LOCAL LIMIT THEOREMS FOR RANDOM WALKS IN A RANDOM ENVIRONMENT ON A STRIP

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ABSTRACT. The paper consists of two parts. In the first part we review recent work on limit theorems for random walks in random environment (RWRE) on a strip with jumps to the nearest layers. In the second part, we prove the quenched Local Limit Theorem (LLT) for the position of the walk in the transient diffusive regime. This fills an important gap in the literature. We then obtain two corollaries of the quenched LLT. The first one is the annealed version of the LLT on a strip. The second one is the proof of the fact that the distribution of the environment viewed from the particle (EVFP) has a limit for a. e. environment. In the case of the random walk with jumps to nearest neighbours in dimension one, the latter result is a theorem of Lally [19]. Since the strip model incorporates the walks with bounded jumps on a one-dimensional lattice, the second corollary also solves the long standing problem of extending Lalley's result to this case.

1. INTRODUCTION.

This paper is devoted to the local limiting behavior of a random walk in random environment on a strip. The study of RWRE was initiated in [29]. The annealed limit theorems for one dimensional RWRE go back to [18] who have analyzed the transient case. In the recurrent case, Sinai proved in [27] a surprising fact: the correct scaling in the limit theorem is $\ln^2 t$.

Results describing the behavior of RWs in quenched (frozen) random environments are relatively recent. The quenched CLT for the simple transient RWRE in the diffusive regime [1] was proved in [14] in 2007 for a wide class of environments (including the iid case) and, independently, for iid environments in [20].

It should be emphasized that in the diffusive regime the CLT holds for almost all environments and that there is a drastic difference between the diffusive and subdiffusive regime. Namely, it has been realized in [21, 24] that in the subdiffusive regime, for almost every environment, the simple RW does not have a distributional limit. In fact, the quenched distribution of the walk turns out to be quite non-trivial, see [7, 22, 23] for detailed

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[1]The precise definition of the diffusivity condition will be explained later (cf. Theorem 2.4(c)).

discussion of this problem (related results were obtained independently in [13]).

The papers [29, 18] as well as [27] relied heavily on the fact that the walk is on a one-dimensional lattice and can jump only to the nearest neighbors (the so called simple walk). Therefore, the natural question asked by Sinai in [27] was how to extend the results about nearest neighbor RWRE on the line to walks with bounded jumps. The key tools needed to achieve such a generalization were developed in [2] which introduced a more general model, namely the RWRE on a strip. This model includes the RWRE with bounded jumps on a line as a special case. The results of [2] were instrumental for proving limit theorems for RWRE on the strip which was done in [15] (transient diffusive walks), [3] (recurrent walks), and [8] (transient subdiffusive walks).

The next step was to prove the local limit theorem in the diffusive regime. In the case of the simple RWRE the quenched LLT was proved in [9]. We would like to mention two corollaries of this result: it implies the annealed LLT, and the LLT is the main ingredient in a new proof of a result of Lalley stating the existence of the limit of the distribution of the environment viewed from the particle (EVFP). Clearly, both the LLT and the EVFP measure characterize the local limiting behaviour of a RW but the strong connection between the two is a newly discovered phenomena. Similar results for recurrent diffusive walks were obtained in [10].

The aim of this paper is twofold. First, we review the recent results about RWRE on the strip placing a particular emphasis on the LLT and on mixing of the EVFP process. Secondly, we extend the main results from [9] to the case of the strip. We prove that in the transient diffusive regime the quenched LLT holds for almost every (a.e.) environment (Theorem 2.8). It may be useful to emphasized that, unlike in the CLT, one has to have an additional random factor ρ_n in front of the exponent which is due to the randomness of the environment. As in [9], the quenched LLT implies the annealed one (Theorem 2.9).

Finally we prove that in the diffusive regime the limit of the distribution of the environment viewed from the particle exists for a. e. environment (Theorem 2.13). Our proof of this result is completely different from the one given in [19].

Since the study of random walks with bounded jumps in dimension one can be reduced to those on a strip with jumps to nearest layers (see [2, 15]), this solves the problem of extending the Lalley's result to the walks on $\mathbb{Z}$ with bounded jumps.

The layout of the paper is the following. Section 2 contains a review of RWRE on the strip. We first introduce the main tools needed for analysis of these walks and then describe the limit theorems in that setting. The new results pertaining to diffusive transient walks are proved in Sections 3–7. Section 3 contains auxiliary facts needed in our analysis. In Section 4 we obtain bounds on moderate deviation of occupation times for diffusive

walks. Occupation times play an important role in the analysis of transient walks since traps which are behind several interesting phenomena related to one dimensional RWRE can be conveniently described as sites with high expected occupation times (see e.g. [8]). Section 5 contains the main ingredient of our analysis–the LLT for the hitting times (Theorem 2.7) which is a new result and an important fact in its own right. Following the approach of [9] we deduce the LLT for the walker position in Section 6 and establish mixing of EVFP process in Section 7.

Since our paper is devoted to the RW on a strip in i.i.d. random environment (see (2.3)), several recent results on RWRE are not discussed here. In particular we do not treat one dimensional RWRE in dependent environment. A review of this subject can be found in [6]. We just note that the first LLT in the quasiperiodic setting is due to Sinai [28]. We also would like to mention a significant progress in the study of LLT for multidimensional RWRE, in both transient [1] and recurrent cases [4]. We note that while there are similarities in the approaches of [1, 9] and the present paper, the details in multidimensional and one dimensional cases are quite different. To a large extent, this difference is due to the fact that in dimension one in the ballistic regime the walker could spend much longer time at an individual site than in similar conditions in higher dimensions.

2. RWRE ON A STRIP: THE MODEL AND REVIEW OF RESULTS.

2.1. **Definition of the model.** The RWRE on a strip $\mathbb{S} \overset{\text{def}}{=} \mathbb{Z} \times \{1, \ldots, m\}$ was introduced in [2]. We shall now recall its definition. The set $L_n \overset{\text{def}}{=} \{(n, j) : 1 \le j \le m\} \subset \mathbb{S}$ is called *layer n* of the strip (or just layer n). The walker can jump from a site in L_n only to a site in L_{n-1}, L_n, or L_{n+1}. Let $\xi_t = (X_t, Y_t)$ be the coordinate of the walk at time t, $t = 0, 1, 2, \ldots$, with $X_t \in \mathbb{Z}$, $1 \le Y_t \le m$. An environment ω on a strip is a sequence of triples of $m \times m$ matrices $\omega = \{(P_n, Q_n, R_n)\}_{n \in \mathbb{Z}}$ with non-negative matrix elements and such that $P_n + Q_n + R_n$ is a stochastic matrix:

$$(2.1) \qquad\qquad (P_n + Q_n + R_n)\mathbf{1} = \mathbf{1},$$

where $\mathbf{1}$ is a vector whose all components are equal to 1. The transition kernel of the walk is given by

$$(2.2) \qquad \mathbb{P}_\omega(\xi_{t+1} = z' | \xi_t = z) = \begin{cases} P_n(i, j) & \text{if } z = (n, i), z' = (n+1, j), \\ Q_n(i, j) & \text{if } z = (n, i), z' = (n-1, j), \\ R_n(i, j) & \text{if } z = (n, i), z' = (n, j) \end{cases}$$

Setting $\xi(0) = z_0 \in L_0$ completes the definition of the Markov chain. Let us introduce the relevant probability spaces. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the space of random environments, and $(\mathfrak{X}_z, \mathcal{F}_{\mathfrak{X}_z}, \mathbb{P}_{\omega,z})$ be the space of the trajectories of the walk starting from $z \in S$; $\mathrm{P}_z := \mathbf{P} \ltimes \mathbb{P}_{\omega,z}$ for the *annealed* probability measure on $(\Omega \times \mathfrak{X}_z, \mathcal{F} \times \mathcal{F}_{\mathfrak{X}_z})$. The expectations over $\mathbb{P}_{\omega,z}$, $\mathbf{P}$, and P_z will

be denoted by $\mathbb{E}_{\omega,z}$, $\mathbf{E}$, and E_z respectively. We use concise notation $\mathbb{P}_\omega$ and P in place of $\mathbb{P}_{\omega,z_0}$ and P_{z_0} when their meaning is obvious from the context.

The above model reduces to the simple RWRE on $\mathbb{Z}$ when $m = 1$. It also incorporates the RWRE on $\mathbb{Z}$ with bounded jumps. The description of the corresponding reduction can be found in [8] (see also [2], [15]) and will not be repeated here.

In most cases we suppose that the following conditions are satisfied:

$$(2.3) \qquad \{(P_n, Q_n, R_n)\}_{n\in\mathbb{Z}} \text{ is an i.i.d. sequence,}$$

(2.4)
There is $\varepsilon > 0$ such that $\mathbf{P}$-almost surely for all $i, j \in [1, m]$

$$\|R_n\| < 1 - \varepsilon, \;\; ((I - R_n)^{-1}P_n)(i, j) > \varepsilon, \;\; ((I - R_n)^{-1}Q_n)(i, j) > \varepsilon,$$

(2.5)
There is a $\kappa > 0$ such that $\mathbf{P}$-almost surely $R_n(i, i) \geq \kappa$ for all $i \in [1, m]$.

