Short-time Fourier transform restriction phenomena and applications to nonlinear dispersive equations

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Chapter 1

Introduction

At the core of this work is the discussion of a robust approach to study regularity properties of solutions to nonlinear dispersive equations. We focus on initial data in L^2-based Sobolev spaces H^s and derivative nonlinearities.

When referring to regularity properties, we mean a priori estimates, existence of solutions and continuous dependence on initial data in a function space $X \hookrightarrow C([0,T], H^s)$ locally in time. We say that the equation is locally well-posed if the data-to-solution mapping exists locally in time and is continuous.

We informally refer to an evolution equation as semilinear if the equation can be solved by the contraction mapping argument and as quasilinear if the equation can not be solved via Picard iteration.

Many examples of Cauchy problems considered below can be written as

$$\left\{ \begin{array}{cl} \partial_t u + P(D)u &= u\partial_{x_1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{K}^n, \\ u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{K}^n), \end{array} \right. \tag{1.1}$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})\}$. $H^s_{\mathbb{R}}$ denotes the isotropic Sobolev space on $\mathbb{K}^n$ of real-valued functions, and $P(D)$ denotes a skew-adjoint Fourier multiplier.

The argument from a seminal work [BS75] by Bona and Smith yields local well-posedness in $H^s_{\mathbb{R}}(\mathbb{K}^n)$ for $s > \frac{n+2}{2}$, but neglects dispersive effects. This approach is commonly referred to as energy method, which we aim to improve in the present work. This will illustrate the regularizing effect of dispersion.

Another motivation to work with less regular initial data is that conserved quantities like mass or energy are typically associated with lower Sobolev regularities. Thus, conserved quantities lead us to physically natural choices for initial data. Furthermore, in many cases a local result for these initial data can be globalized via iteration of the local result.

One-dimensional models are best understood, and the literature is extensive. A more accurate description of the developments in local well-posedness theory is postponed to the sections at the end of this chapter.
One prominent example is the Benjamin-Ono equation (cf. [Ben67, Ono75])

$$\left\{ \begin{array}{cl} \partial_t u + \mathcal{H}\partial_{xx} u &= u\partial_x u, \quad (t,x) \in \mathbb{R} \times \mathbb{K}, \\ u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{K}), \end{array} \right. \tag{1.2}$$

where $\mathcal{H} : L^2(\mathbb{K}) \to L^2(\mathbb{K})$ denotes the Hilbert transform.

This we shall consider as model case to describe some features of the deployed arguments.

On the real line, it is well-known that (1.2) is not locally well-posed in a uniform sense in function spaces embedded into $C([0,T], H^s_{\mathbb{R}})$ for any $s \in \mathbb{R}$ (cf. [MST01, KT05b]).

Another famous example is the Korteweg-de Vries equation (cf. [KdV95])

$$\begin{cases} \partial_t u + \partial_{xxx} u &= u \partial_x u, \quad (t,x) \in \mathbb{R} \times \mathbb{K}, \\ \quad u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{K}). \end{cases} \tag{1.3}$$

Due to higher dispersion than in the Benjamin-Ono case, (1.3) can shown to be locally well-posed in the semilinear sense for sufficiently regular initial data. However, quasilinear behavior is exhibited for $s < -3/4$ on the real line and $s < -1/2$ on the circle.

The above models admit several higher-dimensional generalizations. A multidimensional generalization of the Benjamin-Ono equation (cf. [PS95, Mar02, LRRW19]) is given by

$$\begin{cases} \partial_t u + \partial_{x_1} (-\Delta)^{1/2} u &= u \partial_{x_1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{K}^n, \\ \quad u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{K}^n). \end{cases} \tag{1.4}$$

The same holds for (1.3). One possible higher dimensional version of the Korteweg-de Vries equation is given by the Zakharov-Kuznetsov equation (cf. [ZK74, LS82])

$$\begin{cases} \partial_t u - \partial_{x_1} \Delta u &= u \partial_{x_1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{K}^n, \\ \quad u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{K}^n). \end{cases} \tag{1.5}$$

Other generalizations in two dimensions include the Kadomtsev-Petviashvili equations (cf. [HHK09, IKT08]).

Since P is a Fourier multiplier, we can rewrite the linear part of (1.1) as

$$\begin{cases} i\partial_t u + \varphi(\nabla/i) u &= 0, \quad (t,x) \in \mathbb{R} \times \mathbb{K}^n, \\ \quad u(0) &= u_0 \in H^s(\mathbb{K}^n). \end{cases} \tag{1.6}$$

In the above display $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$ is referred to as dispersion relation, and the Fourier coefficients of solutions evolve by

$$\hat{u}(t,\xi) = e^{it\varphi(\xi)} \hat{u}_0(\xi). \tag{1.7}$$

The L^2-based Sobolev spaces are also natural spaces for initial data as the linear evolution is unitary in these spaces.

Let $U(t)$ denote the unitary group in $H^s(\mathbb{K}^n)$ associated to the linear evolution of (1.1). Seeking for strong solutions to the full equation, we have to consider the following expression

$$u(t) = U(t) u_0 + \int_0^t U(t-s)(u \partial_{x_1} u)(s) ds =: \Phi_{u_0}(u). \tag{1.8}$$

If we prove $\Phi_{u_0} : X \to X$ to be a contraction mapping in a suitable function space $X \hookrightarrow C([0,T], H^s)$, then the aforementioned regularity properties of the data-to-solution mapping will be immediate.

Furthermore, by an infinite dimensional variant of the implicit function theorem, the dependence on the initial data will be as smooth as the nonlinearity. In the above examples, this would imply real analyticity of the data-to-solution mapping. But, by the above, there are models of physical relevance, where the data-to-solution mapping is known to be not even locally uniformly continuous. Thus, the corresponding Cauchy problems are not directly amenable to Picard iteration although well-posedness is expected from scaling arguments.

Control of rough wave interactions via frequency dependent time localization

To understand the problematic interaction disrupting uniform local well-posedness better, we localize frequencies on a dyadic scale. In the following we consider the interaction of a high frequency wave with a low frequency wave in (1.2). Let P_N denote the frequency projector to frequencies around $N \in 2^{\mathbb{N}_0}$. Controlling the energy transfer between high and low frequencies $K \ll N$ involves an estimate of the kind

$$\begin{aligned} &\|\partial_{x_1}(P_N U(t)u_0 P_K U(t)v_0)\|_{L^1([0,T],L^2_x(\mathbb{R}))} \\ &\lesssim NT^{1/2}\|P_N U(t)u_0 P_K U(t)v_0\|_{L^2([0,T],L^2_x(\mathbb{R}))} \\ &\lesssim (NT)^{1/2}\|P_N u_0\|_{L^2}\|P_K v_0\|_{L^2} \end{aligned} \tag{1.9}$$

as one can only expect to recover half of a derivative in a bilinear estimate in the case of Schrödinger interaction as is the case in (1.2).

(1.9) suggests to overcome the remaining derivative loss to consider frequency dependent time localization $T = T(N) = N^{-1}$. This would completely ameliorate the derivative loss. For the Benjamin-Ono equation on the real line this strategy was carried out by Guo et al. in [GPWW11].

To the best of the author's knowledge, the first works, where energy transfer is controlled by considering function spaces with frequency dependent time scales are due to Christ-Colliander-Tao [CCT08], Koch-Tataru [KT07] and Ionescu-Kenig-Tataru [IKT08].

A precursor of the argument can be found in the work [KT03] by Koch and Tzvetkov, where linear Strichartz estimates on frequency dependent time intervals were used to prove local well-posedness of (1.2) for $s > 5/4$.

Earlier, Burq-Gérard-Tzvetkov noticed that dispersive properties of solutions to Schrödinger equations on compact manifolds improve after frequency dependent time localization (cf. [BGT01]). It seems very likely that the stream of research ([KT03, CCT08, IKT08, KT07]) was influenced by the observation of more regular behavior (cf. [BGT01, BGT04]) on frequency dependent time scales.

For the Benjamin-Ono equation better local well-posedness results are available via a gauge transform, see below.
When considering dispersion generalized equations

$$\left\{ \begin{array}{ll} \partial_t u + \partial_x D_x^a u &= u\partial_x u, \quad (t,x) \in \mathbb{R} \times \mathbb{K}, \\ u(0) &= u_0 \in H^s(\mathbb{K}), \end{array} \right. \tag{1.10}$$

arguments involving gauge transform mechanisms are still applicable for $1 < a < 2$ on the real line (cf. [HIKK10]). However, this approach yields severe technical difficulties compared to the Benjamin-Ono case $a = 1$. This is in contrast to improved

dispersive effects for $1 < a < 2$, which makes the solutions exhibit more regular behavior. Moreover, in higher dimensions it is unknown whether there is a gauge transform available at all.
Hence, in order to investigate properties of solutions to more general dispersive PDE at low Sobolev regularities, we choose the approach of frequency dependent time localization.

Another reason is that this approach also works on tori with little adaptations as elaborated on in Chapter 3. This is surprising because in the case of compact manifolds dispersive effects take on a different character than on Euclidean space. Estimates of the kind

$$\|U(t)u_0\|_{L^\infty(M)} \lesssim |t|^{-\theta} \|u_0\|_{L^1(M)} \tag{1.11}$$

for $\theta > 0$ must fail due to conservation of mass on compact manifolds M.

Thus, linear and bilinear Strichartz estimates like discussed above can not hold true. However, it was observed for Schrödinger equations on compact manifolds by Burq-Gérard-Tzvetkov in [BGT04] (see also the work of Staffilani-Tataru [ST02], in which variable-coefficient Schrödinger equations were analyzed) that after localization in time to frequency dependent time intervals (1.11) can be recovered.

Indeed, the necessary time localization has to be chosen to $T = T(N) = N^{-1}$. This can be explained by a simple heuristic argument involving the group velocity, which for Schrödinger equations has modulus proportional to the frequency.

Consequently, a wave packet with frequencies around N should roughly stay in one chart for this time and can display behavior similar to waves in Euclidean space. One obtains the following modification of (1.11) (cf. [ST02, BGT04])

$$\|e^{it\Delta_g} P_N u_0\|_{L^\infty(M)} \lesssim |t|^{-n/2} \|P_N u_0\|_{L^1(M)} \quad 0 < |t| \lesssim N^{-1}, \tag{1.12}$$

where Δ_g denotes the Laplace-Beltrami operator on a compact boundaryless smooth Riemannian manifold M of dimension n, and P_N denotes the orthogonal projector localizing to eigenfunctions of $(-\Delta_g)^{1/2}$ having eigenvalues around $N \in 2^{\mathbb{N}_0}$.

The bilinear estimate can also be recovered for Schrödinger equations on compact manifolds

$$\|P_N e^{it\Delta_g} u_0 P_K e^{it\Delta_g} v_0\|_{L^2_t([0,N^{-1}],L^2_x(M))} \lesssim \left(\frac{K^{n-1}}{N} \right)^{1/2} \|P_N u_0\|_{L^2(M)} \|P_K v_0\|_{L^2(M)} \tag{1.13}$$

which was proved by Hani in [Han12]. Moreover, in the one dimensional case generalized estimates were discussed by Moyua and Vega in [MV08]. Thus, the observation (1.9) remains true on the circle and also the consequence of local well-posedness in $H^s(\mathbb{K})$ for $s > 1$.

In Chapter 3 the above arguments are given in detail, and it is pointed out how frequency dependent time localization can diminish the difference between Euclidean space and tori.

We revisit how bilinear Strichartz estimates follow in a well-known manner from transversality in Euclidean space, and the arguments from [MV08, Han12] are revisited to discuss short-time bilinear Strichartz estimates on compact manifolds. As indicated in (1.9), these estimates are crucial to overcome problematic frequency interactions and effectively improve the energy method by making use of dispersive effects. Also, one can perceive this as a bilinear refinement to the argument from [KT03].

In this chapter we also recall short-time linear Strichartz estimates as proved in [BGT04] and [Din17] in the fractional case. Taking a different approach, it is pointed out how ℓ^2-decoupling (cf. [BD15, BD17a]) leads to new Strichartz estimates for fractional Schrödinger equations on tori. Further, bilinear refinements are proved, and implications for well-posedness of generalized cubic nonlinear Schrödinger equations on tori are given in Section 3.2.1. The derivation of linear and bilinear Strichartz estimates via decoupling was made publicly available in [Sch19b].

Secondly, we shall see how short-time bilinear estimates combine with the idea from [IKT08] of frequency dependent time localization in Euclidean space. This allows us to overcome derivative loss to prove local well-posedness for Cauchy problems with derivative nonlinearities as well in Euclidean space as on tori. In Section 3.6 first applications to infer new well-posedness results are provided.

There have been several previous works where short-time analysis on tori is used to analyze dispersive PDE at low regularities. Among the first ones are contributions by Molinet [Mol12], Zhang [Zha16] and Kwak [Kwa16]. The present work seems to be the first one explicitly relating the results from [BGT04, Din17] and [Han12, MV08] to prove local well-posedness for evolution equations with derivative nonlinearities via frequency dependent time localization.

The improvement of the energy method via short-time bilinear estimates was made publicly available in [Sch18]. Chapter 3 also has a motivational character as the techniques are further refined in the following chapters.

New local well-posedness results for higher-dimensional Benjamin-Ono equations

In Chapter 4 the argument from Chapter 3 is deployed to prove new well-posedness results for Benjamin-Ono equations in higher dimensions as well in Euclidean space as on tori: The improvement stems from deploying bilinear short-time estimates, whereas in previous works (cf. [LPRT19, LRRW19]) only linear short-time estimates were used. The key difficulty is to verify transversality at comparable frequencies which is more involved in case of the higher-dimensional dispersion relations of (1.4) or (1.5). Moreover, we introduce fractional equations to relate the higher dimensional Benjamin-Ono equation from [LRRW19] and (1.5). We refer to Theorems 4.1.1 and 4.1.8 for the results. The analysis was made publicly available in [Sch19c].

New regularity results for dispersive PDE with cubic derivative nonlinearity on the circle

In Chapter 5 new a priori estimates and existence of solutions for one-dimensional dispersive equations with cubic derivative nonlinearity are proved.
The equations under consideration are instances of

$$\left\{ \begin{array}{ll} \partial_t u + \partial_x D_x^a u &= \partial_x (u^3), \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{T}). \end{array} \right. \tag{1.14}$$

For $a = 1$, (1.14) is known as modified Benjamin-Ono equation and for $a = 2$ as modified Korteweg-de Vries equation. The latter equation is well-known to be semilinear provided that $s \geq 1/2$ (cf. [Bou93b]). The periodic modified Benjamin-Ono equation requires a gauge change before it is solvable by Picard iteration (cf. [GLM14]). As well the periodic modified Benjamin-Ono as the modified Korteweg-de Vries equation fail to be C^3-well-posed below $s = 1/2$ although the scaling critical regularities are given by $s = 0$ for $a = 1$ and $s = -1/2$ for $a = 2$. Thus, both equations are expected to be well-posed below $s = 1/2$.

In addition to the arguments from the previous chapters, the analysis of the Sobolev energies of solutions is refined by adding correction terms in the spirit of the I-method (cf. [CKS$^+$02, CKS$^+$03]). This requires a better comprehension of the resonance function than is currently available for higher dimensional Benjamin-Ono equations. We refer to Theorem 5.1.1. For periodic solutions to the Benjamin-Ono equation this gives the first regularity result below $s = 1/2$. By working in Euclidean windows, i.e., frequency dependent time localization given by $T = T(N) = N^{-1}$, we recover the same a priori estimates as were previously shown in Euclidean space (cf. [Guo11]).

A related model is the derivative nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_{xx} u &= i\partial_x(|u|^2 u), \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \\ \qquad u(0) &= u_0 \in H^s(\mathbb{T}). \end{cases} \tag{1.15}$$

(1.15) appears to be very similar to (1.14) for $a = 1$. Sharp C^3 local well-posedness of (1.15) was shown by Herr [Her06]. Here, the same local regularity results below $s < 1/2$ for (1.15) like for the modified Benjamin-Ono equation on the circle are shown. This improves the result of Takaoka [Tak16], which was shown by different means. In [Tak16] a priori estimates and existence of solutions were shown for $s > 12/25$.
However, (1.15) is known to be completely integrable, which is not the case for the modified Benjamin-Ono equation. In order to point out that the method does not depend on complete integrability, we choose to analyze (1.14) for $a = 1$ in detail and point out the necessary modifications to deal with (1.15) as well in Subsection 5.1.5. The analysis of Section 5.1 was made publicly available in [Sch17a].

The modified Korteweg-de Vries equation is known to be completely integrable, too. There are recent results exploiting the integrability and showing a priori estimates up to the scaling critical regularity $s_c = -1/2$ (cf. [KV19, KT18]).

Still, we carry out the perturbative analysis to prove existence of solutions and a priori estimates in Sobolev spaces with positive regularity index as the analysis extends to related models, which fail to be integrable anymore. Here, we choose not to work in Euclidean windows but again with time localization $T = T(N) = N^{-1}$ and perform a more precise multilinear analysis of possible interactions involving the resonance function.

Another motivation to carry out the analysis was to prove existence of solutions. This does not follow from the argument hinging on complete integrability. The result is given in Theorem 5.2.2. This is most interesting in Sobolev spaces with negative regularity index as renormalized versions of the mKdV equation are no longer equivalent, effectively pointing out the only renormalized version admitting existence of solutions.

In related Fourier Lebesgue spaces, this was accomplished by Kappeler and Molnar in [KM17] by arguments relying on complete integrability. We prove existence of solutions to a renormalized version of the modified Korteweg-de Vries equation in Sobolev spaces with negative regularity index and hence, non-existence of the unrenormalized solutions conditional upon conjectured periodic Strichartz estimates for the Airy evolution in Theorem 5.2.3. The results of Section 5.2 were made publicly available in [Sch17b].

Local and global well-posedness for dispersion generalized fractional Benjamin-Ono equations on the circle

In Chapter 6 we revisit dispersion generalized Benjamin-Ono equations (1.10) for $1 < a < 2$ on the circle. Combining the short-time analysis with resonance considerations and correction terms for the energy, we prove new local and global well-posedness results in Theorem 6.1.1. The correction terms are derived from normal form transformations related to the argument from the previous chapter, but without symmetrization.

On the circle, the only results for (1.10) beyond the energy method are global well-posedness for $s \geq 1 - a/2$ by Molinet-Vento [MV15]. Their result was proved by a different approach. For $a > 3/2$ we can prove global well-posedness in $L^2(\mathbb{T})$. The analysis was made publicly available in [Sch19a].

Variable-coefficient decoupling and smoothing estimates

The last section has a different character because no nonlinear evolution equations are considered. Rather, we take a more abstract point of view and discuss regularity results for oscillatory integral operators which come up in the short-time analysis of free solutions to Schrödinger equations on compact manifolds (cf. [BGT04]): these are the Fourier integral operators

$$T^\lambda f(x,t) = \int_{\mathbb{R}^n} e^{i\phi^\lambda(x,t;\xi)} a^\lambda(x,t;\xi) \hat{f}(\xi) d\xi \tag{1.16}$$

for suitable phase functions ϕ. These constitute a variable coefficient generalization of the constant-coefficient phase functions encountered in the classical restriction problem.

There, one considers the operators

$$\mathcal{E} f(x,t) = \int_{\mathbb{R}^n} e^{i(x.\xi + t\phi(\xi))} a(x,t;\xi) f(\xi) d\xi. \tag{1.17}$$

It is well-known (cf. [Wis05],[BG11, Section 6]) that in the context of the restriction problem strictly less $L^p - L^q$-estimates become admissible after changing from constant to variable-coefficient phase functions in (1.17).

However, in Chapter 7 we prove the same decoupling estimates for (1.16) with variable coefficients like in the constant-coefficient case (cf. [BD15]). It is well-known that in the context of wave equations decoupling estimates can be utilized effectively to prove L^p-smoothing estimates (cf. [Wol00, LaW02]), which provided initial motivation to study decoupling estimates. Recently, this was extended to the variable-coefficient context by Beltran-Hickman-Sogge in [BHS18].

In the Schrödinger context we prove new L^p-smoothing estimates for operators (1.16). For this we utilize a variable-coefficient generalization (cf. [Lee06a]) of Tao's bilinear adjoint Fourier restriction theorem [Tao03]. The derived L^p-smoothing estiates extend the constant-coefficient result by Rogers and Seeger from [RS10].

Remarks

In the following we sketch important developments in the exhaustive local well-posedness theory of the Benjamin-Ono equation and the Korteweg-de Vries equation. The equations are well understood, and no new results for these equations are proved in this work. Still, the search for an improved comprehension of these two model cases had been propelling the development of short-time arguments (cf. [GPWW11, Mol12, Liu15]). The description of the well-posedness theory is also given below for a comparison with different approaches.

Well-posedness theory for the Benjamin-Ono equation

The Benjamin-Ono equation was derived by Benjamin in [Ben67] and Ono in [Ono75] to describe internal water waves at great depth.

In [ABFS89] Abdelouhab et al. proved local well-posedness for $s > 3/2$ invoking the energy method (cf. [BS75]). We discuss the situation on the real line first. Molinet-Saut-Tzvetkov proved in [MST01] that the data-to-solution mapping fails to be C^2 in any Sobolev space and Koch-Tzvetkov argued in [KT05b] that the data-to-solution mapping even fails to be uniformly continuous due to the $High \times Low \to High$-interaction described above.

Using linear short-time Strichartz estimates, the same authors proved local well-posedness for $s > 5/4$ in [KT03], which was the first result going beyond 3/2.

A milestone in the well-posedness theory was Tao's proof of global well-posedness in $H^1(\mathbb{R})$ in [Tao04]. In this work, a gauge transform related to the Cole-Hopf transform was used to weaken the derivative loss significantly, and after applying the gauge transform, the equation can be solved by Strichartz estimates.

In [Tao04] only Strichartz estimates were used as the gauge transform requires considerably more careful treatment in Fourier restriction spaces. By these means, Burq-Planchon proved local well-posedness in $H^s(\mathbb{R})$ for $s > 1/4$ in [BP08] and this analysis was further improved by Ionescu and Kenig in [IK07] where global well-posedness in $L^2(\mathbb{R})$ was proved. The original proof was simplified by Molinet and Pilod in [MP12] and Ifrim-Tataru in [IT19]. In [IT19] the use of Fourier restriction spaces was avoided by combining normal form transformations with the gauge transform.
For a recent survey on the Benjamin-Ono equation on the real line, we refer to the work by Saut [Sau18].

On the circle, Molinet pointed out in [Mol07, Mol08] that one can also treat the periodic Benjamin-Ono equation in a perturbative way after gauge transform and fixing the mean of the initial value. This proved the data-to-solution mapping of the original equation to be C^∞ on hypersurfaces of initial data with fixed mean. The argument yields global well-posedness in $L^2(\mathbb{T})$. However, for the original equation and also for dispersion generalized versions, it was checked by Herr in [Her08] that a bilinear estimate controlling the nonlinear wave interaction can not hold true. Thus, this family of equations is not directly amenable to Picard iteration.

Hinging on complete integrability, in a recent preprint by Talbut [Tal18] a priori estimates as well on the real line as on the circle were claimed for $s > -1/2$, that is up to the scaling critical regularity.

For the dispersion generalized equations

$$\begin{cases} \partial_t u + \partial_x D_x^a u &= u \partial_x u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) &= u_0 \in H^s(\mathbb{R}), \end{cases} \tag{1.18}$$

global well-posedness in $L^2(\mathbb{R})$ was proved by Herr et al. in [HIKK10] adapting the gauge transform for $1 < a < 2$. Carrying out this approach brought up substantial technical difficulties due to the strong dependence of the gauge on the frequencies.

Notably, in a previous work by Herr [Her07] was shown that after weakening the problematic $High \times Low \to High$-interaction through introducing a low-frequency weight, (1.18) becomes amenable to Picard iteration for $1 < a < 2$, and sharp local well-posedness results were established.

A much simpler approach than the one from [HIKK10] was pointed out recently by Molinet and Vento in [MV15], where local well-posedness for $s \geq 1 - a/2$ was proved as well on the real line as on the circle. This work constitutes another improvement of the energy method, which relies on understanding the resonance. In higher dimensions this becomes a complicated endeavour.

Well-posedness theory for the Korteweg-de Vries equation

The Korteweg-de Vries equation was derived by Korteweg and de Vries in [KdV95] to describe traveling waves in shallow water and is certainly one of the most important dispersive models. Surprisingly, the solutions to (1.3) possess an infinite number of conserved quantities (cf. [Lax68, MGK68]). This property among others is nowadays perceived as a consequence of complete integrability. However, it turns out that the definition of complete integrability in infinite dimensions is a delicate issue, and we refer to [KV19] for a modern perspective.

(1.3) is linked to the defocusing modified Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_{xxx} u &= u^2 \partial_x u, \quad (t,x) \in \mathbb{R} \times \mathbb{K} \\ u(0) &= u_0 \in H^s(\mathbb{K}) \end{cases} \tag{1.19}$$

via the Miura transform (cf. [Miu68]). Thus, it is not surprising that (1.19) is also completely integrable. However, these properties could not be effectively exploited for the well-posedness theory on the real line until recently (cf. [KT18, KVZ18, KV19]).

The first local well-posedness results on the real line improving the result due to energy methods was established by Kenig-Ponce-Vega in [KPV93] using dispersive effects, in particular smoothing and maximal function estimates.
Breakthrough results were established by applying Picard iteration in Fourier restriction spaces by Bourgain in [Bou93b]. The short-time analysis introduced in [IKT08] builds on Fourier restriction spaces (cf. Chapter 2).
The title of this work is a deliberate homage to the seminal works [Bou93a, Bou93b].

In [Bou93b] global well-posedness in $L^2(\mathbb{T})$ was proved for (1.3) and local well-posedness for (1.19) in $H^{1/2}(\mathbb{T})$. On the circle these were the first results improving on the energy method.
Regarding (1.3), the Fourier restriction approach was refined by Kenig-Ponce-Vega

in [KPV96], where smooth local well-posedness in $H^s(\mathbb{R})$ for $s > -3/4$ and $H^{-1/2}(\mathbb{T})$ was proved.
These results are again sharp as the data-to-solution mapping fails to be C^2 below these regularities.

On the circle the properties following from complete integrability are more accessible due to compactness, and Kappeler and Topalov proved global well-posedness of (1.3) in $H^{-1}(\mathbb{T})$ in [KT06]. Utilizing the Miura transform, global well-posedness of (1.19) in $L^2(\mathbb{T})$ was shown in [KT05a]. These results are sharp (cf. [Mol12]).
We refer to Section 5.2 for further discussion.

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Professor Sanghyuk Lee provided me with a careful introduction to regularity problems related to oscillatory integral operators. In particular, we talked about recent developments regarding decoupling and polynomial partitioning in the context of the restriction problem. From the discussions I could obtain the foundations for the work on Chapter 7.

With Professor Soonsik Kwon I had been discussing normal form transformations and recent works on completely integrable dispersive PDE. Normal form transformations and modified energies play an important role in Chapters 5 and 6, and the time spent at KAIST deepened my insight.
Moreover, I appreciated very much to attend the PDE Summer School 2018 at KAIST, which was also organized by Professor Kwon, with lectures by Professor Kenji Nakanishi and Professor Vera Mikyoung Hur. On this occasion, the physical relevance of dispersion generalized Benjamin-Ono equations, which are studied in Chapter 6, was pointed out to me.

Furthermore, I am deeply indebted to my family for their support and encouragement during the work on my doctoral thesis.

Chapter 2

Notation and function spaces

Purpose of this section is to fix the notation and describe the general setting of the following analysis. We introduce function spaces, into which the solutions will be placed.

The set-up is explained in detail for the Euclidean space. Many basic properties from below remain true when considering periodic domains.

In the following we use the notation $A \lesssim B$ to denote $A \leq CB$ for some harmless constant C, which can change from line to line. To point out dependence on parameters, e.g. p, q, the notation $A \lesssim_{p,q} B$ is used. This is short-hand for $A \leq C(p,q)B$. The symbols $\sim$ or $\gtrsim$ are supposed to be understood likewise.

Furthermore, $s\pm$ refers to $s \pm \varepsilon$ for $\varepsilon > 0$, and $A \lesssim N^{s\pm} B$ is short-hand notation for $A \lesssim_\varepsilon N^{s\pm\varepsilon} B$.

2.1 Schwartz functions and the Fourier transform

Definition 2.1.1. The Schwartz space is defined as

$$\begin{aligned} \mathcal{S}(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C} \mid &f \text{ is smooth and} \\ &\forall \alpha, \beta \in \mathbb{N}_0^n : \|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty \}. \end{aligned} \tag{2.1}$$

References are [SW71, Chapter 1], [Gra14, Chapter 2]. $\mathcal{S}(\mathbb{R}^n)$ becomes a separable Fréchet-space when considering $\| \cdot \|_{\alpha,\beta}$ as a collection of seminorms:

$$d(f,g) = \sum_{\alpha,\beta \in \mathbb{N}_0^n} 2^{-(|\alpha|+|\beta|)} \frac{\|f-g\|_{\alpha,\beta}}{1 + \|f-g\|_{\alpha,\beta}}. \tag{2.2}$$

The topological dual space, whose elements will be referred to as tempered distributions (or simply distributions, when there is no room for confusion), is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

The Fourier transform of an L^1-function $f : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix.\xi} dx, \tag{2.3}$$

where $x.\xi = \sum_{i=1}^n x_i \xi_i$ denotes the standard inner product in $\mathbb{R}^n$.
Here, we follow the conventions of [Sog17, Chapter 0].

The Fourier transform $\hat{f}$ is a homeomorphism on $\mathcal{S}(\mathbb{R}^n)$ and inverted by

$$\check{g}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} g(\xi) e^{ix.\xi} d\xi. \tag{2.4}$$

For $f, g \in L^1(\mathbb{R}^n)$ define convolution by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy \tag{2.5}$$

and for $f, g \in \mathcal{S}(\mathbb{R}^n)$ recall the fundamental relations (cf. [Sog17, Theorem 0.1.8])

$$(2\pi)^n \langle f, g \rangle = (2\pi)^n \int_{\mathbb{R}^n} f(x) \overline{g(x)} dx = \int_{\mathbb{R}^n} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = \langle \hat{f}, \hat{g} \rangle \quad \text{(Parseval)}, \tag{2.6}$$

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} \|f\|_{L^2(\mathbb{R}^n)} \qquad \text{(Plancherel)}, \tag{2.7}$$

$$\widehat{(fg)}(\xi) = (2\pi)^{-n} (\hat{f} * \hat{g})(\xi), \tag{2.8}$$

which imply the Fourier transform to be a unitary operator (up to an irrelevant factor) on $L^2(\mathbb{R}^n)$.

Next, we define Littlewood-Paley projectors in Euclidean space. For a detailed exposition, see [Gra14, Chapter 6].

Let $\rho(\xi)$ be a smooth and radially decreasing function with

$$\rho(\xi) \equiv 1, \quad |\xi| \leq 1 \text{ and supp } \rho \subseteq B(0,2).$$

For $k \in \mathbb{Z}$ define

$$\chi_k(\xi) = \rho(2^{-k}\xi) - \rho(2^{1-k}\xi), \quad \text{supp } \chi_k \subseteq B(0, 2^{k+1}) \backslash B(0, 2^{k-1})$$

and the kth Littlewood-Paley projector is defined by

$$\widehat{(P_k f)}(\xi) = \chi_k(\xi) \hat{f}(\xi), \qquad f \in \mathcal{S}'(\mathbb{R}^n).$$

It follows that $P_k f \in C^\infty(\mathbb{R}^n)$. Occasionally, we write synonymously $P_k = P_K$, where capital letters K denote dyadic numbers and minuscules the dyadic logarithm.

Usually, frequencies less than 1 are considered together:

$$P_{\leq 0} = \sum_{k \leq 0} P_k. \tag{2.9}$$

2.2 Sobolev spaces and Fourier restriction spaces

In the following the function spaces for initial data are typically L^2-based inhomogeneous Sobolev spaces

$$H^s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \|f\|_{H^s} = \|\langle \xi \rangle^s \hat{f}(\xi)\|_{L^2_\xi} < \infty \}, \tag{2.10}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

When we consider solutions $u(t,x) \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^n)$ to evolution equations

$$\begin{cases} i\partial_t u + \varphi(\nabla/i) u &= 0, \quad \varphi \in C^1(\mathbb{R}^n, \mathbb{R}), \\ u(0) &= u_0 \in H^s(\mathbb{R}^n), \end{cases} \tag{2.11}$$

we distinguish time as separate variable, and the space-time Fourier transform is denoted by

$$\mathcal{F}_{t,x}[u](\tau,\xi) = \tilde{u}(\tau,\xi) = \int_{\mathbb{R}\times\mathbb{R}^n} e^{-it\tau} e^{-ix.\xi} u(t,x) dt dx. \tag{2.12}$$

Here, (τ,ξ) denote the dual variables of (t,x).

The space-time Fourier transform is inverted by

$$u(t,x) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}\times\mathbb{R}^n} e^{it\tau} e^{ix.\xi} \tilde{u}(\tau,\xi) d\tau d\xi. \tag{2.13}$$

In (2.11), $\varphi(\nabla/i)$ is supposed to be understood as a Fourier multiplier

$$\widehat{(\varphi(\nabla/i)u)}(\xi) = \varphi(\xi)\hat{u}(\xi).$$

By Stone's theorem, it follows that (2.11) gives rise to a unitary evolution on H^s.
A solution $u \in \mathcal{S}'$ to (2.11) satisfies

$$(\tau - \varphi(\xi))\tilde{u}(\tau,\xi) = 0. \tag{2.14}$$

Thus, the distributional support of $\tilde{u}$ is concentrated on the set $\{\tau = \varphi(\xi)\}$. In the following this will typically be a hypersurface with non-vanishing curvature. It is well-known that the Fourier transform of compactly supported functions on curved surfaces (cf. [Sog17, Chapter 2.2]) leads to the dispersive properties of solutions to (2.11) in Euclidean space.

According to the symbol suggested by (2.14), we define the Fourier restriction spaces

$$X^{s,b}_{\varphi} = \{u \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^n) \mid \|u\|_{X^{s,b}_{\varphi}} = \|\langle \tau - \varphi(\xi)\rangle^b \langle \xi \rangle^s \tilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}} < \infty\}, \tag{2.15}$$

where $s, b \in \mathbb{R}$.

Nonlinear dispersive PDE on tori were systematically studied in [Bou93a] and [Bou93b]. In the context of wave equations in Euclidean space there is the related work by Klainerman-Machedon [KM93], see also the earlier works by Beals and Rauch-Reed [Bea83, RR82].

s is referred to as the variable of spatial regularity and b as variable of modulation regularity.

We have the following lemma that local solutions are $X^{s,b}$-elements:

Lemma 2.2.1 (Free solutions in $X^{s,b}$)**.** *Let $f \in H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. Then, for any Schwartz time cutoff $\eta \in \mathcal{S}(\mathbb{R})$ we find the following estimate to hold:*

$$\|\eta(t) e^{it\varphi(\nabla/i)} u_0\|_{X^{s,b}_{\varphi}(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{\eta,b} \|u_0\|_{H^s(\mathbb{R}^n)}. \tag{2.16}$$

Related to Sobolev embedding, $X^{s,b}$-functions can be written as superposition of free solutions for $b > 1/2$. Consequently, modulation stable properties of free solutions are inherited by the $X^{s,b}$-functions for $b > 1/2$. We have the following lemma:

Lemma 2.2.2 (Transfer principle in Fourier restriction spaces, [Tao06, Lemma 2.9, p. 100])**.** *Let $s \in \mathbb{R}$ and $b > 1/2$. Let Y be a Banach space comprised of functions on $\mathbb{R} \times \mathbb{R}^n$ with the property that*

$$\|e^{it\tau_0} e^{it\varphi(\nabla/i)} f\|_Y \lesssim \|f\|_{H^s(\mathbb{R}^n)} \tag{2.17}$$

for all $f \in H^s(\mathbb{R}^n)$ and $\tau_0 \in \mathbb{R}$. Then, we have the embedding

$$\|u\|_Y \lesssim_b \|u\|_{X^{s,b}_\varphi}. \tag{2.18}$$

E.g. the Strichartz estimates for solutions to the Schrödinger equation (cf. [KT98])

$$\|e^{it\Delta} u_0\|_{L^q_t(\mathbb{R}, L^p_x(\mathbb{R}^n))} \lesssim_{n,p,q} \|u_0\|_{L^2(\mathbb{R}^n)} \quad (2 \leq q, p \leq \infty,\ \frac{2}{q} + \frac{n}{p} = \frac{n}{2}) \tag{2.19}$$

read in the context of $X^{s,b}$-spaces

$$\|e^{it\Delta} u_0\|_{L^q_t(\mathbb{R}, L^p_x(\mathbb{R}^n))} \lesssim_{n,p,q,b} \|u\|_{X^{0,b}_\Delta} \quad (q,p) \text{ like above, } b > 1/2. \tag{2.20}$$

The following linear estimate in $X^{s,b}$-spaces points out how Duhamel's formula generalizes the fundamental theorem of calculus. For a nonlinear equation

$$\begin{cases} i\partial_t u + \varphi(\nabla/i)u & = F(u), \\ \quad u(0) & = u_0 \in H^s(\mathbb{R}^n), \end{cases} \tag{2.21}$$

a function $u \in C([0,T], H^s)$ with $F(u) \in L^1([0,T], H^s)$ is referred to as strong solution to (2.21) provided that u satisfies Duhamel's formula

$$u(t) = e^{it\varphi(\nabla/i)} u_0 - i \int_0^t e^{i(t-s)\varphi(\nabla/i)} F(u(s)) ds. \tag{2.22}$$

In $X^{s,b}$-spaces we have the following linear estimate:

Lemma 2.2.3 ($X^{s,b}$-energy estimate, [Tao06, Proposition 2.12, p. 103])**.** *Let $u \in C^\infty_{t,loc}\mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ be a smooth solution to (2.21). Then, for any $s \in \mathbb{R}$ and $b > 1/2$, and any compactly supported smooth time cutoff $\eta(t)$, we have*

$$\|\eta(t)u\|_{X^{s,b}_\varphi} \lesssim_{\eta,b} \|u(0)\|_{H^s} + \|F(u)\|_{X^{s,b-1}_\varphi}. \tag{2.23}$$

Consequently, to apply a contraction mapping argument in Fourier restriction spaces, one also has to prove a nonlinear (typically multilinear) estimate

$$\|F(u)\|_{X^{s,b-1}_\varphi} \lesssim F(\|u\|)_{X^{s,b}_\varphi}. \tag{2.24}$$

When it comes to large data theory, one can only expect to solve the equation locally in time. At this point the following stability lemma comes into play:

Lemma 2.2.4. *Let $\eta \in \mathcal{S}(\mathbb{R})$ be a Schwartz function in time. Then, we have*

$$\|\eta(t)u\|_{X^{s,b}_\varphi} \lesssim_{\eta,b} \|u\|_{X^{s,b}_\varphi} \tag{2.25}$$

for any $s, b \in \mathbb{R}$ and any function $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$. Furthermore, if $-1/2 < b' \leq b < 1/2$, then for any $0 < T < 1$

$$\|\eta(t/T)u\|_{X^{s,b'}_\varphi} \lesssim_{\eta,b,b'} T^{b-b'} \|u\|_{X^{s,b}_\varphi}. \tag{2.26}$$

2.3 Functions of bounded variation and adaptations for dispersive equations

To maximize the gain in the modulation variable, one would like to apply the contraction mapping argument for $b = 1/2$.
However, $H^{1/2}(\mathbb{R})$ fails to embed into $L^\infty(\mathbb{R})$; thus, $X^{s,1/2}$ does not embed into $L^\infty([0,T], H^s)$. Moreover, the transfer principle fails.

One remedy is to consider a slightly smaller function space using a Besov refinement in the modulation variable. Here, we consider U^p-/V^p-spaces as substitute since these spaces behave well under sharp time cutoff, in contrast to $X^{s,b}$-spaces, where multiplication with a sharp time cutoff is not bounded. This is useful when considering frequency dependent time localization later.
For a detailed exposition on U^p-/V^p-spaces we refer to [HHK09], see also [HHK10]. Below, we collect the most important information to keep the exposition self-contained.

Let $I = [a,b)$, where $-\infty \leq a < b \leq \infty$. The $V^p(I)$-spaces contain functions of bounded p-variation, $p \in [1,\infty)$, which take values in $L^2(\mathbb{T}^n)$ (although the function space properties remain valid for an arbitrary Hilbert space). $U^p(I)$ are atomic spaces, which are predual to the $V^p(I)$-spaces. We let $\mathcal{Z}(I)$ denote the set of all possible partitions of I; these are sequences $a = t_0 < t_1 < \ldots < t_K = b$.

Definition 2.3.1. Let $\{t_k\}_{k=0}^K \in \mathcal{Z}(I)$ and $\{\phi_k\}_{k=0}^{K-1} \subseteq L_x^2$ with $\sum_{k=1}^K \|\phi_{k-1}\|_{L_x^2}^p = 1$. Then, the function

$$a(t) = \sum_{k=1}^K \phi_{k-1}\chi_{[t_{k-1},t_k)}(t) \tag{2.27}$$

is said to be a $U^p(I)$-atom. Further,

$$U^p(I) = \{f : I \to L_x^2(\mathbb{T}^n) \mid \|f\|_{U^p(I)} < \infty\}, \tag{2.28}$$

where

$$\|f\|_{U^p(I)} = \inf\{\|\lambda_k\|_{\ell_k^1} \mid f(t) = \sum_{k=0}^\infty \lambda_k a_k(t),\ a_k - U^p - \text{atom}\}. \tag{2.29}$$

By virtue of the atomic representation, we find elements $u(t) \in U^p(I)$ to be continuous from the right, having left-limits everywhere and admitting only countably many discontinuities (cf. [HHK09, Proposition 2.2, p. 921]). Properties of the spaces with bounded p-variation were already discussed in [Wie79].

Definition 2.3.2. We set

$$V^p(I) = \{v : I \to L_x^2 \mid \|v\|_{V^p(I)} < \infty\},$$

where

$$\|v\|_{V^p(I)} = \sup_{\{t_k\}_{k=0}^{K-1} \in \mathcal{Z}(I)} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L_x^2}^p \right)^{1/p} < \infty.$$

We recall that one-sided limits exist for V^p-functions and again V^p-functions can only have countably many discontinuities (cf. [HHK09, Proposition 2.4, p. 922]). In the following we confine ourselves to consider the subspaces $V_{-,rc}^p \subseteq V^p$ of right-continuous functions vanishing at $-\infty$. For the sake of brevity, we write V^p for $V_{-,rc}^p$.

Definition 2.3.3. We define the following subspaces of V^2, respectively U^2:

$$V_0^2(I) = \{v \in V^2(I) \,|\, v(a) = 0\},$$
$$U_0^2(I) = \{u \in U^2(I) \,|\, u(b) = 0\}.$$

These function spaces behave well with sharp cutoff functions contrary to $X^{s,b}$-spaces, where one has to use smooth cutoff functions. We have the following estimates for sharp cutoffs (cf. [CHT12, Equation (2.2), p. 55]):

$$\|u\|_{U^p(I)} = \|\chi_I u\|_{U^p([-\infty,\infty))},$$
$$\|v\|_{V^p(I)} \leq \|\chi_I u\|_{V^p([-\infty,\infty))} \leq 2\|u\|_{V^p(I)}.$$

We record the following embedding properties:

Lemma 2.3.4. *Let $I = [a, b)$.*

1. *If $1 \leq p \leq q < \infty$, then $\|u\|_{U^q} \leq \|u\|_{U^p}$ and $\|u\|_{V^q} \leq \|u\|_{V^p}$.*
2. *If $1 \leq p < \infty$, then $\|u\|_{V^p} \lesssim \|u\|_{U^p}$.*
3. *If $1 \leq p < q < \infty$, $u(a) = 0$ and $u \in V^p$ is right-continuous, then $\|u\|_{U^q} \lesssim \|u\|_{V^p}$.*
4. *Let $1 \leq p < q < \infty$, E be a Banach space and T be a linear operator with*

$$\|Tu\|_E \leq C_q\|u\|_{U^q}, \ \|Tu\|_E \leq C_p\|u\|_{U^p}, \text{with } 0 < C_p \leq C_q.$$

Then,

$$\|Tu\|_E \lesssim \log\langle\frac{C_q}{C_p}\rangle\|u\|_{V^p}.$$

Proof. The first part follows from the embedding properties of the ℓ^p-norms and the second part from considering U^p-atoms. For the third claim see [HHK09, Corollary 2.6, p. 923] and the fourth claim is proved in [HHK09, Proposition 2.20., p. 930]. □

Definition 2.3.5. We define

$$DU^2(I) = \{\partial_t u \,|\, u \in U^2(I)\} \tag{2.30}$$

with the derivative taken in the sense of tempered distributions.

We observe that for any $f \in DU^2(I)$, the function $u \in U^2(I)$ satisfying $\partial_t u = f$ is unique up to constants. Fixing the right limit to be zero, we can set

$$\|f\|_{DU^2(I)} = \|u\|_{U^2(I)}, \quad f = \partial_t u, \quad u \in U_0^2, \tag{2.31}$$

which makes $DU^2(I)$ a Banach space. We have the following embedding property (cf. [CHT12, p. 56]):

Lemma 2.3.6. *Let $I = [a, b)$. Then,*

$$L^1(I) \hookrightarrow DU^2(I).$$

We have the following lemma on $DU - V$-duality:

Lemma 2.3.7. *[HHK09, Proposition 2.10, p. 925] We have* $(DU^2(I))^* = V_0^2(I)$ *with respect to a duality relation, which for* $f \in L^1(I) \subseteq DU^2(I)$ *is given by*

$$\langle f, v \rangle = \int_a^b \langle f(t), v(t) \rangle_{L_x^2} dt = \int_a^b \int f \overline{v} dx dt.$$

Moreover,

$$\|f\|_{DU^2(I)} = \sup_{\|v\|_{V_0^2}=1} \left| \int_a^b \int f \overline{v} dx dt \right|.$$

For $f \in DU^2(I)$ one can still consider a related mapping, but this requires more careful considerations (cf. [HHK09, Theorem 2.8, p. 924]).
Adapting U^p-/V^p-spaces to the linear propagator $e^{it\varphi(\nabla/i)}$ yields the following function spaces:

$$\begin{aligned} \|u\|_{U_\varphi^p(I;H)} &= \|e^{-it\varphi(\nabla/i)} u\|_{U^p(I;H)}, \\ \|v\|_{V_\varphi^p(I;H)} &= \|e^{-it\varphi(\nabla/i)} v\|_{V^p(I;H)}, \\ \|u\|_{DU_\varphi^2(I;H)} &= \|e^{-it\varphi(\nabla/i)} u\|_{DU^2(I;H)}. \end{aligned}$$

U_φ^p-atoms are piecewise free solutions.

2.4 Function spaces for frequency dependent time localization

The time localization is chosen depending on φ. Let $T \in (0,1]$ and $\alpha = \alpha(\varphi)$. We define the short-time U^2-space, into which we place the solution by

$$\|u\|_{F^s(T)}^2 = \sum_{N \in 2^{\mathbb{N}_0} \cup \{0\}} (1+N)^{2s} \sup_{\substack{|I| = \min(N^{-\alpha}, T), \\ I \subseteq [0,T]}} \|\chi_I P_N u\|_{U_\varphi^2(I;L^2)}^2. \tag{2.32}$$

Here we write $P_0 := P_{\leq 0}$ (cf. (2.9)) for brevity.
The function space N^s, into which we will place the nonlinearity, is given by

$$\|f\|_{N^s(T)}^2 = \sum_{N \in 2^{\mathbb{N}_0} \cup \{0\}} (1+N)^{2s} \sup_{\substack{|I| = \min(N^{-\alpha}, T), \\ I \subseteq [0,T]}} \|\chi_I P_N u\|_{DU_\varphi^2(I;L^2)}^2. \tag{2.33}$$

The frequency dependent time localization erases the dependence on the initial data away from the origin. Instead of a common energy space $C([0,T], H^s)$, we have to consider the following space:

$$\|u\|_{E^s(T)}^2 = \|P_{\leq 0} u(0)\|_{L^2}^2 + \sum_{N \geq 1} N^{2s} \sup_{t \in [0,T]} \|P_N u(t)\|_{L^2}^2. \tag{2.34}$$

This space deviates from the usual energy space logarithmically. The following linear estimate substitutes for the $X^{s,b}$-energy estimate from Lemma 2.2.3.

Lemma 2.4.1. *Let* $T \in (0,1]$ *and* u *be a solution to* (2.21). *Then, we find the following estimate to hold:*

$$\|u\|_{F_\varphi^s} \lesssim \|u\|_{E^s(T)} + \|F(u)\|_{N_\varphi^s(T)}. \tag{2.35}$$

Proof. A proof in the context of a specific evolution equation, which immediately generalizes, is given in [CHT12, Lemma 3.1., p. 59]. □

2.5 Modifications for tori

We turn to a discussion of the necessary modifications in the periodic setting. In some applications one has to consider tori with arbitrary period $\lambda > 0$. We set $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\lambda\mathbb{T} = \mathbb{R}/(2\pi\lambda\mathbb{Z})$. Further, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_{>0}$ we set

$$\lambda\mathbb{T}^n = \times_{i=1}^n (\mathbb{R}/(2\pi\lambda_i)). \tag{2.36}$$

Varying λ one has to keep track of possible dependencies of constants on the spatial scale. The conventions below follow [CKS+03].

For $\lambda \in \mathbb{R}^n_{>0}$ the Fourier transform of a $2\pi\lambda$-periodic L^1-function $f : \lambda_1\mathbb{T} \times \ldots \times \lambda_n\mathbb{T} \to \mathbb{C}$ takes on values in $\mathbb{Z}^n/\lambda := \mathbb{Z}/\lambda_1 \times \ldots \times \mathbb{Z}/\lambda_n$ and is defined by

$$\hat{f}(\xi) = \int_{\lambda_1\mathbb{T}\times\ldots\times\lambda_n\mathbb{T}} f(x)e^{-ix.\xi}dx \quad (\xi \in \mathbb{Z}^n/\lambda). \tag{2.37}$$

Let $(d\xi)_\lambda$ be the normalized counting measure on $\mathbb{Z}^n/\lambda$:

$$\int a(\xi)(d\xi)_\lambda := \prod_{i=1}^n \frac{1}{\lambda_i} \sum_{\xi\in\mathbb{Z}^n/\lambda} a(\xi). \tag{2.38}$$

The Fourier inversion formula is given by

$$f(x) = \frac{1}{(2\pi)^n} \int \hat{f}(\xi)e^{ix\xi}(d\xi)_\lambda. \tag{2.39}$$

We find the usual properties of the Fourier transform to hold:

$$\|f\|_{L^2_x(\lambda\mathbb{T})} = \frac{1}{(2\pi)^{n/2}}\|\hat{f}\|_{L^2_{(d\xi)_\lambda}} \quad \text{(Plancherel)}, \tag{2.40}$$

$$\int_{\lambda\mathbb{T}} f(x)\overline{g(x)}dx = \frac{1}{(2\pi)^n}\int \hat{f}(\xi)\overline{\hat{g}(\xi)}(d\xi)_\lambda \quad \text{(Parseval)}. \tag{2.41}$$

For further properties, see [CKS+03, p. 702]. We define the Sobolev space H^s_λ with norm

$$\|f\|_{H^s_\lambda} = \|\hat{f}(\xi)\langle\xi\rangle^s\|_{L^2_{(d\xi)_\lambda}} \tag{2.42}$$

and like above $H^\infty_\lambda = \bigcap_s H^s_\lambda$.

