector consisting of the geometric mean of the rows of the matrix [9]:

$$w_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}, \quad i = 1, \ldots, n,$$

with an appropriate normalization.

The LLSM is extended, as expected, by considering the (2.8) objective function only for those elements which are not missing. This way we arrive at the Incomplete Logarithmic Least Squares Method (ILLSM). Thus,

$$\min \sum_{\substack{i,j=1 \\ a_{ij} \text{ is given}}}^{n} \left(\log a_{ij} - \log \frac{w_i}{w_j} \right)^2 \qquad (2.9)$$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i > 0, \quad i = 1, \ldots, n.$$

The condition a_{ij} being given is equivalent to that in the corresponding graph $G_{\mathbf{A}}(V, E)$, $(i, j) \in E$.

Bozóki, Fülöp and Rónyai have also proved for the ILLSM that the problem (2.9) has a unique solution if and only if the corresponding graph is connected [7]. It was also shown by Bozóki, Fülöp and Rónyai [7], that the ILLSM can be solved by solving a linear equation system.

2.3 Pareto-efficiency

Pareto-efficiency or Pareto-optimality (or simply efficiency) is a core concept of economics. It means a distribution, activity etc. cannot be trivially improved, or in other words, there can be no improvement in anyone's or anything's status without worsening the status of somebody or something else. In the context of PCMs and weight vectors, we can also introduce the definition of efficiency.

Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in \mathcal{PCM}_n$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector (thus $S = \mathbb{R}^n_{++}$, which is the positive orthant), and n be the number of criteria. Let the objective functions be $f_{ij}(\mathbf{w}) := \left| a_{ij} - \frac{w_i}{w_j} \right|$ for all $i \neq j$, thus there are $M = n^2 - n$ objective functions. Hence the following definition:

Definition 7. A positive $\mathbf{w}$ weight vector is *efficient* (or *Pareto-efficient*), if there exists no other positive $\mathbf{w}' = (w_1', w_2', \ldots, w_n')^\top$ weight vector, such that

$$\left| a_{ij} - \frac{w_i'}{w_j'} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i, j \leq n, \text{ and} \qquad (2.10)$$

$$\left| a_{k\ell} - \frac{w_k'}{w_\ell'} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \leq k, \ell \leq n. \qquad (2.11)$$

The above definition states that a weight vector corresponding to a PCM is efficient, if it is not possible to improve the approximation of a matrix element by changing the elements of the weight vector without worsening the approximation of an other element of the matrix.

Efficiency is a naturally required property of a weight vector. However, Blanquero, Carrizosa and Conde showed, that the principal right eigenvector, which is the weight vector of the Eigenvector Method is not always efficient [5, Section 3]. Bozóki [6] also showed, that efficiency is also not dependent on the extent of inconsistency.

Several necessary and sufficient conditions were examined by Blanquero,

Carrizosa and Conde [5], one of which is of crucial importance here. It uses a directed graph representation as follows:

Definition 8. Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in \mathcal{PCM}_n$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ be a positive weight vector. A directed graph $G = (V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ is defined as follows: $V = \{1, 2, \ldots, n\}$ and

$$\overrightarrow{E} = \left\{ \mathrm{arc}(i \to j) \, \bigg| \, \frac{w_i}{w_j} \geq a_{ij}, i \neq j \right\}.$$

It follows from Definition 8 that if $w_i/w_j = a_{ij}$, then there is a bidirected arc between nodes i, j. The result of Blanquero, Carrizosa and Conde using this representation is as follows:

Theorem 1 ([5, Corollary 10]). *Let* $\mathbf{A} \in \mathcal{PCM}_n$. *A weight vector* $\mathbf{w}$ *is efficient if and only if* $G = (V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ *is a strongly connected digraph, that is, there exist directed paths from i to j and from j to i for all pairs of nodes* i, j.