2.2. **Preparatory statements.**

A fundamental role in the study of RWRE on a strip is played by the *moment Lyapunov exponents* $r(\alpha)$ which were first introduced in [15]. In turn, defining $r(\alpha)$ requires introduction of sequences of matrices ψ_n, ζ_n, and A_n which play a very important role in many aspects of the analysis of RWREs on a strip.

Let ψ_a be an $m \times m$ stochastic matrix. For $n > a$ define $\psi_{n,a}$ recursively:

$$(2.6) \qquad \psi_{n,a} \stackrel{\text{def}}{=} (I - R_n - Q_n\psi_{n-1,a})^{-1}P_n$$

It is easy to see that all $\psi_{n,a}$, $n \geq a$, are stochastic matrices ([2, Lemma 2]).

Theorem 2.1. *([2]) Suppose that condition* (2.4) *is satisfied. Then*

(a) For every sequence ω there exists $\zeta_n = \lim\limits_{a\to-\infty} \psi_{n,a}$, where the convergence is uniform in ψ_a and ζ_n does not depend on the choice of the sequence ψ_a.

(b) The sequence $\zeta_n = \zeta_n(\omega)$, $-\infty < n < \infty$, of $m \times m$ matrices is the unique sequence of stochastic matrices which satisfies the following system of equations

$$(2.7) \qquad \zeta_n = (I - Q_n\zeta_{n-1} - R_n)^{-1}P_n, \quad n \in \mathbb{Z}.$$

Set

$$(2.8) \qquad A_n \stackrel{\text{def}}{=} (I - Q_n\zeta_{n-1} - R_n)^{-1}Q_n.$$

By the Multiplicative Ergodic Theorem the following limit exists a.e. and is independent of ω

$$\boldsymbol{\lambda} = \lim_{n\to\infty} \frac{\ln \|A_{n-1}\ldots A_0\|}{n}.$$

Theorem 2.2. *(a) ξ is recurrent iff $\boldsymbol{\lambda} = 0$;*
(b) $\xi_n \to +\infty$ with probability one iff $\boldsymbol{\lambda} < 0$;
(c) $\xi_n \to -\infty$ with probability one iff $\boldsymbol{\lambda} > 0$.

Lemma 2.3. ([15, Lemma 2]) *Suppose that* (2.3) *and* (2.4) *are satisfied. Then the following limit exists and is finite for all* $\alpha \geq 0$

$$(2.9) \qquad r(\alpha) = \lim_{n \to \infty} \left(\mathbf{E} \| A_n \cdots A_1 \|^{\alpha} \right)^{\frac{1}{n}}.$$

The convergence in (2.9) *is uniform in* $\alpha \in [-\alpha_0, \alpha_0]$ *for any* $\alpha_0 > 0$. *Moreover*

$$r(0) = 1, \quad r'(0) = \boldsymbol{\lambda}.$$

$r(\alpha)$ are called *moment Lyapunov exponents*.

Let (P, Q, R) satisfy (2.4). By the foregoing discussion there exists a unique $\zeta = \zeta(P, Q, R)$ such that $\zeta = (I - R - Q\zeta)^{-1}P$. Let $\lambda(P, Q, R)$ be the leading eigenvalue of $(I - R - Q\zeta(P, Q, R))^{-1}Q$. We say that the environment satisfies *non-arithmeticity condition* if the distribution of the random variable $\ln \lambda(P_0, Q_0, R_0)$ is non arithmetic.

Note that if $m = 1$ then $\zeta_n = 1$, $A_n = \lambda(p_n, q_n) = \dfrac{q_n}{p_n}$, and $r(\alpha) =$ $\mathbf{E}\left(\left(\dfrac{q_0}{p_0} \right)^{\alpha} \right)$.

2.3. Annealed Limit Theorems for RWRE on a strip.

In Sections 2.3–2.4, we consider the transient walks. To fix our notation we assume that $\xi_n \to +\infty$ with probability one, that is $\boldsymbol{\lambda} < 0$.

For $s \in (0, 2)$, let $\mathfrak{L}_s$ denote the cumulative distribution function of the sum

$$(2.10) \qquad \mathfrak{t} = \sum_{n=1}^{\infty} \Theta_n (\Gamma_n - \epsilon_s)$$

where Θ_n is the Poisson process on $\mathbb{R}^+$ with the intensity $\dfrac{s}{\theta^{s+1}}$, Γ_n are i.i.d. exponential random variables with parameter 1, $\{\Theta_n\}$ and $\{\Gamma_n\}$ are independent, and $\varepsilon_s = \begin{cases} 0, s < 1 \\ 1, s \geq 1. \end{cases}$ It is not difficult to see (cf. [7]) that $\mathfrak{t}$ has stable distribution of index s. Let $\mathfrak{F}$ be the cumulative distribution function of the standard normal random variable.

Fix $z \in L_0$. We assume from now on that, unless explicitly stated otherwise, the random walk starts from z. Denote by $T_n = T(n \,|\, \omega, z)$ the hitting time of layer n by the walk starting from z. Below we suppose that conditions (2.3) and (2.4) are satisfied.

Let s denote the positive solution of $r(s) = 1$ where r is definied by (2.9). Note that s can be equal to $+\infty$ (e.g. if the walker has positive drift with probability 1).

Theorem 2.4 (The annealed limit theorem)**.**

(a) If $s < 1$ and the non-arithmeticity condition holds then there is a constant B such that

$$\lim_{N\to\infty} P\left(\frac{T_N}{BN^{1/s}} \le t\right) = \mathfrak{L}_s(t), \quad \lim_{N\to\infty} P\left(\frac{X_N}{N^s} \le t\right) = 1 - \mathfrak{L}_s\left((Bt)^{-1/s}\right).$$

(b) If $1 < s < 2$ and the non-arithmeticity condition holds then there are constants B and v such that

$$\lim_{N\to\infty} P\left(\frac{T_N - N/v}{BN^{1/s}} \le t\right) = \mathfrak{L}_s(t), \quad \lim_{N\to\infty} P\left(\frac{X_N - Nv}{N^s} \le t\right) = 1 - \mathfrak{L}_s\left(-Btv^{1+(1/s)}\right).$$

(c) If $s > 2$ then there are constants B and v such that

$$\lim_{N\to\infty} P\left(\frac{T_N - N/v}{B\sqrt{N}} \le t\right) = \mathfrak{F}(t), \quad \lim_{N\to\infty} P\left(\frac{X_N - Nv}{N^s} \le t\right) = 1 - \mathfrak{F}\left(-Btv^{3/2}\right).$$

For RWRE on $\mathbb{Z}$, Theorem 2.4 was obtained in [18]. For the case of the strip Theorem 2.4(c) was proven in [26]. In cases (a) and (b) the limit theorems for T_N were obtained in [8]. The limit theorems for X_N are simple corollaries of the corresponding results for T_N and the backtracking estimates of Lemma 3.5, see [18].

2.4. **Quenched limit theorems.** Theorem 2.4(c) shows that the walk is diffusive iff $s > 2$. Let

$$(2.11) \qquad\qquad b_n = b_n(\omega) = \min(k : \mathbb{E}_\omega(T_k) \ge n).$$

Theorem 2.5. *If $s > 2$ then there exists $\bar{D} > 0$ such that for almost every ω for all $t \in \mathbb{R}$.*

$$\lim_{N\to\infty} \mathbb{P}_\omega\left(\frac{T_N - \mathbb{E}_\omega T_N}{\sqrt{N}\bar{D}} < t\right) = \mathfrak{F}(t), \quad \lim_{N\to\infty} \mathbb{P}_\omega\left(\frac{X_N - b_N(\omega)}{\sqrt{N}v^{3/2}\bar{D}} < t\right) = \mathfrak{F}(t).$$

Theorem 2.5 was proved in [14] and independently in [20] for the simple RWRE on $\mathbb{Z}$. It was proved for RWRE on a strip in [15].

In contrast, for $s < 2$ the quenched limit theorem fails. To describe the asymptotic behaviour of the walks in this case we need the following notation. Let $\mathfrak{F}_\Theta$ be the conditional distribution function of the sum (2.10) where the sequence $\{\Theta_n\}$ is fixed while Γ are i.i.d. standard exponential random variables.

Theorem 2.6. *Suppose that $s \in (0,2)\setminus\{1\}$. Let $B_s = \begin{cases} 0 & \text{if } s < 1 \\ 1/v & \text{if } s > 1 \end{cases}$ where v is the speed of the walk (see Theorem 2.4(b)).*

Then the process $t \to \mathbb{P}_\omega\left(\frac{T_N - B_s N}{N^{1/s}} \le t\right)$ converges in distribution as $N \to \infty$ to the process $t \to \mathfrak{F}_\Theta(t)$ where $\{\Theta_n\}$ is the Poisson process on $[0,\infty)$ with intensity $\frac{s}{\theta^{s+1}}$.

Theorem 2.6 was obtained independently in [7, 22, 13] for the simple RWRE on $\mathbb{Z}$ and extended to RWRE on a strip in [8]. We note that one can

also obtain a limit of the joint distributions of $T_{\beta_1 N}, T_{\beta_2 N}, \ldots, T_{\beta_k N}$ ([23]) by considering a Poisson process (u_n, Θ_n) on $[0, \infty) \times [0, \infty)$ with the intensity $\frac{s}{\theta^{s+1}}$ and letting

$$\mathfrak{t}_j = \sum_{n:u_n \leq \beta_j} \Theta_n(\Gamma_n - \varepsilon_s).$$

This allows one to describe the quenched distribution of the walker's position using the fact that $\mathbb{P}_\omega(X_N \leq x) \approx \mathbb{P}_\omega(T_x \geq N)$, see [23] for details.