For a $2\pi\lambda$-periodic function $f(t,x)$ with time variable $t \in \mathbb{R}$, we define the space-time Fourier transform

$$\tilde{v}(\tau,\xi) = (\mathcal{F}_{t,x}v)(\tau,\xi) = \int_{\mathbb{R}} dt \int_{\lambda\mathbb{T}^n} dx e^{-ix.\xi}e^{-it\tau}v(t,x) \quad (\xi \in \mathbb{Z}^n/\lambda,\ t \in \mathbb{R}). \tag{2.43}$$

The periodic space-time Fourier transform is inverted by

$$v(t,x) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} d\tau \int_{\mathbb{Z}^n/\lambda} (d\xi)_\lambda e^{ix.\xi}e^{it\tau}\tilde{v}(\tau,\xi). \tag{2.44}$$

We also use short-time U^p-/V^p-function spaces in the periodic case as long as modulation considerations do not play a role.
We contend that in this case the U^p-/V^p-set up yields a simplification compared to

the framework of the classical short-time $X^{s,b}$-spaces introduced in [IKT08] as long as one does not use modulation considerations.

In the latter case, there does not seem to be a simplification and we revisit the well-known construction from [IKT08]. For the proofs of the basic function space properties, which hold true independent of the domain and dispersion relation, we refer to the literature. (NB. The proofs are more involved than in the U^p-framework but well-known in the literature.) The definition requires a partition in the modulation, which we denote differently from the partition of the spatial frequencies.

Let $\eta_0 : \mathbb{R} \to [0,1]$ denote an even smooth function $\mathrm{supp}\,(\eta_0) \subseteq [-8/5, 8/5]$ with $\eta_0 \equiv 1$ on $[-5/4, 5/4]$. For $k \in \mathbb{N}$ we set

$$\eta_k(\tau) = \eta_0(\tau/2^k) - \eta_0(\tau/2^{k-1}).$$

We write $\eta_{\leq m} = \sum_{j=0}^m \eta_j$ for $m \in \mathbb{N}$. For $k \in \mathbb{N}_0$ set $I_0 = [-1,1]$ and $I_k = [-2^k, 2^k] \backslash (-2^{k-1}, 2^{k-1})$.

For $k \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$ set for a dispersion relation $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$

$$\begin{aligned} D_{k,j} &= \{(\tau,\xi) \in \mathbb{R} \times \mathbb{Z} \,|\, \xi \in I_k,\, 2^{j-1} \leq |\tau - \varphi(\xi)| \leq 2^j\}, \\ D_{k,\leq j} &= \{(\tau,\xi) \in \mathbb{R} \times \mathbb{Z} \,|\, \xi \in I_k,\, |\tau - \varphi(\xi)| \leq 2^{j+1}\}. \end{aligned} \tag{2.45}$$

Next, we define an $X^{s,b}$-type space for the Fourier transform of frequency-localized $2\pi\lambda$-functions:

$$\begin{aligned} &X_{k,\lambda} = \{f : \mathbb{R} \times \mathbb{Z}^n/\lambda \to \mathbb{C} \,| \\ &\mathrm{supp}(f) \subseteq \mathbb{R} \times I_k, \|f\|_{X_{k,\lambda}} = \sum_{j=0}^\infty 2^{j/2} \|\eta_j(\tau - \varphi(\xi)) f(\tau,\xi)\|_{L^2_{(d\xi)_\lambda} L^2_\tau} < \infty\}. \end{aligned}$$

Partitioning the modulation variable through a sum over η_j yields the estimate

$$\| \int_{\mathbb{R}} |f_k(\tau',\xi)| d\tau' \|_{L^2_{(d\xi)_\lambda}} \lesssim \|f_k\|_{X_{k,\lambda}}. \tag{2.46}$$

Also, we record the estimate

$$\begin{aligned} &\sum_{j=l+1}^\infty 2^{j/2} \|\eta_j(\tau - \varphi(\xi)) \cdot \int_{\mathbb{R}} |f_k(\tau',\xi)| \cdot 2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \|_{L^2_{(d\xi)_\lambda} L^2_\tau} \\ &+ 2^{l/2} \|\eta_{\leq l}(\tau - \varphi(\xi)) \cdot \int_{\mathbb{R}} |f_k(\tau',\xi)| \cdot 2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \|_{L^2_{(d\xi)_\lambda} L^2_\tau} \\ &\lesssim \|f_k\|_{X_{k,\lambda}}, \end{aligned} \tag{2.47}$$

which is a rescaled version of [GO18, Equation (3.5)].

In particular, we find for a Schwartz-function γ for $k, l \in \mathbb{N}, t_0 \in \mathbb{R}, f_k \in X_{k,\lambda}$ the estimate

$$\|\mathcal{F}[\gamma(2^l(t - t_0)) \cdot \mathcal{F}^{-1}(f_k)]\|_{X_{k,\lambda}} \lesssim_\gamma \|f_k\|_{X_{k,\lambda}}. \tag{2.48}$$

We define the following spaces:

$$E_{k,\lambda} = \{u_0 : \lambda\mathbb{T} \to \mathbb{C} \,|\, \mathrm{supp}(\hat{u}_0) \subseteq I_k,\; \|u_0\|_{E_{k,\lambda}} = \|u_0\|_{L^2_\lambda} < \infty\},$$

which are the spaces for the dyadically localized energy.
Next, we set

$$C_0(\mathbb{R}, E_{k,\lambda}) = \{u_k \in C(\mathbb{R}, E_{k,\lambda}) \mid \operatorname{supp}(u_k) \subseteq [-4, 4]\}$$

and define for a frequency 2^k and $\alpha > 0$ the following short-time $X^{s,b}$-space:

$$F^\alpha_{k,\lambda} = \{u_k \in C_0(\mathbb{R}, E_{k,\lambda}) \mid \|u_k\|_{F^\alpha_{k,\lambda}} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[u_k \eta_0(2^k(t-t_k))]\|_{X_{k,\lambda}} < \infty\}.$$

Similarly, we define the spaces to capture the nonlinearity:

$$\begin{aligned} &N^\alpha_{k,\lambda} = \{u_k \in C_0(\mathbb{R}, E_{k,\lambda}) \mid \\ &\|u_k\|_{N^\alpha_{k,\lambda}} = \sup_{t_k \in \mathbb{R}} \|(\tau - \varphi(\xi) + i2^{\alpha k})^{-1} \mathcal{F}[u_k \eta_0(2^{\alpha k}(t-t_k))]\|_{X_{k,\lambda}} < \infty\}. \end{aligned}$$

We localize the spaces in time in the usual way. For $T \in (0,1]$ we set

$$F^\alpha_{k,\lambda}(T) = \{u_k \in C([-T,T], E_{k,\lambda}) \mid \|u_k\|_{F^\alpha_{k,\lambda}(T)} = \inf_{\tilde{u}_k = u_k \text{in} [-T,T]} \|\tilde{u}_k\|_{F^\alpha_{k,\lambda}} < \infty\}$$

and

$$N^\alpha_{k,\lambda}(T) = \{u_k \in C([-T,T], E_{k,\lambda}) \mid \|u_k\|_{N^\alpha_{k,\lambda}(T)} = \inf_{\tilde{u}_k = u_k \text{in} [-T,T]} \|\tilde{u}_k\|_{N^\alpha_{k,\lambda}} < \infty\}.$$

We assemble the spaces for dyadically localized frequencies in a straight-forward manner using Littlewood-Paley theory: as an energy space for solutions we consider

$$\begin{aligned} &E^s_\lambda(T) = \{u \in C([-T,T], H^\infty_\lambda) \mid \\ &\|u\|^2_{E^s_\lambda(T)} = \|P_{\leq 0} u(0)\|^2_{L^2_\lambda} + \sum_{k \geq 1} \sup_{t_k \in [-T,T]} 2^{2ks} \|P_k u(t_k)\|^2_{L^2_\lambda} < \infty\}. \end{aligned}$$

We define the short-time $X^{s,b}$-space for the solution

$$F^{s,\alpha}_\lambda(T) = \{u \in C([-T,T], H^\infty_\lambda) \mid \|u\|^2_{F^{s,\alpha}_\lambda(T)} = \sum_{k \geq 0} 2^{2ks} \|P_k u\|^2_{F^\alpha_{k,\lambda}(T)} < \infty\},$$

and for the nonlinearity we consider

$$N^{s,\alpha}_\lambda(T) = \{u \in C([-T,T], H^\infty_\lambda) \mid \|u\|^2_{N^{s,\alpha}_\lambda(T)} = \sum_{k \geq 0} 2^{2ks} \|P_k u\|^2_{N^\alpha_{k,\lambda}(T)} < \infty\}.$$

We also make use of k-acceptable time multiplication factors (cf. [IKT08]): for $k \in \mathbb{N}_0$ we set

$$S^\alpha_k = \{m_k \in C^\infty(\mathbb{R}, \mathbb{R}) : \|m_k\|_{S^\alpha_k} = \sum_{j=0}^{10} 2^{-j\alpha k} \|\partial^j m_k\|_{L^\infty} < \infty\}.$$

The generic example is given by time localization on a scale of $2^{-\alpha k}$, i.e., $\eta_0(2^{\alpha k} \cdot)$. The estimates (cf. [IKT08, Eq. (2.21), p. 273])

$$\begin{cases} \|\sum_{k \geq 0} m_k(t) P_k(u)\|_{F^{s,\alpha}_\lambda(T)} \lesssim (\sup_{k \geq 0} \|m_k\|_{S^\alpha_k}) \cdot \|u\|_{F^{s,\alpha}_\lambda(T)}, \\ \|\sum_{k \geq 0} m_k(t) P_k(u)\|_{N^{s,\alpha}_\lambda(T)} \lesssim (\sup_{k \geq 0} \|m_k\|_{S^\alpha_k}) \cdot \|u\|_{N^{s,\alpha}_\lambda(T)} \end{cases} \tag{2.49}$$

follow from integration by parts.
From (2.49) follows that we can assume $F^\alpha_{k,\lambda}(T)$ functions to be supported in time on an interval $[-T-2^{-\alpha k-10}, T+2^{-\alpha k-10}]$.

We record basic properties of the short-time $X^{s,b}_\lambda$-spaces introduced above. The next lemma establishes the embedding $F^{s,\alpha}_\lambda(T) \hookrightarrow C([0,T], H^s_\lambda)$.

Lemma 2.5.1. *(i) We find the estimate*

$$\|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{F^\alpha_{k,\lambda}}$$

to hold for any $u \in F^\alpha_{k,\lambda}$.

(ii) Suppose that $s \in \mathbb{R}$, $T > 0$ and $u \in F^{s,\alpha}_\lambda(T)$. Then, we find the estimate

$$\|u\|_{C([0,T],H^s_\lambda)} \lesssim \|u\|_{F^{s,\alpha}_\lambda(T)}$$

to hold.

Proof. For a proof see [IKT08, Lemma 3.1., p. 274] in Euclidean space and [GO18, Lemma 3.2, 3.3] in the periodic case. □

We state the energy estimate for the above short-time $X^{s,b}$-spaces. The proof, which is carried out on the real line in [IKT08, Proposition 3.2., p. 274] and in the periodic case in [GO18, Proposition 4.1.], is omitted.

Proposition 2.5.2. *Let $T \in (0,1]$, $\alpha > 0$ and $u, v \in C([-T,T], H^\infty_\lambda)$ satisfy the equation*

$$i\partial_t u + \varphi(\nabla/i)u = v \text{ in } \lambda\mathbb{T}^n \times (-T,T).$$

Then, we find the following estimate to hold for any $s \in \mathbb{R}$:

$$\|u\|_{F^{s,\alpha}_\lambda(T)} \lesssim \|u\|_{E^{s,\alpha}_\lambda(T)} + \|v\|_{N^{s,\alpha}_\lambda(T)}.$$

For the large data theory we have to define the following generalizations in terms of regularity in the modulation variable to the X_k-spaces:

$$\begin{aligned} X^b_k = \{ &f : \mathbb{R} \times \mathbb{Z} \to \mathbb{C} \mid \\ &\mathrm{supp}(f) \subseteq \mathbb{R} \times I_k, \|f\|_{X^b_k} = \sum_{j=0}^\infty 2^{bj} \|\eta_j(\tau - \varphi(n)) f(\tau,n)\|_{\ell^2_n L^2_\tau} < \infty \}. \end{aligned}$$

where $b \in \mathbb{R}$. The short-time spaces $F^{b,\alpha}_k$, $F^{b,s,\alpha}(T)$ and $N^{b,\alpha}_k$, $N^{b,s,\alpha}(T)$ are defined following along the above lines with X_k replaced by X^b_k.
Indeed, in a similar spirit to the treatment of $X^{s,b}_T$-spaces, we can trade regularity in the modulation variable for a power of T:

Lemma 2.5.3. *[GO18, Lemma 3.4] Let $T > 0$, $\alpha > 0$ and $b < 1/2$. Then, we find the following estimate to hold:*

$$\|P_k u\|_{F^{b,\alpha}_k} \lesssim T^{(1/2-b)-} \|P_k u\|_{F^\alpha_k}$$

for any function u with temporal support in $[-T,T]$ and implicit constant independent of k.

Below we have to consider the action of sharp time cutoffs in the X_k-spaces. Recall from the usual $X^{s,b}$-space-theory that multiplication with a sharp cutoff in time is not bounded. However, we find the following estimate to hold:

Lemma 2.5.4. *[GO18, Lemma 3.5] Let $k \in \mathbb{Z}$. Then, for any interval $I = [t_1, t_2] \subseteq \mathbb{R}$, we find the following estimate to hold:*

$$\sup_{j \geq 0} 2^{j/2} \| \eta_j(\tau - \varphi(n)) \mathcal{F}_{t,x}[1_I(t) P_k u] \|_{L^2_\tau \ell^2_n} \lesssim \| \mathcal{F}_{t,x}(P_k u) \|_{X_k}$$

with implicit constant independent of k and I.

Chapter 3

Control of rough wave interactions via frequency dependent time localization

In this chapter we give an overview of the approach, which is varied in the following chapters to prove new local regularity results. We reprove in detail local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{T})$ for $s > 1$.
This result does not come close to the global well-posedness result in $L^2(\mathbb{T})$ by Molinet (cf. [Mol08]), which was proved via a gauge transform.
Below we argue how the argument extends to related models, which are no longer easily amenable to a gauge transform.
The chapter also has a motivational character preparing for the more involved arguments, which are deployed in the following chapters.

We return to the example from the introduction, where we had been considering the Benjamin-Ono equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_{xx} u &= u\partial_x u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{T}). \end{cases} \tag{3.1}$$

$\mathcal{H} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the Hilbert transform, which we define as the Fourier multiplier

$$\widehat{(\mathcal{H}f)}(\xi) = -i \operatorname{sgn}(\xi) \hat{f}(\xi).$$

Following the heuristic argument from the introduction, we choose the frequency dependent time localization $T(N) = N^{-1}$ and consider the short-time function spaces

$$\|u\|^2_{F^s_{BO}(T)} = \sum_{N \in 2^{\mathbb{N}_0} \cup \{0\}} (1+N)^{2s} \sup_{\substack{|I| = N^{-1} \wedge T, \\ I \subseteq [0,T]}} \|P_N u\|^2_{U^2_{BO}(I;L^2(\mathbb{T}))}, \tag{3.2}$$

where U^2_{BO} is the U^2-space adapted to the linear propagator of (3.1). $N^s_{BO}(T)$ and $E^s(T)$ are also defined following Section 2.4.

To propagate solutions in $F^s_{BO}(T)$, we prove the following two estimates in addition to the linear estimate (2.35) for $s > 1$:

$$\begin{cases} \|\partial_x(u^2)\|_{N^s_{BO}(T)} & \lesssim \|u\|^2_{F^s_{BO}(T)} \\ \|u\|^2_{E^s(T)} & \lesssim \|u_0\|^2_{H^s} + T\|u\|^3_{F^s_{BO}(T)}. \end{cases} \tag{3.3}$$

To prove continuous dependence, we make use of a known variant of the Bona-Smith approximation (cf. [BS75]). In the context of short-time $X^{s,b}$-spaces on Euclidean space, this was already adapted in [IKT08].
First, we show Lipschitz-continuity in L^2 for initial data with higher regularity. Denoting $v = u_1 - u_2$ for u_i smooth solutions to (3.1) and $s > 1$, we find

$$\begin{cases} \|v\|_{F^0_{BO}(T)} & \lesssim \|v\|_{E^0(T)} + \|\partial_x(v(u_1+u_2))\|_{N^0_{BO}(T)} \\ \|\partial_x(v(u_1+u_2))\|_{N^0_{BO}(T)} & \lesssim \|v\|_{F^0_{BO}(T)}(\|u_1\|_{F^s_{BO}(T)} + \|u_2\|_{F^s_{BO}(T)}) \\ \|v\|^2_{E^0(T)} & \lesssim \|v(0)\|^2_{L^2} \\ & \quad +T\|v\|^2_{F^0(T)}(\|u_1\|_{F^s_{BO}(T)} + \|u_2\|_{F^s_{BO}(T)}), \end{cases} \tag{3.4}$$

and in the second step, the following set of estimates is proved:

$$\begin{cases} \|v\|_{F^s_{BO}(T)} & \lesssim \|v\|_{E^s(T)} + \|\partial_x(v(u_1+u_2))\|_{N^s_{BO}(T)} \\ \|\partial_x(v(u_1+u_2))\|_{N^s_{BO}(T)} & \lesssim \|v\|_{F^s_{BO}(T)}(\|u_1\|_{F^s_{BO}(T)} + \|u_2\|_{F^s_{BO}(T)}) \\ \|v\|^2_{E^s(T)} & \lesssim \|v(0)\|^2_{H^s} + T\|v\|^3_{F^s_{BO}(T)} \\ & \quad +T\|v\|^2_{F^s_{BO}(T)}\|u_2\|_{F^s_{BO}(T)} \\ & \quad +T\|v\|_{F^0_{BO}(T)}\|v\|_{F^s_{BO}(T)}\|u_2\|_{F^{2s}_{BO}(T)} \end{cases} \tag{3.5}$$

The standard bootstrap arguments to conclude local well-posedness are given in Section 3.5.

Important symmetries of (3.1) to prove the above sets of estimates are conservation of mass and the real-valuedness of solutions as already pointed out in [IKT08]. One novel observation is how frequency dependent time localization allows us to overcome the derivative loss on tori via short-time Strichartz estimates.
The argument is modular in the sense that it extends to higher order nonlinearities

$$\begin{cases} \partial_t u + \mathcal{H}\partial_{xx} u & = u^{k-1}\partial_x u, \quad (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u(0) & = u_0 \in H^s_{\mathbb{R}}(\mathbb{T}), \end{cases} \tag{3.6}$$

where $k \in \mathbb{Z}_{\geq 3}$, and dispersion generalizations $\mathcal{H}\partial_{xx} \to \partial_x D^a_x$, $1 < a < 2$.
The corresponding estimates to (3.3) to prove a priori estimates are

$$\begin{cases} \|u\|_{F^s_{BO}(T)} & \lesssim \|u\|_{E^s(T)} + \|\partial_x(u^k)\|_{N^s_{BO}(T)} \\ \|\partial_x(u^k)\|_{N^s_{BO}(T)} & \lesssim \|u\|^k_{F^s_{BO}(T)} \\ \|u\|^2_{E^s(T)} & \lesssim \|u_0\|^2_{H^s} + T\|u\|^{k+1}_{F^s_{BO}(T)}. \end{cases} \tag{3.7}$$

To write down the estimates for differences of solutions to (3.6), we consider

$$\partial_x(u_1^k - u_2^k) = \partial_x(v(Q_k(u_1,u_2)) = \partial_x(v, S_k(v,u_2)), \tag{3.8}$$

and the set of estimates to prove L^2-Lipschitz continuity for initial data in H^s, $s > 1$, is given by

$$\begin{cases} \|v\|_{F^0_{BO}(T)} & \lesssim \|v\|_{E^0(T)} + \|\partial_x(vQ_k(u_1,u_2))\|_{N^0_{BO}(T)} \\ \|\partial_x(vQ_k(u_1,u_2))\|_{N^0_{BO}(T)} & \lesssim \|v\|_{F^0_{BO}(T)}Q_k(\|u_1\|_{F^s_{BO}(T)}, \|u_2\|_{F^s_{BO}(T)}) \\ \|v\|^2_{E^0(T)} & \lesssim \|v(0)\|^2_{L^2} \\ & \quad +T\|v\|^2_{F^0(T)}Q_k(\|u_1\|_{F^s_{BO}(T)}, \|u_2\|_{F^s_{BO}(T)}). \end{cases} \tag{3.9}$$

The corresponding estimates to (3.5) yielding continuous dependence in H^s are

$$\begin{cases} \|v\|_{F^s_{BO}(T)} & \lesssim \|v\|_{E^s(T)} + \|\partial_x(vQ_k(u_1,u_2))\|_{N^s_{BO}(T)} \\ \|\partial_x(vQ_k(u_1,u_2))\|_{N^s_{BO}(T)} & \lesssim \|v\|_{F^s_{BO}(T)} Q_k(\|u_1\|_{F^s_{BO}(T)}, \|u_2\|_{F^s_{BO}(T)}) \\ \|v\|^2_{E^s(T)} & \lesssim \|v(0)\|^2_{H^s} \\ & \quad +T(\|v\|^2_{F^s_{BO}(T)} S^1_k(\|v\|_{F^s_{BO}(T)}, \|u_2\|_{F^s_{BO}(T)})) \\ & \quad +T(\|v\|_{F^0(T)}\|v\|_{F^s(T)}\|u_2\|_{F^{2s}(T)} \\ & \quad\; S^2_k(\|v\|_{F^s_{BO}(T)}, \|u_2\|_{F^s_{BO}(T)})) \end{cases} \tag{3.10}$$

with polynomials S^i_k, $i = 1,2$.
To make the heuristic argument (1.9) precise and carry out the nonlinear and energy estimate rigorously, we start with a discussion of Strichartz estimates.

3.1 Bilinear Strichartz estimates

A Taylor expansion in frequency space suggests that frequency localized solutions $u = e^{it\varphi(\nabla/i)}u_0$, where supp $\hat{u}_0 \subseteq B(\xi_0, \varepsilon)$ for some $\varepsilon \ll 1$ are to first approximation traveling waves with group velocity $-\nabla\varphi(\xi_0)$. The following proposition points out how difference of the group velocities, i.e., transversality of the characteristic surfaces, can lead to bilinear improvements of the linear estimates.

Proposition 3.1.1 (Bilinear Strichartz estimates)**.** *Let U_i be open sets in $\mathbb{R}^n$, $\varphi_i \in C^1(U_i, \mathbb{R})$ and let u_i have Fourier support in balls of radius r, which are contained in U_i for $i = 1,2$. Moreover, suppose that $|\nabla\varphi_1(\xi_1) - \nabla\varphi_2(\xi_2)| \geq N > 0$, whenever $\xi_i \in U_i$, $i = 1,2$.*
Then, we find the following estimate to hold:

$$\|e^{it\varphi_1(\nabla/i)}u_1 e^{it\varphi_2(\nabla/i)}u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \lesssim_n \frac{r^{\frac{n-1}{2}}}{N^{1/2}} \|u_1\|_{L^2(\mathbb{R}^n)}\|u_2\|_{L^2(\mathbb{R}^n)}. \tag{3.11}$$

In Euclidean space this follows from a change of variables (cf. [Bou98]). We omit the proof to avoid repetition because a periodic analog is discussed in detail in the following proposition.
However, in the periodic case one can not expect this estimate to hold globally in time due to lack of dispersion. Instead, we have the following estimate:

Proposition 3.1.2. *Let U_i be open sets in $\mathbb{R}^n$, $\varphi_i \in C^1(U_i, \mathbb{R})$ and let $f_i \in L^2(\mathbb{R} \times \mathbb{Z}^n)$ with*

$$f_i(\tau, \xi) = 0 \quad \text{for } \xi \notin B(\xi_i^*, r) \subseteq U_i, \quad |\tau - \varphi_i(\xi)| \leq 2^{j_i} \tag{3.12}$$

for $i = 1,2$. Moreover, suppose that $|\nabla\varphi_1(\xi_1) - \nabla\varphi_2(\xi_2)| \geq M = 2^m > 0$, whenever $\xi_i \in U_i$, $i = 1,2$.
Then, we find the following estimate to hold:

$$\|f_1 * f_2\|_{L^2_{(d\xi)}L^2_\tau} \lesssim_n (1+r)^{\frac{n-1}{2}} 2^{j_{\min}/2}(1 + 2^{j_{\max}-m})^{1/2}\|f_1\|_{L^2}\|f_2\|_{L^2}. \tag{3.13}$$

Proof. An application of Cauchy-Schwarz gives

$$\begin{aligned} \|f_1 * f_2\|^2_{L^2_\tau L^2_{(d\xi)}} &= \int d\tau \int (d\xi)_1 \left| \int d\tau_1 \int (d\xi_1)_1 f_1(\tau_1, \xi_1) f_2(\tau - \tau_1, \xi - \xi_1) \right|^2 \\ &\lesssim \sup_{\tau,\xi} \text{meas}(B_{\tau,\xi}) \|f_1\|_2^2 \|f_2\|_2^2, \end{aligned}$$

where

$$B_{\tau,\xi} = \{(\tau_1,\xi_1) \mid |\tau_1 - \varphi_1(\xi_1)| \lesssim 2^{j_1},\ |(\tau-\tau_1)-\varphi_2(\xi-\xi_1)| \lesssim 2^{j_2}, \xi_1 \in U_1, \xi-\xi_1 \in U_2\}.$$

In the following let $j_1 \leq j_2$ without loss of generality (since $f_1 * f_2 = f_2 * f_1$). Note that fixing ξ_1 and letting τ_1 vary

$$\operatorname{meas}(B_{\tau,\xi}) \lesssim 2^{j_1} \#(\{\xi_1 \in \operatorname{supp}_\xi f_1 \mid |\tau - \varphi_1(\xi_1) - \varphi_2(\xi - \xi_1)| \lesssim 2^{j_2}\}), \tag{3.14}$$

where $\xi - \xi_1 \in \operatorname{supp}_\xi f_2$. Set $g_\xi(\xi_1) = \varphi_1(\xi_1) + \varphi_2(\xi - \xi_1)$.
Next, we divide $\operatorname{supp}_\xi f_1$ into

$$I_i = \{\xi_1 \in \operatorname{supp}_\xi f_1 \mid |\partial_i g_\xi(\xi_1)| \geq \frac{M}{C_n}\},$$

and choosing C_n as a sufficiently large dimensional constant, we find that $\operatorname{supp}_\xi f_1$ is covered by $(I_i)_{i=1}^n$.
Hence, it is enough to estimate

$$\#(\{\xi_1 \in I_j \mid |\tau - \varphi_1(\xi_1) - \varphi_2(\xi - \xi_1)| \lesssim 2^{j_2}\}) \lesssim (1 + 2^{j_2 - m})(1 + r)^{n-1}. \tag{3.15}$$

The above display follows from counting ξ_{1j} by the lower bound of the derivative and the remaining components by the size of $\operatorname{supp}_\xi f_1$. Taking (3.14) and (3.15) together completes the proof. $\square$

Remark 3.1.3. From the proof is clear that there are variants for general tori, but we will not need them.

Proposition 3.1.2 states that for modulations large relative to the difference of group velocities there is little difference between Euclidean space and the torus. Moreover, the same proof applies in Euclidean space with the difference that the constant in (3.15) is improved to $2^{j_2 - m} r^{n-1}$ because no longer points on a grid are considered but a continuous range.
The localization in time allows us to assume that it is enough to consider regions of modulation, which have a minimal size antiproportional to the frequency dependent time localization (cf. (2.47)). This allows us to obtain enough smoothing to ameliorate the derivative loss.
On the other hand, it is not clear for us how to derive the above estimate directly for solutions at short times. Only after imposing a condition on the dispersion relation, we can derive the corresponding estimate for (3.11):

Definition 3.1.4. We say that a dispersion relation $\varphi : \mathbb{R}^n \to \mathbb{R}$ is of sum type if $\varphi(\xi) = \sum_{i=1}^n \mu(\xi_i)$ with μ slowly varying, i.e., $\mu(x) \sim \mu(2x)$ for any $x \neq 0$.

Proposition 3.1.5. *Let $K \ll N$, suppose that φ is of sum type and satisfies $|\nabla\varphi(\xi)| \sim |\xi|^a$ for some $a > 0$. Then, we find the following estimate to hold:*

$$\begin{aligned} &\|P_N e^{it\varphi(\nabla/i)} u_1 P_K e^{it\varphi(\nabla/i)} u_2\|_{L_t^2([0,N^{-a}],L^2(\mathbb{T}^n))} \\ &\lesssim \frac{K^{\frac{n-1}{2}}}{N^{\frac{a}{2}}} \|P_N u_1\|_{L^2(\mathbb{T}^n)} \|P_K u_2\|_{L^2(\mathbb{T}^n)}. \end{aligned} \tag{3.16}$$

In the one-dimensional case this estimate was proved up to complex conjugation in [MV08, Theorem 4, p. 125], and the below argument follows along its lines.

Proof. We find

$$\begin{aligned} u_1(t) &= \sum_{k_1\in\mathbb{Z}^n} e^{ix.k_1} e^{it\varphi(k_1)} a(k_1), \quad u_2(t) = \sum_{k_2\in\mathbb{Z}^n} e^{ix.k_2} e^{it\varphi(k_2)} b(k_2), \\ u_1u_2(t) &= \sum_{k_1,k_2\in\mathbb{Z}^n} e^{ix.(k_1+k_2)}[e^{it[\varphi(k_1)+\varphi(k_2)]}a(k_1)b(k_2)]. \end{aligned}$$

Consequently, Plancherel's theorem yields

$$\begin{aligned} \|u_1u_2\|_{L^2}^2 &= \sum_{k\in\mathbb{Z}^n}\left|\sum_{k_2\in\mathbb{Z}^n} e^{it(\varphi(k-k_2)+\varphi(k_2))} a(k-k_2)b(k_2)\right|^2 \\ &= \sum_{k\in\mathbb{Z}^n}\sum_{k_2^{(1)},k_2^{(2)}\in\mathbb{Z}^n} e^{it([\varphi(k-k_2^{(1)})+\varphi(k_2^{(1)})]-[\varphi(k-k_2^{(2)})+\varphi(k_2^{(2)})])} \\ &\quad \times a(k-k_2^{(1)})b(k_2^{(1)})\overline{a(k-k_2^{(2)})b(k_2^{(2)})}. \end{aligned} \tag{3.17}$$

Set $\psi_k(k') = \varphi(k-k') + \varphi(k')$. Next, let $\eta_\delta(t) = \eta(t/\delta)$, where η is a suitable bump function and majorize

$$\int_0^{N^{-a}} dt\|u_1u_2(t)\|_{L^2(\mathbb{T}^n)}^2 \le \int \eta_\delta(t)\|u_1u_2(t)\|_{L^2(\mathbb{T}^n)}^2, \quad \delta = N^{-a}$$

and we find

$$\begin{aligned} \int \eta_\delta(t)(3.17)(t)dt = \sum_{k\in\mathbb{Z}^n}\sum_{k_2^{(1)},k_2^{(2)}\in\mathbb{Z}^n} &\hat{\eta}_\delta(\psi_k(k_2^{(1)}) - \psi_k(k_2^{(2)})) \\ &\times a(k-k_2^{(1)})b(k_2^{(1)})\overline{a(k-k_2^{(2)})b(k_2^{(2)})}. \end{aligned} \tag{3.18}$$

The inner sum we will estimate with Young's inequality. Note that

$$\begin{aligned} &\psi_k(k_2^{(1)}) - \psi_k(k_2^{(2)}) \\ &= \int_0^1 \nabla\psi_k(k_2^{(2)} + t(k_2^{(1)} - k_2^{(2)}))(k_2^{(1)} - k_2^{(2)})dt \\ &= \int_0^1 [\nabla\varphi(k_2^{(2)} + t(k_2^{(1)} - k_2^{(2)})) - \nabla\varphi(k - (k_2^{(2)} + t(k_2^{(1)} - k_2^{(2)})))]dt \cdot (k_2^{(1)} - k_2^{(2)}). \end{aligned} \tag{3.19}$$

By assumption, it is easy to see that there is one component of the integral, which is of order N^a independent of t, say the first component. This gives

$$(3.19) = (N^a c_1(k_1, k_{21}^{(1)}, k_{21}^{(2)}))(k_{21}^{(1)} - k_{21}^{(2)}) + \sum_{i=2}^n C_i(k_i, k_{2i}^{(1)}, k_{2i}^{(2)})(k_{2i}^{(1)} - k_{2i}^{(2)}),$$

where, due to our assumptions on μ, there is $C > 0$ so that

$$C^{-1} \le \pm c_1(k_1, k_{21}^{(1)}, k_{21}^{(2)}) \le C.$$

An application of Young's inequality yields

$$(3.18) \lesssim \sum_{k\in\mathbb{Z}^n} \{ \sup_{k_2^{(2)}\in\mathbb{Z}^n} \sum_{k_2^{(1)}\in\mathbb{Z}^n} |\delta\hat{\eta}((\delta N^{\alpha-1} c_1(k_1, k_{21}^{(1)}, k_{21}^{(2)})(k_{21}^{(1)} - k_{21}^{(2)})$$

$$+ \sum_{i=2}^{n} C_i(k_i, k_{2i}^{(1)}, k_{2i}^{(2)})(k_{2i}^{(1)} - k_{2i}^{(2)}))|\} \times \sum_{k_2\in\mathbb{Z}^n} |a(k-k_2)b(k_2)|^2.$$

The sum $\sum_{k_{21}^{(1)}\in\mathbb{Z}} |\hat{\eta}(\ldots)|$ is majorized by $\int |\hat{\eta}(\xi)|d\xi$, and summation over the remaining indices yields a factor K per summation.

Consequently,

$$\begin{aligned}(3.18) &\lesssim \sum_{k\in\mathbb{Z}^n} \delta K^{n-1} \sum_{k_2\in\mathbb{Z}^n} |a(k-k_2)|^2 |b(k_2)|^2 \\ &\lesssim \delta K^{n-1} \|a\|_2^2 \|b\|_2^2,\end{aligned}$$

and the proof is complete. □

Observe how the special form of φ comes into play in the expression (3.19) and the subsequent estimates.

Remark 3.1.6. We illustrate the argument and some of its consequences.

Suppose that $n = 1$ and u_1 and u_2 have Fourier support in intervals I_1 and I_2, respectively, and consider the dispersion relation $\varphi(\xi) = \xi|\xi|^a,\ \xi \in \mathbb{R}$. Suppose that I_1, I_2 do not necessarily belong to dyadically separated annuli, but still satisfy

$$\nabla\varphi(\xi_1) - \nabla\varphi(\xi_2) \sim N^a, \text{ where } \xi_i \in I_i.$$

The Fourier support must be convex so that when we are integrating

$$\int_0^1 \nabla\varphi(\underbrace{k_2^{(2)} + t(k_2^{(1)} - k_2^{(2)})}_{k'})) - \nabla\varphi(k-k'))dt.$$

k' is always an element of I_2 and $k-k'$ is always an element of I_1. This yields the integral to be $\sim N^a$. Then, the proof gives the same estimate like for $High \times Low \to High$-interaction.

We shall see that we can also deal with $High \times High \times High \times Low \times \ldots$-interaction $|\xi_i| \sim N_i,\ i = 1, \ldots, k,\ N_1 \sim N_2 \sim N_3 \gg N_4 \gtrsim N_5 \ldots$ with two bilinear estimates:

There are three frequencies ξ_1, ξ_2, ξ_3 satisfying $|\xi_i| \sim N,\ i = 1, 2, 3$ and we have the convolution constraint $\sum_i^k \xi_i = 0$. We argue that there are $i, j \in \{1, 2, 3\}$: $||\xi_i|^a - |\xi_j|^a| \gtrsim N^a$.

Divide the frequency projector into smaller intervals. We write

$$P_{N_1}u_1 P_{N_2}u_2 P_{N_3}u_3 P_K u_4 \ldots = \sum_{I_1,I_2,I_3} P_{I_1}u_1 P_{I_2}u_2 P_{I_3}u_3 P_K u_4 \ldots. \quad (3.20)$$

Here, I_i denote intervals of length $cN,\ c \ll 1,\ K \ll cN$. With the intervals having a size of cN, there is no loss summing up the different contributions at last. Observe that I_1 and $-I_2$ must be separated due to impossible frequency interaction otherwise. In case I_1 and I_2 are separated, the estimate $||\xi_1|^a - |\xi_2|^a| \gtrsim N^a$ is immediate.

If there is no separation between I_1 and I_2, the intervals are neighbours and $\pm I_3$ will be separated from I_1 and I_2 due to otherwise impossible frequency interaction.

Consequently, we record the estimates

$$\begin{aligned} &\|S_{>N}(P_{N_1} e^{t\partial_x D_x^a} u_1(0) P_{N_2} e^{t\partial_x D_x^a} u_2(0))\|_{L_t^2([0,N^{-a}],L^2)} \\ &\lesssim N^{-\frac{a}{2}} \|P_{N_1} u_1(0)\|_{L^2} \|P_{N_2} u_2(0)\|_{L^2}, \end{aligned}$$

where $S_{>N}$ denotes the part where the modulus of the frequencies is separated of order N and $N_1 \sim N_2 \sim N$. This follows from the interval slicing argument depicted above.

Moreover, rescaling solutions $u(t,x) \to u(\lambda^k t, \lambda x)$ yields the estimate (3.16) with the same constant on a rescaled domain.

We have a look at examples: In the one-dimensional case one can consider the equations:

$$i\partial_t u + D^{a+1} u = 0, \quad a > 0, \quad u : \mathbb{R} \times \mathbb{T} \to \mathbb{C},$$

or, similarly,

$$\partial_t u + \partial_x D_x^a u = 0, \quad u : \mathbb{R} \times \mathbb{T} \to \mathbb{R}.$$

In both cases Proposition 3.1.5 yields for free solutions u_i, $i = 1, 2$

$$\|P_N u_1(t) P_K u_2(t)\|_{L_t^2([0,N^{-a}],L^2(\mathbb{T}))} \lesssim N^{-\frac{a}{2}} \|P_N u_1(0)\|_{L^2(\mathbb{T})} \|P_K u_2(0)\|_{L^2(\mathbb{T})},$$

where $K \ll N$. This becomes useful when we consider fractional Benjamin-Ono equations below.

3.2 Linear Strichartz estimates

In the case of an interaction of comparable frequencies one can not expect to still be able to apply transversality considerations. However, this is exploited in Chapter 4 when dealing with fractional Benjamin-Ono equations even in higher dimensions. Moreover, transversality can also come from angular separation.

One way to derive a nonlinear estimate for comparable frequencies is to apply linear Strichartz estimates, which are related to curvature of the characteristic surface.

In Euclidean space this mechanism is well-understood. Starting point for the derivation of linear estimates is a dispersive estimate

$$\|P_1 e^{it\varphi(\nabla/i)} u_0\|_{L^\infty(\mathbb{R}^n)} \lesssim_n (1 + |t|)^{-\theta} \|P_1 u_0\|_{L^1(\mathbb{R}^n)}. \tag{3.21}$$

For general results relating curvature and decay, see [KT05c]. Due to unitarity of the time evolution, one has the L^2-estimate

$$\|P_1 e^{it\varphi(\nabla/i)} u_0\|_{L^2(\mathbb{R}^n)} \lesssim \|P_1 u_0\|_{L^2(\mathbb{R}^n)} \tag{3.22}$$

and following [GV79], Strichartz estimates for $q \neq 2$ follow from the TT^*-argument ([Tom75])[1].

[1] The endpoints $q = 2, p \neq \infty$ were covered in [KT98] by a more subtle interpolation argument.

3.2.1 Strichartz estimates from decoupling and applications

The situation is less clear in the periodic case. Below, we show how the essentially sharp ℓ^2-decoupling results from [BD15, BD17a] imply Strichartz estimates for more general phase functions.
In the Bourgain-Demeter works this was pointed out for Schrödinger dispersion relation. The following modest generalization of the argument clarifies the role of curvature. We point out how ℓ^2-decoupling implies Strichartz estimates for non-degenerate phase functions on tori $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. These estimates apply to solutions to linear dispersive PDE

$$\begin{cases} i\partial_t u + \varphi(\nabla/i)u &= 0,\ (t,x) \in \mathbb{R} \times \mathbb{T}, \\ \qquad u(0) &= u_0, \end{cases} \tag{3.23}$$

where $\varphi \in C^2(\mathbb{R}^n,\mathbb{R})$.

The eigenvalues of $D^2\varphi(\xi)$ are denoted by $\{\gamma_1(\xi),\dots,\gamma_n(\xi)\}$ and we set

$$\sigma_\varphi(\xi) = \min(\{\#neg.\gamma_i(\xi), \#pos.\gamma_i(\xi)\}).$$

The non-degeneracy hypothesis we assume reads as follows for $\psi : 2^{\mathbb{N}_0} \to \mathbb{R}^{>0}$:

$$\min(|\gamma_i(\xi)|) \sim \max(|\gamma_i(\xi)|) \sim \psi(N),\ |\xi| \in [N,2N), \quad \sigma_\varphi(\xi) \equiv k. \tag{$\mathcal{E}^k(\psi)$}$$

The Strichartz estimates we prove below read

$$\|P_N e^{it\varphi(\nabla/i)} u_0\|_{L^p(I\times\mathbb{T}^n)} \lesssim |I|^{1/p} N^{s(\varphi)} \|P_N u_0\|_{L^2}. \tag{3.24}$$

To prove (3.24), we will use ℓ^2-decoupling (cf. [BD15, BD17a]), more precisely, (variants of) the discrete L^2-restriction theorem.

Proposition 3.2.1. *Suppose that φ satisfies ($\mathcal{E}^k(\psi)$) and let $I \subseteq \mathbb{R}$ be a compact interval with $|I| \gtrsim 1$. Then, we find the following estimates to hold for any $\varepsilon > 0$*

$$\|P_N e^{it\varphi(\nabla/i)} u_0\|_{L^p(I\times\mathbb{T}^n)} \lesssim_\varepsilon |I|^{1/p} \frac{N^{\left(\frac{n}{2}-\frac{n+2}{p}\right)+\varepsilon}}{(\min(\psi(N),1))^{1/p}} \|P_N u_0\|_{L^2} \tag{3.25}$$

provided that $\frac{2(n+2-k)}{n-k} \le p < \infty$.

Proof. Without loss of generality let $I = [0,T]$. First, let $p \ge \frac{2(n+2-k)}{n-k}$ and compute

$$\begin{aligned} \text{lhs}(3.25)^p &= \int_{\substack{0\le x_1,\dots,x_n\le 2\pi,\\ 0\le t\le T}} \Big| \sum_{|\xi|\sim N} e^{i(x.\xi + t\varphi(\xi))} \hat{u}_0(\xi) \Big|^p dxdt \\ &\sim \frac{N^{-(n+2)}}{\psi(N)} \int_{\substack{0\le x_1,\dots,x_n\le N,\\ 0\le t\le TN^2\psi(N)}} \Big| \sum_{|\xi|\sim 1, \xi\in\mathbb{Z}^n/N} e^{i(x.\xi + \frac{t}{N^2\psi(N)}\varphi(N\xi))} \hat{u}_0(N\xi) \Big|^p dxdt. \end{aligned}$$

We distinguish between $\psi(N) \ll 1$ and $\psi(N) \gtrsim 1$. In the latter case, we use periodicity in space to find

$$\sim \frac{N^{-(n+2)}}{(TN\psi(N))^n \psi(N)} \int_{\substack{0\le x_1,\dots,x_n\le TN^2\psi(N),\\ 0\le t\le TN^2\psi(N)}} |\sum_{\substack{|\xi|\sim 1,\\ \xi\in\mathbb{Z}^n/N}} \hat{u}_0(N\xi) e^{i(x.\xi + \frac{t}{N^2\psi(N)}\varphi(N\xi))}|^p dxdt.$$

This expression is amenable to the discrete L^2-restriction theorem [BD15, Theorem 2.1, p. 354] or the variant for hyperboloids because $TN^2\psi(N) \gtrsim N^2$ and the frequency points are separated of size $\frac{1}{N}$ and the eigenvalues of $\frac{\varphi(N\cdot)}{N^2\psi(N)}$ are approximately one.

Hence, we have the following estimate uniform in φ (the dependence is encoded in $\psi(N)$, which drops out in the ultimate estimate)

$$\begin{aligned} &\lesssim_\varepsilon \frac{N^{-(n+2)}}{(TN\psi(N))^n\psi(N)}(TN^2\psi(N))^{n+1}N^{\left(\frac{n}{2}-\frac{n+2}{p}\right)p+\varepsilon}\|P_N u_0\|_2^p \\ &\lesssim_\varepsilon TN^{\left(\frac{n}{2}-\frac{n+2}{p}\right)p+\varepsilon}\|P_N u_0\|_2^p. \end{aligned}$$

Next, suppose that $\psi(N) \ll 1$. In this case we can not avoid loss of derivatives in general. Following along the above lines, we find for $p \geq \frac{2(n+2-k)}{n-k}$

$$\begin{aligned} \text{lhs}(3.25)^p &\sim \frac{N^{-(n+2)}}{\psi(N)} \int_{\substack{0\leq x_1,\dots,x_n\leq N,\\ 0\leq t\leq TN^2\psi(N)}} \left| \sum_{\substack{|\xi|\sim 1,\\ \xi\in\mathbb{Z}^n/N}} e^{i(x.\xi+t\frac{\varphi(N\xi)}{N^2\psi(N)})}\hat{u}_0(N\xi)\right|^p dxdt \\ &\lesssim \frac{N^{-(n+2)}}{(NT)^n\psi(N)} \int_{\substack{0\leq x_1,\dots,x_n\leq TN^2,\\ 0\leq t\leq TN^2}} \left|\sum e^{i(x.\xi+\frac{t\varphi(N\xi)}{N^2\psi(N)})}\hat{u}_0(N\xi)\right|^p dxdt \\ &\lesssim_\varepsilon \frac{T}{\psi(N)}N^{\left(\frac{n}{2}-\frac{n+2}{p}\right)p+\varepsilon}\|P_N u_0\|_2^p, \end{aligned}$$

which yields the claim. □

Recall that certain Strichartz estimates from [Bou93a, BD15, BD17a] are known to be sharp up to endpoints. Since the proposition is a generalization, the Strichartz estimates proved above are also sharp in this sense. Moreover, as in [BD15, BD17a] there are estimates for $2 \leq p \leq \frac{2(n+2-k)}{n-k}$, which follow from interpolation. As an example, we consider Strichartz estimates for the free fractional Schrödinger equation

$$\begin{cases} i\partial_t u + D^a u &= 0, \quad (t,x)\in\mathbb{R}\times\mathbb{T}^n, \\ u(0) &= u_0. \end{cases} \tag{3.26}$$

The phase function $\varphi(\xi) = |\xi|^a,\ 0 < a < 2,\ a \neq 1$ is elliptic, and the lack of differentiability at the origin is not an issue because low frequencies can always be treated with Bernstein's inequality. φ satisfies $(\mathcal{E}^0(\psi))$ with $\psi(N) = N^{a-2}$. Hence, we find by virtue of Proposition 3.2.1

$$\|e^{itD^a}u_0\|_{L^4_{t,x}(I\times\mathbb{T}^n)} \lesssim_{n,a,s} |I|^{1/4}\|u_0\|_{H^s}, \quad s > s_0 = \begin{cases} \frac{2-a}{8}, & n=1, \\ \frac{2-a}{4} + \left(\frac{n}{2}-\frac{n+2}{4}\right), & \text{else.} \end{cases} \tag{3.27}$$

To find the $L^4_{t,x}$-estimate in one dimension, we interpolate the $L^6_{t,x}$-endpoint estimate with the trivial $L^2_{t,x}$-estimate.
In case $n = 1,\ 1 < a < 2$ this recovers the Strichartz estimates from [DET16], and for $0 < a < 1$, this estimate was proved in [Din17].
For $n \geq 2,\ 1 < a < 2$, the estimates seem to be new. In [Din17] short-time arguments were used to derive Strichartz estimates on arbitrary compact manifolds.

These estimates we can improve on tori for $1 < a < 2$ because we do not have to sum up over frequency dependent time intervals.

However, for $p \neq 2$, Proposition 3.2.1 does not yield Strichartz estimates without loss of derivatives. When we want to apply these estimates to prove well-posedness of generalized cubic nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \varphi(\nabla/i)u &= \pm|u|^2 u, \ (t,x) \in \mathbb{R} \times \mathbb{T}^n, \\ u(0) &= u_0 \in H^s(\mathbb{T}^n), \end{cases} \tag{3.28}$$

we will use orthogonality considerations to prove refined bilinear $L^2_{t,x}$-estimates for $High \times Low \to High$-interaction. These estimates have no loss of derivatives in the high frequency, thus allowing us to close the contraction argument.
In [BGT05, Theorem 3, p. 193] was proved the following proposition to derive well-posedness to cubic Schrödinger equations on compact manifolds:

Proposition 3.2.2. *Let $u_0, v_0 \in L^2(\mathbb{T}^n)$, $K, N \in 2^{\mathbb{N}}$. If there exists $s_0 > 0$ such that*

$$\begin{aligned} &\|P_N e^{\pm it\varphi(\nabla/i)} u_0 P_K e^{\pm it\varphi(\nabla/i)} v_0\|_{L^2_{t,x}(I\times\mathbb{T}^n)} \\ &\lesssim |I|^{1/2} \min(N,K)^{s_0} \|P_N u_0\|_{L^2} \|P_K v_0\|_{L^2}, \end{aligned} \tag{3.29}$$

where $I \subseteq \mathbb{R}$ is a compact time interval with $|I| \gtrsim 1$, then the Cauchy problem (3.28) is locally well-posed in H^s for $s > s_0$.

