Applications of tropical combinatorics and monomial modules

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Abstract

We study three aspects of tropical combinatorics and monomial modules. In the first, we consider the tropical geometry specifically arising from convergent Puiseux series in multiple indeterminates. One application is a new view on stable intersections of tropical hypersurfaces. Another one is the study of families of ordinary convex polytopes depending on more than one parameter through tropical geometry.

In the second, we consider matching fields and their connections to combinatorial geometry. We show that the Chow covectors of a linkage matching field define a bijection of lattice points, resolving two open questions from Sturmfels & Zelevinsky. We use a similar method to prove that, given a triangulation of a product of two simplices encoded by a set of bipartite trees, the bijection from left to right degree vectors of the trees is enough to recover the triangulation. As additional results, we show a cryptomorphic description of linkage matching fields and characterise the flip graph of a linkage matching field in terms of its prodsimplicial flag complex.

In the third, we study commutative algebra arising from generalised Frobenius numbers. We define generalised lattice modules, a class of monomial modules whose Castelnuovo–Mumford regularity captures the k-th Frobenius number. We study the filtration of generalised lattice modules providing an explicit characterisation of their minimal generators, and show that there are only finitely many isomorphism classes of generalised lattice modules. As a consequence of our commutative algebraic approach, we prove structural results on the sequence of generalised Frobenius numbers and also construct an algorithm to compute them.

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Chapter 1

Introduction

Tropical geometry is a rapidly developing field in the intersection of combinatorics and algebraic geometry, that has shed light on a number of problems in multiple areas of mathematics. It first caught the attention of the wider community via the work of Mikhalkin, who used tropical curve counting methods to solve multiple problems in enumerative algebraic geometry [65]. It has since been used as a tool in many other geometric contexts, such as toric geometry [68], mirror symmetry [47] and nonarchimedean geometry [16].

The strength of tropical geometry comes from taking geometric problems and reframing them as combinatorial problems. This combinatorial framework lends itself to applications outside of geometry, including several real world applications. The most well known is its application to auction theory [17, 51]; in particular, the work of Klemperer on product-mix auctions, and its implementation by the Bank of England during the 2007 global financial crash [58]. However, there are multiple other fields of applied mathematics it has connections to, including machine learning [88], game theory [1] and optimisation. It is this final connection that shall be a running theme throughout this thesis.

Smale's 9th problem is to find a strongly polynomial time algorithm for linear programming [83]. The solution to this problem may be tropical: Benchimol showed there exists a strongly polynomial simplex algorithm if and only if its tropicalisation is polynomial time [24]. Tropical methods have already been used to show certain classes of linear programming algorithms are not strongly polynomial time [6]. Furthermore, many pathological examples of linear programs have a tropical nature. One such example is the Klee-Minty cube, a family of combinatorial cubes on which the simplex method performs exponentially, for which the whole family can be encoded as a single tropical polytope [55]. For these reasons, there is a great deal of research into tropical linear programming and its insights into the complexity of ordinary linear programming. While not explicitly the topic of research, this is a motivation for Chapters 3 and 4.

In the final chapter, we concern ourselves with another subject in the intersection of combinatorics and algebra: combinatorial commutative algebra. The main focus of this field is to compute algebraic and homological invariants of an object via combinatorial methods. This has opened up a vast number of applications that can be tackled via homological methods, some of which have evolved into fields in their own right [86]. We use these algebraic and homological tools to tackle a problem from discrete optimisation: the Frobenius coin problem.

It is worth remarking that tropical geometry and combinatorial commutative algebra are not incomparable areas; both have the underlying goal of reducing problems from algebraic geometry to combinatorial problems. There are very explicit connections between the two areas as well. Develin and Yu [30] showed that we can associate a tropical polytope to a monomial ideal, and that the lifts of this tropical polytope determine a free resolution of the monomial ideal. There have been multiple other examples of constructing free resolutions of ideals from tropical objects [32, 70]. More recently, Joswig and Loho [54] considered monomial tropical cones, a specific class of tropical cone whose behaviour is similar to that of a monomial ideal. In particular, they show monomial tropical cones exhibit duality analogous to Alexander duality for monomial ideals. In an upcoming paper, Loho and the author further explore the connection between monomial tropical cones and homological invariants of monomial ideals.

1.1 Matching fields and lattice points of simplices

Chapter 3 is concerned with the structure of matching fields. A *matching field* is a set of perfect matchings on bipartite node sets $\sigma \sqcup [d]$, one for each d-subset σ of an n-set L. A natural example is the set of weight minimal matchings of size d in a complete bipartite graph $K_{n,d}$ with generic edge weights. Matching fields arising in this way are *coherent*. Without any further requirements, the set of matchings can be arbitrarily unstructured. Our main objects of study will be *linkage matching fields*. They fulfil the additional property that each subset of matchings defined on a $(d+1)$-subset of L is coherent. Sturmfels & Zelevinsky introduced matching fields in [85] to study the Newton polytope of the product of all maximal minors of an $(n \times d)$-matrix of indeterminates $X = (x_{ji})$. The linkage property occurs as a combinatorial description of the determinantal identity [85, Equation 0.1]

$$\sum_{j \in \tau} (-1)^j x_{ji} X_{\tau \setminus \{j\}} = 0 \text{ for all } i \in [d], \tau \in \binom{[n]}{d+1} ,$$

where X_σ is the minor of the rows labelled by a d-subset $\sigma \subseteq [n]$. This is analogous to the motivation of the exchange axiom of a matroid from the Plücker relations.

Linkage matching fields have already proven to be useful in several contexts. The combination of the results in [25, 85] showed that the maximal minors of an $(n \times d)$-matrix of indeterminates form a universal Gröbner basis of the ideal generated by them. They occur in tropical linear algebra, as tropical determinants are just minimal matchings in a weighted bipartite graph, yielding a matching field in the generic case. This was used in [77] to devise a tropical Cramer's rule. Later, avoiding the genericity assumption, a generalisation called 'matching multifields' was employed to examine the structure of the image of the tropical Stiefel map in [38]. Another recent work uses matching fields to find toric degenerations of Grassmannians [67].

A notion of oriented matroid for tropical geometry was introduced in [11]. Similar to the classical case, tropical oriented matroids have multiple cryptomorphisms with other objects. It was shown in [50] that they are equivalent to the subdivisions of a product of two simplices, and to certain unions of linkage matching fields from [72] in the generic case. As linkage matching fields are the building blocks of tropical oriented matroids, and the linkage axiom is analogous to the basis exchange axiom, we propose to consider them as another matroid-like structure for tropical geometry. A variation of tropical oriented matroids, namely 'signed tropical matroids', was recently proposed to develop an abstraction of tropical linear programming [60] analogous to oriented matroid programming. The algorithm in this paper relies on the interplay of the linkage covectors, see Definition 3.2.6, of certain matching fields derived from a triangulation of a product of two simplices.

A collection of certain graphs associated to a matching field was introduced in [85], which we refer to as the *Chow covectors* of a matching field. They have a combinatorial characterisation as the minimal transversals to a linkage matching field, as shown in [25], and are a key combinatorial tool in the proof of the universal Gröbner basis result mentioned earlier. They were initially introduced as 'brackets' to study the variety of degenerate matrices in $\mathbb{C}^{n \times d}$. In particular, they give insight into the Chow form, a polynomial invariant that determines the variety. Sturmfels & Zelevinsky showed that 'extremal' terms of the Chow form can be recovered from the Chow covectors by taking their product as brackets. However, little was known about the structure of Chow covectors and to what extent they determined the matching field. The tools developed in this chapter allows us to answer both of these questions.

This is where Chapter 3 begins. The main tool for our considerations are *topes*, which occur in the context of tropical oriented matroids [11]. We generalise the concept of *matching fields* to *tope fields*. While matching fields comprise a set of matchings, tope fields can be seen as sets of ordered partitions of a varying ground set. We transfer the crucial *linkage property* from matching fields to tope fields derived from a linkage matching field. This allows us to associate *maximal topes* arising from a linkage matching field to the lattice points in a dilated simplex.

Via tope fields we obtain an explicit construction of the Chow covectors. Our approach leads to a representation derived from the maximal topes of a linkage matching field in Proposition 3.2.22. This yields Theorem 3.2.23, resolving Conjecture 6.10 from [85] which was only resolved for coherent matching fields in [25]. Each bipartite graph induces a pair of lattice points, namely its left and right degree vector. The theorem shows that the set of degree vector pairs for all Chow covectors gives rise to a bijection from $(n-d+1)$-subsets of $[n]$ to lattice points in the dilated simplex $(n-d+1)\Delta_{d-1}$. Naturally, one can now ask if this bijection uniquely defines the matching field. This question is a generalisation of [85, Conjecture 6.8 b)] for linkage matching fields. We answer it positively in Theorem 3.2.29. These results allow us to give a cryptomorphic description of linkage matching fields in the form of *tope arrangements*.

A similar claim for triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ was made after [74, Theorem 12.9]. By describing a triangulation as a collection of trees in the sense of [10, Proposition 7.2], one also obtains a set of pairs of lattice points. For a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$, this yields a subset of $\Delta^{\mathbb{Z}}(n, d-1) \times \Delta^{\mathbb{Z}}(d, n-1)$, which denotes the product of the integer lattice points in the dilated simplices $(d-1)\Delta_{n-1}$ and $(n-1)\Delta_{d-1}$. With essentially the same reasoning as in Theorem 3.2.29, we provide an explicit construction of a triangulation from these lattice point pairs. At the same time, this result was proven also for more general root polytopes in [42]. A comparison of their advances based on *trianguloids* is sketched in Remark 3.2.33.

As additional results, we show how the topes of a linkage matching field are encoded in its flip graph. This leads to a characterisation of its prodsimplicial flag complex (see [59, Section 9.2.1]) as the 'complex of topes' in Theorem 3.2.39. Furthermore, we initiate the study of a *combinatorial Stiefel map*, generalising the construction in [38] and [49], motivated by the topes and trees arising from a linkage matching field.

The overview of the chapter is as follows. We define tope fields and related concepts in Section 3.1. In particular, we introduce the relation with triangulations of products of two simplices and further requirements on the topes.

Section 3.2 is dedicated to our results on linkage matching fields. Our fundamental construction for linkage matching fields is presented in Theorem 3.2.16. It leads to the resolution of two conjectures concerning Chow covectors in Theorem 3.2.23 and Theorem 3.2.29. The cryptomorphism between linkage matching fields and tope arrangements in Theorem 3.2.32 as well as the description of the flip graph of a linkage matching field in Theorem 3.2.39 are further consequences.

In Section 3.3, we deduce the reconstructability of a triangulation from a lattice point bijection in Theorem 3.3.2, analogously to the statement for Chow covectors. We finish with the relation to sets of transversal matroids through a combinatorial Stiefel map in Section 3.4. These last two sections contain several questions for further work.

1.2 Tropical geometry of higher rank

Tropical geometry connects algebraic geometry over some valued field $\mathbb{K}$ with polyhedral geometry over the semifield $\mathbb{T} = (\mathbb{R}, \min, +)$. Often it is less important which field $\mathbb{K}$ is chosen, and a common choice is the field $\mathbb{C}\{\{t\}\}$ of formal Puiseux series with complex coefficients. By taking the convergence of series in $\mathbb{C}\{\{t\}\}$ into account, we obtain a *transfer principle* from $\mathbb{T}$ to $\mathbb{C}$ by first pulling back the valuation map $\mathrm{val} : \mathbb{C}\{\{t\}\} \to \mathbb{T}$ and then substituting t by some complex number. Diagrammatically this can be written as

$$\mathbb{T} \xleftarrow[\mathrm{val}]{} \mathbb{C}\{\{t\}\} \dashrightarrow \mathbb{C} \ . \tag{1.1}$$

Notice that the substitution, which is represented by the dashed arrow, depends on the choice of the complex number substituted. This number must lie within the radius of convergence, and so the dashed arrow is *not* a map defined for all Puiseux series. Nonetheless, this opens up a road for transferring metric information from tropical geometry over $\mathbb{T}$ via algebraic geometry over $\mathbb{C}\{\{t\}\}$ to metric information over $\mathbb{C}$. This idea has been exploited recently to obtain new and surprising complexity results for ordinary linear optimisation [5, 6]. The main motivation of this chapter is to explore generalisations of this transfer principle to tropical geometry of higher rank and its applications.

Tropical geometry of higher rank was pioneered in articles by Aroca [12, 13] and Aroca, Garay and Toghani [14]. Their work is motivated by research on algebraic ways of solving systems of differential equations. This gives a natural notion of a tropical hypersurface of higher rank, and allows for a higher rank version [13, Theorem 8.1] of Kapranov's fundamental theorem of tropical geometry [63, Theorem 3.2.5]. Banerjee [18] and Foster and Ranganathan [40, 41] explored this further by enriching the structure of higher rank tropical hypersurfaces, using methods from algebraic and analytical geometry respectively.

One approach to tropical geometry is the process of tropicalising classical algebraic varieties. Here we consider a variety V over some valued field $\mathbb{K}$, and the *tropicalisation* of V is obtained by applying the valuation map to each point of V coordinatewise. The fundamental theorem of tropical geometry states that this agrees with the description of a tropical variety as the set of solutions to certain tropical polynomials [63, Theorem 3.2.5]. While typically $\mathbb{K}$ is assumed to be algebraically closed, it is worthwhile to consider other fields. One can consider real-closed fields, as working over an ordered field has the advantage that the cancellation of terms, which is the source of many technical challenges in tropical geometry, can be controlled via keeping track of the signs. This leads to Alessandrini's work on the tropicalisation of semialgebraic sets [2] and is essential for applications to optimisation as in [5–7]. Another example is [5, Theorem 4.3], concerning the complexity of the simplex method, that hinges on employing convergent real Puiseux series of higher rank. Despite the fact that the basic idea is simple, the algebraic, topological and analytic properties are somewhat subtle.

This is our point of departure for Chapter 4, and in Section 4.1 we begin with a general description of fields of convergent Puiseux series in more than one indeterminate. To a large extent our exposition follows the fundamental work of van den Dries and Speissegger [87]. However, for what we have in mind we also need to prove some minor extensions to their theory, and this may be of independent interest. An example is Proposition 4.1.5 on partial evaluations of convergent Puiseux series of higher rank, and this gives rise to a higher rank analogue (4.7) of the transfer principle (1.1).

With this we are prepared for the main part of Chapter 4, on tropical hypersurfaces of higher rank, which is Section 4.2. For conciseness, we restrict our attention to rank two; yet all statements admit straightforward generalisations to arbitrarily high rank. Our first contributions are Theorem 4.2.13 and its Corollary 4.2.14 which describe the closure in the Euclidean topology of an arbitrary rank two tropical hypersurface in terms of ordinary polyhedra. These results require us to study sets defined by finitely many linear inequalities with respect to the lexicographic ordering on the rank two tropical semiring, which we call *lex-polyhedra*. A key ingredient in the analysis is the diagram (4.15) which is a consequence of the higher rank transfer principle (4.7).

A fundamental fact of tropical geometry is that intersections of tropical varieties do not need to be tropical varieties, in general. This fact gives rise to technical challenges in proofs in tropical geometry, and the concept of *stable intersection* frequently offers a path towards a solution [63, §3.6]. This is the topic of Section 4.3. Theorem 4.3.6, which is a consequence of our main result, allows us to view stable intersection as an instance of the "symbolic perturbation" paradigm from computational geometry [33, 36]. This should be compared with [5, §3.2] and [8, §5], where a similar idea has been applied to obtain perturbations of rank one tropical linear programs; or with the approach to "genericity by deformation" of monomial ideals [66, §6.3].

In Section 4.4 we follow a completely different strand of tropical geometry: tropical convexity. Tropical cones are precisely $(\min, +)$-semimodules and so this has close ties to $(\min, +)$-linear algebra, a well studied field with numerous applications in optimisation, discrete event systems and other areas [15, 26]. Working over real Puiseux series which are convergent allows us to relate three kinds of objects: ordinary cones over real Puiseux series, tropical cones and ordinary cones over the reals, as expressed by the transfer principle (1.1). The core of this section are Theorems 4.4.9 and 4.4.11. The former gives a decomposition for rank two tropical cones analogous to the covector decomposition for rank one tropical cones [53]; the latter is a tropical convexity analogue to our Theorem 4.2.13 on rank two tropical hypersurfaces. In the rest of the section we study a classical construction of Goldfarb and Sit [45] as an example. They constructed a two parameter family of ordinary d-dimensional combinatorial cubes $G^d(t,u)$ which provide difficult input for certain variants of the simplex method. The entire two parameter family of polytopes can be viewed as *one* polytope over the field of convergent Puiseux series in two indeterminates, and thus admits a rank two tropicalisation.

Section 4.5 ends this article with several concluding remarks. In particular, we hint at generalising our results from rank two to arbitrary rank.

1.3 Commutative algebra of generalised Frobenius numbers

Consider a collection $(a_1,\ldots,a_n)$ of natural numbers such that $\gcd(a_1,\ldots,a_n)=1$. The Frobenius number $F(a_1,\ldots,a_n)$ is the largest natural number that cannot be expressed as a non-negative integral linear combination of $a_1,\ldots,a_n$. Note that this simple problem is surprisingly hard to solve: for $n>2$, there is no "closed form" expression for the Frobenius number, and is NP-hard to compute [75]. It has been studied extensively from several viewpoints including discrete geometry [57], analytic number theory [21] and commutative algebra [78]. Furthermore, it is intimately connected to the integer knapsack problem from integer linear programming, opening up many connections to combinatorial optimisation. There is a vast literature on the Frobenius number, we refer to Alfonsín's book [76] for more information.

The Frobenius number can be rephrased in the language of lattices as follows [81]. We start by letting $L(a_1,\ldots,a_n)$ be a sublattice of the dual lattice $(\mathbb{Z}^n)^\star$ of points that evaluate to zero at $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. The Frobenius number is precisely the largest integer r such that there exists a point $\mathbf{p} \in (\mathbb{Z}^n)^\star$ that evaluates to r at $(a_1, \ldots, a_n)$ and $\mathbf{p}$ does not dominate any point in $L(a_1, \ldots, a_n)$. Here the domination is according to the partial order induced by the standard basis on $(\mathbb{Z}^n)^\star$.

This leads to the following commutative algebraic interpretation of the Frobenius number. Let $\mathbb{K}$ be an arbitrary field and let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients in $\mathbb{K}$. The *lattice module* M_L is the S-module generated by Laurent monomials $\mathbf{x}^{\mathbf{w}}$ over all $\mathbf{w} \in L$. In this framework, the Frobenius number is an invariant of the lattice module.

Theorem 1.3.1. *[20], [78] The Frobenius number $F(a_1, \ldots, a_n)$ is*

$$\mathrm{reg}(M_L) + n - 1 - \sum_{i=1}^{n} a_i$$

where $\mathrm{reg}(M_L)$ *is the Castelnuovo–Mumford regularity of M_L.*

Recently, the following generalisation of the Frobenius number called the k-th Frobenius number has been proposed [22]. Fix a natural number k and a collection $(a_1, \ldots, a_n)$ of natural numbers such that $\gcd(a_1, \ldots, a_n)$ is equal to 1. The k-th Frobenius number $F_k(a_1, \ldots, a_n)$ is the largest natural number that cannot be written as k distinct non-negative integral linear combinations of $a_1, \ldots, a_n$. Hence, the first Frobenius number $F_1(a_1, \ldots, a_n)$ is the Frobenius number of $(a_1, \ldots, a_n)$.

Work on generalised Frobenius numbers has primarily used analytic methods and methods from polyhedral geometry. The work of Beck and Robins [22] uses analytic methods to derive an explicit formula for $n = 2$ and any k. Aliev, Fukshanksy and Henk [4] give bounds for $n > 2$, generalising a theorem of Kannan for the first Frobenius number. They relate the k-th Frobenius number to the k-covering radius of a simplex with respect to the lattice $(a_1, \ldots, a_n)^\perp \cap \mathbb{Z}^n$, giving bounds on the generalised Frobenius number as a corollary. Recent work of Aliev, De Loera and Louveaux [3] takes a more algebraic approach by considered the semigroup of integers with at least k distinct representations.

The goal of Chapter 5 is to develop commutative algebra arising from the k-th Frobenius number. We start by generalising Theorem 1.3.1 to k-th Frobenius numbers. The k-th Frobenius number has an analogous description in the language of lattices. Explicitly, it is the largest integer r such that there exists a point $\mathbf{p} \in (\mathbb{Z}^n)^\star$ that evaluates to r at $(a_1, \ldots, a_n)$ and $\mathbf{p}$ does not dominate k distinct points in $L(a_1, \ldots, a_n)$. This leads to the following generalisation of the lattice module M_L:

Definition 1.3.2. *The k-th lattice module $M_L^{(k)}$ is the S-module generated by Laurent monomials $\mathbf{x}^{\mathbf{w}}$ such that $\mathbf{w}$ dominates at least k lattice points.*

Proposition 1.3.3. *The k-th Frobenius number of $(a_1, \ldots, a_n)$ is given by the formula:*

$$\operatorname{reg}(M_L^{(k)}) + n - 1 - \sum_{i=1}^{n} a_i$$

where $\operatorname{reg}(M_L^{(k)})$ is the Castelnuovo–Mumford regularity of $M_L^{(k)}$.

While Proposition 1.3.3 provides a simple description of generalised Frobenius numbers in terms of $M_L^{(k)}$, we know very little about the structure of generalised lattice modules. For instance, they are naturally related by the filtration:

$$M_L^{(1)} \supseteq M_L^{(2)} \supseteq M_L^{(3)} \ldots \tag{1.2}$$

which is not captured in Proposition 1.3.3. Furthermore, for $k > 1$ we have no description of their minimal generating sets. With this in mind, we delve into a detailed study of generalised lattice modules and their impact on Frobenius numbers.

Chapter 5 is organised as follows. Section 5.1 derives the formulas given in Theorem 1.3.1 and Proposition 1.3.3. Section 5.2 begins an in-depth investigation of generalised lattice modules, in particular we give two structural theorems regarding their minimal generators. Theorem 5.2.8, the Neighbourhood Theorem, shows that all minimal generators occur within an explicit finite neighbourhood of $1_{\mathbb{K}} = \mathbf{x}^{\mathbf{0}}$, while Theorem 5.2.5 gives an inductive characterisation of the minimal generators of $M_L^{(k)}$ in terms of the first syzygies of $M_L^{(k-1)}$. These two results together give a method for computing the minimal generators of $M_L^{(k)}$.

Given these characterisations of the minimal generators of $M_L^{(k)}$, a natural next question is a characterisation of the syzygies of $M_L^{(k)}$. This is the subject of Section 5.3. As a first result in this direction, Theorem 5.3.1 shows there are only finitely many distinct Betti tables for generalised lattice modules of L. We use this to derive Corollary 5.3.9, a structural result for k-th Frobenius numbers. Explicitly, it states there exist a finite set of integers $\{b_1, \ldots, b_t\}$ derived from the Betti tables such that

$$F_k(a_1, \ldots, a_n) = m_k + b_j$$

for some b_j, where m_k is the smallest degree of an element in $M_L^{(k)}$.

With these structural results of $M_L^{(k)}$, in Section 5.4 we turn our attention to the following question:

Problem 1.3.4. (Classification of Frobenius Number Sequences) Given a sequence of natural numbers $\{c_k\}_{k=1}^{\infty}$, does there exist a collection of naturals $(a_1, \ldots, a_n)$ whose sequence of generalised Frobenius numbers is equal to $\{c_k\}_{k=1}^{\infty}$?

To the best of our knowledge, this problem is wide open. As an application of Theorem 5.3.1, we show that the sequence of Frobenius numbers $(F_k)_{k=1}^{\infty}$ is a finite difference progression, a sequence whose set of differences between consecutive terms is finite. This provides a partial answer to Problem 1.3.4. Finally, as an application of our results, we use the Neighbourhood Theorem to construct an algorithm that takes the lattice in terms of a basis and a natural number k as input, and computes the k-th lattice module and the k-th Frobenius number.

Chapter 2

Preliminaries

2.1 Tropical geometry

2.1.1 Polyhedral geometry

We begin by recalling the necessary concepts from polyhedral geometry. All of our polyhedra throughout shall be convex. A *halfspace* H in $\mathbb{R}^d$ is the set of points satisfying an affine inequality of the form

$$a_1 x_1 + \cdots + a_d x_d \leq b \ , \ a_i, b \in \mathbb{R} \ . \tag{2.1}$$

The boundary of H is the set of points where (2.1) is tight.

Definition 2.1.1. *A* polyhedron *P in $\mathbb{R}^d$ is the intersection of finitely many halfspaces. A* face *of P is the intersection of P with any number of boundaries of the halfspaces. A face of codimension one is a* facet *and a face of dimension zero is a* vertex*. A* polyhedral complex *$\mathcal{P} = \{P_j\}_{j \in J}$ is a finite collection of polyhedra such that:*

- *every face of a polyhedron in $\mathcal{P}$ also lies in $\mathcal{P}$,*
- *the intersection of any two polyhedra in $\mathcal{P}$ also lies in $\mathcal{P}$.*

The underlying set $|\mathcal{P}|$ of $\mathcal{P}$ is the set of points contained in $\mathcal{P}$ i.e. we forget the polyhedral structure.

We can specialise these definitions to obtain specific classes of polyhedra. If all of the defining inequalities (2.1) satisfy $b = 0$ then P is a cone. A polyhedral complex whose polyhedra are all cones is a *fan*. If P is bounded, then it is a *polytope*. A polyhedral complex whose polyhedra are all bounded is a *polytopal complex*.

Polytopes and cones have the additional property that they can be described externally as the intersection of halfspaces, or internally as a linear combination of generators. Polytopes can be described as the *convex hull* of finitely many points, while cones can be described as the *positive hull* of finitely many rays. The Minkowski-Weyl Theorem states that any polyhedron can be written as the Minkowski sum of a polytope and a cone, allowing arbitrary polyhedra to also be described internally in terms of generators.

Example 2.1.2 (Newton polytopes)**.** *The following is a key polyhedral construction for tropical geometry. Let $f = \sum c_a x^a \in \mathbb{K}[x_1^{\pm}, \dots, x_d^{\pm}]$ be a (Laurent) polynomial in d variables, each monomial gives rise to a point $a \in \mathbb{Z}^d$ via its exponent vector. The* Newton polytope *of f is the convex hull of exponent vectors of monomials in f*

$$\mathrm{Newt}(f) = \mathrm{conv}\left\{ a \in \mathbb{Z}^d \;\middle|\; c_a \neq 0 \right\} \subset \mathbb{R}^d \ . \tag{2.2}$$

As an example, Figure 2.1 shows the Newton polytope of the bivariate polynomial $x^2y + 3xy^2 + xy - y^2 + 2$.

One can imbue a polytopal complex structure on a polytope via *subdivision.*

Definition 2.1.3. *Let $P = \mathrm{conv}(A)$ be the convex hull of A, a configuration of points in $\mathbb{R}^d$. A* (polyhedral) subdivision *of P is the polytopal complex $\mathcal{P}$ such that $|\mathcal{P}| = |P|$ and the vertices of $\mathcal{P}$ are precisely A.*

There are multiple classes of subdivision; the following are two examples that will be of most use to us.

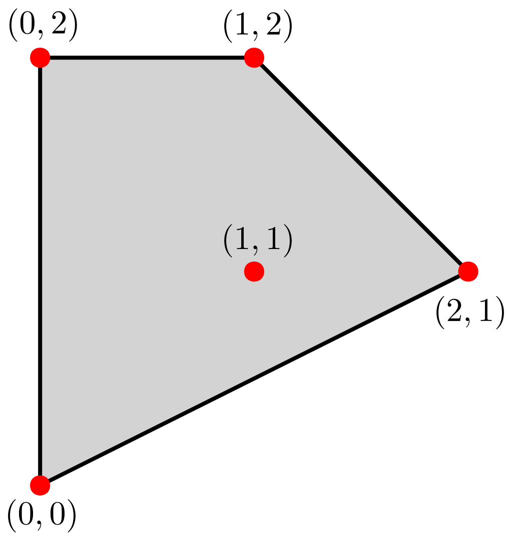

Figure 2.1: The Newton polytope of the polynomial $x^2y + 3xy^2 + xy - y^2 + 2$.

Example 2.1.4 (Regular subdivisions)**.** *One can induce a subdivision of* $P = \mathrm{conv}(A)$ *in the following way. Define a weight function* $w : A \to \mathbb{R}$ *that associates a weight to each element of* A *and consider the polytope*

$$\widehat{P} = \mathrm{conv}\,\{\,(a_1, \ldots, a_d, w(a)) \mid \; a \in A\} \subset \mathbb{R}^{d+1} \quad . \tag{2.3}$$

Informally, "looking at $\widehat{P}$ *from below" gives a copy of* P *that has been subdivided. Formally, we consider the facets of* $\widehat{P}$ *whose (outer) normal vector is negative in the last coordinate, and project these to* $\mathbb{R}^d$ *by deleting the last coordinate. A subdivision that can be induced this way is called* regular*. Note that not every subdivision is regular, see Example 3.2.9.*

Consider the Newton polytope $P \subset \mathbb{R}^2$ *from Figure 2.1. We consider the weight function* w *defined as follows:*

$$w(0,0) = 0\ ,\ \ w(1,1) = 0\ ,\ \ w(2,1) = 1\ ,\ \ w(0,2) = 1\ ,\ \ w(1,2) = 3\ \ . \tag{2.4}$$

The resulting polytope $\widehat{P} \subset \mathbb{R}^3$ *and the regular subdivision of* P *it induces are given in Figure 2.2.*

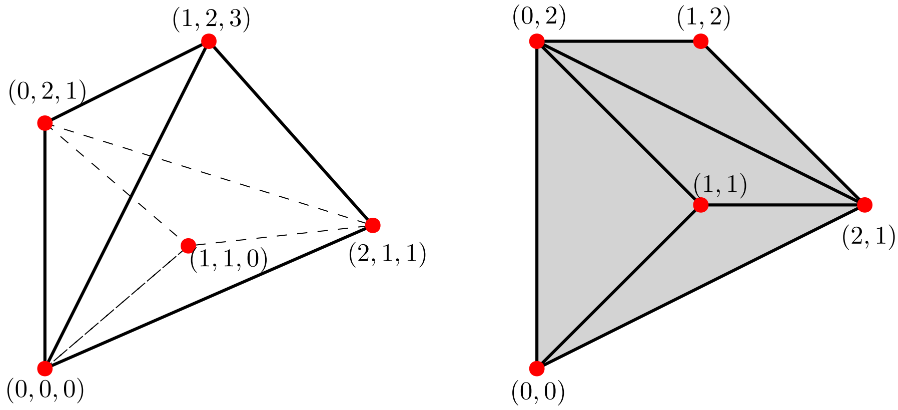

Figure 2.2: $\widehat{P}$ and the regular subdivision of P described in Example 2.1.4.

Note that if w is sufficiently generic, the resulting subdivision will be a triangulation *i.e., all cells are simplices. In this case, the subdivision cannot be subdivided any further and so is called* fine.

Example 2.1.5 (Mixed subdivisions)**.** *The natural addition operation for polyhedra is* Minkowski sum*; given two polyhedra $P, Q \subset \mathbb{R}^d$, their Minkowski sum is*

$$P + Q = \{p + q \mid p \in P \ , \ q \in Q\} \subset \mathbb{R}^d \ . \tag{2.5}$$

Let $P = \sum_{i=1}^{n} P_i$ be the Minkowski sum of n polytopes P_i, we use the structure of Minkowski sum to define the following subdivision of P. A Minkowski cell *of P is a full dimensional polytope $C = \sum_{i=1}^{n} C_i$ where each C_i is a polytope whose vertices are among the vertices of P_i. A* mixed subdivision *is a collection of Minkowski cells $\mathcal{C}$ such that $\mathcal{C}$ covers P and taking intersections of cells of $\mathcal{C}$ commutes with Minkowski sum:*

$$C \cap D = \sum_{i=1}^{n} (C_i \cap D_i) \ . \tag{2.6}$$

A mixed subdivision is fine *if it cannot be subdivided any further.*

Our key example will be mixed subdivisions of the unit $(d-1)$-simplex summed with

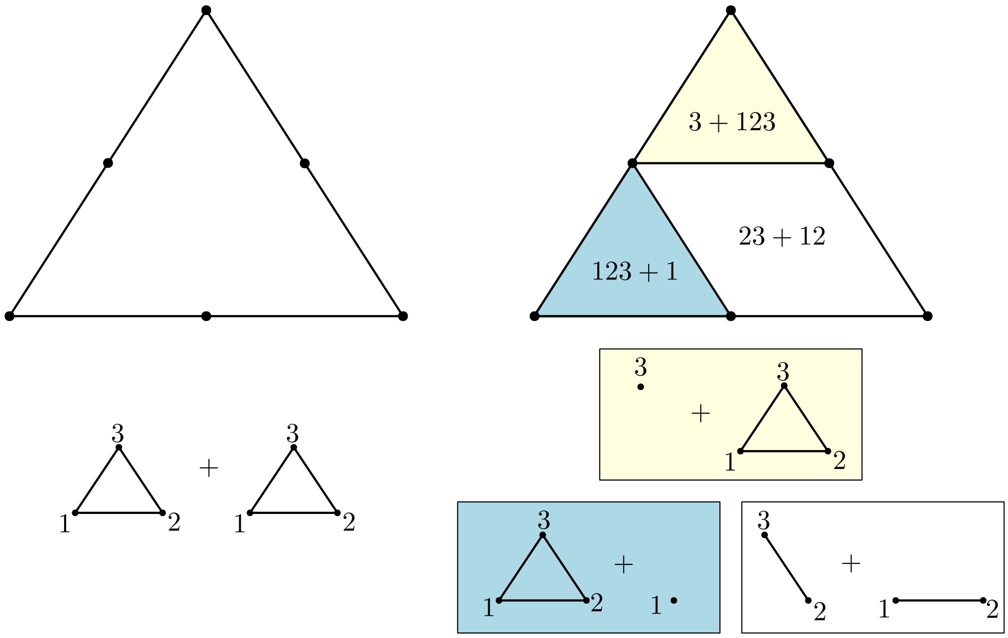

Figure 2.3: A fine mixed subdivision of $2\Delta_2$, labelled with summations that make up its maximal cells.

itself n times:

$$n\Delta_{d-1} = \underbrace{\Delta_{d-1} + \cdots + \Delta_{d-1}}_{n} \ . \tag{2.7}$$

Figure 2.3 gives an example of a mixed subdivision of $2\Delta_2$, and the summations that make up its maximal cells.

Mixed subdivisions of $n\Delta_{d-1}$ are of interest for two reasons: their connection to arrangements of tropical hyperplanes and their bijection with subdivisions of products of simplices:

Theorem 2.1.6 (Cayley Trick)**.** *[28, Theorem 9.2.18] Mixed subdivisions of $n\Delta_{d-1}$ are in bijection with polyhedral subdivisions of the product of simplices $\Delta_{d-1} \times \Delta_{n-1}$.*

Informally, this bijection is given by slicing $\Delta_{d-1} \times \Delta_{n-1}$ through its barycentre with a $(d-1)$-dimensional affine subspace. Figure 2.4 shows the polyhedral subdivision of $\Delta_2 \times \Delta_1$ corresponding to the mixed subdivision of $2\Delta_2$ from Figure 2.3. Mixed subdivisions of $n\Delta_{d-1}$ an subdivisions of products of simplices are considered in more depth in Example

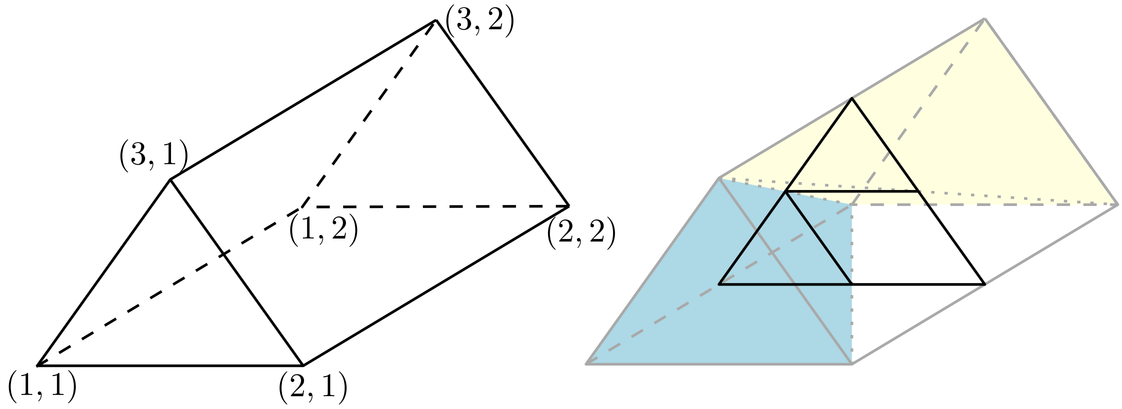

Figure 2.4: The polyhedral subdivision of $\Delta_2 \times \Delta_1$ corresponding to the mixed subdivision of $2\Delta_2$ from Figure 2.3.

2.1.14 and Chapter 3.

The final polyhedral construction we shall require for tropical geometry is the *normal fan* of a polyhedron.

Definition 2.1.7. *Let $P \subset \mathbb{R}^d$ be a polyhedron. The* (inner) normal fan $\mathcal{N}_P$ *of P is the fan in the dual space $(\mathbb{R}^d)^\star$ consisting of cones*

$$\mathcal{N}_P(F) = \left\{ w \in (\mathbb{R}^d)^\star \;\middle|\; w \textit{ is minimised at } F \right\} \tag{2.8}$$

as F varies over all faces of P.

Another way to construct the normal fan is as follows. For each vertex v of P, consider the facets containing v and their inner normal vectors. These vectors generate the maximal cone $\mathcal{N}_P(v)$; ranging over all vertices gives the normal fan. Note that although formally the normal fan lives in the dual space to $\mathbb{R}^d$, we rarely make this distinction.

2.1.2 Tropical hypersurfaces

Tropical hypersurfaces (and varieties) can be constructed from two different viewpoints: an extrinsic way from algebraic hypersurfaces defined by ordinary polynomials, and an intrinsic way as solution sets of tropical polynomials. We shall see that these two viewpoints are equivalent.

The intrinsic viewpoint is the following. The *tropical semiring* $\mathbb{T} = (\mathbb{R}, \oplus, \odot)$ is the idempotent semifield with operations

$$a \oplus b = \min(a, b) \ , \ a \odot b = a + b \ . \tag{2.9}$$

Often it is useful to adjoin $\mathbb{T}$ with a unique maximal element ∞ to give $\mathbb{T}$ an additive identity. However, we remark that $\mathbb{T}$ does not have additive inverses as there does not exist a^{-1} such that $\min(a, a^{-1}) = \infty$ for any $a \in \mathbb{T}$. For simplicity, we do not adjoin ∞ to $\mathbb{T}$ unless explicitly stated otherwise.

A *tropical (Laurent) polynomial* $F \in \mathbb{T}[x_1^{\pm}, \ldots, x_d^{\pm}]$ is a (Laurent) polynomial with operations $\oplus, \odot$ and coefficients in $\mathbb{T}$. It defines a piecewise linear function

$$\begin{aligned} F : \mathbb{T}^d &\longrightarrow \mathbb{T} \\ (p_1, \ldots, p_d) &\longmapsto \bigoplus_{a \in \mathrm{supp}(F)} c_a \odot p^{\odot a} \ , \ c_a \in \mathbb{T} \end{aligned} \tag{2.10}$$

where the support $\mathrm{supp}(F)$ of F is a finite subset of $\mathbb{Z}^d$ and $p^{\odot a}$ denotes $a_1 p_1 + \cdots + a_d p_d$. We shall write tropical polynomials in uppercase, and ordinary polynomials in lowercase to distinguish between the two.

For each $p \in \mathbb{T}^d$, we define the set

$$\mathcal{D}_p(F) = \left\{ a \in \mathrm{supp}(F) \ \middle| \ F(p) = c_a \odot p^{\odot a} \right\} \ , \tag{2.11}$$

i.e, the set of monomials at which $F(p)$ obtains its minimum. The *tropical hypersurface* $\mathcal{T}(F)$ defined by F is the non-linear locus of F in $\mathbb{T}^d$:

$$\mathcal{T}(F) = \left\{ p \in \mathbb{T}^d \ \middle| \ |\mathcal{D}_p(F)| > 1 \right\} \ . \tag{2.12}$$

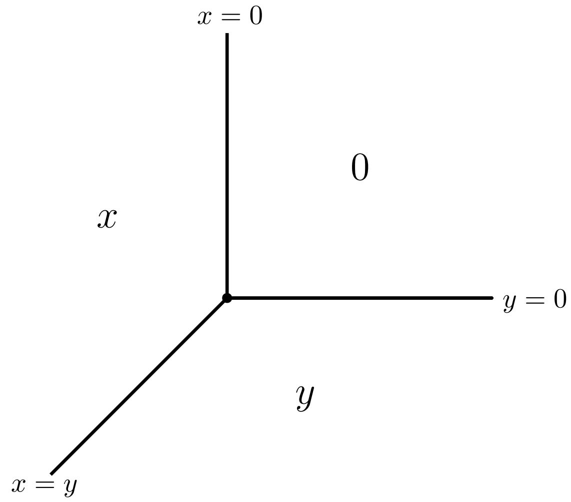

Figure 2.5: A tropical line in $\mathbb{R}^2$.

Note that $\mathbb{T}^d$ is isomorphic to $\mathbb{R}^d$ as a set, therefore we often consider tropical objects in Euclidean space. We shall encounter the topological considerations more in Chapter 4.

Example 2.1.8. *Consider the bivariate tropical polynomial*

$$F = x \oplus y \oplus 0 = \min\{x, y, 0\} \ . \tag{2.13}$$

Its tropical hypersurface, the tropical line shown in Figure 2.5, is a one dimensional fan in $\mathbb{R}^2$ consisting of three rays:

$$\{x = y \leq 0\} \ \cup \ \{x = 0 \leq y\} \ \cup \ \{y = 0 \leq x\} \ , \tag{2.14}$$

intersecting at the zero dimensional cell $x = y = 0$. It decomposes $\mathbb{R}^2$ into three regions, each corresponding to a monomial of F that attains the minimum on that region.

The extrinsic viewpoint is the following. Let $\mathbb{K}$ be a field, a *valuation* on $\mathbb{K}$ is a map $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ such that

1. $\mathrm{val}(a) = \infty \ \Leftrightarrow \ a = 0$

2. $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$

3. $\operatorname{val}(a+b) \geq \min\{\operatorname{val}(a), \operatorname{val}(b)\}$.

Example 2.1.9. *The standard example of a valued field is the field* $\mathbb{C}\{\{t\}\}$ *of* generalised Puiseux series *with coefficients in* $\mathbb{C}$*. The elements of this field are formal power series*

$$\sum_{\alpha \in A} c_\alpha t^\alpha \ , \ A \subset \mathbb{R} \ , \ c_\alpha \in \mathbb{C} \setminus \{0\} \tag{2.15}$$

where A *has a well defined smallest element* α_0 *and no finite accumulation points. The valuation maps a series to* α_0*, its smallest exponent.* $\mathbb{C}\{\{t\}\}$ *is a particularly well-behaved valued field as it is algebraic closed and the valuation map is surjective. This latter property will be particularly important in Chapter 4 when considering higher rank generalisations of this field.*

Given a Laurent polynomial $f = \sum c_a x^a \in \mathbb{K}[x_1^\pm, \ldots, x_d^\pm]$ with support $\operatorname{supp}(f)$, one can construct its *tropicalisation*

$$\begin{aligned} \operatorname{trop}(f) : \mathbb{T}^d &\longrightarrow \mathbb{T} \\ (p_1, \ldots, p_d) &\longmapsto \bigoplus_{a \in \operatorname{supp}(f)} \operatorname{val}(c_a) \odot p^{\odot a} \ . \end{aligned} \tag{2.16}$$

This is a tropical polynomial and, providing our field $\mathbb{K}$ is "large enough", every tropical polynomial can by lifted to an ordinary polynomial in $\mathbb{K}[x_1^\pm, \ldots, x_d^\pm]$. One can construct its corresponding tropical hypersurface $\mathcal{T}(\operatorname{trop}(f))$ as in (2.12).

There is a second construction of the tropical hypersurface associated to f. Let $V(f) \subset (\mathbb{K} \setminus \{0\})^d$ be the algebraic hypersurface defined by f. The tropicalisation of this hypersurface is the set

$$\mathcal{T}(V(f)) = \overline{\{(\operatorname{val}(p_1), \ldots, \operatorname{val}(p_d)) \mid (p_1, \ldots, p_d) \in V(f)\}} \subset \mathbb{R}^d \ ,$$

the closure of the coordinatewise valuation of points of $V(f)$ in the Euclidean topology. Kapranov's theorem states that these two constructions coincide:

Theorem 2.1.10. *(Kapranov's Theorem, [63, Theorem 3.1.3]) Fix a Laurent polynomial $f \in \mathbb{K}[x_1^{\pm}, \ldots, x_d^{\pm}]$. The following sets coincide:*

1. $\mathcal{T}(\mathrm{trop}(f))$*, the tropical hypersurface of the tropical polynomial* $\mathrm{trop}(f)$*.*

2. $\mathcal{T}(V(f))$*, the closure of the coordinatewise valuation of points of $V(f)$ in the Euclidean topology.*

As these two notions coincide, we are free to move between the two and refer to the tropical hypersurface defined by f simply as $\mathcal{T}(f)$. As we shall see in Theorem 4.2.2, Kapranov's Theorem also holds for valuations of higher rank

Tropical hypersurfaces are highly structured sets. We can explicitly describe them in the language of polyhedral subdivisions.

Definition 2.1.11. *Let $f = \sum c_a x^a \in \mathbb{K}[x_1^{\pm}, \ldots, x_d^{\pm}]$ be a d-variate Laurent polynomial with finite support. The* Newton subdivision *of f is the regular subdivision of* $\mathrm{Newt}(f)$ *induced by the weights* $\mathrm{val}(c_a)$ *on the lattice points $a \in \mathrm{supp}(f)$ of* $\mathrm{Newt}(f)$*.*

Proposition 2.1.12. *([63, Proposition 3.1.6]) Let $f \in \mathbb{K}[x_1^{\pm}, \ldots, x_d^{\pm}]$ be a Laurent polynomial. Its tropical hypersurface $\mathcal{T}(f)$ is the support of a codimension one polyhedral complex in $\mathbb{R}^d$. Explicitly, it is the $(d-1)$-skeleton of the polyhedral complex dual to the Newton subdivision of f.*

The dual complex is constructed in the following way. We consider each full dimensional polytope in the subdivision and consider its normal fan. If two polytopes P_1, P_2 share a face F, then we "glue" the cones $\mathcal{N}_{P_1}(F)$ and $\mathcal{N}_{P_2}(F)$ together. Examples 2.1.13 and 2.1.14 demonstrate this construction.

Proposition 4.2.8 is generalisation of Proposition 2.1.12 to valuations of higher rank. Note we do not state it in the language of Newton polytopes, rather in terms of support sets.