2.5. **Local Limit theorems for a RWRE on a strip.** Next we describe local limit theorems for RWRE on a strip.

We begin with the quenched result. The LLT makes sense if the quenched limit of the properly normalized hitting time exists. Thus we need to assume that $s > 2$.

Theorem 2.7. *Suppose that $s > 2$. Then* **P***-almost surely there is a constant $\bar{D} > 0$ such that*

$$\sup_k \left| \bar{D}\sqrt{2\pi n} \mathbb{P}_\omega(T(n \mid \omega, z) = k) - \exp\left(-\frac{(k - \mathbb{E}_{\omega,z} T_n)^2}{2\bar{D}^2 n}\right) \right| \to 0 \text{ as } n \to \infty.$$

Next we discuss quenched LLT for the walkers position. It turns out that in this case an additional local factor is needed in the LLT. Denote

$$(2.12) \qquad \rho_{(k,i)} = \mathbb{E}_\omega \text{Card}(n : \xi_n = (k, i)) \text{ and set } a \stackrel{\text{def}}{=} \mathbf{E}\left(\sum_{i=1}^m \rho_{(k,i)}\right).$$

Theorem 2.8. *(Quenched LLT) Suppose that conditions* (2.3)*,* (2.4)*,* (2.5)*, are satisfied and $r(2) < 1$. Then there is a constant $D > 0$ such that* **P***-almost surely the following holds. For each $\varepsilon, R > 0$ there exists $n_0 = n_0(\omega)$ such that for $n \geq n_0$ uniformly in k satisfying*

$$(2.13) \qquad\qquad |k - b_n(\omega)| \leq R\sqrt{n}$$

we have

$$\left| \frac{\sqrt{2\pi n} Da}{\rho_{(k,i)}} \exp\left[\frac{(k - b_n)^2}{2D^2 n}\right] \mathbb{P}_\omega(\xi_n = (k, i)) - 1 \right| < \varepsilon.$$

Theorem 2.9. *(Annealed LLT) There is a* **D** > 0 *such that for each $\varepsilon, R > 0$ there exists n_0 such that for $n \geq n_0$ uniformly for*

$$(2.14) \qquad\qquad |k - nv| \leq R\sqrt{n}$$

we have

$$\left| \sqrt{2\pi n} \mathbf{D} \exp\left[\frac{(k - nv)^2}{2\mathbf{D}^2 n}\right] \mathbf{P}(\xi_n = (k, i)) - 1 \right| < \varepsilon.$$

For the simple RWRE on $\mathbb{Z}$, Theorems 2.7, 2.8 and 2.9 are due to [9]. In this paper we extend this result to the strip using the approach of [9].

2.6. **Recurrent case.** In this section we review the results for recurrent RWRE on the strip. Even though our paper does not contain new results for recurrent walks, we believe it is useful to state them for the purpose of comparison with the transient case.

For the simple RWRE on $\mathbb{Z}$ (that is with jumps to the nearest neighbors) Sinai has proved [27] that the walk exhibits the following asymptotic behaviour: there is a constant $D > 0$ such that

$$\lim_{N \to \infty} \mathrm{P}\left(\frac{X_N}{D \ln^2 N} \leq t \right) = \mathfrak{H}(t).$$

This is what is called *the Sinai behavior*. Here $\mathfrak{H}$ is the Kesten-Sinai distribution [17] whose density is given by

$$\mathfrak{H}'(t) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2 \pi^2}{8} |t| \right).$$

Theorem 2.10. *Consider recurrent RWRE on a strip and suppose that conditions (2.3) and (2.4) are satisfied. Then either*
   (a) the walk exhibits the Sinai behavior; or
   (b) there is a constant K such that with probability 1 for all n

(2.15) $$\frac{1}{K} \leq ||A_n \dots A_0|| \leq K.$$

Moreover, if (2.15) holds, the environment is stationary and satisfies (2.4) then there is a constant D such that

$$\mathbf{P}\left(\frac{X_N}{D\sqrt{N}} \leq t \right) = \mathfrak{F}(t) \quad \text{and for a.e. } \omega, \ \mathbb{P}_\omega\left(\frac{X_N}{D\sqrt{N}} \leq t \right) = \mathfrak{F}(t).$$

The fact that the walk exhibits the Sinai behavior unless (2.15) holds is due to [3]. The CLT for stationary environments satisfying (2.15) was proved in [10].

Theorem 2.11. *([12]) Suppose that (2.15) holds. Then $\mathbf{P}$-almost surely the following holds. For each $\varepsilon, R > 0$ there exists $n_0 = n_0(\omega)$ such that for $n \geq n_0$ uniformly for $|k| \leq R\sqrt{n}$ we have*

$$\left| \frac{\sqrt{2\pi n} Da}{\rho_{(k,i)}} \exp\left[\frac{k^2}{2D^2 n} \right] \mathbb{P}_\omega(\xi_n = (k,i)) - 1 \right| < \varepsilon.$$

2.7. **Environment seen by the particle.** Let $\mathcal{T}$ be the natural shift on the space of environments. Consider the Markov chain $(\omega^{(N)}, Y_N)$, where $\omega^{(N)} = \mathcal{T}^{X_N}\omega$. This Markov chain is called *environment seen by the particle* (ESP) and it is an effective tool for studying RWRE.

Denote $\tilde{\Omega} = \Omega \times \{1, \dots, m\}$ the phase space of the Markov chain ESP.

Theorem 2.12. *[11] $(\omega^{(N)}, Y_N)$ admits an invariant measure $\mathbf{Q}$ which is absolutely continuous with respect to $\mathbf{P}$ iff one of the following conditions is satisfied*

either (a) the walk is transient and $s > 1$ (that is the walk has positive speed);

or (b) (2.15) holds.

In case (a) we have that for every continuous function $\Phi : \tilde{\Omega} \to \mathbb{R}$

$$\mathbf{Q}(\Phi) = \frac{1}{a} \, \mathbf{E} \left(\sum_{y=1}^{m} \Phi(\omega, y) \rho_{(0,y)}(\omega) \right)$$

where ρ is given by (2.12).

Theorem 2.13. *(a) Suppose that either the walk is transient and $s > 1$ or the walk is recurrent and (2.15) holds. Then for every continuous function $\Phi : \tilde{\Omega} \to \mathbb{R}$*

$$\lim_{N \to \infty} \mathrm{E}(\Phi(\omega^{(N)}, Y_N)) = \mathbf{Q}(\Phi).$$

(b) Suppose that either the walk is transient and $s > 2$ or the walk is recurrent and (2.15) holds. Then for a.e. ω and for every continuous function $\Phi : \tilde{\Omega} \to \mathbb{R}$

(2.16) $$\lim_{N \to \infty} \mathbb{E}_\omega(\Phi(\omega^{(N)}, Y_N)) = \mathbf{Q}(\Phi).$$

For recurrent RWRE both statements are due to [12]. In the transient case, the result was first proven for RWRE on $\mathbb{Z}$ in [16] (part (a)) and [19] (part (b)). For the RWRE on the strip, part (a) is due to [26]. Part (b) is new and will be proven in Section 7 following the approach of [9].

3. Preliminaries.

In the rest of the paper we give the proof of the local limit theorem for transient diffusive RWRE. Therefore unless it is explicitly stated otherwise, we suppose that the walk is transient to the right and that $s > 2$.

3.1. **A Local limit theorem for independent summands.** The following result from [5] provides very general sufficient conditions under which the Local Limit Theorem for sums of independent integer valued random variables holds.

Theorem 3.1. *([5]) Let η_i, $i \geq 1$, be independent integer valued random variables and let $d_i = \sum_j \min[P(\eta_i = j), P(\eta_i = j + 1)]$, $\mathfrak{d}_n = \sum_{i=1}^{n} d_i$. Denote $\Xi_n = \sum_{i=1}^{n} \eta_i$. Suppose that there are numbers $\mathfrak{c}_n > 0$, $\mathfrak{h}_n$, $n \geq 1$, such that, as $n \to \infty$, $\mathfrak{c}_n \to \infty$, $\limsup \mathfrak{c}_n^2/\mathfrak{d}_n < \infty$, and $(\Xi_n - \mathfrak{h}_n)/\mathfrak{c}_n$ is asymptotically normal $\mathcal{N}(0, 1)$. Then*

(3.1) $$\sup_k \left| \mathfrak{c}_n P\left(\Xi_n = k\right) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(k - \mathfrak{h}_n)^2}{2\mathfrak{c}_n^2} \right) \right| \to 0 \ \text{as } n \to \infty.$$

Remark 3.2. The requirement $\mathfrak{d}_n \to \infty$ implies that sufficiently many d_i's are positive. Had this not been the case, then it could happen that Ξ_n would be taking, say, only even values as n becomes large. In our applications the

role of Ξ_n is played by T_n and all the corresponding d_i's are uniformly separated from 0.

3.2. Occupation times. In the proofs of Theorems 2.8–2.13 we assume that $s < \infty$. Proofs become easier if $s = \infty$ and we leave the corresponding modifications to the reader.