2.4   The method of cyclic coordinates

The method of cyclic coordinates [17][pages 253–254.] is used in the two algorithms presented in the 5th Chapter of the dissertation, which is a numerical optimization method. The essence of the method of cyclic coordinates is that in a multivariate optimization problem only one variable is considered changing at a time, while all other variables are fixed on their previously calculated value. The new value of the actually changing variable will be the number where its optimum value is (with fixed values of all other variables), according to the optimization problem. Which variable is considered actually changing is cyclically changed, thus at first it is the first variable, then the second, etc., then after the last variable is reached we jump back and the next one will be the first again, continuing like this until the stopping condition is reached. The nature of the stopping condition depends on the problem.

2.5 Newton's method

Newton's method is a well known extreme value searching algorithm for uni- and multivariate cases. In the univariate case in the rth iteration, the form of the algorithm for a general function $f(x)$ is as follows:

$$x^{(r+1)} = x^{(r)} - \frac{f'(x^{(r)})}{f''(x^{(r)})},$$

where $x^{(r)}$ is the vale of x computed in iteration r.

In the multivariate case, let $L(\mathbf{t})$ be the function which is to be minimized. The multivariate Newton's method in this case is

$$\mathbf{t}^{(r+1)} = \mathbf{t}^{(r)} - \gamma[HL(\mathbf{t}^{(r)})]^{-1}\nabla L(\mathbf{t}^{(r)}),$$

where $HL(\mathbf{t})$ is the Hessian matrix of $L(\mathbf{t})$, $\nabla L(\mathbf{t})$ is its gradient vector and γ is a step size parameter usual in Newton's method. This step size parameter can be used in the univariate case as well.

2.6 The Collatz–Wielandt formula

The original theorem is about irreducible matrices (it is stated like that in the dissertation as well), but in the case of the new results we assume having a positive matrix, which is a special case of irreducible. Because a PCM is always positive, the theorem can be applied to it.

Theorem 2 (Collatz–Wielandt). *Let $\mathbf{A} \geq 0$ be a positive (or more generally an irreducible) $n \times n$ matrix.*

$$\lambda_{\max} = \max_{\mathbf{w} > 0} \min_{i=1,\dots,n} \frac{(\mathbf{A}\mathbf{w})_i}{w_i} = \tag{2.12}$$

$$= \min_{\mathbf{w} > 0} \max_{i=1,\dots,n} \frac{(\mathbf{A}\mathbf{w})_i}{w_i}. \tag{2.13}$$

Chapter 3

New results of the dissertation

3.1 The logarithmic least squares method and the optimal completion

This Proposition can be found in subsection 2.5.2, in the introduction part of the dissertation.

In case of IPCMs, an optimal completion may be required. The EM directly provides a completion which is optimal (minimal) for the CR index. The LLSM however calculates a weight vector without completing the matrix first. It is possible to complete the matrix by writing the appropriate ratios of the weight vector into the missing positions, but it is not apparent if this will be optimal.

In case of the EM, if we recalculate a weight vector from the completed matrix, it will naturally be the same as before. This is the condition the completion with the LLSM ratios has to satisfy, in order to the resulting completion to be optimal with regard to the objective function. The following Proposition is a new result, the statement is from the author, while the proof is from Sándor Bozóki.

Proposition 1. *The Incomplete Logarithmic Least Squares Method (ILLSM)*

provides an optimal completion by substituting $a_{ij} = \frac{w_i^{ILLSM}}{w_j^{ILLSM}}$ in the missing positions.

The proof is not presented here due to space limitations, but it can be found right after the Proposition in the dissertation.

3.2 The Eigenvector Method and Pareto-efficiency

As mentioned earlier, in the case of the Eigenvector Method, the Pareto-efficiency of the weight vector cannot be guaranteed. However, in the case of the two special cases introduced earlier, which are also important in practical applications, the efficiency of the weight vector has been proven.