Example 2.1.13. *Consider the polynomial $f = tx^2y + t^3xy^2 + (1+2t)xy + (t+t^2)y^2 + 1$*

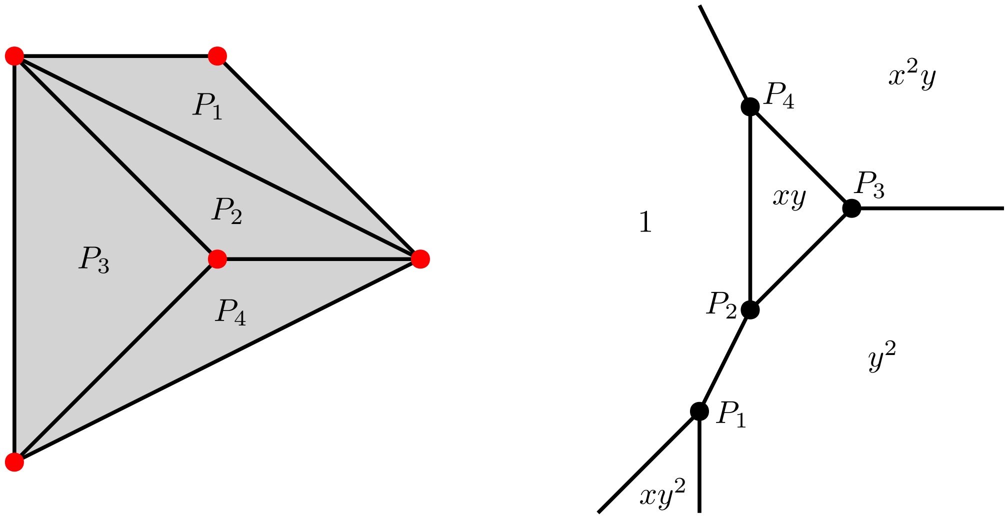

Figure 2.6: The tropical hypersurface $\mathcal{T}(f)$ from Example 2.1.13.

with coefficients in $\mathbb{C}\{\{t\}\}$*. Its Newton subdivision is the subdivided polytope from Figure 2.2. Figure 2.6 shows the corresponding tropical hypersurface* $\mathcal{T}(f)$ *and its duality with the Newton subdivision. The full dimensional polytopes are mapped to zero dimensional cells of the tropical hypersurface, and the regions of the tropical hypersurface are labelled by the monomials that achieve the minimum on that region.*

Example 2.1.14. *Consider the linear form* $f = a_0 + a_1x_1 + \cdots + a_dx_d$ *where* $a_i \in \mathbb{K}$*. Its Newton polytope* $\mathrm{Newt}(f)$ *is the unit* d*-simplex* Δ_d*. As its only lattice points are the vertices of the simplex, its regular subdivision is trivial. Therefore* $\mathcal{T}(f)$ *is a tropical hyperplane, a translate of the* $(d-1)$*-skeleton of the normal fan of the* d*-simplex, whose apex is at the point* $(\mathrm{val}(a_0)-\mathrm{val}(a_1), \ldots, \mathrm{val}(a_0)-\mathrm{val}(a_n))$*. Example 2.1.8 demonstrates this in* $\mathbb{R}^2$*.*

Let $f_1, \ldots, f_n$ *be linear forms. Let* $f = \prod_{i=1}^n f_i$*, its Newton polytope is* $n\Delta_d$*, the unit* d*-simplex dilated by* n*. Its Newton subdivision is non-trivial: it is a regular mixed subdivision of* $n\Delta_d$*. The resulting tropical hypersurface* $\mathcal{T}(f)$ *is the union of the tropical hyperplanes* $\mathcal{T}(f_i)$*. Figure 2.7 gives an explicit example of this.*

Arrangements of n *tropical hyperplanes in* $\mathbb{R}^d$ *are in bijection with regular mixed subdivisions of* $n\Delta_d$*. We can relax this restriction to consider any mixed subdivision*

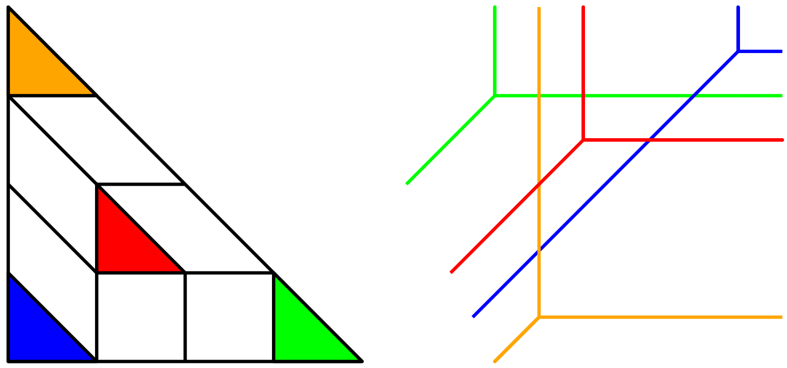

Figure 2.7: An arrangement of 4 tropical hyperplanes in $\mathbb{R}^2$, and its corresponding regular mixed subdivision of $4\Delta_2$.

of $n\Delta_d$, which are in bijection with tropical oriented matroids*, the tropical analogue to oriented matroids [50, 71]. These are again in bijection with subdivisions of $\Delta_d \times \Delta_{n-1}$, and a summary of these bijections are given below. These bijections will be a key tool in Chapter 3.*

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Tropical oriented matroids}\\ \text{(hyperplane arrangements)}\end{array}} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \boxed{\begin{array}{c}\text{(Regular) mixed}\\ \text{subdivisions of } n\Delta_d\end{array}} \arrow[dl, leftrightarrow] \\
& \boxed{\begin{array}{c}\text{(Regular) subdivisions}\\ \text{of } \Delta_d \times \Delta_{n-1}\end{array}} &
\end{tikzcd}$$

Remark 2.1.15. *We have not comprehensively covered all the structural properties of tropical hypersurfaces. The maximal cells are all of the same dimensional and all cells have rational slopes. Furthermore, maximal cells are attached with multiplicities that satisfy a balancing condition. All this additional structure is implied by Proposition 2.1.12 and will not be required for the majority of the content. However, it is worth remarking as some of these properties are not preserved when considering tropical hypersurfaces of higher rank in Chapter 4.*

One can formulate more general tropical varieties than tropical hypersurfaces. However, much more care is required when considering ideals rather than single polynomials.

As we only consider tropical hypersurfaces in the following, we shall not explore this here. However, it is worth remarking that many of the combinatorial and polyhedral properties of tropical hypersurfaces are also held by tropical varieties: they are pure rational polyhedral complexes in $\mathbb{R}^d$ of the same dimension as the underlying algebraic variety that satisfy a balancing condition.

2.1.3 Tropical polytopes

As discussed in the introduction, tropicalisation of semialgebraic sets is a worthwhile endeavour. In particular, there is a rich theory of tropical convexity.

Definition 2.1.16. *Let $V = \{v^{(1)}, \ldots, v^{(n)}\}$ be a subset of points in $\mathbb{R}^d$. A* tropical polytope *generated by V is the tropical convex hull* $\operatorname{tconv}(V)$ *of V, i.e., the set of all points that can be realised as*

$$\lambda_1 \odot v^{(1)} \oplus \cdots \oplus \lambda_n \odot v^{(n)} \ , \ \lambda_i \in \mathbb{T} \ , \ \bigoplus_{i=1}^{n} \lambda_i = 0 \ . \tag{2.17}$$

The structure of tropical polytopes can be described in terms of tropical hyperplane arrangements. Consider the max-*tropical semiring*, where addition is defined to be $a \oplus b = \max(a, b)$. One can define max-tropical hyperplanes (and max-tropical varieties in general) analogously to min-tropical case. Additionally, the identity $\max\{\, a_i \mid i \in I\} = -\min\{\, -a_i \mid i \in I\}$ implies they are simply min-tropical hyperplanes reflected in the ordinary hyperplane $(1, \ldots, 1)^{\perp}$. As stated in Example 2.1.14, a tropical hyperplane with apex a decomposes $\mathbb{R}^d$ into $d + 1$ sectors, one for each monomial in the defining linear form. Over the max-tropical semiring, these sectors are defined by

$$\begin{aligned} S_i(a) &= \left\{ p \in \mathbb{R}^d \ \middle|\ p_i - a_i = \max_{k \in [d]}\{-a_0 \ , \ p_k - a_k\} \right\} \\ S_0(a) &= \left\{ p \in \mathbb{R}^d \ \middle|\ -a_0 = \max_{k \in [d]}\{-a_0 \ , \ p_k - a_k\} \right\} \end{aligned} \ . \tag{2.18}$$

Given an arrangement of max-tropical hyperplanes, they induce a polyhedral cell decomposition of $\mathbb{R}^d$ called a *tropical complex*, whose cells are intersections of sectors. An

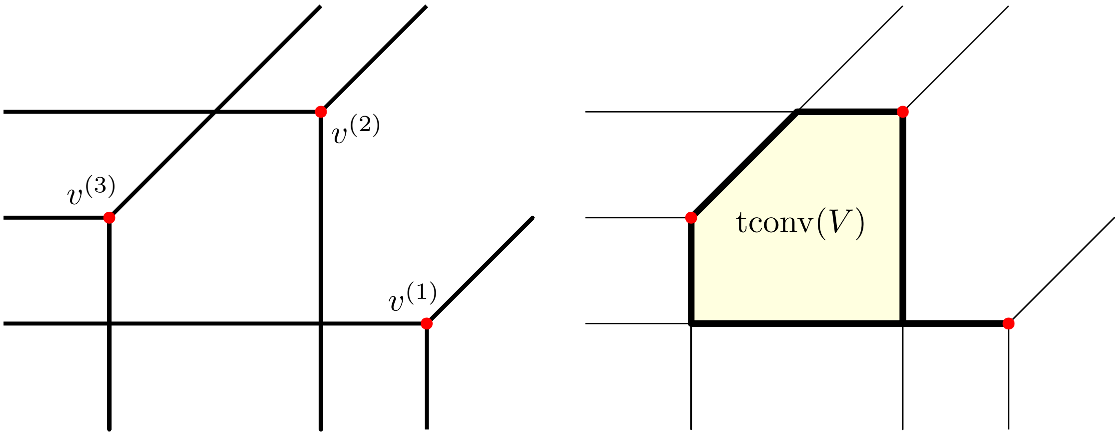

Figure 2.8: An arrangement of three max-tropical hyperplanes, and the corresponding tropical polytope they define.

example of such a tropical complex is given in Figure 2.8. Given a cell C in the tropical complex, we can associate to it a bipartite graph with edges

$$(j,i) \in [n] \times ([d] \cup \{0\}) \iff C \subseteq S_i(v^{(j)}) \ . \tag{2.19}$$

This is the *covector* associated to C.

Theorem 2.1.17. *[29, Theorem 15] Let $V = \{v^{(1)}, \dots, v^{(n)}\} \subset \mathbb{R}^d$. Consider the tropical complex induced by the* max*-tropical hyperplanes with apices $v_1, \dots, v_n$. The* min*-tropical polytope* tconv(V) *is the union of the bounded cells of the tropical complex, or equivalently, those cells whose covector has no isolated nodes.*

Example 2.1.18. *Figure 2.8 shows an arrangement of three* max*-tropical hyperplanes in $\mathbb{R}^2$. The bounded cells of the induced tropical complex form the* min*-tropical polytope given by the tropical convex hull of $v^{(1)}, v^{(2)}, v^{(3)}$. Note that unlike ordinary polytopes, not all maximal cells are of the same dimension.*

Figure 2.9 gives two examples of covectors arising from this tropical complex. The left bipartite graph is the covector associated to the unique full-dimensional bounded cell, while the right bipartite graph is the covector associated to the zero-dimensional cell $v^{(1)}$. The reference hyperplane is included to fix the labelling on the sectors.

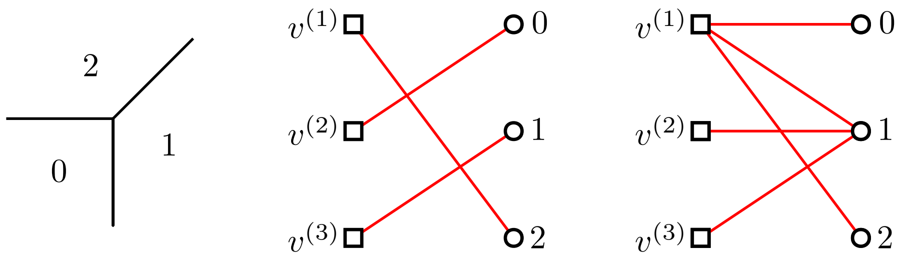

Figure 2.9: The covectors corresponding cells of Figure 2.8. The left covector corresponds to the unique bounded region while the right covector corresponds to $v^{(1)}$. The reference hyperplane is included to fix the labelling on the sectors.

The term covector comes from the language of oriented matroids, where they combinatorialise the cells of an ordinary hyperplane arrangement. Unsurprisingly, we can also associate covectors to tropical oriented matroids, and by extension mixed subdivisions of $n\Delta_d$ and subdivisions of $\Delta_d \times \Delta_{n-1}$. Covectors will play a large role in the construction of matching fields in Chapter 3.

Covectors encode more geometric data than when the cell is bounded. They encode the dimension of the cell via the number of connected components minus one. Furthermore, a cell C is contained in a cell D if and only if its covector is a subgraph of the covector of D. This encoding of geometric data will be exploited in Section 4.4 when we introduce tropical polytopes of higher rank, where certain geometric notions are less clearly defined.

Remark 2.1.19. *While we restrict to $\mathbb{R}^d$ for simplicity, one can also formulate tropical polytopes in $(\mathbb{T} \cup \{\infty\})^d$ with little alteration. Furthermore, Joswig and Loho [53] showed all the previous statements also hold in this more general setting.*

2.2 Commutative algebra of monomial modules

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring in n variables. A *monomial module M* is an S-submodule of the Laurent polynomial ring $\mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$ generated

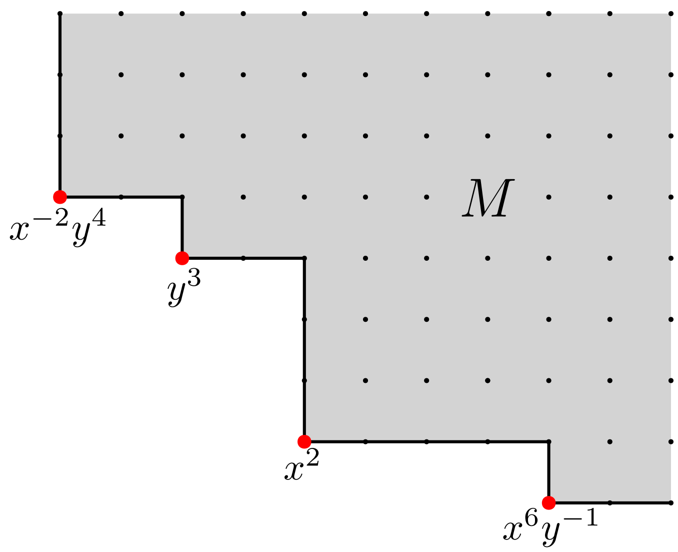

Figure 2.10: The staircase diagram of the monomial module $M = \langle x^6y^{-1}, x^2, y^3, x^{-2}y^4\rangle$. The lattice points in the shaded region exactly encode the monomials in M.

by Laurent monomials

$$M = \langle \mathbf{x}^{\mathbf{u}} \mid \mathbf{u} \in U\rangle \ , \ U \subset \mathbb{Z}^n \ . \tag{2.20}$$

One can represent monomial modules diagrammatically via *staircase diagrams*. One attaches a copy of the positive orthant to each lattice point $u \in U$, each lattice point in this orthant represents a monomial that is hit by the S-action on $\mathbf{x}^{\mathbf{u}}$. The union of these orthants gives the set of lattice points whose corresponding monomials are contained in M. An example is shown in Figure 2.10.

If U is finite, M is finitely generated and isomorphic to a monomial ideal of S. We shall introduce the theory for finitely generated monomial modules then extend it to a special class of infinitely generated modules later.

2.2.1 Finitely generated monomial modules

A *graded ring* R is the direct sum of abelian groups R_i such that $R_iR_j \subseteq R_{i+j}$. The polynomial ring S is a graded ring with graded pieces $S_j = \mathbb{K}\{\mathbf{x}^{\mathbf{u}} \in S \mid \deg(\mathbf{x}^{\mathbf{u}}) = j\}$. This grading extends to M, where M_i is the graded piece $\mathbb{K}\{\mathbf{x}^{\mathbf{u}} \in M \mid \deg(\mathbf{x}^{\mathbf{u}}) = i\}$.

We can define multiple gradings on S by defining the degree of a monomial:

- S has a $\mathbb{Z}^n$-grading by setting $\deg(\mathbf{x}^{\mathbf{u}}) = (u_1, \ldots, u_n)$. This is also referred to as the *fine grading* on S.

- S has a $\mathbb{Z}$-grading by setting $\deg(\mathbf{x}^{\mathbf{u}}) = \sum_{i=1}^{n} u_i$. This is also referred to as the *coarse grading* on S.

Given S with a certain grading, we can "shift" the grading a certain number of steps. The polynomial ring $S(j)$ is the *j-th twist* of S, or S twisted by j. As a module it is isomorphic to S, and its grading is defined by

$$S(j)_i = S_{i+j} \quad . \tag{2.21}$$

Twists of polynomial rings and modules are used frequently when constructing free resolutions.

One way to analyse the structure of M is via the *Hilbert series* of M. The finely graded Hilbert series is defined as

$$H(M;\mathbf{t}) = \sum_{\mathbf{u}\in\mathbb{Z}^n} \dim_{\mathbb{K}}(M_{\mathbf{u}}) \cdot \mathbf{t}^{\mathbf{u}} \ , \ \mathbf{t} = (t_1, \ldots, t_n) \quad . \tag{2.22}$$

where $\dim_{\mathbb{K}}(M_{\mathbf{u}})$ is the dimension of $M_{\mathbf{u}}$ as a $\mathbb{K}$-vector space. Observe that this is particularly simple for monomial modules, as the dimension of $M_{\mathbf{u}}$ is one if $\mathbf{x}^{\mathbf{u}} \in M$, and zero otherwise. Setting $t_i = t$ yields the coarsely graded Hilbert series $H(M; t, \ldots, t)$.

The Hilbert series encodes the dimension of all the graded pieces of M. This is an infinite amount of data, and so at first appears rather intangible. However, the Hilbert series can be expressed as the rational function

$$H(M;\mathbf{t}) = \frac{\mathcal{K}(M;\mathbf{t})}{(1-t_1)\ldots(1-t_n)} \tag{2.23}$$

where

$$\mathcal{K}(M;\mathbf{t}) = \sum_{\substack{i\geq 0\\ \mathbf{u}\in\mathbb{Z}^n}} (-1)^i \beta_{i,\mathbf{u}} \cdot \mathbf{t}^{\mathbf{u}} \tag{2.24}$$

is the *K-polynomial* of M. The coefficients $\beta_{i,\mathbf{u}}$ are the *Betti numbers* of the M. These are key invariants of the module that measure "fluctuations" in the dimension of the graded pieces of M. Note that the K-polynomial only determines the alternating sum of the Betti numbers, we shall see more refined methods later that determines them completely.

The Hilbert series has other invariants attached. Let M be coarsely graded, the function encoding the coefficients of the Hilbert series

$$\begin{aligned} H_M \ : \ \mathbb{Z} &\longrightarrow \mathbb{Z}_{\geq 0} \\ s &\longmapsto \dim_{\mathbb{K}}(M_s) \end{aligned} \tag{2.25}$$

is the *Hilbert function* of M. A surprising corollary of Hilbert's Syzygy Theorem is that for large enough s, this function is a polynomial. An invariant of interest to us is the *Castelnuovo–Mumford regularity*, the value of s at which (under certain conditions) the Hilbert function becomes polynomial. While this is an intuitive way to view regularity, the more practical definition is in terms of Betti numbers.

Definition 2.2.1. *Let M be a $\mathbb{Z}$-graded S-module. M is r-regular if $u - i \leq r$ for all $u \in \mathbb{Z}, i \geq 0$ such that $\beta_{i,u}(M) \neq 0$. The Castelnuovo-Mumford regularity of M is*

$$\operatorname{reg}(M) = \min\{\, r \mid M \textit{ is } r\textit{-regular} \,\} \ . \tag{2.26}$$

Note that Definition 2.2.1 is equivalent to $\operatorname{reg}(M) = \max\{\, u - i \mid \beta_{i,u}(M) \neq 0\}$. It can also be computed as the number of rows of the Betti table minus one, as the Betti table indexes its rows by $u - i$ beginning at zero. We refer to Eisenbud [35, Chapter 4] for more information on this topic. The regularity will be a key invariant in deriving

Frobenius numbers in Chapter 5.

One way to calculate the Betti numbers of M is via *free resolutions* of M. To define free resolutions, we need the following notions. A free S-module of finite rank is a module $F \cong S(-\mathbf{u_1}) \oplus \cdots \oplus S(-\mathbf{u_r})$ isomorphic to the direct sum of r copies of S twisted by degree $-\mathbf{u_i}$. A *complex* $\mathcal{F}$ is a chain of maps between free S-modules

$$\mathcal{F} : 0 \longleftarrow F_0 \xleftarrow[\partial_1]{} F_1 \xleftarrow[\partial_2]{} \ldots \xleftarrow[\partial_k]{} F_k \longleftarrow 0 \tag{2.27}$$

such that $\partial_{i+1} \circ \partial_i = 0$ for all i. A complex is *exact* if $\ker(\partial_i) = \operatorname{im}(\partial_{i+1})$ for all i.

Definition 2.2.2. *A* free resolution $\mathcal{F}$ *of* M *is an exact complex of free* S*-modules such that* $M \cong F_0/\operatorname{im}(\partial_1)$*. A free resolution is* minimal *if for any other free resolution* $\mathcal{F}'$*, there exist maps* π_i *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}' : 0 & F'_0 \arrow[l] \arrow[d, "\pi_0"] & F'_1 \arrow[l, "\partial'_1"'] \arrow[d, "\pi_1"] & \cdots \arrow[l, "\partial'_2"'] & F'_k \arrow[l, "\partial'_k"'] \arrow[d, "\pi_k"] & 0 \arrow[l] \\
\mathcal{F} : 0 & F_0 \arrow[l] & F_1 \arrow[l, "\partial_1"'] & \cdots \arrow[l, "\partial_2"'] & F_k \arrow[l, "\partial_k"'] & 0 \arrow[l]
\end{tikzcd} \quad . \tag{2.28}$$

The length *of a free resolution is the largest* k *such that* $F_k \neq 0$*. The* projective dimension $\operatorname{pdim}(M)$ *of* M *is the length of a minimal free resolution of* M*.*

Free resolutions are not unique, but minimal free resolutions are unique up to isomorphism. Let $\mathcal{F}$ be a minimal free resolution of M, the Betti numbers of M are encoded in $\mathcal{F}$. Explicitly, $F_i \cong \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} S(-\mathbf{u})^{\beta_{i,\mathbf{u}}}$ where $\beta_{i,\mathbf{u}}$ is the i-th Betti number of M in degree $\mathbf{u}$.

Certain classes of monomial modules have methods for constructing minimal free resolutions, however there is no general method that works for all monomial modules. There are constructions that give non-minimal free resolutions for any monomial module, but this only gives bounds on the Betti numbers. One can compute the Betti numbers via other methods [43, 79], or alternatively one can use computer algebra software such as Macaulay2 [46].

Example 2.2.3 (Taylor resolution)**.** *Let* $M = \langle m_1, \ldots, m_r \rangle$ *be an* S*-module minimally generated by monomials* $m_i \in \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$*. The* Taylor complex Δ_M *of* M *is an* $(r-1)$*-dimensional simplex whose* i*-th vertex is labelled by the degree* $\mathbf{u_i}$ *of* m_i*. We label the face* $\sigma \subseteq [r]$ *by the degree* $\mathbf{u}_\sigma$ *of* $\operatorname{lcm}\{m_i \mid i \in \sigma\}$*. The* Taylor resolution *of* M *is the complex* $\mathcal{F}_{\Delta_M}$ *supported on* Δ_M*. Explicitly, this is the complex of free modules and boundary maps*

$$F_i = \bigoplus_{\sigma|=i+1} S(-\mathbf{u}_\sigma) \quad , \quad \partial_i(e_\sigma) = \sum_{\substack{\tau \subset \sigma \\ |\tau|=i}} \operatorname{sgn}(\tau, \sigma) e_\tau$$

where e_σ *is the basis vector of* $S(-\mathbf{u}_\sigma)$ *and* $\operatorname{sgn}(\tau, \sigma)$ *is* $+1$ *if the orientation on* τ *and* σ *agree,* -1 *otherwise. Note that for simplicial complexes, the orientation of* σ *is implicitly given by the ordering on the vertices. However for more general complexes, we have to pick this orientation. Note that any finitely generated monomial module can be resolved by the Taylor complex, but it is usually far from minimal.*

Example 2.2.4 (Hull resolution)**.** *Consider the unbounded convex polyhedron*

$$P_t = \operatorname{conv}\{(t^{u_1}, \ldots, t^{u_n}) \mid \mathbf{x}^{\mathbf{u}} \in M\} \subset \mathbb{R}^n \ , \ t > 0 \ .$$

P_t *has nice properties: the vertices are precisely the set of minimal generators of* M*, and the face poset is invariant for* $t > (n+1)!$*. We define the* hull complex $\operatorname{hull}(M)$ *to be the polyhedral complex of bounded faces of* P_t*, labelled the same way as the Taylor complex. The* hull resolution $\mathcal{F}_{\operatorname{hull}(M)}$ *is the free resolution supported on* $\operatorname{hull}(M)$*.*

As with the Taylor resolution, any finitely generated monomial module can be resolved by the hull resolution, as well as some more general examples we shall see. Furthermore, if M *is generic then* $\mathcal{F}_{\operatorname{hull}(M)}$ *is minimal. However, the trade off is that computing* $\operatorname{hull}(M)$ *is much harder than computing* Δ_M*.*

Remark 2.2.5. *Note that we can construct a hull resolution for any Laurent monomial module, not just finitely generated ones. However, in general the hull complex will have* *infinitely many faces and therefore M will have infinitely many Betti numbers. We shall consider a special class of modules in Section 2.2.2 that have infinite hull resolutions that can be reduced to a finite resolution.*

Our final remark on free resolutions are their connection to relations between generators of M. Given a minimal free resolution $\mathcal{F}$, we can describe $M \cong F_0/\mathrm{im}(\partial_1)$. F_0 is the free module generated by the generators of M, whereas $\mathrm{im}(\partial_1)$ is the module of relations, or *syzygies*, between those generators. This gives rise to the *first syzygy module* $\mathrm{Syz}^1(M) = \mathrm{im}(\partial_1)$. Note that $\mathrm{Syz}^1(M)$ is also an S-module and inherits the same grading as M. Furthermore, we can extend this notion to the *k-th syzygy module* $\mathrm{Syz}^k(M) = \mathrm{im}(\partial_k)$, which encodes higher notions of relations between the generators and their syzygies.

2.2.2 Modules with lattice actions

Chapter 5 considers a class of monomial modules that are infinitely generated, but whose generators are all in the orbit of certain lattice. The methods in the previous section are not immediately amenable as there will be infinitely many Betti numbers in infinitely many degrees. Bayer and Sturmfels [20] developed a framework to condense these modules into a more manageable state, without sacrificing losing information about the module.

Let $L \subset \mathbb{Z}^n$ be a lattice such that $L \cap \mathbb{Z}^n_{\geq 0} = \{\mathbf{0}\}$. We consider the group polynomial ring $S[L]$ where

$$S[L] = \mathbb{K}[\,\mathbf{x}^{\mathbf{u}}\mathbf{z}^{\mathbf{v}} \mid \mathbf{u} \in \mathbb{Z}^n_{\geq 0}\ ,\ \mathbf{v} \in L\,]\ . \tag{2.29}$$

$S[L]$ carries a $\mathbb{Z}^n$-grading via the degree map $\deg(\mathbf{x}^{\mathbf{u}}\mathbf{z}^{\mathbf{v}}) = \mathbf{u} + \mathbf{v}$.

Consider a $\mathbb{Z}^n$-graded S-module M with an equivariant L-action i.e., it commutes with the action of S. Then M is naturally an $S[L]$-module by extending the L-action to $S[L]$. An equivalent statement is that the category of L-equivariant $\mathbb{Z}^n$-graded S-modules is isomorphic to the category of $\mathbb{Z}^n$-graded $S[L]$-modules.

Example 2.2.6. *Consider the S-module*

$$M_L = \langle \mathbf{x}^{\mathbf{u}} \mid \mathbf{u} \in L \rangle \ ,$$

with lattice action $\mathbf{v} \cdot \mathbf{x}^{\mathbf{u}} = \mathbf{x}^{\mathbf{u}+\mathbf{v}}$. As an S-module, M_L is minimally generated by infinitely many Laurent monomials. However these all sit in the same L-orbit and so M_L is generated by a single element as an $S[L]$-module. Furthermore, we can describe it as the quotient of $S[L]$

$$M_L \cong S[L] / \left\langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}\mathbf{z}^{\mathbf{u}-\mathbf{v}} \;\middle|\; \mathbf{u}, \mathbf{v} \in \mathbb{Z}^n_{\geq 0} \ , \ \mathbf{u} - \mathbf{v} \in L \right\rangle \ . \tag{2.30}$$

We call M_L a lattice module*, and will return to it throughout this section and Chapter 5.*

Many of the modules we shall consider are infinitely generated as S-modules but finitely generated as $S[L]$-modules. We would like to keep the finite generation property of the latter, yet continue to work with S-modules. To do this, we quotient out the action of L by identifying $\mathbf{x}^{\mathbf{u}}$ with $\mathbf{x}^{\mathbf{u}}\mathbf{z}^{\mathbf{v}}$ for all $\mathbf{v} \in L$. We observe that for $M = S[L]$, this quotient is

$$\begin{aligned} S[L]/L &\cong S[L] / \left\langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{u}-\mathbf{v}}\mathbf{z}^{\mathbf{v}} \;\middle|\; \mathbf{u} \in \mathbb{Z}^n_{\geq 0} \ , \ \mathbf{v} \in L \right\rangle \\ &\cong S[L] / \left\langle \mathbf{z}^{\mathbf{v}} - 1_{\mathbb{K}} \mid \mathbf{v} \in L \right\rangle \\ &\cong S \ . \end{aligned} \tag{2.31}$$

We remark that this construction affects the grading: $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{u}}\mathbf{z}^{\mathbf{v}}$ should have the same degree after quotienting by L, but this is not true when the grading is $\mathbb{Z}^n$. To ensure they have the same degree, we grade $S[L]/L \cong S$ by $\mathbb{Z}^n/L$.

In the case of an arbitrary $S[L]$-module M, this quotient is achieved by tensoring M

with $S[L]/L$:

$$M/L \cong M \otimes_{S[L]} S[L]/L \cong M \otimes_{S[L]} S \tag{2.32}$$

Informally, one can consider this as setting $\mathbf{z}^{\mathbf{v}}$ to $1_{\mathbb{K}}$. As with the previous construction, this is a $\mathbb{Z}^n/L$-graded module. One can consider this construction as a functor between categories

$$\pi : \{\mathbb{Z}^n\text{-graded } S[L]\text{-modules}\} \to \{\mathbb{Z}^n/L\text{-graded } S\text{-modules}\} \ . \tag{2.33}$$

Example 2.2.7. *Consider the lattice module M_L from Example 2.2.6. Recall that we can describe M_L by*

$$M_L \cong S[L]/\left\langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}\mathbf{z}^{\mathbf{u}-\mathbf{v}} \;\middle|\; \mathbf{u}, \mathbf{v} \in \mathbb{Z}^n_{\geq 0} \ , \ \mathbf{u} - \mathbf{v} \in L \right\rangle \ . \tag{2.34}$$

To compute $M_L/L \cong M_L \otimes_{S[L]} S$, we replace $S[L]$ with S and any occurrences of $\mathbf{z}^{\mathbf{v}}$ with $1_{\mathbb{K}}$. This gives the cyclic $\mathbb{Z}^n/L$-graded S-module

$$\begin{aligned} M_L \otimes_{S[L]} S &\cong S/\left\langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \;\middle|\; \mathbf{u}, \mathbf{v} \in \mathbb{Z}^n_{\geq 0} \ , \ \mathbf{u} - \mathbf{v} \in L \right\rangle \\ &\cong S/I_L \ , \end{aligned} \tag{2.35}$$

where I_L is the lattice ideal *generated by L. Lattice ideals are a well studied class of ideal via toric geometry and commutative algebra; see [66, §2] for more details.*

This functor does not just give us a construction for modules, it also allows us to transfer homological data of the modules.

Theorem 2.2.8. *([20, Theorem 3.2]) The functor π sending M to $M/L \cong M \otimes_{S[L]} S$ is an equivalence of categories.*

A corollary of Theorem 2.2.8 is that π is *exact*; it preserves exact sequences. Therefore for any (minimal) free resolution of a $\mathbb{Z}^n$-graded $S[L]$-modules M, we immediately get a (minimal) free resolution of the $\mathbb{Z}^n/L$-graded S-module M/L. In particular, it allows us to construct Betti numbers for M/L that we can pull back to equivalence classes of Betti numbers of M as an S-module.

Remark 2.2.5 states we can construct the hull resolution for any Laurent monomial module; this includes M as an S-module. As it is not finitely generated, $\mathrm{hull}(M)$ will be an infinite polyhedral complex. However, the hull resolution itself is L-equivariant: the action of L onto M extends to the entire resolution. Therefore the hull resolution of M induces a *finite* free resolution of $S[L]$-modules

$$\mathcal{F}_{\mathrm{hull}(M)} : 0 \longleftarrow S[L]^{\beta_0} \xleftarrow[\partial_1]{} S[L]^{\beta_1} \xleftarrow[\partial_2]{} \dots \xleftarrow[\partial_k]{} S[L]^{\beta_k} \longleftarrow 0 \tag{2.36}$$

where $\beta_i = \sum_{\mathbf{u}\in\mathbb{Z}^n} \beta_{i,\mathbf{u}}$ is the number of i-dimensional faces of $\mathrm{hull}(M)$ modulo the action of L. Applying the functor π gives a $\mathbb{Z}^n/L$-graded free resolution of M/L

$$\mathcal{F}_{\mathrm{hull}(M/L)} : 0 \longleftarrow S^{\beta_0} \xleftarrow[\partial_1]{} S^{\beta_1} \xleftarrow[\partial_2]{} \dots \xleftarrow[\partial_k]{} S^{\beta_k} \longleftarrow 0 \quad . \tag{2.37}$$

Furthermore, $\mathcal{F}_{\mathrm{hull}(M)}$ is minimal if and only if $\mathcal{F}_{\mathrm{hull}(M/L)}$ is by exactness of π.

Example 2.2.9. *Consider the lattice $L = (3,5,8)^\perp \cap \mathbb{Z}^3$. This is the lattice of integer points whose inner product with $(3,5,8)$ is zero. The hull complex of M_L is the infinite polyhedral complex pictured on the left of Figure 2.11. The vertices of $\mathrm{hull}(M_L)$ are the lattice points of L, while the remaining cells form a triangulation of the plane.*

A "section" of $\mathrm{hull}(M_L)$ is pictured on the right of Figure 2.11. Under the action of L, this piece tiles the entire hull complex without overlap, and so its cells describe each of the equivalence classes of cells of $\mathrm{hull}(M_L)$. From this, we compute the free resolution:

$$\mathcal{F}_{\mathrm{hull}(M_L)} : 0 \longleftarrow S[L] \xleftarrow[\partial_1]{} S[L]^3 \xleftarrow[\partial_2]{} S[L]^2 \longleftarrow 0 \tag{2.38}$$

Furthermore, applying π yields a $\mathbb{Z}^n/L$-graded free resolution of S-modules for $\pi(M_L) \cong$

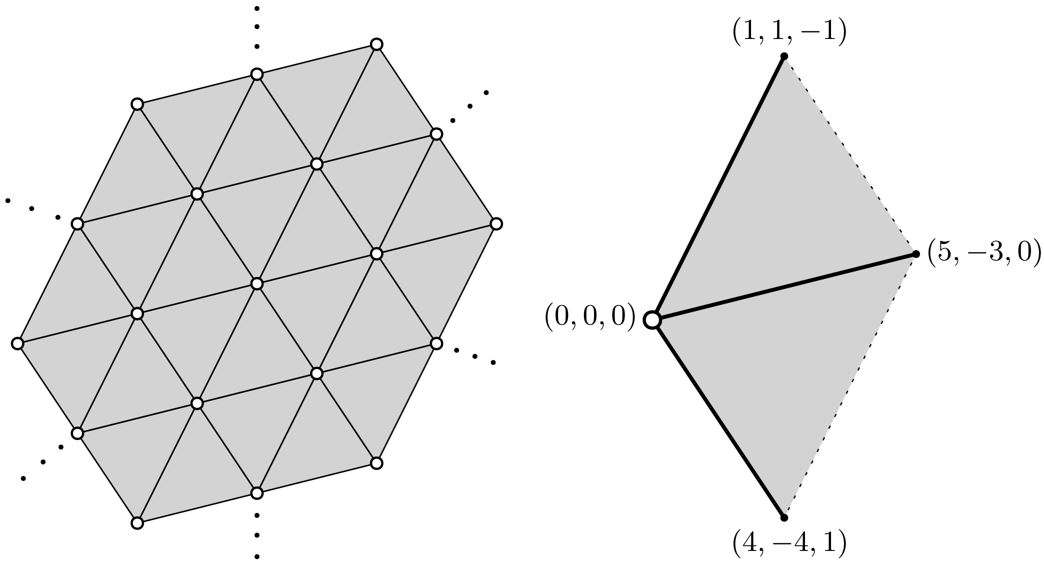

Figure 2.11: The hull complex of M_L from Example 2.2.9, pictured left, and its equivalence classes of cells upto the action of L, pictured right.

S/I_L.

$$0 \longleftarrow S(-(0,0,0)) \longleftarrow \begin{array}{rl} & S(-(1,1,0)) \\ \oplus & S(-(5,0,0)) \\ \oplus & S(-(4,0,1)) \end{array} \longleftarrow \begin{array}{rl} & S(-(5,1,0)) \\ \oplus & S(-(5,0,1)) \end{array} \longleftarrow 0 \quad (2.39)$$

We compute the twists of S by taking the max *of the lattice points contained in each cell. In this case, the free resolution for S/I_L is minimal; note that this is not true in general.*

Chapter 3

Matching fields and lattice points of simplices

The following is based on the paper "Matching fields and lattice points of simplices" by Georg Loho and the author [61].

3.1 Tope fields

Fix a pair (n, d) of positive integers where $n \geq d$. We study bipartite graphs on two node sets L and R, where $|L| = n$ and $|R| = d$. Since they are defined on the same set of nodes, we will often identify them with their set of edges written as pairs of nodes. The elements of L are denoted by $\ell_1, \ldots, \ell_n$, the elements of R by $r_1, \ldots, r_d$. We refer to nodes in L as *left nodes* and nodes in R as *right nodes*. The left degree vector is the tuple of node degrees of $\ell_1, \ldots, \ell_n$; we define the right degree vector analogously for the elements in R.

For two finite sets $A \subseteq B$ we denote the characteristic vector of A in B by e_A^B, where we omit the reference set B if it is clear from the context.

Definition 3.1.1. *Let $(v_1, \ldots, v_d)$ be a tuple of positive numbers with $\sum_{i=1}^{d} v_i = k \leq n$.*

For a k-element subset σ of L, we define a tope *of type $(v_1, \dots, v_d)$ to be a bipartite graph whose right degree vector is its type and the left degree vector is e_σ. An (n, d)*-tope field *of type $(v_1, \dots, v_d)$ is a set of topes $\mathcal{M} = (M_\sigma)$ with a unique tope M_σ for each $\sigma \in \binom{[n]}{k}$. The sum $k = \sum_{i=1}^{d} v_i$ is the* thickness *of the tope field. If the thickness is d, the type is $e_{[d]}$ and the tope field is a* matching field*. If the thickness is n, the tope field has a single tope with left degree vector $e_{[n]}$ that we call* maximal.

Note that our definition of topes differs slightly from the original definition in [11] as we insist that all right nodes must have positive degree. The recent work [42] refers to them as semi-matchings and also allows topes to have isolated right nodes. We shall see that these topes can be considered as lying 'at the boundary'.

There is a natural arbitrariness in the role of L and R. The previous definition is for a *left* tope field, a *right* tope field can be defined analogously for $|L| \leq |R|$. This distinction will become more necessary later. Note that a tope field can also be considered as a set of surjective functions $M_\sigma : \sigma \to R$ where $\left|M_\sigma^{-1}(r_i)\right| = v_i$.

A *sub-tope field* is a tope field which consists of the induced subgraphs on $J \sqcup I$ for subsets $J \subseteq L$ and $I \subseteq R$. Note that in general the induced subgraphs on $J \sqcup I$ do not form a tope field.

Observe that tope fields generalise matching fields in the sense of [85] and each graph is a tope in the sense of [11]. Examples of matching and tope fields are given in Figures 3.1 and 3.2.

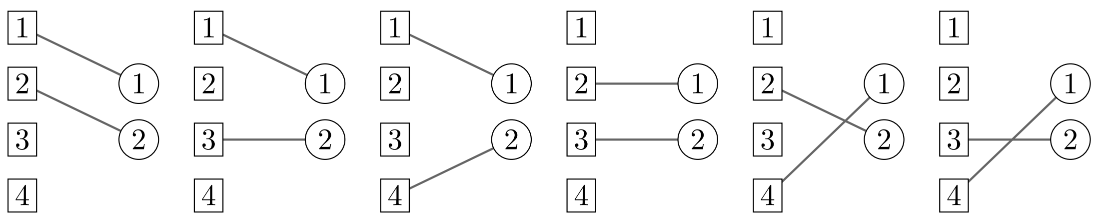

Figure 3.1: A $(4, 2)$-matching field.

Example 3.1.2. *The most natural and well-behaved examples of matching fields are obtained from a generic matrix $A \in \mathbb{R}^{n \times d}$. The minimal matchings in the complete*

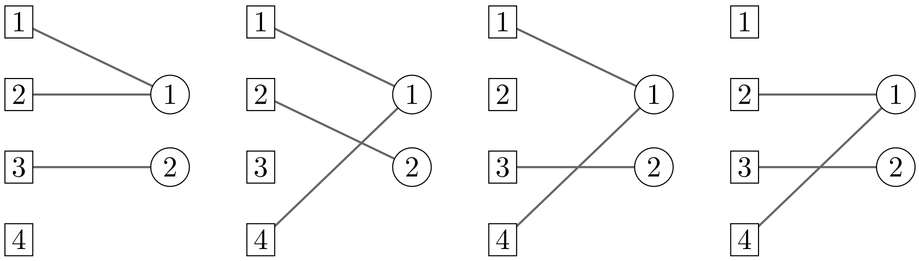

Figure 3.2: A $(4, 2)$-tope field of type $(2, 1)$ with thickness 3.

bipartite graph $K_{n,d}$ weighted by the entries of A give rise to a matching field. Such a matching field is called coherent*, c.f. [85]. If A is not generic, one can slightly perturb the matrix to obtain a unique matching on each d-subset of $[n]$. Alternatively, one could just pick one minimal matching for each d-subset. However, the resulting matching field may be arbitrarily unstructured as one sees if A is just the zero matrix.*

This idea can be extended to obtain coherent *tope fields in a similar fashion. Fix a vector $(v_1, \ldots, v_d) \in \mathbb{Z}^d_{>0}$ with $\sum_{i=1}^d v_i = k \leq n$ and a generic matrix $A \in \mathbb{R}^{n\times d}$. We construct the matrix $\overline{A} \in \mathbb{R}^{n\times k}$ by replacing the column indexed by i in A with v_i copies of itself. Such a matrix gives rise to a complete bipartite graph $K_{n,k}$ with weights $\overline{A}$. The coherent matching field arising from $\overline{A}$ naturally yields a tope field of type $(v_1, \ldots, v_d)$ by setting $M_\sigma^{-1}(r_i)$ to the set of nodes in σ adjacent to a copy of r_i in the matching on $K_{n,k}$.*

The process of duplicating the columns in the definition of a coherent tope field motivates the next construction.

Example 3.1.3. *An (n, d)-tope field $\mathcal{M} = (M_\sigma)$ of type $(v_1, \ldots, v_d)$ gives rise to an (n, k)-matching field $\mathcal{N}$ where $k = \sum_{i=1}^d v_i$. For each node $r_i \in R$ we introduce v_i nodes $r_i^{(1)}, \ldots, r_i^{(v_i)}$. Let $j^{(1)} < \ldots < j^{(v_i)}$ be the increasing list of indices denoting the elements in $M_\sigma^{-1}(r_i)$. By setting $N_\sigma(\ell_{j^{(t)}}) = r_i^{(t)}$ for each $t \in [v_i]$ and all $i \in [d]$, we obtain a matching field $\mathcal{N} = (N_\sigma)$. We call this matching field the* increasing splitting *of the tope field.*

Observe that one could also consider partial splitting from a coarser to a finer tope *field. Furthermore, note that the splitting depends on the ordering of the split copies of the nodes in R. This construction can be seen as a 'refinement' of the tope field, analogous to a refinement of a polyhedral subdivision in [28, Definition 2.3.8]. In particular, the linkage covectors, see Definition 3.2.6, of the increasing splitting can be seen as full dimensional cells in a staircase triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.*

3.1.1 An important example

For the polyhedral background we refer to [28].

The construction in [72] connecting matching fields and triangulations of a product of two simplices motivates us to investigate matching fields and their connection to polyhedral constructions further.

Recall that the maximal simplices in a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ are given by a set of spanning trees on the bipartite node set $L \sqcup R$ with $|L| = n, |R| = d$, which fulfil the axioms given by Ardila & Billey [10]. A simplex with vertices $(e_{j_1}, e_{i_1}), \dots, (e_{j_k}, e_{i_k}) \in \Delta_{n-1} \times \Delta_{d-1}$ corresponds to the bipartite graph with edges $(\ell_{j_1}, r_{i_1}), \dots, (\ell_{j_k}, r_{i_k})$.

Proposition 3.1.4 ([10, Proposition 7.2])**.** *A set of trees encodes the maximal simplices of a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ if and only if:*

1. *Each tree is spanning.*
2. *For each tree G and each edge e of G, either $G - e$ has an isolated node or there is another tree H containing $G - e$.*
3. *If two trees G and H contain perfect matchings on $J \sqcup I$ for $J \subseteq L$ and $I \subseteq R$ with $|J| = |I|$ then the matchings agree.*

We wish to study the connection to triangulations of $\Delta_{n-1} \times \Delta_{d-1}$, therefore all trees we refer to will be spanning trees of the complete bipartite graph on $L \sqcup R$, unless otherwise stated.

Most of our arguments are independent of the embedding of the product of simplices. For the next proposition we choose the canonical embedding

$$\Delta_{n-1} \times \Delta_{d-1} = \operatorname{conv}\{(e_j, e_i) \mid j \in [n], i \in [d]\} \subseteq \mathbb{R}^{n+d} \ .$$

Oh & Yoo introduced in [72] the 'extraction method' which collects the set of all partial matchings occurring in the trees encoding the triangulation. The fact that we obtain a matching field by taking all matchings of size d occurring in the trees can be deduced from the following polyhedral construction.

Proposition 3.1.5. *Let Σ be a triangulation of $\Delta_{d-1} \times \Delta_{d-1}$. Then the bipartite graph G corresponding to the minimal cell (with respect to inclusion) containing the barycentre $g = (\frac{1}{d}, \dots, \frac{1}{d})$ of $\Delta_{d-1} \times \Delta_{d-1}$ is a perfect matching on $[d] \sqcup [d]$.*

Proof. By the first condition of Proposition 3.1.4, the bipartite graph G is a subgraph of a tree, and therefore is a forest. Each point in the cell corresponding to G is a unique convex combination of its vertices, in particular g. Define λ to be the weight function which assigns to each edge of G its coefficient in the representation of g. By minimality, $\lambda \neq 0$ for all the edges. G contains a node of degree 1 as it is a forest. The weight of the incident edge e has to be $\frac{1}{d}$ as it determines the coordinate corresponding to the node of degree 1. Therefore, the other node incident with e has degree 1 as well. By induction, this implies the claim. $\square$

Iterating the construction of Proposition 3.1.5 over all faces of $\Delta_{n-1} \times \Delta_{d-1}$ of the form

$$\operatorname{conv}\{e_j \mid j \in \sigma\} \times \Delta_{d-1} \text{ for } \sigma \in \binom{[n]}{d}$$

gives a perfect matching on each $\sigma \sqcup [d]$, producing a matching field.

3.1.2 Compatibility, trees and topes

Arbitrary matching fields have very little structure, hence we shall study properties of matching fields which occur in connection to polyhedral constructions. The third condition in Proposition 3.1.4 motivates the following notion which was coined in the context of tropical oriented matroids [11].

Definition 3.1.6. *Two forests F_1 and F_2 on the same node set $L \sqcup R$ are* incompatible *if there exist subsets $J \subseteq L$ and $I \subseteq R$ such that F_1 and F_2 contain different perfect matchings on $J \sqcup I$. Otherwise, F_1 and F_2 are* compatible.

Note that we mainly apply this definition to matchings, topes and trees. The next lemma already occurs in [72] but we give a proof to clarify our terminology.

Lemma 3.1.7 ([72, Lemma 3.7])**.** *Let T_1 and T_2 be distinct topes defined on $L \sqcup R$. If they have the same left and right degree vector, then they are incompatible.*

Proof. Consider the symmetric sum of the edges of T_1 and T_2. Direct the edges of T_1 from L to R and of T_2 conversely. In the resulting graph, the indegree and the outdegree of each node are equal. Hence, the graph contains a directed cycle. This consists of two different partial matchings on the same node set in T_1 and T_2. □

There is an analogous statement for trees.

Lemma 3.1.8 ([74, Lemma 12.8])**.** *Let T_1 and T_2 be two different spanning trees on $L \sqcup R$. If they have the same left degree vector or the same right degree vector then they are incompatible.*

Proposition 3.1.9. *Let G be a spanning tree on $L \sqcup R$ with right degree vector $v = (v_1, \dots, v_d)$. For each $r_k \in R$, there is a unique maximal tope with right degree vector $v - e_{[d]\setminus\{k\}}$ contained in G.*

Proof. For each path from r_k to another node in R remove the last edge in that path.