Lemma 3.3. *(a) There is a constant C such that for each $(k,i) \in \mathbb{S}$*

$$\mathbf{P}\left(\rho_{(k,i)} > t\right) \leq Ct^{-s}.$$

(b) For any $\hat{u} > \frac{1}{s}$ for almost every ω there is a constant $C(\omega)$ such that for each $(k,i) \in \mathbb{S}$

$$\rho_{(k,i)} < C(\omega)k^{\hat{u}}.$$

Remark 3.4. In the case $s = \infty$ the statements of Lemma 3.3 read as follows. For any $u, \hat{u} > 0$ we have that $\mathbf{P}(\rho_{(k,i)} > t) \leq Ct^{-u}$ and $\rho_{(k,i)} < C(\omega)k^{\hat{u}}$.

Proof. Part (a) follows from [8, Lemma 3.3]. Part (b) follows from part (a) and the Borel-Cantelli Lemma. $\square$

3.3. Backtracking.

Lemma 3.5. [8, Lemma 3.2] *Suppose that the walk is transient to the right. Then*
(a) There exists $C > 0$ and $\theta < 1$ such that

$$\mathrm{P}(X \text{ visits } k \text{ after } k + m) \leq C\theta^m.$$

(b) Accordingly, for almost every ω there is a constant $K(\omega)$ such that

$$\mathbb{P}_\omega(\exists k < n : X \text{ visits } k \text{ after } T_{k+\ln^2 n}) \leq K(\omega)n^{-100}.$$

4. Occupation times of large segments.

Let $T(n \,|\, \omega, (k,j))$ be the hitting time of layer $n > k$ by the walk starting from $(k,j) \in L_k$. Denote by $e_{k,n}$ the vector whose components are the expectations of these hitting times:

$$(4.1) \qquad\qquad e_{k,n}(j) = \mathbb{E}_\omega T(n \,|\, \omega, (k,j))$$

As has been shown in [15, formula (4.28)]

$$(4.2) \qquad\qquad e_{k,n} = \sum_{j=k}^{n-1} \zeta_k \ldots \zeta_j \left(\sum_{i=0}^{\infty} H_j^i U_{j-i} \mathbf{1} \right).$$

Here $U_j \stackrel{\mathrm{def}}{=} (I - Q_j\zeta_{j-1} - R_j)^{-1}$, $\mathbf{1}$ is a column vector whose components are all equal to 1, we assume that $\zeta_k \ldots \zeta_k = I$,

$$(4.3) \quad H_j^i \stackrel{\mathrm{def}}{=} A_j \ldots A_{j-i+1}, \quad \text{with the convention that} \quad H_j^0 = I, \quad H_j^1 = A_j$$

and the matrices A_n are defined by (2.8). Let y_n be a sequence of m-dimensional probability vectors such that $y_n = y_{n-1}\zeta_n$ for all $n \in \mathbb{Z}$. There

is a unique sequence satisfying these equations (Lemma 1 in [15]). The probabilistic meaning of $y_0 = (y_0(1), ..., y_0(m))$ is the distribution of the starting point of the RW:

$$(4.4) \qquad \qquad \mathbb{P}_\omega\{X(0) = (0, i)\} = y_0(i)$$

(or, equivalently, the distribution of the points in L_0 hit by the RW starting at $-\infty$). In particular $T_n \stackrel{\text{def}}{=} T(n \,|\, \omega, (0, \cdot))$ is the hitting time of L_n by the walk starting from a random initial site in L_0 with the distribution of this site given by (4.4).

Let τ_j be the time the walk takes to reach L_{j+1} after having reached L_j. Then

$$(4.5) \qquad a_j \stackrel{\text{def}}{=} \mathbb{E}_\omega(\tau_j) = y_j \left(\sum_{i=0}^\infty H_j^i U_{j-i} \mathbf{1} \right) \quad \text{and we set} \quad a \stackrel{\text{def}}{=} \mathbf{E}(\mathbb{E}_\omega(\tau_0)).$$

Remark 4.1. The first relation in (4.5) can be derived from the identity $\tau_j = T(j+1 \,|\, \omega, z) - T(j \,|\, \omega, z)$ and (4.2).

Remark 4.2. It is easy to see that $\mathbf{E}(\mathbb{E}_\omega(\tau_0)) = \mathbf{E}\left(\sum_{i=1}^m \rho_{(k,i)} \right)$ and therefore the definitions of a given in (2.12) and (4.5) are consistent. It is natural and convenient to use the two different interpretations of a in our analysis.

Lemma 4.3. *There exists $\varepsilon_0 > 0$ such that almost surely*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \max_{|l| \leq n^{\frac{1+\varepsilon_0}{2}}} |\mathbb{E}_\omega(T_{n+l} - T_n - la)| = 0.$$

The following notations will be used in the rest of this section. If $\mathcal{Y} : \Omega \mapsto \mathbb{R}$ is a random variable then $||\mathcal{Y}||_p = (\mathbf{E}|\mathcal{Y}|^p)^{\frac{1}{p}}$, where $p = 2 + 2\delta$ and $\delta > 0$ is such that $r(2+2\delta) < 1$. Throughout the proof δ is fixed and we write $||\cdot||$ for $||\cdot||_p$ whenever the difference between this norm and the matrix norm is obvious from the context. Set

$$(4.6) \qquad \mathcal{H}(n, \omega) = \sum_{j=0}^{n-1}(a_j - a), \quad \mathcal{H}^*(n, \omega) = \max_{0 \leq s \leq n-1} \left| \sum_{j=0}^s (a_j - a) \right|.$$

The maximal inequality of Lemma 4.4(a) below is the main ingredient of the proof. We also state another maximal inequality (Lemma 4.4(b)) whose proof is quite similar and which will be useful in Section 7.

Let $W(P, Q, R)$ be any $\mathbb{R}^m$ valued function of the triple of non-negative matrices satisfying (2.1) and let $w_n = W(P_n, Q_n, R_n)$. Let

$$(4.7) \qquad \mathfrak{a}_j \stackrel{\text{def}}{=} y_j \left(\sum_{i=0}^\infty H_{j+i}^i U_j w_j \right) \quad \text{and} \quad \mathfrak{a} \stackrel{\text{def}}{=} \lim_{j\to\infty} \mathbf{E}(\mathfrak{a}_j).$$

According to the analysis of [8][Section 3], there are constants $C > 0, \theta < 1$ such that with probability at least $1 - n^{-100}$

$$\left| \mathfrak{a}_j - \sum_{k=1}^{m} \rho_{(n,k)} w_n(k) \right| \leq C\theta^n.$$

Set

$$\mathfrak{H}^*(n, \omega) = \max_{0 \leq s \leq n-1} \left| \sum_{j=0}^{s} (\mathfrak{a}_j - \mathfrak{a}) \right|.$$

Lemma 4.4. *For any $\varepsilon > 0$ there is a C such that the following relations hold:*

(a) $\quad ||\mathcal{H}^*(n)|| \leq Cn^{\frac{1}{2}+\varepsilon};$ (b) $\quad ||\mathfrak{H}^*(n)|| \leq Cn^{\frac{1}{2}+\varepsilon}.$

Remark 4.5. The statements of this Lemma are similar to those of Lemma 4 in [14]. However, in the proof, an extra effort is needed in order to control the dependence between a_j's which in the case of a strip is much stronger than in the case of a simple walk.

Once the maximal inequality of Lemma 4.4(a) is obtained, the proof of Lemma 4.3 proceeds exactly as the derivation of Lemma 5 from Lemma 4 in [14].

Proof of Lemma 4.4. We can suppose without any loss of rigor that n^ε and $n^{1-\varepsilon}$ are integer numbers. One can present $\mathcal{H}(n)$ as

$$\mathcal{H}(n) = \sum_{i=0}^{n^\varepsilon-1} \sum_{j=0}^{n^{1-\varepsilon}-1} (a_{i+n^\varepsilon j} - a).$$

As will be seen below, the members of the inner sum are asymptotically independent random variables (as $n \to \infty$) and this property plays a crucial role in the proof. We can estimate $\mathcal{H}^*(n)$ as

$$\mathcal{H}^*(n) \leq \sum_{i=0}^{n^\varepsilon-1} \max_{0 \leq s \leq n^{1-\varepsilon}} \left| \sum_{j=0}^{s} (a_{i+n^\varepsilon j} - a) \right|.$$

Hence

$$||\mathcal{H}^*(n)|| \leq \sum_{i=0}^{n^\varepsilon-1} \left\| \max_{0 \leq s \leq n^{1-\varepsilon}} \left| \sum_{j=0}^{s} (a_{i+n^\varepsilon j} - a) \right| \right\| = n^\varepsilon \left\| \max_{0 \leq s \leq n^{1-\varepsilon}} \left| \sum_{j=0}^{s} (a_{n^\varepsilon j} - a) \right| \right\|,$$

where the last equality is due to the fact the a_j is a stationary sequence.