Theorem 3 ([2, Theorem 3.4])**.** *The principal right eigenvector of a simple perturbed pairwise comparison matrix is efficient.*

A proof using elementary calculations, and a second proof using Theorem 1 can be found in section 4.1 in the dissertation as well as in the [2] article.

Theorem 4 ([3, Theorem 3])**.** *The principal right eigenvector of a double perturbed pairwise comparison matrix is efficient.*

The quite lengthy proof can be found in its entirety in article [3], as well as a summary in section 4.2 of the dissertation.

3.3 Computing the eigenvector by cyclic coordinates

In this topic two new algorithms are presented. The first one provides an optimal completion, according to the Eigenvector Method, of incomplete pairwise comparison matrices, by Newton's method [1]. The second one is a general method to compute the principal eigenvector and eigenvalue for

positive matrices [4]. The method of cyclic coordinates [17][pages 253–254.] is utilized in both cases.

3.3.1   Optimal completion with Newton's method

The details of the new algorithm presented here can be found in section 5.1. of the dissertation, as well as in article [1]. The goal is to find the completion which corresponds to the minimal eigenvalue, thus our objective function is the principal eigenvalue which has the missing elements as variables. Utilizing Newton's method for the cyclic coordinates starts with setting all the missing elements in the upper triangle of the IPCM to a starting value. Then, by the method of cyclic coordinates, the missing elements are taken one by one and treating only the currently chosen missing element as an actual variable, while fixing all others on their value calculated in the previous step, finding the minimum is done with Newton's method.

Unfortunately optimizing $\lambda_{\max}$ directly in the missing elements is a non-convex problem [7]. In order to guarantee the convergence to a unique global minimum, the problem has to be rescaled to a convex optimization problem. As per the idea of Bozóki, Fülöp and Rónyai [7], let $x_i = e^{t_i}$, $i = 1, \ldots, d$. Thus for the matrix $\mathbf{A}(\mathbf{x}) = \mathbf{A}(x_1, \ldots, x_d)$, $\mathbf{B}(\mathbf{t}) = \mathbf{A}(\mathbf{x})$. The principal eigenvalue, $\lambda_{\max}(\mathbf{B}(\mathbf{t}))$ is a convex function of $\mathbf{t}$ [7]. Harker [15] determined the first and second derivatives of the principal eigenvalue by the matrix elements, and they are only dependent on the position (i, j) of the element. These derivatives though are given for the original x_i, $i = 1, \ldots, d$ elements of the matrix, and not t_i. Hence, the derivatives have to be re-scaled as well. The rescaling can found in the dissertation as well as in article [2]. With knowledge of these rescaled derivatives, the univariate Newton's method can be applied with cyclic coordinates (see subsection 5.1.1 of the dissertation), and also a multivariate Newton's method can be applied instead (see subsection 5.1.2 of the dissertation).

3.3.2 Principal eigenvector of positive matrices by cyclic coordinates

The details of this algorithm can be found in section 5.2 in the dissertation, as well as in manuscript [4]. An iterative algorithm is given for computing the principal eigenvector and eigenvalue of positive matrices. This method works in very general cases, but one application is the calculation of the Eigenvector Method for pairwise comparison matrices.

The algorithm uses form (2.13) to approxiamte $\lambda_{\max}$, but this is an arbitrary choice: the algorithm can be easily adapted to use form (2.12). Later however, both forms will be used to give the stopping condition.

The method of cyclic coordinates will be used in this case as well. The variables are the elements of the right principal eigenvector, $\mathbf{w}$: $w_1, \ldots, w_n$. As discussed earlier, cyclic coordinates considers only one variable as a proper variable in each step. Let the index of this variable be denoted by k, thus in every step w_k will be the actual variable, while the values of all other variables are fixed at their values calculated in the previous step.