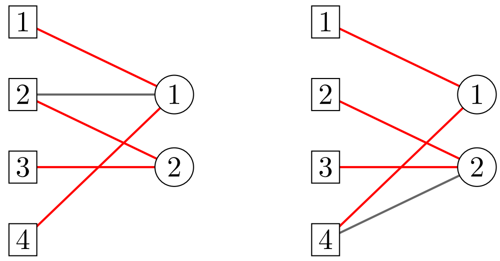

Figure 3.3: Two compatible trees with right degree vectors $(3, 2)$ and $(2, 3)$ respectively. The common (red) edges form a tope with right degree vector $(2, 2)$.

The resulting graph is the desired tope. To show uniqueness, suppose there is another tope with right degree vector $v - e_{[d]\setminus\{k\}}$ contained in G. By Lemma 3.1.7, the two topes are incompatible. This gives a contradiction as the union of the topes contains a cycle and G does not. □

Corollary 3.1.10. *Let T_1 and T_2 be two compatible spanning trees where the first has right degree vector v and the second $v + e_p - e_q$. Then their intersection contains a maximal tope with right degree vector $v - e_{[d]\setminus\{p\}}$. Furthermore, if ℓ_s has degree 1 in both T_1 and T_2 then the trees agree on the edge adjacent to ℓ_s.*

Proof. Proposition 3.1.9 ensures that both trees contain a tope with right degree vector $v - e_{[d]\setminus\{p\}}$. Those topes agree as a consequence of Lemma 3.1.7 because of the compatibility of T_1 and T_2. If ℓ_s has degree 1 in both trees, the edge adjacent to it must be contained in the unique tope in their intersection. □

Proposition 3.1.9 and Corollary 3.1.10 emphasise the structural relationship between topes and trees, in particular how to recover one from the other. Figure 3.3 shows an example of recovering topes from intersections of trees. However, as we shall later see, we can take unions of topes to recover trees.

We can build on these results to find even stronger local conditions on compatible trees. The following lemma captures a combinatorial analogue of certain geometric properties explained in Example 3.1.12.

Lemma 3.1.11. *Let T_1 and T_2 be two compatible spanning trees on $L \sqcup R$ with right degree vectors v and $v + e_p - e_q$ respectively. If (ℓ_s, r_p) is an edge in T_1 then it is also an edge in T_2. Furthermore, the degree δ_1 of ℓ_s in T_1 is bigger or equal to its degree δ_2 in T_2.*

Proof. By Corollary 3.1.10, the intersection $T_1 \cap T_2$ contains a tope with right degree vector $v - e_{[d]\setminus\{p\}}$. The first claim follows directly from this, as if (ℓ_s, r_p) is an edge in T_1, it is contained in this tope, and therefore T_2.

For the second claim, we define an auxiliary graph H as follows. We take the union of the graphs $T_1 \cup T_2$ and make the following alterations. We direct the edges of $T_1 \setminus T_2$ from R to L and direct the edges of $T_2 \setminus T_1$ from L to R. Finally we contract the remaining undirected edges, those in the intersection of T_1 and T_2. Note that these undirected edges form a spanning forest on $L \sqcup R$. The resulting graph is our auxiliary graph H, whose nodes correspond to connected components of $T_1 \cap T_2$. We label each node by the subset of nodes of $L \sqcup R$ in the connected component that has been contracted to that node.

As the node $r_i \in R \setminus \{r_p, r_q\}$ has the same degree in T_1, T_2, every node V such that $r_p, r_q \notin V$ has in-degree equal to out-degree. This implies for every node V such that $r_p \notin V$, we have an outgoing arc. Assume that $\delta_2 > \delta_1$. Then an edge in T_2 incident with ℓ_s gives rise to an out-going arc from the node containing r_p in H. Hence, each node in H has an out-going arc which implies the existence of a cycle in this auxiliary graph. This however contradicts the compatibility of T_1 and T_2.

□

Example 3.1.12. *The intuition behind Lemma 3.1.11 can be seen via arrangements of tropical hyperplanes. A tropical hyperplane H is a translation of the normal fan of Δ_{d-1}, decomposing $\mathbb{R}^{d-1}$ into d sectors $S^{(1)}, \ldots, S^{(d)}$ labelled by the vertices of Δ_{d-1}. Develin and Sturmfels [29] showed that the covectors ('types' in their terminology) la-*

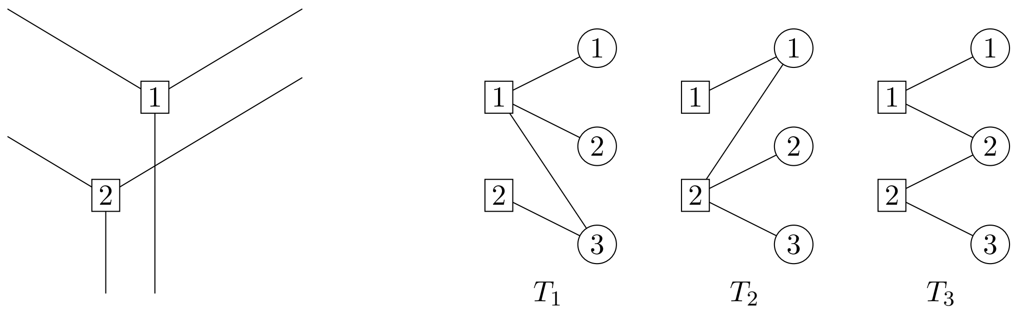

Figure 3.4: An arrangement of two tropical hyperplanes in $\mathbb{R}^2$ and their corresponding trees. T_1, T_2 correspond to the apexes of the hyperplanes, while T_3 is the tree corresponding to their intersection.

belling the cells of a tropical hyperplane arrangement also describe a regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$. In a generic arrangement, this yields a set of trees encoding the arrangement. They can be extracted from the arrangement via the zero dimensional cells: the corresponding tree contains the edge (ℓ_j, r_i) if the cell is contained in $S_j^{(i)}$, sector i of hyperplane j.

Consider the example depicted in Figure 3.4. The trees T_1 and T_2 are the covectors of the apexes of the corresponding hyperplanes H_1 and H_2. The edge (ℓ_1, r_1) in T_1 implies that the corresponding cell is in the sector $S_1^{(1)}$. As r_1 has a larger degree in T_2, walking to the apex of H_2 requires us to move further in the '1-direction', and therefore we do not leave $S_1^{(1)}$. Lemma 3.1.11 is a purely combinatorial description of this behaviour that also covers the non-regular case. This will later motivate our definition of a combinatorial analogue of tropical hyperplane sectors.

3.2 Linkage matching fields

Recall from [85] the definition of the linkage property of a matching field.

Definition 3.2.1. *Let $\mathcal{M} = (M_\sigma)$ be an (n, d)-matching field where $n \geq d$. We say $\mathcal{M}$ is* linkage *if the following* linkage axiom *holds: for every $r_i \in R$ and $(d+1)$-subset $\tau \subseteq L$ there exist two distinct $\ell_j, \ell_{j'} \in \tau$ such that the matchings $M_{\tau \setminus \{\ell_j\}}$ and $M_{\tau \setminus \{\ell_{j'}\}}$ agree everywhere other than on r_i.*

Remark 3.2.2. *When considering a matching field as a set of bipartite graphs, it will be useful to differentiate between* left linkage *and* right linkage*. The previous definition is for left linkage as it describes how matchings on varying subsets of L are linked. Right linkage is defined analogously for matchings of right matching fields, where $|R| \geq |L|$ and one ranges over all $(|L|+1)$-subsets of R.*

Reformulating the conditions from [85, Theorem 2.4(3)] and [85, Corollary 2.12] yields the following.

Lemma 3.2.3. *The linkage axiom is equivalent to the following condition: let τ be a $(d+1)$-element subset of L. Then the union of all matchings on $\tau \sqcup R$ is a tree where all right nodes have degree 2.*

Note that the formerly described trees can also be characterised in terms of 'support sets' in the sense of [38, Proposition 2.6]. This yields another description of matching linkage covectors, see Definition 3.2.6, based on degree conditions.

The linkage axiom can be formulated in even more ways, the following of which we take advantage of:

Lemma 3.2.4. *[25, Theorem 2b] The linkage axiom is equivalent to the following property: for every two distinct d-subsets $\sigma, \sigma' \subset L$ there exists $\ell_{j'} \in \sigma' \setminus \sigma$ such that if $(\ell_{j'}, r_i)$ is an edge of $M_{\sigma'}$ and (ℓ_j, r_i) is an edge of M_σ then the matchings M_σ and $M_{\sigma \setminus \{\ell_j\} \cup \{\ell_{j'}\}}$ agree everywhere other than on r_i.*

The linkage property implies the compatibility of the occurring matchings as the next statement shows.

Proposition 3.2.5 (Weak compatibility)**.** *Let $\mathcal{M}$ be a linkage matching field of type (n, d) with $n \geq d$ and let σ, σ' be two d-element subsets of L. If there are subsets $P \subseteq \sigma \cap \sigma'$ and $Q \subseteq R$ with $|P| = |Q|$ such that $M_\sigma|_{P \sqcup Q}$ and $M_{\sigma'}|_{P \sqcup Q}$ are perfect matchings then those matchings agree.*

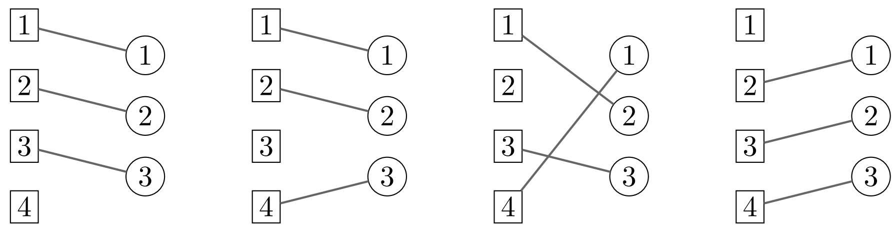

Figure 3.5: A $(4,3)$-matching field whose matchings are compatible. Note that the matching on $\{\ell_1, \ell_3, \ell_4\}$ does not agree with any other matching on two edges, and therefore the matching field does not satisfy the linkage property. Equivalently, the union of the matchings contains a cycle and, hence, it is not a tree.

Proof. The claim follows by induction on the size of the intersection $\sigma \cap \sigma'$. For $\sigma = \sigma'$ the claim is just the fact that there is exactly one matching per d-element subset in the matching field. Otherwise, by Lemma 3.2.4, there is an $\ell_{j'} \in \sigma' \setminus \sigma$ with certain properties. Since $\ell_{j'} \notin P$, the node r_i adjacent to $\ell_{j'}$ in $M_{\sigma'}$ is not an element of Q. Hence, the node ℓ_j adjacent to r_i in M_σ is not in P. By Lemma 3.2.4, the matching on $\sigma'' = \sigma \setminus \ell_j \cup \{\ell_{j'}\}$ is uniquely defined and also contains $M_\sigma|_{P \sqcup Q}$. Furthermore, $|\sigma'' \cap \sigma'| = |\left(\sigma \setminus \ell_j \cup \{\ell_{j'}\}\right) \cap \sigma'| = |\sigma \cap \sigma'| + 1$. This concludes the proof by induction. $\square$

Observe that compatibility is a weaker condition than linkage. Figure 3.5 shows an example of a matching field whose matchings are pairwise compatible but do not satisfy linkage.

We have now gathered the necessary tools to construct a linkage tope field from a linkage matching field.

3.2.1 Tope fields from matching fields

Definition 3.2.6. *We say that an (n,d)-tope field of type $(v_1, \ldots, v_d)$ with thickness k is* linkage *if for all $(k+1)$-subsets τ of L, the union of the topes on τ is a tree. Such a tree is a* (tope) linkage covector*. In particular, an (n,d)-matching field is* linkage *if for all $(d+1)$-subsets τ of L, the union of the matchings on τ is a tree. We call this tree a* (matching) linkage covector*.*

Remark 3.2.7. *The linkage covectors of a matching field are essentially the same as* *linkage trees defined in [85]. One can transform a matching linkage tree to a linkage covector by replacing the edge (j, j') with label i by the edges $(\ell_j, r_i), (\ell_{j'}, r_i)$. Linkage trees will play a role in results on the flip graph of a matching field in Section 3.2.4.*

Example 3.2.8. *Consider the matching field and tope field given in Figures 3.1 and 3.2. Both of these are linkage with the corresponding linkage covectors given in Figure 3.6.*

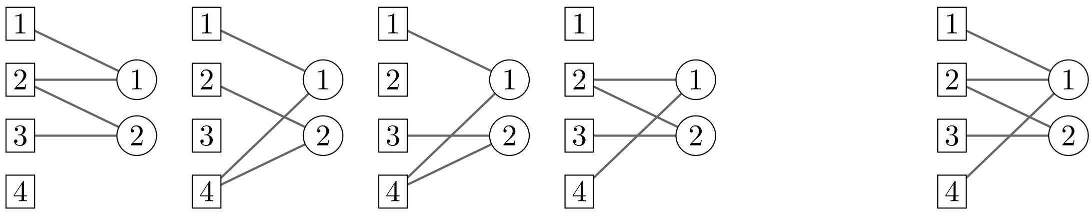

Figure 3.6: The four matching linkage covectors from the $(4, 2)$-matching field and the one tope linkage covector from the $(4, 2)$-tope field.

Example 3.2.9. *We continue to highlight the relationship to triangulations of the polytope $\Delta_{n-1} \times \Delta_{d-1}$ started in Subsection 3.1.1. Let $\tau \subseteq L$ be a $(d+1)$-subset. The triangulation induced on the product*

$$\mathrm{conv}\{e_j \mid \ell_j \in \tau\} \times \Delta_{d-1} \ ,$$

which is a face of $\Delta_{n-1} \times \Delta_{d-1}$, contains a unique maximal simplex whose corresponding tree has right degree vector $2 \cdot \mathrm{e}_{[d]}$ by [74, Theorem 12.9]. This is the linkage covector of the matching field on τ as can be seen from Proposition 3.1.9. This follows as it contains all the matchings on the d-subsets of τ and, hence, their union is just this tree. Hence, the matching field derived from a triangulation is linkage, as also stated in [72]. Restricting to regular subdivisions, this implies that coherent matching fields are linkage, see Example 3.1.2 and [85].

A matching field derived from a non-regular triangulation is depicted in Figure 3.7. The picture on the left shows the non-regular fine mixed subdivision which corresponds to a non-regular triangulation of $\Delta_5 \times \Delta_2$ by the Cayley trick, both of which can be found in [28]. Every cell contains an 'upward' unit simplex, in particular cells who do not share

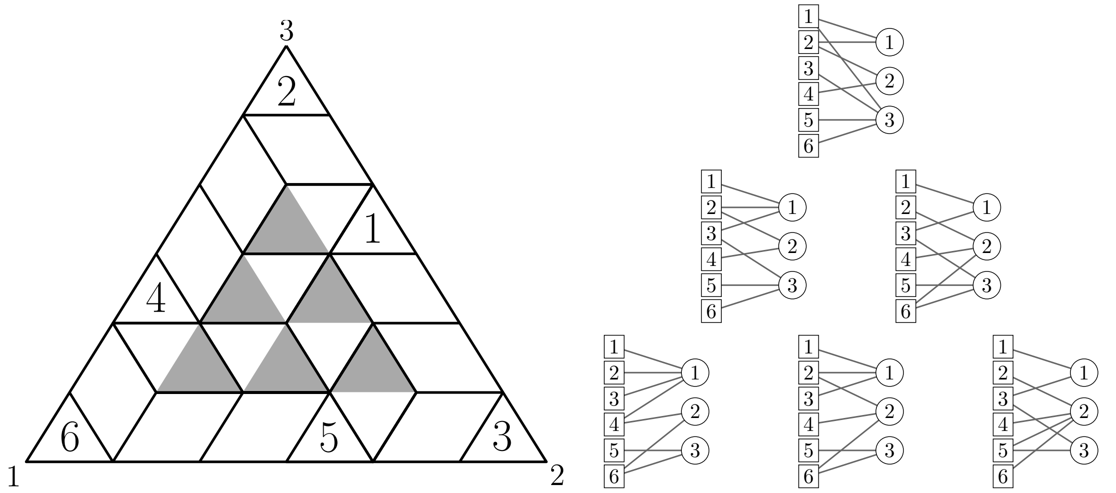

Figure 3.7: The maximal linkage covectors of a matching field extracted from the interior cells of a non-regular subdivision of $6\Delta_2$ (example from [28, Figure 9.53]). Each cell contains a unique grey simplex, the interior one being in bijection with the lattice points of the simplex $2\Delta_2$.

a facet with the boundary have their upward simplices coloured grey. The right side shows the bipartite graphs describing the maximal simplices corresponding to the grey upward simplices on the left. They are in bijection with the lattice points given by the right degree vectors of the trees on the right. Inner cells that do not share a facet with the boundary correspond to those trees whose right degree vector does not contain a 1 *entry.*

The next lemma together with Lemma 3.2.3 shows that Definition 3.2.6 agrees with Definition 3.2.1 for matching fields.

Fix a linkage (n, d)-tope field $\mathcal{M}$ of type v and thickness k. Consider a linkage covector C on $\tau \sqcup R$ with $|\tau| = k + 1$.

Lemma 3.2.10. *For each $r_i \in R$, there are exactly $v_i + 1$ nodes adjacent to r_i in C. Furthermore, for each $\ell_j \in \tau$, C contains the unique tope of the tope field with right degree vector v such that ℓ_j is isolated.*

Proof. Fix a node $r_i \in R$ and consider two different topes which have only isolated nodes in $L \setminus \tau$. Choose the second such that a neighbour of r_i is isolated in the first. This shows that r_i is adjacent to at least $v_i + 1$ nodes. Furthermore, summing up degrees across all nodes shows r_i must have exactly $v_i + 1$ neighbours, else C is not a tree.

For the second claim, the containment of such a tope is guaranteed by the definition of a linkage covector. Now, suppose the tope is not unique; then Lemma 3.1.7 implies that C contains a cycle. □

Our next construction starts to connect tope fields of different types.

Lemma 3.2.11. *For each $r_i \in R$, the tope linkage covector C contains a unique tope with right degree vector $v + e_i$. It is obtained as the union of topes of Lemma 3.2.10.*

Proof. Proposition 3.1.9 gives the first claim. For the second claim, consider $\ell_j, \ell_{j'}$ adjacent to r_i. Removing either of these nodes yields a tope of type v contained in the tope of type $v + e_i$. By Lemma 3.2.10 these topes are unique, therefore we can realise the tope of type $v + e_i$ as the union of them. □

For every $(k+1)$-subset $\tau \subseteq L$, there is a corresponding linkage covector C_τ. Fix some $r_i \in R$. By Lemma 3.2.11, there exists a unique tope G_τ of type $v + e_i$ contained in C_τ.

Definition 3.2.12. *The set of the graphs G_τ is the* i-amalgamation *of the tope field $\mathcal{M}$. This is a tope field of type $v + e_i$.*

Example 3.2.13. *The $(4,2)$-tope field of type $(2,1)$ in Figure 3.2 can be induced from the $(4,2)$-matching field in Figure 3.1 with the construction from Proposition 3.1.9. The topes are obtained from each of the linkage covectors by taking the 1-amalgamation, as we demonstrate in Figure 3.8.*

In the following, we fix a $(k+2)$-subset $\sigma \subseteq L$ and denote the tope $G_{\sigma \setminus \ell_s}$ by G_s.

Lemma 3.2.14. *Let G_j and $G_{j'}$ be two topes in the i-amalgamation of the (n,d)-tope field $\mathcal{M}$. Then the neighbourhood of r_i differs by at most one element in G_j and $G_{j'}$.*

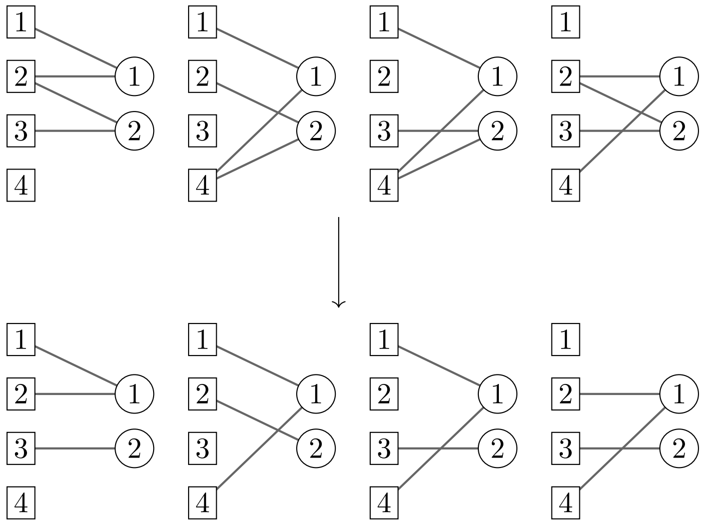

Figure 3.8: The $(4,2)$-tope field in Figure 3.2 arising as the 1-amalgamation of the $(4,2)$-matching field in Figure 3.1.

Proof. The two topes are subgraphs of the linkage covectors $C_j := C_{\sigma \setminus \ell_j}$ and $C_{j'} := C_{\sigma \setminus \ell_{j'}}$ respectively. The two linkage covectors C_j and $C_{j'}$ have a tope in common, namely the tope on $\sigma \setminus \{\ell_j, \ell_{j'}\} \sqcup R$ with right degree vector v, which is unique by Lemma 3.2.10. As G_j contains all edges of C_j incident with $r_i \in R$, in particular those in the common tope, the neighbourhood of r_i differs by at most one element in G_j and $G_{j'}$. □

Now we can prove the crucial observation that the linkage property is preserved under amalgamation.

Proposition 3.2.15. *The i-amalgamation of a linkage tope field of type v is a linkage tope field of type $v + e_i$.*

Proof. Let $\sigma \subseteq L$ be a $(k+2)$-subset and define G as the union of the topes G_τ in the i-amalgamation where $\tau \subset \sigma$ ranges over the $(k+1)$-subsets. We claim that G is a spanning tree with right degree vector $v + e_{[d]} + e_i$. Without loss of generality, we assume $\sigma = \{\ell_1, \ldots, \ell_{k+2}\}$ and $i = 1$. At first, we want to show that the degree of r_1 is $v_1 + 2$.

Assume ℓ_1 is adjacent to r_1 in G. The tope G_1 has $v_1 + 1$ nodes adjacent to r_1, assume these are $\{\ell_2, \ldots, \ell_{v_1+2}\}$. Hence r_1 has at least degree $v_1 + 2$ in G.

Let G_s be a tope containing the edge (ℓ_1, r_1). By Lemma 3.2.14, all other neighbours of r_1 in G_s form a v_1-subset of $\{\ell_2, \ldots, \ell_{v_1+2}\}$. Define ℓ_t to be the unique node such that $\{\ell_1, \ell_2, \ldots, \ell_{v_1+2}\} \setminus \{\ell_t\}$ is the neighbourhood of r_1 in G_s. Consider a tope G_j for $j \in [v_1+2] \setminus \{t\}$. Comparing it with G_1 and G_s, Lemma 3.2.14 shows that the neighbourhood of r_1 in G_j is $\{\ell_1, \ell_2, \ldots, \ell_{v_1+2}\} \setminus \{\ell_j\}$. Comparing G_t with such a G_j and G_1, the same argument yields that the neighbourhood of r_1 in G_t is $\{\ell_1, \ell_2, \ldots, \ell_{v_1+2}\} \setminus \{\ell_t\}$. Analogously, by comparing with G_1 and G_2, we obtain that the neighbourhood of r_1 in G_j for $j > v_1 + 2$ is a subset of $\{\ell_1, \ell_2, \ldots, \ell_{v_1+2}\}$. Therefore r_1 has degree $v_1 + 2$ in G.

Lemma 3.2.10 implies that $G_1, \ldots, G_{v_1+2}$ must agree on $\sigma \setminus \{\ell_1, \ldots, \ell_{v_1+2}\} \sqcup R \setminus \{r_1\}$. Explicitly, for any two topes G_s, G_t, removing the edges (ℓ_t, r_1) and (ℓ_s, r_1) from each graph gives two topes of type v on $\sigma \setminus \{\ell_s, \ell_t\}$. These topes are equal, as they are the common tope between C_s and C_t.

Fix a node $r_i \in R \setminus \{r_1\}$, we want to show that it has degree $v_i + 1$. Assume that this is not the case, let $\ell_{j_1}, \ldots, \ell_{j_{v_i}} \in \sigma$ be the nodes adjacent to r_i in G_m for all $m \in [v_1 + 2]$. Then there are nodes $\ell_{j_p}, \ell_{j_q} \in \sigma \setminus \{\ell_{j_1}, \ldots, \ell_{j_{v_i}}\}$ and $\ell_p, \ell_q \in \sigma$ such that ℓ_{j_p} is adjacent with r_i in G_p and ℓ_{j_q} is adjacent with r_i in G_q.

By Lemma 3.2.14, G_p and G_q agree on at least v_1 edges adjacent to r_1, assume without loss of generality they are $(\ell_1, r_1), \ldots, (\ell_{v_1}, r_1)$. Then G_{v_1+1}, G_p and G_q all contain these edges. Remove the edge (ℓ_1, r_1) from each graph, the resulting graphs are all topes of type v contained in the linkage covector C_1. This implies that r_i is adjacent to $\ell_{j_1}, \ldots, \ell_{j_{v_i}}, \ell_{j_p}, \ell_{j_q}$ in C_1, contradicting the property that r_i has degree $v_i + 1$ in C_1.

Finally, we prove that G is a tree. Assuming the opposite, the degree conditions imply that it is not connected. Then there are disjoint decompositions $\sigma = J \cup \overline{J}$ and $R = I \cup \overline{I}$ such that $r_1 \in I$ and there are no edges between J and $\overline{I}$ as well as between I and $\overline{J}$. For each $\ell_j \in \sigma$, G contains a tope with right degree vector $v + e_1$ such that ℓ_j is isolated, therefore we obtain $|J| \geq 2 + \sum_{r_i \in I} v_i$ and $|\overline{J}| \geq 1 + \sum_{r_i \in \overline{I}} v_i$. This contradicts $|J| + |\overline{J}| = k + 2 = \sum_{r_i \in I} v_i + \sum_{r_i \in \overline{I}} v_i + 2$. $\square$

This allows us to apply sequences of i-amalgamations to obtain a maximal tope from a linkage (n, d)-matching field for any right degree vector $(v_1, \ldots, v_d)$ such that $\sum v_i = n$. We refer to this construction as *iterated amalgamation.*

Theorem 3.2.16 (Iterated amalgamation)**.** *From a linkage matching field, we can construct maximal topes for all positive right degree vectors with sum n. Each one is the unique tope with a given right degree vector that is compatible with the matching field.*

Proof. We can apply Proposition 3.2.15 to obtain linkage tope fields with increasing thickness. By applying an i-amalgamation $v_i - 1$ times iteratively for i from 1 to d, we construct a linkage tope field of type $(v_1, \ldots, v_d)$. Note that a tope field with $\sum_{i=1}^{d} v_i = n$ contains only a single tope.

Moreover, there is exactly one tope with right degree vector $(v_1, \ldots, v_d)$ which is compatible with the original linkage matching field. Assume, on the contrary, that there are two such topes T_1 and T_2. As a result of Lemma 3.2.11, all the matchings in these topes are matchings of the matching field or submatchings of those. By Lemma 3.1.7, these topes differ in a matching. This implies that the contained matchings are not weakly compatible which contradicts Proposition 3.2.5. Finally, this implies the uniqueness and compatibility. □

Remark 3.2.17. *The tope linkage covectors arising from iterated amalgamation do not have to be compatible, see Example 3.2.18. Note, however, that the tope linkage covector is the only tree of that right degree vector which is possibly compatible with the matching field. This follows from Proposition 3.1.9 since it is uniquely determined by the contained maximal topes. In particular, if that tope linkage covector is not compatible with the matching field then there does not exist a tree with the same right degree vector that is. Note that such matching fields cannot be realised by a fine mixed subdivision of $n\Delta_{d-1}$.*

Example 3.2.18. *Consider the $(6, 4)$-linkage matching field $\mathcal{M}$ given by the linkage covectors of matchings in Figure 3.9. Note that there are multiple pairs of trees that are* *not compatible. In particular, the fourth tree contains a matching on* $\{\ell_5, \ell_6\} \sqcup \{r_1, r_2\}$ *which is incompatible with the matching of* $\mathcal{M}$ *on* $\{\ell_3, \ell_4, \ell_5, \ell_6\}$ *that is contained in the fifth and sixth trees. Incompatibility does not affect the amalgamation process and so there is still a unique maximal tope for each* $(v_1, \dots, v_d)$ *such that* $\sum v_i = 6$ *and* $v_i \geq 1$*. These are given in Figure 3.10 by their bijection to the lattice points of* $2\Delta_3$*.*

Figure 3.11 shows the tope linkage covectors that the maximal topes are derived from. They too have a natural lattice point bijection given by their right degree vectors as there is one for each $(v_1, \dots, v_d)$ *such that* $\sum v_i = 9$ *and* $v_i \geq 2$*. However, we note that multiple covectors have the same left degree vectors. By [74, Theorem 12.9], these covectors cannot encode the maximal cells of a triangulation of* $\Delta_5 \times \Delta_3$*, and so this matching field is not realisable by a fine mixed subdivision of* $6\Delta_3$*.*

The bijections between various graphs and lattice points will be explored further in Section 3.2.2.

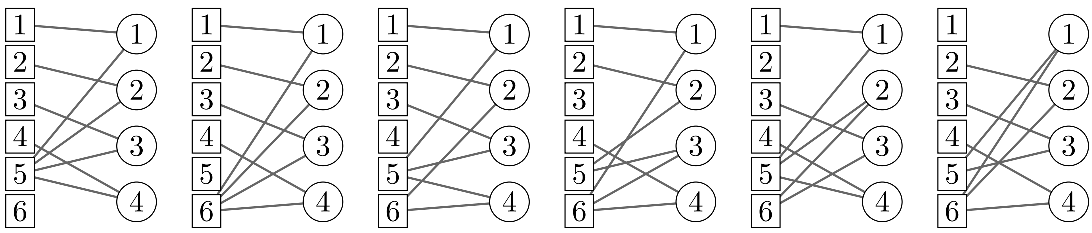

Figure 3.9: The linkage covectors of the linkage matching field discussed in Example 3.2.18.

3.2.2 Chow covectors

The graphs in the next definition were first introduced in [85] in the guise of brackets. They were used to explicitly describe extremal terms of the Chow form of the variety of complex degenerate $(n \times d)$-matrices, as well as to describe a universal Gröbner basis of the ideal generated by the maximal minors of a matrix of indeterminates. We shall define and consider them in purely combinatorial terms as bipartite graphs.

Definition 3.2.19. *Let* $\mathcal{M} = (M_\sigma)$ *be a matching field. For every* $(n-d+1)$*-subset*

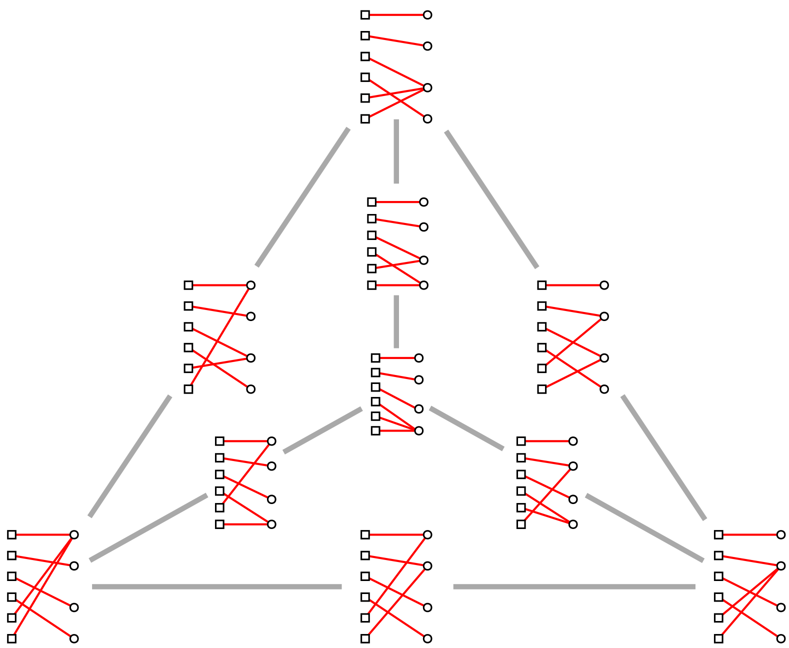

Figure 3.10: The $(6,4)$-tope arrangement of maximal topes derived from the matching field in Example 3.2.18. There is a natural bijection with the lattice points of $2\Delta_3$ via their right degree vector minus $e_{[d]}$.

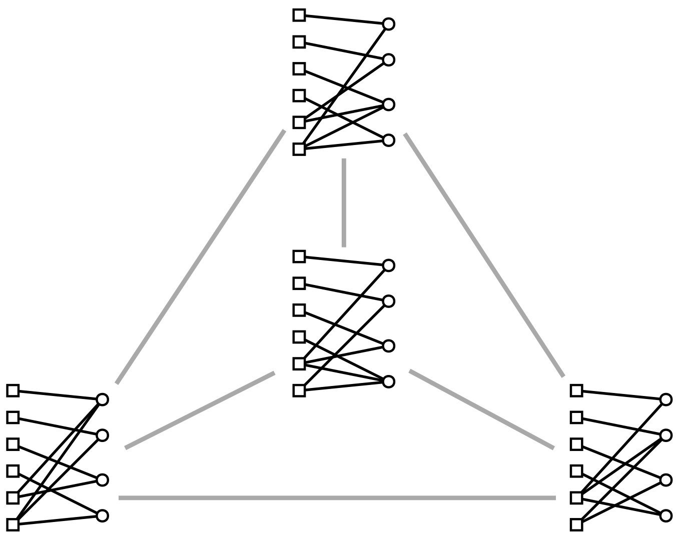

Figure 3.11: The tope linkage covectors that the maximal topes in Figure 3.10 are derived from. There is a natural bijection with the lattice points of Δ_3 via their right degree vector minus $e_{[d]}$.

$\rho \subset L$ *we define the* Chow covector *as the graph of the mapping* $\Omega_\rho \colon \rho \to R$ *with*

$$\Omega_\rho(j) = M_{\bar{\rho}\cup\{j\}}(j)$$

where $\bar{\rho} = L \setminus \rho$.

Remark 3.2.20. *The Chow covectors have a combinatorial characterisation that we shall exploit later. A graph G on $L \sqcup R$ is* transversal *to a matching field $\mathcal{M}$ if $G \cap M \neq \emptyset$ for all $M \in \mathcal{M}$. Bernstein and Zelevinsky [25, Theorem 1] showed that the Chow covectors are the minimal transversals to a matching field.*

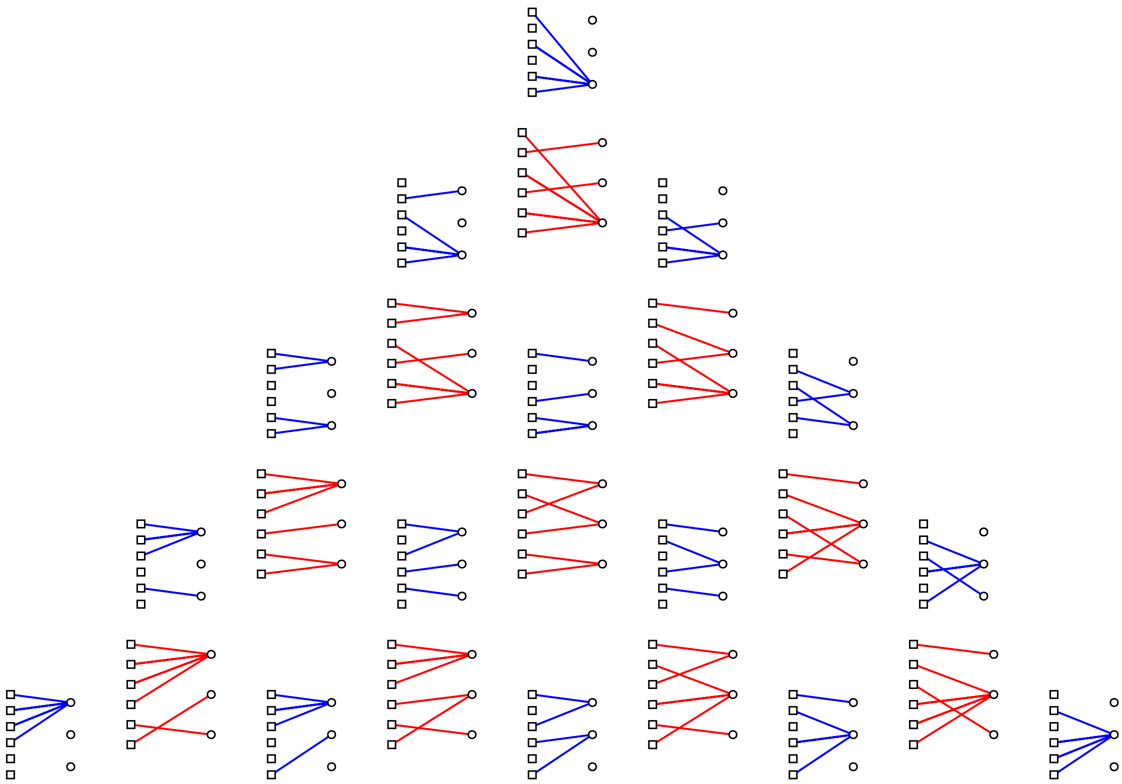

Figure 3.12: The topes and Chow covectors corresponding to the $(6, 3)$-matching field from Figure 3.7, presented via their bijection to $\Delta^{\mathbb{Z}}(d, n-d)$ and $\Delta^{\mathbb{Z}}(d, n-d+1)$ respectively. The topes are coloured red and the Chow covectors are coloured blue.

Again, fix a linkage (n, d)-matching field $\mathcal{M}$. We will derive the Chow covectors of a linkage matching field from its topes. To do this we need a statement similar to Lemma 3.1.11.

Lemma 3.2.21. *Let T_1 and T_2 be two compatible topes with right degree vectors v and $v + e_p - e_q$ (where $v_q \geq 2$). The edges incident with r_q in T_2 are all contained in T_1.*

Proof. We define an auxiliary directed graph H on $L \sqcup R$. The set of arcs is given by the edges of $T_1 \setminus T_2$ directed from L to R and of $T_2 \setminus T_1$ directed from R to L. Observe the following degree properties of H. Every node in L has in-degree equal to out-degree equal to either 1 or 0. Every node in $R \setminus \{r_p, r_q\}$ has in-degree equal to out-degree.

Now, assume that the claim does not hold, which means that r_q has in-degree bigger or equal to 2. As the out-degree of r_q is just 1 less than the in-degree, it has out-degree at least 1. Consequently, since the sum of the right degree of T_1 and T_2 are the same, the out-degree of r_p has to be bigger or equal to 2. Hence, all non-isolated nodes have out-degree at least 1 which means that H contains a directed cycle. This is necessarily alternating between edges of T_1 and T_2. However, this contradicts the compatibility of T_1 and T_2, concluding the claim. □

Proposition 3.2.22. *Given a linkage matching field $\mathcal{M}$, there is a unique Chow covector with right degree vector v associated to $\mathcal{M}$. It can be constructed from the intersection of the maximal topes of $\mathcal{M}$ with right degree vector $v + e_{[d]\setminus\{i\}}$ for $i \in [d]$.*

Proof. Let v be a vector of non-negative integers with coordinate sum $n - d + 1$. We fix a node $r_i \in R$ and consider the graph obtained by intersecting the topes with right degree vectors $v + e_{[d]\setminus\{i\}}$. If $v_i = 0$, there is no tope with right degree vector $v + e_{[d]\setminus\{i\}}$. Instead, we delete any edge adjacent to r_i from the graph. Denote this graph by G. By Lemma 3.2.21, the topes which we intersect agree on the edges of the tope where a given node has degree v_i. For an example, see the red graphs surrounding a blue graph in Figure 3.12. Hence, G has $n - d + 1$ edges, each node $r_i \in R$ has degree v_i and there are $n - d + 1$ nodes in L with degree 1, all others having degree 0.

Let $\rho \subset L$ be the set of left nodes with degree 1 in G. We claim that G is the Chow covector Ω_ρ. Fix an $\ell_j \in \rho$ adjacent to the node r_i. The maximal tope with right degree vector $v + e_{[d]\setminus\{i\}}$ exists, and contains a matching on $(L \setminus \rho) \cup \{\ell_j\}$. Explicitly, it is precisely the union of the edges not contained in G with (ℓ_j, r_i). This has to be a matching of the matching field by Lemma 3.2.11. Hence, Ω_ρ is a subgraph of G and, hence, equal because of the given degrees of G. □

Consider the dilated simplex $n\Delta_{d-1}$ with its canonical embedding into $\mathbb{R}^d$. We denote the set of integer lattice points of $n\Delta_{d-1}$ by $\Delta^{\mathbb{Z}}(d,n)$. Observe that Theorem 3.2.16 gives a bijection between the set of maximal topes of a matching field and $\Delta^{\mathbb{Z}}(d,n-d)$ by mapping a tope with right degree vector v to the lattice point $v - e_{[d]}$. Sturmfels and Zelevinsky [85, Conjecture 6.10] conjectured a similar bijection for Chow covectors. The construction in Proposition 3.2.22 allows us to complete the proof of this conjecture.

Theorem 3.2.23. *The map from the Chow covectors of a linkage (n,d)-matching field to the lattice points of $(n-d+1)\Delta_{d-1}$ is a bijection.*

Proof. For each element of $\Delta^{\mathbb{Z}}(d,n-d+1)$, Proposition 3.2.22 gives us an explicit construction of the Chow covector with that right degree vector, therefore this map is surjective. Furthermore, the set of Chow covectors and $\Delta^{\mathbb{Z}}(d,n-d+1)$ both have cardinality $\binom{n}{n-d+1}$, therefore this map is a bijection. □

Observe that the construction in Proposition 3.2.22 can be inverted.

Corollary 3.2.24. *A linkage matching field is uniquely determined by its set of Chow covectors.*

Proof. Given the set of Chow covectors of a linkage matching field, we can recover the tope with right degree vector v by taking the union of the Chow covectors with right degree vector $v - e_{[d]\setminus\{i\}}$. By the last part of the construction in Proposition 3.2.22, we know that the topes contain a matching on all d-subsets of L. As the topes are all compatible this implies that those matchings form exactly the matching field. □

Example 3.2.25. *Figure 3.12 shows the topes, coloured red, corresponding to the $(6,3)$-matching field derived from the non-regular triangulation illustrated in Figure 3.7 (see Subsection 3.1.1). As there is a unique tope for every possible right degree vector, these form a bijection with $\Delta^{\mathbb{Z}}(3,3)$, the lattice points of the simplex $3\Delta_2$. The Chow covectors, coloured blue, of this matching field can be recovered from the topes via the procedure* *described in the proof of Proposition 3.2.22. There is precisely one for every lattice point in* $\Delta^{\mathbb{Z}}(3,4)$ *as encoded by their right degree vectors. The construction is closely related to the mixed subdivision representation of a triangulation which is given by the Cayley Trick [80].*

As we will see in Section 3.3, the trees encoding a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ are completely determined by the pairs of lattice points given by their left and right degree vectors. We show that this also holds for the bijection associated with the Chow covectors. This is the generalisation of [85, Conjecture 6.8 b)] discussed below their claim. For this, we need two lemmas.

Lemma 3.2.26. *Given the set of Chow covectors of a linkage matching field on* $L \sqcup R$, *the set*

$$\left\{ \Omega_\rho^{(j)} \;\middle|\; \Omega_\rho^{(j)} \text{ restriction of } \Omega_\rho \text{ to } (L \setminus \{\ell_j\}) \sqcup R, \ell_j \in \rho \right\}$$

is the set of Chow covectors of the induced submatching field on $(L \setminus \{\ell_j\}) \sqcup R$.

Proof. A matching μ in the induced submatching field is a matching in the original matching field that does not contain any edges adjacent to ℓ_j. By transversality from [25], $\mu \cap \Omega_\rho$ is non-empty. As μ has no edges adjacent to ℓ_j this implies $\mu \cap \Omega_\rho^{(j)}$ is non-empty as well and, hence, $\Omega_\rho^{(j)}$ is a transversal on the set $\rho \setminus \{\ell_j\}$. As it contains the same number of edges as a Chow covector, by minimality it must be equal to the Chow covector $\Omega_{\rho \setminus \{\ell_j\}}$. Iterating over all ρ, we obtain all Chow covectors on $(L \setminus \{\ell_j\}) \sqcup R$. □

Finally, we need an analogous lemma to Lemmas 3.2.21 and 3.1.11.

Lemma 3.2.27. *Let* T_1 *and* T_2 *be two Chow covectors with right degree vectors* v *and* $v + e_p - e_q$ *(where* $v_q \geq 1$*) constructed from the same linkage matching field. The edges incident with* r_q *in* T_2 *are all contained in* T_1*. Furthermore, if* ℓ_s *has degree 1 in both covectors,* T_1 *and* T_2 *agree on the edge adjacent to* ℓ_s.

Proof. The first claim follows from Lemma 3.2.21 with the construction in Proposition 3.2.22. For the second, consider the tope with right degree vector $v + e_{[d]\setminus p}$. By Proposition 3.2.22, both Chow covectors are contained in this tope and so agree with it on the edge adjacent to ℓ_s. $\square$

Let $\mathcal{M}$ be a linkage matching field. Define $\varphi_{\mathcal{M}} \colon \binom{[n]}{n-d+1} \to \Delta^{\mathbb{Z}}(d, n-d+1)$ to be the bijection mapping $\rho \in \binom{[n]}{n-d+1}$ to the right degree vector of Ω_ρ. We present a proof of Conjecture 6.8 b) of [85] that this map uniquely determines the linkage matching field. To do so, we introduce a combinatorial analogue to sectors of tropical hyperplanes, as discussed in Example 3.1.12.

Definition 3.2.28. *Let $\mathcal{G}$ be a collection of compatible bipartite graphs on $L \sqcup R$ with a bijective map $\mathcal{G} \to \Delta^{\mathbb{Z}}(d, k)$ determined by right degree vectors for some k. The* (open) combinatorial sector $\mathcal{S}_j^{(i)}$ *is defined as follows:*

$$\mathcal{S}_j^{(i)} = \{ G \in \mathcal{G} \mid (\ell_j, r_i) \in G, \ell_j \textit{ has degree } 1 \}$$

We have seen multiple classes of bipartite graphs with a bijection to lattice points of dilated simplices, namely trees, topes and Chow covectors, and so we can define combinatorial sectors for any of them. Furthermore, all have similar local properties (see Lemmas 3.1.11, 3.2.21 and 3.2.27) that give a lot of structure to the combinatorial sectors.

In particular, let $\mathcal{G} = \left\{ \Omega_\rho \;\middle|\; \rho \in \binom{[n]}{n-d+1} \right\}$ where $k = n - d + 1$ and consider the combinatorial sectors of the set of Chow covectors.

Theorem 3.2.29. *A linkage matching field can be uniquely determined by its map φ.*

Proof. We claim that we can reconstruct the Chow covectors from their left and right degree vector pairs. As $\varphi_{\mathcal{M}}$ is given by these degree vectors, the theorem follows from this claim and Corollary 3.2.24. We proceed by induction on $n \geq d$. For $n = d$, the matching field consists of one matching and the n Chow covectors are just the edges of the matching. The degree vector pair of an edge uniquely determines it, which implies the claim for this case.

Assume that the claim is true for all linkage (k, d)-matching fields with $d \leq k < n$ and let $\mathcal{U}$ be the set of degree vector pairs of the Chow covectors for a linkage (n, d)-matching field. We get a non-disjoint decomposition

$$\bigcup_{j \in [n]} \mathcal{L}_j = \mathcal{U} \quad \text{for} \quad \mathcal{L}_j = \{(u, v) \mid u_j = 1\} \ .$$

Now fix a $j \in [n]$. There is a partition of $\mathcal{L}_j$ in the sets

$$\mathcal{L}_j^{(i)} = \{(u, v) \mid (\ell_j, r_i) \in \Omega_\rho, \ \Omega_\rho \text{ has degree vector } (u, v) \text{ with } u_j = 1\} \ .$$

Note that $\mathcal{L}_j^{(i)}$ is the image of $\mathcal{S}_j^{(i)}$ under the map that sends a Chow covector to its degree vector pair.

From $\mathcal{L}_j^{(i)}$ we can construct the set $\overline{\mathcal{L}_j^{(i)}}$ by removing the jth entry of the first component and decreasing the ith entry of the second component for all the pairs in $\mathcal{L}_j^{(i)}$. This corresponds to removing the edge (ℓ_j, r_i). The resulting set

$$\overline{\mathcal{L}_j} = \bigcup_{i \in [d]} \overline{\mathcal{L}_j^{(i)}}$$

is the set of degree pairs of the Chow covectors of the submatching field on $(L \setminus \{\ell_j\}) \sqcup R$, by Lemma 3.2.26.