For $\varphi = \sum_{i=1}^n \alpha_i \xi^2$ (3.29) follows from almost orthogonality and the Galilean transformation (cf. [Bou93a, Wan13]). It turns out that it is enough to require $(\mathcal{E}^k(\psi))$ to hold for some uniform constant $C_\varphi > 0$:

$$\forall \xi \in \mathbb{R}^n : \ \min(|\gamma_i(\xi)|) \sim \max(|\gamma_i(\xi)|) \sim C_\varphi, \quad \sigma_\varphi(\xi) \equiv k. \tag{$\mathcal{E}^k(C_\varphi)$}$$

This will be sufficient to generalize the Galilean transformation and prove the following:

Proposition 3.2.3. *Suppose that $\varphi \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfies $(\mathcal{E}^k(C_\varphi))$. Then, there is $s(n,k)$ such that we find the estimate*

$$\|P_N e^{\pm it\varphi(\nabla/i)} u_0 P_K e^{\pm it\varphi(\nabla/i)} v_0\|_{L^2_{t,x}(I\times\mathbb{T}^n)} \lesssim_{C_\varphi, s} K^{2s} |I|^{1/2} \|P_N u_0\|_{L^2} \|P_K v_0\|_{L^2} \tag{3.30}$$

to hold for $s > s(n,k)$, where $I \subseteq \mathbb{R}$ denotes a compact time interval, $|I| \gtrsim 1$.

Proof. Partition $P_N = \sum_{K_1} R_{K_1}$, where R_K projects to cubes of sidelength K. Then, by means of almost orthogonality,

$$\text{lhs}(3.30)^2 \lesssim \sum_{K_1} \|R_{K_1} e^{it\varphi(\nabla/i)} u_0 P_K e^{it\varphi(\nabla/i)} v_0\|^2_{L^2_{t,x}(I\times\mathbb{T}^n)}.$$

After applying Hölder's inequality, we are left with estimating two $L^4_{t,x}$-norms. Clearly, by Proposition 3.2.1

$$\|P_K e^{it\varphi(\nabla/i)} v_0\|_{L^4_{t,x}(I\times\mathbb{T}^n)} \lesssim_{\varphi,s} K^s \|P_K v_0\|_{L^2}$$

provided that $s > s(n, \sigma_\varphi)$.

To treat the other term, let ξ_0 denote the center of the cube Q_{K_1}, onto which R_{K_1} is projecting in frequency space, and following along the above lines, we write

$$\begin{aligned}
&\|R_{K_1} e^{it\varphi(\nabla/i)} u_0\|^4_{L^4_{t,x}(I\times\mathbb{T}^n)} \\
&= \int_{\substack{0\leq x_1,\ldots,x_n\leq 2\pi,\\ 0\leq t\leq T}} \left| \sum_{\xi\in Q_{K_1}} e^{i(x.\xi+t\varphi(\xi))}\hat{u}_0(\xi)\right|^4 dxdt \\
&= \int_{\substack{0\leq x_1,\ldots,x_n\leq 2\pi,\\ 0\leq t\leq T}} \left| \sum_{|\xi'|\leq K} \hat{u}_0(\xi+\xi') e^{i(x.(\xi_0+\xi')+t\varphi(\xi_0+\xi'))}\right|^4 dxdt \\
&= \int_{\substack{0\leq x_1,\ldots,x_n\leq 2\pi,\\ 0\leq t\leq T}} \left| \sum_{|\xi'|\leq K} e^{i((x+t\nabla\varphi(\xi_0)).\xi'+t\psi_{\xi_0}(\xi'))}\hat{w}_0(\xi')\right|^4 dxdt \\
&= \|P_{\leq K_1} e^{it\psi_{\xi_0}(\nabla/i)} w_0(x+t\nabla\varphi(\xi_0))\|^4_{L^4(I\times\mathbb{T}^n)},
\end{aligned}$$

where $\psi_{\xi_0}(\xi') = \varphi(\xi_0+\xi') - \varphi(\xi_0) - \nabla\varphi(\xi_0).\xi'$.

After breaking $\|P_{\leq K} e^{it\psi_{\xi_0}(\nabla/i)} w_0\|_{L^4_{t,x}(I\times\mathbb{T}^n)} \leq \sum_{1\leq L\leq K} \|P_L e^{it\psi_{\xi_0}(\nabla/i)} w_0\|_{L^4}$, the single expressions are amenable to Proposition 3.2.1. Indeed, the size of the moduli of the eigenvalues of $D^2\psi_{\xi_0}$ are approximately independent of the frequencies.

Hence,

$$\|P_L e^{it\psi_{\xi_0}(\nabla/i)} w_0\|_{L^4_{t,x}(I\times\mathbb{T}^n)} \lesssim_{\varepsilon,C_\varphi} L^{s(n,k)+\varepsilon}\|P_L w_0\|_{L^2},$$

and from carrying out the sum and the relation of u_0 and w_0, we find

$$\|P_{\leq K} e^{it\psi_{\xi_0}(\nabla/i)} w_0\|_{L^4(I\times\mathbb{T}^n)} \lesssim_{\varepsilon,\varphi} K^{s(n,k)+\varepsilon}\|R_{K_1} u_0\|_{L^2}.$$

The claim follows from almost orthogonality, i.e.,

$$\left(\sum_{K_1} \|R_{K_1} u_0\|^2_{L^2}\right)^{1/2} \lesssim \|P_N u_0\|_{L^2}.$$

□

This bilinear improvement can also stem from transversality: Write

$$|\nabla\varphi(\xi_1) \pm \nabla\varphi(\xi_2)| \sim N^\alpha, \text{ whenever } |\xi_1| \sim K,\ |\xi_2| \sim N. \qquad (\mathcal{T}^\alpha)$$

The corresponding short-time estimate from Section 3.1 is sufficient to derive an $L^2_{t,x}$-estimate for finite times by gluing together the short time intervals:

Proposition 3.2.4. *Let $\alpha > 0$, $K \ll N, K, N \in 2^{\mathbb{N}}$ and suppose that φ satisfies $(\mathcal{T}^\alpha)$. Then, we find the following estimate to hold:*

$$\|P_N e^{\pm it\varphi(\nabla/i)} u_0 P_K e^{\pm it\varphi(\nabla/i)} v_0\|_{L^2_{t,x}(I\times\mathbb{T})} \lesssim_\varphi |I|^{1/2}\|P_N u_0\|_{L^2}\|P_K v_0\|_{L^2} \qquad (3.31)$$

provided that $I \subseteq \mathbb{R}$ is a compact time interval with $|I| \gtrsim N^{-\alpha}$.

Proof. Let $I = \bigcup_j I_j$, $|I_j| \sim N^{-\alpha}$, where the I_j are disjoint. Then,

$$\begin{aligned} lhs(3.31)^2 &\lesssim \sum_{I_j} \|P_N e^{\pm it\varphi(\nabla/i)} u_0 P_K e^{\pm it\varphi(\nabla/i)} v_0\|^2_{L^2_{t,x}(I_j \times \mathbb{T})} \\ &\lesssim (\# I_j) N^{-\alpha} \|P_N u_0\|^2_{L^2} \|P_K v_0\|^2_{L^2}, \end{aligned}$$

and the claim follows from $\# I_j \sim |I| N^{\alpha}$. □

Invoking Proposition 3.2.2 together with Propositions 3.2.3 or 3.2.4, the below theorem follows:

Theorem 3.2.5. *Suppose that $\varphi \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfies $(\mathcal{E}^k(C_\varphi))$. Then, there is $s_0(n,k)$ such that* (3.28) *is locally well-posed for $s > s_0(n,k)$.*
Let $n = 1$ and suppose that φ satisfies $(\mathcal{T}^\alpha)$. Then, there is $s_0 = s_0(\varphi)$ such that (3.28) *is locally well-posed for $s > s_0(\varphi)$.*

We give examples: In one dimension we treat the fractional Schrödinger equation

$$\begin{cases} i\partial_t u + D^a u &= \pm |u|^2 u, \; (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u(0) &= u_0 \in H^s(\mathbb{T}), \end{cases} \tag{3.32}$$

where $D = (-\Delta)^{1/2}$.

Theorem 3.2.5 yields uniform local well-posedness for $s > \frac{2-a}{4}$, $1 < a < 2$, which is presumably sharp up to endpoints as discussed in [CHKL15], where the endpoint $s = \frac{2-a}{4}$ was covered by resonance considerations.
For $0 < a < 1$ varying the above arguments, we can also prove local well-posedness for $s > \frac{2-a}{4}$, which was previously proved in [Din17] in the context of Strichartz estimates for fractional Schrödinger equations on compact manifolds.

Moreover, in Euclidean space fractional Schrödinger equations were considered in [HS15]. Key ingredient to well-posedness are linear and bilinear Strichartz estimates, which hold globally in time due to dispersive effects. On the circle we can reach the same regularity up to the endpoint like in [HS15].
It might well be the case that the linear Strichartz estimates are sharp in higher dimensions because they match the estimates from Euclidean space. However, satisfactory bilinear $L^2_{t,x}$-Strichartz estimates appear to be beyond the above arguments and possibly require additional angular decompositions (cf. [CKS+08]).

We also discuss hyperbolic Schrödinger equations. The well-posedness result from [Wan13, GT12] is recovered for the hyperbolic nonlinear Schrödinger equation in two dimensions, which is known to be sharp up to endpoints.
Generalizing the example probing sharpness to higher dimensions indicates that there is only a significant difference between hyperbolic and elliptic Schrödinger equations in low dimensions.
For hyperbolic phase functions, Theorem 3.2.5 recovers the results from [Wan13, GT12], where essentially sharp local well-posedness of

$$\begin{cases} i\partial_t u + (\partial^2_{x_1} - \partial^2_{x_2}) u &= \pm |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^2, \\ u(0) &= u_0 \in H^s(\mathbb{T}^2), \end{cases} \tag{3.33}$$

was proved for $s > 1/2$. Notably, due to subcriticality of the $L^4_{t,x}$-Strichartz estimate, already for the hyperbolic equations

$$\begin{cases} i\partial_t u + (\partial^2_{x_1} - \partial^2_{x_2} + \partial^2_{x_3}) u &= \pm |u|^2 u, \; (t,x) \in \mathbb{R} \times \mathbb{T}^3, \\ u(0) &= u_0, \end{cases} \tag{3.34}$$

and

$$\begin{cases} i\partial_t u + (\partial_{x_1}^2 - \partial_{x_2}^2 + \partial_{x_3}^2 - \partial_{x_4}^2)u &= \pm |u|^2 u, \ (t,x) \in \mathbb{R} \times \mathbb{T}^4, \\ u(0) &= u_0, \end{cases} \tag{3.35}$$

the (essentially sharp) Strichartz estimates yield the same well-posedness results as for the elliptic counterparts:

Firstly, recall the counterexample from [Wan13], which showed C^3-ill-posedness of (3.33) for $s < 1/2$. As initial data consider

$$\phi_N(x) = N^{-1/2} \sum_{|k| \leq N} e^{ikx_1} e^{-ikx_2},$$

which satisfies $\|\phi_N\|_{H^s} \sim N^s$ and $S[\phi_N](t) := e^{it(\partial_{x_1}^2 - \partial_{x_2}^2)} \phi_N = \phi_N$. This implies

$$\left\| \int_0^T |S[\phi_N](s)|^2 S[\phi_N](s) ds \right\|_{H^s} = T \| |\phi_N|^2 \phi_N \|_{H^s} \gtrsim T N^{1+s}.$$

For details on this estimate, see [Wan13].

The validity of the estimate

$$\left\| \int_0^T |S[\phi_N](s)|^2 S[\phi_N](s) ds \right\|_{H^s} \lesssim \|\phi_N\|_{H^s}^3 \quad (T \lesssim 1)$$

requires $s \geq 1/2$.
The same counterexample shows that $s \geq 1/2$ is required for C^3-well-posedness of (3.34). This regularity is reached up to the endpoint in Theorem 3.2.5.
When considering (3.35), we modify the above example to

$$\phi_N(x) = N^{-1} \sum_{|k_1|,|k_2| \leq N} e^{ik_1 x_1} e^{-ik_1 x_2} e^{ik_2 x_3} e^{-ik_2 x_4},$$

which again satisfies $\|\phi_N\|_{H^s} \sim N^s$.
Carrying out the estimate for the first Picard iterate with the necessary modifications yields

$$\left\| \int_0^T |S[\phi_N](s)|^2 S[\phi_N](s) ds \right\|_{H^s} = T \| |\phi_N|^2 \phi_N \|_{H^s} \gtrsim T N^{2+s},$$

which implies C^3-ill-posedness, unless $s \geq 1$. This regularity is again obtained up to the endpoint in Theorem 3.2.5.

Apparently, for other hyperbolic Schrödinger equations, the $L^4_{t,x}$-Strichartz estimate also coincides with the elliptic $L^4_{t,x}$-estimate, and modifications of the above counterexample yield lower thresholds than in the elliptic case. This indicates that the difference between elliptic and hyperbolic Schrödinger equations is only significant in lower dimensions.

3.2.2 Strichartz estimates on compact manifolds

A fairly different approach to derive Strichartz estimates, which does not only work on tori, but on general compact Riemannian manifolds, is to localize time in a frequency dependent way such that the difference between a compact manifold and Euclidean space is no longer significant, and one recovers the dispersive estimate.

As already pointed out in the introduction, this approach was successfully applied in [BGT04] to solutions to the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_g u &= 0, \quad (t,x) \in \mathbb{R} \times M, \\ \quad u(0) &= u_0 \in H^s(M) \end{cases} \tag{3.36}$$

where Δ_g denotes the Laplace-Beltrami operator on a compact boundaryless Riemannian manifold M. The argument admits extension to manifolds with boundary (cf. [BSS08, BSS12]), non-elliptic (cf. [MT15]) and fractional Schrödinger equations on compact manifolds (cf. [Din17]).
We record the last result:

Proposition 3.2.6 ([Din17, Proposition 2.5, p. 8812])**.** *Let $a \in (0,\infty)\backslash\{1\}$, $N \in 2^{\mathbb{N}_0}$ and M an n-dimensional compact Riemannian manifold without boundary. Denoting $D^a = (-\Delta_g)^{a/2}$ we find the following estimate to hold:*

$$\|P_N e^{itD^a} u_0\|_{L_t^q([0,N^{-(a-1)}\wedge 1], L_x^p(M))} \lesssim_{n,p,q} N^s \|P_N u_0\|_{L^2} \tag{3.37}$$

provided that $2 \leq q,p \leq \infty$, $\frac{2}{q} + \frac{n}{p} = \frac{n}{2}$, $(q,p) \neq (2,\infty)$, $s = n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{a}{q}$.

In (3.37) P_N projects to the union of eigenspaces of $-\Delta_g$ consisting of eigenfunctions with eigenvalues comparable to N^2.
These estimates become useful after invoking the transfer principle when one considers fractional Schrödinger equations on the circle for frequency dependent time intervals.

3.3 Short-time nonlinear estimates

With short-time linear and bilinear estimates at our disposal, we come back to the propagation of the nonlinearity of the Benjamin-Ono equation on the circle in short-time function spaces. First, record the following estimates due to Proposition 3.1.5, the transfer principle and the interpolation argument from Lemma 2.3.4:

Proposition 3.3.1. *Let $K, N \in 2^{\mathbb{N}}$, $K \ll N$ and $|I| \lesssim N^{-1}$ be an interval. Suppose that $u, v \in U^2_{BO}(I)$. Then, we find the following estimates to hold:*

$$\|P_N u P_K v\|_{L_t^2([0,N^{-1}],L^2(\mathbb{T}))} \lesssim N^{-1/2} \|P_N u\|_{U^2_{BO}(I)} \|P_K u\|_{U^2_{BO}(I)}, \tag{3.38}$$

$$\|P_N u P_K v\|_{L_t^2([0,N^{-1}],L^2(\mathbb{T}))} \lesssim N^{-1/2} \log^2\langle N\rangle \|P_N u\|_{V^2_{BO}(I)} \|P_K u\|_{V^2_{BO}(I)}. \tag{3.39}$$

The estimates remain true in case of comparable frequencies as long as $||\xi_1| - |\xi_2|| \gtrsim N$ whenever $\xi_1 \in supp_\xi \hat{u}$ and $\xi_2 \in supp_\xi \hat{v}$.

We prove the following proposition:

Proposition 3.3.2. *Let $T \in (0,1]$ and $0 < s < s'$. Then, we find the following estimates to hold:*

$$\|\partial_x(uv)\|_{N^s_{BO}(T)} \lesssim \|u\|_{F^s_{BO}(T)} \|v\|_{F^{s'}_{BO}(T)}, \tag{3.40}$$

$$\|\partial_x(uv)\|_{N^0_{BO}(T)} \lesssim \|u\|_{F^0_{BO}(T)} \|v\|_{F^s_{BO}(T)}. \tag{3.41}$$

Proof. After frequency localization, it is enough to derive short-time $DU^2_{BO} - U^2_{BO}$-estimates by the definition of the function spaces.
In case of $High \times Low \to High$-interaction ($N \sim N_1 \gg N_2$), we use the imbedding $L^1(I) \hookrightarrow DU^2_{BO}(I)$, Hölder in time (recall that $|I| = N^{-1}$) and (3.38) to derive

$$\begin{aligned} \|P_N \partial_x (P_{N_1} u P_{N_2} v)\|_{DU^2_{BO}(I)} &\lesssim N \|P_{N_1} u P_{N_2} v\|_{L^1(I;L^2_x)} \\ &\lesssim N^{1/2} \|P_{N_1} u P_{N_2} v\|_{L^2(I;L^2_x)} \\ &\lesssim \|P_{N_1} u\|_{U^2_{BO}(I)} \|P_{N_2} v\|_{U^2_{BO}(I)}. \end{aligned}$$

For $High \times High \to High$-interaction ($N \sim N_1 \sim N_2$), we use duality to write

$$\|P_N \partial_x (P_{N_1} u P_{N_2} v)\|_{DU^2_{BO}(I;L^2)} \lesssim \sup_{\|w\|_{V^2_{BO}}=1} N \left| \int \int P_{N'} w P_{N_1} u P_{N_2} v dx dt \right|.$$

Since two factors must be frequency separated of order N by Remark 3.1.6[2], we can use a bilinear Strichartz estimate on two factors (say w and u) and use the energy estimate on the remaining factor to find

$$\begin{aligned} N \left| \int_I \int P_{N'} w P_{N_1} u P_{N_2} v dx dt \right| &\lesssim N \|P_{N'} w P_{N_1} u\|_{L^2(I;L^2)} \|P_{N_2} v\|_{L^2(I;L^2)} \\ &\lesssim \log^2 \langle N \rangle \|w\|_{V^2_{BO}(I)} \|P_{N_1} u\|_{V^2_{BO}} \|P_{N_2} v\|_{U^2_{BO}}. \end{aligned}$$

Finally, for $High \times High \to Low$-interaction ($N_1 \sim N_2 \gg N$), we have to partition the interval I, $|I| = N^{-1}$ into N_1^{-1} intervals, which accounts for a factor of N_1/N. Using duality and the bilinear Strichartz estimate, we find with $|I'| = N_1^{-1}$

$$\begin{aligned} \int_{I'} \int P_N w P_{N_1} u P_{N_2} v dx dt &\lesssim \|P_N w P_{N_1} u\|_{L^2(I',L^2_x)} \|P_{N_2} v\|_{L^2(I',L^2_x)} \\ &\lesssim \log^2 \langle \frac{N_1}{N} \rangle \|P_N w\|_{V^2_{BO}(I')} \|P_{N_1} u\|_{V^2_{BO}(I')} \|P_{N_2} v\|_{U^2_{BO}(I')}, \end{aligned}$$

which yields the claim because it implies

$$\begin{aligned} \|P_N \partial_x (P_{N_1} u P_{N_2} v)\|_{DU^2_{BO}(I;L^2)} \lesssim \log^2 \langle \frac{N_1}{N} \rangle &\sup_{\substack{|I'|=N_1^{-1}, \\ I' \subseteq I}} \|P_{N_1} u\|_{U^2_{BO}(I';L^2)} \\ &\times \sup_{\substack{|I'|=N_2^{-1}, \\ I' \subseteq I}} \|P_{N_2} v\|_{U^2_{BO}(I';L^2)}. \end{aligned}$$

□

The proof makes the heuristic argument from the introduction (1.9) precise by using embedding properties of the function spaces and the transfer principle.
We point out that the argument extends to higher nonlinearities:

[2]Strictly speaking, we have to consider a decomposition like in Remark 3.1.6; the details are omitted for the sake of brevity.

Proposition 3.3.3. *Let $s > 1/2$ and $k \geq 3$. Then, we find the following estimates to hold:*

$$\|\partial_x(u_1 \dots u_k)\|_{N^s_{BO}(T)} \lesssim \prod_{l=1}^{k} \|u_l\|_{F^s_{BO}(T)}, \tag{3.42}$$

$$\|\partial_x(u_1 \dots u_k)\|_{N^0_{BO}(T)} \lesssim \|u_1\|_{F^0_{BO}(T)} \prod_{l=2}^{k} \|u_l\|_{F^s_{BO}(T)}. \tag{3.43}$$

Proof. In case of $High \times \dots \to High$-interaction ($N \sim N_1 \gtrsim N_2 \dots \gtrsim N_k$) the following crude estimate suffices:

$$\begin{aligned}\|P_N(\partial_x(P_{N_1}u_1 \dots P_{N_k}u_k)\|_{DU^2_{BO}(I)} &\lesssim N^{1/2}\|P_{N_1}u_1 \dots P_{N_k}u_k\|_{L^1_t(I;L^2_x)} \\ &\lesssim \|P_{N_1}u_1\|_{L^\infty_t L^2_x} \prod_{l=2}^{k} \|P_{N_l}u_l\|_{L^\infty_{t,x}} \\ &\lesssim \|P_{N_1}u_1\|_{U^2_{BO}(I)} \prod_{l=2}^{k} N_l^{1/2}\|P_{N_l}u_l\|_{U^2_{BO}}.\end{aligned} \tag{3.44}$$

For $High \times High \times \dots \to Low$-interaction ($N \ll N_1 \sim N_2, N_2 \gtrsim N_3 \dots \gtrsim N_k$) observe

$$\begin{aligned}&\|P_N(\partial_x(P_{N_1}u_1 P_{N_2}u_2 \dots P_{N_k}u_k))\|_{DU^2(I)} \\ &\lesssim N\frac{N_1}{N} \sup_{\substack{I' \subseteq I, \\ |I'| = N_1^{-1}}} \sup_{\|v\|_{V^2_{BO}}=1} \int_{\substack{I' \subseteq I, \\ |I'| = N_1^{-1}}} \int P_N v P_{N_1}u_1 P_{N_2}u_2 \dots P_{N_k}u_k dxdt \\ &\lesssim N_1 \sup_{\|v\|_{V^2_{BO}}=1} \|P_N v P_{N_1}u_1\|_{L^2_{t,x}} \|P_{N_2}u_2\|_{L^2_{t,x}} \prod_{l=3}^{k+1} \|P_{N_l}u_l\|_{L^\infty_{t,x}} \\ &\lesssim \log\langle \frac{N_1}{N} \rangle \sup_{|I'| = N_1^{-1}} \|P_{N_1}u_1\|_{U^2_{BO}(I')} \|P_{N_2}u_2\|_{U^2_{BO}(I')} \prod_{l=3}^{k+1} N_l^{1/2}\|P_{N_l}u_l\|_{U^2_{BO}(I')},\end{aligned}$$

which is again enough to conclude the estimates (3.42), (3.43). □

3.4 Energy estimates

Purpose of this section is to propagate the energy norm. For solutions to (3.1) the desired estimate reads as

$$\|u\|^2_{E^s(T)} \lesssim \|u_0\|^2_{H^s} + T\|u\|^3_{F^s(T)} \tag{3.45}$$

For solutions to the difference equation, that is for $v = u_1 - u_2$, where u_1, u_2 are solutions to the original equation, we prove two estimates in addition to (3.45). The first one leads to Lipschitz dependence of the data-to-solution mapping in L^2 for initial data in H^s, $s > 1$:

$$\|v\|^2_{E^0(T)} \lesssim \|v(0)\|^2_{L^2} + T\|v\|^2_{F^0(T)}(\|u_1\|_{F^s(T)} + \|u_2\|_{F^s(T)}). \tag{3.46}$$

The second one leads to potentially non-uniform continuous dependence of the data-to-solution mapping in H^s:

$$\|v\|_{E^s(T)}^2 \lesssim \|v\|_{H^s}^2 + T(\|v\|_{F^s(T)}^3 + \|v\|_{F^s(T)}^2 \|u_2\|_{F^s(T)} + \|v\|_{F^0(T)} \|v\|_{F^s(T)} \|u_2\|_{F^{2s}(T)}). \tag{3.47}$$

In case of the Benjamin-Ono equation, we prove the following estimates:

Proposition 3.4.1. *Let $T \in (0,1]$ and $s > 1$.*

(a) For a smooth solution u to (3.1), we find (3.45) to hold.

(b) Let u_1, u_2 be smooth solutions to (3.1) and $v = u_1 - u_2$ be the difference of the two solutions. Then, we find (3.46) and (3.47) to hold.

The building block to prove Proposition 3.4.1 is the following estimate.

Lemma 3.4.2. *Let $T \in (0,1]$ and $N_2 \leq N_1 \sim N$. Then, we find the following estimate to hold:*

$$\int_0^T ds \int dx P_N u_1 P_{N_1} u_2 P_{N_2} u_3 \lesssim T \|P_N u_1\|_{F_{BO}^0(T)} \|P_{N_1} u_2\|_{F_{BO}^0(T)} \|P_{N_2} u_3\|_{F_{BO}^0(T)}. \tag{3.48}$$

Proof. Key ingredient is again the short-time bilinear Strichartz estimate. First, consider the case $N_2 \ll N$. After breaking $[0,T]$ into $\lesssim NT$ intervals I of size N^{-1}, we have to estimate

$$\begin{aligned}
&\int_I ds \int dx P_N u_1 P_{N_1} u_2 P_{N_2} u_3 \\
&\leq \|P_N u_1 P_{N_1} u_2\|_{L^2(I;L^2)} \|P_{N_2} u_3\|_{L^2(I;L^2)} \\
&\lesssim N^{-1} \|P_N u_1\|_{U_{BO}^2(I;L^2)} \|P_{N_1} u_2\|_{U_{BO}^2(I;L^2)} \|P_{N_2} u_3\|_{U_{BO}^2(I;L^2)}
\end{aligned}$$

Since splitting the time interval accounts for a factor of at most TN, the proof is complete. In case $N_2 \sim N$, it is easy to see following Remark 3.1.6 that we can still apply a bilinear Strichartz estimate to one pair. ☐

We show Proposition 3.4.1:

Proof of Proposition 3.4.1. First, we show (3.45). One has to analyze $\sup_{t \in [0,T]} \|P_N u(t)\|_{L^2}^2$ to conclude the estimate of the E^s-norm after carrying out the sum over N with weight N^{2s}.
The fundamental theorem of calculus yields

$$\|P_N u(t)\|_{L^2}^2 = \|P_N u(0)\|_{L^2}^2 + 2 \int_0^t ds \int dx P_N u P_N (\partial_x (uu)).$$

First, consider $High \times Low \to High$-interaction. That means we estimate

$$\int_0^t ds \int dx P_N u P_N [\partial_x (u P_{N_2} u)],$$

where $N_2 \ll N$. In case the derivative hits the high frequency factor, we integrate by parts to derive a more favourable expression: Write

$$\begin{aligned}
&\int_0^t ds \int dx P_N u P_N(\partial_x u P_{N_2} u) \\
&= \int_0^t ds \int dx P_N u P_N(\partial_x u) P_{N_2} u dx \\
&\quad + \int_0^t ds \int dx P_N u [P_N(\partial_x u P_{N_2} u) - P_N(\partial_x u) P_{N_2} u] dx.
\end{aligned}$$

For the first term we find after integration by parts:

$$\int_0^t ds \int dx P_N u P_N(\partial_x u) P_{N_2} u = -\frac{1}{2} \int_0^t ds \int dx P_N u P_N u \partial_x (P_{N_2} u).$$

The second term is a commutator term, which we analyze in Fourier variables:

$$\begin{aligned}
&(P_N[\partial_x u P_{N_2} u] - P_N(\partial_x u) P_{N_2} u)\widehat{\ }(\xi) \\
&= \sum_{\xi = \xi_1 + \xi_2} \chi_N(\xi_1 + \xi_2)(-i\xi_1)\hat{u}(\xi_1)\chi_N(\xi_2)\hat{u}(\xi_2) - \chi_N(\xi_1)(-\xi_1)\hat{u}(\xi_1)\chi_{N_2}(\xi_2)\hat{u}(\xi_2) \\
&= \sum_{\xi = \xi_1 + \xi_2} (\chi_N(\xi_1 + \xi_2) - \chi_N(\xi_1))(-i\xi_1)\chi_N(\xi_2)\hat{u}(\xi_1)\hat{u}(\xi_2) \\
&=: \sum_{\xi = \xi_1 + \xi_2} m(\xi_1, \xi_2)\tilde{\chi}_N(\xi_1)\hat{u}(\xi_1)\tilde{\chi}_{N_2}(\xi_2)\hat{u}(\xi_2).
\end{aligned}$$

It is straight-forward to check that

$$|\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2)| \lesssim_{\alpha_1, \alpha_2} \frac{N_2}{N^{\alpha_1 + \alpha_2}}.$$

In classical short-time Fourier restriction spaces the concluding arguments are detailed in [IKT08, p. 292]. The commutator argument was already used in [BS75] and is crucial to finish the proof.

Here, we point out that one can also change back to position space via expansion of $m(\xi_1, \xi_2)$ into a rapidly converging Fourier series (cf. [Han12, Section 5], [CHT12, Lemma 5.2, p. 68]), which gives

$$\begin{aligned}
&\left| \int_0^t ds \int dx P_N u [P_N(\partial_x u P_{N_2} u) - P_N(\partial_x u) P_{N_2} u] \right| \\
&\lesssim \frac{N_2}{N} \left| \int_0^t ds \int dx P_N u \tilde{P}_N(\partial_x u P_{N_2} u) \right| \\
&\lesssim N_2 \sum_{N_1 \sim N} \left| \int_0^t ds \int dx P_N u P_{N_1} u P_{N_2} u \right|.
\end{aligned}$$

By virtue of (3.48) and the Cauchy-Schwarz inequality, one finds

$$\begin{aligned}
&\sum_{N \geq 1} N^{2s} \sum_{N_1 \sim N} \sum_{1 \leq N_2 \leq N} \left| \int_0^t ds \int dx P_N u P_{N_1} u P_{N_2}(\partial_x u) \right| \\
&\lesssim T \sum_{N \geq 1} N^{2s} \sum_{N_1 \sim N} \sum_{1 \leq N_2 \leq N} N_2 \|P_N u\|_{F^0_{BO}(T)} \|P_{N_1} u\|_{F^0_{BO}(T)} \|P_{N_2} u\|_{F^0_{BO}(T)} \\
&\lesssim T \|u\|^3_{F^s(T)}.
\end{aligned}$$

In the above estimate we did not distinguish between $High \times Low \to High$-interaction or $High \times High \to High$-interaction. When dealing with $High \times High \to High$-interaction, there is no point in integrating by parts, and following Remark 3.1.6, the bilinear Strichartz estimate can still be applied.

In case of $High \times High \to Low$-interaction ($N \ll N_1 \sim N_2$), we again do not integrate by parts but simply use (3.48) to find

$$\begin{aligned}&\sum_{N\geq 1} N^{2s} \sum_{N_1\sim N_2 \gg N} \left| \int_0^t ds \int dx P_N u P_N \partial_x (P_{N_1} u P_{N_2} u) \right| \\ &\lesssim T \sum_{N\geq 1} N^{2s+1} \sum_{N_1 \sim N_2 \gg N} \|P_N u\|_{F^0_{BO}(T)} \|P_{N_1} u\|_{F^0_{BO}(T)} \|P_{N_2} u\|_{F^0_{BO}(T)} \lesssim T \|u\|^3_{F^s_{BO}(T)}\end{aligned}$$

provided that $s > 1$. The proof of estimate (3.45) is complete.

Next, we turn to estimate (3.46): Due to the reduced symmetry, one can not always integrate by parts like above. Again, we invoke the fundamental theorem of calculus to write

$$\|P_N v(t)\|^2_{L^2} = \|P_N v(0)\|^2_{L^2} + 2 \int_0^t ds \int dx P_N v (P_N (\partial_x (v(u_1 + u_2)))).$$

First, consider $High \times Low \to High$-interaction ($N_2 \ll N$). In case the high frequency is on the difference solution, that means we are considering the expression

$$\int_0^t ds \int dx P_N v P_N (\partial_x (v (P_{N_2} (u_1 + u_2)))$$

we can argue like above by integration by parts and the commutator estimate to conclude

$$\begin{aligned}&T \sum_{N \geq 1} \sum_{1 \leq N_2 \ll N} \sum_{N_1 \sim N} \|P_N v\|_{F^0_{BO}} \|P_{N_1} v\|_{F^0_{BO}} N_2 \|P_{N_2} u\|_{F^0_{BO}} \\ &\lesssim T \sum_{N\geq 1} \sum_{N_1 \sim N} \|P_N v\|_{F^0_{BO}(T)} \|P_{N_1} v\|_{F^0_{BO}(T)} \|u\|_{F^s_{BO}(T)} \lesssim T \|v\|^2_{F^0_{BO}} \|u\|_{F^s_{BO}}.\end{aligned}$$

However, when considering the expression $A = \int_0^t ds \int dx P_N v P_N (\partial_x (P_{N_2} v \cdot u))$, $N_2 \ll N$ we can not integrate by parts. Still,

$$\begin{aligned}A &\lesssim T \sum_{N\geq 1} \sum_{1 \leq N_2 \ll N} \sum_{N_1 \sim N} N \|P_N v\|_{F^0_{BO}(T)} \|P_{N_1} u\|_{F^0_{BO}(T)} \|P_{N_2} v\|_{F^0_{BO}(T)} \\ &\lesssim T \|v\|^2_{F^0_{BO}(T)} \|u\|_{F^s_{BO}(T)}\end{aligned}$$

provided that $s > 1$.

In case of $High \times High \to High$- and $High \times High \to Low$-interaction, the argument from the proof of (3.45) applies without modification and yields the desired estimate. This completes the proof of (3.46).

We turn to the proof of (3.47). For this purpose rewrite the equation satisfied by $v = u_1 - u_2$ as

$$\partial_t v + \mathcal{H} \partial_{xx} v = \partial_x (v^2) + 2 \partial_x (v u_2)$$

Using the same strategy like above, we have to focus on $High \times Low \to High$-interaction ($N_2 \ll N$) in the expression $\partial_x (u_2 P_N v)$, where we can not integrate by parts.

More precisely, we have to carry out the estimate

$$\begin{aligned}
&\sum_{N \geq 1} N^{2s} \sum_{1 \leq N_2 \ll N} \int_0^t ds \int dx P_N v P_N (\partial_x u_2 P_{N_2} v) \\
&\lesssim T \sum_{N \geq 1} \sum_{1 \leq N_2 \ll N} \sum_{N \sim N_1} N^{2s+1} \|P_N v\|_{F^0_{BO}(T)} \|P_{N_1} u_2\|_{F^0_{BO}(T)} \|P_{N_2} v\|_{F^0_{BO}(T)} \\
&\lesssim T \|v\|_{F^0_{BO}(T)} \|v\|_{F^s_{BO}(T)} \|u_2\|_{F^{2s}_{BO}(T)}.
\end{aligned}$$

The remaining cases can be treated like above, which completes the proof. □

For smooth solutions u to (3.6) we have to derive

$$\|u\|^2_{E^s(T)} \lesssim \|u_0\|^2_{H^s} + T\|u\|^{k+1}_{F^s_{BO}(T)} \tag{3.49}$$

and for differences of solutions

$$\|v\|^2_{E^0(T)} \lesssim \|v(0)\|^2_{L^2} + T\|v\|^2_{F^0_{BO}(T)} Q_k(\|u_1\|_{F^s_{BO}(T)}, \|u_2\|_{F^s_{BO}(T)}), \tag{3.50}$$

$$\begin{aligned}
\|v\|^2_{E^s(T)} \lesssim \|v\|^2_{H^s} &+ T(\|v\|^2_{F^s_{BO}(T)} S^1_k(\|v\|_{F^s_{BO}(T)}, \|u_2\|_{F^s_{BO}(T)})) \\
&+ T\|v\|_{F^0_{BO}(T)} \|v\|_{F^s_{BO}(T)} \|u_2\|_{F^{2s}_{BO}(T)} S^2_k(\|v\|_{F^s_{BO}(T)}, \|u_2\|_{F^s_{BO}(T)}),
\end{aligned} \tag{3.51}$$

where Q_k and S^i_k, $i = 1, 2$ are polynomials like in (3.8).

The commutator arguments to put the derivative on a low frequency factor extend directly. The second ingredient to generalize Proposition 3.4.1 is the analog of Lemma 3.4.2:

Lemma 3.4.3. *Let $k \in \mathbb{Z}_{\geq 3}$, $(N_i)_{i=1}^{k+1} \subseteq 2^{\mathbb{N}_0}$. Suppose that $N_1 \sim N_2 \geq N_3 \geq \ldots \geq N_{k+1}$ and let $I \subseteq \mathbb{R}$ be an interval with $|I| = N_1^{-1}$. Then, we find the following estimate to hold:*

$$\begin{aligned}
&\int_I \int P_{N_1} u_1 P_{N_2} u_2 \ldots P_{N_{k+1}} u_{k+1} dx dt \\
&\lesssim N_1^{-1} \|P_{N_1} u_1\|_{U^2_{BO}(I)} \|P_{N_2} u_2\|_{U^2_{BO}(I)} \|P_{N_3} u_3\|_{U^2_{BO}(I)} \prod_{i=4}^{k+1} N_i^{1/2} \|P_{N_i} u_i\|_{U^2_{BO}(I)}.
\end{aligned} \tag{3.52}$$

Proof. Suppose that $N_1 \sim N_2 \gtrsim N_3 \gg N_4$. Then, the expression is amenable to two short-time bilinear Strichartz estimates involving the highest to fourth to highest frequencies and using pointwise bounds on the remaining frequencies yields

$$\begin{aligned}
\text{lhs}(3.52) &\lesssim N_1^{-1} \prod_{i=1}^{4} \|P_{N_i} u_i\|_{U^2_{BO}(I)} \prod_{i=5}^{k+1} \|P_{N_i} u_i\|_{L^\infty_t L^\infty_x} \\
&\lesssim N_1^{-1} \prod_{i=1}^{4} \|P_{N_i} u_i\|_{U^2_{BO}(I)} \prod_{i=5}^{k+1} N_i^{1/2} \|P_{N_i} u_i\|_{U^2_{BO}(I)}.
\end{aligned}$$

Suppose that $N_1 \sim N_2 \sim N_3 \sim N_4$. In this case we use Hölder's inequality and Bernstein's inequality to find

$$\begin{aligned}
\text{lhs}(3.52) &\lesssim \prod_{i=1}^{4} \|P_{N_i} u_i\|_{L^4_{t,x}} \prod_{i=5}^{k+1} \|P_{N_i} u_i\|_{L^\infty_{t,x}} \\
&\lesssim N_1^{-1/2} \prod_{i=1}^{4} \|P_{N_i} u_i\|_{L^8_t L^4_x} \prod_{i=5}^{k+1} N_i^{1/2} \|P_{N_i} u_i\|_{L^\infty_t L^2_x}.
\end{aligned}$$

Next, we use the transfer principle for U^p-spaces and Proposition 3.2.6 to find

$$\begin{aligned}
&\lesssim N_1^{-1/2} \prod_{i=1}^{4} \|P_{N_i} u_i\|_{U^8_{BO}(I)} \prod_{i=5}^{k+1} N_i^{1/2} \|P_{N_i} u_i\|_{U^2_{BO}(I)} \\
&\lesssim N_1^{-1} \prod_{i=1}^{4} \|P_{N_i} u_i\|_{U^2_{BO}(I)} \prod_{i=5}^{k+1} N_i^{1/2} \|P_{N_i} u_i\|_{U^2_{BO}(I)}.
\end{aligned}$$

The ultimate estimate follows from $U^2 \hookrightarrow U^8$ (cf. Lemma 2.3.4). This completes the proof. □

We summarize the energy estimates for k-generalized Benjamin-Ono equations:

Proposition 3.4.4. *Let $T \in (0,1]$, $k \in \mathbb{Z}_{\geq 3}$ and $s > 1$.*

(a) For a smooth solution u to (3.6)*, we find* (3.49) *to hold.*

(b) Let u_1, u_2 be smooth solutions to (3.6) *and $v = u_1 - u_2$ be the differences of two solutions. Then, we find* (3.50) *and* (3.51) *to hold.*

3.5 Proof of local well-posedness via Bona-Smith approximation

With the nonlinear estimates and energy estimates at hand, we show how (3.3) allows us to conclude local well-posedness by bootstrap arguments and a variant of the Bona-Smith approximation.
First, we carry out the argument for small initial data and assume like above that $u_0 \in C^\infty(\mathbb{T})$.

The energy method [ABFS89] yields existence of solutions in $C([0,T^*], H^{\tilde{s}}(\mathbb{T}))$ for $\tilde{s} > 3/2$, where $\lim_{t \to T^*} \|u(t)\|_{H^{2\tilde{s}}} = \infty$ whenever $T^* < \infty$. For brevity (and to point out independence of the arguments on the propagator) we write in the following $F^s(T) = F^s_{BO}(T)$.

The set of estimates (3.3) together with (2.35) yields

$$\|u\|^2_{F^s} \lesssim \|u_0\|^2_{H^s} + \|u\|^4_{F^s(T)} + T\|u\|^3_{F^s(T)}.$$

Next, we make use of continuity of $\|u\|_{E^s(T)}$ in T and

$$\lim_{T \to 0} \|u\|_{E^s(T)} \lesssim \|u_0\|_{H^s}, \quad \lim_{T \to 0} \|\partial_{x_1}(u^2)\|_{N^s(T)} = 0.$$

For details, see e.g. [KT07, Section 1, p. 6].

Consequently, the above set of estimates yields

$$\|u\|_{F^s(1)} \lesssim \|u_0\|_{H^s} \tag{3.53}$$

provided that $\|u_0\|_{H^s}$ is chosen sufficiently small.
For $s' > s$ we have

$$\begin{cases}
\|u\|_{F^{s'}(T)} & \lesssim \|u\|_{E^{s'}(T)} + \|\partial_x(u^2)\|_{N^{s'}(T)} \\
\|u\partial_x u\|_{N^{s'}(T)} & \lesssim \|u\|_{F^{s'}(T)} \|u\|_{F^s(T)} \\
\|u\|^2_{E^{s'}(T)} & \lesssim \|u_0\|^2_{H^{s'}} + T\|u\|^2_{F^{s'}(T)} \|u\|_{F^s(T)}.
\end{cases}$$

Together with (3.53), this implies

$$\|u\|_{F^{s'}(1)} \lesssim \|u_0\|_{H^{s'}} \text{ for } s' > s.$$

This a priori estimate for higher regularities together with the blow-up alternative shows that $T^* \geq 1$ provided that $\|u_0\|_{H^s}$ is chosen sufficiently small.

Next, we argue that the set of estimates (3.4) yield an a priori estimate for v in L^2 in dependence of $\|u_i\|_{H^s}$ for $s > 1$: The estimates imply for $\|v\|_{F^0(T)}$

$$\begin{aligned}\|v\|^2_{F^0(T)} \lesssim \|v(0)\|^2_{L^2} &+ \|v\|^2_{F^0(T)}(\|u_1\|^2_{F^s(T)} + \|u_2\|^2_{F^s(T)}) \\ &+ T\|v\|^2_{F^0(T)}(\|u_1\|^2_{F^s(T)} + \|u_2\|^2_{F^s(T)}),\end{aligned}$$

and further, for $\|u_i\|_{H^s} \lesssim \varepsilon$ sufficiently small,

$$\|v\|_{F^0(T)} \lesssim \|v(0)\|_{L^2}.$$

Lastly, the set of estimates (3.5) yields again for $\|u_i\|_{H^s} \lesssim \varepsilon$

$$\|v\|^2_{F^s(T)} \lesssim \|v(0)\|^2_{H^s} + T\|v\|^3_{F^s(T)} + T\|v\|_{F^0(T)}\|v\|_{F^s(T)}\|u_2\|_{F^{2s}(T)}.$$

To prove existence of the data-to-solution mapping by the above display, let u_2 be the solution associated to $P_{\leq N}u_0$ and u_1 be the solution associated to u_0. Due to the difference of initial data consisting only of high frequencies, the gain from estimating $\|v\|_{F^0}$ compensates the loss from

$$\|u_2\|_{F^{2s}(T)} \lesssim \|P_{\leq N}u_0\|_{H^{2s}} \lesssim N^s\|P_{\leq N}u_0\|_{H^s}.$$

Then, the data-to-solution mapping $H^s \to C([0,T],H^s) \cap F^s(T)$ is constructed as an extension of $H^\infty \to C([0,T],H^s) \cap F^s(T)$, a priori only for sufficiently small initial data. Continuity, but no uniform continuity[3] follows likewise because the approximation depends on the distribution of the Sobolev energy along the high frequencies, i.e., $\|P_{\geq N}u_0\|_{H^s}$.

We give the details: Let u_0 be a smooth initial datum and consider $v = S^\infty_T(u_0) - S^\infty_T(P_{\leq N}u_0)$. Observe that

$$\begin{aligned}\|v\|_{F^0(T)} &\lesssim \|u_0 - P_{\leq N}u_0\|_{L^2} \\ &\lesssim \|P_{>N}u_0\|_{L^2} \lesssim N^{-s}\|P_{>N}u_0\|_{H^s}.\end{aligned}$$

Moreover, $P_{>N}u_0$ is the initial datum to v. Consequently,

$$\|P_{>N}u_0\|_{H^s} \lesssim \|v(0)\|_{H^s} \lesssim \|v\|_{F^s(T)}.$$

A variant of the proof of the a priori estimates for solutions yields the bound

$$\|u\|^2_{F^{s'}(T)} \lesssim \|u(0)\|^2_{H^{s'}} + T\|u\|^2_{F^{s'}(T)}\|u\|_{F^s_{BO}(T)} + \|u\|^2_{F^{s'}}$$

for $s' > s$.

This shows $\|u_2\|_{F^{s'}(T)} \lesssim \|u_2(0)\|_{H^{s'}}$ provided that $\|u_2\|_{F^s(T)}$ is sufficiently small. This implies

$$\|u_2\|_{F^{2s}(T)} \lesssim \|P_{\leq N}u_0\|_{H^{2s}} \lesssim N^s\|u_0\|_{H^s}.$$

[3] It is not excluded that the data-to-solution mapping is uniformly continuous as is the case for the Benjamin-Ono evolution on hyperplanes. It simply does not follow from the method of proof.

For the solution to the difference equation we derive the inequality

$$\|v\|^2_{E^s(T)} \lesssim \|v(0)\|^2_{H^s} + T\|v\|^3_{F^s(T)} + T\|v\|^2_{F^s(T)}\|u\|_{H^s}.$$

This allows us to conclude a priori estimates for $v = S_T^\infty(u_0) - S_T^\infty(P_{\leq N} u_0)$ in terms of $P_{>N} u_0$.

Next, we consider a sequence of smooth initial data $(u_n) \subseteq H^\infty$ converging to $u_0 \in H^s, \ s > 1$.
We write

$$\begin{aligned} S_T^\infty(u_n) - S_T^\infty(u_m) = & (S_T^\infty(u_n) - S_T^\infty(P_{\leq N} u_n)) - (S_T^\infty(u_m) - S_T^\infty(P_{\leq N} u_m)) \\ & + (S_T^\infty(P_{\leq N} u_n) - S_T^\infty(P_{\leq N} u_m)), \end{aligned}$$

and by the above considerations,

$$\begin{aligned} \|S_T^\infty(u_n) - S_T^\infty(u_m)\|_{C([0,T],H^s)} \lesssim & \|P_{\geq N} u_n\|_{H^s} + \|P_{\geq N} u_m\|_{H^s} \\ & + \|S_T^\infty(P_{\leq N} u_n) - S_T^\infty(P_{\leq N} u_m)\|_{C([0,T],H^s)}. \end{aligned}$$

For the third term note that

$$\begin{aligned} \|S_T^\infty(P_{\leq N} u_n) - S_T^\infty(P_{\leq N} u_m)\|_{C([0,T],H^s)} & \leq \|S_T^\infty(P_{\leq N} u_n) - S_T^\infty(P_{\leq N} u_m)\|_{C_T H^3} \\ & \leq f(\|P_{\leq N} u_n - P_{\leq N} u_m\|_{H^3}). \end{aligned}$$

We have $f(x) \to 0$ as $x \to 0$ due to continuous dependence in H^3 (cf. [ABFS89]). Since $\|P_{\leq N} u_n - P_{\leq N} u_m\|_{H^3} \lesssim N^{3-s} \|P_{\leq N}(u_n - u_m)\|_{H^s}$, we find that

$$\|S_T^\infty(P_{\leq N} u_n) - S_T^\infty(P_{\leq N} u_m)\|_{C([0,T],H^s)} \to 0 \text{ as } n, m \to \infty$$

for any N. Choosing N so that $\|P_{\geq N} u_n\|_{H^s} + \|P_{\geq N} u_m\|_{H^s} \leq \varepsilon/2$ for any n, m, which is possible due to convergence to u, we infer the existence of the data-to-solution mapping and continuous dependence on the initial data provided that the initial data are sufficiently small.

When dealing with large initial data, we rescale the initial value $u_0 \to \lambda u_0(\cdot/\lambda)$ to consider the Benjamin-Ono equation with small initial data on the rescaled torus $\lambda\mathbb{T}$.
Following Remark 3.1.6, the decisive bilinear Strichartz estimate is scaling invariant, which allows us to rerun the above proof for small initial data on the rescaled torus. But note that this argument does not adapt in a simple manner to the case of higher order nonlinearities because of criticality or supercriticality of the L^2-norm.

3.6 First applications

In this section we give simple applications of the above argument to derive new well-posedness results. The first class of equations we consider are generalized Benjamin-Ono equations on the circle:

$$\begin{cases} \partial_t u + \partial_x D_x^a u & = u^{k-1} \partial_x u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \\ \quad u(0) & = u_0 \in H^s_{\mathbb{R}}(\mathbb{T}), \end{cases} \tag{3.54}$$

where below $1 < a < 2$, $k \in \mathbb{Z}_{\geq 2}$.