Let $\mathcal{S}(n, \varepsilon) = \max_{0 \leq s \leq n^{1-\varepsilon}} \left| \sum_{j=0}^{s} (a_{n^\varepsilon j} - a) \right|$ and present $a_j - a = B(j, k) + D(j, k),$

where

$$(4.8) \qquad B(j, k) \stackrel{\text{def}}{=} y_j \left(\sum_{i=0}^{k} H_j^i U_{j-i} \mathbf{1} \right) - \mathbf{E}\left[y_j \left(\sum_{i=0}^{k} H_j^i U_{j-i} \mathbf{1} \right) \right],$$

$$D(j,k) \stackrel{\text{def}}{=} y_j \left(\sum_{i=k+1}^{\infty} H_j^i U_{j-i} \mathbf{1} \right) - \mathbf{E} \left[y_j \left(\sum_{i=k+1}^{\infty} H_j^i U_{j-i} \mathbf{1} \right) \right],$$

and k is to be specified later, see (4.11). We then have

$$\mathcal{S}(n,\varepsilon) = \max_{0 \le s \le n^{1-\varepsilon}} \left| \sum_{j=0}^{s} (B(n^\varepsilon j, k) + D(n^\varepsilon j, k)) \right|$$

$$\le \max_{0 \le s \le n^{1-\varepsilon}} \left| \sum_{j=0}^{s} B(n^\varepsilon j, k) \right| + \sum_{j=0}^{n^{1-\varepsilon}-1} |D(n^\varepsilon j, k)|.$$

Therefore

$$(4.9) \qquad \begin{aligned} ||\mathcal{S}(n,\varepsilon)|| &\le \left\| \max_{0 \le s \le n^{1-\varepsilon}} \left| \sum_{j=0}^{s} B(n^\varepsilon j, k) \right| \right\| + \sum_{j=0}^{n^{1-\varepsilon}-1} ||D(n^\varepsilon j, k)|| \\ &= \left\| \max_{0 \le s \le n^{1-\varepsilon}} \left| \sum_{j=0}^{s} B(n^\varepsilon j, k) \right| \right\| + n^{1-\varepsilon} ||D(0,k)||, \end{aligned}$$

where the last equality is again due to stationarity. Since by (2.9)

$$\mathbf{E}(||H_j^i||^{2+2\delta}) \le Cr(2+2\delta)^i$$

we have for some constants C_1, C_2, C_3:

$$||D(0,k)|| = \left\| y_j \left(\sum_{i=k+1}^{\infty} H_j^i U_{j-i} \mathbf{1} \right) - \mathbf{E} \left[y_j \left(\sum_{i=k+1}^{\infty} H_j^i U_{j-i} \mathbf{1} \right) \right] \right\|$$

$$\le C_1 \sum_{i=k+1}^{\infty} \| \, ||H_j^i|| \, \|_p + \mathbf{E} \left[y_j \left(\sum_{i=k+1}^{\infty} H_j^i U_{j-i} \mathbf{1} \right) \right] \le C_2 \sum_{i=k+1}^{\infty} r(2+2\delta)^{\frac{i}{2+2\delta}} = C_3 \beta^k,$$

where $\beta \stackrel{\text{def}}{=} r(2+2\delta)^{\frac{1}{2+2\delta}} < 1$.

It remains to estimate $\mathcal{I} \stackrel{\text{def}}{=} \max_{0 \le s \le n^{1-\varepsilon}} \left\| \sum_{j=0}^{s} |B(n^\varepsilon j, k)| \right\|$. To this end we

shall introduce independent random variables $\tilde{B}(n^\varepsilon j, k)$ such that for some n_0 the following holds: if $n > n_0$ then

$$(4.10) \qquad ||B(n^\varepsilon j, k) - \tilde{B}(n^\varepsilon j, k)|| \le \text{Const} \, n^{-100}.$$

This is done as follows.

For each $0 \le j \le n^{1-\varepsilon}$ denote $I_j \stackrel{\text{def}}{=} [n^\varepsilon j, \, n^\varepsilon(j+1)-1]$ and define stochastic matrices $\psi_s^{(j)}$ with $s \in I_j$ as follows: set $\psi_{n^\varepsilon j}^{(j)}(i_1, i_2) = m^{-1}$ and compute $\psi_s^{(j)}$ for all $s > jn^\varepsilon$ as in (2.6), namely $\psi_s^{(j)} \stackrel{\text{def}}{=} (I - R_j - Q_j \psi_{s-1}^{(j)})^{-1} P_s$.

Next, let $y_j^{(j)} = (m^{-1}, \dots, m^{-1})$ and compute $y_s^{(j)} = y_{s-1}^{(j)} \psi_s^{(j)}$ for $s > n^\varepsilon$.

Now define $\tilde{A}_s \stackrel{\text{def}}{=} (I - Q_s \psi_{s-1}^{(j)} - R_s)^{-1} Q_s$, $\tilde{U}_s \stackrel{\text{def}}{=} (I - Q_s \psi_{s-1}^{(j)} - R_s)^{-1}$, and $\tilde{H}_s^i \stackrel{\text{def}}{=} \tilde{A}_s \ldots \tilde{A}_{s-i+1}$.

Finally $\tilde{B}(j, k)$ is defined by (4.8) with the difference that the corresponding A, U, H are replaced by the just defined $\tilde{A}$, $\tilde{U}$, $\tilde{H}$.

From now on we suppose that

(4.11) $$k = K \ln n$$

where K is a large enough constant. It follows from the above definitions that for large n the random variables $\tilde{B}(n^\varepsilon j, k)$, $0 \le j \le n^{1-\epsilon}$, are independent and identically distributed simply because they are (the same) functions of the parts of the environment belonging to pairwise disjoint boxes of the strip. Also, $\mathbf{E}(\tilde{B}(n^\varepsilon j, k)) = 0$.

It follows from results obtained in [8, 15] (see appendix in each of these papers) that there is a $\bar{\beta} < 1$ such that for $s \in I_j$ $\left\| \| \zeta_s - \psi_s^{(j)} \| \right\|_p \le C \bar{\beta}^{s - n^\varepsilon j}$. The derivation of (4.10) follows from this estimate in a standard way.

We can now estimate $\hat{\mathcal{S}} \stackrel{\text{def}}{=} \left\| \max_{0 \le s \le n^{1-\varepsilon}} \left| \sum_{j=0}^s B(n^\varepsilon j, k) \right| \right\|$. Obviously

$$\hat{\mathcal{S}} \le \left\| \max_{0 \le s \le n^{1-\varepsilon}} \left| \sum_{j=0}^s \tilde{B}(n^\varepsilon j, k) \right| \right\| + \left\| \sum_{j=0}^s |B(n^\varepsilon j, k) - \tilde{B}(n^\varepsilon j, k)| \right\|$$

By the Doob inequality for martingales we now have that

$$\left\| \max_{0 \le s \le n^{1-\varepsilon}} \left| \sum_{j=0}^s \tilde{B}(n^\varepsilon j, k) \right| \right\| \le \frac{2 + 2\delta}{1 + 2\delta} \left\| \sum_{j=0}^{n^{1-\varepsilon}} \tilde{B}(n^\varepsilon j, k) \right\|$$

and by the Marcinkiewicz-Zygmund inequality

$$\left\| \sum_{j=0}^{n^{1-\varepsilon}} \tilde{B}(n^\varepsilon j, k) \right\| \le C n^{(1-\varepsilon)/2}$$

Part (a) of Lemma 4.4 is thus proved.

The proof of part (b) is similar. Namely we approximate $\mathfrak{a}_j$ by

$$\mathfrak{a}_j(k) \stackrel{\text{def}}{=} y_j \left(\sum_{i=0}^k H_{j+i}^i U_j w_j \right)$$

and use weak dependence of $a_{j_1}(k)$ and $a_{j_2}(k)$ for $|j_1 - j_2| > 2k$ similarly to the argument of part (a). □

5. Proof of Theorem 2.7.

Proof of Theorem 2.7. We shall use the construction of the enlarged random environment introduced in [15].

For a RW starting from $z \in L_0$ present $T(n \,|\, \omega, z)$ as

(5.1) $$T(n \,|\, \omega, z) = \tau_0 + \tau_1 + \ldots + \tau_{n-1},$$

where τ_j is the time the walk takes to reach L_{j+1} after having reached L_j.

The random variables τ_j are not independent. However, they are conditionally independent, where the condition is that the walk hits the layers $L_1, \ldots, L_{n-1}$ at a given sequence of points. We shall now briefly describe how the measure $\mathbb{P}_{\omega,z}$ on $\mathfrak{X}_z$ can be presented as an integral of the just mentioned conditional measures over a measure on the set of sequences in the strip. The construction of this conditional measure is similar to the one used in [15].

Let $J = \{\, \mathfrak{i} = ((k, i_k))_{0 \leq k < \infty} \,:\, 1 \leq i_k \leq m \,\}$ be the set of all sequences of points in the strip $\mathbb{S}$ which contain exactly one point from each layer and $J_{i_0} \subset J$ be all such sequences starting with $(0, i_0)$.

Let $\zeta_k \equiv \zeta_k(\omega)$ be matrices from Theorem 2.1(a). Note that due to (2.4) there is $\varepsilon > 0$ such that the inequalities $\zeta_k(i, j) \geq \varepsilon$ hold for all k, i, j.

