ON THE CONTINUITY OF THE INTEGRATED DENSITY OF STATES IN THE DISORDER

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Abstract. Recently, Hislop and Marx studied the dependence of the integrated density of states on the underlying probability distribution for a class of discrete random Schrödinger operators, and established a quantitative form of continuity in weak* topology. We develop an alternative approach to the problem, based on Ky Fan inequalities, and establish a sharp version of the estimate of Hislop and Marx. We also consider a corresponding problem for continual random Schrödinger operators on $\mathbb{R}^d$.

1. Introduction

Recently, Hislop and Marx [5] studied the dependence of the integrated density of states (IDS) of random Schrödinger operators on the distribution of the potential.

Let $\{V(n)\}_{n\in\mathbb{Z}^d}$ be independent identically distributed random variables (i.i.d.r.v.) with the common probability distribution μ. Let H be the random Schrödinger operator acting on $\ell^2(\mathbb{Z}^d)$ by

$$H = -\Delta + V, \quad (H\psi)(n) = \sum_{m \text{ is adjacent to } n} (\psi(n) - \psi(m)) + V(n)\psi(n). \tag{1.1}$$

The IDS corresponding to the operator H is the function

$$N(E) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \# \{\text{eigenvalues of } H_\Lambda \text{ in } (-\infty, E]\}, \tag{1.2}$$

where H_Λ is the restriction of H to a finite box $\Lambda \subset \mathbb{Z}^d$, i.e. $H_\Lambda = P_\Lambda H P_\Lambda^*$, where $P_\Lambda : \ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda)$ is the coordinate projection ((1.2) holds with probability 1). The measure with cumulative distribution function N is denoted by ρ.

To discuss the dependence of ρ on the distribution of the potential μ, we introduce two metrics on the space of Borel probability measures on $\mathbb{R}$.

The Kantorovich-Rubinstein (Wasserstein) metric is defined via

$$\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}) = \sup \left\{ \left| \int f \, d\mu - \int f \, d\tilde{\mu} \right| \; : \; f : \mathbb{R} \to \mathbb{R} \text{ is 1-Lipschitz} \right\}. \tag{1.3}$$

By the Kantorovich-Rubinstein duality theorem

$$\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}) = \inf\{\mathbb{E}|X - \tilde{X}|\}, \tag{1.4}$$

where the infimum is taken over $\mathbb{R}^2$-valued random variables $(X, \tilde{X})$, such that $X \sim \mu, \tilde{X} \sim \tilde{\mu}$. Following [5], we also consider the bounded Lipschitz metric, defined by

$$\mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu}) = \sup\left\{ \left| \int f\, d\mu - \int f\, d\tilde{\mu} \right| \,:\, f : \mathbb{R} \to \mathbb{R} \text{ is 1-Lip, } \|f\|_\infty \leq 1 \right\}. \tag{1.5}$$

Our definition differs from [5] by a multiplicative constant. Observe that

$$\mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu}) \leq \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}),$$

and if $\operatorname{supp}\mu, \operatorname{supp}\tilde{\mu} \subset [-A, A]$, then

$$\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}) \leq \max(A, 1)\mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu}).$$

In this notation Theorem 1.1 of [5] (formulated here in slightly less general setting than in the cited work) asserts the following.

Theorem (Hislop–Marx)**.** *Suppose $H, \tilde{H}$ are random Schrödinger operators of the form (1.1) with potentials $\{V(n)\}, \{\tilde{V}(n)\}$ sampled from a probability distributions $\mu, \tilde{\mu}$, respectively. Denote by $N, \tilde{N}$ the IDS corresponding to $H, \tilde{H}$, and let $\rho, \tilde{\rho}$ be the measures with cumulative distribution functions $N, \tilde{N}$, respectively. If $\operatorname{supp}\mu, \operatorname{supp}\tilde{\mu} \subset [-A, A]$, then*

$$\mathrm{d}_{\mathrm{BL}}(\rho, \tilde{\rho}) \leq C_A \mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu})^{1/(1+2d)}, \tag{1.6}$$

$$\sup |N(E) - \tilde{N}(E)| \leq \frac{C_A}{\log_+ \frac{1}{\mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu})}}, \tag{1.7}$$

where C_A depends only on A.