As already stated in the introduction, although we have more dispersion than in the Benjamin-Ono case, these equations seem to be harder to analyze because a

gauge transform is not easily available.
However, the modular approach depicted above via short-time Strichartz estimates allows us to prove well-posedness results, also for polynomial nonlinearities.
The time localization is chosen to $T(N) = N^{-a}$ and the bilinear Strichartz estimate from Proposition 3.1.2 reads

$$\|P_N e^{t\partial_x D_x^a} u_0 P_K e^{t\partial_x D_x^a} v_0\|_{L^2_{t,x}([0,N^{-a}],\mathbb{T})} \lesssim N^{-a/2} \|u_0\|_{L^2(\mathbb{T})} \|v_0\|_{L^2(\mathbb{T})} \quad (K \ll N). \tag{3.55}$$

Then, the arguments from above yield the following theorem:

Theorem 3.6.1. *Let $1 < a < 2$, $k \in \mathbb{Z}_{\geq 2}$ and $s > 1$.*

(a) *There exists a continuous mapping $T = T(\|u_0\|_{H^s}, s)$ so that for any solution u to (3.6), we find the following estimate to hold*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^{s'}_{\mathbb{R}}} \lesssim \|u_0\|_{H^{s'}_{\mathbb{R}}} \tag{3.56}$$

provided that $s' \geq s$ and $u(0)$ is a smooth real-valued initial datum.

(b) *For solutions u_1, u_2 to (3.6) we find the following estimate to hold:*

$$\sup_{t\in[0,T]} \|u_1(t) - u_2(t)\|_{L^2} \lesssim_{\|u_i(0)\|_{H^s}} \|u_1(0) - u_2(0)\|_{L^2} \tag{3.57}$$

provided that $u_i(0)$ are smooth real-valued initial datum for $i = 1, 2$ and $T = T(\max \|u_i(0)\|_{H^s}, s)$ from (a).

(c) *(3.6) is locally well-posed in H^s, i.e., with T like in (a) the data-to-solution mapping $S_T^\infty : H^\infty \to C([0,T], H^\infty)$ admits a unique continuous extension $S_T^s : H^s \to C([0,T], H^s)$.*

Large-data-theory does no longer require rescaling, but, due to the improved short-time bilinear estimate, the corresponding estimates from Proposition 3.3.2 improve with an additional factor T^δ, $\delta > 0$ on the right hand-side.
In the limiting case $a = 1$ the above results can only be proved for small initial data.

Theorem 3.6.1 generalizes the results from [MR09] on generalized Benjamin-Ono equations ($a = 1, k \geq 2$) on the circle up to $s = 1$. In [MR09] the well-posedness result was established by means of a gauge transform recasting the derivative nonlinearity into a milder form. This does not seem to be easily feasible for $1 < a < 2$ or polynomial nonlinearities because even for $a = 1$ the gauge transform changes with the power of the nonlinearity.
Like seen above, the nonlinear estimate can be carried out for $s > 0$ for quadratic nonlinearities. The threshold $s > 1$ stems only from the energy estimate.

In Chapter 6 we shall see how to improve the energy estimate by normal form transformations for $k = 1$. This will give the currently best well-posedness results for dispersion generalized Benjamin-Ono equations on the circle for $1 < a < 2$.

A further equation amenable to the methods of this chapter is the Shrira equation

$$\begin{cases} \partial_t u + \mathcal{H}_{x_1} \Delta u &= u \partial_{x_1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^n, \\ u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{T}^n). \end{cases} \tag{3.58}$$

The above analysis yields well-posedness for $s > (n+1)/2$. In case $n = 2$, this modestly improves the recent result from [BJUM19], where well-posedness for $s > 7/4$ was shown. The results for $n \geq 3$ appear to be the first results below the energy threshold.

Another physically relevant model is the Zakharov-Kuznetsov equation

$$\begin{cases} \partial_t u + \partial_{xxx} u + 3\partial_x \partial_{yy} u &= u\partial_x u, \quad (t,x,y) \in \mathbb{R} \times \mathbb{T} \times \mathbb{T}, \\ \qquad u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{T}^2). \end{cases} \tag{3.59}$$

The linear propagator becomes $\partial_{xxx} + \partial_{yyy}$ after a rotation, which allows us to apply the above arguments and prove corresponding statements for (3.59) like in Theorem 3.6.1 for $s > 3/2$.
In the following chapter we will see how to treat higher-dimensional Benjamin-Ono equations by the above arguments in greater generality.

Chapter 4

New local well-posedness results for higher-dimensional Benjamin-Ono equations

4.1 Introduction to higher-dimensional Benjamin-Ono equations

In this chapter well-posedness of the higher-dimensional fractional Zakharov--Kuznetsov equations

$$\begin{cases} \partial_t u + \partial_{x_1}(-\Delta)^{a/2} u &= u\partial_{x_1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{K}^n \\ u(0) &= u_0 \in H^s(\mathbb{K}^n), \end{cases} \tag{4.1}$$

is discussed, where $n \geq 2$, $1 \leq a \leq 2$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{T}\}$. In the following let $D_x = (-\Delta)^{1/2}$.

In higher dimensions (4.1) yields a generalization of the Benjamin-Ono equation for $a = 1$ (cf. [LRRW19, Mar02, PS95]). For $a = 2$, (4.1) becomes the Zakharov-Kuznetsov equation (cf. [ZK74, LS82]).

The aim is to improve results obtained by the energy method [BS75], which yields well-posedness for $s > \frac{n+2}{2}$. Already in the one-dimensional case, it is well-known that the data-to-solution mapping for dispersion coefficients $1 \leq a < 2$ is not uniformly continuous (cf. [KT05b, HIKK10, MST01]).

Also, in two dimensions it was proved for $a = 1$ in [LRRW19] that the data-to-solution mapping is not C^2. In the same work local well-posedness was proved for $a = 1$ provided that $s > 5/3$ using short-time linear Strichartz estimates (cf. [KT03]).

Here, we improve the local well-posedness for $n = 2$ and relate the cases $a = 1$ and $a = 2$ to recover in the limiting case the local well-posedness result for the Zakharov-Kuznetsov equation for $s > 1/2$ (cf. [GH14, MP15]) in two dimensions and $s > 1$ in three dimensions [MP15, RV12]. Recently, the essentially sharp C^2-well-posedness result for $s > -1/4$ for the Zakharov-Kuznetsov equation in two dimensions was proved by Kinoshita [Kin19] via refined transversality and resonance considerations, crucially making use of the nonlinear Loomis-Whitney inequality. The results obtained in this chapter for dimensions $n \geq 4$ seem to be new for $1 < a \leq 2$.

Following the approach from Chapter 3, we use transversality and localization of time to frequency dependent time intervals to prove the following theorem:

Theorem 4.1.1. *Let $n \geq 2$, $\mathbb{K} = \mathbb{R}$, $1 \leq a < 2$ and $s > \frac{n+3}{2} - a$. Then (4.1) is locally well-posed.*

Remark 4.1.2. As pointed out in Chapter 3, the method of proof extends to generalized Benjamin-Ono-Zakharov-Kuznetsov equations (cf. [Gr4, LP09])

$$\partial_t u + \partial_{x_1} D_x^a u = \partial_{x_1}(u^k), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \quad k \geq 2, \tag{4.2}$$

which we do not cover explicitly.

Furthermore, the proof yields local well-posedness in the Besov space $B^s_{2,1}$ for $s = \frac{n+3}{2} - a$.
In case $a = 2$, the below arguments point out that (4.1) is a semilinear equation for sufficiently large values of s, and no frequency dependent time localization is required to prove local well-posedness. Thus, in the Euclidean case $a = 2$ will not be considered in the following.

For this chapter let $S_a(t)$ denote the linear propagator of (4.1), that is

$$\widehat{S_a(t)u_0}(\xi) = e^{-it\xi_1|\xi|^a} \hat{u}_0(\xi).$$

As already seen above, the most problematic interaction happens in case a low frequency interacts with a high frequency because the derivative nonlinearity

$$\partial_{x_1}(P_N u P_K u) \quad (K \ll N)$$

possibly requires us to recover a whole derivative. The derivative loss is partially ameliorated by the following bilinear Strichartz estimate:

Proposition 4.1.3. *Let $n \geq 2$, $K, N \in 2^{\mathbb{N}_0}$, $K \ll N$. Then, we find the following estimate to hold:*

$$\|P_N S_a(t)u_0 P_K S_a(t)v_0\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \left(\frac{K^{n-1}}{N^a}\right)^{1/2} \|P_N u_0\|_{L^2} \|P_K v_0\|_{L^2}. \tag{4.3}$$

This proposition is a consequence of Proposition 3.1.1 and will be proved in Section 4.4.
Apparently, this is still insufficient to recover the derivative loss for $1 \leq a < 2$. To overcome the gap, we additionally localize time in a frequency dependent way.

In the following we motivate for which frequency dependent time localization we can treat the most problematic $High \times Low \to High$-interaction utilizing (4.3). For $K \ll N$ one finds

$$\begin{aligned} &\|\partial_{x_1}(P_N S_a(t)u_0 P_K S_a(t)v_0)\|_{L^1([0,T];L^2(\mathbb{R}^n))} \\ &\lesssim N T^{1/2} \|P_N S_a(t)u_0 P_K S_a(t)v_0\|_{L^2([0,T];L^2_x(\mathbb{R}^n))} \\ &\lesssim T^{1/2} N \left(\frac{K^{n-1}}{N^a}\right)^{1/2} \|P_N u_0\|_{L^2(\mathbb{R}^n)} \|P_K v_0\|_{L^2_x(\mathbb{R}^n)}. \end{aligned} \tag{4.4}$$

This suggests that for $T(N) = N^{a-2}$ this peculiar interaction can be estimated for $s > (n-1)/2$, which is carried out in Section 4.6. Below, let $F_a^s(T)$ and $N_a^s(T)$ and $E^s(T)$ denote the short-time function spaces adapted to $S_a(t)$ with frequency dependent time localization $T(N) = N^{a-2}$. The precise definitions are given in Section 4.5.

This argument is sufficient to handle $High \times Low \to High$-interactions and $High \times High \to High$-interactions for $n = 2$. For $High \times High \to High$-interactions when $n \geq 3$, we prove a weaker transversality estimate in Proposition 4.4.2, but one can as well utilize linear Strichartz estimates (cf. [LRRW19]):

Proposition 4.1.4. *Let $n \geq 3$, $1 \leq a \leq 2$ and $2 \leq p, q \leq \infty$, $p \neq \infty$. Then, we find the following estimate to hold*

$$\begin{aligned} \|S_a(t)f\|_{L_t^q(\mathbb{R}, L_x^p(\mathbb{R}^n))} &\lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)}, \\ \|S_a(t)f\|_{L_t^q([0,T], L_x^p(\mathbb{R}^n))} &\lesssim_T \|f\|_{H^s(\mathbb{R}^n)} \end{aligned} \tag{4.5}$$

provided that $\frac{2}{q} + \frac{2}{p} = 1$ *and* $s = n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{a+1}{q}$.

These estimates are useful in the proof of local well-posedness results for generalized equations (4.2) as shown in Section 3.5. In Section 4.6 we prove the following proposition:

Proposition 4.1.5. *Let $\mathbb{K} = \mathbb{R}$, $1 \leq a < 2$, $n \geq 2$, $s > (n-1)/2$. Then, we find the following estimates to hold:*

$$\|\partial_x(uv)\|_{N_a^s(T)} \lesssim \|u\|_{F_a^s(T)} \|v\|_{F_a^s(T)}, \tag{4.6}$$

$$\|\partial_x(uv)\|_{N_a^0(T)} \lesssim \|u\|_{F_a^0(T)} \|v\|_{F_a^s(T)}. \tag{4.7}$$

The energy estimates give a worse regularity threshold to close the argument, namely $s > \frac{n+3}{2} - a$. The following proposition is proved in Section 4.7:

Proposition 4.1.6. *Let $\mathbb{K} = \mathbb{R}$, $n \geq 2$, $1 \leq a < 2$ and let u be a smooth solution to* (4.1)*. Then, we find the following estimate to hold*

$$\|u\|_{E^s(T)}^2 \lesssim \|u_0\|_{H^s}^2 + T\|u\|_{F^s(T)}^3 \tag{4.8}$$

provided that $s > s_a := \frac{n+3}{2} - a$.

Corresponding estimates show L^2-Lipschitz dependence for initial values in H^s, $s > s_a$ and continuous dependence like in Chapter 3.
Another Benjamin-Ono-Zakharov-Kuznetsov equation was considered in [RV17]:

$$\begin{cases} \partial_t u - \partial_{x_1} D_{x_1}^a u + \partial_{x_1} \partial_{x_2}^2 u &= u \partial_{x_1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \quad 1 \leq a \leq 2, \\ u(0) &= u_0 \in H^s(\mathbb{R}^n). \end{cases} \tag{4.9}$$

Here, $\widehat{(D_{x_1} f)}(\xi) = |\xi_1| \hat{f}(\xi)$, so that only dispersion in the x_1-component is decreased compared to the Zakharov-Kuznetsov equation. Local and global well-posedness results for (4.9) were also proved via frequency dependent time localization.

Lastly, we remark that the local well-posedness result from Theorem 4.1.1 gives global well-posedness in the energy space $H^{a/2}(\mathbb{R}^2)$ for sufficiently large a in the two-dimensional case due to conservation of energy

$$E(u) = \int_{\mathbb{R}^n} |D_x^{a/2} u|^2 - \frac{1}{3} u^3(t,x) dx.$$

Another conserved quantity is the mass

$$M(u) = \int_{\mathbb{R}^n} u^2(t,x)dx,$$

but a well-posedness result in L^2 seems to be far beyond the methods of this chapter. Iteration of Theorem 4.1.1 for $s = a/2$ yields:

Corollary 4.1.7. *Let $n = 2$, $\mathbb{K} = \mathbb{R}$ and $a > 5/3$. Then, (4.1) is globally well-posed for $s = a/2$.*

We turn to a discussion of the fully periodic case, in which case the following theorem is shown:

Theorem 4.1.8. *Let $\mathbb{K} = \mathbb{T}$, $n \geq 2$, $1 \leq a \leq 2$ and $s > (n+1)/2$. Then, (4.1) is locally well-posed in $H^s(\mathbb{T}^n)$ for sufficiently small initial data.*

In this chapter, we show well-posedness for small periodic initial data on time intervals with fixed size. In Chapter 5 detailed arguments (rescaling the manifold, modulation considerations) are provided to handle large data on small time intervals.

In case $n = 2$ this improves the result from [LPRT19], where local well-posedness was proved in $H^s(\mathbb{T}^2)$ provided that $s > 5/3$ for $a = 2$.
In these works short-time linear Strichartz estimates were used. In this chapter this result is modestly improved by transversality considerations, and corresponding results are proven in higher dimensions.
The results in higher dimensions appear to be the first ones below the energy threshold. However, the covered regularities are still far from the energy space.

To make further progress, one presumably needs a better comprehension of the resonance set, which appears to be more delicate than for the Kadomtsev Petviashvili-equations (cf. [Bou93c, IKT08, Zha16]). The recent work by Kinoshita [Kin19] has the potential to improve the understanding also in the fully periodic case.

Key ingredient in the proof below are bilinear convolution estimates for the space-time Fourier transform of functions, which are localized in frequency and modulation. These are derived in Subsection 4.8.2. Here, the transversality considerations from Euclidean space again comes into play. However, we always have to localize time reciprocally to the highest involved frequency so that transversality becomes observable. Therefore, we can not lower the regularity, for which our method of proof yields local well-posedness, as the dispersion coefficients increase compared to the Euclidean case.

After the derivation of these bilinear convolution estimates, the argument follows Chapter 3.

4.2 Proof of new well-posedness results in Euclidean space

With the short-time nonlinear and energy estimates at hand, we find for smooth solutions u to (1.4) with $1 \leq a < 2$ the following set of estimates to hold, where

$s' \geq s > s_a$:

$$\begin{cases} \|u\|_{F_a^{s'}(T)} & \lesssim \|u\|_{E^{s'}(T)} + \|u\partial_{x_1} u\|_{N_a^{s'}(T)} \\ \|u\partial_{x_1} u\|_{N_a^{s'}(T)} & \lesssim \|u\|_{F_a^s(T)} \|u\|_{F_a^{s'}(T)} \\ \|u\|^2_{E^{s'}(T)} & \lesssim \|u_0\|^2_{H^{s'}} + T\|u\|^2_{F_a^{s'}(T)} \|u\|_{F_a^s(T)}. \end{cases}$$

By the argument from Section 3.5, this yields a priori estimates for solutions in H^s and persistence of regularity. The short-time adaptation of the Bona-Smith argument, which was described in detail in Section 3.5, yields Lipschitz continuity in L^2 for more regular initial data and continuous dependence. Details are omitted to avoid repetition.

4.3 Linear Strichartz estimates

In this section linear Strichartz estimates on Euclidean space are discussed. We start with a dispersive estimate, which was proved for $a = 1$ in [LRRW19, Proposition 3.1.]. The modifications for $a \geq 1$ are straight-forward; we give the details for the sake of completeness.

Proposition 4.3.1. *Let $a \geq 1$, $n \geq 3$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, radial function supported in $B_n(0,2) \backslash B_n(0,1/2)$. Then, we find the following estimate to hold:*

$$\left| \int \psi(|\xi|) e^{i(t\xi_1 |\xi|^a + x.\xi)} d\xi \right| \leq C|t|^{-1} \tag{4.10}$$

with C only depending on n, ψ and a.

Proof. We rewrite the integral in spherical coordinates to find

$$\begin{aligned} I(x,t) &= \int_0^\infty dr \underbrace{r^{n-1}\psi(r)}_{\rho(r)} \int_{\mathbb{S}^{n-1}} d\sigma(\omega) e^{it(r^{a+1}\omega_1 + x_1 r\omega_1 + \ldots + x_n r\omega)} \\ &= \int_0^\infty \rho(r) \hat{\sigma}(y_{t,x}(r)) dr, \end{aligned}$$

where $y_{t,x}(r) = (tr^{a+1} + x_1 r, x_2 r, \ldots, x_n r)$.

Recall the decay (cf. [Sog17, Lemma 8.2.4, p. 262])

$$|\hat{\sigma}(y)| \lesssim (1 + |y|)^{-\frac{n-1}{2}}.$$

This is already enough to prove the claim for $n \geq 4$.

Indeed, partition $supp(\rho) = E_1 \cup E_2$, where $E_1 = \{r \in supp(\rho) | |tr^{a+1} + x_1 r| \leq 1\}$ and $|E_1| \lesssim |t|^{-1}$. To see this, note that $|tr^{a+1} + x_1 r| \leq 1$ implies $|tr^a + x_1| \leq 2$ and, by change of variables,

$$\int_{1/2}^2 1_{\{|tr^a + x_1| \leq 2\}}(r) \rho(r) dr = \int_{r' \sim 1} 1_{\{|tr' + x_1| \leq 2\}} \rho(r') dr' \leq C|t|^{-1},$$

where C depends on ψ, n and a.

Similarly, $E_2 \subseteq \{r \in supp(\rho) | |tr^a + x_1| \geq 2\}$ and consequently,

$$\begin{aligned}\int_{E_2} \rho(r)|\hat{\sigma}(y_{x,t}(r))|dr &\leq \int_{|tr^a+x_1|\geq 2} \rho(r)|tr^{a+1} + x_1 r|^{-\frac{n-1}{2}} dr \\ &\leq C \int_{|tr+x_1|\geq 2} |tr + x_1|^{-\frac{n-1}{2}} dr \\ &= C|t|^{-\frac{n-1}{2}} \int_{|r+x_1/t|\geq 2/|t|} |r + x_1/t|^{-\frac{n-1}{2}} dr.\end{aligned}$$

After a linear change of variables, we estimate the expression by $C|t|^{-1}$.

We turn to $n = 3$. Here, we make use of the asymptotic expansion

$$\hat{\sigma}(y) = c\frac{e^{i|y|}}{|y|} + c\frac{e^{-i|y|}}{|y|} + \mathcal{E}_{t,x}(y),$$

where $|\mathcal{E}_{t,x}(y)| \lesssim |y|^{-2} \quad (|y| \gg 1)$, cp. [Gra09, Example 10.4.3].

Set $\phi(r) = \sqrt{f(r)}$, where $f(r) = (tr^{a+1} + x_1 r)^2 + r^2|x'|^2$ and

$$\begin{aligned}F^1 &= \{r \in supp(\rho) | |tr^{a+1} + x_1 r| \leq 1\} \cap \{r \in supp(\rho) | |f'(r)| \leq |t|\} \supseteq E^1, \\ F^2 &= \{r \in supp(\rho) | |tr^{a+1} + x_1 r| \geq 1, \quad |f'(r)| \geq |t|\} \subseteq E^2.\end{aligned}$$

Below, we see that $|F^1| \lesssim |t|^{-1}$, which means that this contribution is controlled by $|\hat{\sigma}| \lesssim 1$.

Moreover, the contribution of $\mathcal{E}_{t,x}$ when integrating over F^2 is controlled by the higher dimensional argument due to $F^2 \subseteq E^2$ and sufficient decay to run the above argument.

A computation yields

$$\begin{aligned}f'(r) &= 2t^2(a+1)r(r^a - r_-)(r^a - r_+), \\ r_\pm &= -\frac{(a+2)x_1}{2(a+1)t} \pm \sqrt{\left(\frac{a+2}{a+1}\right)^2 \left(\frac{x_1}{t}\right)^2 - \frac{x_1^2}{(a+1)t^2} - \frac{|x'|^2}{(a+1)t^2}}.\end{aligned}$$

We can suppose that $\frac{x_1}{t} \sim 1$ and $\frac{|x'|^2}{t^2} \ll 1$ since otherwise $|f(r)| \gtrsim |t|$. Consequently, the roots are real and separated.

In fact, $|r_\pm| \sim 1$ and $|r_+ - r_-| \sim 1$. Moreover, whenever f' vanishes, then $|f''|$ is still bounded away from zero and thus, $|F^1| \lesssim |t|^{-1}$.

For the estimate of the contribution of $e^{i|y|}/|y|$ over F^2 note that we can write

$$\int \frac{e^{i\phi(r)}}{\phi(r)} \rho(r) dr \sim \int \frac{d}{dr}[e^{i\phi(r)}] \frac{\rho(r)}{f'(r)} dr.$$

Next, the domain of integration is divided up into a finite union of intervals, where ρ/f' is monotone. On each interval, integration by parts yields the desired result. □

Remark 4.3.2. The dispersive estimate follows also from [KT05c, Proposition 4.7].

Proof of Proposition 4.1.4. For $n \geq 3$ the dispersive estimate and conservation of mass give by interpolation

$$\|S_a(t)P_1 f\|_{L^p(\mathbb{R}^n)} \lesssim |t|^{-1+2/p} \|\tilde{P}_1 f\|_{L^{p'}(\mathbb{R}^n)}$$

provided that $2 \leq p \leq \infty$. Combination with the TT^*-argument (cf. [Tom75, GV79, KT98]) proves Strichartz estimates

$$\|S_a(t)P_1 f\|_{L_t^q(\mathbb{R},L_x^p(\mathbb{R}^n))} \lesssim \|\tilde{P}_1 f\|_{L^2(\mathbb{R}^n)}$$

provided that $p, q \geq 2$, $\frac{2}{q} + \frac{2}{p} = 1$, $p \neq \infty$.

A scaling argument gives for p, q like above

$$\|S_a(t)P_N f\|_{L_t^q(\mathbb{R},L_x^p(\mathbb{R}^n))} \lesssim N^s \|\tilde{P}_N f\|_{L^2(\mathbb{R}^n)}, \quad s = n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{a+1}{q},$$

and (4.5) follows from Littlewood-Paley theory. □

4.4 Bilinear Strichartz estimates

Purpose of this section is to prove bilinear Strichartz estimates as stated in Proposition 4.1.3.

Whereas the proof of bilinear estimates is straight-forward in case of separated frequencies, it requires more care to treat the *High* × *High* × *High*-interaction

$$\int\int_{\mathbb{R}^2 \times [0,T]} P_{N_1} S_a(t) u_0 P_{N_2} S_a(t) v_0 P_{N_3} S_a(t) w_0 dx dy dt, \quad N_1 \sim N_2 \sim N_3. \quad (4.11)$$

We shall see that it is still amenable to a bilinear Strichartz estimate.

Both cases follow from Proposition 3.1.1. To apply Proposition 3.1.1, we have to analyze the group velocity $v_a(\xi) = -\nabla \varphi_a(\xi)$, where $\varphi_a(\xi) = \xi_1 |\xi|^a$.

We have

$$\partial_1 \varphi_a(\xi) = |\xi|^a + a\xi_1^2 |\xi|^{a-2}, \quad \partial_2 \varphi_a(\xi) = a\xi_1 \xi_2 |\xi|^{a-2}. \quad (4.12)$$

Proof of Proposition 4.1.3. First, divide $B_{2N} \backslash B_{N/2}$ into finitely overlapping balls of radius K. We denote an associated family of smooth frequency projectors by (R_L). Then, by almost orthogonality

$$\|P_N S_a(t) u_0 P_K S_a(t) v_0\|_{L_{t,x}^2}^2 \lesssim \sum_L \|R_L S_a(t) u_0 P_K S_a(t) v_0\|_{L_{t,x}^2}^2. \quad (4.13)$$

To estimate the terms in the sum, we use Proposition 3.1.1. From (4.12) we find $|\partial_1 \varphi_a(\xi)| \geq (N/2)^a$ for $|\xi| \geq N/2$ and $|\partial_1 \varphi_a(\xi)| \leq (1+a)(2K)^a$ for $|\xi| \leq 2K$ and (3.11) implies

$$\text{rhs}(4.13) \lesssim \sum_L \left(\frac{K^{n-1}}{N^a}\right) \|R_L u_0\|_{L^2}^2 \|P_K v_0\|_{L^2}^2 \lesssim \left(\frac{K^{n-1}}{N^a}\right) \|P_N u_0\|_{L^2}^2 \|P_K v_0\|_{L^2}^2,$$

which completes the proof. □

Next, we turn to the case of three comparable frequencies in the plane as depicted in (4.11). We prove the following proposition:

Proposition 4.4.1. *Let $N \gg 1$ and suppose that $\xi_i \in \mathbb{R}^2, N/16 \leq |\xi_i| \leq 16N$ for $i = 1, 2, 3$ and $\xi_1 + \xi_2 + \xi_3 = 0$. Then, there are $i, j \in \{1, 2, 3\}$ with*

$$|v_a(\xi_i) - v_a(\xi_j)| \gtrsim N^a.$$

Proof. A key observation is that for $\lambda \in \mathbb{R}^2$ with $|\lambda_2| \le c|\lambda|$ or $|\lambda_1| \le c|\lambda|$, where c is a small constant, a Taylor expansion of $|\lambda|$ around the large component reveals

$$\begin{aligned}
\partial_1 \varphi_a(\lambda) &= (1+a)|\lambda_1|^a + O(\lambda_2^2 |\lambda_1|^{a-2}) \\
&= (1+a)|\lambda_1|^a + O(c^2 |\lambda_1|^a) \quad (|\lambda_2| \le c|\lambda|), \\
\partial_1 \varphi_a(\lambda) &= |\lambda_2|^a + O(c^2|\lambda_2|^a) \quad (|\lambda_1| \le c|\lambda|).
\end{aligned}$$

This means that as soon as one component dominates the other one, the propagation into x_1-direction is essentially governed by the group velocity associated to a (fractional) one-dimensional Benjamin-Ono equation, which has been considered in Section 3.6.

To deal with different sizes of the components for $\lambda \in \mathbb{R}^2$, we introduce the notation $\lambda \in (A, B)$, where $A, B \in \{Low, Medium, High\}$.
Low-components λ_i satisfy $|\lambda_i| \le c^3|\lambda|$, $Medium$-components satisfy $|\lambda_i| \in [c^3|\lambda|, c|\lambda|/2]$ and $High$-components $|\lambda_i| \ge c|\lambda|/2$.
Further, write $\lambda \in (X, Y)$, where $X, Y \in \{+, -\}$ to indicate $\lambda_1 \ge 0, \lambda_2 \le 0$.
E.g. $\lambda \in (High(+), Medium(-))$ means $|\lambda_1| \ge \frac{c|\lambda|}{2}$, $|\lambda_2| \in [c^3|\lambda|, \frac{c|\lambda|}{2}]$, $\lambda_1 \ge 0, \lambda_2 \le 0$ or $\lambda \in (Low, High(-))$ means $|\lambda_1| \le c^3|\lambda|$, $|\lambda_2| \ge \frac{c|\lambda|}{2}$, $\lambda_2 \le 0$.
Here, c is a small constant chosen so that the error terms in the above Taylor expansion can be neglected in the following considerations.

We sort the frequencies according to the above system.
Suppose that the components of any frequency are all at least of medium size so that no component of the three frequencies is low.
Then, by (4.12) $|\partial_2 \varphi_a(\xi_i)| \ge c^5 |\xi_i|^a$ for $i = 1, 2, 3$.
Next, observe that for $\xi_i \in (+,+)$ or $\xi_i \in (-,-)$ we have $\partial_2 \varphi_a(\xi_i) \ge c^5|\xi_i|^a$, and in case of mixed signs $\xi_i \in (+,-)$ or $\xi_i \in (-,+)$ we have $\partial_2 \varphi_a(\xi_i) \le -c^5|\xi_i|^a$ and the estimate $|\partial_2\varphi_a(\xi_i) - \partial_2 \varphi_a(\xi_j)| \gtrsim N^a$ is immediate, whenever the components of ξ_i have mixed signs and the components of ξ_j have equal signs.
Next, we turn to the case where all components ξ_{ij}, $j = 1, 2$ and $i = 1, 2, 3$ have size greater than $c^3|\xi_i|$, and all frequencies are of equal signs, i.e., $sgn(\xi_{i1}) = sgn(\xi_{i2})$, $i = 1, 2, 3$; the case of mixed signs follows mutatis mutandis.
Say

$$\begin{aligned}
&\xi_1 \in (High(+), Medium(+)), \quad \xi_2 \in (High(+), High(+)), \\
&\xi_3 \in (High(-), High(-)).
\end{aligned}$$

Write $\xi_{21} = \alpha \xi_{11}$, $\xi_{22} = \beta \xi_{12}$, where $\alpha, \beta \in [c^5, c^{-5}]$, and it follows

$$\begin{aligned}
&|\partial_2 \varphi_a(\xi_1) - \partial_2 \varphi_a(\xi_3)| \\
&= \left| \frac{a \xi_{11} \xi_{12}}{(\xi_{11}^2 + \xi_{12}^2)^{\frac{2-a}{2}}} - \frac{a(1+\alpha)\xi_{11}(1+\beta)\xi_{12}}{((1+\alpha)^2 \xi_{11}^2 + (1+\beta)^2 \xi_{12}^2)^{\frac{2-a}{2}}} \right| \\
&\ge c^5 a \frac{|\xi_{11}\xi_{12}|}{(\xi_{11}^2 + \xi_{12}^2)^{\frac{2-a}{2}}} \gtrsim N^a.
\end{aligned} \tag{4.14}$$

Next, we suppose that there is one low component involved, say $\xi_1 \in (Low, High)$ or $(High, Low)$. Suppose that there is a frequency $\xi_j \in (High, High)$. Then, we find $|\partial_2 \varphi_a(\xi_1)| = O(c^3|\xi|^a)$ and $|\partial_2 \varphi_a(\xi_j)| \gtrsim c^2 |\xi|^a$. Hence, $|\partial_2\varphi_a(\xi_1) - \partial_2\varphi_a(\xi_j)| \gtrsim c^2|\xi|^a$, which yields the desired transversality.

In case $\xi_1 \in (Low, High)$ we have $|\xi_{12}| \sim |\xi|$, and there is another frequency, say ξ_2 with $|\xi_{22}| \sim |\xi|$.
Further, by the above considerations, suppose next that $\xi_2 \in (Low, High)$ or $\xi_2 \in (Medium, High)$.
Either way, $|\xi_{31}| \leq |\xi_{11}|+|\xi_{12}| \leq c|\xi_{11}|$ and we can expand $\partial_1 \varphi(\xi_i)$ in the second component of the frequencies to find that the analysis reduces to the one-dimensional fractional Benjamin-Ono equation and hence, there are ξ_i and ξ_j by Remark 3.1.6 with

$$|\partial_1 \varphi_a(\xi_i) - \partial_1 \varphi_a(\xi_j)| \gtrsim N^a.$$

The same argument applies in case $\xi_1 \in (High, Low)$. In case there is $\xi_j \in (High, High)$ the difference satisfies $|\partial_2 \varphi_a(\xi_1) - \partial_2 \varphi_a(\xi_j)| \gtrsim c^2 |\xi|^a$ and in case there is no $\xi_j \in (High, High)$ we can expand in the first frequency component to reduce the analysis to the one-dimensional fractional Benjamin-Ono equation according to which there are ξ_i, ξ_j such that $|\partial_1 \varphi_a(\xi_i) - \partial_1 \varphi_a(\xi_j)| \gtrsim |\xi|^a$.
The proof is complete. □

These transversality considerations for comparable frequencies do not appear to remain true in higher dimensions. Instead, we revisit the proof of Proposition 4.4.1 to prove the following weaker result in higher dimensions, which is still sufficient for our purposes:

Proposition 4.4.2. *Let $1 \leq a \leq 2$, $n \geq 3$ and $\xi_i \in \mathbb{R}^n$, $i = 1,2,3$ with $\xi_1 + \xi_2 + \xi_3 = 0$ and $|\xi_i| \sim 2^k$. Further, suppose that $|\xi_{i1}| \sim 2^{l_i}$ and set $l = \max_{i=1,2,3} l_i$.*
Then, there are ξ_i and ξ_j such that

$$|v_a(\xi_i) - v_a(\xi_j)| \gtrsim 2^l 2^{k(a-1)}.$$

Proof. First, we deal with the case $n = 3$. To lighten the notation further, we use the less precise notation $\sim$, $\lesssim$ compared to the more carefully defined regions above. The below argument can be made precise borrowing the notation from the proof of Proposition 4.4.1.

By symmetry and convolution constraint, we can suppose that $|\xi_{11}| \sim 2^l$, $|\xi_{21}| \sim 2^l$. If $|\xi_{31}| \ll 2^l$, then there is another component of ξ_3 having size 2^k, by symmetry say $|\xi_{32}| \sim 2^k$.

By the convolution constraint, there is $i \in \{1; 2\}$ such that $|\xi_{i2}| \sim 2^k$. Then, we find

$$|\partial_2 \varphi_a(\xi_i) - \partial_2 \varphi_a(\xi_3)| \gtrsim 2^l 2^{k(a-1)}.$$

Thus, we suppose in the following that $|\xi_{11}| \sim |\xi_{21}| \sim |\xi_{31}| \sim 2^l$.

If there is no component among ξ_{ji}, $j = 1,2,3$, $i = 2, \ldots, n$, which is comparable to 2^k, then the analysis reduces to the one-dimensional fractional Benjamin-Ono equation after expansion of $\partial_1 \varphi_a$.
Thus, we suppose in the following that there is a component say $|\xi_{12}| \sim 2^k$. By convolution constraint, we can suppose further that $|\xi_{22}| \sim 2^k$.
If $|\xi_{32}| \ll 2^k$, then it follows $|\partial_2 \varphi_a(\xi_1) - \partial_2 \varphi_a(\xi_3)| \gtrsim 2^l 2^{(a-1)k}$.
Thus, we suppose in the following that $|\xi_{12}| \sim |\xi_{22}| \sim |\xi_{32}| \sim 2^k$.

Next, we take the third component into account: If $|\xi_{i3}| \ll 2^k$ for $i = 1,2,3$, then the third component can be neglected, and the claim follows from the two-dimensional argument because the third component does not significantly contribute to $\partial_i \varphi_a$, $i = 1,2$.

If $|\xi_{13}| \sim 2^k$, there exists a third component of another frequency of comparable size by convolution constraint, say $|\xi_{23}| \sim 2^k$. If $|\xi_{33}| \ll 2^k$, then we find

$$|\partial_3 \varphi_a(\xi_1) - \partial_3 \varphi_a(\xi_3)| \gtrsim 2^l 2^{(a-1)k}.$$

Thus, we can suppose that $|\xi_{i3}| \sim 2^k$ for $i = 1, 2, 3$.
In the next step, we take the signs into account. If the product of the signs of the first and second or first and third component differs, then the claim follows from the observation that the second, respectively, third component of v_a are of opposite signs.

Thus, we suppose that

$$\begin{aligned} &sgn(\xi_{i1}\xi_{i2}) = sgn(\xi_{j1}\xi_{j2}), \quad i,j \in \{1,2,3\}, \\ &sgn(\xi_{i1}\xi_{i3}) = sgn(\xi_{j1}\xi_{j3}). \end{aligned} \tag{4.15}$$

There are two frequencies, for which the first component has the same sign, say ξ_1 and ξ_2, and one frequency, for which the first component has a different sign, that is ξ_3 in the current setting.
By (4.15) the signs of the other components must also be equal for ξ_1 and ξ_2 and different for ξ_3. Write $\xi_{21} = \alpha\xi_{11}$, $\xi_{22} = \beta\xi_{12}$, $\xi_{23} = \gamma\xi_{13}$, where $\alpha \sim \beta \sim \gamma \sim 1$.

Suppose that $\beta \geq \gamma$. Then, we compute along the lines of (4.14)

$$\begin{aligned} &\frac{(1+\alpha)(1+\beta)}{((1+\alpha)^2\xi_{11}^2 + (1+\beta)^2\xi_{12}^2 + (1+\gamma)^2\xi_{13}^2)^{\frac{2-a}{2}}} \\ \geq &\frac{(1+\alpha)(1+\beta)}{((1+\alpha)^2\xi_{11}^2 + (1+\beta)^2\xi_{12}^2 + (1+\beta)^2\xi_{13}^2)^{\frac{2-a}{2}}} \\ \geq &\frac{1+(\alpha\wedge\beta)}{(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2)^{\frac{2-a}{2}}}, \end{aligned}$$

and consequently,

$$\begin{aligned} &|\partial_2\varphi_a(\xi_1) - \partial_2\varphi_a(\xi_3)| \\ &= |a\xi_{11}\xi_{12}| \left| \frac{(1+\alpha)(1+\beta)}{((1+\alpha)^2\xi_{11}^2 + (1+\beta)^2\xi_{12}^2 + (1+\gamma^2)\xi_{13^2})^{\frac{2-a}{2}}} - \frac{1}{(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2)^{\frac{2-a}{2}}} \right| \\ &\geq |a\xi_{11}\xi_{12}| \frac{\alpha\wedge\beta}{(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2)^{\frac{2-a}{2}}} \gtrsim 2^l 2^{(a-1)k}. \end{aligned}$$

If $\gamma \geq \beta$, then the same computation reveals

$$|\partial_3\varphi_a(\xi_1) - \partial_3\varphi_a(\xi_3)| \gtrsim 2^l 2^{(a-1)k}.$$

This proves the claim for $n = 3$. The above arguments extend to higher dimensions inductively. □

4.5 Function spaces

The U^p-/V^p-spaces are adapted to free solutions in the usual way:

$$\begin{aligned} \|u\|_{U_a^p(I;L^2)} &= \|S_a(-t)u(t)\|_{U^p(I;L^2)}, \\ \|v\|_{V_a^p(I;L^2)} &= \|S_a(-t)v(t)\|_{V^p(I;L^2)}, \\ \|w\|_{DU_a^2(I;L^2)} &= \|S_a(-t)w(t)\|_{DU_a^2(I;L^2)}. \end{aligned}$$

Motivated by (4.4), we choose $T(N) = N^{a-2}$ as frequency dependent time localization.
Below, we shall only deal with the case $1 \leq a < 2$ since for $a = 2$ the localization to small frequency dependent time intervals is no longer necessary, and the analysis comes down to the Fourier restriction analysis without localization in time from [GH14].

Let χ_I denote a sharp cut-off to a time interval I. The short-time U^2-space, into which the solution to (4.1) will be placed, is given by

$$\|u\|^2_{F^s_a(T)} = \|P_{\leq 0} u\|^2_{U^2_a([0,T],L^2)} + \sum_{N \geq 1} N^{2s} \sup_{\substack{|I| = N^{a-2} \wedge T, \\ I \subseteq [0,T]}} \|P_N \chi_I u\|^2_{U^2_a(I;L^2)}.$$

The corresponding spaces for the nonlinearity and the energy space are defined like in Section 2.4.

As in Section 3.1, the estimates for the free solutions from Sections 4.3 and 4.4, the transfer principle and the interpolation argument from Lemma 2.3.4 imply the following linear estimates:

Proposition 4.5.1. *Let $n \geq 3$, $1 \leq a \leq 2$, $N \in 2^{\mathbb{N}_0}$ and I be an interval. Suppose that $2/q + 2/p = 1$, $2 \leq q, p < \infty$. Then, we find the following estimate to hold:*

$$\|P_N u(t)\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} \lesssim N^s \|P_N u_0\|_{U^q_a(I;L^2)}, \tag{4.16}$$

where $s = n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{a+1}{q}$.

For bilinear estimates we record the following:

Proposition 4.5.2. *Let $1 \leq a \leq 2$, $N_1 \gg N_2$ and I be an interval with $|I| = N_1^{a-2}$. Then, we find the following estimates to hold:*

$$\|P_{N_1} u_1 P_{N_2} u_2\|_{L^2_{t,x}(I \times \mathbb{R}^n)} \lesssim \left(\frac{N_2^{n-1}}{N_1^a}\right)^{1/2} \|P_{N_1} u_1\|_{U^2_a(I)} \|P_{N_2} u_2\|_{U^2_a(I)}, \tag{4.17}$$

$$\|P_{N_1} u_1 P_{N_2} u_2\|_{L^2_{t,x}(I \times \mathbb{R}^n)} \lesssim \left(\frac{N_2^{n-1}}{N_1^a}\right)^{1/2} \log^2 \langle N_1 \rangle \|P_{N_1} u_1\|_{V^2_a(I)} \|P_{N_2} u_2\|_{V^2_a(I)}. \tag{4.18}$$

4.6 Short-time nonlinear estimates

This section is devoted to the propagation of the nonlinearity in the short-time function spaces.
The argument is close to Section 3.3 with the difference that we have not adapted the time localization to Euclidean windows because we are not considering compact manifolds. The time localization is chosen such that an application of Hölder's inequality in time together with a bilinear estimate ameliorates the derivative loss.

Proof of Proposition 4.1.5. After using Littlewood-Paley theory, we are reduced to the analysis of $High \times Low \to High$-, $High \times High \to High$- and $High \times High \to Low$-interaction. Carrying out the summation in the short-time function spaces gives (4.6) and (4.7).

Suppose that $N_3 \sim N_1 \gg N_2$ and let I be an interval with $|I| \lesssim N_3^{a-2}$. Then, we compute

$$\begin{aligned}\|P_{N_3}\partial_{x_1}(P_{N_1}uP_{N_2}v)\|_{DU_a^2(I)} &\lesssim N_1\|P_{N_1}uP_{N_2}v\|_{L_I^1L_x^2}\\ &\lesssim N_1N_1^{\frac{a-2}{2}}\|P_{N_1}uP_{N_2}v\|_{L_I^2L_x^2}\\ &\lesssim N_2^{(n-1)/2}\|P_Nu\|_{U_a^2(I)}\|P_Kv\|_{U_a^2(I)}.\end{aligned}$$

Suppose that $N_1 \sim N_2 \sim N_3$ and $n = 2$. Using duality, we find

$$\|P_{N_1}\partial_{x_1}(P_{N_2}uP_{N_3}v)\|_{DU_a^2(I)} = \sup_{\|w\|_{V_a^2}=1}\int\int P_{N_1}w\partial_{x_1}(P_{N_2}uP_{N_3}v)dxdt. \quad (4.19)$$

Now, we use Proposition 4.4.1 to apply a bilinear Strichartz estimate on two factors, say w and u, to find

$$\begin{aligned}(4.19) &\lesssim N_1\sup_{\|w\|_{V_a^2}=1}\|P_{N_1}wP_{N_2}u\|_{L_{t,x}^2}\|P_{N_3}v\|_{L_{t,x}^2}\\ &\lesssim N_1N_2^{\frac{1-a}{2}}\log^2\langle N_2\rangle\|P_{N_2}u\|_{V_a^2(I)}N_3^{a/2-1}\|P_{N_3}v\|_{U_a^2(I)},\end{aligned}$$

which is sufficient.

For $n \geq 3$ we use two $L_{t,x}^4$-Strichartz estimates instead:

$$\begin{aligned}\|P_{N_3}\partial_{x_1}(P_{N_1}uP_{N_2}v)\|_{DU_a^2(I)} &\lesssim N_3\|P_{N_1}uP_{N_2}v\|_{L_I^1L_x^2}\\ &\lesssim N_3N_3^{\frac{a-2}{2}}N_3^{\frac{n-(a+1)}{2}}\|P_{N_1}u\|_{U_a^2(I)}\|P_{N_2}v\|_{U_a^2(I)}\\ &\lesssim N_3^{\frac{n-1}{2}}\|P_{N_1}u\|_{U_a^2(I)}\|P_{N_2}v\|_{U_a^2(I)},\end{aligned}$$

which is again sufficient.

At last, suppose that $N_3 \ll N_1 \sim N_2$. Here, we have to add localization in time which amounts to a factor $(N_1/N_3)^{2-a}$. More concretely, we have to decompose $I = \cup_i J_i$ with $|J_i| \lesssim N_1^{2-a}$. We use duality to write

$$\begin{aligned}\|P_{N_3}\partial_{x_1}(P_{N_1}uP_{N_2}v)\|_{DU_a^2(I)} &\lesssim N_3\sum_i\sup_w\int_{J_i}\int P_{N_3}wP_{N_1}uP_{N_2}vdxdt\\ &\lesssim N_3\sum_i\sup_w\|P_{N_3}wP_{N_1}u\|_{L_{t,x}^2(J_i\times\mathbb{R}^n)}\\ &\quad \|P_{N_2}v\|_{L_{t,x}^2(J_i\times\mathbb{R}^n)}\\ &\lesssim N_3\left(\frac{N_1}{N_3}\right)^{2-a}N_1^{\frac{a-2}{2}}\left(\frac{N_3^{n-1}}{N_1^a}\right)^{1/2}\\ &\quad \log^2\langle N_1\rangle\|P_{N_1}u\|_{F_a^0}\|P_{N_2}v\|_{F_a^0}\\ &\lesssim (N_1/N_3)^{1-a}N_3^{\frac{n-1}{2}}\log^2\langle N_1\rangle\|P_{N_1}u\|_{F_a^0}\|P_{N_2}v\|_{F_a^0},\end{aligned}$$

and carrying out the summation is straight-forward for $s > (n-1)/2$. □

4.7 Energy estimates

First, we turn to the proof of Proposition 4.1.6. Recall $s_a = \frac{n+3}{2} - a$.

Proof of Proposition 4.1.6. The fundamental theorem of calculus yields

$$\|P_N u(t)\|_{L^2}^2 = \|P_N u_0\|_{L^2}^2 + 2\int_0^t ds \int_{\mathbb{R}^n} dx P_N u P_N \partial_{x_1}(u^2).$$

The time integral we treat with Littlewood-Paley decompositions and analyze the possible interactions separately.

Suppose that $N_1 \sim N_3 \gg N_2$. Then, integration by parts and a commutator estimate (cf. Section 3.5) yield after localization in time to intervals of size N_1^{a-2}

$$\begin{aligned}\left|\int_0^t \int_{\mathbb{R}^n} P_{N_1} u \partial_{x_1}(P_{N_2} u P_{N_3} u) dx dt\right| &\lesssim N_2 T N_1^{2-a} \|P_{N_1} u P_{N_2} u\|_{L^2_{t,x}} \|P_{N_3} u\|_{L^2_{t,x}} \\ &\lesssim T N_2 N_1^{2-a} \left(\frac{N_2^{n-1}}{N_1^a}\right)^{1/2} N_1^{\frac{a-2}{2}} \prod_i \|P_{N_i} u\|_{F_{n_i}} \\ &\lesssim T N_2^{s_a} \left(\frac{N_2}{N_1}\right)^{a-1} \prod_i \|P_{N_i} u\|_{F_{n_i}}.\end{aligned}$$

In case $N_1 \ll N_2 \sim N_3$, there is no point to integrate by parts, and the estimate follows like above from the bilinear Strichartz estimate. In case $N_1 \sim N_2 \sim N_3$, one argues like in the proof of the nonlinear estimate, where $High \times High \to High$-interaction is considered. Again, integration by parts is not required. □

Next, we proof the energy estimates, which will yield Lipschitz continuity in L^2 for initial data in H^s, $s > s_a$, and continuity of the data-to-solution mapping after invoking the Bona-Smith approximation.

Proposition 4.7.1. *Let $\mathbb{K} = \mathbb{R}$, $n \geq 2$, $1 \leq a < 2$ and u_1, u_2 be two smooth solutions to* (4.1)*. Denote $v = u_1 - u_2$. Then, we find the following estimates to hold*

$$\|v\|_{E^0(T)}^2 \lesssim \|v(0)\|_{L^2}^2 + T\|v\|_{F_a^0(T)}^2 (\|u_1\|_{F_a^s(T)} + \|u_2\|_{F_a^s(T)}), \tag{4.20}$$

$$\begin{aligned}\|v\|_{E^s(T)}^2 &\lesssim \|v(0)\|_{H^s}^2 \\ &\quad + T(\|v\|_{F_a^s(T)}^3 + \|v\|_{F_a^s(T)}^2 \|u_2\|_{F_a^s(T)} + \|v\|_{F_a^0(T)} \|v\|_{F_a^s(T)} \|u_2\|_{F_a^{2s}(T)})\end{aligned} \tag{4.21}$$

provided that $s > s_a$.

Proof. Performing the same reductions like above, we have to estimate

$$\left|\int \int P_{N_1} v \partial_{x_1}(P_{N_2} u P_{N_3} v) dx dt\right|$$

for $N_1 \sim N_3 \gg N_2$, $N_1 \lesssim N_2 \sim N_3$ and $N_3 \ll N_1 \sim N_2$.
The first case can be dealt with like in the corresponding estimate for solutions because we can still integrate by parts.
The second case does not require integration by parts and can be estimated like above.
For the case $N_3 \ll N_1 \sim N_2$ we estimate

$$\begin{aligned}&\lesssim N_1 T N_1^{2-a} \|P_{N_1} v P_{N_3} v\|_{L^2_{t,x}} \|P_{N_2} u\|_{L^2_{t,x}} \\ &\lesssim T N_1^{2-a} N_3^{\frac{n-1}{2}} \|P_{N_1} v\|_{F_a^0} \|P_{N_2} u\|_{F_a^0} \|P_{N_3} v\|_{F_a^0}.\end{aligned}$$

This yields (4.20) after summation. To prove (4.21), one writes

$$\partial_t v + \partial_{x_1} D_x^a v = v\partial_{x_1} v + \partial_{x_1}(u_2 v).$$

The first term has the same symmetries like the term we encountered when proving a priori estimates for solutions. For the second term the only new estimate one has to carry out (due to impossibility to integrate by parts) is

$$\sum_{1 \leq K \lesssim N} N^{2s} \int_0^T \int_{\mathbb{R}^2} P_N v \partial_{x_1}(P_N u_2 P_K v) dx dt \lesssim T \|v\|_{F_a^0(T)} \|v\|_{F_a^s(T)} \|u_2\|_{F_a^{2s}(T)},$$

which follows by the above means provided that $s > s_a$. □

4.8 Periodic solutions to fractional Zakharov-Kuznetsov equations

Next, the above considerations for short-time nonlinear and energy estimates are extended to the fully periodic case. Firstly, the function spaces are introduced.