From now on, i_0 will be fixed and we shall define a Markov measure on J_{i_0} in the usual way. Namely, for $k \geq 0$ consider a cylinder set in J_{i_0}:

$$\mathrm{C}_k(i_0, \ldots, i_k) \overset{\text{def}}{=} \{\, \mathfrak{i} \in J_{i_0} \,:\, \mathfrak{i}_j = (j, i_j) \ \text{ for all } \ j \in [0, k] \,\},$$

where $\mathfrak{i}_j$ denotes the j-th coordinate of the sequence $\mathfrak{i}$. Set

$$(5.2) \qquad \Lambda_{\omega, i_0}(\mathrm{C}_k(i_0, \ldots, i_k)) \overset{\text{def}}{=} \zeta_0(i_0, i_1) \ldots \zeta_k(i_{k-1}, i_k).$$

Λ_{ω, i_0} defined by (5.2) is a Markov measure with transition probabilities given by stochastic matrices ζ_n. As usual, this measure can be extended to the Borel sigma-algebra generated by the cylinder sets. Let us denote this sigma-algebra by $\mathcal{F}_{i_0}$. We thus have a probability space $(J_{i_0}, \mathcal{F}_{i_0}, \Lambda_{\omega, i_0}(d\mathfrak{i}))$.

We can now define a new probability space $(\tilde{\Omega}, \mathfrak{S}, \mathcal{P})$ where

$$\tilde{\Omega} \overset{\text{def}}{=} \Omega \times J_{i_0} = \{\, \mathrm{w} = (\omega, \mathfrak{i}) \,:\, \omega \in \Omega, \ \mathfrak{i} \in J_{i_0} \,\}$$

with a product sigma-algebra $\mathfrak{S} \overset{\text{def}}{=} \mathcal{F} \otimes \mathcal{F}_{i_0}$ and $\mathcal{P}(d\mathrm{w}) \overset{\text{def}}{=} \mathbf{P}(d\omega) \Lambda_{\omega, i_0}(d\mathfrak{i})$ on $\tilde{\Omega}$.

Definition. *A pair* $\mathrm{w} = (\omega, \mathfrak{i})$ *is called the enlarged random environment. The set $\tilde{\Omega}$ is the collection of enlarged environments with $(\tilde{\Omega}, \mathfrak{S}, \mathcal{P})$ being the corresponding probability space.*

Remark 5.1. The above construction depends on i_0 but the asymptotic behaviour of the walk doesn't which is why in some of the notations above i_0 was dropped.

Next, for $\mathfrak{i} \in J_{i_0}$ denote by $\mathfrak{X}_{i_0}^{\mathfrak{i}}$ the space of trajectories of the walk which start at $(0, i_0)$ and reach each layer L_j, $j \geq 1$, for the first time at (j, i_j). For any $n \geq 0$ and $\xi \in \mathfrak{X}_{i_0}^{\mathfrak{i}}$ denote by $z_0, z_1, \ldots, z_{T_n}$ the values of the trajectory at times $t = 0, 1, \ldots, T_n$, where, as usual, T_n is the hitting time of layer n (with $\xi(0) = (0, i_0)$, $\xi(T_n) = (n, i_n)$). The probability $V_{\omega, \mathfrak{i}}$ on the set of

trajectories from $\mathfrak{X}_{i_0}^{\mathfrak{i}}$ is defined by
(5.3)

$$V_{\omega,\mathfrak{i}}\left(\xi_t = z_t \ \forall \ t \in [0, T_n]\right) \stackrel{\text{def}}{=} \frac{1}{\prod_{k=0}^{n-1} \zeta_k(i_k, i_{k+1})} \prod_{t=0}^{T_n-1} \mathbb{P}_\omega(\xi_{t+1} = z_{t+1}|\xi_t = z_t).$$

It is known ([2], [15]) that if the RW is transient to the right then the probabilistic meaning of matrices ζ_n is given by

$$\zeta_n(i, j) = \mathbb{P}_\omega\left(\text{RW starting from } (n, i) \text{ hits } L_{n+1} \text{ at } (n+1, j)\right).$$

Therefore $\Lambda_{\omega,i_0}(\mathrm{C}_k(i_0, \ldots, i_k))$ is the probability that a RW staring from $(0, i_0)$ proceeds to $+\infty$ so that on its way it hits L_j at (j, i_j) for all $j \in [1, k]$. We obtain that

(5.4) $$\mathbb{P}_{\omega,(0,i_0)}(d\xi) = \int_{J_{i_0}} \Lambda_{\omega,i_0}(d\mathfrak{i}) V_{\omega,\mathfrak{i}}(d\xi).$$

Denote by $T(n|\omega, \mathfrak{i})$ the sum in the right hand side of (5.1) conditioned on the walk hitting each L_j, $j \geq 0$, at $(j, i_j) \in \mathfrak{i}$.

We shall now check that Theorem 3.1 implies that the LLT holds for $T(n|\omega, \mathfrak{i})$ for Λ_{ω,i_0}–a.e. $\mathfrak{i}$ and deduce from here that the LLT holds for $T(n|\omega, (0, i_0))$ by integrating over $\mathfrak{i}$ using (5.4).

Indeed, $T_n = T(n \, | \, \omega, \mathfrak{i})$ is a sum of independent random variables τ_j such that the corresponding $\mathfrak{d}_n$ and $\mathfrak{c}_n$ (from Theorem 3.1) grow linearly and $\mathfrak{c}_n \mathfrak{d}_n^{-1} < \text{Const}$. Also, $T(n|\omega, \mathfrak{i})$ satisfies the CLT by the following result from [15].

Theorem 5.2. *Suppose that conditions (2.3), (2.4), (2.5) are satisfied and $s > 2$. Then for $\mathcal{P}$-a.e. enlarged environment $\mathrm{w} = (\omega, \mathfrak{i})$*

(5.5) $$\lim_{n\to\infty} V_{\omega,\mathfrak{i}} \left\{ \frac{T(n \, | \, \omega, \mathfrak{i}) - E_{\omega,\mathfrak{i}} T(n \, | \, \omega, \mathfrak{i})}{\sqrt{n}} < x \right\} = \frac{1}{\sqrt{2\pi}\hat{\sigma}} \int_{-\infty}^x e^{-\frac{u^2}{2\hat{\sigma}^2}} du,$$

where $E_{\omega,\mathfrak{i}}$ is the expectation with respect to the measure $V_{\omega,\mathfrak{i}}$. Here

(5.6) $$\hat{\sigma}^2 = \lim_{n\to\infty} n^{-1} \mathrm{Var}_{\omega,\mathfrak{i}} T(n \, | \, \omega, \mathfrak{i})$$

and the convergence in (5.6) holds with $\mathcal{P}$-probability 1.

Thus equation (3.1) in our context reads as follows:

(5.7) $$\sqrt{2\pi n}\hat{\sigma} \ V_{\omega,\mathfrak{i}}\left(T_n = k\right) = \exp\left(-\frac{(k - E_{\omega,\mathfrak{i}} T_n)^2}{2n\hat{\sigma}^2}\right) + \varepsilon_n(k, \omega, \mathfrak{i})$$

where for a.e. $(\omega, \mathfrak{i})$

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |\varepsilon_n(k, \omega, \mathfrak{i})| = 0$$

and $\hat{\sigma}$ is the same as in Theorem 5.2.

It remains to carry out the integration of all parts of (5.7) over $\Lambda_{\omega,i_0}(d\mathfrak{i})$. Obviously,

$$\int_{J_{(0,i_0)}} V_{\omega,\mathfrak{i}}\left(T_n = k\right) \Lambda_{\omega,i_0}(d\mathfrak{i}) = \mathbb{P}_{\omega,(0,i_0)}\left(T_n = k\right).$$

In order to control the RHS of (5.7), note that

$$\exp\left(-\frac{(k - E_{\omega,\mathbf{i}}T_n)^2}{2n\hat{\sigma}^2}\right) = \exp\left(-\frac{1}{2}\left((k - \mathbb{E}_{\omega,(0,i_0)}T_n)n^{-\frac{1}{2}}\hat{\sigma}^{-1} - \hat{\sigma}^{-1}\phi_n\right)^2\right)$$

where $\phi_n = (E_{\omega,\mathbf{i}}T_n - \mathbb{E}_{\omega,(0,i_0)}T_n)n^{-\frac{1}{2}}$. By Theorem 7 in [15],

(5.8) ϕ_n has asymptotically normal distribution $\mathcal{N}(0, \tilde{\sigma}^2)$,

where $\tilde{\sigma}$ is a function of the parameters of the model (see [15], formula (3.5)). It may happen that $\tilde{\sigma} = 0$ in which case $\phi_n \to 0$ as $n \to \infty$ and can therefore be ignored. So, from now on we assume that $\tilde{\sigma} > 0$ (which is the only interesting case).

For a given $\varepsilon > 0$, the family of functions $\exp(-(a - c^{-1}\phi)^2)$ with $c \geq \varepsilon$ and arbitrary a is uniformly continuous in ϕ. Therefore the usual property of convergence in distribution implies that

(5.9)
$$\sup_k \left| \int_{J_{(0,i_0)}} \exp\left(-\frac{1}{2}\left(\frac{(k - E_\omega T_n)}{\sqrt{n}\hat{\sigma}} - \hat{\sigma}^{-1}\phi_n\right)^2\right) \Lambda_{\omega,(0,i_0)}(d\mathbf{i}) - \exp\left(-\frac{(k - E_\omega T_n)^2}{2n(\hat{\sigma}^2 + \tilde{\sigma}^2)}\right) \right| \to 0$$

as $n \to \infty$.