We refer to [5] for a discussion of earlier work on the subject, and only mention the result [4] on the continuity of the integrated density of states as a function of the coupling constant.

Hislop and Marx [5] presented several applications, particularly, to the continuity of the Lyapunov exponent of a one-dimensional operator as a function of the underlying distribution of the potential. The proof of the theorem in [5] is based on the approximation of the function f (in (1.5)) by polynomials.

We suggest a different approach to estimates of the form (1.6) using the Ky Fan inequalities. Our first result is the following theorem.

Theorem 1. *Suppose $H, \tilde{H}$ are random Schrödinger operators of the form (1.1), where $\{V(n)\}, \{\tilde{V}(n)\}$ are i.i.d.r.v. distributed accordingly to $\mu, \tilde{\mu}$ respectively. Let $N, \tilde{N}$ be the IDS corresponding to $H, \tilde{H}$, and let $\rho, \tilde{\rho}$ be the measures with cumulative distribution functions $N, \tilde{N}$ respectively. Then,*

$$\mathrm{d}_{\mathrm{KR}}(\rho, \tilde{\rho}) \leq \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}), \tag{1.8}$$

$$\sup_E |N(E) - \tilde{N}(E)| \leq \frac{C}{\log_+ \frac{1}{\mathrm{d}_{\mathrm{KR}}(\mu,\tilde{\mu})}}, \tag{1.9}$$

where $C > 0$ is a numerical constant.

Remark 1.1. *The power* 1 *as well as the prefactor* 1 *in (1.8) are optimal in general.*

Remark 1.2. *This result can be extended to other models in which the potential is of the form*

$$\sum v_j P_j, \tag{1.10}$$

where v_j are i.i.d.r.v. with common Borel distribution supported on a finite interval and P_j are finite rank projections (see [5]).

Remark 1.3. *Theorem 1 can be extended to different underlying lattices, since the proof does not rely on the structure of $\mathbb{Z}^d$.*

In the follow up paper [6], Hislop and Marx prove a version of their results for the continual Anderson model, which is *not* of the form (1.10). A modification of our argument can be applied to the continual setting as well. We illustrate it by the following theorem.

Let H be a random Schrödinger operator acting on $L^2(\mathbb{R}^d)$, defined by

$$H = -\Delta + V, \tag{1.11}$$

where the potential V is of the form

$$V(x) = \sum_{j\in\mathbb{Z}^d} v_j u(x-j), \;\; x \in \mathbb{R}^d, \tag{1.12}$$

where v_j are i.i.d.r.v. distributed accordingly to μ, and u is real-valued continuous compactly supported function: $u \in C_c(\mathbb{R})$. Denote by Λ the cube of side length L around the origin

$$\Lambda = \left[-\frac{L}{2}, \frac{L}{2}\right]^d.$$

Let H_Λ be the restriction of H to $L^2(\Lambda)$ with Dirichlet boundary conditions. Define the IDS corresponding to H similarly to (1.2)

$$N(E) = \lim_{L\to\infty} \frac{1}{L^d} \# \{\text{eigenvalues of}\, H_\Lambda \,\text{in}\, (-\infty, E]\}, \tag{1.13}$$

and let ρ be the measure with cumulative distribution function N.

Theorem 2. *Suppose $H, \tilde{H}$ are random Schrödinger operators of the form (1.11), and suppose that* $\operatorname{supp} \mu, \operatorname{supp} \tilde{\mu} \subset \mathbb{R}_+$ *and $u \geq 0$. Let $N, \tilde{N}$ be the*

IDS corresponding to $H, \tilde{H}$, *and let* $\rho, \tilde{\rho}$ *be the measures with cumulative distribution functions* $N, \tilde{N}$ *respectively. If* $\alpha > \frac{d}{2} - 1$, *then*

$$\left| \int f\left(\frac{1}{(1+E)^\alpha}\right) d\rho(E) - \int f\left(\frac{1}{(1+E)^\alpha}\right) d\tilde{\rho}(E) \right| \leq C(d,u,\alpha) \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}), \tag{1.14}$$

for any 1*-Lipschitz function* f *for which* $\int f\left(\frac{1}{1+E}\right) d\rho(E)$ *converges.*

If $d = 1, 2, 3$ *and* $\operatorname{supp} \mu \subset [0, A]$ *then for any* $E_0 \in \mathbb{R}$

$$\sup_{E \leq E_0} |N(E) - \tilde{N}(E)| \leq \frac{C(d, E_0, A)}{\log_+^{\kappa_d} \frac{1}{\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})}}, \tag{1.15}$$

where $\kappa_1 = 1, \kappa_2 = 1/4, \kappa_3 = 1/8$.