4.8.1 Function spaces in the periodic case

We shall be brief because the function spaces are defined completely analogous to [GO18] with the basic function space properties remaining valid.
For $k \in \mathbb{N}_0$ let

$$A_k = \begin{cases} \{\xi \in \mathbb{R}^n \mid |\xi| \leq 1\}, & k = 0, \\ \{\xi \in \mathbb{R}^n \mid |\xi| \sim 2^k\}, & \text{else.} \end{cases}$$

For $1 \leq a \leq 2$, $k \in \mathbb{N}_0$ define the dyadic $X^{s,b}$-type normed spaces

$$X_{a,k} = \{f \in L^2(\mathbb{R} \times \mathbb{Z}^n) \mid f \text{ is supported in } A_k \times \mathbb{R} \text{ and } \|f\|_{X_{a,k}} < \infty\}, \text{ where}$$

$$\|f\|_{X_{a,k}} = \sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - \varphi_a(\xi)) f\|_{L^2_{(d\xi)_1, \tau}}.$$

The basic properties from Section 2.5 remain valid: if $k, l \in \mathbb{N}_0$ and $f_k \in X_{a,k}$, then

$$\begin{aligned} &\sum_{j=l+1}^{\infty} 2^{j/2} \|\eta_j(\tau - \varphi_a(\xi)) \int |f_k(\tau', \xi)| 2^{-l} (1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \|_{L^2_{(d\xi)_1, \tau}} \\ &+ 2^{l/2} \|\eta_{\leq l}(\tau - \varphi_a(\xi)) \int |f_k(\tau', \xi)| 2^{-l} (1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \|_{L^2_{(d\xi)_1, \tau}} \lesssim \|f_k\|_{X_{a,k}}. \end{aligned} \tag{4.22}$$

Consequently, for $f \in X_{a,k}$ we find for $l \in \mathbb{N}_0$, $t_0 \in \mathbb{R}$, $\gamma \in \mathcal{S}(\mathbb{R})$

$$\|\mathcal{F}_{t,x}[\gamma(2^l(t - t_0)) \mathcal{F}^{-1}_{t,x}(f)]\|_{X_{a,k}} \lesssim_\gamma \|f\|_{X_{a,k}}.$$

The frequency localized spaces are defined like in Section 2.5, replacing X_k with $X_{a,k}$. The spaces $F_a^s(T)$, $N_a^s(T)$, $E^s(T)$ are again assembled by Littlewood-Paley theory. Let $C = C([-T,T], H^\infty(\mathbb{T}^n))$ and define

$$F_a^s(T) = \{u \in C | \|u\|^2_{F_a^s(T)} = \sum_{k \in \mathbb{N}_0} 2^{2sk} \|P_k u\|^2_{F_{a,k}(T)} < \infty\},$$
$$N_a^s(T) = \{u \in C | \|u\|^2_{N_a^s(T)} = \sum_{k \in \mathbb{N}_0} 2^{2sk} \|P_k u\|^2_{N_{a,k}(T)} < \infty\},$$
$$E^s(T) = \{u \in C | \|u\|^2_{E^s(T)} = \|P_{\leq 0} u(0)\|^2_{H^s} + \sum_{k \geq 1} \sup_{t_k \in [-T,T]} 2^{2sk} \|P_k u(t_k)\|^2_{E_k} < \infty\}.$$

4.8.2 Bilinear estimates

Next, we point out bilinear convolution estimates for space-time Fourier transforms of functions localized in frequency and modulation.
For the remainder of this section suppose that $1 \leq a \leq 2$ and for $k, j \in \mathbb{N}_0$ let

$$D^a_{k,\leq j} = \{(\tau, \xi) \quad | \quad \xi \in A_k, \ |\tau - \varphi_a(\xi)| \leq 2^j\}.$$

Lemma 4.8.1. *Let $k_i, j_i \in \mathbb{N}$, $f_i : \mathbb{R} \times \mathbb{Z}^n \to \mathbb{R}_+$, $f_i \in L^2(\mathbb{R} \times \mathbb{Z}^n)$, supp$(f_i) \subseteq D^a_{k_i, \leq j_i}$ for $i \in \{1,2,3\}$.*

(a) Suppose that $k_2 \leq k_1 - 5$. Then, we find the following estimate to hold:

$$\int_{\mathbb{R} \times \mathbb{Z}^n} (f_1 * f_2) f_3 \lesssim (1 + 2^{j_1 - a k_1})^{1/2} \|f_1\|_2 2^{j_2/2} 2^{(n-1)k_2/2} \|f_2\|_2 \|f_3\|_2. \quad (4.23)$$

(b) Suppose that $|k_1 - k_2| \leq 10, |k_2 - k_3| \leq 10$ and $j_i \geq k_i$. Further, suppose that $f_i(\xi, \tau) = 0$ for $|\xi_1| \notin [2^{l_i}, 2^{l_i+1})$, where $i = 1,2,3$ and $l = \max_{i=1,2,3} l_i$. Then, we find the following estimate to hold:

$$\int_{\mathbb{R} \times \mathbb{Z}^n} (f_1 * f_2) f_3 \lesssim 2^{-l/2} 2^{(n-2)k_1/2} \prod_{i=1}^3 2^{j_i/2} \|f_i\|_{L^2}. \quad (4.24)$$

(c) The estimate

$$\int_{\mathbb{R} \times \mathbb{Z}^n} (f_1 * f_2) f_3 \lesssim 2^{n k_{\min}/2} 2^{j_{\min}/2} \prod_{i=1}^3 \|f_i\|_{L^2} \quad (4.25)$$

holds true.

Proof. (a) and (b) are consequences of the considerations from Section 4.4.
By almost orthogonality, we can suppose that the f_i are supported in balls of radius 2^{k_2}. Then, the estimate (4.23) follows from Hölder's inequality

$$\int_{\mathbb{R} \times \mathbb{Z}^n} (f_1 * f_2) f_3 \leq \|f_3\|_{L^2} \|f_1 * f_2\|_{L^2_{\tau,(d\xi)_1}},$$

and invoking Proposition 3.1.2. (4.24) follows likewise.
(c) follows from two applications of Cauchy-Schwarz inequality without using the resonance function. ☐

4.8.3 Short-time nonlinear estimates

Proposition 4.8.2. *Let $n \geq 2$ and $T \in (0, T_0]$. We find the following estimates to hold:*

$$\|\partial_{x_1}(uv)\|_{N_a^{s'}(T)} \lesssim \|u\|_{F_a^s(T)} \|v\|_{F_a^{s'}(T)}, \tag{4.26}$$

$$\|\partial_{x_1}(uv)\|_{N_a^0(T)} \lesssim \|u\|_{F_a^0(T)} \|v\|_{F_a^s(T)} \tag{4.27}$$

provided that $n/2 < s \leq s'$.

Remark 4.8.3. The argument below yields nonlinear estimates up to $s > (n-1)/2$. The regularity threshold $s > (n+1)/2$ comes from carrying out energy estimates.

Proof. Choose by the definition of the function spaces $\tilde{u}, \tilde{v} \in C(\mathbb{R}, H^{n+2})$ such that

$$\|P_k \tilde{u}\|_{F_{a,k}} \leq 2\|P_k u\|_{F_{a,k}(T)} \text{ and } \|P_k \tilde{v}\|_{F_{a,k}} \leq 2\|P_k v\|_{F_{a,k}(T)}$$

for $k \in \mathbb{N}$. Set $u_k = P_k \tilde{u}$ and $v_k = P_k \tilde{v}$. Then, it suffices to consider the interactions $High \times Low \to High$:

$$\|P_k(\partial_{x_1}(u_{k_1} v_{k_2})\|_{N_{a,k}} \lesssim 2^{(n-1)k_2/2} \|u_{k_1}\|_{F_{a,k_1}} \|v_{k_2}\|_{F_{a,k_2}} \quad (k_2 \leq k-5) \tag{4.28}$$

$High \times High \to High$:

$$\|P_k \partial_{x_1}(u_{k_1} v_{k_2})\|_{N_{a,k}} \lesssim 2^{((n-1)/2+)k} \|u_{k_1}\|_{F_{a,k_1}} \|v_{k_2}\|_{F_{a,k_2}} \quad (|k_1-k| \leq 10, |k_2-k| \leq 10) \tag{4.29}$$

$High \times High \to Low$:

$$\|P_k \partial_{x_1}(u_{k_1} v_{k_2})\|_{N_{a,k}} \lesssim 2^{((n-1)/2+)k_1} \|u_{k_1}\|_{F_{a,k_1}} \|v_{k_2}\|_{F_{a,k_2}} \quad (k \leq k_1 - 5) \tag{4.30}$$

Then the claim follows from the definition of the function spaces by summing over the frequencies.

We start with $High \times Low \to High$-interaction. By the definition of $N_{a,k}$ and $F_{a,k}$-spaces, it suffices to show the estimate

$$2^k \sum_{j \geq k} 2^{-j/2} \|1_{D_{k,\leq j}}(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim 2^{(n-1)k_2/2} \prod_{i=1}^{2} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2}. \tag{4.31}$$

Here,

$$f_{k_i,j_i}(\xi,\tau) = \begin{cases} \eta_{j_i}(\tau - \varphi_a(\xi))\mathcal{F}_{t,x}[u_i], & j_i > k, \\ \eta_{\leq j_i}(\tau - \varphi_a(\xi))\mathcal{F}_{t,x}[u_i], & j_i = k. \end{cases}$$

To prove (4.31), use duality and apply estimate (4.23) to find

$$\begin{aligned} \|1_{D^a_{k,\leq j}}(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} &= \sup_{\|g_{k,j}\|_{L^2}=1} \int_{\mathbb{Z}^n \times \mathbb{R}} g_{k,j}(f_{k_1,j_1} * f_{k_2,j_2}) \\ &\lesssim 2^{(n-1)k_2/2} 2^{-k/2} \prod_{i=1}^{2} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2}. \end{aligned}$$

For the $High \times High \to High$-interaction, we split the sum over the output modulation variable into $k \leq j \leq 2k$ and $j \geq 2k$. Further, we introduce an additional frequency localization in the x_1-variable to find for frequency localized pieces

$$\begin{aligned}
&2^{l^*}\sum_{k\leq j\leq 2k}2^{-j/2}\|1_{D^a_{k,l,\leq j}}(f^{l_1}_{k_1,j_1}*f^{l_2}_{k_2,j_2})\|_{L^2}\\
&\lesssim 2^{l^*}\sum_{k\leq j\leq 2k}2^{-j/2}2^{j/2}2^{-l^*/2}2^{(n-2)k/2}\prod_{i=1}^2 2^{j_i/2}\|f^{l_i}_{k_i,j_i}\|_{L^2}\\
&\lesssim 2^{((n-1)k/2)+}\prod_{i=1}^2 2^{j_i/2}\|f^{l_i}_{k_i,j_i}\|_{L^2}
\end{aligned}$$

after applying duality and estimate (4.24), where $f^{l_i}_{k_i,j_i}(\xi_i,\tau)=0$ for $|\xi_{i1}|\notin[2^{l_i},2^{l_i+1})$.
Summation over l and l_i only gives a logarithmic factor, not changing the estimate effectively.
For the high modulation output, apply duality and estimate (4.25) to find

$$\begin{aligned}
2^k\sum_{j\geq 2k}2^{-j/2}\|1_{D^a_{k,\leq j}}(f_{k_1,j_1}*f_{k_2,j_2})\|_{L^2}&\lesssim 2^k\sum_{j\geq 2k}2^{-j/2}2^{(n-1)k/2}\prod_{i=1}^2 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2}\\
&\lesssim 2^{(n-1)k/2}\prod_{i=1}^2 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2}.
\end{aligned}$$

For the $High\times High\to Low$-interaction, we argue similarly: Taking into account the additional time localization, it suffices to prove

$$2^{k_1}\sum_{j\geq k}2^{-j/2}\|1_{D^a_{k,\leq j}}(f_{k_1,j_1}*f_{k_2,j_2})\|_{L^2}\lesssim 2^{(n-1)k_1/2}\prod_{i=1}^2 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2},\tag{4.32}$$

where $\mathrm{supp}(f_{k_i,j_i})\subseteq D^a_{k_i,\leq j_i}$, $j_i\geq k_1$ for $i=1,2$.
Again, the sum over j is split into $k\leq j\leq 2k_1$, $j\geq 2k_1$.
In the first case, we use duality and apply (4.23) to find

$$\begin{aligned}
&2^{k_1}\sum_{k\leq j\leq 2k_1}2^{-j/2}2^{j/2}2^{(n-1)k/2}2^{-k_1}\prod_{i=1}^2 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2}\\
&\lesssim (2k_1-k)2^{(n-1)k/2}\prod_{i=1}^2 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2}.
\end{aligned}$$

In the second case, estimate (4.25) yields

$$\begin{aligned}
&2^{k_1}\sum_{j\geq 2k_1}2^{-j/2}\|1_{D^a_{k,\leq j}}(f_{k_1,j_1}*f_{k_2,j_2})\|_{L^2}\\
&\lesssim 2^{k_1}\sum_{j\geq 2k_1}2^{-j/2}2^{nk/2}2^{-k_1/2}\prod_{i=1}^2 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2}\\
&\lesssim 2^{(n-1)k/2}\prod_{i=1}^2 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2}.
\end{aligned}\tag{4.33}$$

The proof is complete. □

4.8.4 Energy estimates

Purpose of this section is to propagate the energy norm of solutions and differences of solutions in terms of short-time norms. We prove the following proposition:

Proposition 4.8.4. *Let $n \geq 2$, $1 \leq a \leq 2$, $T \in (0, T_0]$, $s > (n+1)/2$ and $u \in C([-T,T], H^\infty(\mathbb{T}^n))$ be a smooth solution to* (4.1)*. Then, we find the following estimate to hold:*

$$\|u\|^2_{E^s(T)} \lesssim \|u_0\|^2_{H^s} + T\|u\|^3_{F^s_a(T)}. \tag{4.34}$$

For two solutions to (4.1) *$u_i \in C([-T,T], H^\infty_0)$ the function $v = u_1 - u_2$ satisfies the estimates*

$$\|v\|^2_{E^0(T)} \lesssim \|v(0)\|^2_{L^2} + T_0\|v\|^2_{F^0_a(T)}(\|u_1\|_{F^s_a(T)} + \|u_2\|_{F^s_a(T)}), \tag{4.35}$$

$$\begin{aligned} \|v\|^2_{E^s(T)} \lesssim \|v(0)\|^2_{H^s} + T_0\|v\|^3_{F^s_a(T)} + T_0\|v\|_{F^0_a(T)}\|v\|_{F^s_a(T)}\|u_2\|_{F^{2s}_a(T)} \\ + T_0\|v\|^2_{F^s_a(T)}\|u_2\|_{F^s_a(T)}. \end{aligned} \tag{4.36}$$

Like in the Euclidean case we find for the evolution of the L^2-norm of the frequencies

$$\|P_k u(t_k)\|^2_{L^2} = \|P_k u(0)\|^2_{L^2} + 2\int_{\mathbb{T}^n \times [0,t_k]} P_k u \partial_{x_1} P_k(uv) dx dt,$$

where u solves the following forced equation:

$$\partial_t u + \partial_{x_1} D^a_x u = \partial_{x_1}(uv).$$

The key estimates are carried out in the following lemma, the rest follows from integration by parts and commutator estimates (cf. Lemma 3.4.2, Section 3.5):

Lemma 4.8.5. *Let $0 < T \leq T_0$, $u_i \in F_{a,k_i}(T)$, $i = 1,2,3$ and $k_{max} \geq k_0$.*

(a) Suppose that $k_2 \leq k_1 - 5$. Then, we find the following estimate to hold:

$$\left| \int_{[0,T]\times\mathbb{T}^n} u_1 u_2 u_3 dx dt \right| \lesssim T 2^{(n-1)k_2/2} \prod_{i=1}^3 \|u_i\|_{F_{a,k_i}(T)}. \tag{4.37}$$

(b) Suppose that $|k_1 - k_2| \leq 10$, $|k_1 - k_3| \leq 10$ and $\mathcal{F}_{t,x}(u_i)(\xi,\tau) = 0$, whenever $|\xi| \notin [2^{l_i}, 2^{l_i+1})$. Set $l^* = \max_{i=1,2,3} l_i$. Then, the following estimate holds:

$$\left| \int_{[0,T]\times\mathbb{T}^n} u_1 u_2 u_3 dx dt \right| \lesssim T_0 2^{-l^*/2} 2^{nk_2/2} \prod_{i=1}^3 \|u_i\|_{F_{a,k_i}(T)}. \tag{4.38}$$

(c) Suppose that $k_1 \leq k - 5$. Then, we find the following estimate to hold:

$$\begin{aligned} &\left| \int_{[0,T]\times\mathbb{T}^n} P_k u \partial_{x_1} P_k(u P_{k_1} v) dx dt \right| \\ &\lesssim T_0 2^{(n+1)k_1/2} \|v\|_{F_{a,k_1}(T)} \sum_{|k'-k|\leq 10} \|P_{k'} u\|^2_{F_{a,k'}(T)}. \end{aligned} \tag{4.39}$$

Proof. By symmetry we can assume that $k_1 \leq k_2 \leq k_3$. Let $\tilde{u}_i \in F_{a,k_i}$ with $\|\tilde{u}_i\|_{F_{a,k_i}} \leq 2\|u_i\|_{F_{a,k_i}(T)}$, $i = 1,2,3$ by the definitions.
The $\tilde{u}_i$ will be denoted by u_i to lighten the notation. To estimate the functions in the short-time function spaces, time has to be localized according to the highest frequency. Let $\gamma : \mathbb{R} \to [0,1]$ be a smooth function supported in $[-1,1]$ with

$$\sum_{n \in \mathbb{Z}} \gamma^3(x-n) = 1 \quad \forall x \in \mathbb{R}.$$

The left hand-side of (4.37) is dominated by

$$\begin{aligned} &\sum_{|n| \leq C T_0 2^{k_3}} \Big| \sum_{j_i \geq k_i} \int_{\mathbb{Z}^n \times \mathbb{R}} \eta_{j_1}(\tau - \varphi_a(\xi)) \mathcal{F}_{t,x}(u_1 \gamma(2^{k_3} t - n) 1_{[0,T]}(t)) \\ &\quad (\eta_{j_2}(\tau - \varphi_a(\xi)) \mathcal{F}_{t,x}[u_2 \gamma(2^{k_3} t - n)]) * (\eta_{j_3} \mathcal{F}_{t,x}[u_3 \gamma(2^{k_3} t - n)])(d\xi)_1 d\tau \Big| \\ &= \sum_{n \in A} (\dots) + \sum_{n \in B} (\dots), \end{aligned} \tag{4.40}$$

where

$$\begin{aligned} A &= \{n \in \mathbb{Z} | \gamma(2^{k_3} \cdot - n) 1_{[0,T]} \neq \gamma(2^{k_3} \cdot - n)\}, \\ B &= \{n \in \mathbb{Z} | \gamma(2^{k_3} \cdot - n) 1_{[0,T]} = \gamma(2^{k_3} \cdot - n)\}. \end{aligned}$$

In (4.40) read $\eta_{j_i} = \eta_{\leq j_i}$ for $j_i = k_i$; it is sufficient to derive bounds for this modulation variable decomposition according to (4.22).
Apparently, $|A| \leq 10$, $|B| \leq C_0 T 2^{k_3}$. The main contribution of B is handled first.
Denote

$$f_{k_i,j_i} = \eta_{j_i}(\tau - \varphi_a(\xi)) \mathcal{F}_{t,x}[u_1 \gamma(2^{k_3} t - n) 1_{[0,T]}(t)], \quad i = 1,2,3.$$

We do not distinguish between different values of n because the following estimates are independent of n.

In case $k_1 \leq k_2 - 5$ an application of (4.23) yields

$$\begin{aligned} \sum_{n \in B} (..) &\lesssim T 2^{k_3} \sum_{j_i \geq k_i} 2^{j_1/2} 2^{k_1/2} (1 + 2^{\frac{j_2 - k_3}{2}}) \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2} \\ &\lesssim T 2^{k_1/2} \prod_{i=1}^{3} \sum_{j_i \geq k_i} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2} \end{aligned}$$

because $j_3 \geq k_3$.

In case $|k_1 - k_2| \leq 10$, $|k_2 - k_3| \leq 10$ an application of (4.24) gives

$$\begin{aligned} \sum_{n \in B} (..) &\lesssim T 2^{n k_3/2} 2^{-l^*/2} \prod_{i=1}^{3} \sum_{j_i \geq k_i} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2} \\ &\lesssim T 2^{n k_3/2} 2^{-l^*/2} \prod_{i=1}^{3} \sum_{j_i \geq k_i} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2}. \end{aligned}$$

For the boundary terms note that sharp cutoffs in time are almost bounded in $X_{a,k}$: for an interval $I \subseteq \mathbb{R}$, $k \in \mathbb{N}_0$, $f_k \in X_{a,k}$ and $f_k^I = \mathcal{F}(1_I(t) \mathcal{F}^{-1}(f_k))$ (cf. [IKT08, p. 291]) we find

$$\sup_{j \in \mathbb{N}} 2^{j/2} \|\eta_j(\tau - \varphi_a(\xi)) f_k^I\|_{L^2} \lesssim \|f_k\|_{X_{a,k}}.$$

An application of Cauchy-Schwarz gives

$$\sum_{n \in B} (..) \lesssim \sum_{j_i \geq k_i} 2^{j_1/2} 2^{nk_1/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}$$
$$\lesssim 2^{nk_1/2} 2^{-k_3/2} \prod_{i=1}^{2} \sum_{j_i \geq k_i} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2} \sup_{j \in \mathbb{N}} 2^{j/2} \|f_j\|_{L^2},$$

which yields the claim.

For the proof of (4.39), we integrate by parts (cf. [IKT08]) to find

$$\begin{aligned} &|\int_{\mathbb{T}^n \times [0,T]} P_k u P_k (\partial_{x_1} u \tilde{P}_{k_1} v) dx dy dt| \\ &\leq |\int_{\mathbb{T}^n \times [0,T]} P_k u \tilde{P}_k (\partial_{x_1} u) P_{k_1} v dx dy dt| + C \sum_{i=1}^{n} |\int_{\mathbb{Z}^n \times \mathbb{R}} (d\xi)_1 d\tau \mathcal{F}(P_k u)(\xi, \tau) \\ &\quad \times \int_{\mathbb{Z}^n \times \mathbb{R}} (d\xi_1)_1 d\tau_1 \mathcal{F}(P_{k_1} \partial_{x_i} v)(\xi_1, \tau_1) \mathcal{F} v(\xi - \xi_1, \tau - \tau_1) \psi_i(\xi, \xi_1)|, \end{aligned}$$

where ψ_i, $i = 1, \ldots, n$ are bounded and regular multipliers. The resulting expressions can be handled by (4.23). □

We are ready to prove Proposition 4.8.4.

Proof of Proposition 4.8.4. Following the remark after Proposition 4.8.4, we find for a solution to (4.1)

$$\|P_k u(t_k)\|_{L^2}^2 = \|P_k u(0)\|_{L^2}^2 + 2 \int_0^{t_k} ds \int_{\mathbb{T}^n} dx P_k u P_k (\partial_{x_1} u^2).$$

For the integral we consider the following interactions:
High $\times$ *Low* $\to$ *High*:

$$\int_0^{t_k} ds \int_{\mathbb{T}^n} dx P_k u P_k \partial_{x_1} (u P_{k_1} u) \quad (k_1 \leq k - 5) \tag{4.41}$$

High $\times$ *High* $\to$ *High*:

$$\int_0^{t_k} ds \int_{\mathbb{T}^n} dx P_k u P_k \partial_{x_1} (u P_{k_1} u) \quad (|k - k_1| \leq 5) \tag{4.42}$$

High $\times$ *High* $\to$ *Low*:

$$\int_0^{T} ds \int_{\mathbb{T}^n} dx P_k u \partial_{x_1} (P_{k_1} u P_{k_2} u) \quad (k \leq k_1 - 5) \tag{4.43}$$

High $\times$ *Low* $\to$ *High*-interaction is estimated by (4.39) to

$$(4.41) \lesssim T 2^{(n+1)k_1/2} \sum_{|m-k| \leq 5} \|P_m u\|_{F_m(T)}^2 \sum_{|m_1 - k_1| \leq 5} \|P_{m_1} u\|_{F_{m_1}(T)},$$

and summing over $k_1 \leq k - 10$ and square summing over k gives (4.34).

In case of $High \times High \to High$-interaction, the functions are additionally partitioned in the first component of the frequencies. Estimate (4.38) is used to obtain

$$(4.42) \lesssim T 2^{(n+1)k/2} \sum_{|m-k|\leq 5} \|P_m u\|^2_{F_m(T)} \sum_{|m_1-k_1|\leq 10} \|P_{m_1} u\|_{F_{m_1}(T)}$$

and square summing over k gives (4.34). For $High \times High \to Low$-interaction, there is no reason to integrate by parts, and the argument for $High \times Low \to High$-interaction is used.

To prove (4.39), we write

$$\|P_k v(t_k)\|^2_{L^2} = \|P_k v(0)\|^2_{L^2} + 2\int_0^{t_k} ds \int_{\mathbb{T}^n} dx P_k v P_k \partial_{x_1}(v(u_1+u_2))$$

and estimate $High \times High \to High$-interaction and $High \times High \to Low$-interaction like above to obtain (4.39). In case of $High \times Low \to High$-interaction, one finds two different terms:

$$\int_0^T ds \int_{\mathbb{T}^n} dx P_k v P_k \partial_{x_1}(v P_{k_1}(u_1+u_2)) \quad (k_1 \leq k-5) \tag{4.44}$$

and

$$\int_0^T ds \int_{\mathbb{T}^n} dx P_k v P_k \partial_{x_1}((u_1+u_2) P_{k_1} v) \quad (k_1 \leq k-5). \tag{4.45}$$

(4.44) is estimated following along the above lines because we can integrate by parts to arrange the derivative on the smallest frequency.
For (4.45) we use estimate (4.37) instead to find

$$\begin{aligned}(4.45) \lesssim\ & T 2^k 2^{(n-1)k_1/2} \\ & \sum_{|m-k|\leq 5} \|P_m v\|_{F_{a,m}(T)} \sum_{|m_1-k_1|\leq 5} \|P_{m_1} v\|_{F_{a,m_1}(T)} \\ & \sum_{|m-k|\leq 5} (\|P_m u_1\|_{F_{a,m}(T)} + \|P_m u_2\|_{F_{a,m}(T)})\end{aligned}$$

and square summing in k and summing over $k_1 \leq k-5$ gives (4.35).

To prove (4.36), the solution to the difference equation is rewritten as

$$\partial_t v + \partial_{x_1} D^a_x v = \partial_{x_1}(v^2) + \partial_{x_1}(v u_2).$$

When estimating $\|v\|_{E^s(T)}$ for $s > (n+1)/2$, the contribution of $\partial_{x_1}(v^2)$ can be handled like in the proof of (4.34), which gives

$$\sum_k 2^{2ks} \int_0^T ds \int_{\mathbb{T}^n} dx P_k v P_k \partial_{x_1}(v^2) \lesssim T \|v\|^3_{F^s_a(T)}.$$

The contribution of $\partial_{x_1}(v u_2)$ can be treated like in the proof of (4.34) except for the interaction

$$\int_0^T ds \int_{\mathbb{T}^n} dx P_k v P_k \partial_{x_1}(u_2 P_{k_1} v) \quad (k_1 \leq k-5)$$

because here we can not integrate by parts like above. Instead, estimate (4.37) and square summing in k and summation in $k_1 \leq k-5$ gives

$$\sum_{k_1 \leq k-5} 2^{2ks} \int_0^T ds \int_{\mathbb{T}^n} dx P_k v P_k \partial_{x_1}(u_2 P_{k_1} v) \lesssim T \|v\|_{F_a^s(T)} \|u_2\|_{F_a^{2s}(T)} \|v\|_{F_a^0(T)}.$$

□

4.8.5 Proof of new well-posedness results on tori

Proof of Theorem 4.1.8. Fix $s > (n+1)/2$. We only demonstrate the proof of a priori estimates for smooth initial values. The additionally required arguments to construct the data-to-solution mapping are like in the proof of Theorem 4.1.1. For $0 < T \leq T_0$ we find for a smooth solution

$$\begin{cases} \|u\|_{F_a^s(T)} & \lesssim \|u\|_{E^s(T)} + \|\partial_{x_1}(u^2)\|_{N_a^s(T)} \\ \|u\partial_{x_1} u\|_{N_a^s(T)} & \lesssim \|u\|^2_{F_a^s(T)} \\ \|u\|^2_{E^s(T)} & \lesssim \|u_0\|^2_{H^s} + T_0 \|u\|^3_{F_a^s(T)} \end{cases}$$

This implies

$$\|u\|^2_{F_a^s(T)} \lesssim \|u_0\|^2_{H^s} + T_0^{1/4} \|u\|^2_{F_a^s(T)} + T_0 \|u\|^3_{F_a^s(T)}$$

and since

$$\lim_{T \to 0} \|u\|_{E^s(T)} \lesssim \|u_0\|_{H^s}, \quad \lim_{T \to 0} \|\partial_{x_1}(u^2)\|_{N_a^s(T)} = 0$$

like in Section 3.5, we can choose $T_0 = 1$ for sufficiently small initial data and prove a priori estimates by a bootstrap argument. L^2-Lipschitz continuity for initial data in H^s and continuity and existence of the data-to-solution mapping are derived like in Section 3.4. □

Chapter 5

New regularity results for dispersive PDE with cubic derivative nonlinearities on the circle

In this chapter we prove local existence of solutions and new a priori estimates for solutions to dispersive PDE with cubic derivative nonlinearity posed on the circle. The model cases we discuss in detail are the modified Benjamin-Ono equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_{xx} u &= \partial_x(u^3), \quad (t,x) \in \mathbb{R} \times \lambda\mathbb{T}, \\ u(0) &= u_0 \in H^s_{\mathbb{R}}(\lambda\mathbb{T}) \end{cases} \tag{5.1}$$

and the modified Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_{xxx} u &= \pm\partial_x(u^3), \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u(0) &= u_0 \in H^s_{\mathbb{R}}(\mathbb{T}). \end{cases} \tag{5.2}$$

The sign of the nonlinearity is irrelevant for our local-in-time analysis.
The arguments extend to dispersion generalizations interpolating between the two equations or the closely related derivative nonlinear Schrödinger equation.

Frequency dependent time localization will be used to introduce extra smoothing, and we add a correction term to control the energy transfer at lower regularities. The argument differs in the following point: In case of quadratic dispersion relation, we will work in Euclidean windows $T = T(N) = N^{-1}$ allowing us to recover dispersive properties from Euclidean space. Then, one is in the position to utilize the analysis on the real line from [Guo11] and to prove local existence and a priori estimates for the same regularity $s > 1/4$. This is the first regularity result for (5.1) on the circle below $s = 1/2$.

For the modified Korteweg-de Vries equation, one finds stronger dispersive properties to hold compared to (5.1) due to a cubic dispersion relation. We do not work in Euclidean windows but with $T = T(N) = N^{-1}$. This will be sufficient to cover $s > 0$. To go below L^2, we have to increase localization in time.
We can prove existence of solutions after renormalization conditional upon conjectured Strichartz estimates on tori for negative Sobolev regularities.

5.1 Quadratic dispersion relations

In this section we discuss the existence and a priori estimates of periodic solutions to cubic one-dimensional Schrödinger-like equations with derivative nonlinearity like (5.1).

It turns out that the following derivative nonlinear Schrödinger equation (dNLS) is also amenable to the employed methods:

$$\begin{cases} i\partial_t u + \partial_{xx} u &= i\partial_x(|u|^2 u), \quad (t,x) \in \mathbb{R} \times \lambda\mathbb{T}, \\ u(0,x) &= u_0(x). \end{cases} \tag{5.3}$$

From the point of view of dispersive equations, the models look very similar. However, (5.3) is completely integrable (cf. [KN78]) in contrast to (5.1), which is not known to be completely integrable. Since it is useful to point out that the approach does not hinge on complete integrability in a crucial manner, we choose to analyze (5.1) in detail and give the modifications for (5.3) in Subsection 5.1.5.

On the real line, the equations share the scaling symmetry

$$u(t,x) \to \lambda^{-1/2} u(\lambda^{-2} t, \lambda^{-1} x), \tag{5.4}$$

which makes the scaling critical regularity of these equations $s_c = 0$, but it is known that the data-to-solution mapping fails to be C^3 for $s < 1/2$.

On the real line, (5.1) has been analyzed by Guo in [Guo11]: In [Guo11] it was proved that the Cauchy problem (5.1) on the real line is locally well-posed with uniform continuity of the data-to-solution mapping as long as $s \geq 1/2$ and provided that the L^2-norm of the initial data is sufficiently small, see also the earlier work [MR04] and references therein.

Furthermore, for smooth and real-valued solutions, a priori estimates have been established for $s > 1/4$ in [Guo11]. The analog local well-posedness result in $H^{1/2}(\mathbb{T})$ was shown by Guo-Lin-Molinet in [GLM14].

Takaoka showed in [Tak99] that (5.3) on the real line is locally well-posed in $H^{1/2}(\mathbb{R})$ making use of the Fourier restriction spaces and a gauge transform to remedy the problematic nonlinear term $|u|^2 \partial_x u$. Global well-posedness was later shown employing the I-method by Colliander et al. in [CKS$^+$02] on the real line ($s > 1/2$) and in [Mos17] on the circle ($s \geq 1/2$).

Adapting the Fourier restriction spaces and the gauge transform to the periodic setting, Herr showed in [Her06] that the Cauchy problem is locally well-posed in $H^{1/2}(\mathbb{T})$. Again, the data-to-solution mapping fails to be C^3 below $H^{1/2}(\mathbb{T})$ and even fails to be uniformly continuous below $H^{1/2}(\mathbb{T})$ (cf. [Mos17]).

In [Gr5] Grünrock proved that (5.3) on the real line is locally well-posed in Fourier Lebesgue spaces, which scale almost like L^2, and in [GH08], Grünrock-Herr showed that (5.3) on the circle is locally well-posed in Fourier Lebesgue spaces, which scale like $H^{1/4}$. In a recent work [DNY19] by Deng-Nahmod-Yue, the extension to almost scaling critical Fourier Lebesgue spaces is claimed.

Unconditional uniqueness for $s > 1/2$ was proved by Kishimoto on the circle (cf. [Kis12]) and by Mosincat-Yoon ([MY18]) on the real line.

Takaoka showed in [Tak16] the existence of weak solutions and a priori estimates for $s > 12/25$ combining the analysis from [CKS$^+$02] and [Her06]. The gauge transform plays an important role in [Tak16] so that the $High \times Low \times Low \to High$-interaction can be estimated in $X^{s,b}$-spaces.

Our approach is different: After frequency dependent time localization, basic ingredients to carry out the estimates are short-time linear and bilinear Strichartz estimates discussed in Chapter 3. The time localization allows us to control the derivative nonlinearity in conservative form and we will not use a gauge transform. This allows us to obtain the same regularity $s > 1/4$ for a priori estimates like in the Euclidean case. We prove the following theorem:

Theorem 5.1.1. *Let $s > 1/4$ and $u_0 \in H^s(\mathbb{T})$. There are mappings $\mu = \mu(s) > 0$, $T = T(s, \|u_0\|_{H^s}) > 0$ so that there is a solution $u \in C([-T,T], H^s(\mathbb{T}))$ to (5.1) with $\lambda = 1$ in the sense of generalized functions, and we find the a priori estimate*

$$\sup_{t \in [-T,T]} \|u(t)\|_{H^s} \leq C(s, \|u_0\|_{H^s}) \|u_0\|_{H^s} \tag{5.5}$$

to hold provided that $\|u_0\|_{L^2} \leq \mu_s$. Moreover, we have $C(s, \|u_0\|_{H^s}) \leq C_s$ and $T(s, \|u_0\|_{H^s}) \geq 1$ as $\|u_0\|_{H^s} \to 0$.

For $s > 1/4$ we will show the bounds[1]

$$\begin{cases} \|u\|_{F^s_\lambda(T)} & \lesssim \|u\|_{E^s_\lambda(T)} + \|\partial_x(u^3/3)\|_{N^s_\lambda(T)} \\ \|\partial_x(u^3/3)\|_{N^s_\lambda(T)} & \lesssim \|u\|^3_{F^s_\lambda(T)} \\ \|u\|^2_{E^s_\lambda(T)} & \lesssim \|u_0\|^2_{H^s_\lambda} + T\|u\|^6_{F^s(T)}. \end{cases}$$

The proof of the a priori estimate will be concluded by a continuity argument. To infer existence of weak solutions, we use a compactness argument: a smoothing effect in the energy estimates admits the construction of solutions.

5.1.1 Function spaces and Strichartz estimates

Following Section 2.4, we define the following short-time function spaces for arbitrary periods $\lambda > 0$ for frequency localized functions $\text{supp}_\xi(\hat{u}) \subseteq I_k$:

$$\begin{aligned} \|u\|_{F_{k,\lambda}(T)} &= \sup_{\substack{|I|=2^{-k}\wedge T, \\ I \subseteq [0,T]}} \|u\|_{U^2_{BO}(I;L^2_\lambda)}, \\ \|v\|_{N_{k,\lambda}(T)} &= \sup_{\substack{|I|=2^{-k}\wedge T, \\ I \subseteq [0,T]}} \|u\|_{DU^2_{BO}(I;L^2_\lambda)}, \\ E_{k,\lambda} &= \{u_0 : \lambda\mathbb{T} \to \mathbb{C} \mid \text{supp}_\xi(\hat{u}) \subseteq I_k, \|u_0\|_{E_{k,\lambda}} = \|u_0\|_{L^2_\lambda} < \infty\}, \end{aligned} \tag{5.6}$$

and the spaces $F^s_\lambda(T)$, $N^s_\lambda(T)$ and $E^s_\lambda(T)$ are assembled by Littlewood-Paley theory like in Section 2.4.

When we analyze (5.3), the function spaces will be adapted to the Schrödinger propagator. The adapted U^2-spaces are denoted by U^2_Δ. The short-time function spaces, where U^2_{BO} is replaced with U^2_Δ, will also be denoted by $F_{k,\lambda}$, $N_{k,\lambda}$. The adaptation will be clear from context. For the short-time nonlinear estimates, the frequency projectors can be taken to be sharp, but for a technical reason the frequency projectors have to be chosen smooth for the energy estimate.
Building blocks for the nonlinear and energy estimate are linear and bilinear estimates:

[1] The precise form of the energy estimate is slightly more complicated.

Proposition 5.1.2. *Let $u \in U^2_{BO}(I; L^2_\lambda)$ with $P_k u = u$ and $|I| \lesssim 2^{-k}$. Then, we find the following estimate to hold:*

$$\|u\|_{L^6_t(I;L^6_\lambda)} \lesssim \|u\|_{U^2_{BO}(I;L^2_\lambda)}. \tag{5.7}$$

Further, let $v \in U^2_{BO}(I; L^2_\lambda)$ with $P_m v = v$ for some $m \leq k-5$ or $||\xi_1| - |\xi_2|| \gtrsim 2^n$ provided that $\xi_1 \in \operatorname{supp}_\xi \hat{u}$ and $\xi_2 \in \operatorname{supp}_\xi \hat{v}$. Then, we find the following estimates to hold:

$$\|uv\|_{L^2(I;L^2_\lambda)} \lesssim 2^{-k/2} \|u\|_{U^2_{BO}(I;L^2_\lambda)} \|v\|_{U^2_{BO}(I;L^2_\lambda)}, \tag{5.8}$$

$$\|uv\|_{L^2(I;L^2_\lambda)} \lesssim k^2 2^{-k/2} \|u\|_{V^2_{BO}(I;L^2_\lambda)} \|v\|_{V^2_{BO}(I;L^2_\lambda)}. \tag{5.9}$$

For the latter estimate, it is enough to assume $u, v \in V^2_{BO}(I; L^2_\lambda)$.

Proof. It is enough to verify the claims for $\lambda = 1$ as the general case follows from rescaling. (5.7) is a consequence of Proposition 3.2.6 and the transfer principle after considering positive and negative frequencies separately.
(5.8) follows from the transfer principle and the short-time bilinear Strichartz estimate from Proposition 3.1.5, see also the subsequent remark.
(5.9) follows from interpolating (5.8) with linear estimates, see Property (iv) from Lemma 2.3.4. □

We record the corresponding estimates in case of Schrödinger interaction, which are proved like in the previous proposition:

Proposition 5.1.3. *Let $u \in U^2_\Delta(I; L^2_\lambda)$ with $P_k u = u$ and $|I| \lesssim 2^{-k}$. Then, we find the following estimate to hold:*

$$\|u\|_{L^6_t(I;L^6_\lambda)} \lesssim \|u\|_{U^2_\Delta(I;L^2_\lambda)}. \tag{5.10}$$

Further, let $v \in U^2_\Delta(I; L^2_\lambda)$ with $P_m v = v$ for some $m \leq k-5$ or $||\xi_1| - |\xi_2|| \gtrsim 2^k$ provided that $\xi_1 \in \operatorname{supp}_\xi \hat{u}$ and $\xi_2 \in \operatorname{supp}_\xi \hat{v}$. Then, we find the following estimates to hold:

$$\|uv\|_{L^2(I;L^2_\lambda)} \lesssim 2^{-k/2} \|u\|_{U^2_\Delta(I;L^2_\lambda)} \|v\|_{U^2_\Delta(I;L^2_\lambda)}, \tag{5.11}$$

$$\|uv\|_{L^2(I;L^2_\lambda)} \lesssim k^2 2^{-k/2} \|u\|_{V^2_\Delta(I;L^2_\lambda)} \|v\|_{V^2_\Delta(I;L^2_\lambda)}. \tag{5.12}$$

The estimates remain true after replacing v by $\overline{v}$.
For the latter estimate, it is enough to assume $u, v \in V^2_\Delta(I; L^2_\lambda)$.

5.1.2 Short-time trilinear estimate

Aim of this section is to derive a short-time trilinear estimate:

Proposition 5.1.4. *Suppose that $s > 1/4$, $T \in (0,1]$ and $u, v, w \in F^s_\lambda(T)$. Then, we find the following estimate to hold:*

$$\|\partial_x(uvw)\|_{N^s_\lambda(T)} \lesssim \|u\|_{F^s_\lambda(T)} \|v\|_{F^s_\lambda(T)} \|w\|_{F^s_\lambda(T)}. \tag{5.13}$$

We perform decompositions with respect to frequency, essentially reducing the estimate (5.13) from above to

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4,\lambda}} \lesssim \underbrace{\alpha(k_1,k_2,k_3,k_4)}_{\alpha(\underline{k})} \|u_{k_1}\|_{F_{k_1,\lambda}}\|v_{k_2}\|_{F_{k_2,\lambda}}\|w_{k_3}\|_{F_{k_3,\lambda}}. \tag{5.14}$$

We prove (5.14) using the estimates from Proposition 5.1.2. In order to structure the proof, we list each possible frequency interaction: in any case, we find estimate (5.13) to hold for regularities $s > 1/4$.

(i) *High* $\times$ *Low* $\times$ *Low* $\to$ *High*-interaction: This interaction will be estimated by Lemma 5.1.5.

(ii) *High* $\times$ *High* $\times$ *Low* $\to$ *High*-interaction: This interaction will be estimated by Lemma 5.1.6.

(iii) *High* $\times$ *High* $\times$ *High* $\to$ *High*-interaction: This interaction will be estimated by Lemma 5.1.7.

(iv) *High* $\times$ *High* $\times$ *Low* $\to$ *Low*-interaction: This interaction will be estimated by Lemma 5.1.8.

(v) *High* $\times$ *High* $\times$ *High* $\to$ *Low*-interaction: This interaction will be estimated by Lemma 5.1.9.

(vi) *Low* $\times$ *Low* $\times$ *Low* $\to$ *Low*-interaction: This interaction will be estimated by Lemma 5.1.10.

We start with *High* $\times$ *Low* $\times$ *Low* $\to$ *High*-interaction:

Lemma 5.1.5. *Suppose that* $k_4 \geq 20$ *and* $k_1 \leq k_2 \leq k_3 - 5$*. Then, we find* (5.14) *to hold with* $\alpha = 2^{k_1/2}$*.*

Proof. We use the embedding $L^1(I) \hookrightarrow DU^2_{BO}(I)$ and Hölder in time to find for $|I| = 2^{-k_4}$

$$\begin{aligned}
&\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{DU^2_{BO}(I;L^2_\lambda)} \\
&\lesssim \|\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{L^1(I;L^2_\lambda)} \\
&\lesssim 2^{k_4/2}\|u_{k_1}v_{k_2}w_{k_3}\|_{L^2(I;L^2_\lambda)} \\
&\lesssim 2^{k_4/2}\|u_{k_1}\|_{L^\infty(I;L^\infty_\lambda)}\|v_{k_2}w_{k_3}\|_{L^2(I;L^2_\lambda)} \\
&\lesssim 2^{k_1/2}\|u_{k_1}\|_{U^2_{BO}(I;L^2_\lambda)}\|v_{k_2}\|_{U^2_{BO}(I;L^2_\lambda)}\|w_{k_3}\|_{U^2_{BO}(I;L^2_\lambda)}.
\end{aligned}$$

The ultimate estimate follows from (5.8) and Bernstein's inequality. The claim follows from the definition of the function spaces. ☐

Next, *High* $\times$ *High* $\times$ *Low* $\to$ *High*-interaction is considered:

Lemma 5.1.6. *Suppose that* $k_4 \geq 20$, $k_1 \leq k_2 \leq k_3$ *and* $k_1 \leq k_2 - 20$, $|k_2 - k_4| \leq 10$*. Then,* (5.14) *holds with* $\alpha(\underline{k}) = 2^{(0+)k_4}$*.*

Proof. Let I be an interval with $|I| \lesssim 2^{-k_4}$. We use duality to write

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{DU^2_{BO}(I;L^2_\lambda)} \lesssim 2^{k_4} \sup_{\|f\|_{V^2_{BO}}=1} \int_I \int_{\lambda\mathbb{T}} P_{k_4} f u_{k_1} v_{k_2} w_{k_3} dx dt. \quad (5.15)$$

Among $P_{k_4}f$, v_{k_2}, w_{k_3}, there is a pair amenable to a bilinear Strichartz estimate as pointed out in Section 3.1. Say this is $P_{k_4}fw_{k_3}$. Then, we find from (5.8) and (5.9) the following

$$\begin{aligned}(5.15) &\lesssim 2^{k_4}\|P_{k_4}fw_{k_3}\|_{L^2(I;L^2_\lambda)}\|u_{k_1}v_{k_2}\|_{L^2(I;L^2_\lambda)} \\ &\lesssim k_4^2\left(\sup_{\|f\|_{V^2_{BO}}=1}\|P_{k_4}f\|_{V^2_{BO}}\right)\|w_{k_3}\|_{V^2_{BO}(I;L^2_\lambda)}\|u_{k_1}\|_{U^2_{BO}(I;L^2_\lambda)}\|v_{k_2}\|_{U^2_{BO}(I;L^2_\lambda)} \\ &\lesssim 2^{(0+)k_4}\|u_{k_1}\|_{U^2_{BO}(I;L^2_\lambda)}\|v_{k_2}\|_{U^2_{BO}(I;L^2_\lambda)}\|w_{k_3}\|_{U^2_{BO}(I;L^2_\lambda)},\end{aligned}$$

where the ultimate step follows from the embedding properties of U^p-/V^p-spaces. The claim follows from the definition of the function spaces. □

We estimate the interaction, which gives the $s = 1/2$-threshold of uniform local well-posedness, that is $High \times High \times High \to High$-interaction:

Lemma 5.1.7. *Suppose that $k_4 \geq 20$, $k_1 \leq k_2 \leq k_3$ and $|k_i - k_4| \leq 30$ for any $i \in \{1,2,3\}$. Then,* (5.14) *holds with $\alpha(\underline{k}) = 2^{k_4/2}$.*

Proof. We use the embedding $L^1(I) \hookrightarrow DU^2_{BO}(I)$, Hölder in time and (5.7) to find for $|I| \lesssim 2^{-k_4}$

$$\begin{aligned}&\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{DU^2_{BO}(I;L^2)} \\ &\lesssim 2^{k_4}\|u_{k_1}v_{k_2}w_{k_3}\|_{L^1(I;L^2_\lambda)} \\ &\lesssim 2^{k_4/2}\|u_{k_1}v_{k_2}w_{k_3}\|_{L^2(I;L^2_\lambda)} \\ &\lesssim 2^{k_4/2}\|u_{k_1}\|_{L^6(I;L^6_\lambda)}\|v_{k_2}\|_{L^6(I;L^6_\lambda)}\|w_{k_3}\|_{L^6(I;L^6_\lambda)} \\ &\lesssim 2^{k_4/2}\|u_{k_1}\|_{U^2_{BO}(I;L^2_\lambda)}\|v_{k_2}\|_{U^2_{BO}(I;L^2_\lambda)}\|w_{k_3}\|_{U^2_{BO}(I;L^2_\lambda)},\end{aligned}$$

which yields the claim. □

In the following interactions one input frequency is significantly larger than the output frequency. This requires us to add localization in time. We consider the contribution from $High \times High \times Low \to Low$-interaction.