Here, we can apply induction and deduce that $\overline{\mathcal{L}_j}$ uniquely defines the Chow covectors with the contained degree vectors. From the partition into the $\overline{\mathcal{L}_j^{(i)}}$ we can recover to which node ℓ_j is incident in the original Chow covector. Therefore, we can construct all Chow covectors for which ℓ_j has degree 1. Ranging over all $j \in [n]$, we get all the Chow covectors.

It remains to show how to construct the set $\mathcal{L}_j^{(i)}$ for each $r_i \in R$, which we now demonstrate. Assume without loss of generality that $i = 1$ and apply Algorithm 1.

Algorithm 1 Construct the degree pairs of a combinatorial sector of Chow covectors

1: **if** $u_j = 1$ for (u, v) with $v_1 = n - d + 1$ **then**
2: $\quad \mathcal{K}_j \leftarrow \{(u, v)\}$
3: **else**
4: $\quad$ terminate
5: **end if**
6: $h \leftarrow n - d$
7: **while** $h \geq 0$ **do**
8: $\quad$ **for all** $(u, v) \in \mathcal{L}_j$ with $v_1 = h$ **do**
9: $\quad\quad$ **if** $\exists k \in [d]\colon v_k > 1\colon \exists\, w^{(k)}\colon (w^{(k)}, v + e_1 - e_k) \in \mathcal{K}_j$ **then**
10: $\quad\quad\quad \mathcal{K}_j \leftarrow \mathcal{K}_j \cup (u, v)$
11: $\quad\quad$ **end if**
12: $\quad\quad h \leftarrow h - 1$
13: $\quad$ **end for**
14: **end while**

Claim: $\mathcal{K}_j = \mathcal{L}_j^{(1)}$.

Proof by induction There is a unique Chow covector Ω_{ρ_0} with the right degree vector $(n - d + 1)e_1$. If $u_j = 1$ then ℓ_j is adjacent to r_1 because of the structure of the right degree vector. Line 2 in the algorithm guarantees that Ω_{ρ_0} is in $\mathcal{K}_j$. Furthermore, the edge (ℓ_j, r_1) shows that it is also contained in $\mathcal{L}_j^{(1)}$.

Now, assume that $\mathcal{K}_j$ and $\mathcal{L}_j^{(1)}$ agree in all elements whose first entry of the second component is $h + 1 \leq n - d + 1$.

Let $(u, v) \in \mathcal{L}_j$ such that $v_1 = h$ and $(w, v + e_1 - e_k) \in \mathcal{K}_j$ an element fulfilling the condition in Line 9. These two vectors are the right degree vectors of two Chow covectors Ω_{ρ_1} and Ω_{ρ_2}. Note that $\mathcal{K}_j \subseteq \mathcal{L}_j$. As, by the induction hypothesis, (ℓ_j, r_1) is an edge of Ω_{ρ_2} we can deduce with Lemma 3.2.27 that this is also an edge of Ω_{ρ_1}. Hence, (u, v) is an element of $\mathcal{L}_j^{(1)}$.

Conversely, let $(u, v) \in \mathcal{L}_j^{(1)}$ be with $v_1 = h$. Also by Lemma 3.2.27, there is a $k \in [d]$ and a $w = e_\rho$ for some ρ such that in the Chow covector with degree pair $(w, v + e_1 - e_k)$ the node ℓ_j is a leaf and it is adjacent to r_1. The induction hypothesis implies that

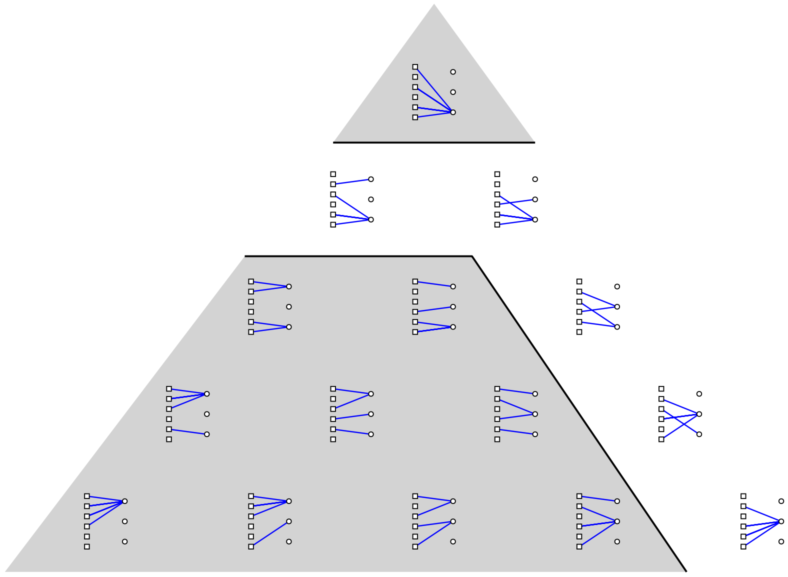

Figure 3.13: Decomposition of the Chow covectors from Example 3.2.25 depending on the neighbouring vertex of ℓ_1; it is either adjacent to r_3, isolated or adjacent to r_1. The grey regions are the combinatorial sectors $S_1^{(1)}$ and $S_1^{(3)}$. Note that $S_1^{(2)}$ is empty as the edge (ℓ_1, r_2) appears in no Chow covector.

$(w, v + e_1 - e_k) \in \mathcal{K}_j$. Now, Line 9 shows that also (u, v) is an element of $\mathcal{K}_j$. □

3.2.3 A cryptomorphic description

In Section 3.2.1, we saw how one can construct a set of compatible topes in bijection with $\Delta^{\mathbb{Z}}(d, n-d)$ from a linkage (n, d)-matching field. A slight generalisation of the proof of Proposition 3.2.22 and Corollary 3.2.24 leads to a cryptomorphic description of linkage matching fields in terms of topes.

Definition 3.2.30. *An (n, d)-*tope arrangement *is a set of compatible topes in bijection with $\Delta^{\mathbb{Z}}(d, n-d)$ via the map that sends a tope with right degree vector v to $v - e_{[d]}$.*

Example 3.2.31. *Figure 3.10 shows a $(6, 4)$-tope arrangement which cannot arise from a triangulation of $\Delta_5 \times \Delta_3$.*

For the construction of the Chow covectors, one only needs Lemma 3.2.21. Analogously to Corollary 3.2.24 we get a matching for each d-subset. Combining this with Theorem 3.2.16 yields the following.

Theorem 3.2.32. *Linkage (n, d)-matching fields and (n, d)-tope arrangements are cryptomorphic.*

Proof. As tope arrangements satisfy the conditions of Lemma 3.2.21 and are in bijection with $\Delta^{\mathbb{Z}}(d, n-d)$, we can construct the graphs Ω_ρ for all $\rho \in \binom{[n]}{n-d+1}$ via Proposition 3.2.22. The Chow covector with left degree vector ρ gives rise to the matchings on $[n] \setminus \rho \cup \{j\}$ for all $j \in \rho$. As the topes are compatible, each ρ occurs exactly once.

These graphs have the same properties as Chow covectors, in particular that they yield the existence of a perfect matching for every d-subset $\sigma \subset L$ contained in some tope in the tope arrangement in the same way as the construction at the end of the proof of Proposition 3.2.22. Note that these matchings are unique as the topes are compatible, therefore the tope arrangement induces a matching field. It remains to show that the matching field is linkage.

Let $\sigma, \sigma' \subset L$ be distinct d-subsets and $\ell_{j'} \in \sigma' \setminus \sigma$. Consider a tope T that contains the matching on σ and consider the node r_i adjacent to $\ell_{j'}$. There exists some node $\ell_j \in \sigma$ adjacent to r_i in the matching on σ, therefore the matching on $\sigma \setminus \{\ell_j\} \cup \{\ell_{j'}\}$ agrees with the matching on σ outside of r_i. This is equivalent to the linkage axiom by Lemma 3.2.4. $\square$

Remark 3.2.33. *Tope arrangements have similar structural properties to trianguloids, which were introduced very recently in [42]. The trianguloid axioms $(T1)$ and $(T2)$ essentially state that the graph associated to every lattice point in $\Delta^{\mathbb{Z}}(d, n-d)$ is a tope. Furthermore, axiom $(T3)$ comprises the combinatorial sector condition which we exhibit for general linkage matching fields in Lemma 3.2.21. However, for trianguloids there is no global compatibility assumption, rather there is the Hexagon axiom $(T4)$. This replaces the need for pairwise compatibility with a more manageable local condition. It may be possible to replace the need for compatibility in the definition of tope arrangement with a similar local condition.*

Note that while structurally similar, tope arrangements are a more general class of objects than trianguloids. This is immediate from the fact that trianguloids are in bijection with triangulations of $\Delta_{n-1} \times \Delta_{d-1}$, while there are examples of linkage matching fields that are not realisable as a fine mixed subdivision of $n\Delta_{d-1}$, see Example 3.2.18. The question of realisability of linkage matching fields is addressed further in Section 3.4. Our notion of "matching field completion" leads to the notion of tope arrangement which allows "boundary" topes. This would give a cryptomorphic description of trianguloids and shed light on the problem of extendibility addressed in Section 3.4.

3.2.4 Matching field polytopes and the flip graph

The notion of a *matching field polytope* first occurs in [67]. It is the convex hull of the characteristic vectors of the matchings of an (n, d)-matching field in $\mathbb{R}^{n\times d}$.

This is a natural analogue of the matroid polytope, as in some sense matching fields play the role of a matroid for tropical linear algebra. However, unlike matroid polytopes, their vertex-edge graph is not the flip graph of the matchings as we demonstrate in Example 3.2.34.

Here, the nodes of the *flip graph* are the matchings and two matchings are adjacent if and only if they differ in one edge.

Example 3.2.34. *We consider a $(5, 3)$-linkage matching field as shown in Figure 3.14. The matchings are encoded by words of length three where the matching contains (ℓ_j, r_i) if j is in the ith position. Two matchings differ by a flip if the words differ in precisely one position. We note that the flip graph can be decomposed into the linkage trees for each 4-subset of the set $[5]$, each one represented by a different colour.*

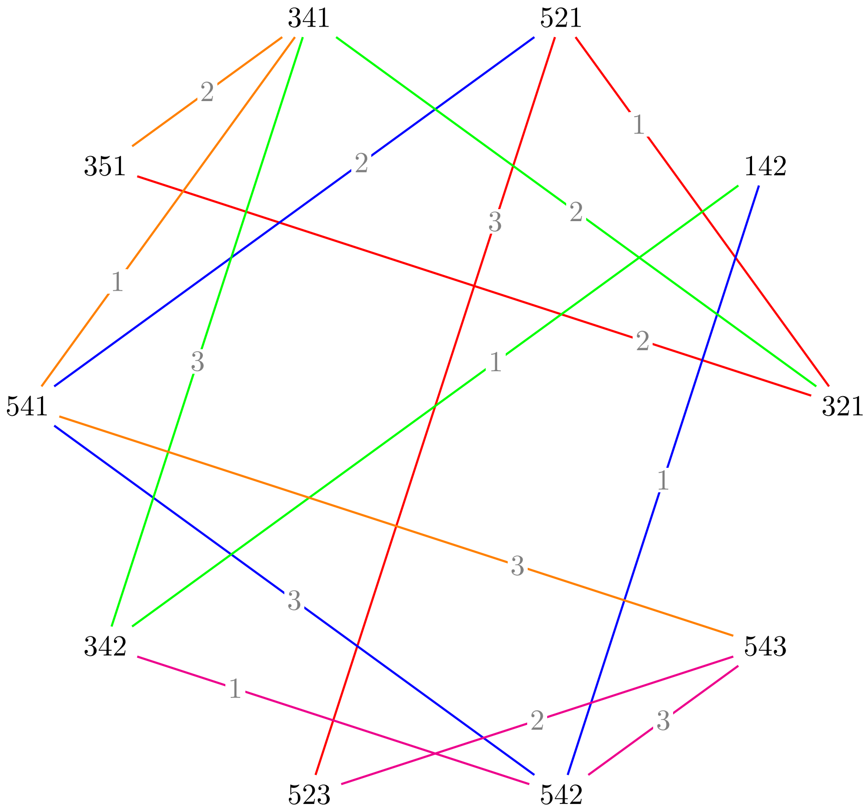

Figure 3.14: The flip graph of a $(5,3)$-matching field. Each coloured subgraph is the embedding of the linkage tree of the 4-subset. Each edge is labelled by the deviating position.

$$
\begin{array}{rccccccccccccccc}
0: & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1: & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
2: & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
3: & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
4: & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
5: & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
6: & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
7: & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
8: & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
9: & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}
$$

The convex hull of these vectors is the matching field polytope of the $(5,3)$*-matching* *field. It has* 35 *edges and* f*-vector* $(10, 35, 61, 59, 32, 9)$*. The adjacencies of its vertex edge graph are*

$$\begin{array}{lllllllll}
0: & 1 & 2 & 3 & 6 & 7 & 8 & & \\
1: & 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2: & 0 & 1 & 5 & 6 & 8 & 9 & & \\
3: & 0 & 1 & 4 & 5 & 6 & 7 & 8 & \\
4: & 1 & 3 & 5 & 6 & 7 & 8 & 9 & \\
5: & 1 & 2 & 3 & 4 & 6 & 8 & 9 & \\
6: & 0 & 1 & 2 & 3 & 4 & 5 & 8 & \\
7: & 0 & 1 & 3 & 4 & 8 & 9 & & \\
8: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 \\
9: & 1 & 2 & 4 & 5 & 7 & 8 & &
\end{array}.$$

This was computed with `polymake` *[44]. In particular, the graph in Figure 3.14 is a proper subgraph of the vertex-edge graph.*

For a linkage matching field $\mathcal{M}$, the flip graph has some nice properties.

Lemma 3.2.35. *The flip graph of an* (n, d)*-linkage matching field has* $\binom{n}{d+1} \cdot d$ *edges.*

Proof. The edges correspond to those topes whose right degree vector is a permutation of the partition $(2^1, 1^{d-1})$. By Theorem 3.2.16, there are exactly d such topes for each linkage covector, each of which are distinct. Since there are $\binom{n}{d+1}$ linkage covectors, the claim follows. □

More generally we obtain a characterisation of the topes in terms of subgraphs of the flip graph.

Proposition 3.2.36. *A tope with right degree vector* v *is the union of the* $v_1 \cdots v_d$ *matchings on the sets* $N_1 \times \cdots \times N_d$*, where* N_i *are the nodes adjacent with* r_i *in the tope.*

Conversely, let U *be a subset of the matchings such that the induced subgraph of the flip graph on* U *is the vertex-edge graph of a product of simplices* $\Delta_{v_1-1} \times \cdots \times \Delta_{v_d-1}$*,*

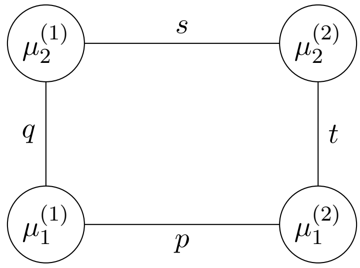

Figure 3.15: Quadrangle of matchings from Lemma 3.2.37.

where $v_i \geq 1$ for all $i \in [d]$. Then the union of the matchings in U is a tope with right degree vector $(v_1, \ldots, v_d)$.

Lemma 3.2.37. *Let $\mu_1^{(1)}, \mu_1^{(2)}, \mu_2^{(1)}, \mu_2^{(2)}$ be matchings such that the induced subgraph on their corresponding vertices in the flip graph is a quadrangle. Then there exist two distinct nodes $r_p, r_q \in R$ such that $\mu_m^{(1)}, \mu_m^{(2)}$ agree outside of r_p and $\mu_1^{(m)}, \mu_2^{(m)}$ agree outside of r_q for $m = 1, 2$.*

Proof. By definition of the flip graph, each pair of adjacent matchings agree outside of a single node. Let Figure 3.15 be the induced subgraph of the matchings where the edge labels denote which node they differ in. We deduce that $\mu_1^{(1)}, \mu_2^{(2)}$ must agree outside of two nodes, specifically $\{r_p, r_q\}$ and $\{r_s, r_t\}$, therefore these two sets must be equal. Observe that any two adjacent edges in the flip graph must correspond to different right nodes, else they form a 3-cycle between their vertices, contradicting the quadrangle as our induced subgraph. Therefore $r_p = r_s$ and $r_q = r_t$.

□

Proof. (Proposition 3.2.36) Given a tope on $\sigma \subseteq L$, the restriction of the linkage matching field to σ is also linkage. The first part follows directly from Theorem 3.2.16 by applying it to the linkage matching field restricted to σ.

We prove the second part by induction. If the induced subgraph on U is the vertex-edge graph of a product of simplices with only one non-trivial factor, it is the vertex-edge graph of a simplex and so all the matchings in U can only differ in the edges incident with the same node. Hence, their union is a tope.

Assume that the induced subgraph is the vertex-edge graph of a product of simplices where $k \geq 2$ factors are non-trivial. Without loss of generality let these be the first k factors. Then U decomposes into v_k disjoint sets $U_1, \ldots, U_{v_k}$ corresponding to faces of the product such that the induced subgraph on U_i is isomorphic to the graph of $\prod_{j \in [k-1]} \Delta_{v_j - 1}$ for all $i \in [v_k]$. By induction the union of the matchings in U_i forms a tope T_i whose right degree vector is a permutation of $(v_1, \ldots, v_{k-1}, 1, \ldots, 1)$. In particular, there is a $(k-1)$-set σ_i, such that every matching contained in T_i agrees on $R \setminus \sigma_i$. Note that T_i and U_i contain exactly the same matchings as subgraphs and as elements respectively.

We claim that for all $i, j \in [v_k]$ the topes T_i, T_j differ in a single node of degree one. As the induced subgraph of $U_i \cup U_j$ is isomorphic to $(\prod_{j \in [k-1]} \Delta_{v_j - 1}) \times \Delta_1$, for any matchings $\mu_i^{(1)}, \mu_i^{(2)} \in U_i$ that differ by a flip, there exists $\mu_j^{(1)}, \mu_j^{(2)} \in U_j$ such that the induced subgraph on their corresponding vertices in the flip graph is a quadrangle. By Lemma 3.2.37, we draw two conclusions: that $\mu_i^{(1)}, \mu_j^{(1)}$ and $\mu_i^{(2)}, \mu_j^{(2)}$ differ in the same node and that $\mu_i^{(1)}, \mu_i^{(2)}$ and $\mu_j^{(1)}, \mu_j^{(2)}$ do also. Iterating this over all pairs of matchings that differ by a flip, the first statement implies that T_i, T_j differ in one node, while the second implies it must be a node of degree one. If this was not the case, a node of degree two would form a 3-cycle with any pair of matchings in U_i that agree outside of that node, breaking the quadrangle.

We obtain that the topes T_i all differ in a flip of an edge incident with the same node. As each T_i has a different neighbour, the union of the T_i gives a tope with right degree vector $(v_1, \ldots, v_d)$. $\square$

The occurrence of all the vertex-edge graphs of products of simplices in the flip graph can be used to define an interesting cell complex.

Definition 3.2.38 ([59, Section 9.2.1])**.** *Let G be an arbitrary graph. The* prodsimplicial flag complex $PF(G)$ *of* G *is defined as follows: the graph* G *is taken to be the* 1*-skeleton of* $PF(G)$*, and the higher-dimensional cells are taken to be all those products of simplices whose* 1*-dimensional skeleton is contained in the graph* G*.*

The prodsimplicial complexes is an object from combinatorial algebraic topology that allow for more flexibility that simplicial complexes but more structure than an arbitrary cell complex. For example, they generalise both simplicial and cubical complexes. Furthermore, the prodsimplicial flag complex can be viewed as a direct generalisation of the clique complex of G.

Proposition 3.2.36 and Theorem 3.2.16 directly imply the next statement.

Theorem 3.2.39. *The prodsimplicial flag complex of the flip graph of a linkage* (n, d)*-matching field has the same face lattice as the set of topes derived from it. In particular, the maximal cells of the prodsimplicial flag complex are in bijection with* $\Delta^{\mathbb{Z}}(d, n-d)$*.*

3.3 Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and pairs of lattice points

We denote the set of triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ by $\mathcal{TS}(n, d)$. By [28, Theorem 6.2.13] there are exactly $K = \binom{n+d-2}{n-1}$ full-dimensional simplices in such a triangulation.

Given a set of trees encoding a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$, consider the map that sends a tree T to its left and right degree vector pair (u, v). By Lemma 3.1.8, this map is injective. As each left and right degree vector can be identified with a lattice point in $\Delta^{\mathbb{Z}}(n, d-1)$ and $\Delta^{\mathbb{Z}}(d, n-1)$ by subtracting $e_{[n]}$ and $e_{[d]}$ respectively, this map can be written as

$$\phi_{n,d} \colon T \mapsto (u - e_{[n]}, v - e_{[d]}) \in \Delta^{\mathbb{Z}}(n, d-1) \times \Delta^{\mathbb{Z}}(d, n-1) \ .$$

This induces the map

$$\Phi_{n,d} \colon \mathcal{TS}(n, d) \to \binom{\Delta^{\mathbb{Z}}(n, d-1) \times \Delta^{\mathbb{Z}}(d, n-1)}{K} \ , \tag{3.1}$$

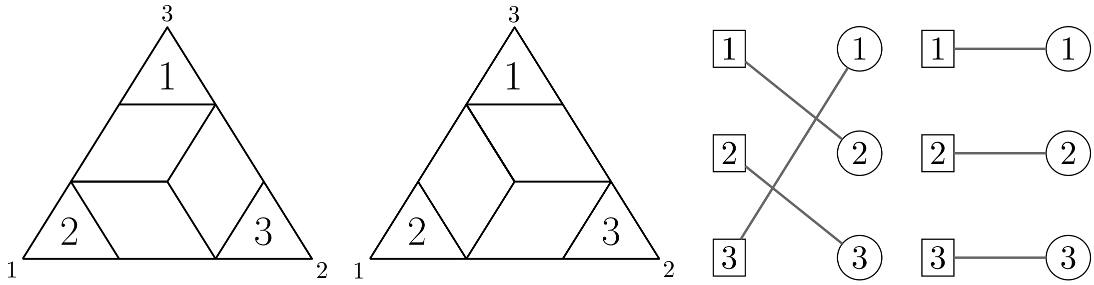

Figure 3.16: The mixed subdivisions corresponding to two triangulations of $\Delta_2 \times \Delta_2$. Both have the same dual graph, and the same non-maximal matchings. However their unique maximal matchings are different.

where each tree describing a full-dimensional simplex in the triangulation is mapped to its left and right degree vector pair minus $e_{[n]}$ and $e_{[d]}$ respectively.

After [74, Theorem 12.9], Postnikov asked whether the map defined in (3.1) is injective. His question is posed for root polytopes in general. As discussed earlier, parallel to our work, Galashin, Nenashev and Postnikov derived an affirmative answer to this question in [42]. Whereas their approach is based on the newly introduced notion of trianguloids, we present an independent proof for the case of the full product of two simplices (but not root polytopes in general) by exploiting the same structure as for Theorem 3.2.29.

A natural approach is to reconstruct the triangulation using the structure of the dual graph. However, this does not determine the triangulation as we illustrate.

Example 3.3.1 (Triangulation not determined by dual graph)**.** *Figure 3.16 shows two triangulations of $\Delta_2 \times \Delta_2$ whose dual graphs are the same. Furthermore, the trees encoding their triangulations contain all of the same non-maximal matchings. However, their unique maximal matchings are not equal, and so the triangulations are not equal, even up to symmetry.*

Definition 3.2.28 introduces combinatorial sectors for a collection of compatible bipartite graphs $\mathcal{G}$ with a bijective map to lattice points of a simplex. The trees encoding a triangulation are all compatible and have a natural bijection to both $\Delta^{\mathbb{Z}}(n, d-1)$ and

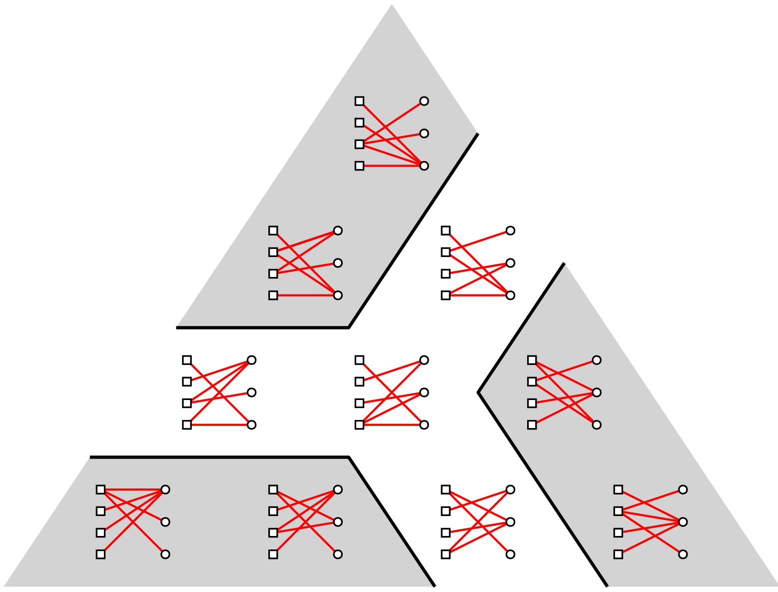

Figure 3.17: The trees encoding a triangulation of $\Delta_3 \times \Delta_2$ arranged by their bijection to $\Delta^{\mathbb{Z}}(2,4)$. The grey regions are the (open) combinatorial sectors $S_4^{(1)}$, $S_4^{(2)}$, $S_4^{(3)}$.

$\Delta^{\mathbb{Z}}(d, n-1)$ determined by their left and right degree vectors. When $\mathcal{G}$ is a triangulation $\mathcal{T}$, combinatorial sectors are a direct analogue to (open) sectors of tropical hyperplane arrangements as described in Example 3.1.12, see Figure 3.17 for an example. We use this notion alongside an iterative method, analogous to the proof of Theorem 3.2.29, to construct the triangulation inductively from the triangulations of its faces.

Theorem 3.3.2. *The map $\Phi_{n,d}$ is injective.*

Proof. We proceed by induction on $n + d$. For $n = d = 1$ there is only one tree. Now consider a triangulation $\mathcal{T}$ represented by a collection of trees for $n+d > 2$. Assume that any triangulation of $\Delta_{j-1} \times \Delta_{i-1}$ is uniquely determined by the degree vector pairs of its trees for $j + i < n + d$. We show the case $n \geq d$, the case $n \leq d$ is entirely analogous. Each left degree vector contains an entry equal to 1 since

$$2 \cdot (n-1) + 1 = 2n - 2 + 1 = 2n - 1 \geq n + d - 1 \ .$$

Hence, we get a non-disjoint decomposition

$$\bigcup_{j\in[n]} \mathcal{L}_j = \Phi_{n,d}(\mathcal{T}) \quad \text{for} \quad \mathcal{L}_j = \{(u,v) \mid u_j = 1\} \ .$$

Now fix a $j \in [n]$. There is a partition of $\mathcal{L}_j$ in the sets

$$\mathcal{L}_j^{(i)} = \{(u,v) \mid (\ell_j, r_i) \in G,\ G \in \mathcal{T} \text{ has degree vector } (u,v) \text{ with } u_j = 1\}$$

where $\mathcal{L}_j^{(i)}$ is the image of $\mathcal{S}_j^{(i)}$ in $\Phi_{n,d}$.

From $\mathcal{L}_j^{(i)}$ we can construct a set $\overline{\mathcal{L}_j^{(i)}}$ by removing the jth entry of the first component and decreasing the ith entry of the second component for all the pairs. This corresponds to removing the leaf edge (ℓ_j, r_i) of the trees. The resulting set

$$\overline{\mathcal{L}_j} = \bigcup_{i\in[d]} \overline{\mathcal{L}_j^{(i)}}$$

is the set of degree vectors of the deletion of $\mathcal{T}$ with respect to j. Hence, we can apply induction and deduce that $\overline{\mathcal{L}_j}$ uniquely defines the trees with the contained degree vectors. From the partition into the $\overline{\mathcal{L}_j^{(i)}}$ we can recover to which node ℓ_j is incident in the original tree. Therefore, we can construct all trees for which ℓ_j has degree 1. Ranging over all $\ell_j \in L$, we get all trees of $\mathcal{T}$.

It remains to show how to construct the set $\mathcal{L}_j^{(i)}$ for each $r_i \in R$, which we now demonstrate. Assume without loss of generality that $i = 1$ and apply Algorithm 2.

Claim: $\mathcal{K}_j = \mathcal{L}_j^{(1)}$.

Proof by induction There is a unique tree T_0 with the right degree vector $e_{[d]} + (n-1)e_1$. If $u_j = 1$ then ℓ_j is a leaf and it is adjacent to r_1 because of the structure of the right degree vector. Line 2 in the algorithm guarantees that T_0 is in $\mathcal{K}_j$. Furthermore, the edge (ℓ_j, r_1) shows that it is also contained in $\mathcal{L}_j^{(1)}$.

Algorithm 2 Construct the degree pairs of a combinatorial sector of trees

1: **if** $u_j = 1$ for (u, v) with $v_1 = n$ **then**
2: $\quad \mathcal{K}_j \leftarrow \{(u, v)\}$
3: **else**
4: $\quad$ terminate
5: **end if**
6: $h \leftarrow n - 1$
7: **while** $h > 0$ **do**
8: $\quad$ **for all** $(u, v) \in \mathcal{L}_j$ with $v_1 = h$ **do**
9: $\quad\quad$ **if** $\exists k \in [d] \colon v_k > 1 \colon \exists\, w^{(k)} \colon (w^{(k)}, v + e_1 - e_k) \in \mathcal{K}_j$ **then**
10: $\quad\quad\quad \mathcal{K}_j \leftarrow \mathcal{K}_j \cup (u, v)$
11: $\quad\quad$ **end if**
12: $\quad\quad h \leftarrow h - 1$
13: $\quad$ **end for**
14: **end while**

Now, assume that $\mathcal{K}_j$ and $\mathcal{L}_j^{(1)}$ agree in all elements whose first entry of the second component is $h + 1 \leq n$.

Let $(u, v) \in \mathcal{L}_j$ such that $v_1 = h$ and $(w, v + e_1 - e_k) \in \mathcal{K}_j$ an element fulfilling the condition in Line 9. These two vectors are the right degree vectors of two compatible trees T_1 and T_2. Note that $\mathcal{K}_j \subseteq \mathcal{L}_j$. As, by the induction hypothesis, (ℓ_j, r_1) is an edge of T_2 we can deduce with Corollary 3.1.10 that this is also an edge of T_1. Hence, (u, v) is an element of $\mathcal{L}_j^{(1)}$.

Conversely, let $(u, v) \in \mathcal{L}_j^{(1)}$ be with $v_1 = h$. By Lemma 3.1.11, there is a $k \in [d]$ and a $w \in \Delta^{\mathbb{Z}}(n, d-1)$ such that in the tree with degree pair $(w + e_{[n]}, v + e_1 - e_k)$ the node ℓ_j is a leaf and it is adjacent to r_1. The induction hypothesis implies that $(w + e_{[n]}, v + e_1 - e_k) \in \mathcal{K}_j$. Now, Line 9 shows that also (u, v) is an element of $\mathcal{K}_j$. $\square$

This map is far from surjective. By [74, Theorem 12.9], every possible lattice point in $\Delta^{\mathbb{Z}}(n, d-1)$ and $\Delta^{\mathbb{Z}}(d, n-1)$ must appear precisely once, the only remaining choice is how to pair them up. This gives the following immediate corollary.

Corollary 3.3.3. *The number of all triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ is bounded from above by $\binom{n+d-2}{d-1}!$.*

Remark 3.3.4. *Note that this bound is tight for $n = 2$ as by [28, Proposition 6.2.3] triangulations of $\Delta_1 \times \Delta_{d-1}$ are in bijection with permutations of $[d]$. Theorem 5.4 and Corollary 5.5 in [80] give upper bounds for regular subdivisions but, to the knowledge of the authors, this is the first upper bound on the number of all triangulations. Recall from [28, Theorem 6.2.19] that non-regular triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ exist if and only if $(n-2)(d-2) \geq 4$. Even if there are many non-regular triangulations, the bound might be very coarse as we do not use the structure of the lattice points.*

An eventual goal would be to give an axiom system for lattice point pairs, in a similar vein to Proposition 3.1.4. We state some necessary conditions that lattice point pairs must satisfy to induce a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.

We denote a lattice point pair of type (n, d) by $p = (u, v) \in \Delta^{\mathbb{Z}}(n, d-1) \times \Delta^{\mathbb{Z}}(d, n-1)$. We say two lattice point pairs p, p' are *adjacent* if u, u' and v, v' differ by one in precisely two coordinates and differ nowhere else. Note that every pair of trees that differ by a flip induce adjacent lattice point pairs, but the converse is not true. By the second condition of Proposition 3.1.4, the number of neighbours a tree has in the flip graph is equal to the number of edges in the tree that are not leaves.

We defined the sets $\mathcal{L}_j = \{(u, v) \mid u_j = 1\}$ in the proof of Theorem 3.3.2 for all $j \in [n]$. We construct a *deletion* of this set $\overline{\mathcal{L}_j}$ by removing the jth entry of the first component and decreasing an entry in the second component of all lattice point pairs. Note the distinction between left and right is arbitrary and we can analogously define the set $\mathcal{L}_i = \{(u, v) \mid v_i = 1\}$ and its deletion $\overline{\mathcal{L}_i}$. In the proof we construct a specific deletion, but we just need to ensure a well behaved one exists. This, along with [74, Theorem 12.9], gives the following necessary conditions on pairs of lattice points:

Corollary 3.3.5. *Let $\mathcal{P}$ be a set of lattice point pairs that induces a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$. Then $\mathcal{P}$ satisfies the following conditions:*

- *The projections from $\mathcal{P}$ onto $\Delta^{\mathbb{Z}}(n, d-1)$ and $\Delta^{\mathbb{Z}}(d, n-1)$ are bijections.*

- *Each $p \in \mathcal{P}$ has at least $n + d - 1 - l(p)$ adjacent lattice point pairs in $\mathcal{P}$, where $l(p)$ is the number of coordinates in p whose entry is 1.*
- *For every $\mathcal{L}_k \subset \mathcal{P}$, there exists a deletion $\overline{\mathcal{L}_k}$ satisfying the first two conditions.*

Recall that triangulations of $\Delta_{n-1} \times \Delta_1$ are in bijection with the permutations in S_n. By Theorem 3.3.2, such a triangulation is also determined by a certain collection of lattice point pairs. Let μ be the permutation corresponding to the triangulation. Then the lattice point pairs are given by

$$([k, n+1-k], [1^{\mu(k)-1}, 2, 1^{n-\mu(k)}]) \qquad \text{for all } k \in [n] \ .$$

The lattice point on the left corresponds to an element $k \in [n]$, the lattice point on the right corresponding to the element $\mu(k)$ that it is mapped to. This establishes the bijection with S_n.

Theorem 3.3.2 allows us to generalise this construction. We fix orderings on $\Delta^{\mathbb{Z}}(n, d-1)$ and $\Delta^{\mathbb{Z}}(d, n-1)$ to establish a correspondence with $[K]$, where $K = \binom{n+d-2}{n-1}$. Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ now correspond to elements of the symmetric group S_K, where each lattice point pair determines an element of $[K]$ and where it is mapped to. Note that we have natural S_n and S_d actions given by changing the ordering on the lattice points, therefore each triangulation gives rise to a subset of a conjugacy class of S_K. It would be interesting to study this link to the symmetric group in more detail.

3.4 Matching stacks and transversal matroids

In some sense, matching fields contain complementary information to transversal matroids. While transversal matroids encode on which subsets of the nodes a graph contains matchings, a matching field contains a matching for all d-subsets of R and one is interested in the interplay of the matchings.

Definition 3.4.1. *A* matching stack *on $L \sqcup R$ is a map which assigns to each pair* (J, I)

with $J \subseteq L, I \subseteq R$ and $|J| = |I|$ a perfect matching on $J \sqcup I$.

A matching stack is a matching ensemble *if the matchings for all fixed $J \subseteq L$ and for all fixed $I \subseteq R$ form linkage matching fields. Matching ensembles were first studied in [72].*

The main result in [72] is the equivalence of triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and matching ensembles on $L \sqcup R$. We present an intermediate result to demonstrate further research directions.

The tropical Stiefel map, extensively studied in [38], and more generally the polyhedral construction in [49, Theorem 7] essentially do the following. Given a collection of bipartite graphs $\mathcal{G}$ defined on the node set $L \sqcup R$, they map each graph $G \in \mathcal{G}$ to the transversal matroid M_G of G with ground set L. In the first reference, these graphs are the bipartite graphs corresponding to the full-dimensional simplices of a regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$. The second reference starts with the bipartite graphs corresponding to the full-dimensional simplices in a subdivision of $\Delta_{k-1} \times \Delta_{d-1}$ where $k = n - d$. They augment the vertex set of the bipartite graphs by d dummy nodes. By connecting all nodes in R of degree 1 in each graph G to the corresponding dummy node, they ensure that the transversal matroid on L has no loops. This Stiefel map was recently used in [82] to obtain new results for the coarsest non-trivial subdivisions of $\Delta_{n-d-1} \times \Delta_{d-1}$.

Definition 3.4.2. *Let $\mathcal{G}$ be a collection of bipartite graphs on the same node set $L \sqcup R$. The* combinatorial Stiefel map *associates to each graph in $\mathcal{G}$ its corresponding transversal matroid.*

With the construction for Theorem 3.2.16, we can start with a linkage matching field and construct trees in a natural way for all degree vectors. Recall that the topes are compatible with the matching field but the tope linkage covectors may not be. Taking the combinatorial Stiefel map of the collection of the maximal tope linkage covectors results in a collection of transversal matroids. Inspired by [38, Corollary 5.6] and [49, Theorem 7] we conjecture the following.

Conjecture 3.4.3. *A linkage matching field is determined by the collection of transversal matroids associated to the maximal tope linkage covectors by the combinatorial Stiefel map.*

Remark 3.4.4. *As maximal topes arising from a linkage matching field play an important role, one should keep in mind that the combinatorial Stiefel image of a tope is just a partition matroid. However, in the tropical Stiefel map which gives rise to a matroid subdivision of the hypersimplex, they do not correspond to maximal cells.*

We motivate this conjecture by a result connecting matching fields, matching ensembles and triangulations of $\Delta_{n-1} \times \Delta_{d-1}$.

Let $\mathcal{S}$ be a matching stack on $L \sqcup R$. We introduce d dummy nodes to get the left node set $\hat{L} = L \cup \{\ell_{n+i} \mid i \in [d]\}$. To the matching μ on $J \sqcup I$ in $\mathcal{S}$ we associate the matching $\hat{\mu}$ defined as follows:

$$\hat{\mu}(\ell_j) = \mu(\ell_j) \quad \text{for } \ell_j \in J \quad \text{and} \quad \hat{\mu}(\ell_{n+h}) = r_h \quad \text{for all } r_h \in R \setminus I \enspace .$$

This yields a matching field on $\hat{L} \sqcup R$, which we call the *matching field completion* of $\mathcal{S}$. This is a pointed matching field in the sense of [85, Example 1.4].

Theorem 3.4.5. *Let $\mathcal{M}$ be the matching field completion of the matching stack $\mathcal{S}$. Then $\mathcal{M}$ fulfils left linkage if and only if $\mathcal{S}$ fulfils left linkage.*

Proof. Consider a linkage covector C of $\mathcal{S}$ on the node set $J \sqcup I$ such that $|J| = |I| + 1$. We will construct the linkage covector D of $\mathcal{M}$ on the node set $\hat{J} \sqcup R$ where $\hat{J} = J \cup \{\ell_{n+i} \mid r_i \notin I\}$. The matching M_j on $\hat{J} \setminus \{\ell_j\}$, where $\ell_j \in J$, is the union of the matching in C obtained by isolating ℓ_j with the edges $\{(\ell_{n+i}, r_i) \mid r_i \notin I\}$. The matching M_{n+i} on $\hat{J} \setminus \{\ell_{n+i}\}$, where $r_i \notin I$, is the union of the matching in $\mathcal{S}$ on the node set $J \sqcup (I \cup \{r_i\})$ with the edges $\{(\ell_{n+k}, r_k) \mid r_k \notin I \cup \{r_i\}\}$. Note that if the edge (ℓ_j, r_i) is in M_{n+i}, the submatching obtained by deleting it is contained in C. Therefore D is the union of C and pairs of edges $\{(\ell_j, r_i), (\ell_{n+i}, r_i) \mid r_i \notin I, (\ell_j, r_i) \in M_{n+i}\}$, so it is a tree with degree two on all right nodes. Any linkage covector in $\mathcal{M}$ can be constructed this way and so it must satisfy the linkage property.

Conversely, consider a linkage covector of $\mathcal{M}$ on the node set $\hat{J} \sqcup R$. Removing the edges incident with the dummy nodes yields a tree on $J \sqcup R$ in which nodes in I have degree 2 and nodes in $R \setminus I$ have degree 1. Deleting the edges adjacent to nodes in $R \setminus I$ gives the union of matchings on $J \sqcup I$ arising in $\mathcal{S}$. As we have only removed leaves from a tree, this resulting graph is also a tree with all right nodes degree 2. □

We say that a matching field $\mathcal{M}$ satisfies the *compatible right submatching property* if and only if the following holds:

Let $\mu_1, \dots, \mu_r$ be submatchings of matchings in $\mathcal{M}$ on $J \sqcup I_1, \dots, J \sqcup I_r$ where $I := \bigcup_{[r]} I_s \subseteq R$ with $|J| + 1 = |I|$. Then $T = \bigcup_{[r]} \mu_s$ is a forest on $J \sqcup I$ and each matching μ of size $|J|$ in T is compatible with the matchings in $\mathcal{M}$.

Question 3.4.6. *We say a matching field $\mathcal{M}$ can be* extended *to a matching ensemble if there exists a matching ensemble whose matchings of size d are the matchings in $\mathcal{M}$. When can this occur? By construction it has to be linkage but this is not enough.*

Conjecture 3.4.7. *The maximal tope linkage covectors of a left linkage matching field are compatible if and only if it has the compatible right submatching property.*

Example 3.4.8. *Continuation from Example 3.2.18. The 2×2 matchings on $\{\ell_5, \ell_6\}$ given in Figure 3.18 are submatchings of matchings in $\mathcal{M}$. Therefore, any matching stack that extends $\mathcal{M}$ must contain them. However, the 2×3 linkage covectors on $\{r_1, r_2, r_3\}$ and $\{r_2, r_3, r_4\}$ contain cycles and so any matching stack extending $\mathcal{M}$ cannot satisfy right linkage.*

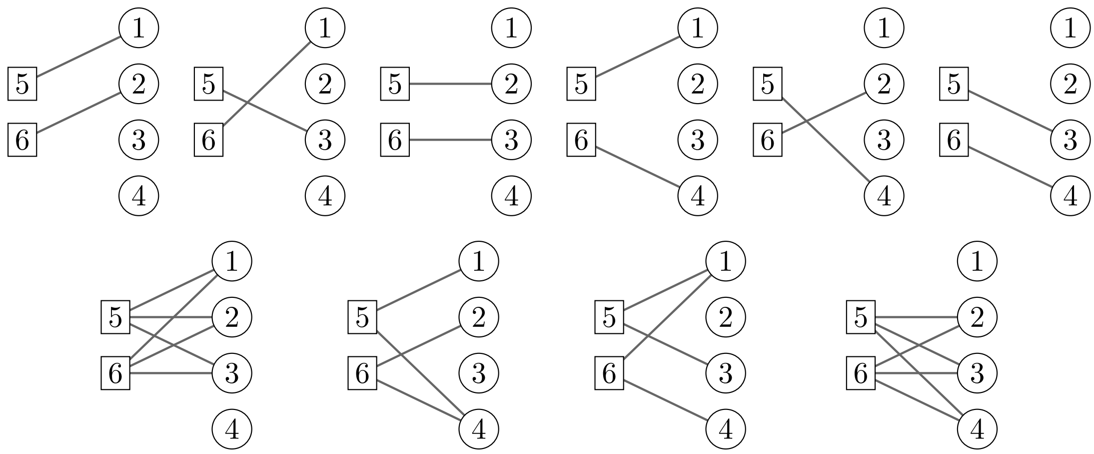

Figure 3.18: The submatching field on $\{\ell_5, \ell_6\} \sqcup \{r_1, \ldots, r_4\}$ with all possible linkage covectors of matchings. Two of the linkage covectors contain cycles and so are not right linkage.

Chapter 4

Convergent Puiseux series and tropical geometry of higher rank

The following is based on the paper "Convergent Puiseux series and tropical geometry of higher rank" by Michael Joswig and the author [56].

4.1 Convergent generalised Puiseux series

Following van den Dries and Speissegger [87] we consider a tuple $T = (t_1, \ldots, t_m)$ of m indeterminates and formal power series of the form

$$\gamma \ = \ \gamma(T) \ = \ \sum_{\alpha} c_\alpha T^\alpha \ , \tag{4.1}$$

where the multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ lies in $[0, \infty)^m$, the coefficient c_α is a real number, and T^α is the formal monomial $t_1^{\alpha_1} \cdots t_m^{\alpha_m}$. We further assume that the *support* $\operatorname{supp}(\gamma) = \{\alpha \in [0,\infty)^m \mid c_\alpha \neq 0\}$ is contained in the Cartesian product $A_1 \times \cdots \times A_m$, where $A_i \subseteq [0, \infty)$ such that $A_i \cap [0, \lambda]$ is finite for all positive real numbers λ. The coefficient-wise addition and the usual convolution product yield an algebra, which we denote as $\mathbb{R}[\![T^*]\!]$. More generally, given a ring R one can define $R[\![T^*]\!]$ analogously, where

c_α is an element of R.

Here comes a big caveat: our series will have "good support" in the sense of [87, §4.1] throughout; see also [87, §10.2]. That is, we use their notation with a slightly different interpretation. We can afford this simplification in the presentation as we will not study limits of series in $\mathbb{R}[\![T^*]\!]$. Observe that, formally, $m = 0$ with $T = ()$ and $\mathbb{R}[\![T^*]\!] = \mathbb{R}$ makes sense; cf. Remark 4.1.9 below.

A vector $r = (r_1, \ldots, r_m)$ with positive real numbers r_i is a *polyradius*. This gives rise to the *r-norm*

$$\|\gamma\|_r \ := \ \sum_\alpha |c_\alpha| r^\alpha \tag{4.2}$$

of $\gamma \in \mathbb{R}[\![T^*]\!]$, which is infinite if that series does not converge. The series with finite r-norm form the normed subalgebra $\mathbb{R}\{T^*\}_r$; cf. [87, §5.2]. Each series $\gamma(T) \in \mathbb{R}\{T^*\}_r$ yields a continuous function

$$\rho \mapsto \gamma(\rho) \ = \ \sum_\alpha c_\alpha \rho^\alpha \tag{4.3}$$

which is defined on the set $[0, r_1] \times \cdots \times [0, r_m]$ and which is analytic on the interior $(0, r_1) \times \cdots \times (0, r_m)$. The union

$$\mathbb{R}\{T^*\} \ := \ \bigcup_r \mathbb{R}\{T^*\}_r$$

is a local ring with maximal ideal $\{\gamma \in \mathbb{R}\{T^*\} \mid \gamma(0) = 0\}$; cf. [87, Corollary 5.6]. Its field of fractions is the field of *convergent generalised Puiseux series* $\mathbb{R}\{\{T^*\}\}$ (with real coefficients). Note that "convergence" here means absolute convergence in view of (4.2). Furthermore, the map which sends a series $\mathbb{R}\{T^*\}_r$ to the continuous function (4.3) is injective; cf. [87, Lemma 6.4].

Here is where we deviate from [87] by equipping the set $[0, \infty)^m$ with the lexicographic ordering. As a consequence the support $\operatorname{supp}(\gamma)$ of γ is a countable well ordered set, and the *order*

$$\operatorname{val}(\gamma) \ := \ \min \operatorname{supp}(\gamma)$$

of $\gamma \in \mathbb{R}\{T^*\}$ is defined, unless $\gamma = 0$. The *leading term* $\mathrm{lt}(\gamma)$ is $c_\alpha T^\alpha$, when $\alpha = \mathrm{val}(\gamma)$, and the *leading coefficient* $\mathrm{lc}(\gamma)$ is c_α. For any nonzero $\delta \in \mathbb{R}\{\{T^*\}\}$ there exist nonzero $\gamma, \gamma' \in \mathbb{R}\{T^*\}$ such that $\delta = \gamma/\gamma'$. In this way the order map extends to $\mathbb{R}\{\{T^*\}\} \setminus \{0\}$ by letting

$$\mathrm{val}(\delta) \ := \ \mathrm{val}(\gamma) - \mathrm{val}(\gamma') \ ,$$

and this is well defined. A nonzero convergent generalised Puiseux series is *positive* if the signs of the leading coefficients $\mathrm{lc}(\gamma)$ and $\mathrm{lc}(\gamma')$ agree. This definition turns $\mathbb{R}\{\{T^*\}\}$ into an ordered field.