Remark 1.4. *The following example shows that the condition* $\alpha > \frac{d}{2} - 1$ *is optimal in general, and in particular one can not expect a result of the same form as in the discrete case (which would correspond to* $\alpha = -1$*).*

Assume $\alpha \leq \frac{d}{2} - 1$. *Let* u *be such that* $\sum_{j \in \mathbb{Z}^d} u(x - j) \equiv 1$, *and let* $v_j \equiv 0, \tilde{v}_j \equiv \delta, f_\epsilon(x) = \max((x - \epsilon), 0)$. *The integrated density of states of the free Lapacian is given by*

$$d\rho(\lambda) = C_d \lambda^{\frac{d}{2}-1} d\lambda.$$

Therefore we have

$$\int f((1+\lambda)^{-\alpha}) d\rho(\lambda) = C_d \int_0^{\epsilon^{-1/\alpha}-1} ((1+\lambda)^{-\alpha} - \epsilon) \lambda^{\frac{d}{2}-1} d\lambda, \tag{1.16}$$

$$\int f((1+\lambda)^{-\alpha}) d\tilde{\rho}(\lambda) = C_d \int_0^{\epsilon^{-1/\alpha}-1-\delta} ((1+\lambda+\delta)^{-\alpha} - \epsilon) \lambda^{\frac{d}{2}-1} d\lambda, \tag{1.17}$$

and for any $\delta > 0$ *we obtain*

$$\begin{aligned}
&\liminf_{\epsilon \to 0} \left| \int f((1+\lambda)^{-\alpha}) d\rho(\lambda) - \int f((1+\lambda)^{-\alpha}) d\tilde{\rho}(\lambda) \right| \\
&\geq \liminf_{\epsilon \to 0} C_d \int_0^{\epsilon^{-1/\alpha}-1-\delta} ((1+\lambda)^{-\alpha} - (1+\lambda+\delta)^{-\alpha}) \lambda^{\frac{d}{2}-1} d\lambda \\
&\geq \liminf_{\epsilon \to 0} C_d \alpha \delta \int_0^{\epsilon^{-1/\alpha}-1-\delta} (1+\lambda+\delta)^{-\alpha-1} \lambda^{\frac{d}{2}-1} d\lambda = \infty.
\end{aligned}$$

Remark 1.5. *The restrictions on the dimension and on* V *in the second part of Theorem 2 come from the work of Bourgain and Klein [2] which we use to deduce (1.15) from (1.14).*

Remark 1.6. *The restriction* $\operatorname{supp} \mu \subset [0, A]$*, also coming from [2], can be relaxed using the work of Klein and Tsang [7, Theorem 1.3].*

Remark 1.7. *Theorem 2 formally implies a similar result for sign-indefinite* V *bounded from below.*

2. Preliminaries

2.1. **Discrete case.** The main ingredient of the proof of Theorem 1 is the Ky Fan inequality [8]:

Assume that A, B, and $\tilde{A} = A + B$ are linear self-adjoint operators that act on n-dimensional Euclidean space. Let $\lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_1$, $e_n \le e_{n-1} \le \cdots \le e_1$, $\tilde{\lambda}_n \le \tilde{\lambda}_{n-1} \le \cdots \le \tilde{\lambda}_1$ be the eigenvalues of A, B, and $\tilde{A}$ respectively. Then, for any continuous convex function $\phi : \mathbb{R} \to \mathbb{R}$

$$\sum_{j=1}^{n} \phi(\tilde{\lambda}_j - \lambda_j) \le \sum_{j=1}^{n} \phi(e_j). \tag{2.1}$$

In particular,

$$\sum_{j=1}^{n} |\tilde{\lambda}_j - \lambda_j| \le \sum_{j=1}^{n} |e_j|. \tag{2.2}$$