Lemma 5.1.8. *Suppose that $k_3 \geq 20$, $k_1 \leq k_2 \leq k_3$, $k_1 \leq k_2 - 10$ and $k_4 \leq k_2 - 10$. Then,* (5.14) *holds with $\alpha(\underline{k}) = 2^{(0+)k_3}$.*

Proof. We use duality to write for $|I| \lesssim 2^{-k_4}$

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{DU^2_{BO}(I;L^2_\lambda)} \lesssim 2^{k_4} \sup_{\|f\|_{V^2_{BO}}=1} \int_I \int_{\lambda\mathbb{T}} P_{k_4} f u_{k_1} v_{k_2} w_{k_3} dx dt. \quad (5.16)$$

To estimate u_{k_1} and v_{k_2} in F_λ, we have to divide up I into intervals J with $|J| \lesssim 2^{-k_3}$ and write

$$\begin{aligned}
(5.16) &\lesssim \sum_{\substack{J\subseteq I,\\ |J|\lesssim 2^{-k_3}}} 2^{k_4} \sup_{\|f\|_{V^2_{BO}}=1} \int_J \int_{\lambda\mathbb{T}} P_{k_4} f u_{k_1} v_{k_2} w_{k_3} dxdt \\
&\lesssim 2^{k_4} \sum_{J\subseteq I} \sup_{\|f\|_{V^2_{BO}}=1} \|P_{k_4} f v_{k_2}\|_{L^2(J;L^2_\lambda)} \|u_{k_1} w_{k_3}\|_{L^2(J;L^2_\lambda)} \\
&\lesssim 2^{k_4} \sum_{J\subseteq I} k_3^2 2^{-k_3/2} \|v_{k_2}\|_{U^2_{BO}(J;L^2_\lambda)} 2^{-k_3/2} \|u_{k_1}\|_{U^2_{BO}(J;L^2_\lambda)} \|w_{k_3}\|_{U^2_{BO}(J;L^2_\lambda)} \\
&\lesssim 2^{(0+)k_3} \sup_{\substack{J\subseteq I,\\ |J|\lesssim 2^{-k_3}}} \left(\|u_{k_1}\|_{U^2_{BO}(J;L^2_\lambda)} \|v_{k_2}\|_{U^2_{BO}(J;L^2_\lambda)} \|w_{k_3}\|_{U^2_{BO}(J;L^2_\lambda)} \right),
\end{aligned}$$

where the penultimate estimate follows from (5.8) and (5.9). The ultimate estimate follows from partitioning I with intervals of length 2^{-k_1} giving a factor of $2^{k_1-k_4}$. The claim follows from the definition of the function spaces. □

Next, we deal with $High \times High \times High \to Low$-interaction:

Lemma 5.1.9. *Suppose that $k_3 \geq 20$, $k_1 \leq k_2 \leq k_3$, $k_4 \leq k_1 - 10$ and $k_3 - k_1 \leq 10$. Then, we find* (5.14) *to hold with $\alpha(\underline{k}) = 2^{(0+)k_1}$.*

Proof. By the above argument we write

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{DU^2_{BO}(I;L^2_\lambda)} \lesssim 2^{k_4} \sum_{\substack{J\subseteq I,\\ |J|\lesssim 2^{-k_4}}} \sup_{\|f\|_{V^2_{BO}}=1} \int_J \int_{\lambda\mathbb{T}} P_{k_4} f u_{k_1} v_{k_2} w_{k_3} dxdt. \tag{5.17}$$

Now we observe that among the high frequencies there must be one pair say u_{k_1}, v_{k_2} with $||\xi_1| - |\xi_2|| \gtrsim 2^{k_1}$ provided that $\xi_1 \in \text{supp}_\xi\, \hat{u}_{k_1}$ and $\xi_2 \in \text{supp}_\xi\, \hat{v}_{k_2}$.
To this pair, we can apply a bilinear Strichartz estimate from Proposition 5.1.2 to find

$$\begin{aligned}
(5.17) &\lesssim 2^{k_4} \sum_{J\subseteq I} \sup_{\|f\|_{V^2_{BO}}=1} \|P_{k_4} f w_{k_3}\|_{L^2(J;L^2_\lambda)} \|u_{k_1} v_{k_2}\|_{L^2(J;L^2_\lambda)} \\
&\lesssim 2^{k_4} \sum_{J\subseteq I} k_1^2 2^{-k_1/2} \|w_{k_3}\|_{U^2_{BO}(J;L^2_\lambda)} 2^{-k_1/2} \|u_{k_1}\|_{U^2_{BO}(J;L^2_\lambda)} \|v_{k_2}\|_{U^2_{BO}(J;L^2_\lambda)} \\
&\lesssim 2^{(0k_1)+} \sup_{J\subseteq I} \left(\|u_{k_1}\|_{U^2_{BO}(J;L^2_\lambda)} \|v_{k_2}\|_{U^2_{BO}(J;L^2_\lambda)} \|w_{k_3}\|_{U^2_{BO}(J;L^2_\lambda)} \right).
\end{aligned}$$

□

At last, we record the $Low \times Low \times Low \to Low$-estimate which is straightforward from Young's inequality and Bernstein's inequality:

Lemma 5.1.10. *Suppose that $\max_{i=1,2,3,4} k_i \leq 100$. Then, we find estimate* (5.14) *to hold with $\alpha(\underline{k}) = 1$.*

5.1.3 Energy estimates

In the following the energy norm is propagated in terms of the short-time Fourier restriction norm. More precisely, we shall show the estimate

$$\|u\|_{E^s_\lambda(T)}^2 \lesssim \|u_0\|_{H^s_\lambda}^2 + T\|u\|_{F_\lambda^{s-\tilde{\varepsilon}}(T)}^6 \tag{5.18}$$

for $s > 1/4$, small enough $\|u_0\|_{L^2_\lambda}$ and $\tilde{\varepsilon} = \tilde{\varepsilon}(s) > 0$. A similar estimate was proved on the real line in [Guo11, Proposition 8.1., p. 1124].

Proposition 5.1.11. *Let $T \in (0,1]$ and $u \in C([-T,T], H^\infty_\lambda)$ be a real-valued solution to* (5.1)*. Then, for $s > 1/4$, there exists $\tilde{\varepsilon}(s) > 0$ and $\delta(s) > 0$ such that we find* (5.18) *to hold provided that*

$$\|u_0\|_{L^2_\lambda} \leq \delta(s). \tag{5.19}$$

In order to prove Proposition 5.1.11, we employ a variant of the I-method (cf. [CKS$^+$02, CKS$^+$03]):
Symmetrized energy quantities are considered, which come into play after integration by parts in the time variable. In the context of short-time norms, this strategy was previously used in [KT07, KT12].
The following analysis is close to the arguments on the real line from [Guo11]. In fact, we see from the proof that one can treat the Euclidean and periodic case simultaneously. However, we prefer to use multilinear estimates than linear estimates as was done in [Guo11].
We also make use of the following definition from [KT07]:

Definition 5.1.12. Let $\varepsilon > 0$ and $s \in \mathbb{R}$. Then S^s_ε is the set of real-valued spherically symmetric and smooth functions (symbols) with the following properties:

(i) Slowly varying condition: For $\xi \sim \xi'$ we have

$$a(\xi) \sim a(\xi'),$$

(ii) symbol regularity,

$$\forall \alpha \in \mathbb{N}_0 : |\partial^\alpha a(\xi)| \lesssim a(\xi)(1+\xi^2)^{-\alpha/2},$$

(iii) growth at infinity, for $|\xi| \gg 1$ we have

$$s - \varepsilon \leq \frac{\log a(\xi)}{\log(1+\xi^2)} \leq s + \varepsilon.$$

Note that since a and expressions involving a are going to act as a Fourier multiplier for $2\pi\lambda$-periodic functions, the actually relevant domain of a is $\mathbb{Z}/\lambda$. However, in order to derive favourable pointwise estimates extended versions to the real line are used. Furthermore, if we only wanted to control the H^s-norm of u, then we would just have to take into account the symbols $a(\xi) = (1+\xi^2)^s$. But since we have to derive estimates uniform in time, we have to allow a slightly larger class of symbols following [KT07]. This makes up for the difference between $E^s_\lambda(T)$ and $C([0,T], H^s_\lambda)$. The proof of Proposition 5.1.11 is concluded choosing symbols which allow us to derive suitable estimates for frequency localized energies.

To derive the estimate (5.18), we are going to analyze the following generalized energy $E_0^{a,\lambda}$ for a smooth, real-valued solution to (5.1):

$$E_0^{a,\lambda}(u) = \int_{\xi_1+\xi_2=0} a(\xi_1)\hat{u}(\xi_1)\hat{u}(\xi_2) d\Gamma_2^\lambda \left(= \frac{1}{\lambda} \sum_{\xi_1 \in \mathbb{Z}/\lambda} a(\xi_1)\hat{u}(\xi_1)\hat{u}(-\xi_1) \right).$$

The following symmetrization and integration by parts arguments can be found almost verbatim in [Guo11]. Again, there is the difference that the computations in [Guo11] were carried out for a continuous frequency range.

We use the following notation for the $d-1$-dimensional grid in d-dimensional space:

$$\Gamma_d^\lambda = \{\xi_1 + \xi_2 + \ldots + \xi_d = 0 \mid \xi_i \in \mathbb{Z}/\lambda\},$$

and the measure is given as follows:

$$\int_{\Gamma_d^\lambda} f(\xi_1) \ldots f(\xi_d) d\Gamma_d^\lambda(\xi_1, \ldots, \xi_d) = \frac{1}{\lambda^{d-1}} \sum_{\xi_1+\ldots+\xi_d=0} f(\xi_1)\ldots f(\xi_d).$$

We find for the derivative of $E_0^{a,\lambda}(u)$ after symmetrization

$$\begin{aligned} \frac{d}{dt} E_0^{a,\lambda}(u) &= R_4^{a,\lambda}(u) \\ &= \frac{1}{2} \int_{\Gamma_4^\lambda} i[\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)]\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4) d\Gamma_4^\lambda. \end{aligned}$$

The symmetrization argument fails for differences of solutions. This leads to the well-known breakdown of uniform continuity of the data-to-solution mapping below $H^{1/2}$.

Next, we consider the correction term

$$E_1^{a,\lambda}(u) = \int_{\Gamma_4^\lambda} b_4^a(\xi_1,\xi_2,\xi_3,\xi_4)\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4) d\Gamma_4^\lambda,$$

where we require the multiplier b_4^a to satisfy the following identity on Γ_4^λ:

$$\begin{aligned} &(\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4)) b_4^a(\xi_1,\xi_2,\xi_3,\xi_4) \\ &= \frac{-i}{2}(\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)). \end{aligned}$$

Consequently, we achieve a cancellation

$$\begin{aligned} R_6^{a,\lambda}(u) &= \frac{d}{dt}(E_0^{a,\lambda}(u) + E_1^{a,\lambda}(u)) \\ &= C \int_{\Gamma_6^\lambda} b_4^a(\xi_1,\xi_2,\xi_3,\xi_4+\xi_5+\xi_6)(\xi_4+\xi_5+\xi_6) \prod_{i=1}^{6} \hat{u}(\xi_j). \end{aligned}$$

We have the following proposition on choosing the multiplier b_4^a smooth and extending it off diagonal, which allows us to separate variables easier later on. We follow ideas from [KT12] and [CHT12].

Proposition 5.1.13. *Let $a \in S^s_\varepsilon$. Then, for each dyadic $\lambda \leq \beta \leq \mu$, there is an extension of $\tilde{b}^a_4$ from the diagonal set*

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma^\lambda_4 : |\xi_1| \sim \lambda, |\xi_2| \sim \beta, |\xi_3|, |\xi_4| \sim \mu\}$$

to the full dyadic set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : |\xi_1| \sim \lambda, |\xi_2| \sim \beta, |\xi_3|, |\xi_4| \sim \mu\},$$

which satisfies

$$|\tilde{b}^a_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim a(\mu)\mu^{-1}$$

and

$$|\partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \partial_4^{\alpha_4} \tilde{b}^a_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim_\alpha a(\mu)\mu^{-1} \lambda^{-\alpha_1} \beta^{-\alpha_2} \mu^{-(\alpha_3+\alpha_4)}.$$

with the implicit constant depending on α, but not on λ, β, μ.

Proof. In the following we can assume that $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) \neq 0$ as long as we show b_4 to be smooth because it is easy to see that $\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4) = 0$ whenever $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = 0$.
Furthermore, due to symmetry we can assume that $\xi_3 > 0$, $\xi_4 < 0$. First, we check the cases $|\xi_2| \ll |\xi_3|$, $|\xi_1| \ll |\xi_3|$.

Suppose that $\xi_1, \xi_2 > 0$. In this case we have $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = -2(\xi_1\xi_2 + (\xi_1 + \xi_2)\xi_3)$, and we consider

$$Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1\xi_2 + (\xi_2 + \xi_1)\xi_3} + \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{(\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3)}.$$

The size and regularity properties of the first term follow from the size and regularity properties of a. For the second term we multiply with $1 = -(\xi_1 + \xi_2)/(\xi_3 + \xi_4)$. We set

$$q(\xi, \eta) = \frac{\xi a(\xi) + \eta a(\eta)}{\xi + \eta},$$

which is a smooth function. Since q satisfies the bounds $|q| \lesssim a(N)$ and $|\partial^a_\xi \partial^b_\eta q| \lesssim a(N) N^{-(a+b)}$ for $|\xi| \sim |\eta| \sim N$, the conclusion follows also for the second term

$$\frac{(\xi_1 + \xi_2)(\xi_3 a(\xi_3) + \xi_4 a(\xi_4))}{(\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3)(\xi_3 + \xi_4)} = \frac{\xi_1 + \xi_2}{\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3} q(\xi_3, \xi_4).$$

In the case $\xi_1 < 0, \xi_2 > 0$ we find $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = -2(\xi_1 + \xi_2)(\xi_1 + \xi_3)$. Hence,

$$\begin{aligned} Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{(\xi_1 + \xi_3)(\xi_1 + \xi_2)} + \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{(\xi_1 + \xi_3)(\xi_3 + \xi_4)} \\ &= \frac{1}{\xi_1 + \xi_3} q(\xi_1, \xi_2) - \frac{1}{\xi_1 + \xi_3} q(\xi_3, \xi_4), \end{aligned}$$

which satisfies the required bounds because $|\xi_1| \ll |\xi_3|$.

In case $|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim |\xi_4|$ we can assume $\xi_4 < 0, \xi_2 < 0$ and $\xi_1, \xi_3 > 0$ and write

$$\begin{aligned} Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{a(\xi_1)\xi_1 + a(\xi_2)\xi_2}{(\xi_1 + \xi_2)(\xi_2 + \xi_3)} + \frac{a(\xi_3)\xi_3 + a(\xi_4)\xi_4}{(\xi_1 + \xi_2)(\xi_2 + \xi_3)} \\ &= \frac{q(\xi_1, \xi_2) - q(\xi_3, -\xi_1 - \xi_2 - \xi_3)}{\xi_2 + \xi_3} \\ &= \frac{q(\xi_1, \xi_2) - q(\xi_1 + (\xi_2 + \xi_3), \xi_2 - (\xi_2 + \xi_3))}{\xi_2 + \xi_3}. \end{aligned}$$

Now the bounds follow from the size and regularity of q. $\square$

After smoothly extending the symbol on a dyadic scale $\{(\xi_1,\xi_2,\xi_3,\xi_4) \in \Gamma_4 : |\xi_1| \sim \lambda, |\xi_2| \sim \beta, |\xi_3|, |\xi_4| \sim \mu\}$ off diagonal we can separate variables without restriction (possibly after an additional partition of unity):

$$b_4^a(\xi_1,\xi_2,\xi_3,\xi_4) \sim b_4^a(N_1,N_2,N_3,N_4)\chi_1(\xi_1)\chi_2(\xi_2)\chi_3(\xi_3)\chi_4(\xi_4) \tag{5.20}$$

with regular bump functions χ of size $\lesssim 1$ localized at $|\xi_i| \lesssim N_i$ so that we can absorb the bump functions into the frequency projectors and return to position space.
For details on the separation of variables by expanding b_4^a into a rapidly converging Fourier series see [Han12, Section 5].
The boundary term $E_1^{a,\lambda}(u)$ can be estimated in a favourable way in terms of regularity. But since the boundary term does not depend on the length of the time interval, it is not surprising that the scaling invariant L^2-norm comes into play:

Proposition 5.1.14. *Let $a \in S^s_\varepsilon$. Then, we have*

$$|E_1^{a,\lambda}(u(t))| \lesssim \|u(t)\|^2_{L^2_\lambda} E_0^{a,\lambda}(u(t)).$$

Proof. We use a dyadic decomposition of Γ_4^λ and the expansion (5.20) to write

$$\begin{aligned}
|E_1^{a,\lambda}(u)| &= \left| \int_{\Gamma_4} b_4^a(\xi_1,\xi_2,\xi_3,\xi_4)\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4) d\Gamma_4 \right| \\
&\leq \sum_{N_1 \leq N_2 \leq N_3 \sim N_4} \left| \int_{\Gamma_4^\lambda : |\xi_i| \sim N_i} b_4^a(\xi_1,\xi_2,\xi_3,\xi_4)\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4) d\Gamma_4 \right| \\
&\lesssim \sum_{N_1 \leq N_2 \leq N_3 \sim N_4} |b_4^a(N_1,N_2,N_3,N_4)| \left| \int_{\lambda\mathbb{T}} P_{n_1}uP_{n_2}uP_{n_3}uP_{n_4}u dx \right|.
\end{aligned} \tag{5.21}$$

The normalization of $d\Gamma_4^\lambda$ allows us to switch back to position space with an estimate independent of λ.
The size estimate of b_4^a and applications of Hölder's and scale-invariant Bernstein's inequality imply

$$\begin{aligned}
(5.21) &\lesssim \sum_{N_1 \leq N_2 \leq N_3 \sim N_4} a(N_4)N_4^{-1} \|P_{n_1}u\|_{L^\infty_\lambda} \|P_{n_2}u\|_{L^\infty_\lambda} \|P_{n_3}u\|_{L^2_\lambda} \|P_{n_4}u\|_{L^2_\lambda} \\
&\lesssim \sum_{N_1 \leq N_2 \leq N_3 \sim N_4} a(N_4) \frac{(N_1N_2)^{1/2}}{N_4} \|P_{n_1}u\|_{L^2_\lambda} \|P_{n_2}u\|_{L^2_\lambda} \|P_{n_3}u\|_{L^2_\lambda} \|P_{n_4}u\|_{L^2_\lambda} \\
&\lesssim \|u\|^2_{L^2_\lambda} E_0(u),
\end{aligned}$$

which yields the claim. $\square$

Now we estimate the remainder. Since the localization in time yields a behaviour of solutions very similar to the real line case, some of the arguments from the proof below can already be found in the proof of the real line analog [Guo11, Proposition 8.5., p. 1127]. Because the short-time smoothing estimate seems to be logarithmically inferior to the real line case, we avoid its use by using short-time bilinear Strichartz estimates.

Proposition 5.1.15. *Let $s > 1/4$ and $T \in (0,1]$. There exists $\varepsilon = \varepsilon(s) > 0$ and $\tilde{\varepsilon}(s) > 0$ so that*

$$\left| \int_0^T R_6^{a,\lambda}(u) \right| \lesssim T \|u\|^6_{F_\lambda^{s-\tilde{\varepsilon}}(T)} \tag{5.22}$$

holds true for any $u \in C([-T,T], H^\infty_\lambda)$ and $a \in S^s_\varepsilon$.

Proof. We have to estimate

$$\int_0^T \int_{\Gamma_6^\lambda} [b_4^a(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6)] \prod_{j=1}^6 \hat{u}(\xi_j, t) d\Gamma_6^\lambda dt. \tag{5.23}$$

Smoothly divide the frequencies into dyadic blocks and use the notation $|\xi_j| \sim 2^{k_j} = K_j$. Due to symmetry, we can assume that $K_1 \leq K_2 \leq K_3, \; K_4 \leq K_5 \leq K_6$. We write $\xi_{456} = \xi_4 + \xi_5 + \xi_6$. Temporarily, we introduce an additional frequency projector $\tilde{P}_K$ for ξ_{456}, which is also required to be smooth.
We find

$$(5.23) \lesssim \sum_{K_j, K} \left| \int_0^T \int_{\Gamma_6 : |\xi_i| \sim K_i, |\xi_{456}| \sim K} b_4(\xi_1, \xi_2, \xi_3, \xi_{456}) \xi_{456} \chi_K(\xi_{456}) \prod_{j=1}^6 \hat{u}(\xi_j) dt \right|. \tag{5.24}$$

In order to derive estimates in terms of the short-time norms, we have to localize time reciprocally to the highest occuring frequency.
We bound the dyadically localized expression (5.24) in several cases:
Case 1: $K_5 \sim K_6 \sim K_{\max}, K_3 \lesssim K_5$: Write $C_1 = \{(K_1, \ldots, K_6) : \; K_5 \sim K_6 \sim K_{\max}, K_3 \lesssim K_5\}$ and estimate this part of (5.24) by

$$\sum_{K_j \in C_1, K \lesssim K_3} T K_6 \sup_{|I| = K_6^{-1}} | \int_I \int_{\Gamma_6^\lambda} b_4(\xi_1, \xi_2, \xi_3, \xi_{456}) \xi_{456} \chi_K(\xi_{456}) \prod_{j=1}^6 \hat{u}(t, \xi_j) d\Gamma_6^\lambda dt|, \tag{5.25}$$

where $I \subseteq (0,T]$ denotes an interval of length K_6^{-1}.
Expand

$$\chi_K(\xi_{456}) = \int_{\mathbb{R}} e^{-ix\xi_{456}} f_K(x) dx = \int_{\mathbb{R}} e^{-ix\xi_4} e^{-ix\xi_5} e^{-ix\xi_6} f_K(x) dx, \tag{5.26}$$

and it is easy to see that for $K \geq 1$ we can choose f_K as rescaled versions of each other, yielding a uniformly in K bounded L^1_x-norm.
Plugging in the expression (5.20) in addition and absorbing the factors stemming from (5.26) into the $\hat{u}_i$, we are left with estimating

$$\begin{aligned} &\sum_{\underline{K} \in C_1, K \lesssim K_3} T K_6 |b_4^a(K_1, K_2, K_3, K) K| \sup_{|I| = K_6^{-1}} \int_I \int_{\Gamma_6^\lambda} \prod_{j=1}^6 \hat{u}_{k_j}(t, \xi_j) d\Gamma_6^\lambda dt \\ &\lesssim T \sum_{\underline{K} \in C_1, K \lesssim K_3} |b_4(K_1, K_2, K_3, K) K| K_6 \\ &\quad \sup_{|I| = K_6^{-1}} | \int_I \int_{\lambda \mathbb{T}} P_{k_1} u_1 P_{k_2} u_2 \ldots P_{k_6} u_6 dx dt|, \end{aligned}$$

where we have changed back to position space at last.
Using the pointwise estimate of b_4^a, we find

$$\sum_{K \lesssim K_3} |b_4^a(K_1, K_2, K_3, K)K| \lesssim a(K_3).$$

We use the short-time estimates from Section 5.1.1 to derive suitable estimates for the expression

$$\left| \int_I dt \int_{\lambda \mathbb{T}} dx P_{k_1} u_1 \dots P_{k_6} u_6 \right|. \tag{5.27}$$

The bounds for (5.27) are derived according to the separation of the involved frequencies. Let $K_1^* \leq \ldots \leq K_6^*$ denote the increasing rearrangement of $\{K_1, \ldots, K_6\}$ and write u_{k_i} instead of $P_{k_i} u_i$ in the following.
Subcase 1a: $K_4^* \ll K_6^*$:
In this case we can use two bilinear Strichartz estimates. Say K_1 and K_2 are the lowest and second-to-lowest frequencies and K_5 and K_6 are the highest and second-to-highest frequencies. We arrange $u_{k_4} u_{k_5}$ and $u_{k_3} u_{k_6}$ in pairs for two bilinear Strichartz estimates and use Bernstein's inequality on u_{k_1} and u_{k_2}. We find together with an application of the transfer principle

$$\begin{aligned}
(5.27) &\lesssim \|u_{k_1}\|_{L_t^\infty L_\lambda^\infty} \|u_{k_2}\|_{L_t^\infty L_\lambda^\infty} \|u_{k_4} u_{k_5}\|_{L_t^2 L_\lambda^2} \|u_{k_3} u_{k_6}\|_{L_t^2 L_\lambda^2} \\
&\lesssim \frac{(K_1^* K_2^*)^{1/2}}{K_6^*} \|u_{k_1}\|_{F_{\lambda,k_1}} \|u_{k_2}\|_{F_{\lambda,k_2}} \dots \|u_{k_6}\|_{F_{\lambda,k_6}}.
\end{aligned}$$

Taking all estimates together, we have proved

$$\begin{aligned}
\left| \int_0^T R_{6,C_{1a}}^{a,\lambda}(u) \right| &\lesssim T \sum_{K_i} (K_4^*)^{2(s+\varepsilon)} (K_1^* K_2^*)^{1/2} \prod_{i=1}^6 \|P_{k_i} u\|_{F_{k_i}^\lambda} \\
&\lesssim T \prod_{i=1}^6 \|u\|_{F_\lambda^{s-\tilde{\varepsilon}}(T)},
\end{aligned}$$

where the last step follows from carrying out the summations and choosing ε and $\tilde{\varepsilon}$ sufficiently small.
Subcase 1b: $K_3^* \ll K_4^* \sim K_5^* \sim K_6^*$: In this case it is easy to see that there is still one pair of highest frequencies which is separated of order K_6^* in the frequency supports following Section 4.8.2. Say K_1 and K_2 are the smallest frequencies. Following along the above lines, we are led to the estimate:

$$\begin{aligned}
&T \sum_{K_i^*} (K_1^* K_2^*)^{1/2} (K_4^*)^{2(s+\varepsilon)} \prod_{i=1}^6 \|P_{k_i} u\|_{F_{k_i,\lambda}} \\
&\lesssim T \prod_{i=1}^6 \|u\|_{F_\lambda^{s-\tilde{\varepsilon}}(T)},
\end{aligned}$$

where carrying out the summations is straight-forward.
Subcase 1c: $K_2^* \ll K_3^* \sim K_6^*$: In this case we use a bilinear estimate on $u_{k_2^*} u_{k_6^*}$, three $L_{t,x}^6$-linear Strichartz estimates on $u_{k_3^*}$, $u_{k_4^*}$, $u_{k_5^*}$ and one pointwise bound $u_{k_1^*}$.

This gives

$$T \sum_{K_1^* \leq K_2^* \ll K_3^* \sim K_6^*} (K_1^* K_6^*)^{1/2} (K_6^*)^{2(s+\varepsilon)} \prod_{i=1}^{6} \|P_{k_i} u\|_{F_{k_i,\lambda}}$$
$$\lesssim T \prod_{i=1}^{6} \|P_{k_i} u\|_{F_\lambda^{s-\tilde{\varepsilon}}(T)}.$$

Subcase 1d: $K_1^* \ll K_2^* \sim K_6^*$: The argument from Subcase 1c is applicable because there is a pair of high frequencies with $\text{dist}(|\xi_i|, |\xi_j|) \gtrsim K_6^*$ for $\xi_i \in \text{supp}_\xi \hat{u}_{k_i}$, due to otherwise impossible frequency interaction.
Subcase 1e: $K_1^* \sim K_6^*$. Here, no multilinear estimates are used, but six $L_{t,x}^6$-Strichartz estimates to find

$$|\int_0^T R_{6,C_{1e}}^{a,\lambda}| \lesssim T \sum_{K_1^* \sim K_6^*} (K_6^*)^{2(s+\varepsilon)} K_6^* \prod_{i=1}^{6} \|P_{k_i} u\|_{F_{k_i,\lambda}} \lesssim T \prod_{i=1}^{6} \|u\|_{F_\lambda^{s-\tilde{\varepsilon}}(T)}.$$

Case 2: $K_2 \sim K_3 \sim K_{\max}, K_6 \lesssim K_2$: Introduce the notation

$$C_2 = \{(K_1, \ldots, K_6) \mid K_2 \sim K_3 \sim K_{\max}, K_6 \lesssim K_2\}$$

and suppose in the following $K \lesssim K_6$. We have to bound

$$T \sum_{\underline{K} \in C_2, K \lesssim K_6} K_3 \sup_{|I| = K_3^{-1}} \left| \int_I \int_{\Gamma_6^\lambda} b_4^a(\xi_1, \xi_2, \xi_3, \xi_{456}) \xi_{456} \chi_K(\xi_{456}) \prod_{j=1}^{6} \hat{u}_{k_j}(t, \xi_j) d\Gamma_6^\lambda dt \right|. \tag{5.28}$$

Following along the above lines, we are led to the estimate

$$(5.28) \lesssim T \sum_{\underline{K} \in C_2, K \lesssim K_6} |b_4(K_1, K_2, K_3, K) K| K_3 \sup_{|I| = K_3^{-1}} \left| \int_I \int_{\lambda \mathbb{T}} P_{k_1} u \ldots P_{k_6} u dx dt \right|$$
$$\lesssim T \sum_{\underline{K} \in C_2} K_6 K_3^{2(s+\varepsilon)} \sup_{|I| = K_3^{-1}} \left| \int_I \int_{\lambda \mathbb{T}} P_{k_1} u \ldots P_{k_6} u dx dt \right|.$$

The product is estimated according to the separation of the frequencies like in Case 1.
Subcase 2a, 2b: $K_4^* \ll K_5^* \sim K_6^*, \quad K_3^* \ll K_4^* \sim K_6^*$: Here, one can use two bilinear Strichartz estimates on the highest frequencies leading to a gain of $(K_6^*)^{-1}$ and two pointwise bounds on the lowest frequencies, which gives a factor $(K_1^* K_2^*)^{1/2}$, and one has to sum

$$T \sum_{K_i^*} K_4^* (K_6^*)^{2(s+\varepsilon)} (K_1^* K_2^*)^{1/2} (K_6^*)^{-1} \prod_{i=1}^{6} \|P_{k_i} u\|_{F_{k_i,\lambda}} \lesssim T \prod_{i=1}^{6} \|u\|_{F_\lambda^{s-\tilde{\varepsilon}}(T)}.$$

Subcase 2c, 2d: $K_2^* \ll K_3^* \sim K_6^*$, $K_1^* \ll K_2^* \sim K_6^*$: In this case one uses again one bilinear estimate, three $L_{t,x}^6$-Strichartz estimates and one pointwise bound to find

$$T \sum_{K_i^*} (K_1^*)^{1/2} K_6^* (K_6^*)^{-1/2} (K_6^*)^{2(s+\varepsilon)} \prod_{i=1}^{6} \|P_{k_i} u\|_{F_{k_i,\lambda}} \lesssim T \prod_{i=1}^{6} \|P_{k_i} u\|_{F_\lambda^{s-\tilde{\varepsilon}}(T)}.$$

Subcase 2e: $K_1^* \sim K_6^*$: After using six $L^6_{t,x}$-Strichartz estimates, the estimate is concluded like in Subcase 1e.
Case 3: $K_3 \sim K_6 \sim K_{max}$; $K_2, K_5 \ll K_3$: In this case the above argument is enhanced with an additional symmetrization, which corresponds to a further integration by parts: Note that

$$\begin{aligned} &\int_0^T \int_{\Gamma_6^\lambda} b_4^a(\xi_1,\xi_2,\xi_3,\xi_{456})\xi_{456} \prod_{i=1}^6 \hat{u}(t,\xi_i) d\Gamma_6^\lambda dt \\ &= \int_0^T \int_{\Gamma_6^\lambda} b_4^a(\xi_4,\xi_5,\xi_6,\xi_{123})\xi_{123} \prod_{i=1}^6 \hat{u}(t,\xi_i) d\Gamma_6^\lambda dt \\ &= \int_0^T \int_{\Gamma_6^\lambda} -b_4^a(\xi_4,\xi_5,\xi_6,-\xi_{456})\xi_{456} \prod_{i=1}^6 \hat{u}(t,\xi_i) d\Gamma_6^\lambda dt \\ &= -\int_0^T \int_{\Gamma_6^\lambda} b_4^a(-\xi_4,-\xi_5,-\xi_6,\xi_{456})\xi_{456} \prod_{i=1}^6 \hat{u}(t,\xi_i) d\Gamma_6^\lambda dt \end{aligned} \tag{5.29}$$

and thus, it is enough to estimate

$$\int_0^T \int_{\Gamma_6^\lambda} [b_4^a(\xi_1,\xi_2,\xi_3,\xi_{456}) - b_4^a(-\xi_4,-\xi_5,-\xi_6,\xi_{456})]\xi_{456} \prod_{i=1}^6 \hat{u}(t,\xi_i) d\Gamma_6^\lambda dt.$$

By the mean value theorem and regularity of b_4^a, we find the symmetrized expression to be regular likewise. Further, we have the size estimate

$$|b_4^a(\xi_1,\xi_2,\xi_3,\xi_{456}) - b_4^a(-\xi_4,-\xi_5,-\xi_6,\xi_{456})| \lesssim \frac{(K_6^*)^{2(s+\varepsilon)}}{(K_6^*)^2} K_4^*.$$

As $K_4^* \ll K_5^* \sim K_6^*$, we can use two bilinear Strichartz estimates and two pointwise bounds like in the Subcases a,b to finish the proof. $\square$

To conclude the proof of the energy estimate, we derive thresholds of the frequency localized energy. Recall the following lemma from [KT07], which was only proved on the real line; however, the proof carries over to $\lambda\mathbb{T}$.

Lemma 5.1.16. *[KT07, Lemma 5.5., p. 26] For any $u_0 \in H^s(\lambda\mathbb{T})$ and $\varepsilon > 0$, there is a sequence $(\beta_n)_{n\in\mathbb{N}_0}$ satisfying the following conditions:*

(a) $2^{2ns}\|P_n u_0\|^2_{L^2_\lambda} \leq \beta_n \|u_0\|^2_{H^s_\lambda}$,

(b) $\sum_n \beta_n \lesssim 1$,

(c) (β_n) satisfies a log-Lipschitz condition, which is given by

$$|\log_2 \beta_n - \log_2 \beta_m| \leq \frac{\varepsilon}{2}|n-m|.$$

By this, we conclude the proof of Proposition 5.1.11 in a similar spirit to [KT07, Section 5].

Proof of Proposition 5.1.11. We choose $\varepsilon > 0$ and $\tilde{\varepsilon} > 0$ in dependence of $s > 1/4$ so that the estimate (5.22) becomes true for any $a \in S^s_\varepsilon$ by virtue of Proposition 5.1.15.
Let $k_0 \in \mathbb{N}_0$ and let (β_n) be an envelope sequence from Lemma 5.1.16 for the initial data u_0. We prove

$$\sup_{t \in [-T,T]} 2^{2ks} \|P_k u(t)\|^2_{L^2_\lambda} \lesssim \beta_k (\|u_0\|^2_{H^s_\lambda} + T\|u\|^6_{F^{s-\tilde{\varepsilon}}_\lambda(T)}), \tag{5.30}$$

from which follows (5.18) after carrying out the summation over k, due to property (b) from Lemma 5.1.16.

We consider $\tilde{a}^{k_0}_k = 2^{2ks} \max(1, \beta^{-1}_{k_0} 2^{-\varepsilon|k-k_0|})$ and we find

$$\begin{aligned} \sum_{k \geq 0} \tilde{a}^{k_0}_k \|P_k u_0\|^2_{L^2_\lambda} &\leq \sum_k 2^{2ks} \|P_k u_0\|^2_{L^2_\lambda} + 2^{2ks} 2^{-\frac{\varepsilon}{2}|k-k_0|} \beta^{-1}_k \|P_k u_0\|^2_{L^2_\lambda} \\ &\lesssim_\varepsilon \|u_0\|^2_{H^s_\lambda}, \end{aligned}$$

due to the slowly varying condition and property (i) from Lemma 5.1.16.
The implicit constant in the estimate above does not depend on k_0, but only on ε. Smoothing out a linearly interpolated version, we can find a symbol $a^{k_0}(\xi) \in S^s_\varepsilon$ so that

$$a^{k_0}(\xi) \sim \tilde{a}^{k_0}_k, \quad |\xi| \sim 2^k.$$

For details on this procedure, see [OW18, Subsection 2.3].
Next, following the computations from the beginning of this subsection, we find by means of Proposition 5.1.14 and 5.1.15

$$\|u(t)\|^2_{H^a} \lesssim_s \|u_0\|^2_{H^a} + \|u_0\|^2_{L^2_\lambda} \|u_0\|^2_{H^a} + \|u_0\|^2_{L^2_\lambda} \|u(t)\|^2_{H^a} + T\|u\|^6_{F^{s-\tilde{\varepsilon}}_\lambda(T)}.$$

Requiring $\|u_0\|_{L^2}$ to be sufficiently small, this implies

$$\|u(t)\|^2_{H^a} \lesssim_s \|u_0\|^2_{H^a} + T\|u\|^6_{F^{s-\tilde{\varepsilon}}_\lambda(T)} \lesssim_\varepsilon \|u_0\|^2_{H^s} + T\|u\|^6_{F^{s-\tilde{\varepsilon}}_\lambda(T)}$$

with the second estimate following from $\|u_0\|^2_{H^a} \lesssim_\varepsilon \|u_0\|^2_{H^s}$. At last, since $\|u\|^2_{H^a} \sim \sum_{k \geq 0} \tilde{a}^{k_0}_k \|P_k u(t)\|^2_{L^2_\lambda}$, we arrive at

$$\sup_{t \in [0,T]} \left(\sum_{k \geq 0} \tilde{a}^{k_0}_k \|P_k u(t)\|^2_{L^2_\lambda} \right) \lesssim_s \|u_0\|^2_{H^s} + T\|u\|^6_{F^{s-\tilde{\varepsilon}}_\lambda(T)}.$$

Restricting the sum to k_0 implies (5.30). The proof is complete. □

5.1.4 Proof of new regularity results for the modified Benjamin-Ono equation

As typical for the construction of solutions, we prove a priori estimates for smooth solutions first. In the second step, we use a compactness argument to construct solutions. For this we will use a smoothing effect in the energy estimates. Our first aim is to prove the following proposition:

Proposition 5.1.17. *Let $s > 1/4$ and $u_0 \in H^\infty_{\mathbb{R}}(\mathbb{T})$. There is a constant $\mu_s > 0$ depending on s and a function $T = T(s, \|u_0\|_{H^s})$ so that we find the estimate*

$$\sup_{t \in [-T,T]} \|u(t)\|_{H^s} \leq C(s, \|u_0\|_{H^s}) \|u_0\|_{H^s} \tag{5.31}$$

to hold for the unique smooth solution to (5.1)*, where $\lambda = 1$ and provided that $\|u_0\|_{L^2} \leq \mu_s$. Moreover, we find $T \geq 1$ and $C(s, \|u_0\|_{H^s}) = D(s)$ as $\|u_0\|_{H^s} \to 0$.*

We control the $F^s(T)$-norm of the solution. This suffices to conclude an a priori bound for the Sobolev norm due to $F^s(T) \hookrightarrow L^\infty_T H^s$.
Continuity and limit properties of $T' \mapsto \|u\|_{E^s_\lambda(T')}, \|u\|_{F^s_\lambda(T')}$ as $T' \to 0$ to carry out the bootstrap argument were discussed in Section 3.5.
We are ready to prove Proposition 5.1.17.

Proof of Proposition 5.1.17. First, we assume that $\|u_0\|_{H^s} \leq \tilde{C}_s \ll 1$. $\tilde{C}_s$ will be specified below, and we shall see how the general case follows from rescaling. To be in the position to invoke Proposition 5.1.11, we have to assume that $\|u_0\|_{L^2} \leq \mu_s$. Then, we find the following estimates to hold from Propositions 2.5.2, 5.1.4 and 5.1.11:

$$\begin{cases} \|u\|_{F^s(T)} & \leq C_1(\|u\|_{E^s(T)} + \|\partial_x(u^3/3)\|_{N^s(T)}) \\ \|\partial_x(u^3/3)\|_{N^s(T)} & \leq C_{2,s} \|u\|^3_{F^s(T)} \\ \|u\|^2_{E^s(T)} & \leq C_{3,s}(\|u_0\|^2_{H^s} + T\|u\|^6_{F^s(T)}). \end{cases}$$

Following [KT07, Section 1], set $X(T) = \|u\|_{E^s(T)} + \|u\|_{F^s(T)}$ and derive a bound on $X(T)$ from a continuity argument: Firstly, we find $\lim_{T' \to 0} X(T') \leq 2\|u_0\|_{H^s}$. Secondly, we note that from the above estimates we find

$$X(T) \leq C_s(\|u_0\|_{H^s} + X(T)^3) \tag{5.32}$$

with $C_s = C_s(C_{1,s}, C_{2,s}, C_{3,s}, T) > 1$, which we can argue to be uniform in T on $(0, 1]$.
From continuity of $X(T)$, we have

$$X(T) \leq 4C_s \|u_0\|_{H^s}$$

for $T' \in (0, \tilde{T}]$. However, we find from (5.32) the improvement

$$X(T) \leq 2C_s \|u_0\|_{H^s}$$

choosing $\tilde{C}_s$ sufficiently small in dependence of C_s, e.g. $\tilde{C}_s = (4C_s)^{-3/2}$.
This proves

$$\sup_{t \in [0,1]} \|u(t)\|_{H^s} \leq 2C_s \|u_0\|_{H^s}$$

provided that $\|u_0\|_{H^s} \leq \tilde{C}_s$.

Next, we consider the case of initial data large in H^s_λ.
We rescale $u_0 \to \lambda^{-1/2} u_0(\lambda^{-1} \cdot) =: u_0^\lambda$, which also changes the underlying manifold $\mathbb{T} \to \lambda\mathbb{T}$. For the rescaled initial data, we have $\|u_0^\lambda\|_{H^s} \to \|u_0\|_{L^2} \leq \mu_s$ as $\lambda \to \infty$ and $\|u_0^\lambda\|_{L^2} = \|u_0\|_{L^2}$ is small enough.
On the other hand, we have the following set of inequalities for the emanating solutions u^λ:

$$\begin{cases} \|u^\lambda\|_{F^s_\lambda(T)} & \leq C_1(\|u^\lambda\|_{E^s_\lambda(T)} + \|\partial_x((u^\lambda)^3/3)\|_{N^s_\lambda(T)} \\ \|\partial_x((u^\lambda)^3/3)\|_{N^s_\lambda(T)} & \leq C_{2,s} \|u^\lambda\|^3_{F^s_\lambda(T)} \\ \|u^\lambda\|^2_{E^s_\lambda(T)} & \leq C_{3,s}(\|u_0^\lambda\|^2_{H^s_\lambda} + T\|u\|^6_{F^s_\lambda(T)}). \end{cases}$$

By the above means, we find

$$X(T) \le C_s(\|u_0^\lambda\|_{H^s_\lambda}^2 + X(T)^3)$$

and further,

$$\sup_{t \in [0,1]} \|u^\lambda(t)\|_{H^s_\lambda} \le 2C_s \|u_0\|_{H^s_\lambda}$$

provided that $\|u_0\|_{H^s_\lambda} \le \tilde{C}_s$.

Scaling back, we find the following a priori estimate

$$\sup_{t \in [0,\lambda^{-2}]} \|u(t)\|_{H^s} \le C(s, \|u_0\|_{H^s}) \|u_0\|_{H^s}$$

to hold, where the dependence of C on $\|u_0\|_{H^s}$ stems from an insufficient control over low frequencies when scaling back and forth. Since we can choose

$$\lambda = \frac{\|u_0\|_{\dot{H}^s}}{\tilde{C}_s},$$

the proof is complete. □

We turn to the proof of existence of solutions. For $u_0 \in H^s_{\mathbb{R}}(\mathbb{T})$ with $\|u_0\|_{L^2} \le \mu_s$, we denote $u_{0,n} = P_{\le n} u_0$ for $n \in \mathbb{N}$. With $u_{0,n} \in H^\infty(\mathbb{T})$ and $\|u_{0,n}\|_{L^2} \le \mu_s$, there is an emanating sequence of smooth solutions u_n to (5.1) by the energy method (cf. Section 3.5) with $u_n(0) = u_{0,n}$, and we can already give the a priori estimate

$$\sup_{t \in [0,T_0]} \|u_n(t)\|_{H^s} \le C(s, \|u_0\|_{H^s}) \|u_{0,n}\|_{H^s} \le C(s, \|u_0\|_{H^s}) \|u_0\|_{H^s}$$

with T_0 and C independent of n. Next, we prove precompactness of (u_n):

Lemma 5.1.18. *Let $u_0 \in H^s_{\mathbb{R}}(\mathbb{T})$ for $s > 1/4$ and denote by (u_n) the sequence of solutions to* (5.1) *with $u_n(0) = u_{0,n}$, where $u_{0,n} = P_{\le n} u_0$. Then, we find the sequence (u_n) to be precompact in $C([-T,T], H^s(\mathbb{T}))$ for $T \le T_0 = T_0(s, \|u_0\|_{H^s})$.*

Proof. By the a priori estimate, we have a bound for $\|u_n\|_{C([-T,T],H^s)}$ uniform in n for $T \le T_0$. In addition, we prove the following uniform tail estimate: For any $\varepsilon > 0$, there is $n_0 = n_0(u_0)$ so that we find the estimate

$$\|P_{\ge n_0} u_n\|_{C([-T,T],H^s)} < \varepsilon \tag{5.33}$$

for all $n \in \mathbb{N}$.

This is a consequence of the smoothing effect of the energy estimates from Section 5.1.3: We consider symbols resembling

$$a(m) = \begin{cases} \langle m \rangle^{2s}, \ |m| \ge 2^{n_0}, \\ 0, \ \text{else} \end{cases} \tag{5.34}$$

to derive the estimate

$$\Big| \|P_{>k} u_n\|^2_{E^s(T)} - \|P_{>k} u_{0,n}\|^2_{H^s} \Big| \le C(s, \|u_0\|_{H^s}) 2^{-2\varepsilon k} \to 0 \text{ as } k \to \infty,$$

and consequently,

$$\|P_{>k} u_n\|^2_{C([-T,T],H^s)} \le \|P_{>k} u_0\|^2_{H^s} + C(s, \|u_0\|_{H^s}) 2^{-2\varepsilon k} \to 0 \text{ as } k \to \infty.$$

Hence, it is enough to prove the precompactness of $(P_{\leq n_0} u_n)$ to conclude that of (u_n). From Duhamel's formula and the boundedness of the linear propagator on low frequencies, we find

$$\begin{aligned} &\|P_{\leq n_0} u_n(t+\delta) - P_{\leq n_0} u_n(t)\|_{H^s} \\ &\leq \|(e^{i\delta\partial_x^2} - 1) P_{\leq n_0} u_n(t)\|_{H^s} + \| \int_t^{t+\delta} e^{i(t+\delta-t')\partial_x^2} P_{\leq n_0}(\partial_x (u_n(t')^3/3)) dt' \|_{H^s} \\ &\lesssim e^{\delta N_0^2} \|u_n\|_{C([-T,T],H^s)} + N_0^{s+1/2} \delta \|u_n\|^3_{C([-T,T],H^s)} \\ &\lesssim_{N_0, \|u_0\|_{H^s}} \delta. \end{aligned} \tag{5.35}$$

For the penultimate estimate we use Bernstein's inequality and Sobolev imbedding $H^{1/4} \hookrightarrow L^3$ to write

$$\begin{aligned} \| \int_t^{t+\delta} e^{i(t+\delta-t')\partial_x^2} P_{\leq n_0}(\partial_x (u_n(t')^3/3)) dt' \|_{H^s} &\lesssim N_0^{s+1/2} \|(u_n)^3\|_{L^1([t,t+\delta],L^1)} \\ &\lesssim N_0^{s+1/2} \delta \|u_n\|^3_{L_t^\infty L_x^3} \\ &\lesssim N_0^{s+1/2} \delta \|u_n\|^3_{L_t^\infty H^s}. \end{aligned}$$

The ultimate estimate in (5.35) follows from choosing δ small enough in dependence of n_0 and the a priori estimate. The equicontinuity of the small frequencies together with the uniform tail estimate (5.33) implies precompactness by the Arzelà-Ascoli criterion. This completes the proof. □

We are ready to finish the proof of the main result:

Proof of Theorem 5.1.1. As described above for $s > 1/4$, we consider $u_0 \in H^s_{\mathbb{R}}(\mathbb{T})$ with small L^2-norm and denote by $(u_n)_n$ the smooth solutions emanating from $P_{\leq n} u_0$. By Lemma 5.1.18, there is a subsequence $(u_{n_k})_k$ which converges to a function $u \in C([-T,T], H^s)$. For the sake of brevity we denote u_{n_k} again with u_n.

It remains to check that the limit object satisfies the a priori estimate and the equation (5.1) in the sense of generalized functions. The estimate

$$\sup_{t \in [0,T_0]} \|u(t)\|_{H^s} \leq C(s, \|u_0\|_{H^s}) \|u_0\|_{H^s}$$

is immediate from the convergence in the H^s-norm.

Furthermore, for any $n \in \mathbb{N}$ and $\varphi \in C_c^\infty([-T,T], C^\infty(\mathbb{T}))$ we find the identity

$$\int_{-T}^T \int_{\mathbb{T}} \partial_t u_n(t,x) \varphi(t,x) dx dt + \int_{-T}^T \int_{\mathbb{T}} \partial_x^2 u_n \varphi dx dt = \pm \int_{-T}^T \int_{\mathbb{T}} \partial_x ((u_n)^3/3) \varphi dx ds \tag{5.36}$$

to hold.

Integration by parts gives

$$- \int_{-T}^T \int_{\mathbb{T}} u_n \partial_t \varphi dx dt + \int_{-T}^T \int_{\mathbb{T}} u_n \partial_x^2 \varphi dx dt = \mp \int_{-T}^T \int_{\mathbb{T}} (u_n^3/3) \partial_x \varphi dx dt. \tag{5.37}$$

From Hölder's inequality we find

$$\text{lhs}(5.37) \to - \int_{-T}^T \int_{\mathbb{T}} u \partial_t \varphi dx dt + \int_{-T}^T \int_{\mathbb{T}} u \partial_x^2 \varphi dx dt,$$

and using in addition Sobolev embedding $H^{1/4}(\mathbb{T}) \hookrightarrow L^4(\mathbb{T})$, we find

$$\text{rhs}(5.37) \to \mp \int_{-T}^{T} \int_{\mathbb{T}} (u^3/3) \partial_x \varphi dx dt.$$

This completes the proof. $\square$

5.1.5 Modifications for the derivative nonlinear Schrödinger equation

In this paragraph we sketch the necessary modifications to show that the assertions from Theorem 5.1.1 on periodic solutions to (5.1) extend to periodic solutions to (5.3).

We show a corresponding short-time trilinear estimate and an energy estimate with smoothing effect after adapting the short-time function spaces to the Schrödinger flow. More precisely, it is shown

$$\begin{cases} \|\partial_x(|u|^2 u)\|_{N^s_\lambda(T)} & \lesssim \|u\|^3_{F^s_\lambda(T)} \\ \|u\|^2_{E^s_\lambda(T)} & \lesssim \|u_0\|^2_{H^s_\lambda} + T\|u\|^6_{F^{s-\varepsilon}_\lambda(T)} \end{cases} \tag{5.38}$$

provided that $\lambda \geq 1$, $s > 1/4$ and $\varepsilon = \varepsilon(s) > 0$, $\|u_0\|_{L^2_\lambda} \leq \mu_s$. We start with the short-time trilinear estimate:

Proposition 5.1.19. *Let $\lambda \geq 1$, $T \in (0,1]$, $1/4 < s < 1/2$ and suppose that $u, v, w \in F^s_\lambda(T)$. Then, we find the following estimate to hold:*

$$\|\partial_x(u\overline{v}w)\|_{N^s_\lambda(T)} \lesssim \|u\|_{F^s_\lambda(T)} \|v\|_{F^s_\lambda(T)} \|w\|_{F^s_\lambda(T)}. \tag{5.39}$$

Proof. The strategy is the same like in the proof of Proposition 5.1.4. The claim follows from revisiting the proof of Proposition 5.1.4, and whenever one applies an estimate from Proposition 5.1.2, the corresponding estimate from Proposition 5.1.3 is applied.