Definition 4.1.1. *We equip $\mathbb{R}^m$ with the lexicographic total ordering. We denote the* rank m tropical semifield *by $\mathbb{T}_m := (\mathbb{R}^m, \min, +)$, where* min *is the minimum with respect to the lexicographic ordering.*

We use $\mathbb{R}^m$ when the underlying set is equipped with the Euclidean topology, and $\mathbb{T}_m$ when the underlying set is equipped with the order topology. Note that $\mathbb{R}^m$ and $\mathbb{T}_m$ agree as sets, however it will be useful throughout to differentiate between their topologies.

Remark 4.1.2. *Restricting the order map to convergent generalised Puiseux series which are positive gives a homomorphism $\mathrm{val} : \mathbb{R}\{\{T^*\}\}_{>0} \to \mathbb{T}_m$ of semirings, which reverses the ordering; i.e., $\delta \leq \delta'$ implies $\mathrm{val}(\delta) \geq \mathrm{val}(\delta')$.*

Remark 4.1.3. *The valuation map on $\mathbb{R}\{\{T^*\}\}$ is surjective, which will be crucial for our definition of higher rank tropical hypersurfaces. Unlike the rank one case where one can simply take topological closures, tropical geometry over non-surjective higher rank valuations is delicate. Remark 4.2.16 highlights the issues that arise in this case.*

Let us consider a second tuple $U = (u_1, \ldots, u_n)$ of n indeterminates. The roles of $T_1, \ldots, T_m$ and of $U_1, \ldots, U_n$ are symmetric. Extending the above construction, we arrive at the field of convergent generalised Puiseux series $\mathbb{R}\{\{(T, U)^*\}\}$ in $m+n$ indeterminates, and $\mathbb{R}\{\{T^*\}\}$ as well as $\mathbb{R}\{\{U^*\}\}$ are subfields.

Lemma 4.1.4. *Fix a polyradius (r, s). Each series $\gamma(T, U) = \sum_{\alpha,\beta} c_{\alpha,\beta} T^\alpha U^\beta$ in the normed algebra $\mathbb{R}\{(T, U)^*\}_{(r,s)}$ can be written as $\sum_\alpha(\sum_\beta c_{\alpha,\beta} U^\beta)T^\alpha$, which is an element of $\mathbb{R}\{U^*\}_s\{T^*\}_r$. Similarly, $\gamma(T, U)$ can also be written as an element of $\mathbb{R}\{T^*\}_r\{U^*\}_s$.*

Proof. By [87, Lemma 4.6] we have

$$\gamma(T,U) \;=\; \sum_{\alpha,\beta\geq 0} c_{\alpha,\beta} T^\alpha U^\beta \;=\; \sum_{\alpha\geq 0} \underbrace{\Bigg(\sum_{\beta\geq 0} c_{\alpha,\beta} U^\beta\Bigg)}_{*} T^\alpha \tag{4.4}$$

in the ring $\mathbb{R}[\![(T, U)^*]\!]$. That is, the claimed equality holds formally, without considering aspects of convergence. This shows that $\mathbb{R}[\![(T, U)^*]\!]$ is a subring of $\mathbb{R}[\![U^*]\!][\![T^*]\!]$. It follows that $\gamma(T, U)$ lies in $\mathbb{R}[\![U^*]\!]_s[\![T^*]\!]_r$, and due to absolute convergence within the polyradius (r, s) we may reorder the terms arbitrarily. As a consequence, for any fixed $\alpha_0 \geq 0$, we get

$$r^{\alpha_0} \sum_{\beta\geq 0} |c_{\alpha_0,\beta}| s^\beta \;=\; \sum_{\beta\geq 0} |c_{\alpha_0,\beta}| r^{\alpha_0} s^\beta \;\leq\; \sum_{\alpha,\beta\geq 0} |c_{\alpha,\beta}| r^\alpha s^\beta \;<\; \infty \;.$$

The term r^{α_0} does not vanish, and hence $\sum_{\beta\geq 0}|c_{\alpha_0,\beta}|s^\beta$ is finite. In particular, each starred coefficient in (4.4) is contained in the normed subalgebra $\mathbb{R}\{U^*\}_s$. The roles of T and U can be exchanged. ☐

The elements in $\mathbb{R}[\![(T, U)^*]\!]$ are *mixed series* in the sense of [87, §4.15]. It should be stressed that there are elements in $\mathbb{R}[\![U^*]\!][\![T^*]\!]$ which do not arise via Lemma 4.1.4; in loc. cit. $\sum_{k=1}^\infty T_1^{1/k} U_1^k$ is given as one example. The *T-support* of $\gamma \in \mathbb{R}[\![(T, U)^*]\!]$ is the support of γ seen as a series in T with coefficients in $\mathbb{R}[\![U^*]\!]$ as in the first statement of Lemma 4.1.4; the *U-support* of γ is defined by interchanging the roles of T and U. The subsequent observation should be compared with [87, Lemma 9.4].

Proposition 4.1.5. *Let $\delta(T, U) \in \mathbb{R}\{\{(T, U)^*\}\}$ be a generalised Puiseux series which converges in the polyradius $(r, s) = (r_1, \ldots, r_m, s_1, \ldots, s_n)$. Then the partial evaluations of $U = (u_1, \ldots, u_n)$ at constants $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i \in (0, s_i]$ yields a convergent generalised Puiseux series $\delta(T, \sigma) \in \mathbb{R}\{\{T^*\}\}$. Moreover, its order* $\operatorname{val}(\delta(T, \sigma))$ *does not* *depend on σ, provided that σ is admissible.*

A similar result holds for the partial evaluations of $T = (t_1, \ldots, t_m)$.

The precise definition of *admissible* will be given in the proof below; see also Remark 4.1.6.

Proof. We can write $\delta = \gamma/\gamma'$ with $\gamma, \gamma' \in \mathbb{R}\{(T,U)^*\}$ and $\gamma' \neq 0$. By assumption the evaluation

$$\delta(\rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_n) \;=\; \frac{\gamma(\rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_n)}{\gamma'(\rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_n)}$$

is defined and finite for all $0 < \rho_i \leq r_i$ and $0 < \sigma_j \leq s_j$. Lemma 4.1.4 gives the equality

$$\gamma(T,U) \;:=\; \sum_{\alpha,\beta \geq 0} c_{\alpha,\beta} T^\alpha U^\beta \;=\; \sum_{\alpha \geq 0} \left(\sum_{\beta \geq 0} c_{\alpha,\beta} U^\beta \right) T^\alpha \tag{4.5}$$

in the algebra $\mathbb{R}\{(T,U)^*\}$. A similar computation holds for the denominator $\gamma' = \sum c'_{\alpha,\beta} T^\alpha U^\beta$. Thus the partial evaluation at sufficiently small values is defined, and it reads

$$\delta(T,\sigma) \;=\; \frac{\sum_{\alpha \geq 0} \left(\sum_{\beta \geq 0} c_{\alpha,\beta} \sigma^\beta \right) T^\alpha}{\sum_{\alpha \geq 0} \left(\sum_{\beta \geq 0} c'_{\alpha,\beta} \sigma^\beta \right) T^\alpha} \;.$$

This yields

$$\operatorname{val}(\delta(T,\sigma)) = \min\left\{ \alpha \;\middle|\; \sum_{\beta \geq 0} c_{\alpha,\beta} \sigma^\beta \neq 0 \right\} - \min\left\{ \alpha \;\middle|\; \sum_{\beta \geq 0} c'_{\alpha,\beta} \sigma^\beta \neq 0 \right\}. \tag{4.6}$$

Now the functions $\sigma \mapsto \sum_{\beta \geq 0} c_{\alpha,\beta} \sigma^\beta$ and $\sigma \mapsto \sum_{\beta \geq 0} c'_{\alpha,\beta} \sigma^\beta$ map the set $(0, s_1] \times \cdots \times (0, s_n]$ analytically to $\mathbb{R}$. We call σ *admissible* if it is sent to zero by neither of these two functions for any α in the union of the T-supports of γ and γ'. In this case the expression (4.6) does not depend on σ.

As in Lemma 4.1.4 the roles of T and U can be exchanged. □

Remark 4.1.6. *Our notion of admissibility given in the proof above depends on the* *representation $\delta = \gamma/\gamma'$. For example, we can multiply γ and γ' by $(1 - t_1)$ without changing δ, but this would exclude $\sigma_1 = 1$ from the admissible values. So when we say that σ is* admissible *for δ we mean that there exists some representation $\delta = \gamma/\gamma'$ such that σ is admissible with respect to γ and γ'.*

Restricting the order maps from convergent generalised Puiseux series to the respective sub-semirings of positive series yields the following diagram of ordered semirings; see also Remark 4.1.2.

$$\begin{tikzcd}
\mathbb{R}\{\{T^*\}\}_{>0} \arrow[r, "\iota"] \arrow[d, "\mathrm{val}_m"] & \mathbb{R}\{\{(T,U)^*\}\}_{>0} \arrow[r, dashed, "\pi_u"] \arrow[d, "\mathrm{val}_{m+n}"] & \mathbb{R}\{\{T^*\}\}_{>0} \arrow[d, "\mathrm{val}_m"] \\
\mathbb{T}_m \arrow[r, "\iota_*"] & \mathbb{T}_{m+n} \arrow[r, "\pi_{u*}"] & \mathbb{T}_m
\end{tikzcd} \tag{4.7}$$

A few remarks are in order. Whenever we wish to distinguish between the various valuation maps we add the appropriate index to the symbol "val". The embedding $\iota : \mathbb{R}\{\{T^*\}\} \to \mathbb{R}\{\{(T,U)^*\}\}$ is induced by the mapping $\iota_* : \mathbb{R}^m \to \mathbb{R}^{m+n}$ which sends the exponent α to $(\alpha, 0)$. The dashed arrow labelled π_u in the diagram (4.7) is a subtle point. We would like to define $\pi_u(\delta(T,U))$ as the partial evaluation $\delta(T,\sigma)$. The latter expression depends on σ (and its admissibility), and so there is no way to extend such a map to the entire field $\mathbb{R}\{\{(T,U)^*\}\}$. However, by Proposition 4.1.5, for each $\delta \in \mathbb{R}\{\{(T,U)^*\}\}$ there is a polyradius in which the partial evaluation at admissible values is defined, and the order of the resulting series in $\mathbb{R}\{\{T^*\}\}$ does not depend on that polyradius or the specific choice of σ. The map π_{u*} is the projection $(\alpha,\alpha') \mapsto \alpha$ onto the first coordinate. In this sense the diagram (4.7) commutes, despite the fact that π_u is not globally defined.

Example 4.1.7. *Let us look at the series*

$$\begin{aligned}
\gamma(t,u) &= \sum_{\alpha\in\mathbb{N},\,\beta\in\mathbb{N}\setminus\{0\}} t^\alpha u^\beta = \sum_{\alpha\in\mathbb{N}} \Bigg(\sum_{\beta\in\mathbb{N}\setminus\{0\}} u^\beta \Bigg) t^\alpha \\
&= \Bigg(\sum_{\alpha\in\mathbb{N}} t^\alpha \Bigg) \Bigg(\sum_{\beta\in\mathbb{N}\setminus\{0\}} u^\beta \Bigg) = \Bigg(\sum_{\alpha\in\mathbb{N}} t^\alpha \Bigg) \Bigg(u \cdot \sum_{\beta\in\mathbb{N}} u^\beta \Bigg) ,
\end{aligned}$$

which is a positive element in the algebra $\mathbb{R}\{(t,u)^*\}$*, which we identify with a subalgebra of* $\mathbb{R}\{u^*\}\{t^*\}$*. That is, we are in the case* $m = n = 1$ *with* $T = (t_1)$*,* $t_1 = t$ *and* $U = (u_1)$*,* $u_1 = u$*. For the polyradius of convergence we may pick, e.g.,* $(\frac{3}{4}, \frac{3}{4})$*.*

The partial evaluation $u \mapsto \frac{1}{2}$ *is defined, and we arrive at*

$$\pi_{u \mapsto \frac{1}{2}}(\gamma(t,u)) \;=\; \gamma(t, \tfrac{1}{2}) \;=\; \frac{1}{1-t} \cdot \tfrac{1}{2}\frac{1}{1-\frac{1}{2}} \;=\; \frac{1}{1-t} \ ,$$

which is an element in the quotient field $\mathbb{R}\{\{t^*\}\}$ *of* $\mathbb{R}\{t^*\}$*. Clearly, other partial evaluations yield other results, such as, e.g.,*

$$\pi_{u \mapsto \frac{1}{3}}(\gamma(t,u)) \;=\; \gamma(t, \tfrac{1}{3}) \;=\; \frac{1}{1-t} \cdot \tfrac{1}{3}\frac{1}{1-\frac{1}{3}} \;=\; \tfrac{1}{2}\frac{1}{1-t} \ .$$

Yet $\operatorname{val}_2(\gamma) = (0,1)$ *and*

$$\operatorname{val}\bigl(\gamma(t, \tfrac{1}{2})\bigr) \;=\; \operatorname{val}\bigl(\gamma(t, \tfrac{1}{3})\bigr) \;=\; 0 \;=\; \pi_{u*}(\operatorname{val}_2(\gamma)) \ .$$

In this example all real numbers in the open interval $(0,1)$ *are admissible.*

In Lemma 4.1.4 and Proposition 4.1.5 the roles of the T-variables and the U-variables are symmetric. Yet the definition of val_2 breaks this symmetry. The following example shows that T and U cannot be exchanged in (4.7). Nonetheless the notation "$\pi_{t \mapsto \rho}$" and "π_{t*}" makes sense; the map π_{t*} is the projection $(\alpha, \alpha') \mapsto \alpha'$ onto the second coordinate.

Example 4.1.8. *For* $\gamma(t,u) = tu^3 + t^2u^{-1}$ *in* $\mathbb{R}\{\{(t,u)^*\}\}$ *we have* $\operatorname{val}_2(\gamma) = (1,3)$*. According to* (4.7) *we have the equality*

$$\operatorname{val}(\pi_{u \to 1}(\gamma)) \;=\; \operatorname{val}(t + t^2) \;=\; 1 \;=\; \pi_{u*}(1,3) \ .$$

Yet, here the roles of t *and* u *cannot be exchanged:*

$$\operatorname{val}(\pi_{t \to 1}(\gamma)) \;=\; \operatorname{val}(u^{-1} + u^3) \;=\; -1 \;\neq\; \pi_{t*}(1,3) \ .$$

Remark 4.1.9. *It is worth noting that the case $m = 0$ and $n = 1$ does make sense in* (4.7). *Then we have $T = ()$ and $U = (u)$, leading to $\mathbb{R}\{\{T^*\}\} \cong \mathbb{R}$ and $\mathbb{T}_0 = \{0\}$; the map ι sends $c \in \mathbb{R}_{>0}$ to the constant Puiseux series $c \cdot u^0 \in \mathbb{R}\{\{u^*\}\}$, and val_0 is the trivial valuation on the positive reals. The right half of the diagram now degenerates to the transfer principle from* (1.1) *as:*

$$\begin{tikzcd} \mathbb{R}\{\{u^*\}\}_{>0} \arrow[r, dashed, "\pi_u"] \arrow[d, "\mathrm{val}"] & \mathbb{R}_{>0} \\ \mathbb{T} & \end{tikzcd} \tag{4.8}$$

In fact, this can be exploited to pull back metric information from the semimodule $\mathbb{T}^k$ and project it to (the positive orthant of) the real vector space $\mathbb{R}^k$, for arbitrary k. This is a key idea behind [6], where this approach was used to show that standard versions of the interior point method cannot solve ordinary linear programs in strongly polynomial time.

For a single indeterminate $T = (t_1)$, $t_1 = t$, van den Dries and Speissegger prove that $\mathbb{R}\{\{t^*\}\}$ is a real closed field; cf. [87, Corollary 9.2]. This does not hold for more than one indeterminate. Instead we have the following.

Proposition 4.1.10. *The field $\mathbb{R}\{\{u^*\}\}\{\{t^*\}\}$ is real closed, and its algebraic closure is $\mathbb{C}\{\{u^*\}\}\{\{t^*\}\}$.*

Proof. We mimic the proofs of [87, Lemma 9.1] and [87, Corollary 9.2] with $\mathbb{R}\{\{u^*\}\}$ instead of $\mathbb{R}$. Consider a univariate polynomial

$$f(t,u,w) \;=\; w^n + \Big(\sum_\alpha\big(\sum_\beta c^{n-1}_{\alpha,\beta}u^\beta\big)t^\alpha\Big)w^{n-1} + \cdots + \Big(\sum_\alpha\big(\sum_\beta c^{0}_{\alpha,\beta}u^\beta\big)t^\alpha\Big)$$

in $\mathbb{R}\{\{u^*\}\}\{t^*\}[w]$ with $f(0,0,0) = 0$ and $(\partial f/\partial w)(0,0,0) \neq 0$. That is, the coefficients of f lie in the local ring $\mathbb{R}\{\{u^*\}\}\{t^*\}$ of convergent series with coefficients in the real closed field $\mathbb{R}\{\{u^*\}\}$. We need to show that there is a series $\gamma(t,u) \in \mathbb{R}\{\{u^*\}\}\{t^*\}$ with $\gamma(0,0) = 0$ and $f(t,u,\gamma(t,u)) = 0$. We may view $f(t,u,w)$ as an element of $\mathbb{R}\{\{u^*\}\}\{t^*,w\}$.

Applying the Weierstrass Preparation [87, 5.10] with respect to the real closed field $A = \mathbb{R}\{\{u^*\}\}$ yields a unit $g \in \mathbb{R}\{\{u^*\}\}\{t^*, w\}$ and some $\gamma(t,u) \in \mathbb{R}\{\{u^*\}\}\{t^*\}$ with $f(t,u,w) = g(t,u,w)(w - \gamma(t,u))$. It follows that this γ is as required, i.e., we have $\gamma(0,0) = 0$ and $f(t,u,\gamma(t,u)) = 0$. We infer that the local ring $\mathbb{R}\{\{u^*\}\}\{t^*\}$ is *Henselian*. As in [87, Corollary 9.2] this entails that the field of fractions $\mathbb{R}\{\{u^*\}\}\{\{t^*\}\}$ is real closed; cf. [37, Theorem 4.3.7]. The claim on the algebraic closure is a direct consequence. □

Note that we cannot invoke standard model theory arguments, e.g., Tarski's principle, in the previous proof as [87, Lemma 9.1] is not a first order statement over the reals. This is because, in general, a convergent series in $\mathbb{R}\{u^*\}$ has infinitely many nonzero coefficients. The "replacement" is the more technical Weierstrass Preparation [87, 5.10], which is sufficiently general.

Remark 4.1.11. *The argument in the proof of Proposition 4.1.10 can be iterated to show that the tower $\mathbb{R}\{\{t_1^*\}\}\cdots\{\{t_m^*\}\}$ of convergent generalised Puiseux series is also real closed. However, we prefer to stick to rank two from now on in order to minimise the technical overhead.*

The rank two valuation map $\mathrm{val}_2 : \mathbb{R}\{\{(t,u)^*\}\} \to \mathbb{T}_2$ admits an extension to the real closed field $\mathbb{R}\{\{u^*\}\}\{\{t^*\}\}$ as follows. A typical nonzero element is of the form $\gamma(t,u) = \sum_\alpha(\sum_\beta c_{\alpha,\beta}u^\beta)t^\alpha$. As a convergent series in t this has a leading term (i.e., a term of lowest order), say, $(\sum_\beta c_{\alpha_0,\beta}u^\beta)t^{\alpha_0}$. The leading coefficient is a nonzero convergent series in u. This again has a leading term, say, $c_{\alpha_0,\beta_0}u^{\beta_0}$. Now we have

$$\mathrm{val}_2(\gamma(t,u)) \;=\; (\alpha_0, \beta_0) \ . \tag{4.9}$$

In the abstract field $\mathbb{R}\{\{(t,u)^*\}\}$ the role of the two indeterminates, t and u, is symmetric. Yet the valuation map that we chose, val_2, prefers t before u. So, while the field $\mathbb{R}\{\{t^*\}\}\{\{u^*\}\}$, obtained from interchanging the roles of t and u, is abstractly isomorphic to $\mathbb{R}\{\{u^*\}\}\{\{t^*\}\}$, our valuation map val_2 defined on $\mathbb{R}\{\{(t,u)^*\}\}$ only extends to $\mathbb{R}\{\{u^*\}\}\{\{t^*\}\}$.

4.2 Rank two tropical hypersurfaces

As in Example 4.1.7, in the sequel we will be investigating the special case where $m = n = 1$. That is, we consider the field $\mathbb{R}\{\{(t,u)^*\}\}$ of convergent generalised Puiseux series in two indeterminates, t and u. This contains the subfields $\mathbb{R}\{\{t^*\}\}$ and $\mathbb{R}\{\{u^*\}\}$, and we have $\mathbb{R}\{\{(t,u)^*\}\} \subset \mathbb{R}\{\{u^*\}\}\{\{t^*\}\}$ by Lemma 4.1.4. All these fields are ordered. From Proposition 4.1.10 we know that $\mathbb{R}\{\{u^*\}\}\{\{t^*\}\}$ is real closed and the rank two valuation map from $\mathbb{R}\{\{(t,u)^*\}\}$ extends; cf. (4.9). Formally, in the sequel we could also replace the field $\mathbb{R}\{\{u^*\}\}\{\{t^*\}\}$ by the real closure of $\mathbb{R}\{\{(t,u)^*\}\}$, which is smaller. However, we prefer to work with a "somewhat reasonable" field and hope that this adds improved readability.

Since we now want to look at hypersurfaces, it is natural to pass to algebraically closed fields. Picking an imaginary unit $i = \sqrt{-1}$ we obtain

$$\mathbb{C}\{\{(t,u)^*\}\} \;=\; \mathbb{R}\{\{(t,u)^*\}\} + i\,\mathbb{R}\{\{(t,u)^*\}\} \;\subseteq\; \mathbb{C}\{\{u^*\}\}\{\{t^*\}\} \ ,$$

and the latter field, which we abbreviate as $\mathbb{L}$, is algebraically closed due to Proposition 4.1.10. Both, $\mathbb{C}\{\{(t,u)^*\}\}$ and $\mathbb{L} = \mathbb{C}\{\{u^*\}\}\{\{t^*\}\}$ are equipped with the rank two valuation map val_2. Note that evaluating a series in $\mathbb{C}\{\{(t,u)^*\}\}$, within its polyradius of convergence, is only defined for admissible positive *real* values, despite that the coefficients are allowed to be complex numbers. This yields a real-analytic function, which is not holomorphic, in general. We will come back to convergent Puiseux series with real coefficients in Section 4.4 below.

The following is based on [12] and [13]. Given a Laurent polynomial $f = \sum \gamma_s x^s \in \mathbb{L}[x_1^{\pm}, \ldots, x_d^{\pm}]$, the *rank two tropicalisation of* f is the tropical polynomial obtained from f by applying val_2 to each coefficient and replacing addition and multiplication with their tropical counterparts. This induces the tropical polynomial map

$$\begin{aligned} \operatorname{trop}_2(f) : (\mathbb{T}_2)^d &\longrightarrow \mathbb{T}_2 \\ p &\longmapsto \min\{\operatorname{val}_2(\gamma_s) + \langle s, p\rangle \mid s \in \operatorname{supp}(f)\} \quad , \end{aligned}$$

where $\langle s, p\rangle$ is the pairing

$$\begin{aligned} \langle -, -\rangle \,:\, \mathbb{Z}^d \times (\mathbb{T}_2)^d &\longrightarrow \mathbb{T}_2 \\ \big((s_1, \ldots, s_d), (p_1, \ldots, p_d)\big) &\longmapsto \sum_{i=1}^{d} (s_i p_{1i}, s_i p_{2i}) \quad . \end{aligned} \tag{4.10}$$

For every $p \in (\mathbb{T}_2)^d$ there exists at least one term of the polynomial where $\operatorname{trop}_2(f)$ attains its minimum, and hence the set

$$\mathcal{D}_p(f) \;=\; \left\{ s \in \mathbb{Z}^d \;\middle|\; \operatorname{trop}_2(f)(p) = \operatorname{val}_2(\gamma_s) + \langle s, p\rangle \right\}$$

is not empty.

Definition 4.2.1. *The* rank two tropical hypersurface *of f is the set*

$$\mathcal{T}_2(f) \;=\; \left\{ p \in (\mathbb{T}_2)^d \;\middle|\; |\mathcal{D}_p(f)| > 1 \right\} \quad .$$

As with rank one tropical hypersurfaces, this construction commutes with taking the coordinatewise valuation of the zero set of f. Here it is essential that $\mathbb{L}$ is algebraically closed and that the valuation map is surjective onto $\mathbb{T}_2$.

Theorem 4.2.2 ([13, Theorem 8.1])**.** *Let $f \in \mathbb{L}[x_1^{\pm}, \ldots, x_d^{\pm}]$. The rank two tropical hypersurface of f is the set of pointwise valuations of the zero set of f, i.e.,*

$$\mathcal{T}_2(f) \;=\; \left\{ \big(\operatorname{val}_2(p_1), \ldots, \operatorname{val}_2(p_d)\big) \;\middle|\; p \in \mathbb{L}^d,\ f(p) = 0 \right\} \quad .$$

As rank one tropical hypersurfaces are ordinary polyhedral complexes, we would like an analogous structure for rank two tropical hypersurfaces. As sets $\mathbb{T}_2$ and $\mathbb{R}^2$ are equal, but the order topology (on $\mathbb{T}_2$) is strictly finer than the Euclidean topology (on $\mathbb{R}^2$); recall that the open intervals form a basis of the order topology. Similarly $(\mathbb{T}_2)^d$ and $(\mathbb{R}^2)^d$ are equal as sets but the respective product topologies are distinct. In particular, $(\mathbb{R}^2)^d$ is homeomorphic with $\mathbb{R}^{2d}$, and we use the latter notation for readability. However, we shall write point coordinates as $(p_{11}, p_{21}; \dots; p_{1d}, p_{2d})$ to emphasise that points are d-tuples of elements of $\mathbb{R}^2$ or $\mathbb{T}_2$.

Example 4.2.3. *For the bivariate linear polynomial $f = x_1 + tx_2 + t^2u \in \mathbb{L}[x_1, x_2]$ its rank two tropical hypersurface is the following subset of $(\mathbb{T}_2)^2$.*

$$\begin{aligned}
\mathcal{T}_2(f) = &\{ (p_{11}, p_{21}; p_{12}, p_{22}) \mid (0,0) + (p_{11}, p_{21}) = (2,1) \leq (1,0) + (p_{12}, p_{22})\} \\
&\cup \{ (p_{11}, p_{21}; p_{12}, p_{22}) \mid (1,0) + (p_{12}, p_{22}) = (2,1) \leq (0,0) + (p_{11}, p_{21})\} \\
&\cup \{ (p_{11}, p_{21}; p_{12}, p_{22}) \mid (0,0) + (p_{11}, p_{21}) = (1,0) + (p_{12}, p_{22}) \leq (2,1)\} \\
= &\{ (2,1;1,1) + (0,0;\lambda_1, \lambda_2) \mid (\lambda_1, \lambda_2) \geq (0,0)\} \\
&\cup \{ (2,1;1,1) + (\lambda_1, \lambda_2; 0,0) \mid (\lambda_1, \lambda_2) \geq (0,0)\} \\
&\cup \{ (2,1;1,1) + (-\lambda_1, -\lambda_2; -\lambda_1, -\lambda_2) \mid (\lambda_1, \lambda_2) \geq (0,0)\}
\end{aligned}$$

Recall that "$\leq$" and "$\geq$" refers to the lexicographic ordering. Due to this ordering, $\mathcal{T}_2(f)$ is not closed in the Euclidean topology. For example, consider the sequence of points $(2,1;1+c_k,0)$ where $c_k \to 0$ is a sequence of positive reals converging to zero. Each of these points are contained in $\mathcal{T}_2(f)$ but its limit $(2,1;1,0)$ is not.

Example 4.2.3 highlights that rank two tropical hypersurfaces are not closed in the Euclidean topology, therefore they do not have the structure of a polyhedral complex as rank one tropical hypersurfaces do. However, we can consider polyhedral-like structures with respect to the lex-order topology on $\mathbb{T}_2$.

We recall the following notions from [40, 41]. There is a natural pairing (4.10) which arises from considering the abelian group $\mathbb{T}_2$ as a $\mathbb{Z}$-module. A *lex-halfspace* in $(\mathbb{T}_2)^d$ is a set of the form

$$\mathbf{H}_{s,q} \;=\; \left\{ p \in (\mathbb{T}_2)^d \;\middle|\; \langle s, p \rangle \le q \right\}$$

for some fixed *slope* $s \in \mathbb{Z}^d$ and *affine constraint* $q \in \mathbb{R}^2$. Its *boundary* is

$$\left\{ p \in (\mathbb{T}_2)^d \;\middle|\; \langle s, p \rangle = q \right\} \;=\; \mathbf{H}_{s,q} \cap \mathbf{H}_{-s,q} \quad . \tag{4.11}$$

Note that the slopes are integral vectors as we are considering Laurent polynomials (whose exponents lie in $\mathbb{Z}^d$) with coefficients in $\mathbb{L}$, which is equipped with a rank two valuation that is *not* discrete. Thus $\mathbb{Z}^d$ arises as a factor of the domain of the pairing map (4.10).

Definition 4.2.4. *A* lex-polyhedron $\mathbf{P}$ *in* $(\mathbb{T}_2)^d$ *is any intersection of finitely many lex-halfspaces*

$$\mathbf{P} \;=\; \mathbf{H}_{s_1,q_1} \cap \cdots \cap \mathbf{H}_{s_r,q_r} \quad . \tag{4.12}$$

A face *of* $\mathbf{P}$ *is the intersection with any number of boundaries of the lex-halfspaces defining* $\mathbf{P}$*. Its* relative interior $\mathrm{int}(\mathbf{P})$ *is the set of points contained in* $\mathbf{P}$ *but in no face of* $\mathbf{P}$*. A* lex-polyhedral complex *in* $(\mathbb{T}_2)^d$ *is a finite collection* $\{\mathbf{P}_j\}_{j \in J}$ *of lex-polyhedra in* $(\mathbb{T}_2)^d$ *such that every face of* $\mathbf{P}_j$ *also lies in the collection and the intersection of any two lex-polyhedra also lies in the collection.*

Note that [40, 41] simply refer to these as “polyhedra”. As we are also working with ordinary and tropical polyhedra, we use the prefix “lex” to stress the underlying lexicographical ordering, and use a bold typeface to differentiate it. By (4.11), boundaries of lex-halfspaces and thus faces are lex-polyhedra. It is worth keeping in mind that the notions defined above depend on the choice of the representation (4.12); cf. Question 4.5.2 below. Lex-polyhedra are necessarily closed in the order topology.

Given some subset $S \subseteq \mathrm{supp}(f)$, we define the *support cell*

$$\mathbf{P}_S(f) = \left\{ p \in (\mathbb{T}_2)^d \;\middle|\; S \subseteq \mathcal{D}_p(f) \right\} , \quad \text{for } S \subseteq \mathrm{supp}(f) \ . \tag{4.13}$$

By definition, $\mathbf{P}_S = \mathbf{P}_S(f)$ is cut out by lex-halfspaces defined by the inequalities of the form

$$\mathrm{val}_2(\gamma_s) + \langle s, p \rangle \leq \mathrm{val}_2(\gamma'_s) + \langle s', p \rangle, \quad \text{for } s \in S \ , \ s' \in \mathrm{supp}(f) \tag{4.14}$$

and so has the structure of a lex-polyhedron.

Note that for a non-generic polynomial f, there may exist S such that $\mathrm{trop}_2(f)$ does not obtain its minimum at precisely S when evaluated at any point in $\mathbf{P}_S$. Equivalently, there may exist S, T such that $S \neq T$ but their support cells are equal as sets, i.e., $\mathbf{P}_S = \mathbf{P}_T$. Any point in the support cells satisfies $S, T \subseteq \mathcal{D}_p(f)$ and so they are equal to $\mathbf{P}_S \cup \mathbf{P}_T$ as a set. This implies any support cell can be labelled by a unique maximal set, which we call the *support set* i.e., S is a support set of f if $\mathbf{P}_S(f) = \mathbf{P}_T(f)$ implies $T \subseteq S$. Note that the rank one analogue of support cells in $\mathbb{T}^d$ are ordinary polyhedra; see [63, Proposition 3.1.6] and Question 4.5.1 below. Support cells have some nice combinatorial properties:

Lemma 4.2.5. *Let S, T be support sets.*

1. $\mathbf{P}_S \cap \mathbf{P}_T = \mathbf{P}_{S \cup T}$.
2. *$S \subset T$ if and only if $\mathbf{P}_T$ is a face of $\mathbf{P}_S$.*

Proof. Denote inequalities of the form (4.14) by $\alpha_{s,s'}$. Consider the intersection $\mathbf{P}_S \cap \mathbf{P}_T$, it is cut out by the union of inequalities defining $\mathbf{P}_S$ and $\mathbf{P}_T$. These are precisely the inequalities $\alpha_{s,s'}$ for $s \in S \cup T$, and is therefore equal to $\mathbf{P}_{S \cup T}$. Furthermore, as S, T are support sets, their union also is.

Any face of $\mathbf{P}_S$ is defined by setting certain inequalities of (4.14) to equalities, or equivalently by adding the inequality $\alpha_{s',s}$. If $T \supset S$ is the set of elements of $\mathrm{supp}(f)$

contained in an equality, then $\alpha_{s,s'}$ holds for all $s \in T$ and $s' \in \mathrm{supp}(f)$. Therefore T is a support set and $\mathbf{P}_T$ is the corresponding face of $\mathbf{P}_S$. □

Remark 4.2.6. *Lemma 4.2.5 has two important consequences. The first is that by associating support cells with their unique support set, each support cell has a canonical halfspace description via* (4.14)*. Furthermore, as faces of support cells are themselves support cells, this extends to a canonical inequality description of each face. The second consequence is that as the faces of* $\mathbf{P}_S$ *are the points* p *such that* $S \subsetneq \mathcal{D}_p(f)$*, the relative interior of* $\mathbf{P}_S$ *is the set*

$$\mathrm{int}(\mathbf{P}_S) \;=\; \left\{ p \in (\mathbb{T}_2)^d \;\middle|\; S = \mathcal{D}_p(f) \right\} \;.$$

Note that this is not true if S *is not a support set.*

Remark 4.2.7. *In topology the term "cell" is typically used for subsets of* $\mathbb{R}^{2d}$ *which are homeomorphic with some closed Euclidean ball. Here we deviate slightly based on the topology that we are using. When working with* $\mathbb{R}^{2d}$ *and the Euclidean topology, our cells will be convex polyhedra, whereas when working with* $(\mathbb{T}_2)^d$ *and the order topology, our cells will be lex-polyhedra. Note that in both cases, cells may be unbounded.*

[41, Theorem 2.5.2] and [69, Proposition 1.2] show $\mathcal{T}_2(f)$ carries the structure of a lex-polyhedral complex. The following shows that this lex-polyhedral complex is labelled by subsets of monomials of f.

Proposition 4.2.8. *The rank two tropical hypersurface* $\mathcal{T}_2(f)$ *is a lex-polyhedral complex whose cells are of the form* $\mathbf{P}_S$*, where* S *is a support set of cardinality greater than one.*

Proof. Define the collection of lex-polyhedra

$$\mathbf{\Sigma} \;=\; \{ \mathbf{P}_S \mid S \text{ support set },|S| > 1\} \;.$$

By definition $\mathbf{\Sigma}$ and $\mathcal{T}_2(f)$ are equal as sets; it remains to show $\mathbf{\Sigma}$ is a lex-polyhedral complex. By Lemma 4.2.5, $\mathbf{\Sigma}$ is closed under taking intersections and restricting to faces, therefore it is a lex-polyhedral complex. $\square$

Example 4.2.9. *We return to the polynomial $f = x_1 + tx_2 + t^2u$ from Example 4.2.3. Its support is* $\operatorname{supp}(f) = \{(0,0),(1,0),(0,1)\}$*, and so $\mathcal{T}_2(f)$ is a lex-polyhedral complex in $(\mathbb{T}_2)^2$ with three maximal lex-polyhedral cells:*

$$\begin{aligned}
\mathbf{P}_{\{(0,0),(1,0)\}} &= \{(2,1;1,1) + (0,0;\lambda_1,\lambda_2) \mid (\lambda_1,\lambda_2) \geq (0,0)\} \\
\mathbf{P}_{\{(0,0),(0,1)\}} &= \{(2,1;1,1) + (\lambda_1,\lambda_2;0,0) \mid (\lambda_1,\lambda_2) \geq (0,0)\} \\
\mathbf{P}_{\{(1,0),(0,1)\}} &= \{(2,1;1,1) + (-\lambda_1,-\lambda_2;-\lambda_1,-\lambda_2) \mid (\lambda_1,\lambda_2) \geq (0,0)\} \ .
\end{aligned}$$

Their intersection is the common face $\mathbf{P}_{\{(0,0),(1,0),(0,1)\}} = \{(2,1;1,1)\}$.

While Proposition 4.2.8 gives a concrete description of rank two tropical hypersurfaces, the structure of lex-polyhedra is not as well understood as ordinary polyhedra; cf. Question 4.5.2. Furthermore, as we shall show later, the projections π_{u*} and π_{t*} map $\mathcal{T}_2(f)$ to ordinary polyhedral complexes. Theorem 4.2.13 and Corollary 4.2.14 exploit this structure to give an explicit description of $\mathcal{T}_2(f)$ in terms of ordinary polyhedra. To do so, we introduce the following notation.

The higher rank transfer principle (4.7) naturally extends to the following commutative diagram of Laurent polynomial (semi-)rings.

$$\begin{tikzcd}
\mathbb{C}\{\{t^*\}\}[\boldsymbol{x}^{\pm}] \arrow[r, "\iota"] \arrow[d, "\mathrm{trop}"] & \mathbb{C}\{\{(t,u)^*\}\}[\boldsymbol{x}^{\pm}] \arrow[r, dashed, "\pi_u"] \arrow[d, "\mathrm{trop}_2"] & \mathbb{C}\{\{t^*\}\}[\boldsymbol{x}^{\pm}] \arrow[d, "\mathrm{trop}"] \\
\mathbb{T}[\boldsymbol{x}^{\pm}] \arrow[r, "\iota_*"] & \mathbb{T}_2[\boldsymbol{x}^{\pm}] \arrow[r, "\pi_{u*}"] & \mathbb{T}[\boldsymbol{x}^{\pm}]
\end{tikzcd} \tag{4.15}$$

Here $\boldsymbol{x}^{\pm}$ is shorthand for $x_1^{\pm},\ldots,x_d^{\pm}$. Furthermore, $\iota,\iota_*,\pi_u,\pi_{u*}$ are the same as in (4.7), applied coefficientwise. As before, π, the partial evaluation at admissible and sufficiently small real constants, is not globally defined. Again we also use π_t and π_{t*} despite the fact that the roles of t and u are not interchangeable in (4.15); cf. Example 4.1.8. Note that these partial evaluations for a given polynomial must be admissible simultaneously for all its coefficients.

Remark 4.2.10. *Let $f \in \mathbb{C}\{\{(t,u)^*\}\}[x_1^{\pm}, \ldots, x_d^{\pm}]$ be a rank two polynomial with partial evaluation $\pi_{u\mapsto\sigma}(f)$ at an admissible value $\sigma > 0$. Then $\operatorname{supp}(f) = \operatorname{supp}(\pi_{u\mapsto\sigma}(f))$ as subsets of $\mathbb{Z}^d$. This is immediate from the definition of admissibility. From now on we assume that all choices of σ and ρ are admissible real values.*

Example 4.2.11. *Consider the rank two bivariate polynomial $f = x_1 + tx_2 + t^2u$ in $\mathbb{C}\{\{(t,u)^*\}\}[x_1, x_2]$ from Example 4.2.3. Its coefficients converge to nonzero values for any positive evaluation. For instance, this gives the rank one polynomials*

$$\begin{aligned} \pi_{u\mapsto 1}(f) &= x_1 + tx_2 + t^2 && \in \mathbb{C}\{\{t^*\}\}[x_1, x_2] \quad \text{and} \\ \pi_{t\mapsto 1}(f) &= x_1 + x_2 + u && \in \mathbb{C}\{\{u^*\}\}[x_1, x_2]\ , \end{aligned}$$

obtained from evaluating at $u = 1$ and $t = 1$. Their rank one tropical hypersurfaces both are tropical lines in $\mathbb{R}^2$.

For clarity, we use $\mathcal{T}$ rather than $\mathcal{T}_2$ to denote tropical hypersurfaces where the underlying field has rank one valuation. As $\pi_{u\mapsto\sigma}(f)$ and $\pi_{t\mapsto\rho}(f)$ are polynomials over an algebraically closed field with a rank one valuation, their tropical hypersurfaces $\mathcal{T}(\pi_{u\mapsto\sigma}(f))$ and $\mathcal{T}(\pi_{t\mapsto\rho}(f))$ are ordinary polyhedral complexes. However, the underlying fields are different and so these tropical hypersurfaces sit in different ambient spaces that we denote by $\mathbb{R}^d_t$ and $\mathbb{R}^d_u$ respectively. Using Theorem 4.2.2 and the commutative diagram (4.15), we may view the entire space

$$\mathbb{R}^{2d} \;=\; \pi_{u*}(\mathbb{R}^{2d}) + \pi_{t*}(\mathbb{R}^{2d}) \;=\; \mathbb{R}^d_t + \mathbb{R}^d_u$$

as their Cartesian product, or as the Minkowski sum of orthogonal spaces.

As noted previously, $\mathcal{T}_2(f)$ is not closed in the Euclidean topology and so is not a polyhedral complex. However, we can still use the additional structure of $\mathcal{T}(\pi_{u\mapsto\sigma}(f))$

and $\mathcal{T}(\pi_{t\mapsto\rho}(f))$ to describe $\mathcal{T}_2(f)$.

Definition 4.2.12. *The* (relative) interior *of an ordinary polyhedron P is the set of points* $\mathrm{int}(P)$ *contained in P but no face of P. Equivalently, it is the set cut out by the defining equalities and inequalities of P, where any proper inequalities are changed to strict inequalities.*

By removing its boundary, the interior of a polyhedron is not closed in the Euclidean topology, and so this is what we shall use to describe $\mathcal{T}_2(f)$. Note that the interior of a polyhedron is open if and only if it is full dimensional.

Let $f = \sum \gamma_s x^s$. For $S \subseteq \mathrm{supp}(f)$, we denote the restriction of f to the monomials labelled by S by $f_S = \sum_{s\in S} \gamma_s x^s$.

Theorem 4.2.13. *Let $f \in \mathbb{C}\{\{(t,u)^*\}\}[x_1^{\pm}, \dots, x_d^{\pm}]$ be a d-variate Laurent polynomial with admissible partial evaluations $t \mapsto \rho$ and $u \mapsto \sigma$. The rank two tropical hypersurface $\mathcal{T}_2(f)$ is the finite disjoint union*

$$\mathcal{T}_2(f) \;=\; \bigsqcup_{S} \bigsqcup_{T\supseteq S} \big(\mathrm{int}(Q_T) + \mathrm{int}(R_S)\big)$$

of interiors of polyhedra in $\mathbb{R}^{2d}$, where T, S are support sets of $\pi_{u\mapsto\sigma}(f)$ and $\pi_{t\mapsto\rho}(f_T)$ respectively, and Q_T, R_S are support cells of their rank one tropical hypersurfaces $\mathcal{T}(\pi_{u\mapsto\sigma}(f))$ in $\mathbb{R}^d_t$ and $\mathcal{T}(\pi_{t\mapsto\rho}(f_T))$ in $\mathbb{R}^d_u$ respectively.

Proof. By Proposition 4.2.8, $\mathcal{T}_2(f)$ is a lex-polyhedral complex of support cells $\mathbf{P}_S$ as S runs over all support sets of f of cardinality greater than one. In particular, this becomes a disjoint union if we restrict to the relative interiors of $\mathbf{P}_S$; by Remark 4.2.6 these are the points p such that $\mathrm{trop}_2(f)(p)$ attains its minimum at precisely the monomials labelled by S. We claim that $\mathrm{int}(\mathbf{P}_S) = \bigsqcup_{T\supseteq S} (\mathrm{int}(Q_T) + \mathrm{int}(R_S))$.

The point p is contained in $\mathrm{int}(\mathbf{P}_S)$ if and only if $\mathrm{trop}_2(f)(p)$ attains its minimum at precisely the monomials labelled by S i.e.,

$$\mathrm{val}_2(\gamma_s) + \langle s, p\rangle \ \leq \ \mathrm{val}_2(\gamma_{s'}) + \langle s', p\rangle \ , \text{ for all } s \in S \text{ and } s' \in \mathrm{supp}(f)$$

with equality if and only if $s' \in S$. Considering the lexicographical ordering on $\mathbb{T}_2$ and its coordinates separately, this is equivalent to the following two conditions:

$$\begin{aligned} &\pi_{u*}(\mathrm{val}_2(\gamma_s)) + \pi_{u*}(\langle s, p\rangle) \leq \pi_{u*}(\mathrm{val}_2(\gamma_{s'})) + \pi_{u*}(\langle s', p\rangle) \\ \Leftrightarrow \ &\mathrm{val}(\pi_{u\mapsto\sigma}(\gamma_s)) + \sum_{i=1}^{d} s_i p_{1i} \leq \mathrm{val}(\pi_{u\mapsto\sigma}(\gamma_{s'})) + \sum_{i=1}^{d} s'_i p_{1i} \ , \text{ for } s \in T \end{aligned} \tag{4.16}$$

for some $T \supseteq S$, with equality if and only if $s' \in T$.

$$\begin{aligned} &\pi_{t*}(\mathrm{val}_2(\gamma_s)) + \pi_{t*}(\langle s, p\rangle) \leq \pi_{t*}(\mathrm{val}_2(\gamma_{s'})) + \pi_{t*}(\langle s', p\rangle) \\ \Leftrightarrow \ &\mathrm{val}(\pi_{t\mapsto\rho}(\gamma_s)) + \sum_{i=1}^{d} s_i p_{2i} \leq \mathrm{val}(\pi_{t\mapsto\rho}(\gamma_{s'})) + \sum_{i=1}^{d} s'_i p_{2i} \ , \text{ for } s \in S, s' \in T \end{aligned} \tag{4.17}$$

with equality if and only if $s' \in S$. Condition (4.16) is equivalent to $\pi_{u*}(p)$ being contained in the interior of the support cell Q_T of $\mathcal{T}(\pi_{u\mapsto\sigma}(f))$. Condition (4.17) is equivalent to $\pi_{t*}(p)$ being contained in the interior of the support cell R_S of $\mathcal{T}(\pi_{t\mapsto\rho}(f_T))$.

It remains to show each part of the disjoint union is the interior of a polyhedron, or explicitly that $\mathrm{int}(Q_T + R_S) = \mathrm{int}(Q_T) + \mathrm{int}(R_S)$. As Q_T and R_S are in orthogonal ambient spaces, the union of their defining equalities and inequalities cut out $Q_T + R_S$. Changing the inequalities to strict inequalities gives the desired result. □

Since the order topology is finer than the Euclidean topology, the Euclidean closure becomes larger.

Corollary 4.2.14. *With the notation of Theorem 4.2.13: the closure of $\mathcal{T}_2(f)$ in the*

Euclidean topology is the finite union

$$\overline{\mathcal{T}_2(f)} \;=\; \bigcup_{S} \bigcup_{T \supseteq S} (Q_T + R_S)$$

of polyhedra in $\mathbb{R}^{2d}$*.*

Proof. As $Q_T + R_S = \overline{\operatorname{int}(Q_T) + \operatorname{int}(R_S)}$, the result follows from Theorem 4.2.13 using the fact that the closure of a finite union of sets equals the union of their closures. □

Remark 4.2.15. *Building on Theorem 4.2.13 and Corollary 4.2.14, one can give a slightly different characterisation of* $\mathcal{T}_2(f)$ *and its closure. Letting* T *range over support sets of* $\pi_{u\mapsto\sigma}(f)$ *and* S *over support sets of* $\pi_{t\mapsto\rho}(f_T)$*, we get*

$$\begin{aligned}
\mathcal{T}_2(f) \;&=\; \bigsqcup_{S} \bigsqcup_{T \supseteq S} \big(\operatorname{int}(Q_T) + \operatorname{int}(R_S)\big) \\
&= \bigsqcup_{T} \Big(\operatorname{int}(Q_T) + \bigsqcup_{S \subseteq T} \operatorname{int}(R_S)\Big) \\
&= \bigsqcup_{T} \big(\operatorname{int}(Q_T) + \mathcal{T}(\pi_{t\mapsto\rho}(f_T))\big) \;.
\end{aligned}$$

Taking the closure in the Euclidean topology gives the expression $\overline{\mathcal{T}_2(f)} = \bigcup_T \big(Q_T + \mathcal{T}(\pi_{t\mapsto\rho}(f_T))\big)$*. These alternative characterisations will be of use for Section 4.3.*

Remark 4.2.16. *Foster and Ranganathan [41] and Banerjee [18] both study notions of higher rank tropical geometry; in both cases the group of values is* $\mathbb{T}_m$ *(or a discrete subgroup). Banerjee considers the tropicalisation of subvarieties of the torus over* m*-dimensional local fields with discrete valuation, while Foster and Ranganathan consider a generalisation of Berkovich analytification. We note that our tropicalisation is not comparable to Banerjee's as* $\mathbb{C}\{\{(t,u)^*\}\}$ *is not a higher dimensional local field in the sense of [18, Definition 3.1]. However, both are special cases of the tropicalisation of Foster and Ranganathan. In particular, for* $m = 2$ *our* $\mathcal{T}_2(f)$ *from Definition 4.2.1 is covered in [41].*

There is also a conceptual difference between the approach of Foster and Ranganathan and Banerjee's approach. Banerjee begins with small fields and discrete valuations and then takes algebraic and topological closures to "fill in gaps", while Foster and Ranganathan begin with larger fields, via Hahn analytification, to avoid taking topological closures. Our approach via generalised Puiseux series is in the same spirit as Foster and Ranganathan's. While either approach behaves well for $m = 1$, the following shows that topological closure operations go awry when $m > 1$ and thus need to be dealt with carefully.