To deduce (1.9) from (1.8) (similarly to [5]) we shall use the following result due to Craig and Simon [3]. Denote by

$$\omega(\delta) = \sup\{|\rho(E) - \rho(E')| \ : \ E' < E \le E + \delta\}, \tag{2.3}$$

the modulus of continuity of ρ. Then ([3]) the measure ρ with the cumulative distribution function N (the IDS) of any ergodic Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ is log-Hölder continuous, namely, for any $\delta \in (0, \frac{1}{2}]$

$$\omega(\delta) \le \frac{C}{\log \frac{1}{\delta}}, \tag{2.4}$$

where $C > 0$ is a universal constant.

2.2. **Continual case.** First, recall that for $1 \le p < \infty$ the Schatten class S_p is the class of all compact operators in a given Hilbert space such that

$$\|A\|_p = \left(\sum_{n=1}^{\infty} s_n(A)^p \right)^{1/p} < \infty,$$

where $\{s_n(A)\}$ is the sequence of all singular values of the operator A enumerated with multiplicities taken into account. The class S_∞ consists of all compact operators.

The main ingredient in the proof of Theorem 2 is the following version of the Ky Fan inequality (see Markus [9]).

If $A \in S_1$, $B \in S_\infty$ that are self-adjoint, and $\tilde{A} = A + B$, $\lambda_1 \ge \lambda_2 \ge \cdots$, $e_1 \ge e_2 \ge \cdots$, $\tilde{\lambda}_1 \ge \tilde{\lambda}_2 \ge \cdots$, are the eigenvalues of A, B, and $\tilde{A}$ respectively, then, for any continuous convex function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$

$$\sum_{j=1}^{\infty} \phi(\tilde{\lambda}_j - \lambda_j) \le \sum_{j=1}^{\infty} \phi(e_j). \tag{2.5}$$

In particular,

$$\sum_{j=1}^{\infty} |\tilde{\lambda}_j - \lambda_j| \le \sum_{j=1}^{\infty} |e_j| = \|B\|_1. \tag{2.6}$$

To deduce (1.15) from (1.14) we will need the following result due to Bourgain and Klein [2].

Theorem (BK)**.** *Assume that H as in (1.11)–(1.12) on $L^2(\mathbb{R}^d)$, $d = 1, 2, 3$, with* $\operatorname{supp} \mu \subset [-A, A]$*. Let N be the corresponding IDS. Then, given $E_0 \in \mathbb{R}$, for all $E \le E_0$ and $\delta \le 1/2$*

$$|N(E) - N(E+\delta)| \le \frac{C(d, E_0, A)}{\log^{\kappa_d} \frac{1}{\delta}}, \tag{2.7}$$

where $C(d, E_0, A) > 0$, and $\kappa_1 = 1, \kappa_2 = 1/4, \kappa_3 = 1/8$.

3. Proof of Theorem 1 and Theorem 2

3.1. **Proof of Theorem 1.** Denote by $\Lambda \subset \mathbb{Z}^d$ a finite box and let (in the notation of Ky Fan's inequality)

$$A = H_\Lambda = (-\Delta + V)_\Lambda, \;\; \tilde{A} = \tilde{H}_\Lambda = (-\Delta + \tilde{V})_\Lambda,$$

be the restrictions of the operators H and $\tilde{H}$ to the box Λ. Then,

$$\begin{aligned} |\mathrm{tr} f(A) - \mathrm{tr} f(\tilde{A})| &= |\sum_{j=1}^{|\Lambda|} f(\lambda_j) - \sum_{j=1}^{|\Lambda|} f(\tilde{\lambda}_j)| \\ &\le \sum_{j=1}^{|\Lambda|} |f(\lambda_j) - f(\tilde{\lambda}_j)| \le \sum_{j=1}^{|\Lambda|} |\lambda_j - \tilde{\lambda}_j| \\ &\le \sum_{j=1}^{|\Lambda|} |e_j| = \sum_{x \in \Lambda} |V(x) - \tilde{V}(x)|, \end{aligned} \tag{3.1}$$

where the second inequality holds since f is 1-Lipschitz and the last inequality follows from (2.1).