Recall the possible frequency interactions, which were enumerated for the proof of Proposition 5.1.4 and remain the same. We give the details in case of $High \times Low \times Low \to High$-interaction and $High \times High \times High \to High$-interaction. In the first case, under the same assumptions like in Lemma 5.1.5, let I be an interval of length 2^{-k_4} and we compute by Hölder in time, a short-time bilinear Strichartz estimate and Bernstein's inequality

$$\begin{aligned} \|P_{k_4}\partial_x(u_{k_1}\overline{v}_{k_2}w_{k_3})\|_{DU^2_\Delta(I;L^2_\lambda)} &\lesssim 2^{k_4}\|u_{k_1}\overline{v}_{k_2}w_{k_3}\|_{L^1(I;L^2_\lambda)} \\ &\lesssim 2^{k_4/2}\|u_{k_1}\|_{L^\infty(I;L^\infty_\lambda)}\|\overline{v}_{k_2}w_{k_3}\|_{L^2(I;L^2_\lambda)} \\ &\lesssim 2^{k_1/2}\|u_{k_1}\|_{U^2_\Delta(I;L^2_\lambda)}\|v_{k_2}\|_{U^2_\Delta(I;L^2_\lambda)}\|w_{k_3}\|_{U^2_\Delta(I;L^2_\lambda)}. \end{aligned}$$

By the above means, the corresponding estimate to Lemma 5.1.5 follows from the definition of the function spaces.

In the second case we use Hölder in time and three $L^6_{t,x}$-Strichartz estimates to find

$$\begin{aligned} \|P_{k_4}\partial_x(u_{k_1}\overline{v}_{k_2}w_{k_3})\|_{DU^2_\Delta(I;L^2_\lambda)} &\lesssim 2^{k_4}\|u_{k_1}\overline{v}_{k_2}w_{k_3}\|_{L^1(I;L^2_\lambda)} \\ &\lesssim 2^{k_4/2}\|u_{k_1}\overline{v}_{k_2}w_{k_3}\|_{L^2(I;L^2_\lambda)} \\ &\lesssim 2^{k_4/2}\|u_{k_1}\|_{L^6(I;L^6_\lambda)}\|v_{k_2}\|_{L^6(I;L^6_\lambda)}\|w_{k_3}\|_{L^6(I;L^6_\lambda)} \\ &\lesssim 2^{k_4/2}\|u_{k_1}\|_{U^2_\Delta(I;L^2_\lambda)}\|v_{k_2}\|_{U^2_\Delta(I;L^2_\lambda)}\|w_{k_3}\|_{U^2_\Delta(I;L^2_\lambda)}. \end{aligned}$$

The other cases follow like in Section 5.1.2, too. □

The energy estimate is more involved due to the reduced symmetry. If one wants to stick to the use of linear and bilinear short-time Strichartz estimates, one has to integrate by parts a second time in one case of the remainder estimate. Alternatively, the claim follows from a refined trilinear estimate. Below we do the extra work of a second integration by parts to point out that the second correction satisfies better bounds.

Proposition 5.1.20. *Let $T \in (0,1]$, $s > 1/4$ and suppose that $u \in C([-T,T], H^\infty_\lambda)$ is a smooth solution to* (5.3)*. Then, there exists $\tilde{\varepsilon}(s)$ and $\delta(s) > 0$ such that we find the estimate*

$$\|u\|^2_{E^s_\lambda(T)} \lesssim_s \|u_0\|^2_{H^s_\lambda} + T(\|u\|^6_{F^{s-\tilde{\varepsilon}}_\lambda(T)} + \|u\|^8_{F^{s-\tilde{\varepsilon}}_\lambda(T)}) \tag{5.40}$$

to hold provided that

$$\|u_0\|_{L^2_\lambda} \leq \delta(s). \tag{5.41}$$

We analyze the following generalized energy $E_0^{a,\lambda}$ for a smooth solution to (5.3):

$$E_0^{a,\lambda} = \int_{\Gamma_2^\lambda} a(\xi_1)\hat{u}(\xi_1)\hat{\bar{u}}(\xi_2) d\Gamma_2^\lambda. \tag{5.42}$$

In the following we carry out the program from above. We have to take care of the change of dispersion relation and that the solutions are no longer real-valued. It turns out that the symmetrized expression when computing $\frac{d}{dt}E_0^{a,\lambda}$ is still close to the corresponding expression from Section 5.1.3:

$$\begin{aligned}
\frac{d}{dt}E_0^{a,\lambda} &= \int_{\xi_1+\xi_2=0} a(\xi_1)(i\xi_1) \int_{\xi_1=\xi_{11}+\xi_{12}+\xi_{13}} \hat{u}(\xi_{11})\hat{\bar{u}}(\xi_{12})\hat{u}(\xi_{13}) d\Gamma_3^\lambda \hat{\bar{u}}(\xi_2) d\Gamma_2^\lambda \\
&\quad + \int_{\xi_1+\xi_2=0} a(\xi_1)\hat{u}(\xi_1)(i\xi_2) \int_{\xi_2=\xi_{21}+\xi_{22}+\xi_{23}} \hat{\bar{u}}(\xi_{21})\hat{u}(\xi_{22})\hat{\bar{u}}(\xi_{23}) d\Gamma_3^\lambda d\Gamma_2^\lambda \\
&= -\frac{i}{2}\int_{\Gamma_4^\lambda} (a(\xi_1)\xi_1 + a(\xi_2)\xi_2 + a(\xi_3)\xi_3 + a(\xi_4)\xi_4)\hat{u}(\xi_1)\hat{\bar{u}}(\xi_2)\hat{u}(\xi_3)\hat{\bar{u}}(\xi_4) d\Gamma_4^\lambda.
\end{aligned}$$

Like above we consider the correction term

$$E_1^{a,\lambda}(u) = \int_{\Gamma_4^\lambda} b_4^a(\xi_1,\xi_2,\xi_3,\xi_4)\hat{u}(\xi_1)\hat{\bar{u}}(\xi_2)\hat{u}(\xi_3)\hat{\bar{u}}(\xi_4) d\Gamma_4^\lambda, \tag{5.43}$$

and we require the multiplier b_4^a to satisfy the following identity on Γ_4^λ:

$$(-i)(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2) b_4^a(\xi_1,\xi_2,\xi_3,\xi_4) = \frac{i}{2}(\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)) \tag{5.44}$$

so that we have:

$$
\begin{aligned}
R_6^{a,\lambda} &= \frac{d}{dt}(E_0^{a,\lambda} + E_1^{a,\lambda}) \\
&= 2\int_{\Gamma_6^\lambda} b_4^a(\underbrace{\xi_{11}+\xi_{12}+\xi_{13}}_{\xi_1}, \xi_2, \xi_3, \xi_4)(i\xi_1)\hat{u}(\xi_{11})\hat{\bar{u}}(\xi_{12})\hat{u}(\xi_{13})\hat{\bar{u}}(\xi_2)\hat{u}(\xi_3)\hat{\bar{u}}(\xi_4)d\Gamma_6^\lambda \\
&+ 2\int_{\Gamma_6^\lambda} b_4^a(\xi_1, \underbrace{\xi_{21}+\xi_{22}+\xi_{23}}_{\xi_2}, \xi_3, \xi_4)\hat{u}(\xi_1)(i\xi_2)\hat{\bar{u}}(\xi_{21})\hat{u}(\xi_{22})\hat{\bar{u}}(\xi_{23})\hat{u}(\xi_3)\hat{\bar{u}}(\xi_4)d\Gamma_6^\lambda \\
&= C\Im\int_{\Gamma_6^\lambda} b_4^a(\xi_{11}+\xi_{12}+\xi_{13}, \xi_2, \xi_3, \xi_4)\xi_1\hat{u}(\xi_{11})\hat{\bar{u}}(\xi_{12})\hat{u}(\xi_{13})\hat{\bar{u}}(\xi_2)\hat{u}(\xi_3)\hat{\bar{u}}(\xi_4)d\Gamma_6^\lambda.
\end{aligned}
$$

We show the same size and regularity estimates for the symbol b_4^a from (5.44) like in Section 5.1.3:

Proposition 5.1.21. *Let $a \in S_\varepsilon^s$. Then, for each dyadic $\lambda \leq \beta \leq \mu$, there is an extension $\tilde{b_4^a}$ of b_4^a from the diagonal set*

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4^\lambda \mid |\xi_1^*| \sim \lambda, |\xi_2^*| \sim \beta, |\xi_3^*| \sim |\xi_4^*| \sim \mu\} \tag{5.45}$$

to the full dyadic set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \mid |\xi_1^*| \sim \lambda, |\xi_2^*| \sim \beta, |\xi_3^*| \sim |\xi_4^*| \sim \mu\}, \tag{5.46}$$

which satisfies

$$|\tilde{b}_4^a| \lesssim a(\mu)\mu^{-1}$$

and

$$|\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}\partial_4^{\alpha_4}\tilde{b}_4^a| \lesssim_\alpha a(\mu)\mu^{-1}N_1^{-\alpha_1}N_2^{-\alpha_2}N_3^{-\alpha_3}N_4^{-\alpha_4},$$

where $|\xi_i| \sim N_i$ and $|\xi_1^*| \leq \ldots \leq |\xi_4^*|$ denotes an increasing rearrangement of ξ_i, $i = 1, \ldots, 4$.

Proof. We prove the proposition through Case-by-Case analysis: Note the symmetries between ξ_1 and ξ_3, ξ_2 and ξ_4 and the pairs $\{\xi_1, \xi_3\}$ and $\{\xi_2, \xi_4\}$. Moreover, we dispose of irrelevant factors below.

Case 1 ($|\xi_3^*| \ll |\xi_1^*|$):

Subcase 1a ($|\xi_1| \sim |\xi_2| \gg |\xi_3|, |\xi_4|$):

In this subcase we find $|\xi_2 + \xi_3| \sim |\xi_1|$ and decompose

$$b_4^a(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{(\xi_1 + \xi_2)(\xi_2 + \xi_3)} + \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{(\xi_2 + \xi_3)(\xi_1 + \xi_2)}.$$

Using the notation from the proof of Proposition 5.1.13, we have

$$b_4^a(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{q(\xi_1, \xi_2)}{\xi_2 + \xi_3} - \frac{q(\xi_3, \xi_4)}{\xi_2 + \xi_3},$$

and the size and regularity estimates follow from the size and regularity estimates of q. These estimates were already discussed in Section 5.1.3.

Subcase 1b ($|\xi_1| \sim |\xi_3| \gg |\xi_2|, |\xi_4|$):

In this subcase we find for the resonance function $|\Omega| \sim |\xi_1|^2$, and the size and regularity estimates for an extension of b_4^a follow from considering the trivial decomposition

$$\sum_{i=1}^{4} \frac{\xi_i a(\xi_i)}{(\xi_1+\xi_2)(\xi_2+\xi_3)}. \tag{5.47}$$

Case 2 ($|\xi_1^*| \sim |\xi_3^*| \gg |\xi_4^*|$):

In this case it is clear again that the resonance function is of size $|\xi_1^*|^2$, and a suitable extension is provided through (5.47).

Case 3 ($|\xi_1^*| \sim |\xi_4^*|$):

Subcase 3a ($|\xi_1+\xi_2|, |\xi_2+\xi_3| \ll |\xi_1^*|$):

We compute

$$\begin{aligned} b_4^a(\xi_1,\xi_2,\xi_3,\xi_4) &= \frac{a(\xi_1)\xi_1 + a(\xi_2)(\xi_2)}{(\xi_1+\xi_2)(\xi_2+\xi_3)} + \frac{a(\xi_3)\xi_3 + a(\xi_4)(\xi_4)}{(\xi_1+\xi_2)(\xi_2+\xi_3)} \\ &= \frac{q(\xi_1,\xi_2) - q(\xi_3,-\xi_1-\xi_2-\xi_3)}{\xi_2+\xi_3} \\ &= \frac{q(\xi_1,\xi_2) - q(\xi_1+(\xi_2+\xi_3),\xi_2-(\xi_2+\xi_3))}{\xi_2+\xi_3}, \end{aligned}$$

and the claim follows from the size and regularity properties of q.

Subcase 3b ($|\xi_1+\xi_2| \ll |\xi_1^*|, |\xi_2+\xi_3| \sim |\xi_1^*|$):

We use the decomposition

$$\begin{aligned} b_4^a(\xi_1,\xi_2,\xi_3,\xi_4) &= \frac{a(\xi_1)\xi_1 + a(\xi_2)\xi_2}{(\xi_1+\xi_2)(\xi_2+\xi_3)} + \frac{a(\xi_3)\xi_3 + a(\xi_4)\xi_4}{(\xi_1+\xi_2)(\xi_2+\xi_3)} \\ &= \frac{q(\xi_1,\xi_2)}{\xi_2+\xi_3} + \frac{q(\xi_3,\xi_4)}{\xi_2+\xi_3}, \end{aligned}$$

and the claim follows from the considerations of Subcase 1a. In case $|\xi_1+\xi_2| \sim |\xi_1^*|$ and $|\xi_2+\xi_3| \sim |\xi_1^*|$ we argue mutatis mutandis.

Subcase 3c ($|\xi_1+\xi_2| \sim |\xi_2+\xi_3| \sim |\xi_1^*|$):

The claim follows again from considering the decomposition (5.47). □

In the following estimates we have decreased symmetry compared to Section 5.1.3, but we still have the same frequency interactions. With Proposition 5.1.3 playing the role of Proposition 5.1.2, we can argue in most of the cases like above.

We record the estimate for the boundary term, which is derived like in Proposition 5.1.14:

Proposition 5.1.22. *Let $E_i^{a,\lambda}$, $i=1,2$ be like above. Then, we find the following estimate to hold:*

$$|E_1^{a,\lambda}(u(t))| \lesssim E_0^{a,\lambda}(u(t)) \|u_0\|_{L^2_\lambda}^2 \tag{5.48}$$

For the remainder we derive the following estimate:

Proposition 5.1.23. *We find the following estimate to hold:*

$$\int_0^T R_{a,\lambda}^6(u)ds \lesssim (E_0^{a,\lambda}(u(T)) + E_1^{a,\lambda}(u(0)))\|u_0\|_{L^2_\lambda}^4 + T(\|u\|^6_{F_\lambda^{s-\tilde{\varepsilon}}(T)} + \|u\|^8_{F^{s-\tilde{\varepsilon}}(T)}). \tag{5.49}$$

Proof. We consider dyadic frequency ranges $|\xi_{1i}| \sim M_i$, $i = 1, 2, 3$ and $|\xi_i| \sim L_i$ for $i = 2, 3, 4$ with the increasing rearrangements $M_1^* \leq M_2^* \leq M_3^*$, $L_2^* \leq L_3^* \leq L_4^*$. Further, let K_i^* denote the increasing rearrangement of the union of M_i and L_j.
Case 1: $M_2^* \sim M_3^* \sim K_6^*$, $L_4^* \lesssim M_2^*$;
Case 2: $L_3^* \sim L_4^* \sim K_6^*$, $M_3^* \lesssim L_3^*$: In both cases the argument from the proof of Proposition 5.1.15 applies because it depends only on short-time Strichartz estimates and the symbol size and regularity.
Case 3: $L_4^* \sim M_3^* \sim K_6^*$; $L_3^*, M_2^* \ll K_6^*$: This case needs more care as, due to the reduced symmetry, we can not always argue like in Case 3 from the proof of Proposition 5.1.15.

If $L_3 \sim K_6^*$, we have an improved estimate for the symbol, namely $((K_6^*)^{2(s+\varepsilon)} K_4^*)/(K_6^*)^2$. In this case the claim can be concluded by two bilinear Strichartz estimates involving the high frequencies and two pointwise bounds. This gives

$$R^6_{a,\lambda}(M_i, L_j) \lesssim T(K_1^* K_2^*)^{1/2} \frac{(K_6^*)^{2(s+\varepsilon)} K_4^*}{K_6^*} \prod_{i=1}^{6} \|u_{k_i^*}\|_{F_{k_i^*,\lambda}}$$

with a straight-forward summation over the frequency blocks.

Note the symmetry between M_1, M_3 and L_2, L_4. Suppose that $L_2 \sim K_6^*$. We write out the imaginary part to find

$$\begin{aligned} R^6_{a,\lambda} = C \int_{\Gamma_6^\lambda} &[b_4^a(\xi_{11} + \xi_{12} + \xi_{13}, \xi_2, \xi_3, \xi_4) - b_4^a(\xi_{11} + \xi_{12} + \xi_{13}, -\xi_{11}, -\xi_{12}, -\xi_{13})] \\ &\times (\xi_{11} + \xi_{12} + \xi_{13}) \hat{u}(\xi_{11}) \hat{\bar{u}}(\xi_{12}) \hat{u}(\xi_{13}) \hat{\bar{u}}(\xi_2) \hat{u}(\xi_3) \hat{\bar{u}}(\xi_4) d\Gamma_6^\lambda. \end{aligned}$$

If $L_2 \sim M_1$, the same argument from Case 3 from the proof of Proposition 5.1.15 is applicable as we find a more favourable bound for the difference of the multipliers after using the mean value theorem.

If $L_2 \sim M_2$, this is not the case, but the second resonance function is very favourable:

$$\Omega^{(2)}(\xi_{11}, \xi_{12}, \xi_{13}, \xi_2, \xi_3, \xi_4) = -\xi_{11}^2 + \xi_{12}^2 - \xi_{13}^2 + \xi_2^2 - \xi_3^2 + \xi_4^2 \gtrsim (K_6^*)^2.$$

Then, another integration by parts gives

$$\begin{aligned} R^6_{s,a}(M_i, L_j) = &[\Im \int_{\Gamma_6^\lambda, |\xi_i| \sim M_i, L_j} \frac{b_4^a(\xi_{11} + \xi_{12} + \xi_{13}, \xi_2, \xi_3, \xi_4)}{\Omega^{(2)}(\xi_{11}, \xi_{12}, \xi_{13}, \xi_2, \xi_3, \xi_4)} (\xi_{11} + \xi_{12} + \xi_{13}) \\ &\times (\hat{u}(\xi_{11}) \hat{\bar{u}}(\xi_{12}) \hat{u}(\xi_{13}) \hat{\bar{u}}(\xi_2) \hat{u}(\xi_3) \hat{\bar{u}}(\xi_4) d\Gamma_6^\lambda]_0^T \\ &+ [\Im \int_{\substack{|\xi_{12}| \sim |\xi_2| \sim K_6^*, \\ M_5^*, L_3^* \ll K_6^*}} \frac{b_4^a(\xi_{11} + \xi_{12} + \xi_{13}, \xi_2, \xi_3, \xi_4)}{\Omega^{(2)}(\xi_{11}, \xi_{12}, \xi_{13}, \xi_2, \xi_3, \xi_4)} (\xi_{11} + \xi_{12} + \xi_{13}) \\ &\times \hat{u}(\xi_{11}) \hat{\bar{u}}(\xi_{12}) \hat{u}(\xi_{13}) i\xi_2 (\hat{\bar{u}}(\xi_{21}) \hat{u}(\xi_{22}) \hat{\bar{u}}(\xi_{23})) \hat{u}(\xi_3) \hat{\bar{u}}(\xi_4) d\Gamma_8^\lambda + \dots] \\ =: &E_2^{a,\lambda}(u(T)) - E_2^{a,\lambda}(u(0)) + R^8_{s,a,\lambda}(u), \end{aligned}$$

where we did not record the terms coming up in case the derivative hits another factor than $\hat{\bar{u}}(\xi_2)$. This case we have singled out as $|\xi_2| \sim K_6^*$ so the factor ξ_2 presumably gives the main contribution. From the estimates we shall see that further terms are lower order indeed.

Since $\Omega^{(2)}$ is bounded from below, we still have the necessary regularity to argue like in the proof of Proposition 5.1.15. Further, we have the size estimate

$$\left|\frac{b_4^a(\xi_{11}+\xi_{12}+\xi_{13},\xi_2,\xi_3,\xi_4)}{\Omega^{(2)}(\xi_{11},\xi_{12},\xi_{13},\xi_2,\xi_3,\xi_4)}(\xi_{11}+\xi_{12}+\xi_{13})\right| \lesssim \frac{a(K_6^*)}{(K_6^*)^2}.$$

Hence, estimating

$$\left|\int_{\lambda\mathbb{T}} P_{k_{11}} u P_{k_{12}} \overline{u} P_{k_{13}} u P_{k_2} \overline{u} P_{k_3} u P_{k_4} u dx\right| \lesssim \|P_{k_6^*} u\|_{L^2_\lambda} \|P_{k_5^*} u\|_{L^2_\lambda} \prod_{i=1}^4 \|P_{k_i^*} u\|_{L^\infty_\lambda}$$

gives together with the pointwise bound after carrying out the sum over K_i

$$|E_2^{a,\lambda}(u(t))| \lesssim E_0^{a,\lambda}(u(t)) \|u(t)\|^4_{L^2_\lambda}.$$

Let $N_1^* \leq \ldots \leq N_8^*$ denote the increasing rearrangement of the dyadic sizes of the occurring frequencies. We shall estimate the expression like above according to the separation of the involved frequencies.

Case 1: $N_6^* \ll N_7^* \sim N_8^*$, $N_5^* \ll N_6^* \sim N_8^*$. In both cases we apply two bilinear Strichartz estimates. Note that the time localization amounts to a factor of TN_8^* and the two bilinear Strichartz estimates yield a gain of $(N_8^*)^{-1}$, the four pointwise bounds give a contribution $\prod_{i=1}^4 (N_i^*)^{1/2}$.

<u>Subcase 1a</u>: $N_8^* \sim K_6^*$. Summing the derivatives $\xi_{11}+\xi_{12}+\xi_{13}$, ξ_2 with the same argument like in the proof of Proposition 5.1.15 yields a contribution of $(N_8^*)^2$. Further, the multiplier is estimated by $(N_8^*)^{2(s+\varepsilon)}/(N_8^*)^3$. This gives

$$|R^8_{s,a,\lambda}(N_1^*,\ldots,N_8^*)| \lesssim T\frac{(N_8^*)^{2(s+\varepsilon)}}{N_8^*} \prod_{i=1}^4 (N_i^*)^{1/2} \prod_{i=1}^8 \|u\|_{F_{n_i^*,\lambda}},$$

which is actually summable for $s > 1/6$.

Next, suppose that $N_8^* \gg K_6^*$. This implies $N_6^* \sim K_6^*$ or $K_6^* \sim N_5^* \ll N_6^*$. In the first scenario, the above estimate yields the following bound

$$|R^8_{s,a,\lambda}(N_1^*,\ldots,N_8^*)| \lesssim TN_8^*(N_8^*)^{-1}\frac{(N_6^*)^{2(s+\varepsilon)}}{N_6^*}\frac{(N_6^*)^2}{(N_6^*)^2} \prod_{i=1}^4 (N_i^*)^{1/2} \prod_{i=1}^8 \|u\|_{F_{n_i^*,\lambda}}$$

and in the second case

$$|R^8_{s,a,\lambda}(N_1^*,\ldots,N_8^*)| \lesssim T\frac{(N_5^*)^{2(s+\varepsilon)}}{N_5^*} \prod_{i=1}^4 (N_i^*)^{1/2} \prod_{i=1}^8 \|u\|_{F_{n_i^*,\lambda}}.$$

In both cases summing over dyadic frequencies yields

$$|R^8_{s,a,\lambda}| \lesssim T\|u\|^8_{F_\lambda^{s-\tilde{\varepsilon}}(T)}$$

for $s > 1/6$.

Case 2: $N_4^* \ll N_5^* \sim \ldots \sim N_8^*$, $N_3^* \ll N_4^* \sim \ldots \sim N_8^*$. From the constraint on the initial frequencies $N_3^* \sim N_8^*$ is ruled out and it has to hold $N_8^* \sim K_6^*$.
Either way, one applies one bilinear Strichartz estimate on $u_{n_4^*} u_{n_8^*}$ (in the first subcase this is clear, in the latter, since there is an odd number of high frequencies,

one pair is amenable to one bilinear Strichartz estimate) and three $L^6_{t,x}$-Strichartz estimates on the remaining high frequencies $u_{n_5^*}$, $u_{n_6^*}$, $u_{n_7^*}$; the other frequencies are estimated by pointwise bounds. Here, we do not have to take into account complex conjugation because we argue by Proposition 5.1.3. This gives

$$|R^8_{s,a,\lambda}(N_1^*, \ldots, N_8^*)| \lesssim T N_8^* (N_8^*)^{-1/2} \frac{(N_8^*)^{2(s+\varepsilon)}}{(N_8^*)^3} (N_8^*)^2 \prod_{i=1}^{3} (N_i^*)^{1/2} \prod_{i=1}^{8} \|u_{n_i^*}\|_{F_{n_i^*,\lambda}},$$

which, after summation, gives the estimate

$$|R^8_{s,a,\lambda}| \lesssim T \|u\|^8_{F_\lambda^{s-\varepsilon}(T)}$$

for $s > 1/6$. The proof is complete. □

With the bound for the remainder terms and the boundary terms at disposal, the energy estimate is carried out like for the modified Benjamin-Ono equation. The concluding arguments from Subsection 5.1.4 adapt mutatis mutandis.

5.2 Cubic dispersion relation

In this section we consider the Cauchy problem for the modified Korteweg-de Vries (mKdV) equation (5.2) posed on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with real-valued initial data u_0.

Note that on the real line, there is the scaling symmetry

$$u(t,x) \to \lambda u(\lambda^3 t, \lambda x), \; u_0(x) \to \lambda u_0(\lambda x),$$

which leads to the scaling invariant homogeneous Sobolev space $\dot{H}^{-1/2}(\mathbb{R})$.

The energy is given by

$$E[u] = \int_{\mathbb{T}} \frac{(\partial_x u)^2}{2} \pm \frac{u^4}{12}, \tag{5.50}$$

where the signs from (5.2) match the signs in (5.50). Hence, the positive sign gives rise to the defocusing and the negative sign gives rise to the focusing modified Korteweg-de Vries equation. The mKdV equation is closely related to the classical Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_{xxx} u &= \partial_x(u^2)/2, \; (t,x) \in \mathbb{R} \times \mathbb{T}, \\ \quad u(0) &= u_0. \end{cases} \tag{5.51}$$

We do not give a complete description of previous works on (5.2) (see also the section in Chapter 1) and (5.51), but rather an excerpt of work more closely related to the following considerations. The reader is also referred to the list of literature therein.

We stress that although several of the symmetries of the mKdV equation are certainly used in the proof of the main result, in particular that real-valued initial data give rise to real-valued solutions, the method does not depend on complete integrability.

Likewise, it is possible to prove a priori estimates and existence of solutions to the KdV-mKdV-equation (cf. [Mol12])

$$\begin{cases} \partial_t u + \partial_{xxx} u &= \partial_x (u^2)/2 + \partial_x (u^3)/3, \ (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u(0) &= u_0 \in H^s(\mathbb{T}). \end{cases}$$

The analysis can be adapted to consider generalized KdV equations (cf. [Sta97, CKS$^+$04]) like

$$\partial_t u + \partial_{xxx} u = \pm u^{k-1} \partial_x u, \ (t,x) \in \mathbb{R} \times \mathbb{T}$$

or dispersion generalized equations like

$$\partial_t u + \partial_x D_x^{a-1} u = \pm u^2 \partial_x u, \ (t,x) \in \mathbb{R} \times \mathbb{T},$$

where $1 < a < 2$, which are no longer amenable to inverse scattering techniques.

Exploiting the integrability properties and the inverse scattering transform, Kappeler and Topalov showed (1.19) in the defocusing case to be globally well-posed in $L^2(\mathbb{T})$ (cf. [KT05a]) with a notion of solutions defined through smooth approximations. From Sobolev embedding one finds that these solutions satisfy the mKdV equation in the sense of generalized functions as soon as $s \geq 1/6$.
Unconditional well-posedness of the mKdV equation by means of normal form reduction was shown by Kwon and Oh in [KO12] for $s \geq 1/2$.

Since the mKdV equation is completely integrable, there is an infinite number of conserved quantities of solutions: In addition to the conservation of energy, we record the conservation of mass for real-valued solutions, i.e.,

$$\int_{\mathbb{T}} u^2 dx = \int_{\mathbb{T}} u_0^2 dx$$

because this provides us with an L^2-a priori estimate $\sup_{t \in \mathbb{R}} \|u(t)\|_{L^2(\mathbb{T})} \lesssim \|u_0\|_{L^2(\mathbb{T})}$ for smooth solutions.

It is known that the data-to-solution map fails to be C^3 below $s < 1/2$ (cf. [Bou97]) and even fails to be uniformly continuous (cf. [CCT03, BGT02]) because of the resonant term on the diagonal.

Non-diagonal resonant interactions can be removed by changing to the renormalized modified Korteweg-de Vries equation:

$$\partial_t u + \partial_{xxx} u = (u^2 - \frac{1}{2\pi} \int_{\mathbb{T}} u^2) \partial_x u = \mathfrak{N}(u). \tag{5.52}$$

The solution to (5.52) is given in terms of the solution to (5.2) as follows:

$$v(t,x) = u(t, x - C(\int_0^t \int_{\mathbb{T}} u^2(x',t') dx' dt')) = u(t, x - Ct \|u_0\|_{L^2}^2). \tag{5.53}$$

The norm of the solution to (5.52) for positive Sobolev regularities equals the one of the solution to (5.2), and most of the well-posedness results were in fact shown for the renormalized mKdV equation. Removing the off-diagonal resonant interactions introduces a drift term governed by the L^2-norm, which breaks uniform continuity of the unrenormalized mKdV equation for initial data with variable L^2-norm.

For negative Sobolev regularities one should define the nonlinear interaction $\mathfrak{N}$ for (5.52) in Fourier variables, see below. For the technical reason of having the non-diagonal resonant interactions removed, we also work with the renormalized version (5.52). However, by the above considerations, the Cauchy problems are essentially equivalent in Sobolev spaces with non-negative regularity index.

There are further results not relying on complete integrability: Employing a nonlinear modification of the Fourier restriction spaces instead, Nakanishi et al. showed in [NTT10] local well-posedness of (1.19) for $s > 1/3$ and a priori estimates for $s > 1/4$ (see also the previous work [TT04] by Takaoka and Tsutsumi).

Combining the normal form approach from [KO12] and the nonlinear ansatz from [NTT10], Molinet-Pilod-Vento proved unconditional well-posedness for $s \geq 1/3$ in [MPV19].

Moreover, in another recent work [Mol12] by Molinet, it was shown that the Kappeler-Topalov solutions satisfy the defocusing mKdV equation in $L^2(\mathbb{T})$ in the distributional sense.

In the focusing case, relying on the conservation of mass and using short-time Fourier restriction, was shown the existence of global distributional solutions in $C_w(\mathbb{R}; L^2(\mathbb{T}))$. This means that the solutions are continuous curves in $L^2(\mathbb{T})$ endowed with the weak topology.

In this work it was also proved that the data-to-solution map fails to be continuous from $L^2(\mathbb{T})$ to $\mathcal{D}'([0,T])$ for non-constant initial data $u_0 \in H^\infty(\mathbb{T})$. We revisit the analysis and see that one can control the nonlinear interaction below L^2 for suitable time localization.

On the real line, (5.2) is better behaved than on the torus because of stronger dispersive effects: In [KPV93] it was shown that (1.19) is locally well-posed for $s > 1/4$ by a Picard iteration scheme in a resolution space capturing the dispersive effects. This result was recovered in the framework of Fourier restriction spaces by Tao in [Tao01]. Global well-posedness for $s > 1/4$ was also shown in [CKS$^+$02].

Christ et al. showed in [CHT12] a priori estimates for smooth solutions on the real line for $-1/8 < s \leq 1/4$ making use of the short-time Fourier restriction spaces.

Again heavily relying on complete integrability, Koch and Tataru showed a priori estimates for $s > -1/2$ in [KT18].

When we refer to existence of solutions in the following, we work with the following definition:

Definition 5.2.1. We say that there exist solutions to an evolution equation if there is a data-to-solution mapping $S : H^s \to C([-T,T], H^s)$, where $T = T(\|u_0\|_{H^s}) > 0$, with the following properties:

(i) $S(u_0)$ satisfies the equation in the distributional sense and $S(u_0)(0) = u_0$.

(ii) There exists a sequence of smooth global solutions (u_n) such that $u_n \to S(u_0)$ in $C([-T,T], H^s)$ as $n \to \infty$.

This notion was introduced by Guo-Oh in [GO18] to discuss existence of solutions to the nonlinear Schrödinger equation on the circle for low regularities.

We recall why the second property is natural for two reasons following [GO18]: Local well-posedness requires continuity of the data-to-solution map, but also from a practical point of view the construction of solutions typically requires at least one approximating sequence of smooth global solutions.

One purpose of this chapter is to show the existence of solutions and a priori estimates below $H^{1/2}(\mathbb{T})$ up to $L^2(\mathbb{T})$ relying on localization in time of the Fourier restriction spaces. The frequency dependent localization in time introduces extra smoothing, which allows us to overcome the loss of derivative at low regularities.

Essentially[2], we show the following three estimates for $T \in (0,1]$ and $s > 0$:

$$\begin{cases} \|u\|_{F^{s,\alpha}(T)} & \lesssim \|u\|_{E^s(T)} + \|\mathfrak{N}(u)\|_{N^{s,\alpha}(T)} \\ \|\mathfrak{N}(u)\|_{N^{s,\alpha}(T)} & \lesssim T^\theta \|u\|^3_{F^{s,\alpha}(T)} \\ \|u\|^2_{E^s(T)} & \lesssim \|u_0\|^2_{H^s} + T^\theta \|u\|^6_{F^{s,\alpha}(T)}. \end{cases} \tag{5.54}$$

With the above set of estimates at disposal, bootstrap and compactness arguments allow us to prove the following theorem:

Theorem 5.2.2. *Let $s > 0$. Given $u_0 \in H^s(\mathbb{T})$, there is a function $T = T(\|u_0\|_{H^s})$ so that there exists a local solution $u \in C([-T,T], H^s(\mathbb{T}))$ to (1.19). Furthermore, we find the a priori estimate*

$$\sup_{t \in [-T,T]} \|u(t)\|_{H^s} \leq C \|u_0\|_{H^s} \tag{5.55}$$

to hold.

There is also the recent work [KVZ18] by Killip et al. relying on complete integrability, where a priori estimates for smooth periodic initial data are shown, too. In fact, for a solution to (5.2) with smooth initial data u_0, the a priori estimate

$$\|u(t)\|_{H^s(\mathbb{T})} \lesssim \|u_0\|_{H^s} (1 + \|u_0\|^2_{H^s})^{\frac{|s|}{1-2|s|}}$$

is proved in [KVZ18] for $-1/2 < s < 1/2$. By means of the transformation (5.53), the a priori estimate extends to smooth solutions to (5.52).

Notably, in [KVZ18] are also proved a priori estimates for smooth solutions to the cubic NLS

$$\begin{cases} i\partial_t u + \partial_{xx} u & = \pm |u|^2 u, \ (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u(0) & = u_0, \end{cases} \tag{5.56}$$

in the same range $-1/2 < s < 1$, but in [GO18] it was shown that because the data-to-solution mapping can be constructed with the aid of compactness arguments for a renormalized version of the cubic NLS for $-1/8 < s < 0$, it can not exist for (5.56).

In the context of Fourier Lebesgue spaces which scale like Sobolev spaces with negative regularity index this program was carried out by Kappeler and Molnar in [KM17] for (5.2) relying on complete integrability.

We conjecture that the renormalized version (5.52) is the correct formulation of the mKdV equation for negative Sobolev regularities because of the reduced resonances in view of the ill-posedness result from [Mol12].
Another purpose of the following computations is to point out the critical interactions which require further comprehension to clarify non-existence of solutions:

For the non-linear estimate we shall see that localizing time higher than reciprocal to the frequency allows us to control the renormalized nonlinear interaction for negative Sobolev regularity. The situation for the energy estimate is more delicate: the critical interactions in the energy estimate occur for small second resonance.

[2]For the actually slightly more complicated energy estimate see Subsection 5.2.4.

These are the interactions we can not estimate below L^2 in this work without the currently unproved $L^8_{t,x}$-Strichartz estimate

$$\|u\|_{L^8_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0+,1/2+}_{\text{Airy}}}. \tag{5.57}$$

This estimate was conjectured by Bourgain in [Bou93b]. Although there has been considerable progress regarding Strichartz estimates on tori (cf. [HL13, BD15]), this estimate seems to be out of reach at the moment.

Theorem 5.2.3. *Suppose that* (5.57) *holds. Then, there is* $s' < 0$ *so that for* $s' < s < 0$ *there exists* $T = T(\|u_0\|_{H^s})$ *such that there exists a local solution* $u \in C([-T,T], H^s(\mathbb{T}))$ *to* (5.52)*, and we find the a priori estimate*

$$\sup_{t\in[-T,T]} \|u(t)\|_{H^s(\mathbb{T})} \leq C\|u_0\|_{H^s}$$

to hold.
Furthermore, solutions to (5.2) *do not exist for* $s' < s < 0$.

In Subsection 5.2.1 we introduce notation, and in Subsection 5.2.5 we finish the proof of the a priori estimates relying on a short-time trilinear estimate from Subsection 5.2.3 and energy estimates from Subsection 5.2.4. Preliminary multilinear estimates to prove the short-time trilinear estimate are discussed in Section 5.2.2.

5.2.1 Function spaces and more notation

Regarding the conventions for the Fourier transform, Littlewood-Paley theory and function space properties, we refer to Section 2.5. The dispersion relation for the Airy equation is given by $\varphi(\xi) = \xi^3$. For the remainder of this chapter, the $X^{s,b}$-spaces are adapted to the Airy dispersion relation. Further, we confine ourselves to unit periods $\lambda = 1$ and λ is omitted from the notation in the following.
Also, the spaces E_k, F^α_k and N^α_k are defined like in Section 2.5 with dispersion relation $\varphi(\xi) = \xi^3$.

Localization of the function spaces in time is carried out like in Section 2.5 yielding spaces $F^\alpha_k(T)$ and $N^\alpha_k(T)$. Littlewood-Paley assembly yields the spaces E^s, $E^s(T)$, $F^s_\alpha(T)$ and $N^s_\alpha(T)$.

We also use the spaces $F^{b,\alpha}_k$ with generalized modulation regularity $b < 1/2$ to propagate large data. This is accomplished by Lemma 2.5.3 gaining a factor $T^{(1/2-b)-}$ when we increase modulation regularity to $1/2$.

Throughout this chapter, we work with the renormalized version (5.52) of (1.19). We use the following notation for the trilinear interaction in (5.52):

$$\mathfrak{N}(u,v,w)\hat{}(n) = \underbrace{in\hat{u}(n)\hat{v}(-n)\hat{w}(n)}_{\mathcal{R}(u,v,w)\hat{}(n)} + \underbrace{in \sum_{\substack{n_1+n_2+n_3=n,\\ (n_1+n_2)(n_1+n_3)(n_2+n_3)\neq 0}} \hat{u}(n_1)\hat{v}(n_2)\hat{w}(n_3)}_{\mathcal{N}(u,v,w)\hat{}(n)}. \tag{5.58}$$

We abbreviate the condition $(n_1+n_2)(n_1+n_3)(n_2+n_3) \neq 0$ in the sum for the non-resonant interaction $\mathcal{N}$ with $(*)$. In Fourier variables we write

$$(f_1 * f_2 * f_3)\tilde{}(n) = \sum_{\substack{n_1+n_2+n_3=n,\\ (*)}} f_1(n_1)f_2(n_2)f_3(n_3), \quad f_i : \mathbb{Z} \to \mathbb{C}. \tag{5.59}$$

5.2.2 Multilinear estimates

In the following we recall and derive multilinear estimates for periodic functions with support of the space-time Fourier transform adapted to the Airy equation. We denote the frequency ranges with k_i and the modulation ranges with j_i. The decreasing arrangements are denoted by $k_1^* \geq k_2^* \geq \ldots$ or $j_1^* \geq j_2^* \geq \ldots$, respectively. We recall the following linear Strichartz estimates going back to Bourgain (cf. [Bou93a]):

Lemma 5.2.4. *For $u \in X^{0,1/3}$ we find the following estimate to hold:*

$$\|u\|_{L_t^4(\mathbb{R},L_x^4(\mathbb{T}))} \lesssim \|u\|_{X^{0,1/3}}. \tag{5.60}$$

For $u_0 \in L^2(\mathbb{T})$ with $\text{supp}(\hat{u}_0) \subseteq I$ we find

$$\|S(t)u_0\|_{L_{t,x}^6(\mathbb{T}^2)} \lesssim C_\varepsilon |I|^\varepsilon \|u_0\|_{L^2(\mathbb{T})}. \tag{5.61}$$

Proof. Estimate (5.60) is proved in [Bou93b, Proposition 7.15., p. 211] and (5.61) is proved in Proposition 3.2.1 (for Schrödinger and Airy dispersion this estimate was already proven in [Bou93a, Bou93b] by a more direct method). $\square$

From the above displays the following estimates are consequences of Hölder's inequality and almost orthogonality:

Lemma 5.2.5. *For $u \in L^2(\mathbb{R}\times\mathbb{T})$ with $\text{supp}(\mathcal{F}_{t,x}(u_i)) \subseteq D_{k_i,j_i}$ we find the following estimates to hold:*

$$\int_{\mathbb{R}\times\mathbb{T}} u_1 u_2 u_3 u_4 dx dt \lesssim \prod_{i=1}^4 2^{j_i/3} \|\mathcal{F}_{t,x}(u_i)\|_{L_\tau^2 \ell_n^2}, \tag{5.62}$$

$$\int_{\mathbb{R}\times\mathbb{T}} u_1 u_2 u_3 u_4 dx dt \lesssim 2^{-j_1^*/2} 2^{\varepsilon k_3^*} \prod_{i=1}^4 2^{j_i/2} \|\mathcal{F}_{t,x}(u_i)\|_{L_\tau^2 \ell_n^2}. \tag{5.63}$$

Proof. Estimate (5.62) follows from an application of Hölder's inequality and (5.60), and for a proof of (5.63), see for instance [GO18, Equation (5.5)]. $\square$

Already in [Bou93b] was conjectured that the estimate

$$\|u\|_{L_{t,x}^8(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0+,1/2+}} \tag{5.64}$$

holds true. Interpolation with (5.60) gives

$$\|u\|_{L_{t,x}^6(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0+,4/9+}}. \tag{5.65}$$

This estimate would provide us with smoothing in any short-time F^α-space, which seems to be necessary to carry out energy estimates for negative Sobolev regularities. From the short-time estimates from Proposition 3.2.6 we find the estimate

$$\|P_n e^{t\partial_x^3} u_0\|_{L_t^6([0,2^{-2n}],L^6(\mathbb{T}))} \lesssim 2^{-n/6} \|P_n u_0\|_{L^2}. \tag{5.66}$$

Thus, we know that the $L_{t,x}^6$-Strichartz estimate loses no derivatives in F^1-space by the following computation:

$$\begin{aligned}\|P_n e^{t\partial_x^3} u_0\|_{L_t^6([0,2^{-n}],L^6(\mathbb{T}))} &\lesssim \left(\sum_{I:|I|=2^{-2n}} \|P_n e^{t\partial_x^3} u_0\|^6_{L_t^6(I,L^6(\mathbb{T}))} \right)^{1/6} \\ &\lesssim \|P_n u_0\|_{L^2}.\end{aligned}$$

The smoothing obtained in F^{α}-spaces for $\alpha > 1$ by (5.66) is not sufficient to prove energy estimates for negative Sobolev regularities.

We recall the following bilinear estimates from [Mol12].

Lemma 5.2.6 ([Mol12, Equation (3.7), p. 1906]). *Let $f_1, f_2 \in L^2(\mathbb{R} \times \mathbb{Z})$ with the following support properties*

$$(\tau, n) \in \text{ supp } (f_i) \Rightarrow \langle \tau - n^3 \rangle \lesssim 2^{j_i}, \quad i = 1, 2,$$

where $j_1 \leq j_2$.
Then, for any $N > 0$ we find the following estimate to hold:

$$\|f_1 * f_2\|_{L^2_\tau \ell^2_n(|n| \geq N)} \lesssim 2^{j_1/2} \left(\frac{2^{j_2/4}}{N^{1/4}} + 1 \right) \|f_1\|_{L^2} \|f_2\|_{L^2}. \tag{5.67}$$

In case of separated frequencies we can refine the above estimates.
The following lemma is adapted to the nonlinear interaction dictated by the modified Korteweg-de Vries equation in the following sense: The nonlinear interaction takes place between u_1, u_2, u_3, where u_4 will serve as a dual test function.

If there is one frequency much smaller than the remaining three, the resonance is very favourable, and we do not need a refined estimate. Thus, we only consider the case where two frequencies are smaller than the remaining two. This is relevant for $High \times Low \times Low \to High$-interaction:

Lemma 5.2.7. *Suppose that $k_4 \geq 20$, $k_1 \leq k_2 \leq k_3 - 5$. Moreover, suppose that $j_a \geq [\alpha k_1^*]$ for $a \in \{1, 2, 3, 4\}$ with $\alpha \leq 2$ and $supp(\tilde{u}_i) \subseteq \tilde{D}_{k_i, j_i}$ and that $supp_\xi \tilde{u}_i \subseteq I_i$ where $|I_i| \lesssim 2^l$.*
Then, we find the following estimate to hold:

$$\int_{\mathbb{R} \times \mathbb{T}} u_1(t,x) u_2(t,x) u_3(t,x) u_4(t,x) dt dx \lesssim M \prod_{i=1}^{4} 2^{j_i/2} \|\mathcal{F}_{t,x}(u_i)\|_{L^2_\tau \ell^2_n}, \tag{5.68}$$

where $M = 2^{l/2} 2^{-j_1^*/2} 2^{-[\alpha k_1^*]/2}$.

Proof. We denote the space-time Fourier transform of $u_i : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$ with $f_i : \mathbb{R} \times \mathbb{Z} \to \mathbb{C}$, $\mathcal{F}_{t,x}(u_i)(\tau, n) = f_i(\tau, n)$. Further, we consider the shifted function $g_i(\tau, n) = f_i(\tau + n^3, n)$ and observe $(\tau, n) \in \text{supp}(g_i) \Leftrightarrow (\tau + n^3, n) \in \text{supp}(f_i)$ so that $g_i(\tau, n) = 0$, unless $|\tau| \lesssim 2^{j_i}$.
Case A: Suppose that $j_1^* = j_2$: That is a low frequency carries a high modulation. We shall see that the computation below can also deal with the case $j_1^* = j_3$ by exchanging the roles of g_2 and g_3.

Changing into Fourier space, we find after change of variables

$$\begin{aligned}
&\int_{\mathbb{T}} dx \int_{\mathbb{R}} dt u_1(t,x) u_2(t,x) u_3(t,x) u_4(t,x) \\
&= \int_{\Gamma_4(\tau)} \int_{\Gamma_4(n)} f_1(\tau_1, n_1) f_2(\tau_2, n_2) f_3(\tau_3, n_3) f_4(\tau_4, n_4) \\
&= \int_{\tau_1, \tau_2, \tau_4} \sum_{n_1, n_2, n_4} g_1(\tau_1, n_1) g_2(\tau_2, n_2) \\
&\quad g_4(\tau_4, n_4) g_3(h(\tau_1, \tau_2, \tau_4, n_1, n_2, n_4), -n_1 - n_2 - n_4).
\end{aligned}$$

By means of the resonance function

$$h(\tau_1, \tau_2, \tau_3, n_1, n_2, n_3) = -\tau_1 - \tau_2 - \tau_3 + 3(n_1 + n_2)(n_1 + n_3)(n_2 + n_3),$$

we can compute the effective supports in the modulation variables.

Set $E_{34} = \{n_4 \in \mathbb{Z} | \, |h(\tau_1, \tau_2, \tau_4, n_1, n_2, n_4)| \lesssim 2^{j_3}\}$ and with the third variable distinguished, we denote $h(\tau_1, \tau_2, \tau_4, n_1, n_2, n_4)$ as h_3 and compute $\partial_{n_4} h_3 = (n_1 + n_2)(n_4 - n_3)$, which gives $|\partial_{n_4} h_3| \gtrsim 2^{2k_1^*}$.
Thus, an application of the Cauchy-Schwarz inequality yields $|E_{34}| \lesssim (1 + 2^{j_3 - 2k_1^*})$, and we find the estimate

$$\begin{aligned}
&\sum_{n_2, n_1} \int d\tau_2 d\tau_1 d\tau_4 |g_1(\tau_1, n_1)||g_2(\tau_2, n_2)| \sum_{n_4} |g_3(h_3, -n_1 - n_2 - n_4)||g_4(\tau_4, n_4)| \\
&\lesssim (1 + 2^{j_3 - 2k_1^*})^{1/2} \sum_{n_2, n_1} \int d\tau_2 |g_2(\tau_2, n_2)| \int d\tau_1 \int d\tau_4 |g_1(\tau_1, n_1)| \times \\
&(\sum_{n_4} |g_3(h_3, -n_1 - n_2 - n_4)|^2 |g_4(\tau_4, n_4)|^2)^{1/2}.
\end{aligned} \tag{5.69}$$

By repeated applications of the Cauchy-Schwarz inequality, we find:

$$\begin{aligned}
&(5.69) \lesssim (1 + 2^{j_3 - 2k_1^*})^{1/2} \sum_{n_2, n_1} \int d\tau_1 |g_1(\tau_1, n_1)| \int d\tau_4 \\
&\left(\int d\tau_2 |g_2(\tau_2, n_2)|^2 \right)^{1/2} \left(\sum_{n_4} \|g_3(h_3, -n_1 - n_2 - n_4)\|^2_{L^2_{\tau_2}} |g_4(\tau_4, n_4)|^2 \right)^{1/2} \\
&\lesssim (1 + 2^{j_3 - 2k_1^*})^{1/2} 2^{l/2} \|g_2\|_{L^2_\tau \ell^2_n} \sup_{n_2} \int d\tau_1 \int d\tau_4 \\
&\sum_{n_1} |g_1(\tau_1, n_1)| \left(\sum_{n_4} \|g_3(h_3, -n_1 - n_2 - n_4)\|^2_{L^2_{\tau_2}} |g_4(\tau_4, n_4)|^2 \right)^{1/2} \\
&\lesssim (1 + 2^{j_3 - 2k_1^*})^{1/2} 2^{l/2} \|g_2\|_{L^2_\tau \ell^2_n} \sup_{n_2} \int d\tau_1 \int d\tau_4 \\
&\left(\sum_{n_1} |g_1(\tau_1, n_1)|^2 \right)^{1/2} \left(\sum_{n_4} |g_4(\tau_4, n_4)|^2 \sum_{n_1} \|g_3(h_3, -n_1 - n_2 - n_4)\|^2_{L^2_{\tau_2}} \right)^{1/2} \\
&\lesssim (1 + 2^{j_3 - 2k_1^*})^{1/2} 2^{l/2} \|g_2\|_{L^2_\tau \ell^2_n} \|g_3\|_{L^2_\tau \ell^2_n} \\
&\quad \times \sup_{n_2} \int d\tau_1 \int d\tau_4 \left(\sum_{n_1} |g_1(\tau_1, n_1)|^2 \right)^{1/2} \left(\sum_{n_4} |g_4(\tau_4, n_4)|^2 \right)^{1/2} \\
&\lesssim (1 + 2^{j_3 - 2k_1^*})^{1/2} 2^{l/2} 2^{j_1/2} 2^{j_4/2} \prod_{i=1}^4 \|g_i\|_{L^2_\tau \ell^2_n}.
\end{aligned}$$

In case $j_3 \geq 2k_1^*$ we find (5.68) to hold with $M = 2^{l/2} 2^{-j_1^*/2} 2^{-k_1^*}$. If $j_3 \leq 2k_1^*$, we find (5.68) to hold with $M = 2^{l/2} 2^{-j_1^*/2} 2^{-[\alpha k_1^*]/2}$, which is the worse bound. This proves (5.68) in Case A.