To see this, first let us very briefly describe the setup of [18]. Any m-dimensional local field $\mathbb{K}$*, in the sense of [18, Definition 3.1], admits a valuation* $\nu^{\mathbb{K}} : \mathbb{K}^\times \to \Gamma^{\mathbb{K}}$ *where* $\Gamma^{\mathbb{K}} \cong \mathbb{Z}^m$ *with the lexicographical ordering. For any finite field extension* $\mathbb{L}$ *of* $\mathbb{K}$*, this valuation extends to a valuation* $\nu^{\mathbb{L}} : \mathbb{L}^\times \to \Gamma^{\mathbb{L}}$*. This allows us to extend* $\nu^{\mathbb{K}}$ *to the algebraic closure of* $\mathbb{K}$*, becoming the surjective map* $\nu : (\mathbb{K}^{\mathrm{al}})^\times \to \Gamma_{\mathbb{Q}} \cong \mathbb{Q}^m$ *where* $\Gamma_{\mathbb{Q}}$ *is the direct limit of all groups* $\Gamma^{\mathbb{L}}$ *taken over all finite field extensions* $\mathbb{L}$ *of* $\mathbb{K}$*. Finally, we let* $\Gamma_{\mathbb{R}} := \Gamma_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^m$ *and extend the codomain of* ν *to* $\Gamma_{\mathbb{R}}$*. One then considers subvarieties of the* d*-dimensional algebraic torus over* $\mathbb{K}$ *and their images in* ν*.*

Banerjee's notion of a tropical hypersurface is the same as Aroca's [12], and this agrees with Definition 4.2.1. Now [18, Theorem 5.3] claims that $\mathcal{T}_m(f)$ *is equal to*

$$\overline{\{\nu(p) \mid p \in \mathcal{X}_f\}} \ ,$$

where $\mathcal{X}_f$ *is the hypersurface in the algebraic torus defined by* f*. Unfortunately, in which topology the closure is taken in is not specified. The discussion in [41, Section 2.3] erroneously assumes it is the Euclidean topology. However, the resulting set contains* $\mathcal{T}_m(f)$ *but is too large and contains points where* $\operatorname{trop}_m(f)$ *is linear. Note that Banerjee's definition of a polyhedron [18, Notation 4.1.(v)] generalises our definition of a lex-polyhedron slightly by replacing* $\mathbb{Z}^m$ *by any totally order group* Γ*. Furthermore, [18, Example 5.11] is a computation of a rank two tropical hypersurface, similar to our Example 4.2.3, and* *is not closed in the Euclidean topology.*

However, it is worth noting that taking the order topology does not fix the claim made in [18, Theorem 5.3]. The image of the valuation ν is isomorphic to $\mathbb{Q}^m$ with the lexicographical ordering. In the order topology, $\mathbb{Q}^m$ is not dense in $\mathbb{R}^m$, as its closure does not contain any elements of the form $(a_1, \ldots, a_m)$ where a_1 is irrational. Therefore the closure in the order topology is contained in $\mathcal{T}_m(f)$ but is too small.

To close this section, we give two examples to demonstrate that rank two tropical hypersurfaces are quite different from their rank one counterparts, even when taking their closure in the Euclidean topology. Example 4.2.17 demonstrates the closure of a rank two tropical hypersurface is not a polyhedral complex, as polyhedra may not intersect at their faces. Example 4.2.18 shows the closure of a rank two tropical hypersurfaces do not satisfy a purity condition, as the polyhedra that are maximal with respect to inclusion may not be of the same dimension.

Example 4.2.17. *We return to the rank two tropical hypersurface of the polynomial $f = x_1 + tx_2 + t^2u$ from Examples 4.2.3, 4.2.9 and 4.2.11. As its coefficients are monomials in t and u, the partial evaluations of f are defined at the admissible values $\rho = \sigma = 1$. Let $T = \{(0,0),(0,1)\}$, and consider the support cell*

$$Q_T \;=\; \{\,(2+\lambda_1, 1) \mid \lambda_1 \geq 0\}$$

of the tropical line $\mathcal{T}(\pi_{u\mapsto 1}(f))$ in $\mathbb{R}^2_t$. The polynomial $\pi_{t\mapsto 1}(f_T) = x_2 + u$ defines a rank 1 tropical hypersurface with a single support cell

$$R_S \;=\; \{\,(\lambda_2, 1) \mid \lambda_2 \in \mathbb{R}\} \quad ,$$

in $\mathbb{R}^2_u$, where $S = \{(0,0),(0,1)\}$. By Corollary 4.2.14, the sum of these two polyhedra

$$Q_T + R_S \;=\; \{\,(2,1;1,1) + (\lambda_1, \lambda_2; 0, 0) \mid \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\} \subset \mathbb{R}^4$$

is a polyhedron in $\overline{\mathcal{T}_2(f)}$*. Ranging over all support sets* S *and* T*, the closure of* $\mathcal{T}_2(f)$ *in the Euclidean topology is the union*

$$\begin{aligned}\overline{\mathcal{T}_2(f)} &= \{(2,1;1,1) + (\lambda_1,\lambda_2;0,0) \mid \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\}\\ &\cup \{(2,1;1,1) + (0,0;\lambda_1,\lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\}\\ &\cup \{(2,1;1,1) + (\lambda_1,\lambda_2;\lambda_1,\lambda_2) \mid \lambda_1 \leq 0, \lambda_2 \in \mathbb{R}\}\end{aligned}$$

of three ordinary halfplanes in $\mathbb{R}^4$*. Note that this is not an ordinary polyhedral complex as the polyhedra do not intersect at faces. The joint intersection of the three ordinary halfplanes is the point* $(2,1;1,1)$*, but this is not a (zero-dimensional) face of any of them.*

Example 4.2.18. *Consider the polynomial* $f = ux_1x_2 + x_1 + x_2 + 1$*, whose vanishing locus is a conic. The closure of its rank two tropical hypersurface is the union of ordinary polyhedra:*

$$\begin{aligned}\overline{\mathcal{T}_2(f)} &= \{(\lambda_1,\lambda_2;0,0) \mid \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\}\\ &\cup \{(0,0;\lambda_1,\lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\}\\ &\cup \{(0,\lambda;0,\lambda) \mid \lambda \in [-1,0]\}\\ &\cup \{(\lambda_1,\lambda_2;0,-1) \mid \lambda_1 \leq 0, \lambda_2 \in \mathbb{R}\}\\ &\cup \{(0,-1,\lambda_1,\lambda_2) \mid \lambda_1 \leq 0, \lambda_2 \in \mathbb{R}\} \;.\end{aligned}$$

We say a finite union of polyhedra is pure *if all its maximal polyhedra (with respect to inclusion) have the same dimension. This generalises a notion commonly used for polyhedral complexes; in fact, it is the same if applied to the polyhedral complex obtained by taking the common refinement of the finitely many given polyhedra. Observe that* $\overline{\mathcal{T}_2(f)}$ *is not pure, as the maximal polyhedra are all two-dimensional, except for the line segment* $\{(0,\lambda;0,\lambda) \mid \lambda \in [-1,0]\}$*. This can be decomposed as the direct sum of support cells*

$$Q_T + R_S = \{(0,0)\} + \{(\lambda,\lambda) \mid \lambda \in \mathbb{R}\}$$

where $T = \{(0,0),(1,0),(0,1),(1,1)\}$ *and* $S = \{(1,0),(0,1)\}$*. In particular,* $S \subset T$

implies $\dim(Q_T) < \dim(R_S)$*. However, the pairs of support cells in the decomposition of the other maximal polyhedra have equal support sets, and therefore the same dimension.a*

4.3 Stable intersection

In this section, we use the higher rank machinery developed so far to obtain a new description of the stable intersection of rank one tropical hypersurfaces. To do so, we must first consider the structure of rank two tropical hypersurfaces determined by polynomials with coefficients in $\mathbb{C}\{\{t^*\}\}$.

We recall the following polyhedral definition. Fix some polyhedral complex Σ and let P be a cell in Σ. The *star* of P is the fan spanned by the cells of Σ containing P; more precisely,

$$\operatorname{star}(P) \;=\; \bigcup_{Q\in\Sigma,\, Q\supseteq P} \{\, \lambda(q-p) \mid \lambda \geq 0,\, p\in P,\, q\in Q\} \quad . \tag{4.18}$$

Let f be a Laurent polynomial in $\mathbb{C}\{\{t^*\}\}[x_1^\pm,\ldots,x_d^\pm]$. Under the embedding ι, we can also consider f as a polynomial in $\mathbb{C}\{\{(t,u)^*\}\}[x_1^\pm,\ldots,x_d^\pm]$ with an associated rank two tropical hypersurface. We arrive at another consequence of Theorem 4.2.13.

Corollary 4.3.1. *Let* $f \in \mathbb{C}\{\{t^*\}\}[x_1^\pm,\ldots,x_d^\pm]$ *be a* d*-variate Laurent polynomial. The rank two tropical hypersurface* $\mathcal{T}_2(f)$ *is the disjoint finite union*

$$\mathcal{T}_2(f) \;=\; \bigsqcup_S \big(\operatorname{int}(P_S) + \operatorname{star}(P_S)\big)$$

in $\mathbb{R}^{2d}$*, where* P_S *is a support cell of* $\mathcal{T}(f)$ *in* $\mathbb{R}^d_t$ *and* $\operatorname{star}(P_S)$ *is embedded in* $\mathbb{R}^d_u$*.*

Proof. Clearly, this is a special case of Remark 4.2.15 where f agrees with $\pi_{u\mapsto\sigma}(f)$. We infer that $\mathcal{T}_2(f)$ is the disjoint union $\operatorname{int}(P_S)+\mathcal{T}(\pi_{t\mapsto\rho}(f_S))$. Since $\pi_{t\mapsto\rho}(f_S)$ has constant coefficients its tropical hypersurface is a fan. By [63, Theorem 3.5.6] this is the *recession fan* of $\mathcal{T}(f_S)$, and in this case it agrees with $\operatorname{star}(P_S)$. $\square$

Corollary 4.3.2. *The closure of $\mathcal{T}_2(f)$ in the Euclidean topology is the finite union*

$$\overline{\mathcal{T}_2(f)} \;=\; \bigcup_S (P_S + L_S)$$

of polyhedra in $\mathbb{R}^{2d}$, where P_S is a maximal support cell of $\mathcal{T}(\pi_{u\mapsto\sigma}(f))$ in $\mathbb{R}^d_t$ and L_S is the linear space equal to the affine span of P_S translated to the origin in $\mathbb{R}^d_u$.

Proof. Remark 4.2.15 and Corollary 4.3.1 imply that $\overline{\mathcal{T}_2(f)}$ equals the union $\bigcup\big(P_S + \operatorname{star}(P_S)\big)$. Each cell of $\operatorname{star}(P_S)$ is labelled by some $T \subseteq S$ corresponding to $P_T \supseteq P_S$. Note that if P_S is a maximal support cell of $\mathcal{T}(\pi_{u\mapsto\sigma}(f))$, $\operatorname{star}(P_S)$ is simply the linear space L_S. Furthermore, if P_S is not a maximal support cell of $\mathcal{T}(\pi_{u\mapsto\sigma}(f))$, then the maximal cell of $\operatorname{star}(P_S)$ labelled by $T \subset S$ is contained in L_T. Therefore we can restrict the union to just the maximal support cells, giving the desired result. □

Example 4.3.3. *Consider the degree three polynomial*

$$f \;=\; 1 + t(x+y) + t^3xy + t^5(x^2+y^2) + t^9(x^2y+xy^2) + t^{15}(x^3+y^3)$$

in $\mathbb{C}\{\{t^*\}\}[x,y]$. It describes an elliptic curve, whose rank one tropicalisation is shown in Figure 4.1. When we view f as a polynomial with coefficients in $\mathbb{C}\{\{(t,u)^*\}\}$, Corollary 4.3.1 describes the resulting rank two tropical curve. The partial evaluation $\pi_{u\mapsto\sigma}(f)$ equals f, and $\pi_{t\mapsto\rho}(f)$ has constant coefficients, for any ρ and σ. For instance, let us look at the cell marked "P_S" in Figure 4.1 where $S = \{(0,1),(1,1)\}$, we get $f_S = ty + t^3xy$. It follows that $L_S = \mathcal{T}(f_S)$ is the y-axis, and this is also the only cell in that tropical hypersurface.

To develop a new description of stable intersection, we introduce the following notion of perturbation on the level of Puiseux series.

Definition 4.3.4. *Let $\beta > 0$ be a fixed transcendental number. The u-*perturbation *of f by β is the polynomial $f^u \in \mathbb{C}\{\{(t,u)^*\}\}[x_1^{\pm},\ldots,x_d^{\pm}]$ obtained from f by the d linear*

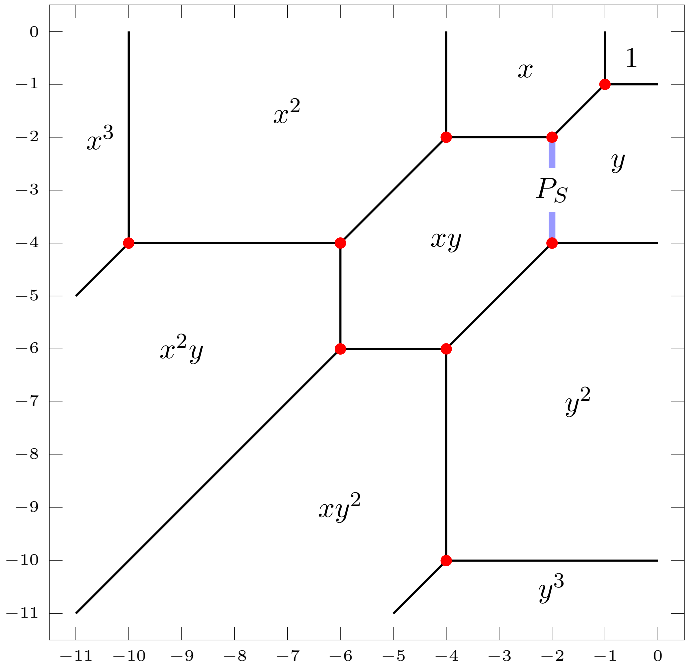

Figure 4.1: Tropical elliptic curve with the one-dimensional cell P_S marked; cf. Example 4.3.3. Each region is labelled with its supporting monomial.

substitutions $x_k \mapsto u^{\beta^k} x_k$.

We are interested in the effect of the u-perturbation to the tropicalisation of f. As $\mathrm{val}(u) < \mathrm{val}(t)$, the variable u can be considered an infinitesimal perturbation to the coefficients of f. Explicitly, the u-perturbation of the term $\gamma_s x^s$, which is a d-variate Laurent monomial whose single coefficient γ_s lies in $\mathbb{C}\{\{t^*\}\}$, equals

$$\gamma_s u^{s_1\beta + s_2\beta^2 + \cdots + s_d\beta^d} x^s \quad .$$

Its rank two tropicalisation is

$$\left(\mathrm{val}(\gamma_s), \sum s_i\beta^i\right) + s_1x_1 + \cdots + s_dx_d \quad .$$

Since β is transcendental, the expression $\sum s_i\beta^i$ does not vanish, unless $s_1 = \cdots = s_d = 0$. In particular, we have $u^{s_1\beta+s_2\beta^2+\cdots+s_d\beta^d} \neq 1$, and it follows that no nonconstant term of f^u has a coefficient which lies in the subfield $\mathbb{C}\{\{t^*\}\}$. Yet the partial evaluation

$\pi_{u\mapsto\sigma}(f)$ is defined for all $\sigma > 0$. Moreover, $\mathrm{supp}(f^u) = \mathrm{supp}(f)$.

The following lemma describes the u-perturbation as a translation at the level of rank two tropical hypersurfaces.

Lemma 4.3.5. *Let $f \in \mathbb{C}\{\{t^*\}\}[x_1^\pm, \ldots, x_d^\pm]$ be a d-variate Laurent polynomial. Then*

$$\mathcal{T}_2(f) \ = \ \mathcal{T}_2(f^u) + (0, \ldots, 0; \beta, \ldots, \beta^d) \ .$$

Moreover, the same holds for the closures in the Euclidean topology, i.e.,

$$\overline{\mathcal{T}_2(f)} \ = \ \overline{\mathcal{T}_2(f^u)} + (0, \ldots, 0; \beta, \ldots, \beta^d) \ .$$

Proof. Let $p = (p_{11}, p_{21}; \ldots; p_{1d}, p_{2d}) \in \mathcal{T}_2(f)$. Then there exist distinct s and s' in $\mathrm{supp}(f)$ with $\mathrm{val}_2(\gamma_s) + \langle s, p\rangle = \mathrm{val}_2(\gamma_{s'}) + \langle s', p\rangle$, where $\mathrm{val}_2(\gamma_s) = (\mathrm{val}(\gamma_s), 0)$ and $\mathrm{val}_2(\gamma_{s'}) = (\mathrm{val}(\gamma_{s'}), 0)$. Hence

$$\begin{aligned} &\big(\mathrm{val}(\gamma_s), \textstyle\sum s_i\beta^i\big) + s_1(p_{11}, p_{21} - \beta) + \cdots + s_d(p_{1d}, p_{2d} - \beta^d) \\ &= \ \mathrm{val}_2(\gamma_s) + \langle s, p\rangle \ = \ \mathrm{val}_2(\gamma_{s'}) + \langle s', p\rangle \\ &= \ \big(\mathrm{val}(\gamma_{s'}), \textstyle\sum s_i\beta^i\big) + s_1'(p_{11}, p_{21} - \beta) + \cdots + s_d'(p_{1d}, p_{2d} - \beta^d) \ . \end{aligned} \tag{4.19}$$

In other words, as $\mathrm{supp}(f^u) = \mathrm{supp}(f)$, the point $p - (0, \ldots, 0; \beta, \ldots, \beta^d)$ lies in $\mathcal{T}_2(f^u)$, and this proves one inclusion. The argument can be reversed, and the claim on $\mathcal{T}_2(f)$ follows.

The explicit computation in (4.19) carries over to the topological closure by continuity of the arithmetic operations. ☐

We recall the following concepts from [63, §3.6]. Let f and g be Laurent polynomials in $\mathbb{C}\{\{t^*\}\}[x_1^\pm, \ldots, x_d^\pm]$. The (polyhedral) *stable intersection* of their tropical hypersurfaces is the polyhedral complex

$$\mathcal{T}(f) \cap_{\mathrm{st}} \mathcal{T}(g) \;=\; \bigcup_{\dim(P+Q)=d} (P \cap Q) \tag{4.20}$$

where P and Q are cells of $\mathcal{T}(f)$ and $\mathcal{T}(g)$, respectively. This is a coarser notion than stable intersection of tropical varieties as it does not remember the multiplicities of the varieties. Unless explicitly stated, we restrict purely to polyhedral stable intersection from now on.

Theorem 4.3.6. *Let $f, g \in \mathbb{C}\{\{t^*\}\}[x_1^{\pm}, \ldots, x_d^{\pm}]$. The stable intersection of $\mathcal{T}(f)$ and $\mathcal{T}(g)$ is given by projecting the set theoretic intersection of (the closures of) the rank two tropical hypersurfaces $\overline{\mathcal{T}_2(f)}$ and $\overline{\mathcal{T}_2(g^u)}$; more precisely,*

$$\mathcal{T}(f) \cap_{\mathrm{st}} \mathcal{T}(g) \;=\; \pi_{u*}\big(\overline{\mathcal{T}_2(f)} \cap \overline{\mathcal{T}_2(g^u)}\big) \;\; .$$

Proof. Let $p_1 \in \mathcal{T}(f) \cap_{\mathrm{st}} \mathcal{T}(g) \subset \mathbb{R}^d_t$. Then there are maximal support cells P_S and P_T of $\mathcal{T}(f)$ and $\mathcal{T}(g)$, respectively, containing p_1 with $\dim(P_S + P_T) = d$. Corollary 4.3.2 says that $P_S + L_S$ and $P_T + L_T$ are maximal polyhedra in $\overline{\mathcal{T}_2(f)}$ and $\overline{\mathcal{T}_2(g)}$, respectively. We have $\overline{\mathcal{T}_2(g)} = \overline{\mathcal{T}_2(g^u)} + (0, \ldots, 0; \beta, \ldots, \beta^d)$ by Lemma 4.3.5. From $\dim(P_S + P_T) = d$, we infer $L_S + L_T = \mathbb{R}^d_u$. Thus there are $q_S \in L_S$ and $q_T \in L_T$ with $q_T - q_S = (\beta, \ldots, \beta^d)$. Hence, setting $p_2 := q_S = q_T - (\beta, \ldots, \beta^d)$ and $p := p_1 + p_2$, yields

$$p \in (P_S + L_S) \cap (P_T + (L_T - (\beta, \ldots, \beta^d))) \;\; ,$$

which is contained in $\overline{\mathcal{T}_2(f)} \cap \overline{\mathcal{T}_2(g^u)}$, and $\pi_{u*}(p) = p_1$.

Conversely let $p \in \overline{\mathcal{T}_2(f)} \cap \overline{\mathcal{T}_2(g^u)} \subset \mathbb{R}^{2d}$. Then there are maximal support cells P_S and P_T of $\mathcal{T}(f)$ and $\mathcal{T}(g)$, respectively, such that $\pi_{u*}(p) \in P_S \cap P_T$ and $\pi_{t*}(p) \in L_S \cap (L_T - (\beta, \ldots, \beta^d))$. We need to show that $\dim(P_S + P_T) = d$. As P_S and P_T are both maximal, we have $\dim P_S = \dim L_S = \dim L_T = \dim P_T = d - 1$. Suppose that $\dim(P_S + P_T) < d$. Then $\dim(P_S + P_T) = d - 1$, and the linear subspaces $L_S = L_T$

must be equal. As a consequence the linear subspace L_S and the parallel affine subspace $L_T - (\beta, \ldots, \beta^d)$ are disjoint. Yet this contradicts that $\pi_{t*}(p)$ lies in their intersection. We conclude that $\dim(P_S + P_T) = d$, and $\pi_{u*}(p)$ is contained in the stable intersection. $\square$

The stable intersection of $\mathcal{T}(f)$ and $\mathcal{T}(g)$ can also be obtained by perturbing $\mathcal{T}(g)$ generically and taking the limit of its intersection with $\mathcal{T}(f)$ [63, Proposition 3.6.12], i.e.,

$$\mathcal{T}(f) \cap_{\mathrm{st}} \mathcal{T}(g) \;=\; \lim_{\epsilon \to 0} \big(\mathcal{T}(f) \cap (\mathcal{T}(g) + \epsilon v)\big) \tag{4.21}$$

for any generic $v \in \mathbb{R}^d$. In this way, Theorem 4.3.6 can be seen as a version of (4.21) based on the "symbolic perturbation" paradigm common in computational geometry; e.g., see [33] and [36].

Example 4.3.7. *Consider the two bivariate polynomials*

$$f = xy + x + y + 1 \quad \textit{and} \quad g = x + ty + t$$

with coefficients in $\mathbb{C}\{\{t^*\}\}$*. Their corresponding tropical hypersurfaces are shown in Figure 4.2. The intersection of their corresponding rank one tropical hypersurfaces is a ray and a point*

$$\mathcal{T}(f) \cap \mathcal{T}(g) \;=\; \{\,(\lambda + 1, 0) \mid \lambda \geq 0\} \cup \{(0, -1)\} \ .$$

That is, the intersection at the origin is not transverse *in the sense of [63, Definition 3.4.9].*

We consider f *and* g *as polynomials with coefficients in* $\mathbb{C}\{\{(t, u)^*\}\}$*. The* u*-perturbation of* g *is*

$$g^u \;=\; u^\beta x + t u^{\beta^2} y + t \ .$$

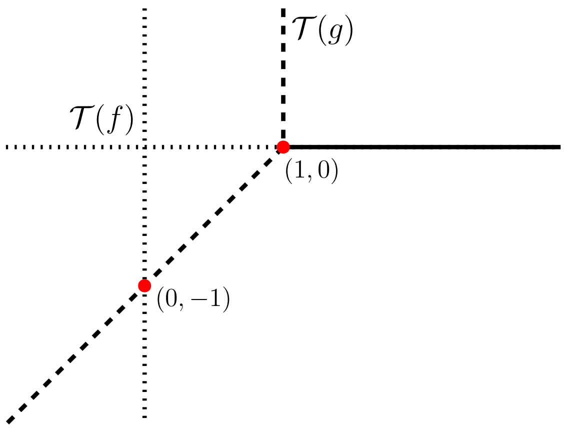

Figure 4.2: The tropical hypersurfaces $\mathcal{T}(f), \mathcal{T}(g)$ from Example 4.3.7. The red points correspond to their stable intersection, while the solid ray only appears in their set-theoretic intersection.

The closure of their rank two tropical hypersurfaces in $\mathbb{R}^4$ *read as follows:*

$$
\begin{aligned}
\overline{\mathcal{T}_2(f)} &= \{(\lambda_1, \lambda_2; 0, 0) \mid \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\} \\
&\cup \{(\lambda_1, \lambda_2; 0, 0) \mid \lambda_1 \leq 0, \lambda_2 \in \mathbb{R}\} \\
&\cup \{(0, 0; \lambda_1, \lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\} \\
&\cup \{(0, 0; \lambda_1, \lambda_2) \mid \lambda_1 \leq 0, \lambda_2 \in \mathbb{R}\} \\
\overline{\mathcal{T}_2(g^u)} &= \left\{(1+\lambda_1, \lambda_2; 0, -\beta^2) \;\middle|\; \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\right\} \\
&\cup \{(1, -\beta; \lambda_1, \lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\} \\
&\cup \left\{(1-\lambda_1, -\beta+\lambda_2; -\lambda_1, -\beta^2+\lambda_2) \;\middle|\; \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}\right\} \quad .
\end{aligned}
$$

Their intersection is the three points $(1, -\beta; 0, 0)$, $(1, \beta^2 - \beta; 0, 0)$ *and* $(0, 0; -1, \beta - \beta^2)$. *Projecting them via* π_{u*} *yields* $(1, 0)$ *and* $(0, -1)$ *in* $\mathbb{R}^2$. *These two points form the stable intersection of* $\mathcal{T}(f)$ *and* $\mathcal{T}(g)$.

4.4 Rank two tropical convexity

Now we switch back to Puiseux series with real coefficients. We start out with a closer look at the ordering on $\mathbb{R}\{\{(t,u)^*\}\}$, which is induced by the lexicographic ordering of the exponents. The map

$$\mathrm{val}_2 : \mathbb{R}\{\{(t,u)^*\}\} \setminus \{0\} \longrightarrow \mathbb{T}_2$$

is a rank two valuation. It sends a convergent generalised Puiseux series $\gamma(t,u)$ to its smallest exponent vector. The restriction to positive series is an order reversing homomorphism of ordered semirings onto $\mathbb{T}_2$, which is equipped with the lexicographic ordering; cf. (4.7). For instance, we have the following strict inequalities

$$t^9 \ < \ t^2 \ < \ tu^{1000} \ < \ tu$$

of positive monomials, and these are equivalent to the reverse inequalities

$$(9,0) \ > \ (2,0) \ > \ (1,1000) \ > \ (1,1)$$

of the exponents. An example involving more general series, which are not necessarily positive, is

$$\begin{aligned} \mathrm{val}_2(t^9 - 3t^{10}) = (9,0) \ &> \ \mathrm{val}_2(-t^2 + 5t^4u^2 + t^{17}) = (2,0) \\ &> \ \mathrm{val}_2(tu^{1000}) = (1,1000) \ > \ \mathrm{val}_2(tu) = (1,1) \ . \end{aligned}$$

It is useful to extend $\mathbb{T}_2$ by the additional element ∞ which is neutral with respect to the tropical addition min, absorbing with respect to the tropical multiplication + and larger than any element in $\mathbb{T}_2$. By letting $\mathrm{val}_2(0) = \infty$ this yields an extension of the rank two valuation map. This is continuous with respect to the respective order topologies. Recall that the order topology on $\mathbb{T}_2$, which agrees with $\mathbb{R}^2$ as a set, is finer than the Euclidean topology.

In the subfield $\mathbb{R}\{\{t^*\}\}$ we have the inequalities $0 < u < c$ for any real number c, and we write this as $0 < u \ll 1$. By the same token we have

$$0 \; < \; t \; \ll \; u \; \ll \; 1 \tag{4.22}$$

in $\mathbb{R}\{\{(t,u)^*\}\}$; cf. Figure 4.3. Since our valuation prefers terms of *minimal* order we say that the indeterminate t *dominates* u.

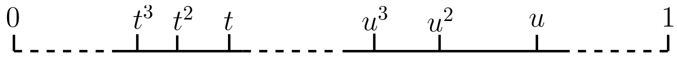

Figure 4.3: The relation between infinitesimals $t \ll u$.

The purpose of this section is to study the interplay between three notions of convexity: *ordinary convexity* with respect to the ordered field $\mathbb{R}\{\{(t,u)^*\}\}^d$, *rank two tropical convexity* with respect to tropical semifield $\mathbb{T}_2$, and *lex-convexity* with respect to the lexicographic ordering on $\mathbb{T}_2$.

An *(ordinary) cone* in $\mathbb{R}\{\{(t,u)^*\}\}^d$ is a nonempty subset K which satisfies $\lambda p + \mu q \in K$ for all $p, q \in K$ and $\lambda, \mu \geq 0$. It is *polyhedral* if it is finitely generated. By definition a cone in $\mathbb{R}\{\{(t,u)^*\}\}^d$ is exactly the same as a submodule with respect to the semiring $\mathbb{R}\{\{(t,u)^*\}\}_{\geq 0}$ of nonnegative elements. We now make use of the notation '$\oplus$' instead of 'min' and '$\odot$' instead of '$+$' to stress the connection between tropical and ordinary linear algebra .

Definition 4.4.1. *A* rank two tropical cone *in* $(\mathbb{T}_2 \cup \{\infty\})^d$ *is a nonempty subset* M *which satisfies*

$$(\lambda \odot p) \oplus (\mu \odot q) \; = \; \min(\lambda + p, \mu + q) \in M$$

for all $p, q \in M$ *and* $\lambda, \mu \in \mathbb{T}_2 \cup \{\infty\}$. *A rank two tropical cone is* polyhedral *if it is finitely generated.*

The following is a rank two analogue of a result by Develin and Yu [30, Proposition 2.1].

Proposition 4.4.2. *Let* K *be an ordinary cone in* $\mathbb{R}\{\{(t,u)^*\}\}^d_{\geq 0}$. *Then* $\mathrm{val}_2(K)$ *is a* *rank two tropical cone in* $(\mathbb{T}_2 \cup \{\infty\})^d$*, and conversely each rank two tropical cone arises in this way. Furthermore, if* K *is polyhedral then* $\mathrm{val}_2(K)$ *is also, and conversely each rank two tropical polyhedral cone is the image of a polyhedral cone in the valuation map.*

Proof. As val_2 is a homomorphism of semirings if restricted to positive convergent Puiseux series it follows that $\mathrm{val}_2(K)$ is a rank two tropical cone. Another consequence of this is that if K is polyhedral then $\mathrm{val}_2(K)$ is also.

It remains to show that, for a rank two tropical cone M in $(\mathbb{T}_2 \cup \{\infty\})^d$, there is a cone K in $\mathbb{R}\{\{(t,u)^*\}\}^d_{\geq 0}$ with $\mathrm{val}_2(K) = M$. We set

$$K \ := \ \{ (t^{p_{11}}u^{p_{21}}, \ldots, t^{p_{1d}}u^{p_{2d}}) \mid (p_{11}, p_{21}; \ldots; p_{1d}, p_{2d}) \in M \} \ ,$$

where we use the convention $t^a u^b = 0$ for $(a,b) = \infty$. The fact that M is a rank two tropical cone implies that K is a cone, again because $\mathrm{val}_2 : \mathbb{R}\{\{(t,u)^*\}\}_{\geq 0} \to \mathbb{T}_2 \cup \{\infty\}$ is a homomorphism of semirings. As a further consequence, if M is polyhedral then K must be also. $\square$

A subset K of $\mathbb{R}\{\{(t,u)^*\}\}^d$ is *(ordinary) convex* if $\lambda p + \mu q \in K$ for all $p, q \in K$ and $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. It is an *(ordinary) polytope* if it is finitely generated.

Definition 4.4.3. *A subset* M *of* $(\mathbb{T}_2 \cup \{\infty\})^d$ *is* rank two tropically convex *if* $(\lambda \odot p) \oplus (\mu \odot q) \in M$ *for all* $p, q \in M$ *and* $\lambda, \mu \in \mathbb{T}_2 \cup \{\infty\}$ *with* $\lambda \oplus \mu = (0,0)$*. It is a* rank two tropical polytope *if it is finitely generated.*

Corollary 4.4.4. *Let* K *be a convex set in the positive orthant* $\mathbb{R}\{\{(t,u)^*\}\}^d_{\geq 0}$*. Then* $\mathrm{val}_2(K)$ *is a rank two tropically convex set in* $(\mathbb{T}_2 \cup \{\infty\})^d$*, and conversely each rank two tropically convex set arises in this way. Furthermore, if* K *is an ordinary polytope then* $\mathrm{val}_2(K)$ *is a rank two tropical polytope, and conversely every rank two tropical polytope is the image of a polytope in the valuation map.*

Proof. All the claims follow from Proposition 4.4.2 by homogenisation. Indeed, consider the cone K' generated by the vectors $(1,p) \in \mathbb{R}\{\{(t,u)^*\}\}_{\geq 0}^{d+1}$ for $p \in K$. Then $\mathrm{val}_2(K')$ is a rank two tropical cone. The set M of points $q \in (\mathbb{T}_2 \cup \{\infty\})^d$ such that $((0,0),q) \in \mathrm{val}_2(K')$ is rank two tropically convex and $\mathrm{val}_2(K) = M$. □

None of the above is a special property of fields of (convergent) Puiseux series with real coefficients. In fact, this generalises to any ordered field $\mathbb{K}$ with a valuation map which is surjective onto some totally ordered abelian group G. Yet, combining the higher rank transfer principle (4.7) with Proposition 4.4.2 we get a third diagram, this time of modules over semirings, i.e., cones. As before π_u is not globally defined. This makes sense for convergent Puiseux series only.

$$\begin{tikzcd}
\mathbb{R}\{\{t^*\}\}_{\geq 0}^d \arrow[r, "\iota"] \arrow[d, "\mathrm{val}"] & \mathbb{R}\{\{(t,u)^*\}\}_{\geq 0}^d \arrow[r, dashed, "\pi_u"] \arrow[d, "\mathrm{val}_2"] & \mathbb{R}\{\{t^*\}\}_{\geq 0}^d \arrow[d, "\mathrm{val}"] \\
(\mathbb{T} \cup \{\infty\})^d \arrow[r, "\iota_*"] & (\mathbb{T}_2 \cup \{\infty\})^d \arrow[r, "\pi_{u*}"] & (\mathbb{T} \cup \{\infty\})^d
\end{tikzcd} \tag{4.23}$$

As the tropicalisation of any ordinary cone or polytope in $\mathbb{R}\{\{(t,u)^*\}\}_{\geq 0}^d$ is a rank two tropical cone or polytope, any results on the latter objects hold also for the former. Additionally, any results for rank two tropical cones give analogous results for rank two tropical polytopes by homogenisation. Therefore for simplicity, we shall work only with rank two tropical cones for the remainder of the section.

Rank one tropical cones have an explicit description as a polyhedral complex in terms of their *covector decomposition*; cf. [63, §5.2] and [53]. As with rank two tropical hypersurfaces, rank two tropical cones are not closed in the Euclidean topology; cf. Figure 4.4, therefore they do not have a polyhedral decomposition in the ordinary sense. However, we can construct an analogous decomposition in terms of lex-polyhedra by building on the corresponding notions in rank one.

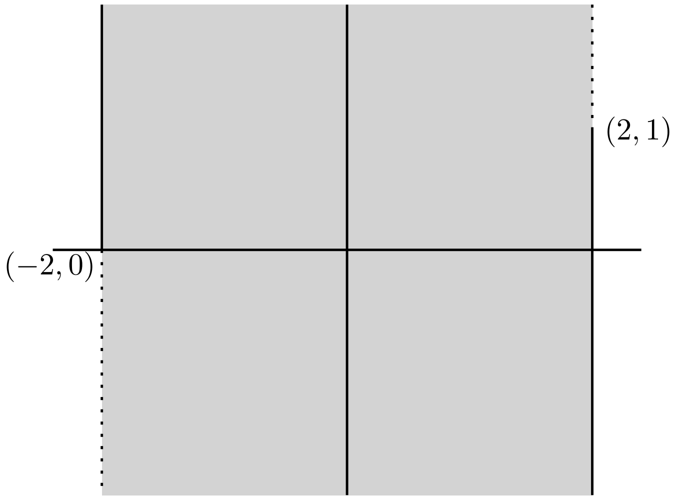

Figure 4.4: The tropicalisation of the ordinary interval $[t^2 u, t^{-2}]$ in $\mathbb{R}\{\{(t,u)^*\}\}$ as a subset of $\mathbb{T}_2$. It is a tropically convex set generated by $\{(-2,0),(2,1)\}$. Note that it is not closed under the Euclidean topology as the dotted boundary is not part of the interval.

Given a point $u \in (\mathbb{T}_2 \cup \{\infty\})^d$ with $u_i \neq \infty$, we define its *ith sector*

$$\begin{aligned} \mathbf{Z}_i(u) &= \bigcap_{k \in [d], u_k \neq \infty} \left\{ p \in \mathbb{T}_2^d \;\middle|\; p_k - p_i \leq u_k - u_i \right\} \\ &= \bigcap_{k \in [d], u_k \neq \infty} \mathbf{H}_{e_k - e_i, u_k - u_i} \end{aligned}$$

where $e_1, \dots, e_d \in \mathbb{Z}^d$ are the standard unit vectors. Observe that by definition each sector is a lex-polyhedron.

Remark 4.4.5. *As the two operations behave isomorphically, one can choose tropical addition to be* min *or* max. *The rank m tropical max-plus semiring $\mathbb{T}_m^{\max} = (\mathbb{R}^m, \max, +)$ is appended with the additive identity element $-\infty$, the smallest element under the lexicographical ordering. This allows us to give some geometric intuition to the sectors $\mathbf{Z}_i(u)$.*

Given a point $u \in (\mathbb{T}_2 \cup \{\infty\})^d$, consider the max-tropical linear form

$$F_u = \max\{ x_i - u_i \mid i \in [d] \;,\; u_i \neq \infty \} \;. \tag{4.24}$$

Its support is the set of standard unit vectors $\operatorname{supp}(F_u) = \{ e_i \mid i \in [d] \ , \ u_i \neq \infty\}$. *As with min-tropical hypersurfaces, its max-tropical hypersurface* $\mathcal{T}_2(F_u)$ *is the locus of points at which* F_u *is non-linear. The results of Section 4.2 hold for* $\mathcal{T}_2(F_u)$*, in particular it induces a decomposition of* $\mathbb{T}_2^d$ *in terms of support cells. Comparing definitions implies the sector* $\mathbf{Z}_i(u)$ *is the precisely the set of points in the support cell* $\mathbf{P}_{e_i}$ *induced by* $\mathcal{T}_2(F_u)$*. Furthermore, these sectors can be considered translated lex-cones, where a* lex-cone *is the intersections of linear lex-halfspaces. Therefore the lex-polyhedral cell complex* $\mathcal{T}_2(F_u)$ *induced is a translated lex-polyhedral fan (i.e., it consists of translated lex-cones) whose apex is the point* u.

In the sequel let $\mathcal{K}$ be a rank two tropical cone, equipped with a fixed system of (labelled) generators $V = (v^{(1)}, \ldots, v^{(n)})$, where $v^{(j)} \in (\mathbb{T}_2 \cup \{\infty\})^d$.

Lemma 4.4.6. *A point* $p \in (\mathbb{T}_2)^d$ *is contained in* $\mathcal{K}$ *if and only if for each* $i \in [d]$*, there exists some* $j \in [n]$ *such that* $p \in \mathbf{Z}_i(v^{(j)})$.

Proof. The proof of [53, Lemma 27] generalises directly. □

As in [63, §5.2] and [53] Lemma 4.4.6 inspires the following combinatorial data. Given a point $p \in (\mathbb{T}_2)^d$, we define its *covector* $S_p = S_p(V)$ to be the bipartite graph on the node set $[d] \sqcup [n]$ where $(i, j) \in S_p$ if and only if $p \in \mathbf{Z}_i(v^{(j)})$. We say a covector is *bounded* if no node in $[d]$ is isolated. With this, we can restate Lemma 4.4.6 as $p \in \mathcal{K}$ if and only if S_p is bounded.

By definition, the points with a given covector S satisfy the inequalities

$$p_k - p_i \leq v_k^{(j)} - v_i^{(j)} \quad \text{for all } k \in \operatorname{supp}(v^{(j)}) \text{ where } (i, j) \in S \ . \tag{4.25}$$

Note that these are also satisfied by any point whose covector contains S. We define the *covector cell*

$$\mathbf{C}_S(V) \ = \ \left\{ p \in (\mathbb{T}_2)^d \ \middle| \ S \subseteq S_p \right\} \ ,$$

and immediately note that $\mathbf{C}_S = \mathbf{C}_S(V)$ is a lex-polyhedron, as it is cut out by lex-halfspaces defined by the family of inequalities (4.25). We can define $\mathbf{C}_S(V)$ for any bipartite graph S on $[d] \sqcup [n]$ with no node isolated in $[n]$. As with support cells, there may be bipartite graphs S, T such that $\mathbf{C}_S = \mathbf{C}_T$, but the maximal bipartite graph defining the cell is the smallest covector containing S and T. Note that covectors can be defined analogously in $\mathbb{T}^d$, where C_S are ordinary polyhedra; cf. [53].

Lemma 4.4.7. *The covector cell $\mathbf{C}_S$ is rank two tropically convex.*

Proof. Let p and q be points in $\mathbf{C}_S(V)$. It suffices to show that for $\mu \in \mathbb{T}_2$ with $\mu \geq (0,0)$ we have $p \oplus (\mu \odot q) \in \mathbf{C}_S(V)$. This follows from

$$\begin{aligned}(p_k \oplus (\mu \odot q_k)) - (p_i \oplus (\mu \odot q_i) \ &= \ \min(p_k, \mu + q_k) - \min(p_i, \mu + q_i) \\ &= \ \min(p_k - p_i, p_k - \mu - q_i, \mu + q_k - p_i, q_k - q_i) \\ &\leq \ \min(p_k - p_i, q_k - q_i) \ \leq \ v_k^{(j)} - v_i^{(j)} \quad \text{for all } k \in \operatorname{supp}(v^{(j)}) \ .\end{aligned}$$

□

This means that the covector cells $\mathbf{C}_S$ are both lex-polyhedra and rank two tropically convex; i.e., they form rank two analogues of the polytropes in [52]. Covector cells $\mathbf{C}_S$ have some further nice combinatorial properties, analogous to support cells:

Lemma 4.4.8. *Let S, T be bipartite graphs on $[d] \sqcup [n]$ such that no node of $[n]$ is isolated.*

1. *$\mathbf{C}_S \cap \mathbf{C}_T = \mathbf{C}_{S \cup T}$.*
2. *$S \subseteq T$ if and only if $\mathbf{C}_T$ is a face of $\mathbf{C}_S$.*

Proof. Both claims are immediate generalisations of existing results. The first is [29, Corollary 11], and the second is [29, Corollary 13]. Note that [29] only addresses rank one tropical convexity in $\mathbb{T}^d$, i.e., without ∞ as a coordinate. □

The second statement of Lemma 4.4.8 implies that given a covector cell $\mathbf{C}_S$, its relative interior, denoted $\mathrm{int}(\mathbf{C}_S)$, is the set of points whose covector is precisely S. We recall that as $\mathbf{C}_S$ is a lex-polyhedron, $\mathrm{int}(\mathbf{C}_S)$ is open in the order topology but not in the Euclidean topology.

The following generalises the covector decomposition of rank one tropical cones from [53, §3.2]; the latter generalises the earlier result [29, Theorem 15] for tropical cones in $\mathbb{T}^d$; see also [63, §5.2].

Theorem 4.4.9. *The intersection $\mathcal{K} \cap (\mathbb{T}_2)^d$ decomposes as a lex-polyhedral complex whose cells are of the form $\mathbf{C}_S$ where S is a bounded covector with respect to the generating system V.*

Proof. Lemma 4.4.6 shows that the collection of lex-polyhedra

$$\mathbf{\Sigma} \;=\; \{\mathbf{C}_S \mid S \text{ bounded covector}\}$$

covers $\mathcal{K} \cap (\mathbb{T}_2)^d$. Lemma 4.4.8 shows that $\mathbf{\Sigma}$ is closed under intersections and taking faces, and therefore is a lex-polyhedral complex. □

Remark 4.4.10. *Recall from Remark 4.4.5 that the the rank two max-tropical hyperplane $\mathcal{T}_2(F_u)$ induces a decomposition of $\mathbb{T}_2^d$ into a lex-polyhedral fan. Furthermore, the maximal lex-cones are the sectors $\mathbf{Z}_i(u)$ equal to the support cell $\mathbf{P}_{e_i}$. Given the generating set $V = \{v^{(1)}, \ldots, v^{(n)}\}$, the covector cell $\mathbf{C}_S$ is equal to the finite intersection*

$$\mathbf{C}_S \;=\; \bigcap_{(i,j)\in S} \mathbf{Z}_i(v^{(j)}) \;\; .$$

Therefore the covector decomposition is precisely the common refinement of the lexicographical fan structures induced by the max-tropical hyperplanes $\mathcal{T}_2(F_{v^{(j)}})$. Moreover, taking the product of the max-tropical linear forms gives the rank two max-tropical multilinear form $F_V = \bigodot F_{v^{(j)}}$. The support sets of F_V are precisely the covectors induced by V, implying covectors are a special case of support sets. This generalises the known *connection of the fundamental theorem of tropical geometry [63, Theorem 3.2.5] with the mentioned result by Develin and Yu [30, Proposition 2.1] in rank one; cf. [53, Remark 32] and [51, §4].*

For a rank two tropical cone $\mathcal{K}$ generated by $V = \{v^{(1)}, \ldots, v^{(n)}\}$ and a covector T, we let $\mathcal{K}_T$ denote the rank two tropical cone generated by $V_T = \{v_T^{(1)}, \ldots, v_T^{(n)}\}$ where

$$(v_T^{(j)})_i \;=\; \begin{cases} v_i^{(j)} & \text{if } (i,j) \in T \\ \infty & \text{otherwise} \end{cases} .$$

The following results give decompositions for rank two tropical cones in terms of the interiors of polyhedra and ordinary polyhedra, analogous to Theorem 4.2.13 and Corollary 4.2.14.

Theorem 4.4.11. *Let $\mathcal{K}$ be a rank two tropical cone generated by $V = \{v^{(1)}, \ldots, v^{(n)}\} \subset (\mathbb{T}_2 \cup \{\infty\})^d$. The intersection $\mathcal{K} \cap (\mathbb{T}_2)^d$ is the finite disjoint union*

$$\mathcal{K} \cap \mathbb{T}_2^d \;=\; \bigsqcup_{S} \bigsqcup_{T \supseteq S} \big(\operatorname{int}(A_T) + \operatorname{int}(B_S)\big)$$

of interiors of polyhedra in $\mathbb{R}^{2d}$, where A_T and B_S are covector cells of the rank one tropical cones $\pi_{u*}(\mathcal{K})$ in $\mathbb{R}_t^d$ and $\pi_{t*}(\mathcal{K}_T)$ in $\mathbb{R}_u^d$ respectively.