It remains to show that for $\mathbf{P}$–almost every ω

(5.10)
$$\int_{J_{(0,i_0)}} \varepsilon_n(k, \omega, \mathbf{i}) \Lambda_{\omega,(0,i_0)}(d\mathbf{i}) \to 0$$

uniformly in k. To this end we note that due to (5.7)

$$|\varepsilon_n| \leq \sqrt{2\pi n}\hat{\sigma} \, V_{\omega,\mathbf{i}}(T_n = k) + 1.$$

We claim that the first summand on the RHS is uniformly bounded. Indeed, T_n conditioned on $(\omega, \mathbf{i})$ is a sum of independent random variables τ_j. Moreover, due to (2.5) there exists $\epsilon > 0$ such that for each $\mathbf{i}$ for each l the probability

$$V_{\omega,\mathbf{i}}(\tau_j = l \,|\, \mathbf{i}) \leq 1 - \epsilon$$

(in fact, one can take $\epsilon = \frac{1}{1+\kappa}$ where κ is from (2.5)). Now Theorem 3 from [25, Part III, §2] implies that there is a constant C such that

(5.11) $\sqrt{2\pi n}\hat{\sigma} \, V_{\omega,\mathbf{i}}(T_n = k) \leq C.$

Therefore $|\varepsilon_n|$ is bounded above and (5.10) follows from the dominated convergence theorem.

Theorem 2.7 is proved (with $\bar{D}^2 = \hat{\sigma}^2 + \tilde{\sigma}^2$) where $\hat{\sigma}$ is from (5.7) and $\tilde{\sigma}$ is from (5.8). $\square$

6. LLT for the walker's position.

6.1. The quenched LLT.

Proof of Theorem 2.8. Take $\frac{1}{s} < u < \frac{1}{2}$. Let $\ell = \ln^2 n$, $\bar{k} = k - \ell$. We claim that for $\mathbf{P}$-almost all ω

(6.1) $\mathbb{P}_\omega(\exists k \leq n \ \exists m \in \mathbb{N} : X_m = k \text{ and } T_{\bar{k}} < m - n^u) \leq \dfrac{\bar{C}(\omega)}{n^{100}}.$

Indeed, if $X_m = k$ and $m > T_{\bar{k}} + n^u$ then one of the following events takes place:

$A_1 = \{X_t \in [k - 2\ell, k + \ell] \text{ for all } t \in [T_{\bar{k}}, T_{\bar{k}} + n^u]\}$,

$A_2 = \{\exists t \in [T_{\bar{k}}, T_{\bar{k}} + n^u] \text{ such that } X_t < k - 2\ell\}$,

$A_3 = \{\exists t \in [T_{\bar{k}}, T_{\bar{k}} + n^u] \text{ s. t. } X_t > k + \ell \text{ and then } X \text{ backtracks to } k\}$.

$\mathbb{P}_\omega(A_2)$ and $\mathbb{P}_\omega(A_3)$ are $O(n^{-100})$ by Lemma 3.5(b). Take $\frac{1}{s} < u' < u'' < u$. If A_1 happens then there exists $k^* \in [k - 2\ell, k + \ell]$ which is visited more than $n^{u''}$ times. However the number of visits to k^* has geometric distribution with mean $\rho_{\bar{k}} < C(\omega) n^{u'}$. Thus $\mathbb{P}_\omega(A_1) \leq n^{-100}$ proving (6.1).

Next we claim that given R we can take $\bar{R}$ so large that if k satisfies (2.13) then

$$(6.2) \qquad\qquad |n - \mathbb{E}_\omega T_{\bar{k}}| \leq \bar{R}\sqrt{k}.$$

Indeed

$$(6.3) \qquad n - \mathbb{E}_\omega T_{\bar{k}} = (n - \mathbb{E}_\omega T_{b_n}) + (\mathbb{E}_\omega T_{b_n} - \mathbb{E}_\omega T_{\bar{k}}).$$

Observe that by definition $\mathbb{E}_\omega T_{b_n - 1} < n \leq \mathbb{E}_\omega T_{b_n}$ and by Lemma 4.3

$$\mathbb{E}_\omega(T_{b_n} - T_{b_n - 1}) = o(\sqrt{n})$$

so that the first term in (6.3) is $o(\sqrt{n})$. Next, Lemma 4.3 also implies that

$$\mathbb{E}_\omega(T_{b_n} - T_{\bar{k}}) = a(b_n - \bar{k}) + o(\sqrt{n}).$$

This implies (6.2) and shows moreover that

$$(6.4) \qquad\qquad \frac{(n - \mathbb{E}_\omega T_{\bar{k}})^2}{k} \approx \frac{a^3(b_n - \bar{k})^2}{n}.$$

Thus for $j \in [0, n^u]$ we have due to Theorem 2.7

$$\mathbb{P}_\omega(T_{\bar{k}} = n - j) \approx \frac{1}{\sqrt{2\pi \bar{k} \bar{D}}} \exp - \left(\frac{(n - \mathbb{E}_\omega T_{\bar{k}})^2}{2\bar{D}^2 \bar{k}} \right).$$

On the other hand for $\mathbf{P}$–almost every ω and all sufficiently large n

$$\sum_{j=0}^{n^u} \mathbb{P}_\omega(\xi_j = (k, i)|X_0 = \bar{k}) = \rho_{(k,i)} + O\left(n^{-100}\rho_{(k,i)}\right) = \rho_{(k,i)} + O\left(n^{-99}\right).$$

Thus

$$(6.5) \qquad \mathbb{P}_\omega(\xi_n = (k, i)) \approx \frac{\rho_{(k,i)}}{\sqrt{2\pi \bar{k} \bar{D}}} \exp - \left(\frac{(n - \mathbb{E}_\omega T_{\bar{k}})^2}{2\bar{D}^2 \bar{k}} \right).$$

Combining this with (6.4) we get

$$\mathbb{P}_\omega(\xi_n = (k, i)) \approx \frac{\sqrt{a}\rho_{(k,i)}}{\sqrt{2\pi n}\bar{D}} \exp - \left(\frac{(k - b_n)^2 a^3}{2\bar{D}^2 n} \right)$$

$$= \frac{\rho_{(k,i)}}{\sqrt{2\pi n}Da} \exp - \left(\frac{(k - b_n)^2}{2D^2 n} \right),$$

where $D = \bar{D}/a^{3/2}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.2. The annealed LLT.

Given k denote $\bar{k} = k - \ln^2 n$, $\hat{k} = \frac{k+\bar{k}}{2}$.

Let $\tilde{\xi}(n)$ be equal to $\xi(n)$ if the walker does not backtrack to $L_{\bar{k}}$ during the time $[T_{\hat{k}}, n]$ and $\tilde{\xi}(n)$ be the place of the first return to $L_{\bar{k}}$ after $T_{\hat{k}}$ otherwise. In other words, we stop our walk if it returns to $L_{\bar{k}}$ after $T_{\hat{k}}$. By Lemma 3.5, $P(\tilde{\xi}(n) \neq \xi(n)) = O\left(n^{-100}\right)$ and Theorem 2.8 remains true with ξ replaced by $\tilde{\xi}$. We need the following estimate.

Lemma 6.1. *The family of random variables*

$$\left\{ \sqrt{n} \mathbb{P}_\omega(\tilde{\xi}(n) = z) \right\}_{n \in \mathbb{N}, z \in \mathbb{S}}$$

is uniformly integrable.

Proof. Let $I_j = \{l : l - \mathbb{E}_\omega(T_{\bar{k}}) \in [j\sqrt{n}, (j+1)\sqrt{n}\}$. Then

$$\mathbb{P}_\omega(\tilde{\xi}(n) = z) = \sum_l \mathbb{P}_\omega(T_{\bar{k}} = l) \mathbb{P}_\omega(\tilde{\xi}(n) = z | T_{\bar{k}} = l)$$

$$\leq \sum_j \max_{l \in I_j} \mathbb{P}_\omega(T_k = l) \max_{y \in \{1 \dots m\}} \sum_{t = j\sqrt{n}}^{(j+1)\sqrt{n}} \mathbb{P}(\tilde{\xi}(t) = z | \tilde{\xi}(0) = (\bar{k}, y))$$

$$(6.6) \qquad \leq \sum_j \max_{l \in I_j} \mathbb{P}_\omega(T_k = l) \tilde{\rho}(z)$$

where $\tilde{\rho}(z)$ is the maximum over $\hat{z} \in L_{\hat{k}}$ of the expected total number of visits to site z by the walk $\tilde{\xi}$ started from $\hat{z}$.