By (1.4) there is a realization of V and $\tilde{V}$ on a common probability space such that

$$\mathbb{E}|V(x) - \tilde{V}(x)| \le \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}).$$

Thus, using (3.1) for any 1-Lipschitz function f, we obtain

$$|\mathbb{E}\,\mathrm{tr}\, f(A) - \mathbb{E}\,\mathrm{tr}\, f(\tilde{A})| \le \mathbb{E} \sum_{x \in \Lambda} |V(x) - \tilde{V}(x)| \le |\Lambda| \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}). \tag{3.2}$$

Since

$$\int f d\rho = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathbb{E}\,\mathrm{tr}\, f(A),$$

we obtain by passing to the limit $\Lambda \nearrow \mathbb{Z}^d$

$$\mathrm{d_{KR}}(\rho,\tilde{\rho}) \leq \mathrm{d_{KR}}(\mu,\tilde{\mu}), \tag{3.3}$$

thus concluding the proof of (1.8).

To deduce (1.9), we choose

$$f(x) = \begin{cases} \delta, & x \leq E \\ -x+E+\delta, & E \leq x \leq E+\delta \\ 0, & x \geq E+\delta, \end{cases} \tag{3.4}$$

for $\delta > 0$. Then, by definition of the IDS, we get for any $E \in \mathbb{R}$

$$\delta N(E) \leq \int f(E)d\rho(E) \leq \delta N(E+\delta), \tag{3.5}$$

$$\delta\tilde{N}(E) \leq \int f(E)d\tilde{\rho}(E) \leq \delta\tilde{N}(E+\delta). \tag{3.6}$$

Since

$$\int f(E)d\tilde{\rho}(E) = \int f(E)d\rho(E) + \int f(E)d(\tilde{\rho}-\rho)(E),$$

combining (3.3), (3.5), and (3.6), we obtain

$$\delta\tilde{N}(E) \leq \delta N(E+\delta) + \mathrm{d_{KR}}(\mu,\tilde{\mu}),$$

namely

$$\tilde{N}(E) \leq N(E+\delta) + \frac{\mathrm{d_{KR}}(\mu,\tilde{\mu})}{\delta}. \tag{3.7}$$

In the same way we get

$$\tilde{N}(E) \geq N(E-\delta) - \frac{\mathrm{d_{KR}}(\mu,\tilde{\mu})}{\delta}. \tag{3.8}$$

Let ω be the modulus of continuity of N. Combining (2.4), (3.7), and (3.8), we obtain

$$\sup_E |N(E)-\tilde{N}(E)| \leq \inf_\delta \left(\omega(\delta) + \frac{\mathrm{d_{KR}}(\mu,\tilde{\mu})}{\delta}\right) \leq \frac{C}{\log_+ \frac{1}{\mathrm{d_{KR}}(\mu,\tilde{\mu})}}, \tag{3.9}$$

where $C > 0$ is a constant and we choose $\delta = \mathrm{d_{KR}}(\mu,\tilde{\mu})/\omega(\mathrm{d_{KR}}(\mu,\tilde{\mu}))$. This finishes the proof of (1.9).

□

Remark 3.1. *If the operator H is such that the modulus of continuity ω satisfies*

$$\omega(\delta) \leq C\delta^a,$$

for some $C, a > 0$, then (3.9) implies that

$$\sup_E |N(E) - \tilde{N}(E)| \leq \inf_\delta \left(C\delta^a + \frac{\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})}{\delta} \right) \leq \tilde{C}\, \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})^{1/(1+a)}.$$

3.2. **Proof of Theorem 2.** Let

$$H_\Lambda = (-\Delta + V)_\Lambda, \ \tilde{H}_\Lambda = (-\Delta + \tilde{V})_\Lambda,$$

be the restrictions of the operators H and $\tilde{H}$ to a finite box $\Lambda \in \mathbb{R}^d$ with Dirichlet boundary conditions. Let (in the notation of Ky Fan's inequality)

$$A = (H_\Lambda + 1)^{-\alpha}, \ \tilde{A} = (\tilde{H}_\Lambda + 1)^{-\alpha}.$$