Proof. By Theorem 4.4.9, $\mathcal{K} \cap \mathbb{T}_2^d$ is the union of lex-polyhedral cells $\mathbf{C}_S$ as S runs over all covectors. Furthermore, the second statement of Lemma 4.4.8 implies this union becomes disjoint if we restrict to the interiors of $\mathbf{C}_S$. Note that each $\operatorname{int}(\mathbf{C}_S)$ is a lex-open polyhedron. We claim that $\operatorname{int}(\mathbf{C}_S) = \bigsqcup_{T \supseteq S} \big(\operatorname{int}(A_T) + \operatorname{int}(B_S)\big)$.

The point p is contained in $\operatorname{int}(\mathbf{C}_S)$ if and only if for each $v^{(j)}$:

$$p_k - v_k^{(j)} \leq p_i - v_i^{(j)} \quad \text{for all } k \in \operatorname{supp}(v^{(j)}) \text{ where } (i,j) \in S .$$

with equality if and only if $(k,j) \in S$. Considering the lexicographical ordering on $\mathbb{T}_2$ and its coordinates separately, this is equivalent to the following two conditions:

$$\pi_{u*}(p_k) - \pi_{u*}(v_k^{(j)}) \leq \pi_{u*}(p_i) - \pi_{u*}(v_i^{(j)}) \ , \tag{4.26}$$

for all $k \in \mathrm{supp}(v^{(j)})$ and $(i,j) \in T$ for some $T \supseteq S$, with equality if and only if $(k,j) \in T$.

$$\pi_{t*}(p_k) - \pi_{t*}((v_T^{(j)})_k) \leq \pi_{t*}(p_i) - \pi_{u*}((v_T^{(j)})_i) \ , \tag{4.27}$$

for all $k \in \mathrm{supp}(v_T^{(j)})$ and $(i,j) \in S$, with equality if and only if $(k,j) \in S$. Condition (4.26) is equivalent to $\pi_{u*}(p)$ being contained in the relative interior of the covector cell A_T of $\pi_{u*}(\mathcal{K})$. Condition (4.27) is equivalent to $\pi_{t*}(p)$ being contained in the relative interior of the covector cell B_S of $\pi_{t*}(\mathcal{K}_T)$.

It remains to show each part of the disjoint union is the interior of a polyhedron. The proof is identical to the end of the proof of Theorem 4.2.13. □

Corollary 4.4.12. *With the notation of Theorem 4.4.11: the closure of $\mathcal{K} \cap \mathbb{T}_2^d$ in the Euclidean topology is the finite union*

$$\overline{\mathcal{K} \cap \mathbb{T}_2^d} \ = \bigcup_S \bigcup_{T \supseteq S} (A_T + B_S)$$

of polyhedra in $\mathbb{R}^{2d}$.

Proof. As $A_T + B_S = \overline{\mathrm{int}(A_T) + \mathrm{int}(B_S)}$, the result follows from Theorem 4.4.11 and that the closure of a finite union of sets equals the union of their closures. □

Recall that Diagram (4.23) says π_{u*} and $\pi_{u \mapsto \sigma}$ (and π_{t*} and $\pi_{t \mapsto \rho}$) commute with the valuation map. Therefore if $\mathcal{K} = \mathrm{val}_2(K)$ for some ordinary cone $K \subset \mathbb{R}\{\{(t,u)*\}\}$, we can obtain an analogous result to Theorem 4.4.11 in terms of the covector decompositions of $\mathrm{val}(\pi_{u \mapsto \sigma}(K))$ and $\mathrm{val}(\pi_{t \mapsto \rho}(K))$.

As with Corollary 4.4.4, we can obtain an analogous statement to Theorem 4.4.11 and Corollary 4.4.12 for tropical polytopes by dehomogenisation. Explicitly, given some generating set $V \subset (\mathbb{T}_2 \cup \{\infty\})^d$ for a convex polytope $\mathcal{K}$, we can consider the cone $\mathcal{K}' \subset (\mathbb{T}_2 \cup \{\infty\})^{d+1}$ generated by

$$\left\{ ((0,0), v^{(j)}) \;\middle|\; v^{(j)} \in V \right\} \ .$$

Then $\mathcal{K}$ inherits the structure of $\mathcal{K}'$ intersected with the hyperplane $\{x_0 = (0,0)\}$. Note that Diagram (4.23) implies we can do this dehomogenisation in $\mathbb{R}\{\{(t,u)^*\}\}^{d+1}_{\geq 0}$.

Example 4.4.13. *The following is a construction of Goldfarb and Sit [45] in the version of [55, Example 2]. For $d \geq 2$ consider the polyhedron $G^d(t,u)$ given by the $2d$ linear inequalities*

$$\begin{array}{rclrcll} 0 & \leq & x_1 \,, & x_1 & \leq & t^{d-1} & \\ x_{j-1} & \leq & ux_j \,, & x_{j-1} & \leq & t^{d-j}u & \text{for } 2 \leq j \leq d \end{array} \tag{4.28}$$

in $\mathbb{R}\{\{(t,u)^*\}\}^d$. It has 2^d vertices which are obtained by solving systems of linear equations arising from picking one of each of the d pairs of linear inequalities in (4.28)*, taken as equalities. Thus $G^d(t,u)$ is a bounded polytope which is combinatorially equivalent to the d-dimensional cube. As each feasible point x satisfies $0 \leq x_1 \leq ux_2 \leq u^2x_3 \leq \cdots \leq u^{d-1}x_d$, the polyhedron $G^d(t,u)$ is contained in the positive orthant. Hence, by Proposition 4.4.2, its rank two tropicalisation $\mathrm{val}_2(G^d(t,u))$ is given by the rank two tropical linear inequalities*

$$\begin{array}{rclrcll} \infty & \geq & x_1 \,, & x_1 & \geq & (d-1,0) & \\ x_{j-1} & \geq & (0,1) + x_j \,, & x_{j-1} & \geq & (d-j,1) & \text{for } 2 \leq j \leq d \ . \end{array} \tag{4.29}$$

The inequalities (4.29) *are dehomogenised versions of the inequalities in* (4.25)*. We infer that $\mathrm{val}_2(G^d(t,u))$ is a lex-polyhedron and thus a rank two polytrope.*

The partial substitution $G^d(t,\frac{1}{2})$ obtained from $u \mapsto \frac{1}{2}$ is a polyhedron over $\mathbb{R}\{\{t^*\}\}$

defined by

$$\begin{array}{lllllll} 0 & \leq & x_1\,, & x_1 & \leq & t^{d-1} & \\ 2x_{j-1} & \leq & x_j\,, & 2x_{j-1} & \leq & t^{d-j} & \quad \textit{for } 2 \leq j \leq d\ . \end{array} \tag{4.30}$$

Its (rank one) tropicalisation $\operatorname{val}(G^d(t, \frac{1}{2}))$ *is given by the tropical linear inequalities*

$$\begin{array}{lllllll} \infty & \geq & x_1\,, & x_1 & \geq & d-1 & \\ x_{j-1} & \geq & x_j\,, & x_{j-1} & \geq & d-j & \quad \textit{for } 2 \leq j \leq d\ . \end{array} \tag{4.31}$$

Since the rank one tropical linear inequalities in (4.31) *happen to be ordinary linear inequalities over* $\mathbb{R}$*, too, the rank one tropical polytope* $\operatorname{val}(G^d(t, \frac{1}{2}))$ *is convex in the ordinary sense; i.e., it is a* polytrope *in the sense of [52]. Substituting t in $G^d(t, \frac{1}{2})$ for a sufficiently small value, e.g., $t \mapsto \frac{1}{8}$ gives the combinatorial d-cube $G^d(\frac{1}{8}, \frac{1}{2})$ in $\mathbb{R}^d$ of Goldfarb and Sit [45]. The interest in this construction stems from the fact that the simplex method with the "steepest edge" pivoting strategy (for a suitable objective function and starting at the origin) visits all the 2^d vertices; cf. [55, Example 2].*

4.5 Concluding remarks and open questions

The proof of the crucial Proposition 4.1.10 can be iterated to show that $\mathbb{R}\{\{t_1^*\}\}\cdots\{\{t_m^*\}\}$ is real closed for arbitrary $m \geq 1$; cf. Remark 4.1.11. This opens up a path to study tropical hypersurfaces and tropical cones of arbitrarily high finite rank. To avoid cumbersome notation in this article, which is technical already, we decided to restrict our exposition to the rank two case. Yet the characterisations of rank two tropical hypersurfaces and cones can be generalised to arbitrary finite rank by recursively exploiting the structure of tropical hypersurfaces and cones of corank one. This entails a generalisation of Theorem 4.3.6 to the simultaneous stable intersection of any finite number of tropical hypersurfaces. We leave the details to the reader.

A rank one tropical hypersurface, given by a tropical polynomial F, is dual to the regular subdivision of the point configuration given by the monomials of F, where the coefficients yield the height function; cf. [63, Proposition 3.1.6].

Question 4.5.1. *How does this generalise to higher rank?*

This should be related to the *regular refinement* of subdivisions in the sense of [27, Definition 2.3.17].

In Sections 4.2 and 4.4, we gave several descriptions of rank two tropical hypersurfaces and cones. Each of these have their benefits and flaws, which motivates our next question.

Proposition 4.2.8 and Theorem 4.4.9 describe rank two objects as a lex-polyhedral complex, and moreover gives a canonical inequality description for each. Lex-polyhedra do not have a canonical inequality description in general, and Definition 4.2.4 formally depends on the representation (4.12). It would be interesting to find out if this is unavoidable. Furthermore, we can extend this question by considering the dimension of a lex-polyhedron. One can formulate a combinatorial notion of dimension, by considering flags of faces in the face poset, and compare it to a more geometric notion of dimension on $(\mathbb{T}_2)^d$.

Question 4.5.2. *Does the face poset of a lex-polyhedron depend on the representation? Is there a coherent notion of dimension for lex-polyhedra?*

It is worth noting that the proofs in [5], [8] and [6] circumvent answering the same question for rank one tropical polyhedra by working with fixed exterior descriptions.

Finally, our current setup for rank two tropical hypersurfaces is purely polyhedral, and so this does not capture any arithmetic properties.

Question 4.5.3. *What is the proper notion of multiplicity for tropical hypersurfaces of higher rank?*

In this context it could be interesting to investigate the recent work of Gwoździewicz and Hejmej on the factorisation of formal power series of higher rank [48].

Chapter 5

Commutative algebra of generalised Frobenius numbers

The following is based on the paper “Commutative algebra of generalised Frobenius numbers” by Madhusudan Manjunath and the author [64]. We acknowledge the computer algebra system Macaulay2 [46] for both investigation and preparation of examples.

5.1 A homological formula for $F_k(a_1, \ldots, a_n)$

Definition 5.1.1. *Let $(a_1, \ldots, a_n)$ be a collection of natural numbers with no common divisor. The* Frobenius number *$F(a_1, \ldots, a_n)$ is the largest natural number that cannot be expressed as a non-negative integral linear combination of $a_1, \ldots, a_n$.*

Example 5.1.2. *When $n = 2$, we have a closed formula for the Frobenius number, due to Sylvester:*

$$F(a_1, a_2) = a_1a_2 - a_1 - a_2 \quad . \tag{5.1}$$

For $n > 2$, no such closed form expression exists.

Example 5.1.3. *Consider the collection of naturals $(3, 5, 8)$, their Frobenius number is*

$F(3,5,8) = 7$. *We note that as* 8 *can be expressed as a non-negative integral combination of* 3 *and* 5*, one can compute this via*

$$F(3,5,8) = F(3,5) = 3 \cdot 5 - 3 - 5 = 7$$

The finiteness of the Frobenius number follows from Equation (5.1) and that $F(a_1, \ldots, a_n)$ is upper bounded by $F(a_i, a_j)$, the Frobenius number of any pair of integers. Note that this does not hold if $\gcd(a_1, \ldots, a_n) > 1$, as any non-negative integral combination will necessarily be a multiple of $\gcd(a_1, \ldots, a_n)$.

The Frobenius number can be rephrased in the language of lattices as follows [81]. We define

$$\begin{aligned} L = L(a_1, \ldots, a_n) &= \left\{ \mathbf{v} \in \mathbb{Z}^n \;\middle|\; \sum_{i=1}^{n} a_i v_i = 0 \right\} \\ &= (a_1, \ldots, a_n)^{\perp} \cap \mathbb{Z}^n \end{aligned} \tag{5.2}$$

to be the sublattice of $\mathbb{Z}^n$ of points whose inner product with $(a_1, \ldots, a_n)$ is zero. Fix some integer r, and suppose there exists a point $\mathbf{p} \in \mathbb{Z}^n$ such that $\sum_{i=1}^{n} a_i p_i = r$ and $\mathbf{p}$ dominates a lattice point $\mathbf{v} \in L$. This gives a condition on r being representable, as $\mathbf{p} - \mathbf{v} \in \mathbb{Z}^n_{\geq 0}$ and

$$(a_1, \ldots, a_n) \cdot (\mathbf{p} - \mathbf{v}) = \sum_{i=1}^{n} a_i (p_i - v_i) = r \ . \tag{5.3}$$

In this framework, the Frobenius number is precisely the largest integer r such that there exists a point $\mathbf{p} \in \mathbb{Z}^n$ whose inner product with $(a_1, \ldots, a_n)$ is r, and $\mathbf{p}$ does not dominate any point in L. Here the domination is according to the partial order induced by the standard basis on $\mathbb{Z}^n$.

Note that one can also view L as the sublattice of the dual lattice $(\mathbb{Z}^n)^\star$ of points that evaluate to zero at $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. As there is a standard isomorphism between $(\mathbb{Z}^n)^\star$ and $\mathbb{Z}^n$, these two viewpoints are equivalent.

This leads to a commutative algebraic interpretation of the Frobenius number. Let $\mathbb{K}$ be an arbitrary field and $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients in $\mathbb{K}$.

Definition 5.1.4. *[20] Let L be a lattice such that $L \cap \mathbb{Z}^n_{\geq 0} = \{\mathbf{0}\}$. The* lattice module

$$M_L = \langle \mathbf{x}^{\mathbf{u}} \mid \mathbf{u} \in L \rangle \tag{5.4}$$

is the $\mathbb{Z}^n$-graded S-submodule of the Laurent polynomial ring $T = \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$ generated by $\{\mathbf{x}^{\mathbf{u}} \mid \mathbf{u} \in L\}$.

Recall from Section 2.2.2 that as an S-module M_L is not finitely generated. Example 2.2.6 shows that we can view M_L as an $S[L]$-module, and it is finitely generated in this setting.

The functor π defined in Equations (2.32) and (2.33) is an equivalence of categories from $\mathbb{Z}^n$-graded $S[L]$-modules to $\mathbb{Z}^n/L$-graded S-modules. The $\mathbb{Z}^n/L$-grading has a notable interpretation in our setting. The kernel of the map that sends $\mathbf{p} \in \mathbb{Z}^n$ to $\sum a_i p_i$ is precisely L, and so

$$\begin{aligned} \mathbb{Z}^n/L &\longrightarrow \mathbb{Z} \\ \mathbf{p} + L &\longmapsto \sum_{i=1}^{n} a_i p_i \end{aligned} \tag{5.5}$$

is an isomorphism. Therefore we consider the degree of a monomial under the $\mathbb{Z}^n/L$-grading as

$$\deg(\mathbf{x}^{\mathbf{u}}) = \sum_{i=1}^{n} a_i u_i \ . \tag{5.6}$$

We also refer to this grading as the $(a_1, \ldots, a_n)$-*weighted grading* or simply the weighted grading. Note that we can consider M_L with the weighted grading, although $\pi(M_L)$ is more amenable to computation as every graded part of M_L is infinite dimensional.

A Laurent monomial $\mathbf{x}^{\mathbf{u}}$ is contained in M_L if and only if there exists a lattice point

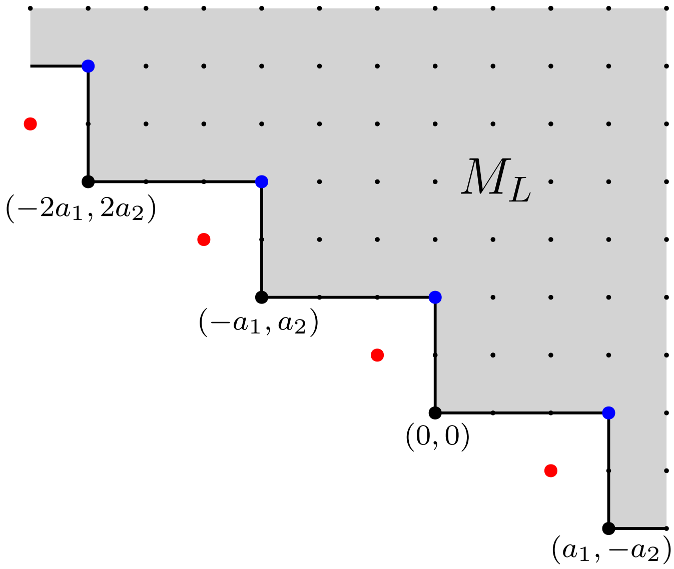

Figure 5.1: The staircase diagram of the lattice module induced by the lattice $L(a_1, a_2)$. The Frobenius number $F(a_1, a_2)$ equals the degree of the monomials corresponding to the red points, which are the blue corner points translated by $(-1, \ldots, -1)$.

$\mathbf{v} \in L$ such that $\mathbf{u} \geq \mathbf{v}$. Therefore, the Frobenius number is an invariant of M_L:

$$F(a_1, \ldots, a_n) = \max \{ r \in \mathbb{Z} \mid \deg(\mathbf{x}^{\mathbf{u}}) = r \ , \ \mathbf{x}^{\mathbf{u}} \notin M_L \} \ . \tag{5.7}$$

Theorem 5.1.5. *[20], [78] The Frobenius number $F(a_1, \ldots, a_n)$ is*

$$\mathrm{reg}(M_L) + n - 1 - \sum_{i=1}^{n} a_i$$

where $L := L(a_1, \ldots, a_n)$ and $\mathrm{reg}(M_L)$ is the Castelnuovo–Mumford regularity of M_L with respect to its $(a_1, \ldots, a_n)$-weighted grading.

We will later prove a generalisation of this result, Proposition 5.1.13, but we can give some intuition for it via staircase diagrams. The Frobenius number is largest degree of an element not contained in M_L. This should be represented by one of the red points in Figure 5.1, an integer point not in M_L but increasing any of its coordinates puts it inside M_L. These are obtained by finding a blue "corner point" and then translating by the vector $(-1, \ldots, -1)$. Homologically, these blue corner points correspond to maximal syzygies of M_L, and translating by $(-1, \dots, -1)$ corresponds to subtracting $\sum a_i$ when graded. The final step will be to show the maximal degree of a syzygy is given by the regularity of M_L plus $n-1$.

Recall that for a sublattice L of $\mathbb{Z}^n$, the *lattice ideal* I_L is the ideal generated by all binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\mathbf{u} - \mathbf{v} \in L$ and $\mathbf{u},\ \mathbf{v} \in \mathbb{Z}^n_{\geq 0}$. Example 2.2.7 shows that under the functor π, M_L becomes

$$\pi(M_L) = M_L/L \cong S/I_L \ . \tag{5.8}$$

As a cyclic S-module, S/I_L is easier to work with than M_L in general, especially from a computational point of view.

Example 5.1.6. *Consider the lattice $L = (3,5,8)^{\perp} \cap \mathbb{Z}^3$. Via Macaulay2 [46], we calculate its corresponding lattice ideal $I_L = \langle x_3 - x_1x_2, x_2^3 - x_1^5 \rangle$ and its Betti numbers. The Betti table corresponding to the minimal free resolution of S/I_L has 22 rows and 3 columns, hence* $\operatorname{reg}(S/I_L) = 21$ *and* $F(3,5,8) = 21 + 2 - 16 = 7$.

5.1.1 Generalised Frobenius numbers

Definition 5.1.7. *[22] Let $(a_1, \dots, a_n)$ be a collection of natural numbers with no common divisor. For a natural number k, the k-th* Frobenius number *$F_k(a_1, \dots, a_n)$ is the largest natural number that cannot be written as k distinct non-negative integral linear combinations of $a_1, \dots, a_n$.*

Note that the first Frobenius number $F_1(a_1, \dots, a_n)$ is the Frobenius number of $(a_1, \dots, a_n)$.

Example 5.1.8. *As with the Frobenius number, $F_k(a_1, a_2)$ has a closed form expression [22]*

$$F_k(a_1, a_2) = ka_1a_2 - a_1 - a_2 \ . \tag{5.9}$$

Again, no such closed formula exists for $n > 2$.

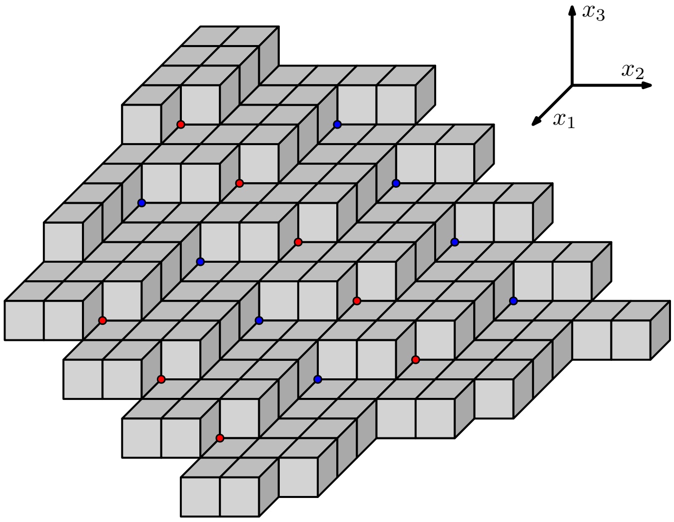

Figure 5.2: The staircase diagram corresponding to the 3rd lattice module $M^{(3)}_{L(3,4,11)}$. The polynomial ring $S = \mathbb{K}[x_1, x_2, x_3]$ has the $(3, 4, 11)$-weighted grading. The blue dots correspond to minimal generators of degree 15, red dots correspond to minimal generators of degree 20.

Example 5.1.9. *Consider the collection of naturals* $(3, 5, 8)$*. Unlike Example 5.1.3,* $F_k(3, 5, 8) \neq F_k(3, 5)$ *for* $k > 1$*. A brute force calculation reveals* $F_2(3, 5, 8) = 12$*, as* 12 *can only be represented by* $4 \cdot 3 + 0 \cdot 5 + 0 \cdot 8$.

The finiteness of $F_k(a_1, \ldots, a_n)$ for all natural numbers k follows by an argument similar to the one for $F(a_1, \ldots, a_n)$.

We would like a homological formula for $F_k(a_1, \ldots, a_n)$ in the vein of Theorem 5.1.5. We first rephrase the k-th Frobenius number in the language of lattices. In this framework, the k-th Frobenius number is the largest integer r such that there exists a point $\mathbf{p} \in \mathbb{Z}^n$ whose inner product with $(a_1, \ldots, a_n)$ is r and $\mathbf{p}$ does not dominate k distinct lattice points of L. This leads to the following generalisation of lattice modules.

Definition 5.1.10. *The* k-th lattice module $M_L^{(k)}$ *is the* S*-module generated by Laurent monomials* $\mathbf{x}^{\mathbf{w}}$ *where* $\mathbf{w}$ *is an element in* $\mathbb{Z}^n$ *that dominates at least* k *points in* L*. Formally,*

$$M_L^{(k)} = \langle \mathbf{x}^{\mathbf{w}} \mid \mathbf{w} \in \mathbb{Z}^n \textit{ dominates at least } k \textit{ points in } L \rangle \tag{5.10}$$

Figure 5.2 shows the staircase diagram corresponding to the 3rd lattice module $M_L^{(3)}$ of the lattice $L(3,4,11)$. By construction, the first lattice module $M_L^{(1)}$ is the lattice module M_L.

The module $M_L^{(k)}$ can also be viewed as a $\mathbb{Z}^n$-graded $S[L]$-module. Furthermore, while it is not finitely generated as an S-module, it is finitely generated as an $S[L]$-module.

Proposition 5.1.11. *For any natural number k, the k-th lattice module $M_L^{(k)}$ is a finitely generated $S[L]$-module.*

Proof. By the action of $S[L]$ on $M_L^{(k)}$, it suffices to consider orbit representatives of the L-action on $M_L^{(k)}$ that dominate the origin. These representatives are monomials in S (rather than Laurent monomials) and define a monomial ideal in the polynomial ring S. By the Gordan–Dickson Lemma, this monomial ideal is finitely generated and hence $M_L^{(k)}$ is finitely generated as an $S[L]$-module. □

The proof of Proposition 5.1.11 is based on an argument in [3]. Note that it is not constructive in the sense that it gives little indication how to characterise a minimal generating set of $M_L^{(k)}$ or even bounds on the degrees. The methods in Sections 5.2 and 5.3 address both of these points.

Remark 5.1.12. *A recent work of Aliev, De Loera and Louveaux [3] considers the semigroup*

$$\mathrm{Sg}_{\geq k}((a_1,\ldots,a_n)) = \{b : \exists\, \mathbf{x_1},\ldots,\mathbf{x_k} \in \mathbb{Z}^n_{\geq 0} \text{ such that } (a_1,\ldots,a_n)\cdot \mathbf{x_i} = b\}$$

where $\mathbf{x_1},\ldots,\mathbf{x_k}$ are distinct. In this framework, the k-th Frobenius number is the largest non-negative integer $b \notin \mathrm{Sg}_{\geq k}((a_1,\ldots,a_n))$. They study this semigroup by considering the monomial ideal $I^{(k)}((a_1,\ldots,a_n))$ of monomials whose $(a_1,\ldots,a_n)$-weighted degrees are contained in $\mathrm{Sg}_{\geq k}((a_1,\ldots,a_n))$ [3, Theorem 1]. This monomial ideal $I^{(k)}((a_1,\ldots,a_n))$ is the ideal used in the proof of Proposition 5.1.13, and the intersection of $M_L^{(k)}$ with the *polynomial ring S. We note that this ideal $I^{(k)}((a_1,\ldots,a_n))$ does not carry an L-action and this seems to make it less amenable to study compared to $M_L^{(k)}$.*

$M_L^{(k)}$ is a $\mathbb{Z}^n$-graded $S[L]$-module, therefore we can apply the exact functor π to obtain an equivalent $\mathbb{Z}^n/L$-graded S-module. This allows us to extend our weighted grading to $M_L^{(k)}$, giving the following commutative algebraic characterisation of generalised Frobenius numbers in terms of the generalised lattice modules.

Proposition 5.1.13. *The k-th Frobenius number of $(a_1,\ldots,a_n)$ is given by the formula:*

$$\operatorname{reg}(M_L^{(k)}) + n - 1 - \sum_{i=1}^{n} a_i$$

where $L := L(a_1,\ldots,a_n)$ and $\operatorname{reg}(M_L^{(k)})$ is the Castelnuovo–Mumford regularity of the $S[L]$-module $M_L^{(k)}$ with respect to its $(a_1,\ldots,a_n)$-weighted grading.

To prove this, we must introduce some notions from commutative algebra.

Definition 5.1.14. *Given an S-module M, we can write its $(a_1,\ldots,a_n)$-graded Hilbert series as*

$$H(M;t^{a_1},\ldots,t^{a_n}) = \frac{P(t)}{(1-t)^d}\ ,\ P(1) \neq 0\ , \tag{5.11}$$

for some polynomial $P(t)$. The exponent d in (5.11) *is the* Krull dimension $\dim(M)$ *of M. The Krull dimension and projective dimension are related by the inequality*

$$\dim(M) + \operatorname{pdim}(M) \geq n\ . \tag{5.12}$$

A module is Cohen–Macaulay *if and only if this inequality is tight.*

We remark that these are non-standard definitions, however they suffice for technical simplicity. The definition of Krull dimension given follows from [34, Exercise 10.13c]. Equation (5.12) and the definition of Cohen–Macaulay given follows from the Auslander–Buchsbaum formula [35, Theorem A2.15] and the fact that depth is upper bounded by dimension.

Proof. The proof is split into two: we first show that F_k is equal to the highest degree of the Betti numbers of $\pi(M_L^{(k)})$ minus $\sum a_i$. We then show that the highest degree of the Betti numbers is precisely $\mathrm{reg}(M_L^{(k)}) + n - 1$.

Consider the $(a_1, \ldots, a_n)$-graded Hilbert function of $\pi(M_L^{(k)})$. By [66, Lemma 8.19], it can be written in terms of the Hilbert functions of the free modules in its minimal free resolution. We compute the Hilbert series of $\pi(M_L^{(k)})$ to be

$$\frac{K(t)}{\prod_i (1 - t^{a_i})} \ , \ K(t) = \sum_{\substack{i \geq 0 \\ u \in \mathbb{Z}_{\geq 0}}} (-1)^i \beta_{i,u} t^u \ . \tag{5.13}$$

Alternatively, as the dimension of each graded piece can only be zero or one, the Hilbert series of $\pi(M_L^{(k)})$ is the sum of monomials t^m where m has at least k distinct non-negative integral combinations of $a_1, \ldots, a_n$. Define $\theta_k(t) = \sum t^m$ as the finite sum of monomials whose exponents $m \in \mathbb{Z}_{\geq 0}$ do not have k distinct non-negative integral combinations of $a_1, \ldots, a_n$. Note that by definition, the k-th Frobenius number is the degree of θ_k. We can rewrite the Hilbert series of $\pi(M_L^{(k)})$ as

$$\prod_{i=1}^{n} \frac{1}{(1 - t^{a_i})} - \theta_k(t) \ , \tag{5.14}$$

the Hilbert series of S minus θ_k, the monomials whose exponents are not representable. Equating expressions (5.13) and (5.14) and rearranging, we find

$$F_k = \deg(\theta_k) = \deg(K) - \sum_i a_i \ . \tag{5.15}$$

If $\pi(M_L^{(k)})$ is Cohen–Macaulay, there are two implications. First, it is a (rather weak) corollary of Boij–Soderberg theory [39] that $\deg(K) = \max_i \{u \in \mathbb{Z} \mid \beta_{i,u} \neq 0\}$. Furthermore, by [35, Exercise 4.5] the regularity equals $\max_i \{u \in \mathbb{Z} \mid \beta_{i,u} \neq 0\}$ minus the projective dimension. It remains to show $\pi(M_L^{(k)})$ is Cohen-Macaulay and its projective dimension is $n-1$.

We first prove the Krull dimension of $\pi(M_L^{(k)})$ is one via a Hilbert series computation. For $k=1$, $\pi(M_L^{(1)}) = S/I_L$ has Krull dimension one as L is a codimension one lattice [66, Proposition 7.5]. For general k, we note that $M_L^{(k)}$ differs from $M_L^{(1)}$ in finitely many graded pieces up to the action to the lattice. Therefore the Hilbert series of $\pi(M_L^{(k)})$ is the Hilbert series of $\pi(M_L^{(1)})$ minus finitely many terms. As we can write $\pi(M_L^{(1)})$ as a rational function $P(t)/(1-t)$, we can also do so for $\pi(M_L^{(k)})$.

Equation (5.12) implies the projective dimension is lower bounded by $n-1$. The hull complex is a polyhedral complex of dimension $n-1$, making the length of the hull resolution $n-1$. Therefore this lower bound is tight, and the module is Cohen–Macaulay. The final remark is that the regularity of $\pi(M_L^{(k)})$ carries over to $M_L^{(k)}$ by exactness of π. □

Remark 5.1.15. *One can extend the definition of generalised lattice module to any finite index sublattice H of $L(a_1, \ldots, a_n)$. This is the level of generality that all of our structural results hold at. However, it is worth noting that the $\mathbb{Z}^n/H$-grading is not isomorphic to the $(a_1, \ldots, a_n)$-weighted grading as H is not a saturated lattice. As elements of $\mathbb{Z}^n/H$ are not integers, we have to coarsen to the weighted grading to associate a Frobenius number to H.*

5.2 Minimal generators of generalised lattice modules

In this section, we describe the minimal generators of generalised lattice modules in detail. For $k=1$, $M_L^{(1)}$ is a cyclic $S[L]$-module with a single minimal generator $1_{\mathbb{K}}$. For $k \geq 2$, the lattice modules $M_L^{(k)}$ are in general not cyclic $S[L]$-modules, furthermore we have no simple characterisation for their minimal generators.

For $k=2$, we give a simple description of a minimal generating set of $M_L^{(2)}$ in terms of the first syzygies of $M_L^{(1)}$ (Theorem 5.2.2). For $k \geq 3$, a generalisation of this result is more involved and is the content of Theorem 5.2.5. One source of complication is that for $k \geq 2$, the lattice modules $M_L^{(k)}$ have exceptional generators i.e., those that dominate strictly greater than k points in L, whereas $M_L^{(1)}$ does not have exceptional generators. Another complication is that for $k \geq 3$, we may get minimal generators of $M_L^{(k)}$ that do not arise as a syzygy between two minimal generators of $M_L^{(k-1)}$, rather as a "syzygy between a minimal generator and a lattice point". This will motivate us to consider the syzygies of a modification of $M_L^{(k)}$.

5.2.1 Inductive characterisation of $M_L^{(2)}$

We start with a description of the minimal generators of the simplest generalised lattice module $M_L^{(2)}$. For each minimal generator $\mathbf{x}^{\mathbf{u}}$ of $M_L^{(1)}$, let $\mathrm{Syz}^1_{\mathbf{x}^{\mathbf{u}}}(M_L^{(1)})$ be the $\mathbb{K}$-vector space generated by syzygies of the form:

$$m \cdot (0, \dots, 0, \underbrace{\mathrm{lcm}(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})/\mathbf{x}^{\mathbf{u}}}_{\mathbf{x}^{\mathbf{u}}}, \dots, -\underbrace{\mathrm{lcm}(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})/\mathbf{x}^{\mathbf{v}}}_{\mathbf{x}^{\mathbf{v}}}, 0, \dots, 0) \tag{5.16}$$

where $\mathbf{x}^{\mathbf{v}} \neq \mathbf{x}^{\mathbf{u}}$ is a minimal generator of $M_L^{(1)}$ and m is a monomial in S. Note that multiplication by m is the standard multiplication on S.

We define a map $\phi_S^{(1)}$ on this basis of $\mathrm{Syz}^1_{\mathbf{x}^{\mathbf{u}}}(M_L^{(1)})$ and extend it $\mathbb{K}$-linearly. The map $\phi_S^{(1)} : \bigoplus_{\mathbf{u} \in L} \mathrm{Syz}^1_{\mathbf{x}^{\mathbf{u}}}(M_L^{(1)}) \to M_L^{(2)}$ takes the element s of the form (5.16) to $\mathbf{x}^{\deg_{\mathbb{Z}^n}(s)}$ where $\deg_{\mathbb{Z}^n}(s)$ is the $\mathbb{Z}^n$-graded degree of s. In fact, $\deg_{\mathbb{Z}^n}(s) = \max(\mathbf{u}, \mathbf{v}) \cdot \deg_{\mathbb{Z}^n}(m)$ where max is the coordinate-wise maximum. Furthermore, $\mathbf{x}^{\deg_{\mathbb{Z}^n}(s)} \in M_L^{(2)}$ since the point $\max(\mathbf{u}, \mathbf{v})$ dominates at least two lattice points, namely $\mathbf{u}$ and $\mathbf{v}$. In the following, we note that the map $\phi_S^{(1)}$ is surjective.

Proposition 5.2.1. *The map $\phi_S^{(1)}$ is surjective.*

Proof. It suffices to prove that every Laurent monomial in $M_L^{(2)}$ can be realised as the image of an element in $\mathrm{Syz}^1_{\mathbf{x}^{\mathbf{u}}}(M_L^{(1)})$ for some minimal generator $\mathbf{x}^{\mathbf{u}}$ of $M_L^{(1)}$. To see this, consider a Laurent monomial $\mathbf{x}^{\mathbf{w}}$ in $M_L^{(2)}$. By the definition of $M_L^{(2)}$, the point $\mathbf{w}$

dominates at least two points in L. Consider any two points $\mathbf{u_1}$ and $\mathbf{u_2}$ in L that $\mathbf{w}$ dominates and consider the Laurent monomial lcm $(\mathbf{x}^{\mathbf{u_1}}, \mathbf{x}^{\mathbf{u_2}})$. This is contained in $M_L^{(2)}$ and is the image of $(\mathrm{lcm}(\mathbf{x}^{\mathbf{u_1}}, \mathbf{x}^{\mathbf{u_2}})/\mathbf{x}^{\mathbf{u_1}}, -\mathrm{lcm}(\mathbf{x}^{\mathbf{u_1}}, \mathbf{x}^{\mathbf{u_2}})/\mathbf{x}^{\mathbf{u_2}}) \in \mathrm{Syz}^1_{\mathbf{x}^{\mathbf{u_1}}}(M_L^{(1)})$ under $\phi_S^{(1)}$. Hence, by multiplying this syzygy by the monomial $\mathbf{x}^{\mathbf{w}}/\mathrm{lcm}(\mathbf{x}^{\mathbf{u_1}}, \mathbf{x}^{\mathbf{u_2}})$ we conclude that $\mathbf{x}^{\mathbf{w}}$ is also in the image of $\phi_S^{(1)}$. □

Proposition 5.2.1 is not directly amenable for computational purposes since $M_L^{(2)}$ is not finitely generated as an S-module. However, $M_L^{(2)}$ is finitely generated as an $S[L]$-module. Note that there is a natural L-action on $\oplus_{\mathbf{u}\in L}\mathrm{Syz}^1_{\mathbf{x}^{\mathbf{u}}}(M_L^{(1)})$ and a surjective map between the first syzygy module of $M_L^{(1)}$ as an $S[L]$-module and the piece $\mathrm{Syz}^1_{\mathbf{x}^{\mathbf{0}}}(M_L^{(1)})$. Composing this with $\phi_S^{(1)}$ gives a surjective map $\phi_{S[L]}^{(1)}$ between the first syzygy module of $M_L^{(1)}$ and $M_L^{(2)}$ as $S[L]$-modules.

To explicitly describe the map $\phi_{S[L]}^{(1)}$, we first note that the first syzygy module of $M_L^{(1)}$ as an $S[L]$-module has a $\mathbb{K}$-vector space basis of the form:

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}\mathbf{z}^{\mathbf{u}-\mathbf{v}} \tag{5.17}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n_{\geq 0}$ and $\mathbf{u} - \mathbf{v} \in L$. The map $\phi_{S[L]}^{(1)}$ takes $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}\mathbf{z}^{\mathbf{u}-\mathbf{v}}$ to $\mathbf{x}^{\mathbf{u}} \in M_L^{(2)}$. As the functor π takes $M_L^{(1)}$ to S/I_L and $\mathrm{Syz}^1(S/I_L) = I_L$ (along with the categorical equivalence between $\mathbb{Z}^n$-graded $S[L]$-modules and $\mathbb{Z}^n/L$-graded S-modules), this induces a map from any binomial minimal generating set of I_L to $M_L^{(2)}$, which we also refer to as $\phi_{S[L]}^{(1)}$. We obtain the following.

Theorem 5.2.2. *The lattice module $M_L^{(2)}$ as an $S[L]$-module is generated by the image of $\phi_{S[L]}^{(1)}$ on a binomial minimal generating set of the lattice ideal I_L.*

Example 5.2.3. *Consider the lattice $(3,5,8)^{\perp} \cap \mathbb{Z}^3$ from Example 5.1.6 with corresponding lattice ideal $I_L = \langle x_3 - x_1x_2, x_2^3 - x_1^5 \rangle$. The minimal first syzygies of $M_L^{(1)}$, up to the action of L, are of the form $(lcm(\mathbf{x}^{\mathbf{u}}, 1_{\mathbb{K}})/1_{\mathbb{K}}, -lcm(\mathbf{x}^{\mathbf{u}}, 1_{\mathbb{K}})/\mathbf{x}^{\mathbf{u}})$ where $\mathbf{u}$ are lattice points of L "sufficiently close" to $\mathbf{0} \in L$. The precise notion of sufficiently close will be* *addressed in Section 5.2.3. In this case it suffices to take the lattice points*

$$\{\pm(1,1,-1), \pm(5,-3,0)\} \subset L \ .$$

The map $\phi_S^{(1)}$ sends the minimal first syzygies to $\{x_3, x_1x_2, x_2^3, x_1^5\}$, precisely the monomials in each minimal binomial of I_L. Note that x_3, x_1x_2 and x_2^3, x_1^5 are in the same L-orbit. This gives us an explicit description of $M_L^{(2)} = \langle x_3, x_2^3 \rangle$ as an $S[L]$-module and a minimal generating set.

5.2.2 Inductive characterisation of $M_L^{(k)}$

We generalise Proposition 5.2.1 to generalised lattice modules to obtain an induction characterisation of $M_L^{(k)}$. As mentioned at the beginning of the section, certain minimal generators of $M_L^{(k)}$ arise as a "syzygy between a minimal generator and a lattice point", motivating the following definition.

The *modification* $M_{L,\text{mod}}^{(k)}$ of $M_L^{(k)}$ is the S-module generated by every element of $M_L^{(k)}$ and the element $1_{\mathbb{K}}$. Formally,

$$M_{L,\text{mod}}^{(k)} = \left\langle 1_{\mathbb{K}}, m \;\middle|\; m \in M_L^{(k)} \right\rangle_S \ . \tag{5.18}$$

Note that for $k = 1$, the modification process has no effect i.e., $M_{L,\text{mod}}^{(1)} = M_L$. Note that for $k > 1$, $M_{L,\text{mod}}^{(k)}$ is naturally an S-module but not an $S[L]$-module i.e. $M_{L,\text{mod}}^{(k)}$ does not inherit the natural L-action. By the construction of $M_{L,\text{mod}}^{(k)}$, we have the following characterisation of its minimal generators.

Proposition 5.2.4. *The (Laurent) monomial minimal generating set of $M_{L,\text{mod}}^{(k)}$ consists of precisely $1_{\mathbb{K}}$ and every (Laurent) monomial minimal generator of $M_L^{(k)}$ that is not divisible by $1_{\mathbb{K}}$ (in other words, whose exponent does not dominate the origin).*

Our construction proceeds similarly to the characterisation of $M_L^{(2)}$. For each minimal generator g_1 of $M_{L,\text{mod}}^{(k)}$, let $\text{Syz}_{g_1}^1(M_{L,\text{mod}}^{(k)})$ be the $\mathbb{K}$-vector space generated by syzygies of the form:

$$m \cdot (0, \dots, 0, \underbrace{\mathrm{lcm}(g_1, g_2)/g_1}_{g_1}, 0, \dots, 0, \underbrace{-\mathrm{lcm}(g_1, g_2)/g_2}_{g_2}, 0, \dots, 0) \tag{5.19}$$

where $g_2 \neq g_1$ is a minimal generator of $M_{L,\mathrm{mod}}^{(k)}$ and m is a monomial in S. We define a map $\phi_S^{(k)}$ from $\oplus_g \mathrm{Syz}_g^1(M_{L,\mathrm{mod}}^{(k)})$ to $M_L^{(k+1)}$ by first defining the map from the canonical basis of each piece $\mathrm{Syz}_g^1(M_{L,\mathrm{mod}}^{(k)})$ as follows:

$$\phi_S^{(k)}(s) = \mathbf{x}^{\deg_{\mathbb{Z}^n}(s)} \tag{5.20}$$

where $\deg_{\mathbb{Z}^n}(\cdot)$ is the $\mathbb{Z}^n$-graded degree of s. We extend this map $\mathbb{K}$-linearly to define $\phi_S^{(k)}$.

Note that the image of $\phi_S^{(k)}$ is an element of $M_L^{(k+1)}$. This is because s is of the form $m \cdot (\mathrm{lcm}(g_1, g_2)/g_2, -\mathrm{lcm}(g_1, g_2)/g_1)$ for two distinct minimal generators of $M_{L,\mathrm{mod}}^{(k)}$. By construction, $\phi_S^{(k)}(s) = m \cdot \mathrm{lcm}(g_1, g_2)$. By Proposition 5.2.4, we have the following two cases: either both g_1 and g_2 are minimal generators of $M_L^{(k)}$ or one of them, say g_1, is equal to $1_{\mathbb{K}}$ and g_2 is a minimal generator of $M_L^{(k)}$ that is not divisible by $1_{\mathbb{K}}$. In both cases, the support of $\mathrm{lcm}(g_1, g_2)$ contains at least $(k+1)$ points in L (by support of Laurent monomial, we mean the set of points in L that its exponent dominates). It contains (potentially among others) the unions of the supports of g_1 and g_2. Hence, the image of $\phi_S^{(k)}$ is in $M_L^{(k+1)}$. Theorem 5.2.5 is the converse to this.

Suppose that $\mathbf{x}^{\mathbf{w}} \in M_L^{(k+1)}$ is a minimal generator and let $U = \{\mathbf{u_1}, \dots, \mathbf{u_r}\}$ be the set of points in L that $\mathbf{w}$ dominates. For a subset $T \subset L$ of size k, let ℓ_T be the least common multiple of the Laurent monomials associated to points in T.

Theorem 5.2.5. *Up to the action of L, any minimal generator $\mathbf{x}^{\mathbf{w}}$ of $M_L^{(k+1)}$ is either in the image of $\phi_S^{(k)}$ or is an exceptional generator of $M_L^{(k)}$. Furthermore, we have the following classification of minimal generators of $M_L^{(k+1)}$.*

1. *If ℓ_T is the same for every subset T of U of size k, then $\mathbf{x}^{\mathbf{w}}$ is an exceptional generator of $M_L^{(k)}$.*

2. *If there exist subsets T_1 and T_2 of U of size k such that their least common multiples do not divide each other, then $\mathbf{x}^{\mathbf{w}}$ is the image of a syzygy between two minimal generators of $M_L^{(k)}$ in $\phi_S^{(k)}$.*

3. *Otherwise, $\mathbf{x}^{\mathbf{w}}$ is the image of a syzygy between a minimal generator of $M_L^{(k)}$ and $1_{\mathbb{K}}$ in $\phi_S^{(k)}$.*

Proof. By definition, $\mathbf{w}$ dominates at least $(k+1)$ points in L. Consider the $\binom{r}{k}$ subsets of U of size k and note that $\binom{r}{k} \geq 2$. For each subset T of size k, let ℓ_T be the least common multiple of the set of points in T. If the least common multiple ℓ_T is the same for all subsets T of size k, then we claim that $\mathbf{x}^{\mathbf{w}}$ is an exceptional generator of $M_L^{(k)}$. To see this, note that $\mathbf{x}^{\mathbf{w}} \in M_L^{(k)}$ and any minimal generator ℓ of $M_L^{(k)}$ that divides $\mathbf{x}^{\mathbf{w}}$ dominates every point in some subset of U of size k and ℓ is the least common multiple of the Laurent monomials corresponding to points in U. However, this least common multiple is $\mathbf{x}^{\mathbf{w}}$. Hence, $\ell = \mathbf{x}^{\mathbf{w}}$ and is an exceptional generator of $M_L^{(k)}$.

Otherwise, consider two subsets T_1 and T_2 of U of size k such that their least common multiples ℓ_{T_1} and ℓ_{T_2} respectively, are different. There are two cases:

Either ℓ_{T_1} and ℓ_{T_2} do not divide each other. Then both ℓ_{T_1} and ℓ_{T_2} are not equal to $\mathbf{x}^{\mathbf{w}}$ but divide it. Their supports (the set of lattice points that their exponents dominate) are precisely T_1 and T_2 respectively (otherwise, this would contradict $\mathbf{x}^{\mathbf{w}}$ being a minimal generator of $M_L^{(k+1)}$). Hence, ℓ_{T_1} and ℓ_{T_2} are minimal generators of $M_L^{(k)}$ as any Laurent monomial that divides either ℓ_{T_1} or ℓ_{T_2} must have strictly smaller support. The map $\phi_S^{(k)}$ takes their syzygy to a monomial m that divides $\mathbf{x}^{\mathbf{w}}$. Furthermore, since this monomial m is in $M_L^{(k+1)}$ and $\mathbf{x}^{\mathbf{w}}$ is a minimal generator of $M_L^{(k+1)}$, we conclude that $m = \mathbf{x}^{\mathbf{w}}$. Finally, note that by Proposition 5.2.4 there is a lattice point $\mathbf{q} \in L$ such that $\ell_{T_1} \cdot \mathbf{x}^{-\mathbf{q}}$ and $\ell_{T_2} \cdot \mathbf{x}^{-\mathbf{q}}$ are minimal generators of $M_{L,\mathrm{mod}}^{(k)}$. Their syzygy maps to an element in the same orbit as $\mathbf{x}^{\mathbf{w}}$ under the action of L.