Next we claim that there is a constant C such that for all $\omega \in \Omega$ and all $l \in I_j(\omega)$ we have

$$(6.7) \qquad \mathbb{P}_\omega(\tilde{\xi}(n) = l) \leq \frac{C \operatorname{Var}_\omega(T_{\bar{k}})}{j^2 n^{3/2}}$$

Indeed, let $T' = T_{\bar{k}/2}$, $T'' = T_{\bar{k}} - T_{\bar{k}/2}$. Clearly

$$\mathbb{P}_\omega(\tilde{\xi}(n) = l) \leq$$

$$\mathbb{P}_\omega\left(\tilde{\xi}(n) = l, |T' - \mathbb{E}_\omega(T')| \geq \frac{j\sqrt{n}}{2}\right) + \mathbb{P}_\omega\left(\tilde{\xi}(n) = l, |T'' - \mathbb{E}_\omega(T'')| \geq \frac{j\sqrt{n}}{2}\right).$$

We will estimate the first term, the second one is similar.

$$\mathbb{P}_\omega\left(\tilde{\xi}(n) = l, |T' - \mathbb{E}_\omega(T')| \geq \frac{j\sqrt{n}}{2}\right) \leq$$

$$\mathbb{P}_\omega\left(|T' - \mathbb{E}_\omega(T')| \geq \frac{j\sqrt{n}}{2}\right) \mathbb{P}_\omega\left(\tilde{\xi}(n) = l \big| |T' - \mathbb{E}_\omega(T')| \geq \frac{j\sqrt{n}}{2}\right).$$

The first factor is smaller than

$$\frac{4\operatorname{Var}_\omega(T')}{j^2 n} \leq \frac{4\operatorname{Var}_\omega(T_{\bar{k}})}{j^2 n}$$

due to Chebyshev inequality. On the other hand the proof of (5.11) shows that the second factor is smaller than $\frac{C}{\sqrt{n}}$ proving (6.7).

Rewriting (6.7) as

$$\max_{l \in I_j(\omega)} \mathbb{P}_\omega(T_{\bar{k}} = l) \le \frac{C \, \mathrm{Var}_\omega(T_k)}{j^2 n^{3/2}}$$

and summing over j in (6.6) we obtain that

$$\sqrt{n}\mathbb{P}_\omega(\tilde{\xi}(n) = z) \le C \, \frac{\mathrm{Var}_\omega(T_{\bar{k}})}{n} \tilde{\rho}(z).$$

Now the uniform integrability of the LHS follows from the following facts:

(a) $\dfrac{\mathrm{Var}_\omega(T_{\bar{k}})}{n}$ and $\tilde{\rho}(z)$ are independent (since the first variable depends only on environment to the left of $\bar{k}$ the second variable depends only on environment to the right of $\bar{k}$);

(b) $\dfrac{\mathrm{Var}_\omega(T_{\bar{k}})}{n}$ is uniformly integrable as follows easily from the explicit expression for $\mathrm{Var}_\omega(T_{\bar{k}})$ given in [15, Equation (4.28)];

(c) $\{\tilde{\rho}(z)\}_{z \in \mathbb{S}}$ are uniformly integrable since $\tilde{\rho}(z) \le C\rho(z)$ and $\{\tilde{\rho}(z)\}_{z \in \mathbb{S}}$ are uniformly integrable due to Lemma 3.3(a). □

Proof of Theorem 2.9. Once the uniform integrability of $\{\sqrt{n}\mathbb{P}_\omega(\tilde{\xi}(n) = z)\}$ is established, the derivation of Theorem 2.9 from Theorem 2.8 is similar to derivation of Theorem 2.7 from the LLT in the enlarged environment which is explained in Section 5 so we just sketch the argument leaving the details to the reader. The proof consists of the following steps.

(I) We have

$$\mathrm{P}(\xi_n = (k, i)) = \mathbf{E}(\mathbb{P}_\omega(\xi_n = (k, i))).$$

Theorem 2.8 and Lemma 6.1 allow us to replace the above expectation by

$$\mathbf{E}\left(\frac{\rho_{(k,i)}}{\sqrt{2\pi n D a}} \exp - \left[\frac{(k - b_n(\omega))^2}{2D^2 n}\right]\right).$$

(II) Asymptotic independence of b_n and $\rho_{k,i}$ (which comes from the fact that $\rho_{(k,i)}$ can be well approximated by a variable which depends only on the environment to the right of $n - \ln^2 n$ while b_n can be well approximated by a variable which depends only on the environment from the left of $n - \ln^2 n$) allows us to replace the last expectation by the product

(6.8) $$\mathbf{E}\left(\frac{\rho_{(k,i)}}{a}\right) \mathbf{E}\left(\frac{1}{\sqrt{2\pi n D}} \exp - \left[\frac{(k - b_n(\omega))^2}{2D^2 n}\right]\right).$$

(III) The first factor in (6.8) equals to 1 due to Remark 4.2.

(IV) Asymptotic normality of $b_n(\omega)$ shows that the second factor in (6.8) is asymptotic to

$$\mathbf{E}\left(\frac{1}{\sqrt{2\pi n \mathbf{D}}} \exp - \left[\frac{(k - b_n(\omega))^2}{2\mathbf{D}^2 n}\right]\right)$$

where $\mathbf{D^2} = D^2 + \hat{D}^2$ and $\hat{D}$ is the limiting variance of $\dfrac{b_n(\omega)}{\sqrt{n}}$ (cf. the derivation of (5.9) in Section 5). $\qquad\qquad\square$

7. Mixing for environment seen by the particle.

Proof of Theorem 2.13(b). As has been explained in Section 2.7, the result is known in the recurrent case, so it remains to consider the transient case.

Denote $\omega_n \overset{\text{def}}{=} (P_n, Q_n, R_n)$. It suffices to prove (2.16) for a dense set of functions, in particular, it is enough to consider the case when $\Phi(\omega, k)$ depends only on ω_n for $|n| \leq M$ for some finite M. Below we shall give the proof in the case $M = 0$. The case of arbitrary finite n requires routine modifications which are left to the reader. Thus $\Phi = \Phi(\omega_0, Y)$. Let w_n be the vector with components $w_n(k) = \Phi(\omega_n, k)$. We have

$$\mathbb{E}_\omega\left(\Phi\left(\omega^{(N)}, Y_N\right)\right) = \sum_{k=1}^{m} \sum_{n=-\infty}^{\infty} \mathbb{P}_\omega(X_N = n, Y_N = k)w_n(k).$$

By the quenched CLT for each ε there exists R such that for all sufficiently large N, the sum over n such that $|n - b_N| \geq R\sqrt{N}$ is less than ε where $b_N(\omega)$ is defined by (2.11).

Next, fix small constants δ_1, δ_2 and let

$$N_l = l^{2-\delta_1}, \quad I_{j,l} = \left[N_l + jN_l^{\frac{1}{2}-\delta_2}, \ N_l + (j+1)N_l^{\frac{1}{2}-\delta_2}\right]$$

and let $x_{l,j}$ be the center of $I_{l,j}$. Let

$$\mathfrak{r}_{l,j} = \sum_{n \in I_{l,j}} \sum_{k=1}^{m} \rho_{n,k} w_n(k) = \sum_{n \in I_{l,j}} \sum_{k=1}^{m} \rho_{n,k} w_n(k) + \mathcal{O}\left(\theta^{N_l}\right).$$

Lemma 4.4 shows that

$$\mathbf{P}\left(\left|\frac{\mathfrak{r}_{l,j}}{N_l^{1/2-\delta_2}} - \mathfrak{a}\right| > \varepsilon\right) \leq C\frac{1}{\varepsilon^{2+\delta}N_l^{(1/2-\delta_2)(1+\delta/2)}} = \frac{C}{\varepsilon^{2+\delta}l^{(1/2-\delta_2)(1+\delta/2)(2-\delta_1)}}.$$

Thus if

$$\left(\frac{1}{2} - \delta_2\right)\left(1 + \frac{\delta}{2}\right)(2 - \delta_1) - 2(2 - \delta_1)\delta_2 > 1$$

then

$$\sum_{l=1}^{\infty} \sum_{|j| < N_l^{2\delta_2}} \mathbf{P}\left(\left|\frac{\mathfrak{r}_{l,j}}{N_l^{1/2-\delta_2}} - \mathfrak{a}\right| > \varepsilon\right) < \infty,$$

so Borel–Cantelli Lemma gives that

(7.1) $$\frac{\mathfrak{r}_{l,j}}{N_l^{1/2-\delta_2}} \to \mathfrak{a}$$

uniformly for $|j| < N_l^{2\delta_2}$. Given N let l be such that $N_l \leq N < N_{l+1}$. Now the quenched LLT (Theorem 2.11) gives that

$$\sum_{n=b_N-R\sqrt{N}}^{b_N+R\sqrt{N}} \sum_{k=1}^{m} \mathbb{P}_\omega \left(\xi_N = (n,k) \right) w_n(k) \approx \sum_j \exp\left[-\frac{(x_{l,j}-b_N)^2}{2D^2 N} \right] \sum_{n\in I_{l,j}} \sum_{k=1}^{m} \frac{\rho_{(n,k)}}{\sqrt{2\pi N}Da} w_n(k)$$

where the first summation is over j such that $I_{l,j} \cap \left[b_N - R\sqrt{N}, b_N + R\sqrt{N} \right] \neq$

$\emptyset$ and "$\approx$" holds since $\exp\left[-\frac{(x-b_N)^2}{2D^2 N} \right]$ is approximately constant when x varies over $I_{l,j}$.

Next, (7.1) allows us to replace the sums over n and k by $N_l^{1/2-\delta_2} \mathfrak{a}$. Performing summation over j we obtain the Riemann sum for

$$\mathfrak{a} \int_{-R}^{R} \frac{1}{\sqrt{2\pi}D} e^{-x^2/2D^2} dx.$$

Since R can be chosen arbitrarily large, (2.16) follows. $\square$

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