We have the following

Claim 3.2. *If $\alpha > \frac{d}{2} - 1$, then*

$$\|A - \tilde{A}\|_1 \leq C(d, u, \alpha) \sum_{j \in 2\Lambda \cap \mathbb{Z}^d} |v_j - \tilde{v}_j| \ .$$

Proof. Let us number

$$2\Lambda \cap \mathbb{Z}^d = \{j_1, \ldots, j_n\}, \ \ n \leq C|\Lambda| \ ,$$

and let $H_\Lambda^{(k)}$, $0 \leq k \leq n+1$, be the (restricted) operator corresponding to the potential

$$V(x) = \sum_{l<k} \tilde{v}_{j_l} u(x - j_l) + \sum_{l \geq k} v_{j_l} u(x - j_l) \ .$$

Observe that

(3.10)
$$\|(H_\Lambda^{(k)} + 1)^{-\alpha} - (H_\Lambda^{(k+1)} + 1)^{-\alpha}\|_1 \leq$$
$$|\alpha| \left(\left\| \frac{1}{(H_\Lambda^{(k)} + 1)^{\alpha+1}} (H_\Lambda^{(k)} - H_\Lambda^{(k+1)}) \right\|_1 + \left\| (H_\Lambda^{(k)} - H_\Lambda^{(k+1)}) \frac{1}{(H_\Lambda^{(k+1)} + 1)^{\alpha+1}} \right\|_1 \right) \ ,$$

as follows, for example, from the Birman-Solomyak formula [1, Theorem 8.1] (note that for $\alpha = 1$ it suffices to use the second resolvent identity). Then we have

$$\begin{aligned}\left\|\frac{1}{(H_\Lambda^{(k)}+1)^{\alpha+1}}u_{j_k}\right\|_1 &= \left\|\frac{1}{(H_\Lambda^{(k)}+1)^{\frac{\alpha+1}{2}}}\sqrt{u_{j_k}}\sqrt{u_{j_k}}\frac{1}{(H_\Lambda^{(k)}+1)^{\frac{\alpha+1}{2}}}\right\|_1\\ &= \left\|\sqrt{u_{j_k}}\frac{1}{(H_\Lambda^{(k)}+1)^{\frac{\alpha+1}{2}}}\right\|_2^2\\ &= \left\|{u_{j_k}}^{1/4}\frac{1}{(H_\Lambda^{(k)}+1)^{\frac{\alpha+1}{2}}}{u_{j_k}}^{1/4}\right\|_2^2\\ &\le \left\|{u_{j_k}}^{1/4}\frac{1}{(-\Delta_\Lambda+1)^{\frac{\alpha+1}{2}}}{u_{j_k}}^{1/4}\right\|_2^2\\ &\le \|u\|_\infty\left\|\mathbf{1}_{\text{supp }u}\frac{1}{(-\Delta_\Lambda+1)^{\frac{\alpha+1}{2}}}\mathbf{1}_{\text{supp }u}\right\|_2^2,\end{aligned}$$

where the first inequality follows from the positivity of the potential. A similar bound holds for the second term of (3.10). By Weyl's law the last norm is bounded by $C(d,u,\alpha)$ (uniformly in Λ) whenever $\alpha+1>\frac{d}{2}$. Thus, we obtain

$$\|(H_\Lambda^{(k)}+1)^{-\alpha}-(H_\Lambda^{(k+1)}+1)^{-\alpha}\|_1\le C(d,u,\alpha)|v_{j_k}-\tilde v_{j_k}|.$$

□

By the Kantorovich-Rubinstein duality (1.4) there is a realization of v and $\tilde v$ on a common probability space such that

$$\mathbb{E}|v_j-\tilde v_j|\le \mathrm{d}_{\mathrm{KR}}(\mu,\tilde\mu).$$

Thus, using Claim 3.2, we get for $\alpha>\frac{d}{2}-1$

$$\begin{aligned}\mathbb{E}\|A-\tilde A\|_1 &\le C(d,u,\alpha)\mathbb{E}\sum_{j\in 2\Lambda\cap\mathbb{Z}^d}|v_j-\tilde v_j| = C(d,u,\alpha)\sum_{j\in 2\Lambda\cap\mathbb{Z}^d}\mathbb{E}|\tilde v_j-v_j|\\ &\le C(d,u,\alpha)|\Lambda|\mathrm{d}_{\mathrm{KR}}(\mu,\tilde\mu).\end{aligned}$$