Thus, as described earlier, in each step we are looking for the value of w_k for which the following is true:

$$w_k = \arg\min_{w_k} \max_{i=1,\ldots,n} \frac{(\mathbf{A}\mathbf{w})_i}{w_i}. \tag{3.1}$$

Because all other w_j, $j \neq k$ values are fixed, for all i (3.1) is only dependent on w_k. Thus the following notation can be introduced:

$$f_i(w_k) = \frac{(\mathbf{A}\mathbf{w})_i}{w_i}, \quad i = 1, \ldots, n. \tag{3.2}$$

Therefore, what we are searching for is the $w_k > 0$ value, for which $w_k = \arg\min_{w_k} \max_{i=1,\ldots,n} f_i(w_k)$, or in other words where the maximum function of f_i has the minimum point. The $f_i(w_k)$ value will be the approximation (upper bound) of $\lambda_{\max}$. It can be shown, that the f_i functions for $i \neq k$ are linear, while for $i = k$, $f_k(w_k)$ is a hyperbolic function. It can also be shown,

that it is sufficient to calculate only the intersection points of f_k with each f_i, $i \neq k$. Because of the strict monotonic descent of f_k, the value of $w_k > 0$ which satisfies (3.1), will be the smallest w_k, which is in the intersection of the hyperbolic and a linear function. The intersection point itself can be calculated by the quadratic formula.

An opportunity for faster running arises if we consider that the calculation of all intersection points is unnecessary. Those linear functions that have no common points with the maximum of the linear functions can be ignored.

The stopping condition is when the estimation of the expression $\min_{\mathbf{w}>0} \max_{i=1,\dots,n} \frac{(\mathbf{A}\mathbf{w})_i}{w_i}$ in the Collatz–Wielandt formula (the minimum point of the maximum function), and the estimation of the expression $\max_{\mathbf{w}>0} \min_{i=1,\dots,n} \frac{(\mathbf{A}\mathbf{w})_i}{w_i}$ (the maximum point of the minimum function) are closer to each other than a predefined threshold, the algorithm stops.

For starting values, any positive vector is acceptable. A possible simple starting value is 1 for all variables, $w_i^{(0)} = 1, i = 1,\dots,n$. In case of PCMs though, the principal eigenvector (the weight vector of the eigenvector method) is close to the row-wise geometric mean (the weight vector for the logarithmic least squares method) [16]. Therefore, the starting values in this case should be

$$w_i^{(0)} = \prod_{j=1}^{n} \sqrt[n]{a_{ij}}. \tag{3.3}$$

This starting value set can also be used in case of general positive matrices.

The algorithm presented above is a new method for calculating the principal eigenvector and eigenvalue, which is tailored for speed on large matrices, and its simplicity comes from the method of cyclic coordinates and the arithmetically simple calculations.

Chapter 4

List of publications

Scientific articles in English

1. K. Ábele-Nagy. Minimization of the Perron eigenvalue of incomplete pairwise comparison matrices by Newton iteration. *Acta Universitatis Sapientiae, Informatica*, 7(1):58–71, 2015.

2. K. Ábele-Nagy and S. Bozóki. Efficiency analysis of simple perturbed pairwise comparison matrices. *Fundamenta Informaticae*, 144:279–289, 2016.

3. K. Ábele-Nagy, S. Bozóki, and Ö. Rebák. Efficiency analysis of double perturbed pairwise comparison matrices. *Journal of the Operational Research Society*, 69(5):707–713, 2018.

Manuscript in English

4. K. Ábele-Nagy and J. Fülöp. On computing the principal eigenvector of positive matrices by the method of cyclic coordinates. *kézirat*, 2019.

Master's theses in Hungarian

5. K. Ábele-Nagy. Nem teljesen kitöltött páros összehasonlítás mátrixok a többszempontú döntésekben. Diplomamunka, Eötvös Loránd Tudományegyetem, 2010.

6. K. Ábele-Nagy. Nem teljesen kitöltött páros összehasonlítás mátrixok aggregálása. MA Szakdolgozat, Budapesti Corvinus Egyetem, 2012.

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