Suppose that for every pair ℓ_{T_1} and ℓ_{T_2} one divides the other. Assume that ℓ_{T_1} is a proper divisor of ℓ_{T_2} and ℓ_{T_1} dominates exactly k points in L. Then ℓ_{T_2} along with the least common multiple of any other subset of size k other than T_1 is precisely $\mathbf{x}^{\mathbf{w}}$ (this is because $\mathbf{x}^{\mathbf{w}}$ is a minimal generator for $M_L^{(k+1)}$). Hence, the least common multiple of the set of Laurent monomials with exponents in $T_1 \cup \{\mathbf{q}\}$ is $\mathbf{x}^{\mathbf{w}}$ for any $\mathbf{q} \in T_2 \setminus T_1$. The map $\phi_S^{(k)}$ takes the syzygy between the minimal generators $\ell_{T_1} \cdot \mathbf{x}^{-\mathbf{q}}$ and $1_{\mathbb{K}}$ of $M_{L,\mathrm{mod}}^{(k)}$ to an element in the same orbit of $\mathbf{x}^{\mathbf{w}}$ under the action of the lattice L.

□

Remark 5.2.6. *Note that the proof of Theorem 5.2.5 also shows that any element in the image of $\phi_S^{(k)}$ satisfies Case 3 in Theorem 5.2.5 i.e. it is also in its image under a syzygy between a minimal generator of $M_L^{(k)}$ and $1_{\mathbb{K}}$. However, those that satisfy Case 2 also carry an L-action and hence, we have included this as a separate item in Theorem 5.2.5.*

Example 5.2.7. *Consider the lattice $L = (3,4,11)^{\perp} \cap \mathbb{Z}^3$. Theorem 5.2.5 gives us the basis for an algorithm to compute its lattice modules (we discuss this more in Section 5.4.2). We compute its 4th lattice module $M_L^{(4)}$; as an $S[L]$-module it is minimally generated by*

$$\langle x_3^2\ ,\ x_1^{-1}x_2x_3^2\ ,\ x_1^3x_2x_3 \rangle\ .$$

The minimal generator $x_1^{-1}x_2x_3^2$ dominates the lattice points

$$\{(-1,-2,1),(-2,-4,2),(-6,-1,2),(-5,1,1)\}\ .$$

Note that there exists two 3-subsets whose least common multiples are distinct and proper divisors of $x_1^{-1}x_2x_3^2$. We observe that these subsets consist of the first three and last three *lattice points, and give the following minimal generators of* $M^{(3)}_{L,mod}$*:*

$$x_1^{-1}x_2^{-1}x_3^2 = \operatorname{lcm}(x_1^{-1}x_2^{-2}x_3, x_1^{-2}x_2^{-4}x_3^2, x_1^{-6}x_2^{-1}x_3^2)$$
$$x_1^{-2}x_2x_3^2 = \operatorname{lcm}(x_1^{-2}x_2^{-4}x_3^2, x_1^{-6}x_2^{-1}x_3^2, x_1^{-5}x_2x_3)$$

Therefore $x_1^{-1}x_2x_3^2$ *equals* $\phi_S^{(3)}((x_1^{-1}x_2^{-1}x_3^2, x_1^{-2}x_2x_3^2))$ *and so is realised as the image of a syzygy between two minimal generators of* $M_L^{(3)}$*, see Figure 5.3.*

The minimal generator x_3^2 *cannot be constructed in this way. It dominates the lattice points* $\{(0,0,0),(-1,-2,1),(-2,-4,2),(-6,-1,2)\}$ *where only the least common multiple of the last three lattice points gives a proper divisor of* x_3^2*, specifically* $x_1^{-1}x_2^{-1}x_3^2$*. This is a minimal generator of* $M^{(3)}_{L,\mathrm{mod}}$ *and so* x_3^2 *equals* $\phi_S^{(3)}((x_1^{-1}x_2^{-1}x_3^2, 1_{\mathbb{K}}))$*, a syzygy between a minimal generator of* $M_L^{(3)}$ *and* $1_{\mathbb{K}}$*, as shown in Figure 5.4.*

For an example of an exceptional generator, we look at the lattice $L = (2,5,10)^\perp \cap \mathbb{Z}^3$*. The corresponding lattice ideal is* $I_L = \langle x_3 - x_1^5, x_3 - x_2^2 \rangle$*, therefore as an* $S[L]$*-module* $M_L^{(2)}$ *has generators* x_1^5, x_3, x_2^2*. These all lie in the same L-orbit and so* $M_L^{(2)}$ *is minimally generated by a single element* x_3*. However* x_3 *dominates 3 lattice points* $\{(0,0,0),(-5,0,1),(0,-2,1)\}$*. Therefore,* x_3 *is an exceptional generator of* $M_L^{(2)}$*, as shown in Figure 5.5. Indeed, note that the least common multiple of Laurent monomials corresponding to every pair of lattice points is also* x_3*.*

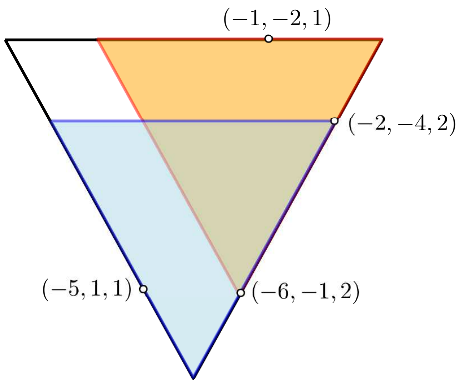

Figure 5.3: Minimal generator of $M_L^{(4)}$ realised as a syzygy between two minimal generators of $M_L^{(3)}$.

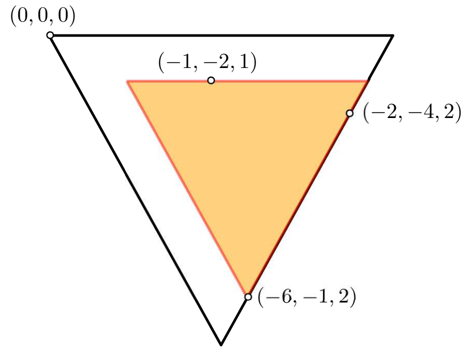

Figure 5.4: Minimal generator of $M_L^{(4)}$ realised as a syzygy between a minimal generator of $M_L^{(3)}$ and $1_{\mathbb{K}}$.

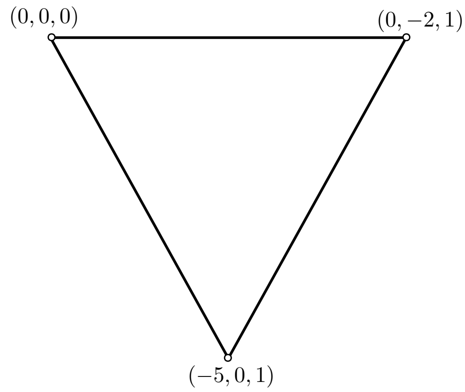

Figure 5.5: Exceptional generator of $M_{L(2,5,10)}^{(2)}$.

Note that $M_{L,\mathrm{mod}}^{(k)}$ is not finitely generated as an S-module and is also not an $S[L]$-module. This makes Theorem 5.2.5 somewhat unwieldy to compute $M_L^{(k)}$. In the following, we use Theorem 5.2.5 to prove the Neighbourhood Theorem that is computationally more amenable.

5.2.3 Neighbourhood Theorem

We associate a graph G_L to a lattice L as follows. Fix a binomial minimal generating set B of I_L. There is an edge between points $\mathbf{w_1}$ and $\mathbf{w_2}$ in L if there exists a minimal generator $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in B$ such that $\mathbf{u} - \mathbf{v} = \mathbf{w_1} - \mathbf{w_2}$. Let d_{G_L} be the metric on L induced by the graph G_L. For a point $\mathbf{w} \in L$, we define $N^{(k)}(\mathbf{w})$ to be the set of all points in L in the ball of radius k with respect to the metric d_{G_L} and with $\mathbf{w}$ as its center.

Theorem 5.2.8. (**Neighbourhood Theorem**) *Any minimal generator of $M_L^{(k)}$ as an $S[L]$-module is the least common multiple of Laurent monomials corresponding to k lattice points in $N^{(k-1)}(\mathbf{0})$, one of which is the origin. Equivalently, for any minimal generator of $M_L^{(k)}$ as an S-module, there is a point $\mathbf{q} \in L$ such that this minimal generator is the least common multiple of Laurent monomials corresponding to k lattice points in $N^{(k-1)}(\mathbf{q})$, one of which is $\mathbf{q}$.*

Theorem 5.2.8 makes computing generalised lattice modules amenable. Explicitly, to compute $M_L^{(k)}$ we only need to consider points in the finite neighbourhood $N^{(k-1)}(\mathbf{0})$. This is the notion of "sufficiently close" that we were lacking in Example 5.2.3.

In order to prove the theorem, we study certain "local pieces" of G_L called the fiber graph.

Definition 5.2.9. *([84]) Let $A = (a_1, \ldots, a_n)$. For each non-negative integer b we define the set $\mathcal{F}_b = \{\mathbf{u} \in \mathbb{Z}^n_{\geq 0} : A \cdot \mathbf{u} = b\}$ to be the* fiber *of A over b.*

For any lattice point $\mathbf{u} \in L$, we can express it uniquely as the difference of positive and negative parts $\mathbf{u}^+ - \mathbf{u}^-$, where the i-th coordinate of $\mathbf{u}^+$ equals u_i if $u_i > 0$ and equals 0 otherwise. Since L is contained in $(a_1, \ldots, a_n)^\perp$, we have $\mathbf{u}^+ \in \mathcal{F}_b$ if and only if $\mathbf{u}^- \in \mathcal{F}_b$.

We induce a natural graph on the fiber, denoted the fiber graph G_b. Fix a binomial minimal generating set B of I_L. The vertices of the graph are the elements of the fiber $\mathcal{F}_b$ with an edge between $\mathbf{w_1}$ and $\mathbf{w_2}$ if there exists a minimal generator $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in B$ such that $\mathbf{u} - \mathbf{v} = \mathbf{w_1} - \mathbf{w_2}$. We note that G_b is a finite graph that can be embedded into G_L. The following lemma generalises the statement [84, Theorem 5.3] that if I_L is a prime ideal (equivalently, if L is a saturated lattice) then $\mathcal{F}_b$ is connected.

Lemma 5.2.10. *Let $\mathbf{u}, \mathbf{v} \in \mathcal{F}_b$. The difference $\mathbf{u} - \mathbf{v}$ is a lattice point in L if and only if $\mathbf{u}, \mathbf{v}$ are in the same connected component of G_b.*

Proof. Suppose $\mathbf{u} - \mathbf{v} \in L$, then by definition $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_L$ and so can be represented as an S-linear combination of the minimal generators:

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \sum_{i=1}^{N} \mathbf{x}^{\mathbf{w_i}} \cdot (\mathbf{x}^{\mathbf{g_i^+}} - \mathbf{x}^{\mathbf{g_i^-}}) \tag{5.21}$$

We will show by induction on N there exists a path in G_b between $\mathbf{u}$ and $\mathbf{v}$. For $N = 1$, expression (5.21) is equivalent to saying that $\mathbf{u} - \mathbf{v} = \mathbf{g_i}$ and so they must be connected by an edge.

Assume the induction hypothesis holds for all $N < N'$, consider expression (5.21) for $N = N'$. We have $\mathbf{x}^{\mathbf{u}} = \mathbf{x}^{\mathbf{w_i}} \cdot \mathbf{x}^{\mathbf{g_i^+}}$ for some i, so without loss of generality we say that $\mathbf{u} = \mathbf{w_1} + \mathbf{g_1^+}$, implying $\mathbf{u}$ and $\mathbf{w_1} + \mathbf{g_1^-}$ are connected by an edge. Subtracting $\mathbf{x}^{\mathbf{w_1}} \cdot (\mathbf{x}^{\mathbf{g_1^+}} - \mathbf{x}^{\mathbf{g_1^-}})$ from (5.21) gives us an expression of length $N' - 1$ for $\mathbf{x}^{\mathbf{w_1}+\mathbf{g_1^-}} - \mathbf{x}^{\mathbf{v}}$. By the induction hypothesis, these exponents are connected and so $\mathbf{u}$ and $\mathbf{v}$ must also be connected.

Conversely, assume that $\mathbf{u}, \mathbf{v}$ are in the same connected component of G_b. Then there exists some path $\mathbf{u} = v^{(0)}, v^{(1)}, \ldots, v^{(N)} = \mathbf{v}$ in G_b. We can write the binomial

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \sum_{i=1}^{N} \mathbf{x}^{v^{(i-1)}} - \mathbf{x}^{v^{(i)}}$$

where each binomial $\mathbf{x}^{v^{(i-1)}} - \mathbf{x}^{v^{(i)}}$ is an element of I_L, as $v^{(i-1)}, v^{(i)}$ are connected by an edge. Therefore, $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_L$ and so $\mathbf{u} - \mathbf{v} \in L$. □

Lemma 5.2.11. *Let* $\mathbf{v}$ *be a lattice point with* $\mathbf{v}^+, \mathbf{v}^- \in F \subseteq \mathcal{F}_b$*, where* F *is a subset of the fiber* $\mathcal{F}_b$ *consisting of all elements in the same connected component of* G_b*. The exponent of the least common multiple* $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, 1_{\mathbb{K}})$ *dominates precisely* $|F|$ *lattice points, specifically those of the form* $\mathbf{v}^+ - \mathbf{u}$ *where* $\mathbf{u} \in F$*.*

Proof. We first observe that $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, 1_{\mathbb{K}}) = \mathbf{x}^{\mathbf{v}^+}$. Let $\mathbf{u} \in F$, then by Lemma 5.2.10 we deduce that $\mathbf{v}^+ - \mathbf{u} \in L$. As $\mathbf{u} \in \mathbb{Z}^n_{\geq 0}$, we see $\mathbf{v}^+ \geq \mathbf{v}^+ - \mathbf{u}$. This holds for every $\mathbf{u} \in F$

and so the exponent of $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, 1_{\mathbb{K}})$ dominates at least $|F|$ lattice points. Conversely, suppose that for some $\mathbf{p} \in L$, $\mathbf{v}^+ \geq \mathbf{p}$. Let $\mathbf{u} = \mathbf{v}^+ - \mathbf{p} \in \mathbb{Z}^n_{\geq 0}$. Then $\mathbf{v}^+ - \mathbf{u} \in L$, hence by Lemma 5.2.10 $\mathbf{u} \in F$. □

Lemma 5.2.12. *Let* $\mathbf{u}, \mathbf{v} \in L, d_{G_L}(\mathbf{u}, \mathbf{v}) = k$. *There exists a path of length at least* k *in* G_L *from* $\mathbf{u}$ *to* $\mathbf{v}$ *such that the exponent of* $\mathrm{lcm}(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})$ *dominates every lattice point on the path.*

Proof. As G_L is invariant under translation by L, it suffices to prove the case where $\mathbf{u} = \mathbf{0}$. Suppose that $\mathbf{v}^+, \mathbf{v}^- \in \mathcal{F}_b$, by Lemma 5.2.10 they lie in the same connected component of G_b and so there exists a path in G_b given by $\mathbf{v}^+ = v^{(0)}, v^{(1)}, \ldots, v^{(n)} = \mathbf{v}^-$. We can embed this path into G_L by the embedding $\mathbf{v}^+ - v^{(i)}$. This gives us a path from $\mathbf{0}$ to $\mathbf{v}$ in G_L and by Lemma 5.2.11 the exponent of $\mathrm{lcm}(1_{\mathbb{K}}, \mathbf{x}^{\mathbf{v}})$ dominates each of the lattice points on this path. As $d_{G_L}(\mathbf{0}, \mathbf{v}) = k$, this path must be at least length k. □

Proof. (Proof of Theorem 5.2.8) We proceed by induction on k. For the base case of $k = 1$, the lattice module $M_L^{(1)} = M_L$ has a single generator $1_{\mathbb{K}}$ corresponding to the single lattice point in $N^{(0)}(\mathbf{0})$. Assume the statement is true for all $k \leq k_0$. Let $\mathbf{x}^{\mathbf{u}}$ be a minimal generator of $M_L^{(k_0+1)}$, then by Theorem 5.2.5 this is either in the image of the map $\phi_S^{(k_0)}$ or is an exceptional generator of $M_L^{(k_0)}$.

Suppose that it is an exceptional generator of $M_L^{(k_0)}$, then by the inductive hypothesis $\mathbf{x}^{\mathbf{u}}$ can be expressed as the least common multiple of Laurent monomials corresponding to a set of precisely k_0 lattice points, which we denote as $P_{\mathbf{u}}$. Note that $P_{\mathbf{u}}$ is a proper subset of the support of $\mathbf{x}^{\mathbf{u}}$. By lattice translation, we assume that $P_{\mathbf{u}}$ is contained in $N^{(k_0-1)}(\mathbf{0})$ and contains $\mathbf{0}$. It suffices to show that $\mathbf{u}$ dominates another lattice point in $N^{(k_0)}(\mathbf{0})$.

As an exceptional generator $\mathbf{x}^{\mathbf{u}}$ must dominate at least k_0+1 lattice points, so consider a lattice point $\mathbf{p} \notin P_{\mathbf{u}}$ that is dominated by $\mathbf{u}$. If $\mathbf{p} \in N^{(k_0)}(\mathbf{0})$, we are done. Suppose $\mathbf{p} \in N^{(r)}(\mathbf{0}), r > k_0$. By Lemma 5.2.12 there exists a path from $\mathbf{p}$ to $\mathbf{0}$ in G_L such that every lattice point in the path is dominated by the exponent of $\mathrm{lcm}(\mathbf{x}^{\mathbf{p}}, 1_{\mathbb{K}})$. Therefore there exists some lattice point $\mathbf{q}$ in this path with $d_{G_L}(\mathbf{0}, \mathbf{q}) = k_0$ that is dominated by the exponent of $\mathrm{lcm}(\mathbf{x}^{\mathbf{p}}, 1_{\mathbb{K}})$. Furthermore as $P_{\mathbf{u}}$ is contained in $N^{(k_0-1)}(\mathbf{0})$, $\mathbf{q} \notin P_{\mathbf{u}}$. As $\mathrm{lcm}(\mathbf{x}^{\mathbf{p}}, 1_{\mathbb{K}})$ divides $\mathbf{x}^{\mathbf{u}}$, it must also dominate all lattice points along this path. Therefore $\mathbf{x}^{\mathbf{u}}$ can be written as the least common multiple of the Laurent monomials corresponding to the lattice points $P_{\mathbf{u}} \cup \{\mathbf{q}\}$ whose cardinality is $k_0 + 1$.

Suppose that $\mathbf{x}^{\mathbf{u}}$ is in the image of $\phi_S^{k_0}$. According to Remark 5.2.6, $\mathbf{x}^{\mathbf{u}}$ is the image of a syzygy between one minimal generator of $M_L^{(k_0)}$ as an S-module and $1_{\mathbb{K}}$. This minimal generator is in the same L-orbit as $\mathbf{x}^{\mathbf{v}}$, a minimal generator of $M_L^{(k_0)}$ satisfying the induction hypothesis. More precisely, there exists a set $P_{\mathbf{v}}$ of k_0 lattice points whose least common multiple of Laurent monomials equals $\mathbf{x}^{\mathbf{v}}$ that is contained in $N^{(k_0-1)}(\mathbf{0})$ and contains $\mathbf{0}$. Hence, $\mathbf{x}^{\mathbf{u}}$ is in the same L-orbit as $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$ for some lattice point $\mathbf{p}$. It suffices to show that $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$ satisfies the statement of the theorem.

Let $\mathbf{p} \in N^{(r)}(\mathbf{0})$, if $r \leq k_0$ then we are done. Suppose $r > k_0$, by Lemma 5.2.12 there exists a path from $\mathbf{0}$ to $\mathbf{p}$ in G_L such that every lattice point in the path is dominated by the exponent of $\mathrm{lcm}(1_{\mathbb{K}}, \mathbf{x}^{\mathbf{p}})$. By the same argument as the previous case, there exists a lattice point $\mathbf{q}$ on this path with $d_{G_L}(\mathbf{0}, \mathbf{q}) = k_0$, that is necessarily dominated by $\mathbf{u}$ and not contained in $P_{\mathbf{v}}$. Therefore $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{q}})$ is the least common multiple of the Laurent monomials corresponding to $k_0 + 1$ lattice points $P_{\mathbf{v}} \cup \{\mathbf{q}\}$. The monomial $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{q}})$ divides $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$, and so is equal to it by the minimality of $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$. Therefore $\mathrm{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$ is the least common multiple of $k_0 + 1$ Laurent monomials corresponding to $P_{\mathbf{v}} \cup \{\mathbf{q}\}$ contained in $N^{(k_0)}(\mathbf{0})$.

□

5.3 Finiteness results

In this section, we show that after suitable twists there are only finitely many isomorphism classes of generalised lattice modules. More precisely, we show the following:

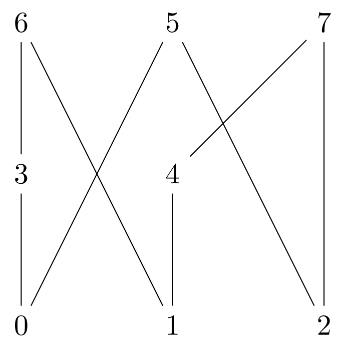

Figure 5.6: The structure poset of $L(3,5,8)$.

Theorem 5.3.1. *Let L be a lattice of the form $(a_1,\dots,a_n)^\perp \cap \mathbb{Z}^n$. For each $k \in \mathbb{N}$, let $\mathbf{x}^{\mathbf{u_k}}$ be any element $M_L^{(k)}$ of the smallest $(a_1,\dots,a_n)$-weighted degree. There are finitely many classes among the generalised lattice modules $\{M_L{}^{(k)}(\mathbf{u_k})\}_{k\in\mathbb{N}}$ up to isomorphism of both $\mathbb{Z}^n$-graded $S[L]$-modules and $\mathbb{Z}^n$-graded S-modules.*

The main ingredient of the proof of Theorem 5.3.1 is the *structure poset of L.*

Definition 5.3.2. *The* structure poset of L *is the poset on the elements of $\mathbb{Z}^n/L$ of $(a_1,\dots,a_n)$-weighted degree in the range $[0,F_1]$ where F_1 is the first Frobenius number of L. The partial order in this poset is defined as follows: for elements $[\mathbf{a}]$, $[\mathbf{b}]$ we say that $[\mathbf{a}] \geq [\mathbf{b}]$ if for every representative $\mathbf{a} \in \mathbb{Z}^n$ of $[\mathbf{a}]$ there exists a representative $\mathbf{b}$ of $[\mathbf{b}]$ such that $\mathbf{a} \geq \mathbf{b}$.*

We note that $[\mathbf{a}] \geq [\mathbf{b}]$ if and only if $[\mathbf{a}-\mathbf{b}] \geq [\mathbf{0}]$. Hence, the structure poset of L can be constructed from the set of all elements $[\mathbf{a}] \geq [\mathbf{0}]$ in $\mathbb{Z}^n/L$ whose $(a_1,\dots,a_n)$-weighted degree is in the range $[0,F_1]$. This observation is useful to compute the structure poset.

Example 5.3.3. *Let $(a_1,a_2,a_3) = (3,5,8)$ and hence, $L(3,5,8) = (3,5,8)^\perp \cap \mathbb{Z}^3$. The first Frobenius number is 7. Hence, the structure poset of L consists of eight elements labelled 0 to 7. The poset relations can be determined from the set of all elements that dominate 0, in this case they are $3,5,6$. The Hasse diagram of the structure poset is shown in Figure 5.6.*

Let m_k be the minimum $(a_1,\dots,a_n)$-weighted degree of any element of $M_L^{(k)}$. A key observation is that $M_L^{(k)}$ is determined (up to isomorphism of $\mathbb{Z}^n$-graded $S[L]$-modules) by the elements in $\mathbb{Z}^n/L$ of weighted degree $[m_k, m_k+F_1]$ that dominate at least k points in L. We can see this by considering the submodule $\mathbf{x}^{\mathbf{u}} \cdot M_L$ of $M_L^{(k)}$ where $\mathbf{x}^{\mathbf{u}} \in M_L^{(k)}$ has weighted degree m_k. Any element of weighted degree greater than $m_k + F_1$ dominates an element with weighted degree m_k in $\mathbf{x}^{\mathbf{u}} \cdot M_L$ and so also dominates k lattice points. Using this observation, we can associate a structure poset to $M_L^{(k)}$ that is a subposet of the structure poset of L.

Definition 5.3.4. *The* structure poset of $M_L^{(k)}$ *is the poset on the elements of*

$$\left\{ r \in \mathbb{Z}^n/L \;\middle|\; \textit{there exists } \mathbf{x}^{\mathbf{u}} \in M_L^{(k)} \textit{ such that } \deg(\mathbf{x}^{\mathbf{u}}) = r \right\}$$

of $(a_1, \ldots, a_n)$*-weighted degree in the range* $[m_k, m_k + F_1]$*. The partial order is the same as on the structure poset of* L*.*

By mapping $m_k + i$ to i, the structure poset of $M_L^{(k)}$ is a subposet of L. Note that the minimal generators of $M_L^{(k)}$ correspond to the minimal elements of its structure poset.

Proof. (Proof of Theorem 5.3.1) Note that for any k, the $(a_1, \ldots, a_n)$-weighted degree of the minimal generators of $M_L^{(k)}$ are in the range $[m_k, m_k + F_1]$. Furthermore, the structure poset of $M_L^{(k)}$ as a subposet of the structure poset of L determines $M_L^{(k)}(\mathbf{u_k})$ up to isomorphism of $\mathbb{Z}^n$-graded $S[L]$-modules (and $\mathbb{Z}^n$-graded S-modules). More precisely, if $M_L^{(k_1)}$ and $M_L^{(k_2)}$ have the same structure poset, then multiplying $M_L^{(k_1)}(\mathbf{u_{k_1}})$ by the Laurent monomial $\mathbf{x}^{\mathbf{u_2}}/\mathbf{x}^{\mathbf{u_1}}$ is an isomorphism between $M_L^{(k_1)}(\mathbf{u_{k_1}})$ and $M_L^{(k_2)}(\mathbf{u_{k_2}})$ (as both $\mathbb{Z}^n$-graded $S[L]$-modules and $\mathbb{Z}^n$-graded S-modules). In particular, this map induces a bijection between the (monomial) minimal generating set of $M_L^{(k_1)}(\mathbf{u_{k_1}})$ and the (monomial) minimal generating set of $M_L^{(k_2)}(\mathbf{u_{k_2}})$ and preserves degrees. Since the structure poset of L is finite, it has only finitely many subposets. Hence, there are only finitely many $\mathbb{Z}^n$-graded isomorphism classes of the twisted generalised lattice modules $\{M_L^{(k)}(\mathbf{u_k})\}_{k \in \mathbb{N}}$. □

Theorem 5.3.1 and its proof also generalises to finite index sublattices of $L(a_1, \dots, a_n)$. The only additional subtlety is that the structure poset of $M_L^{(k)}$ will have precisely as many embeddings into the structure poset of L as the number of elements of weighted degree m_k in $M_L^{(k)}$. If $M_L^{(k_1)}$ and $M_L^{(k_2)}$ have the same embedding into the structure poset of L, then we have exactly the same isomorphism as in the proof of Theorem 5.3.1. There are still only finitely many subposets of the structure poset of L.

Remark 5.3.5. *It is worth noting that not all subposets of the structure poset of L can be realised as the structure poset of some $M_L^{(k)}$. If an element is contained in the structure poset of some $M_L^{(k)}$, all elements greater than it according to the partial order must also be contained in it. Therefore the poset is completely determined by its set of minimal elements, which form an antichain of the structure poset of L. As a result, the number of subposets realisable as the structure poset of some $M_L^{(k)}$ is upper bounded by the number of antichains of the structure poset of L. For counting the number of antichains, tools such as Dilworth's theorem [31] are useful.*

Remark 5.3.6. *The data of the structure poset of $M_L^{(k)}$ where $L = L(a_1, \dots, a_n)$ is encoded in the Hilbert series of the polynomial ring S with the $(a_1, \dots, a_n)$-weighted grading. The elements of $M_L^{(k)}$ are those j such that the Hilbert coefficient h_j is at least k. This Hilbert series is also referred to as the restricted partition function in [23, Page 6] and is a useful tool for explicitly computing the structure poset. Note that for a finite index sublattice L of $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$, this data is encoded in the Hilbert series of S with the $\mathbb{Z}^n/L$-grading.*

Example 5.3.7. *In the following we compute the structure poset of $M_L^{(k)}$ where $L = L(3,5,8)$ for k from 1 to 6. The Hilbert series of the polynomial ring with the $(a_1, \dots, a_n)$-weighted grading is given by the rational function*

$$H(S; t^{a_1}, \dots, t^{a_n}) = \prod_{i=1}^{n} \frac{1}{(1-t)^{a_i}} \quad . \tag{5.22}$$

Using this information, we determine $m_1, \dots, m_6$ to be $0, 8, 16, 21, 24, 29$. The other *elements of the structure poset of* $M_L^{(k)}$ *are the integers* i *in the interval* $[0,7]$ *such that* $h_{m_k+i} \geq k$. *The corresponding structure posets are shown in Figure 5.7.*

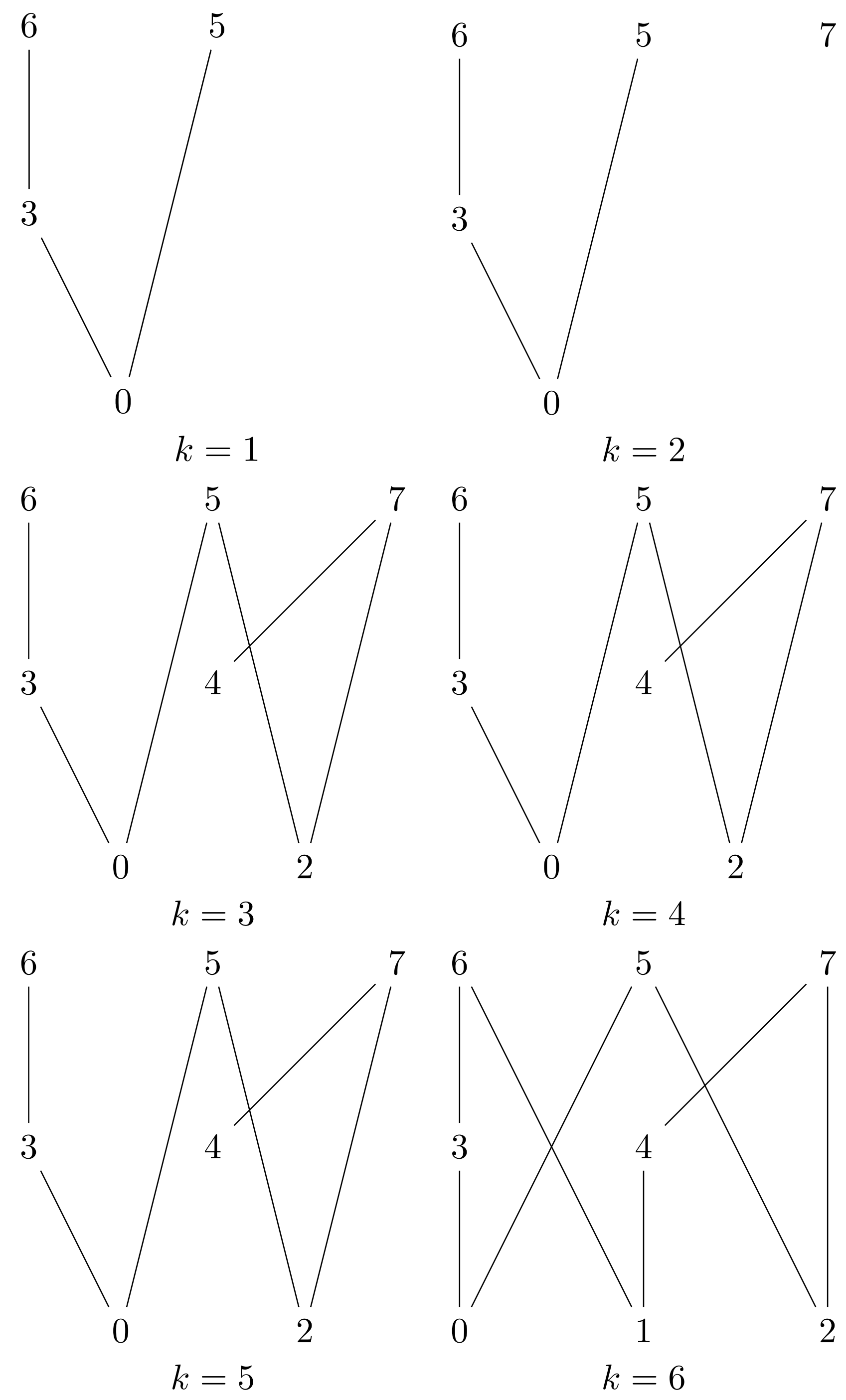

Figure 5.7: The structure posets of $M^{(k)}_{L(3,5,8)}$ for k from 1 to 6.

Based on the same ideas as in Theorem 5.3.1, we obtain the following upper bounds on generalised Frobenius numbers and the number of minimal generators of $M_L^{(k)}$.

Proposition 5.3.8. *The k-th Frobenius number F_k is upper bounded by $m_k + F_1$. The number $\beta_1(M_L^{(k)})$ of minimal generators of $M_L^{(k)}$ as an $S[L]$-module is upper bounded by the maximum length of an antichain in the structure poset of L.*

Furthermore, we have the following corollary to Theorem 5.3.1.

Corollary 5.3.9. *There exists a finite set of integers $\{b_1, \ldots, b_t\} \subset \mathbb{Z}_{\geq 0} \cup \{-1\}$ such that for every k there exists a natural number j such that the k-th Frobenius number can be written as:*

$$F_k = m_k + b_j$$

where m_k is the minimum $(a_1, \ldots, a_n)$-weighted degree of an element in $M_L^{(k)}$. This finite set $\{b_1, \ldots, b_t\}$ is the precisely the set of integers that can be realised as $\mathrm{reg}(M_L^{(k)}(\mathbf{u_k})) + n - 1 - \sum_{i=1}^{n} a_i$.

5.4 Applications

5.4.1 The sequence of generalised Frobenius numbers

We prove that the sequence of generalised Frobenius numbers form a finite difference progression.

Definition 5.4.1. *A sequence $(c_k)_{k=1}^{\infty}$ is called a* finite difference progression *if there exists a finite set of differences such that for every $k \in \mathbb{N}$ the difference $c_{k+1} - c_k$ is contained in this set. The* rank *of the progression is defined to be the cardinality of this set.*

Theorem 5.4.2. *For any finite index sublattice L of $(a_1, \ldots, a_n)^{\perp} \cap \mathbb{Z}^n$, the sequence of generalised Frobenius numbers $(F_k)_{k=1}^{\infty}$ is a finite difference progression.*

We note that this follows immediately from Corollary 5.3.9 once we show that the sequence $(m_k)_{k=1}^{\infty}$ is also a finite difference progression.

Lemma 5.4.3. *For any finite index sublattice L of $(a_1, \ldots, a_n)^{\perp} \cap \mathbb{Z}^n$, the sequence $(m_k)_{k=1}^{\infty}$ is a finite difference progression.*

Proof. We show the difference of successive terms is bounded by $0 \leq m_{k+1} - m_k \leq m_2$, and therefore the set of successive differences is finite. The inequality $m_{k+1} - m_k \geq 0$ follows by construction.

To prove the other bound, we construct an element of degree at most $m_k + m_2$ in $M_L^{(k+1)}$ and hence, conclude that $m_{k+1} - m_k \leq m_2$. Consider a minimal generator of $M_L^{(2)}$ of weighted degree m_2 that dominates the origin and another lattice point $\mathbf{p}$. Note that this minimal generator is $\mathbf{x}^{\mathbf{p}^+}$.

Consider a minimal generator $\mathbf{x}^{\mathbf{q}}$ of $M_L^{(k)}$ of weighted degree m_k, such that the origin is in its support and $\mathbf{p}$ is not in its support. Note that such a generator exists by the following lattice translation argument. Take any minimal generator $\mathbf{x}^{\mathbf{q}'}$ of $M_L^{(k)}$ of weighted degree m_k and maximise the linear functional $\mathbf{p} \cdot \mathbf{x}$ over its support. Suppose that $\mathbf{r}$ is a point in the support at which this functional is maximised, multiply the minimal generator by $\mathbf{x}^{-\mathbf{r}}$. The resulting minimal generator contains the origin but does not contain the point $\mathbf{p}$ in its support. This is because the origin maximises the functional $\mathbf{p} \cdot \mathbf{x}$ over the support of $\mathbf{x}^{\mathbf{q}'} \cdot \mathbf{x}^{-\mathbf{r}}$ and the inner product of $\mathbf{p}$ with the origin is zero whereas its inner product with itself is strictly positive.

The monomial $\operatorname{lcm}(\mathbf{x}^{\mathbf{p}^+}, \mathbf{x}^{\mathbf{q}})$ is contained in $M_L^{(k+1)}$ as its support contains the union of supports of $\mathbf{x}^{\mathbf{q}}$ and $\mathbf{x}^{\mathbf{p}}$, and has weighted degree at most $m_2 + m_k$. As an element of $M_L^{(k+1)}$ it must have weighted degree at least m_{k+1} and therefore $m_{k+1} - m_k \leq m_2$. $\square$

The sequence $(m_k)_{k=1}^{\infty}$ inherits much of the structure of $M_L^{(k)}$ given by its inductive characterisation (Theorem 5.2.5). This additional structure makes it more natural to derive bounds on successive differences rather than $(F_k)_{k=1}^{\infty}$ directly.

Recall that the rank of the finite difference progression is defined as the cardinality of its set of successive differences. Note that the rank is equal to one when the sequence is an arithmetic progression. Given the sequence of k-th Frobenius numbers $(F_k)_{k=1}^{\infty}$ with associated $\{b_1, \ldots, b_t\}$ such that $b_t > b_{t-1} > \cdots > b_1$ (as defined in Corollary 5.3.9), we derive two upper bounds on its rank from Lemma 5.4.3 and Corollary 5.3.9.

Proposition 5.4.4. *The rank of the finite difference progression $(F_k)_{k=1}^{\infty}$ is upper bounded by:*

$$\operatorname{rank}((F_k)_{k=1}^{\infty}) \leq m_2 + b_t - b_1 + 1 \tag{5.23}$$

$$\operatorname{rank}((F_k)_{k=1}^{\infty}) \leq \left(\binom{t}{2} + 1\right)(m_2 + 1) \tag{5.24}$$

Proof. Bound (5.23) is derived from the fact that the largest possible difference between successive terms is $m_2 + b_t - b_1$. This is possible when $F_k = m_k + b_1$ and $F_{k+1} = m_{k+1} + b_t$ where $m_{k+1} - m_k = m_2$, the largest possible difference as shown in the proof of Lemma 5.4.3. All possible differences are in the interval $[0, m_2 + b_t - b_1]$ and so the rank is upper bounded by its cardinality.

Bound (5.24) is derived as follows. By Corollary 5.3.9, we can express the difference $F_{k+1} - F_k = (m_{k+1} - m_k) + (b_j - b_i)$ for some $b_i, b_j \in \{b_1, \ldots, b_t\}$. Recall from Lemma 5.4.3 that the set of differences $\{m_{k+1} - m_k\}_{k \in \mathbb{N}}$ is a subset of $[0, m_2]$. We consider the following two cases:

Case 1 **($b_j > b_i$):** There are $\binom{t}{2}$ choices of b_i, b_j that satisfy $b_j > b_i$ and so the number of differences $\{b_j - b_i\}_{i<j}$ is upper bounded by $\binom{t}{2}$. Therefore the number of differences $\{F_{k+1} - F_k\}$ is upper bounded by $\binom{t}{2}(m_2 + 1)$.

Case 2 **($b_j \leq b_i$):** Here $0 \leq F_{k+1} - F_k \leq m_{k+1} - m_k$, therefore the set of differences is a subset of $[0, m_2]$.

Summing up the upper bounds over both cases, we get the bound $\operatorname{rank}((F_k)_{k=1}^{\infty}) \leq (\binom{t}{2} + 1)(m_2 + 1)$ □

Corollary 5.4.5. *A geometric progression with common ratio strictly greater than one cannot occur as a sequence of generalised Frobenius numbers of any finite index sublattice of* $(a_1, \ldots, a_n)^{\perp} \cap \mathbb{Z}^n$.

Proof. By Theorem 5.4.2, a sequence of generalised Frobenius numbers $(F_k)_{k=1}^{\infty}$ is a finite difference progression. Hence, the difference $F_{k+1} - F_k$ is uniformly upper bounded. On the other hand, since the common ratio of the geometric progression is greater than one, the difference between successive terms goes to infinity with k. Hence, such a geometric progression cannot occur as a sequence of generalised Frobenius numbers. □

Remark 5.4.6. *Another reason to expect Corollary 5.4.5 is that the sequence of generalised Frobenius numbers of lattices of dimension at least two usually contains plenty of repetitions. However, Theorem 5.4.2 implies a stronger statement that even after removing the repetitions the resulting sequence cannot be a geometric progression of common ratio strictly greater than one.*

5.4.2 Algorithms for generalised Frobenius numbers

We use the Neighbourhood Theorem (Theorem 5.2.8) to give an algorithmic construction of generalised lattice modules and via Proposition 5.1.13 compute generalised Frobenius numbers.

Remark 5.4.7. *A method for computing the lattice ideal given a basis for that lattice is presented in [66]. One method to compute the Castelnuovo–Mumford regularity of* $\pi(M_L^{(k)})$ *is to construct a free presentation of* $\pi(M_L^{(k)})$*, for instance via the hull complex of* $M_L^{(k)}$*. We can use this as the input to the algorithm presented in [19] to compute the Castelnuovo–Mumford regularity.*

Example 5.4.8. *In the following example, we illustrate our algorithm with the inputs* $L = L(3, 4, 11) = (3, 4, 11)^{\perp} \cap \mathbb{Z}^3$ *and* $k = 3$*. Figure 5.2 shows the monomial staircase for this lattice module.*

Algorithm 3 Generalised Lattice Modules

Input: A basis of a finite index sublattice L of $(a_1, \ldots, a_n)^{\perp} \cap \mathbb{Z}^n$ where $(a_1, \ldots, a_n) \in \mathbb{N}^n$ and a natural number $k \in \mathbb{N}$.

Output: A minimal generating set of $M_L^{(k)}$ as an $S[L]$-module and the k-th Frobenius number F_k of L.

1: Compute the lattice ideal I_L.
2: Compute all lattice points in $N^{(k-1)}(\mathbf{0})$.
3: For each k-subset $P \subseteq N^{(k-1)}(\mathbf{0})$ containing $\mathbf{0}$, calculate the least common multiple $\ell_P = \operatorname{lcm}(\mathbf{x}^{\mathbf{p_i}} \mid \mathbf{p_i} \in P)$.
4: Construct $M_L^{(k)} = \left\langle \ell_P \mid P \subseteq N^{(k-1)}(\mathbf{0}) \,,\ |P| = k,\ \mathbf{0} \in P \right\rangle_{S[L]}$.
5: Pick a representative that is minimal under divisibility for each L-orbit and declare the resulting set to be a minimal generating set of $M_L^{(k)}$.
6: Compute the $\mathbb{Z}^n/L$-graded S-module $\pi(M_L^{(k)}) := M_L^{(k)} \otimes_{S[L]} S$ and its Castelnuovo–Mumford regularity $\operatorname{reg}(\pi(M_L^{(k)}))$.
7: Set the k-th Frobenius number F_k to $\operatorname{reg}(\pi(M_L^{(k)})) + n - 1 - \sum_{i=1}^{n} a_i$.

The set $\{(1,2,-1),(4,-3,0)\}$ *is a basis for* L*. The binomials corresponding to this basis generate the ideal* $J = \left\langle x_1x_2^2 - x_3, x_1^4 - x_2^3 \right\rangle$*. The lattice ideal* I_L *is given by the saturation of* J *with respect to the product of all the variables, and so*

$$I_L = \langle J : \langle x_1x_2x_3 \rangle^{\infty} \rangle = \left\langle x_1x_2^2 - x_3, x_1^4 - x_2^3 \right\rangle.$$

In this case, the lattice ideal does not have any new binomials.

The lattice points $(1,2,-1),(4,-3,0)$ *along with their negative and the origin* $\mathbf{0}$*, give the first neighbourhood* $N^{(1)}(\mathbf{0})$*. Next, we compute* $N^{(k-1)}(\mathbf{0})$ *by taking all* k*-subsets of* $N^{(1)}(\mathbf{0})$ *and taking their sum. This computation gives us*

$$\begin{aligned} N^{(2)}(\mathbf{0}) = &\{(0,0,0),(1,2,-1),(4,-3,0),(-1,-2,1),(-4,3,0),(8,-6,0),(3,-5,1), \\ &(5,-1,-1),(-2,-4,2),(2,4,-2),(-5,1,1),(-3,5,-1),(-8,6,0)\}. \end{aligned}$$

For each 3*-subset of* $N^{(2)}(\mathbf{0})$*, we take the least common multiple of the corresponding monomials and denote the* $S[L]$*-module generated by these monomials as* M_{con}*. By the Neighbourhood Theorem,* M_{con} *is equal to* $M_L^{(3)}$*. Note that this requires computing* $\binom{12}{2} = 66$ *monomials.*

To calculate a minimal generating set of $M_L^{(3)}$, we choose the monomials from this set that do not dominate any other monomial in $M_L^{(3)}$. In our case, this gives the following list of generators

$$M_L^{(3)} = \left\langle x_1^5, x_1^4x_2^2, x_1x_2^3, x_1^3x_3, x_2^5, x_2x_3 \right\rangle_{S[L]}.$$

All minimal generators with the same $\mathbb{Z}^n/L$-degree must be in the same L-orbit. Hence, we pick representatives for each degree to give a minimal generating set of $M_L^{(3)}$. All minimal generators are in degree 15 or 20, and so $M_L^{(3)} = \langle x_1^5, x_1^4x_2^2 \rangle_{S[L]}$. We compute the Castelnuovo–Mumford regularity of $\pi(M_L^{(3)}) = 33$. Therefore, we calculate F_3 to be

$$33 + 2 - 3 - 4 - 11 = 17$$

5.5 Future directions

We organise potential future directions into three items with the first two closely related.

- **Classification of Sequences of Generalised Frobenius Numbers:** We have shown that the sequence of generalised Frobenius numbers form a finite difference progression, however there is still information that we have not fully utilised. For instance, we have not used the filtration of the generalised lattice modules and the inductive characterisation provided by Theorem 5.2.8. Can this information be used to study sequences of generalised Frobenius numbers? For instance, by studying the sequence of Castelnuovo–Mumford regularity of modules in a filtration.

- **Syzygies of Generalised Lattice Modules:** Our finiteness result shows that for any finite index sublattice of $(a_1, \ldots, a_n)^\perp \cap \mathbb{Z}^n$ there are only finitely many isomorphism classes of generalised lattice modules. What are the possible Betti tables that can occur as Betti tables of generalised lattice modules? How are they related? Note that this is closely related to the previous item since the Castelnuovo–Mumford regularity of $M_L^{(k)}$ is the number of rows of its Betti table minus one and this is essentially the k-th Frobenius number (Proposition 5.1.13). This problem is also closely related to the problem of classifying structure posets of generalised lattice modules (see Section 5.3 for more details).

Peeva and Sturmfels [73] define a notion of lattice ideals associated to generic lattices and show that the Scarf complex minimally resolves lattice ideals associated to generic lattices. For any fixed k and a generic lattice L, is there a generalisation of the Scarf complex to a complex that minimally resolves $M_L^{(k)}$ as an $S[L]$-module?

- **Generalised Frobenius Numbers of Laplacian Lattices:** Let G be a labelled graph. Recall that the Laplacian matrix $Q(G)$ is the matrix $D - A$ where D is the diagonal matrix $\mathrm{diag}(\mathrm{val}(v_1), \ldots, \mathrm{val}(v_n))$ where $\mathrm{val}(v_i)$ is the valency of the vertex v_i and A is the vertex-vertex adjacency matrix. The Laplacian lattice L_G of G is the lattice generated by the rows of the Laplacian matrix. This is a finite index sublattice of the root lattice $A_{n-1} = (1, \ldots, 1)^{\perp} \cap \mathbb{Z}^n$ of index equal to the number of spanning trees of G. We know from [9] that the first Frobenius number of L_G is equal to the genus of the graph. The genus of the graph is its first Betti number as a simplicial complex of dimension one and is equal to $m - n + 1$ where m is the number of edges. Is this there a generalisation of this interpretation to generalised Frobenius numbers?

  Arithmetical graphs are generalisations of graphs motivated by applications from arithmetic geometry, see Lorenzini [62]. Lorenzini associated a Laplacian lattice to an arithmetical graph and defines its genus as the first Frobenius number of its Laplacian lattice. He studies it in the context of the Riemann–Roch theorem. The generalised Frobenius numbers of Laplacian lattices associated to arithmetical graphs seems another fruitful future direction.

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