The eigenvalues of A are exactly $\frac{1}{(1+\lambda_j)^\alpha}$, where λ_j are the eigenvalues of H_Λ, thus using (2.6), we obtain for $\alpha>\frac{d}{2}-1$ and for any 1-Lipschitz function f for which $\int f\left(\frac{1}{(1+E)^\alpha}\right)d\rho(E)$ converges (in particular, $f(0)=0$)

$$\begin{aligned}
&\left|\sum_{j=1}^{\infty} f\left(\frac{1}{(1+\lambda_j)^\alpha}\right) - \sum_{j=1}^{\infty} f\left(\frac{1}{(1+\tilde{\lambda}_j)^\alpha}\right)\right| \\
&\leq \sum_{j=1}^{\infty} \left| f\left(\frac{1}{(1+\lambda_j)^\alpha}\right) - f\left(\frac{1}{(1+\tilde{\lambda}_j)^\alpha}\right)\right| \\
&\leq \sum_{j=1}^{\infty} \left| \frac{1}{(1+\lambda_j)^\alpha} - \frac{1}{(1+\tilde{\lambda}_j)^\alpha}\right| \leq C(u,d,\alpha)\,|\Lambda|\,\mathrm{d}_{\mathrm{KR}}(\mu,\tilde{\mu}),
\end{aligned} \tag{3.11}$$

where the last step follows from the Ky Fan inequality. Using the definition (1.13) of ρ and passing to the limit $\Lambda \nearrow \mathbb{R}^d$, we conclude that if $\alpha > \frac{d}{2} - 1$, then

$$\left|\int f\left(\frac{1}{(1+E)^\alpha}\right) d\rho(E) - \int f\left(\frac{1}{(1+E)^\alpha}\right) d\tilde{\rho}(E)\right| \leq C(d,u,\alpha)\mathrm{d}_{\mathrm{KR}}(\mu,\tilde{\mu}) \ . \tag{3.12}$$

Thus we complete the proof of (1.14).

To deduce (1.15), we define

$$f(x) = \begin{cases} \frac{\delta}{2(1+E)^2}, & x \geq \frac{1}{1+E} \\ \frac{1+E+\delta}{2(1+E)}x - \frac{1}{2(1+E)}, & \frac{1}{1+E+\delta} \leq x \leq \frac{1}{1+E} \\ 0, & x \leq \frac{1}{1+E+\delta}, \end{cases} \tag{3.13}$$

for $\delta > 0$. Then, by the definition of the IDS (1.13) we get for a fixed $E_0 \in \mathbb{R}$ in the same way as in the proof of Theorem 1

$$\tilde{N}(E_0) \leq N(E_0+\delta) + \frac{2\mathrm{d}_{\mathrm{KR}}(\mu,\tilde{\mu})\,(1+E_0)^2}{\delta}, \tag{3.14}$$

$$\tilde{N}(E_0) \geq N(E_0-\delta) - \frac{2\mathrm{d}_{\mathrm{KR}}(\mu,\tilde{\mu})\,(1+E_0)^2}{\delta}. \tag{3.15}$$

Let ω be the modulus of continuity of N. Then, by (2.7) for any $E \leq E_0$ and $\delta \leq 1/2$

$$|N(E) - N(E+\delta)| \leq \frac{C(d,E_0,A)}{\log_+^{\kappa_d}\frac{1}{\delta}}.$$

Thus, choosing $\delta = \frac{C(d,E_0,A)\,\mathrm{d}_{\mathrm{KR}}(\mu,\tilde{\mu})}{\omega(C(d,E_0,A)\,\mathrm{d}_{\mathrm{KR}}(\mu,\tilde{\mu}))}$, we obtain

$$\sup_{E\leq E_0} |\tilde{N}(E) - N(E)| \leq \frac{\tilde{C}(d,E_0,A)}{\log_+^{\kappa_d}\frac{1}{\mathrm{d}_{\mathrm{KR}}(\mu,\tilde{\mu})}},$$

therefore completing the proof.

□

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