Case B: In case $j_1 = j_1^*$, that is a high frequency carrying a high modulation we estimate the low frequency g_3 by Cauchy-Schwarz to find:

$$\begin{aligned} &2^{j_3/2} 2^{l/2} \|g_3\|_{L^2_\tau \ell^2_n} \sup_{n_3, \tau_3} \int d\tau_1 \int d\tau_4 \sum_{n_1} |g_1(\tau_1, n_1)| \\ &\sum_{n_2} |g_2(\tau_2, n_2)| |g_4(h_4, -n_1 - n_2 - n_3)| \end{aligned} \tag{5.70}$$

We consider the set $E_{42} = \{n_2 \in \mathbb{Z} \,|\, h_4 = \mathcal{O}(2^{j_4})\}$. Since $\partial_{n_2} h_4 = 3(n_1+n_3)(n_2-n_4)$ we find $|\partial_{n_2} h_4| \gtrsim 2^{2k_1^*}$ and further $|E_{42}| \lesssim (1 + 2^{j_4 - 2k_1^*})^{1/2}$. We find after several applications of the Cauchy-Schwarz inequality:

$$\begin{aligned} (5.70) &\lesssim (1 + 2^{j_4 - 2k_1^*})^{1/2} 2^{j_3/2} 2^{l/2} \|g_3\|_{L^2_\tau \ell^2_n} \sup_{n_3, \tau_3} \int d\tau_1 \int d\tau_2 \\ &\quad \sum_{n_1} |g_1(\tau_1, n_1)| \left(\sum_{n_2} |g_2(\tau_2, n_2)|^2 |g_4(h_4, -n_1 - n_2 - n_3)|^2 \right)^{1/2} \\ &\lesssim 2^{j_3/2} 2^{l/2} (1 + 2^{j_4 - 2k_1^*})^{1/2} \|g_3\|_{L^2_\tau \ell^2_n} \sup_{n_3, \tau_3} \int d\tau_2 \sum_{n_1} \left(\int d\tau_1 |g_1(\tau_1, n_1)|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{n_2} |g_2(\tau_2, n_2)|^2 \|g_4(h_4, -n_1 - n_2 - n_3)\|^2_{L^2_{\tau_1}} \right)^{1/2} \\ &\lesssim 2^{j_3/2} 2^{l/2} (1 + 2^{j_4 - 2k_1^*})^{1/2} \|g_3\|_{L^2_{\tau_3} \ell^2_{n_3}} \sup_{n_3, \tau_3} \int d\tau_2 \|g_1\|_{L^2_\tau \ell^2_n} \\ &\quad \times \left(\sum_{n_2} |g_2(\tau_2, n_2)|^2 \sum_{n_1} \|g_4(h_4, -n_1 - n_2 - n_3)\|^2_{L^2_\tau} \right)^{1/2} \\ &\lesssim 2^{j_3/2} 2^{l/2} (1 + 2^{j_4 - 2k_1^*})^{1/2} 2^{j_2/2} \prod_{i=1}^4 \|g_i\|_{L^2_\tau \ell^2_n}. \end{aligned}$$

Clearly, an adapted computation shows the claim if $j_1^* = j_4$.
The estimate (5.68) follows from the same considerations as in Case A. □

The estimate for *High* × *High* × *Low* → *Low*-interaction is related, but the minimal size of the support of the modulation variable is different. This is taken into account in the following sections.

5.2.3 Short-time trilinear estimates

Recall for $k \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$ the regions $D_{k,j}$, $D_{k,\leq j}$ in Fourier space defined in (2.45).
Our aim is to prove estimates of the following kind for all possible frequency interactions:

$$\|P_{k_4} \mathcal{N}(u_1, u_2, u_3)\|_{N^\alpha_{k_4}} \lesssim \underbrace{D(\alpha, k_1, k_2, k_3, k_4)}_{D(\alpha, \underline{k})} \|u_1\|_{F^{1/2-,\alpha}_{k_1}} \|u_2\|_{F^{1/2-,\alpha}_{k_2}} \|u_3\|_{F^{1/2-,\alpha}_{k_3}}. \tag{5.71}$$

In fact, the resonant interaction can be perceived as a special case of $High \times High \times High \to High$-interaction, see below. Hence, we only estimate the non-resonant part.
The trilinear estimate

$$\|\mathfrak{N}(u_1,u_2,u_3)\|_{N^{s,\alpha}(T)} \lesssim T^{\theta}\|u_1\|_{F^{s,\alpha}(T)}\|u_2\|_{F^{s,\alpha}(T)}\|u_3\|_{F^{s,\alpha}(T)} \tag{5.72}$$

then follows from splitting up the frequency support of the functions and Lemma 2.5.3. Note that it will be enough to estimate one function in (5.71) at a modulation slightly below 1/2 to derive (5.72).

Below, we only derive (5.71) for $F^{\alpha}_{k_i}$-spaces in detail. The systematic modification to find (5.71) to hold with one modulation regularity strictly less than 1/2 follows by accepting a slight loss in the highest modulation in modulation localized estimates.

We start with $High \times Low \times Low \to High$-interaction:

Lemma 5.2.8. *Let $k_4 \geq 20$ and $k_1 \leq k_2 \leq k_3 - 5$ and suppose that $P_{k_i} u_i = u_i$ for $i \in \{1,2,3\}$. Then, we find the estimate (5.71) to hold with $D(\alpha,\underline{k}) = 2^{-(\alpha/2-\varepsilon)k_4}$ for any $\varepsilon > 0$.*

Proof. Let $\gamma : \mathbb{R} \to [0,1]$ be a smooth function with $\mathrm{supp}(\gamma) \subseteq [-1,1]$ and

$$\sum_{m\in\mathbb{Z}} \gamma^3(x-m) \equiv 1.$$

We find the left hand-side in (5.71) to be dominated by

$$\begin{aligned} &C2^{k_4}\sum_{m\in\mathbb{Z}}\sup_{t_{k_4}\in\mathbb{R}} \|(\tau - n^3 + i2^{[\alpha k_4]})^{-1} 1_{I_{k_4}}(n) \\ &(\mathcal{F}_{t,x}[\eta_0(2^{[\alpha k_4]}(t-t_{k_4}))\gamma(2^{[\alpha k_1^*]}(t-t_{k_4})-m)u_1] \\ &* \mathcal{F}_{t,x}[\gamma(2^{[\alpha k_1^*]}(t-t_{k_4})-m)u_2] * \mathcal{F}_{t,x}[\gamma(2^{[\alpha k_1^*]}(t-t_{k_4})-m)u_3])\tilde{}\|_{X_{k_4}}. \end{aligned}$$

We observe that $\#\{m \in \mathbb{Z} | \eta_0(2^{[\alpha k_4]}(\cdot - t_k))\gamma(2^{[\alpha k_1^*]}(\cdot - t_k) - m) \neq 0\} = \mathcal{O}(1)$. Consequently, it is enough to prove

$$\begin{aligned} &C2^{k_4}\sup_{t_{k_4}\in\mathbb{R}} \|(\tau - n^3 + i2^{[\alpha k_4]})^{-1} 1_{I_{k_4}}(n)(\mathcal{F}_{t,x}[\eta_0(2^{[\alpha k_4]}(t-t_k))\gamma(2^{[\alpha k_1^*](t-t_k)-m)}u_1] \\ &* \mathcal{F}_{t,x}[\gamma(2^{[\alpha k_1^*]}(t-t_k)-m)u_2] * \mathcal{F}_{t,x}[\gamma(2^{[\alpha k_1^*]}(t-t_k))u_3])\tilde{}\|_{X_{k_4}} \\ &\lesssim 2^{-(\alpha/2-\varepsilon)k_4}\|u_1\|_{F^{\alpha}_{k_1}}\|u_2\|_{F^{\alpha}_{k_2}}\|u_3\|_{F^{\alpha}_{k_3}}. \end{aligned}$$

We write $f_{k_i} = \mathcal{F}_{t,x}[\eta_0(2^{[\alpha k_4]}(t-t_k)\gamma(2^{[\alpha k_1^*]}(t-t_k)-m)u_i]$ and with additional localization in modulation we use the notation

$$f_{k_i,j_i} = \begin{cases} \eta_{\leq j_i}(\tau - n^3) f_{k_i},\ j_i = [\alpha k_1^*], \\ \eta_{j_i}(\tau - n^3) f_{k_i},\ j_i > [\alpha k_1^*]. \end{cases}$$

By means of the definition of $F^{\alpha}_{k_i}$ and (2.48), it is further enough to prove

$$\begin{aligned} &\sum_{j_4 \geq [\alpha k_4]}\sum_{j_1,j_2,j_3\geq[\alpha k_1^*]} 2^{-j_4/2}\|1_{D_{k_4,(\leq)j_4}}(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\tilde{}\|_{L^2_\tau \ell^2_n} \\ &\lesssim 2^{-(\alpha/2-\varepsilon)k_1}\prod_{i=1}^3 \sum_{j_i\geq[\alpha k_1^*]} 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n}. \end{aligned} \tag{5.73}$$

We see that (5.73) follows from (5.68). In fact, the resonance function, giving a lower bound for the minimal modulation in (5.73), is given by

$$\Omega = (k_1 + k_2 + k_3)^3 - k_1^3 - k_2^3 - k_3^3 = 3(k_1 + k_2)(k_1 + k_3)(k_2 + k_3).$$

Thus, $2^{2k_1^*} \lesssim |\Omega| \lesssim 2^{2k_1^* + k_3^*}$. To derive effective estimates, we localize $|\Omega| \sim 2^{2k_4 + l}$. This is equivalent to prescribe $|k_2 + k_3| \sim 2^l$, and the contribution to (5.73) will be denoted by

$$\| P_\Omega^l 1_{D_{k_4,(\leq) j_4}} (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\tilde{} \|_{L^2_\tau \ell^2_n}.$$

We write D_{k_4,j_4} for $D_{k_4,(\leq)j_4}$ in the following.

In the above display we split the frequency support of f_{k_2,j_2} up into intervals of length 2^l, that is $f_{k_2,j_2} = \sum_I f^I_{k_2,j_2}$ and, due to the localization of Ω, this also gives a decomposition of f_{k_3,j_3} so that the above display is dominated by

$$\sum_I \| P_\Omega^l 1_{D_{k_4,j_4}} (f_{k_1,j_1} * f^I_{k_2,j_2} * f^I_{k_3,j_3})\tilde{} \|_{L^2_\tau \ell^2_n}.$$

Further, we split after decomposition in $0 \leq l \leq k_3^*$ the sum over j_4 into $j_4 \leq 2k_1^* + l$ and $j_4 \geq 2k_1^* + l$. For fixed l we find from (5.68)

$$\begin{aligned}
&2^{k_4} \sum_{\substack{[\alpha k_4] \leq j_4 \leq 2k_1^* + l, \\ j_i \geq [\alpha k_1^*],\, i=1,2,3}} 2^{-j_4/2} \sum_I \| P_\Omega^l 1_{D_{k_4,j_4}} (f_{k_1,j_1} * f^I_{k_2,j_2} * f^I_{k_3,j_3})\tilde{} \|_{L^2_\tau \ell^2_n} \\
&\lesssim 2^{k_4} \sum_{[\alpha k_4] \leq j_4 \leq 2k_1^* + l} 2^{-j_4/2} 2^{-j_1^*/2} 2^{l/2} 2^{-[\alpha k_1]/2} 2^{j_4/2} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2} \\
&\lesssim k_1^* 2^{-[\alpha k_1]/2} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n},
\end{aligned}$$

where $f^I_{k_i,j_i}$ for $i = 2,3$ were reassembled by means of almost orthogonality and Cauchy-Schwarz inequality.

For the second part $j_4 \geq 2k_1^* + l$, we take $2^{-j_1^*/2} \leq 2^{-j_4/2}$ to find similarly

$$\begin{aligned}
&2^{k_4} \sum_{j_4 \geq 2k_1^* + l} \sum_{j_1,j_2,j_3 \geq [\alpha k_1^*]} 2^{-j_4/2} \sum_I \| P_\Omega^l 1_{D_{k_4,j_4}} (f_{k_1,j_1} * f^I_{k_2,j_2} * f^I_{k_3,j_3})\tilde{} \|_{L^2_\tau \ell^2_n} \\
&\lesssim 2^{k_4} \sum_{j_4 \geq 2k_1^* + l} 2^{-j_4/2} 2^{l/2} 2^{-[\alpha k_1]/2} \prod_{i=1}^3 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2} \\
&\lesssim 2^{-[\alpha k_1]/2} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n}.
\end{aligned}$$

An estimate with one modulation strictly less than 1/2 follows from slight loss in the highest modulation. We omit the details. □

We turn to $High \times High \times Low \to High$-interaction.

Lemma 5.2.9. *Let $k_4 \geq 20$, $k_1 \leq k_2 \leq k_3$, $k_1 \leq k_2 - 15$, $|k_2 - k_4| \leq 10$ and suppose that $P_{k_i} u_i = u_i$ for $i \in \{1,2,3\}$. Then, we find estimate (5.71) to hold with $D(\alpha, \underline{k}) = 2^{-(1/2-\varepsilon)k_4}$ for any $\varepsilon > 0$.*

Proof. With the reduction steps and notation from above, we have to prove

$$\begin{aligned} &2^{k_4} \sum_{j_4 \geq [\alpha k_4]} 2^{-j_4/2} \sum_{j_1,j_2,j_3 \geq [\alpha k_1^*]} \| 1_{D_{k_4,j_4}} (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\tilde{} \|_{L^2_\tau \ell^2_n} \\ &\lesssim_\varepsilon 2^{-(1/2-\varepsilon)k_4} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n}. \end{aligned} \tag{5.74}$$

We use (5.63) to find

$$\| 1_{D_{k_4,j_4}} (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\tilde{} \|_{L^2_\tau \ell^2_n} \lesssim 2^{-j_1^*/2} 2^{\varepsilon k_1^*} 2^{j_4/2} \prod_{i=1}^3 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n}. \tag{5.75}$$

We find from the resonance relation that $j_1^* \geq 3k_1^* - 10$. Now the estimate follows in a similar spirit to the computation above: Splitting up the sum over j_4 into $[\alpha k_4] \leq j_4 \leq 3k_1^*$ and $j_4 \geq 3k_1^*$, we find

$$\begin{aligned} &2^{k_4} \sum_{[\alpha k_4] \leq j_4 \leq 3k_1^*} 2^{-j_4/2} \sum_{j_1,j_2,j_3 \geq [\alpha k_1^*]} 2^{\varepsilon k_1^*} 2^{-3k_1^*/2} 2^{j_4/2} \prod_{i=1}^3 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n} \\ &\lesssim_\varepsilon 3k_1^* 2^{-k_1^*/2 + (\varepsilon/2)k_1^*} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n} \\ &\lesssim_\varepsilon 2^{-(1/2-\varepsilon)k_4} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n}. \end{aligned}$$

For the remaining part we argue like above

$$\begin{aligned} &\sum_{j_4 \geq 3k_1^*} 2^{-j_4/2} 2^{(\varepsilon/2)k_1^*} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n} \\ &\lesssim 2^{-k_4/2 + \varepsilon k_4} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n}, \end{aligned}$$

and the claim follows. The variant with one function in a strictly less modulation regularity than 1/2 follows from the same considerations like in the previous lemma. ◻

We turn to $High \times High \times High \to High$-interaction. In this case, we do not use a multilinear argument but only the bilinear estimate from Lemma 5.2.6. In the special case $\alpha = 1$, this becomes the analysis from [Mol12]. Additionally, the computation reveals that the interaction under consideration can be estimated for negative Sobolev regularities for $\alpha > 1$.

Lemma 5.2.10. *Let $k_4 \geq 20$ and $|k_i - k_4| \leq 20$ for any $i \in \{1,2,3\}$ and suppose that $P_{k_i} u_i = u_i$ for $i \in \{1,2,3\}$. Then, we find (5.71) to hold with $D(\alpha, \underline{k}) = 2^{-(\alpha/2-1/2)k_4}$ whenever $\alpha \geq 1$.*

Proof. The usual reduction steps lead us to the remaining estimate:

$$\begin{aligned}&\sum_{j_4\geq[\alpha k_4]} 2^{-j_4/2}2^{k_4} \sum_{j_1,j_2,j_3\geq[\alpha k_1^*]} \|1_{D_{k_4,j_4}}(f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3})\tilde{}\|_{L^2_\tau \ell^2_n} \\ &\lesssim 2^{-(\alpha/2-1/2)k_4} \prod_{i=1}^{3} \sum_{j_i\geq[\alpha k_1^*]} 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n}.\end{aligned} \tag{5.76}$$

We use duality to write

$$\|1_{D_{k_4,j_4}}(f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3})\|_{L^2_\tau \ell^2_n} = \sup_{\|u_4\|_{L^2_{t,x}=1}} \int\int u_1u_2u_3u_4 dxdt.$$

where $u_i = \mathcal{F}^{-1}_{t,x}[f_{k_i,j_i}]$ for $i=1,2,3$.

After splitting the expression according to $P_\pm u_i$, where $P_\pm$ projects to only positive, respectively negative frequencies, it is easy to see that two bilinear estimates are applicable.
Indeed, the same sign must appear twice. A pair of this kind is amenable to (5.67) as the output frequency must be of order $2^{k_1^*}$, and the two remaining factors are also amenable to a bilinear estimate.

Say we can apply bilinear estimates to u_4u_1 and u_2u_3. This gives

$$\begin{aligned}&\|1_{D_{k_4,j_4}}(f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3})\|_{L^2_\tau \ell^2_n} \\ &\lesssim 2^{j_1/2}2^{(j_4-k_4)/2}2^{j_2/2}2^{(j_3-k_4)/4}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n} \\ &\lesssim 2^{-k_4/2}2^{j_4/4}2^{-\alpha k_4/4}\prod_{i=1}^{3}2^{j_1/2}\|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n}.\end{aligned}$$

The claim follows after summation over j_4. □

The resonant interaction is a special case of $High \times High \times High \to High$-interaction, but we mention that in this case the same estimate like above can be proved by elementary means.
Next, we deal with $High \times High \times Low \to Low$-interaction:

Lemma 5.2.11. *Let $k_1 \geq 20$, $k_1 \leq k_2 \leq k_3$, $k_1 \leq k_2 - 5$, $k_4 \leq k_2 - 5$ and suppose that $P_{k_i}u_i = u_i$ for $i \in \{1,2,3\}$. Then, we find (5.71) to hold with $D(\alpha,\underline{k}) = 2^{(\alpha/2-1+\varepsilon)k_1}2^{(1-\alpha)k_4}$ for any $\varepsilon > 0$.*

Proof. Contrary to the previous cases, we have to add localization in time in order to estimate u_{k_1} and u_{k_2} in $F^\alpha_{k_1}$ or $F^\alpha_{k_2}$, respectively.
For this purpose let $\gamma : \mathbb{R} \to [0,1]$ be a smooth function supported in $[-1,1]$ with the property

$$\sum_{m\in\mathbb{Z}} \gamma^3(x-m) \equiv 1.$$

We find the left hand-side to be dominated by

$$\begin{aligned}&C2^{k_4} \sum_{|m|\lesssim 2^{[\alpha(k_1-k_4)]}} \sup_{t_{k_4}\in\mathbb{R}} \|\mathcal{F}_{t,x}[u_1\eta_0(2^{\alpha k_4}(t-t_{k_4}))\gamma(2^{\alpha k_1}(t-t_{k_4})-m)]* \\ &\mathcal{F}_{t,x}[u_2\gamma(2^{\alpha k_1}(t-t_k)-m)]*\mathcal{F}_{t,x}[u_3\gamma(2^{\alpha k_1}(t-t_k)-m)]\tilde{}\|_{X_{k_4}}.\end{aligned}$$

With the additional localization in time available, we can annex the modulation variable for $j_i \leq [\alpha k_1^*]$, $i = 1,2,3$ and denote $f_{k_i} = \mathcal{F}[u_i \gamma(2^{k_1}(t-t_k)-m)]$ and with additional localization in modulation we write

$$f_{k_i,j_i} = \begin{cases} \eta_{\leq j_i}(\tau - n^3) f_{k_i}, \ j_i = [\alpha k_1^*], \\ \eta_{j_i}(\tau - n^3) f_{k_i}, \ j_i > [\alpha k_1^*]. \end{cases}$$

With the reduction steps from above, we have to prove

$$\begin{aligned} &2^{\alpha(k_1-k_4)} 2^{k_4} \sum_{j_4 \geq [\alpha k_4]} 2^{-j_4/2} \sum_{j_1,j_2,j_3 \geq [\alpha k_1^*]} \| 1_{\tilde{D}_{k_4,j_4}} (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\tilde{} \|_{L^2_\tau \ell^2_n} \\ &\lesssim 2^{(\alpha/2-1+\varepsilon)k_1} 2^{(1-\alpha)k_4} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n}. \end{aligned} \tag{5.77}$$

Like in the proof of Lemma 5.2.8, the resonance is localized to

$$2^{2k_1^*} \lesssim |\Omega| \lesssim 2^{2k_1^*+k_3^*},$$

and we introduce additional localization P_Ω^l for $|\Omega| \sim 2^{2k_1^*+l}$, where $0 \leq l \leq k_3^*$. Correspondingly, we decompose f_{k_1,j_1} into intervals I of length 2^l, which allows an almost orthogonal decomposition of the output.

Lastly, we split the sum over j_4 into $j_4 \leq 2k_1^* + l$ and $j_4 \geq 2k_1^* + l$. For fixed l we find from (5.68)

$$\begin{aligned} &2^{\alpha(k_1-k_4)} 2^{k_4} \sum_{\substack{[\alpha k_4] \leq j_4 \leq [2k_1^*+l], \\ j_i \geq [\alpha k_1^*]}} 2^{-j_4/2} \\ &\times \left(\sum_I \| P_\Omega^l 1_{\tilde{D}_{k_4,j_4}} (f_{k_1,j_1} * f_{k_2,j_2} * f^I_{k_3,j_3})\tilde{} \|^2_{L^2_\tau \ell^2_n} \right)^{1/2} \\ &\lesssim 2^{k_4} 2^{\alpha(k_1-k_4)} \sum_{\substack{[\alpha k_4] \leq j_4 \leq [2k_1^*+l], \\ j_i \geq [\alpha k_1^*]}} 2^{-j_4/2} 2^{-j_1^*/2} 2^{l/2} 2^{-[\alpha k_1]/2} 2^{j_4/2} \prod_{i=1}^3 2^{j_i/2} \| f_{k_i,j_i} \|_2 \\ &\lesssim k_1^* 2^{\alpha k_1/2} 2^{(1-\alpha)k_4} 2^{-k_1} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_2 \\ &\lesssim 2^{(\alpha/2-1+\varepsilon/2)k_1} 2^{(1-\alpha)k_4} \prod_{i=1}^3 \sum_{j_i} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2}. \end{aligned}$$

Likewise, we find for the contribution of $j_4 \geq 2k_1^* + l$ the bound

$$2^{(\alpha/2-1+\varepsilon/2)k_1} 2^{(1-\alpha)k_4} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i,j_i} \|_2.$$

Summation over l yields the claim. □

At last, we turn to *High* × *High* × *High* → *Low*-interaction:

Lemma 5.2.12. *Let $k_1 \geq 20$, $k_1 \leq k_2 \leq k_3$, $k_1 \geq k_3 - 10$, $k_4 \leq k_1 - 10$ and suppose that $P_{k_i} u_i = u_i$ for $i \in \{1,2,3\}$. Then, we find* (5.71) *to hold with $D(\alpha, \underline{k}) = 2^{(\alpha - 3/2 + \varepsilon)k_1} 2^{(1-\alpha)k_4}$ for any $\varepsilon > 0$.*

Proof. As in the proof of Lemma 5.2.11, we have to add localization in time according to k_1^*. With the notation and conventions from Lemma 5.2.11, we have to show the estimate

$$\begin{aligned} &2^{k_4} 2^{\alpha(k_1 - k_4)} \sum_{j_4 \geq [\alpha k_4]} 2^{-j_4/2} \sum_{j_1, j_2, j_3 \geq [\alpha k_1^*]} \| 1_{\tilde{D}_{k_4, j_4}} (f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3})\tilde{} \|_{L^2_\tau \ell^2_n} \\ &\lesssim_\varepsilon 2^{(1-\alpha)k_4} 2^{(\alpha - 3/2 + \varepsilon)k_1} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i, j_i} \|_{L^2_\tau \ell^2_n}. \end{aligned} \tag{5.78}$$

The resonance function implies $j_1^* \geq 3k_1^* - 15$. We split the sum in (5.78) over j_4 up into $[\alpha k_4] \leq j_4 \leq 3k_1^*$ and $j_4 \geq 3k_1^*$.
The first part is estimated by an application of (5.63)

$$\begin{aligned} &2^{k_4} 2^{\alpha(k_1 - k_4)} \sum_{[\alpha k_4] \leq j_4 \leq 3k_1^*} 2^{-j_4/2} \sum_{j_i \geq [\alpha k_1^*]} \| 1_{\tilde{D}_{k_4, j_4}} (f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3})\tilde{} \|_{L^2_\tau \ell^2_n} \\ &\lesssim_\varepsilon 2^{\alpha k_1} 2^{(1-\alpha)k_4} \sum_{[\alpha k_4] \leq j_4 \leq 3k_1^*} 2^{-j_4/2} \sum_{j_1, j_2, j_3 \geq [\alpha k_1^*]} 2^{-j_1^*/2} 2^{\varepsilon k_1^*} 2^{j_4/2} \prod_{i=1}^3 2^{j_i/2} \| f_{k_i, j_i} \|_{L^2_\tau \ell^2_n} \\ &\lesssim 2^{(\alpha + \varepsilon - 3/2)k_1} 2^{k_4(1-\alpha)} (3k_1^*) 2^{(\alpha - 2)k_1/2} \prod_{i=1}^3 \sum_{j_i \geq [\alpha k_1^*]} 2^{j_i/2} \| f_{k_i, j_i} \|_{L^2_\tau \ell^2_n}. \end{aligned}$$

The estimate for $j_4 \geq 3k_1^*$ follows similarly. This proves the claim together with the standard modification of lowering the modulation regularity slightly. $\square$

We record the estimate for the interaction of low frequencies, which follows in a straight-forward manner from Cauchy-Schwarz inequality.

Lemma 5.2.13. *Let $k_1, \ldots, k_4 \leq 200$. Then, we find* (5.71) *to hold with $D(\alpha, \underline{k}) = 1$.*

We summarize the regularity thresholds for which we can show the trilinear estimate (5.72) by splitting up the frequencies and using the estimate (5.71)

1. *High* $\times$ *Low* $\times$ *Low* $\to$ *High*-interaction: Lemma 5.2.8 provides us with the regularity threshold $s = -(\alpha/4)+$.
2. *High* $\times$ *High* $\times$ *Low* $\to$ *High*-interaction: Lemma 5.2.9 provides us with the regularity threshold $s = -(1/2)+$.
3. *High* $\times$ *High* $\times$ *High* $\to$ *High*-interaction: Lemma 5.2.10 provides us with the regularity threshold $s = (1-\alpha)/4$.
4. *High* $\times$ *High* $\times$ *Low* $\to$ *Low*-interaction: Lemma 5.2.11 provides us with the regularity threshold $s = -(1/6)+$ for $\alpha = 1$.
5. *High* $\times$ *High* $\times$ *High* $\to$ *Low*-interaction: Lemma 5.2.12 provides us with the regularity threshold $s = -(1/6)+$ for $\alpha = 1$.

6. *Low* × *Low* × *Low* → *Low*-interaction: There is no threshold.

This proves the following proposition:

Proposition 5.2.14. *Let $T \in (0,1]$. For $0 < s < 1/2$, there is $\alpha(s) < 1$ and $\theta = \theta(s) > 0$ or for $s = 0$, $\alpha = 1$ and $\theta = 0$ we find the following estimate holds:*

$$\|\mathfrak{N}(u_1, u_2, u_3)\|_{N^{s,\alpha}(T)} \lesssim T^\theta \prod_{i=1}^{3} \|u_i\|_{F^{s,\alpha}(T)}.$$

Furthermore, there is $\delta' > 0$ so that for any $0 < \delta < \delta'$ there is $s = s(\delta) < 0$ and $\theta > 0$ so that

$$\|\mathfrak{N}(u_1, u_2, u_3)\|_{N^{s,1+\delta}(T)} \lesssim T^\theta \prod_{i=1}^{3} \|u_i\|_{F^{s,1+\delta}(T)}.$$

We do not quantify the estimates in detail for negative Sobolev regularity because in this case, we can only prove a conditional result.

5.2.4 Energy estimates

In order to close the iteration, we have to propagate the energy norm. First aim of this section is to prove the following proposition:

Proposition 5.2.15. *If $s > 1/4$, then we find the following estimate to hold*

$$\|u\|^2_{E^s(T)} \lesssim \|u_0\|^2_{H^s} + T\|u\|^4_{F^{s,\alpha}(T)}. \tag{5.79}$$

Furthermore, there are $\theta > 0$ and non-negative functions $c(s)$, $d(s)$ so that we find for any $M \in 2^{\mathbb{N}}$ the estimate

$$\|u\|^2_{E^s(T)} \lesssim \|u_0\|^2_{H^s} + T^\theta M^{c(s)} \|u\|^4_{F^{s,1}(T)} + M^{-d(s)} \|u\|^4_{F^{s,1}(T)} + T^\theta \|u\|^6_{F^{s,1}(T)} \tag{5.80}$$

to hold whenever $s > 0$.

To prove the above estimates, we analyze the energy functional $\|u(t)\|^2_{H^s} = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\hat{u}(t,k)|^2$. Like in Section 5.1.3 we have to take into account the larger class of symbols from Definition 5.1.12.

It is admissible to choose $\varepsilon = \varepsilon(s) > 0$ in the following, but the subsequent estimates must be uniform in ε. We also write $\tilde{a}(2^{k_1^*}) = 2^{2k_1^*(s+\varepsilon)}$ because the expression safely estimates linear combinations of $a(2^{k_i})$.
For $a \in S^s_\varepsilon$ we set

$$\|u(t)\|^2_{H^a} = \sum_{n \in \mathbb{Z}} a(n) |\hat{u}(t,n)|^2$$

and for a solution to (5.52) we compute

$$\begin{aligned}
\partial_t \|u(t)\|_{H^a}^2 &= 2\Re(\sum_{n\in\mathbb{Z}} a(n)(\partial_t \hat{u}(t,n))\hat{u}(t,-n)) \\
&= 2\Re(\sum_{n\in\mathbb{Z}} a(n) i n^3 |\hat{u}(t,n)|^2 + i n a(n) |\hat{u}(t,n)|^2 \hat{u}(t,n)\hat{u}(t,-n) \\
&\quad + i\frac{n}{3} a(n) \sum_{\substack{n=n_1+n_2+n_3,\\ (*)}} \hat{u}(t,n_1)\hat{u}(t,n_2)\hat{u}(t,n_3)\hat{u}(t,-n)) \\
&= C\Im(\sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*)}} \underbrace{(a(n_1)n_1 + a(n_2)n_2 + a(n_3)n_3 + n_4 a(n_4))}_{\psi_{s,a}(\overline{n})} \\
&\quad \times \hat{u}(t,n_1)\hat{u}(t,n_2)\hat{u}(t,n_3)\hat{u}(t,n_4)),
\end{aligned}$$

where the last step follows from a symmetrization argument, which fails for the difference equation. This is a consequence of the lack of uniform continuous dependence for $s < 1/2$.

We set

$$\begin{aligned}
&R_4^{s,a}(T, u_1, u_2, u_3, u_4) \\
&= \sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*)}} \int_0^T \psi_{s,a}(\overline{n}) \hat{u}_1(t,n_1)\hat{u}_2(t,n_2)\hat{u}_3(t,n_3)\hat{u}_4(t,n_4) dt,
\end{aligned}$$

and we write $R_4^{s,a}(T,u) := R_4^{s,a}(T,u,u,u,u)$.

Above we have found that $\|u(t)\|_{H^a}^2 = \|u_0\|_{H^a}^2 + C R_4^{s,a}(t,u)$, and we shall see that the above expression can be estimated as long as $s > 1/4$ in $F^{s,1}(T)$-spaces. However, to go below $s = 1/4$, we have to add a correction term in a similar spirit to the I-method like in Section 5.1.3. But the boundary term will not depend on the length of the time interval anymore. To remedy this, we do not differentiate by parts all of $R_4^{s,a}$ but only the part which contains high frequencies. Precisely, we set for a large frequency $M \in 2^{\mathbb{N}}$

$$R_4^{s,a,M}(T,u) = C \int_0^T \sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*), |n_j| \leq M}} \psi_{s,a}(\overline{n}) \hat{u}(t,n_1)\hat{u}(t,n_2)\hat{u}(t,n_3)\hat{u}(t,n_4) dt$$

and decompose $R_4^{s,a}(t,u) = R_4^{s,a,M}(t,u) + (R_4^{s,a}(t,u) - R_4^{s,a,M}(t,u))$. The frequency cutoff M will be later chosen depending on the norm of the initial value.

Next, we differentiate by parts: We have

$$\partial_t \hat{u}(t,n) + (in)^3 \hat{u}(t,n) = in|\hat{u}(t,n)|^2 \hat{u}(t,n) + \frac{in}{3} \sum_{\substack{n=n_1+n_2+n_3,\\ (*)}} \hat{u}(t,n_1)\hat{u}(t,n_2)\hat{u}(t,n_3),$$

and after changing to interaction picture $\hat{v}(t,n) = e^{-in^3 t}\hat{u}(t,n)$, we find

$$\partial_t \hat{v}(t,n) = in|\hat{v}(n)|^2 \hat{v}(n) + \frac{in}{3} \sum_{\substack{n=n_1+n_2+n_3,\\ (*)}} e^{it\Omega(\overline{n})} \hat{v}(t,n_1)\hat{v}(t,n_2)\hat{v}(t,n_3).$$

The resonance function is given by

$$\Omega(\overline{n}) = n_1^3 + n_2^3 + n_3^3 + n_4^3 = -3(n_1+n_2)(n_1+n_3)(n_2+n_3) \quad (n_1,\ldots,n_4) \in \Gamma_4.$$

Differentiation by parts is possible because the resonance function does not vanish for the terms in $R_4^{s,a}$:

$$\begin{aligned}
R_4^{s,a}(T) &= \int_0^T ds \sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*)}} \psi_{s,a}(\overline{n})\hat{u}(t,n_1)\hat{u}(t,n_2)\hat{u}(t,n_3)\hat{u}(t,n_4) \\
&= \sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*)}} \psi_{s,a}(\overline{n}) \int_0^T dt e^{it\Omega(n_1,n_2,n_3,n_4)} \prod_{i=1}^4 \hat{v}(n_i,t) \\
&= \sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*)}} \psi_{s,a}(\overline{n}) \int_0^T dt \partial_t \left(\frac{e^{it\Omega(n_1,n_2,n_3,n_4)}}{i\Omega(n_1,n_2,n_3,n_4)} \right) \prod_{i=1}^4 \hat{v}(t,n_i) \\
&= [\sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*)}} \frac{\psi_{s,a}(\overline{n})}{i\Omega(\overline{n})} \hat{u}(t,n_1)\hat{u}(t,n_2)\hat{u}(t,n_3)\hat{u}(t,n_4)]_{t=0}^T \\
&\quad + 4 \sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*)}} \frac{\psi_{s,a}(\overline{n})}{i\Omega(\overline{n})} \int_0^T dt (\partial_t \hat{v}(t,n_1))\hat{v}(t,n_2)\hat{v}(t,n_3)\hat{v}(t,n_4) \\
&= B_4(0;T) + I(T) + II(T),
\end{aligned}$$

where

$$I(T) = C \int_0^T ds \sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*)}} \frac{\psi_{s,a}(\overline{n}) n_1}{\Omega(\overline{n})} |\hat{u}(t,n_1)|^2 \hat{u}(t,n_1)\hat{u}(t,n_2)\hat{u}(t,n_3)\hat{u}(t,n_4)$$

and

$$\begin{aligned}
II(T) = C \int_0^T ds \sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*)}} \frac{\psi_{s,a}(\overline{n}) n_1}{\Omega(\overline{n})} \sum_{\substack{n_1+n_5+n_6+n_7=0,\\(*)}} \hat{u}(t,n_2)\hat{u}(t,n_3)\hat{u}(t,n_4) \\
\times \hat{u}(t,n_5)\hat{u}(t,n_6)\hat{u}(t,n_7).
\end{aligned}$$

If we differentiate only $R_4^{s,a}(t,u) - R_4^{s,a,M}(t,u)$, then one of the initial frequencies has to be larger than M.

The following lemma provides us with a pointwise bound on $|\psi_{s,a}|$:

Lemma 5.2.16. *Let $s > 0$, $n_i \in I_{k_i}$ for $i \in \{1,2,3,4\}$ and $a \in S_\varepsilon^s$. Then, we find the following estimate to hold:*

$$|\psi_{s,a}(\overline{n})| \lesssim \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} |\Omega(\overline{n})|. \tag{5.81}$$

The tools we use to derive the pointwise bound are the mean value theorem and the double mean value theorem. To avoid confusion, we state the double mean value theorem.

Lemma 5.2.17. *If y is controlled by z and $|\eta|, |\lambda| \ll |\xi|$, then*

$$y(\xi + \eta + \lambda) - y(\xi + \eta) - y(\xi + \lambda) + y(\xi) = \mathcal{O}(|\eta||\lambda|\frac{z(\xi)}{\xi^2}).$$

Proof. Cf. [CKS$^+$03, Lemma 4.2., p. 715]. □

We are ready to prove Lemma 5.2.16.

Proof of Lemma 5.2.16. We prove the bound by Case-by-Case analysis.
Case 1: $|n_1| \sim |n_2| \sim |n_3| \sim |n_4| \sim 2^{k_1^*}$.

Subcase a: Two of the factors $|n_2 + n_3|, |n_2 + n_4|, |n_3 + n_4|$ are much smaller than $|n_1^*|$ (note that one factor must be of size $|n_1^*|$ because at least two numbers will be of the same sign).
For definiteness suppose in the following that $|n_2 + n_3|, |n_1 + n_2| \ll |n_1^*|$ and from this assumption follows $|\Omega(\overline{n})| \sim |n_1^*||n_2 + n_3||n_1 + n_2|$.
We set $\xi = n_1$, $\xi + \eta = -n_2$, $\xi + \lambda = -n_4$, $\xi + \eta + \lambda = n_3$ in order to check that the assumptions of the double-mean value theorem for the function $a(\cdot)\cdot$ are fulfilled. Hence, from property (ii) of the symbol we find $|\psi_{s,a}| \lesssim \frac{\tilde{a}(2^{k_1^*})}{2^{k_1^*}}|n_2 + n_3||n_1 + n_2|$. Consequently, we find (5.81) to hold in this case.

Subcase b: Next, suppose that one of the factors $|n_2 + n_3|, |n_2 + n_4|, |n_3 + n_4|$ is much smaller than $2^{k_1^*}$ whereat the others are comparable to $2^{k_1^*}$. For definiteness suppose that this is $|n_1 + n_2|$. For the resonance function follows $|\Omega(\overline{n})| \sim (n_1^*)^2|n_1 + n_2|$. We invoke the mean value theorem to find

$$|\psi_{s,a}(\overline{n})| \lesssim \tilde{a}(2^{k_1^*})|n_1 + n_2| \sim \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}}|\Omega(\overline{n})|,$$

which proves the claim in this subcase.

Subcase c: Suppose that all three factors $|n_2 + n_3|, |n_2 + n_4|, |n_3 + n_4|$ are comparable to n_1^*. This gives $|\Omega(\overline{n})| \sim 2^{3k_1^*}$. Using the trivial bound $|\psi_{s,a}(\overline{n})| \lesssim \tilde{a}(2^{k_1^*})2^{k_1^*}$ we find (5.81) to hold also in this subcase.
Case 2: $|n_1| \ll |n_2| \sim |n_3| \sim |n_4| \sim 2^{k_1^*}$.
In this case we have $|\Omega(\overline{n})| \sim 2^{3k_1^*}$; together with the trivial bound the estimate (5.81) is immediate.
Case 3: $|n_1|, |n_2| \ll |n_3| \sim |n_4| \sim |n_1^*|$.
In this case we find $|\Omega(\overline{n})| \sim |n_1 + n_2|(n_1^*)^2$, and an application of the mean value theorem yields

$$|\psi_{s,a}(\overline{n})| \lesssim |n_1 + n_2|\tilde{a}(2^{k_1^*}),$$

which yields the claim in this case. □

Remark 5.2.18. In Sobolev spaces with negative regularity index a related estimate was proven in [CHT12, Lemma 5.2, p. 59]. There, also regularity of the extension was proven. This will allow us to separate variables in Sobolev spaces with negative regularity index.

The second important ingredient to find the bound for $R_4^{s,a}$ is the following improvement of the L^6-Strichartz estimate. We remark that following along the lines of Section 5.2.2 one can derive stronger estimates in some cases. But since this would not improve the overall analysis, we just record the simplified version below.

Lemma 5.2.19. *Suppose that $supp(\mathcal{F}_{t,x}(u_i)) \subseteq \tilde{D}_{k_i,j_i}$, where $j_i \geq [\alpha k_1^*]$ for $1 \leq \alpha \leq 2$ and $k_4^* \leq k_1^* - 10$. Then, we find the following estimate to hold:*

$$\|P_{k_1}\mathcal{N}(u_2,u_3,u_4)\|_{L^2_{t,x}} \lesssim 2^{-k_1^*/2}2^{k_4^*/2}\prod_{i=2}^{4} 2^{j_i/2}\|u_i\|_{L^2_{t,x}}.$$

Proof. After changing to Fourier space and using duality, we rewrite the left hand-side by means of the notation and conventions from Subsection 5.2.2

$$\sup_{\substack{\|\tilde{v}\|_{L^2_\tau \ell^2_n}=1,\\ 1_{I_{k_1}}\tilde{v}=\tilde{v}}} \int d\tau_1 \int d\tau_2 \int d\tau_3 \sum_{n_1,n_2,n_3} \tilde{v}(\tau_1,n_1)g_2(\tau_2,n_2)g_3(\tau_3,n_3)g_4(h_4,n_1+n_2+n_3).$$

Case A: $k_3^* \leq k_1^* - 10$. Suppose that v and g_2 are at low frequencies and further, $|k_3^* - k_4^*| \leq 5$.
In the following we use the same notation like above in Subsection 5.2.3. Observing that $|\partial h_4/\partial n_3| = 3|(n_1+n_2+2n_3)(n_1+n_2)| \gtrsim 2^{k_1^*}$, applications of the Cauchy-Schwarz inequality in n_3 and τ_1 yield

$$\begin{aligned}
&\int d\tau_1 \int d\tau_2 \int d\tau_3 \sum_{n_1,n_2,n_3} \tilde{v}(\tau_1,n_1)g_2(\tau_2,n_2)g_3(\tau_3,n_3)g_4(h_4,n_1+n_2+n_3)\\
&\lesssim \int d\tau_1 \int d\tau_2 \sum_{n_1,n_2} \tilde{v}(\tau_1,n_1)g_2(\tau_2,n_2)(1+2^{j_4-k_1^*})^{1/2}\\
&\quad \times \left(\sum_{n_3} |g_3(\tau_3,n_3)|^2 |g_4(h_4,n_1+n_2+n_3)|^2\right)^{1/2}\\
&\lesssim \int d\tau_2 \int d\tau_3 \sum_{n_2,n_1} g_2(\tau_2,n_2)\left(\int d\tau_1 |\tilde{v}(\tau_1,n_1)|^2\right)^{1/2}\\
&\quad \times \left(\sum_{n_3} |g_3(\tau_3,n_3)|^2 \int d\tau_1 |g_4(h_4,n_1+n_2+n_3)|^2\right)^{1/2}.
\end{aligned}$$

Next, suppose that $k_2 \leq k_1$. Then, an application of the Cauchy-Schwarz inequality in n_2 gives

$$\begin{aligned}
&\lesssim (1+2^{j_4-k_1^*})^{1/2} \int d\tau_2 \int d\tau_3 \sum_{n_1} \left(\int d\tau_1 |\tilde{v}(\tau_1,n_1)|^2\right)^{1/2} \left(\sum_{n_2} |g_2(\tau_2,n_2)|^2\right)^{1/2}\\
&\quad \times \left(\sum_{n_3} |g_3(\tau_3,n_3)|^2\right)^{1/2} \|g_4\|_{L^2_\tau \ell^2_n}\\
&\lesssim (1+2^{j_4-k_1^*})^{1/2} 2^{j_2/2}2^{j_3/2}2^{k_4^*/2}\|\tilde{v}\|_{L^2_\tau \ell^2_n} \prod_{i=2}^{4} \|g_i\|_{L^2_\tau \ell^2_n}
\end{aligned}$$

with the ultimate estimate following by applications of the Cauchy-Schwarz inequality in n_1, τ_2 and τ_3.

Suppose that $|k_3^* - k_4^*| \leq 3$. Since $|\partial^2 h_3 / \partial n_2^2| \gtrsim 2^{k_1^*}$, we find by the above means

$$\begin{aligned}
&\int d\tau_1 \int d\tau_2 \int d\tau_4 \sum_{n_1,n_2,n_4} \tilde{v}(\tau_1,n_1) g_4(\tau_4,n_4) g_2(\tau_2,n_2) g_3(h_3, n_1+n_2+n_4) \\
&\lesssim \int d\tau_1 \int d\tau_2 \int d\tau_4 \sum_{n_1,n_4} \tilde{v}(\tau_1,n_1) g_4(\tau_4,n_4) (1+2^{j_3-k_1^*})^{1/4} \\
&\quad \times \left(\sum_{n_2} |g_2(\tau_2,n_2)|^2 |g_3(h_3,n_1+n_2+n_4)|^2 \right)^{1/2},
\end{aligned}$$

and the proof is concluded by further applications of Cauchy-Schwarz like above.

Next, suppose that v is among the high frequencies and $k_2 = k_4^*$. Then, we can estimate due to $|\partial^2 h_3 / \partial n_4^2| \sim 2^{k_1^*}$

$$\begin{aligned}
&\int d\tau_1 \sum_{n_1} \tilde{v}(\tau_1,n_1) \int d\tau_2 \int d\tau_4 \sum_{n_2} g_2(\tau_2,n_2) \sum_{n_4} g_4(\tau_4,n_4) g_3(h_3,n_1+n_2+n_4) \\
&\lesssim \int d\tau_2 \sum_{n_2} g_2(\tau_2,n_2) \int d\tau_4 \int d\tau_1 \sum_{n_1} \tilde{v}(\tau_1,n_1) (1+2^{j_3-k_1^*})^{1/4} \\
&\quad \times \left(\sum_{n_4} |g_4(\tau_4,n_4)|^2 |g_3(h_3,n_1+n_2+n_4)|^2 \right)^{1/2} \\
&\lesssim (1+2^{j_3-k_1^*})^{1/4} \int d\tau_2 \sum_{n_2} g_2(\tau_2,n_2) \int d\tau_4 \sum_{n_1} \left(\int d\tau_1 |\tilde{v}(\tau_1,n_1)|^2 \right)^{1/2} \\
&\quad \times \left(\sum_{n_4} |g_4(\tau_4,n_4)|^2 \int d\tau_1 |g_3(h_3,-n_1-n_2-n_4)|^2 \right)^{1/2},
\end{aligned}$$

and the claim follows from further applications of the Cauchy-Schwarz inequality in n_1, τ_4, n_2 and τ_2.

Next, we turn to the case, where $k_3 \leq k_1 - 10$, $|k_1 - k_2|, |k_1 - k_4| \leq 5$.
Like above we estimate by the Cauchy-Schwarz inequality

$$\begin{aligned}
&\int d\tau_1 \int d\tau_2 \int d\tau_3 \sum_{n_1,n_3} |g_3(\tau_3,n_3)| |\tilde{v}(\tau_1,n_1)| \sum_{n_2} |g_4(h_4,-n_1-n_2-n_3)| |g_2(\tau_2,n_2)| \\
&\lesssim (1+2^{j_4-k_1^*})^{1/2} \int d\tau_1 \int d\tau_2 \int d\tau_3 \sum_{n_1,n_3} |g_3(\tau_3,n_3)| |\tilde{v}(\tau_1,n_1)| \\
&\quad \times \left(\sum_{n_2} |g_4(h_4,-n_1-n_2-n_3)|^2 |g_2(\tau_2,n_2)|^2 \right)^{1/2},
\end{aligned}$$

where the estimate follows from $|\partial_{n_2} h_4| \gtrsim 2^{k_1^*}$ provided that $n_1 + 2n_2 + n_3 \neq 0$. There is at most one n_2 satisfying this relation. The degeneration of the sum in this special case implies the claim in general.

We continue like above with applications of the Cauchy-Schwarz inequality in

τ_1, n_1, τ_2, τ_3 and n_3 to find

$$\begin{aligned}
&\lesssim (1+2^{j_4-k_1^*})^{1/2} \int d\tau_2 \int d\tau_3 \sum_{n_1,n_3} |g_3(\tau_3,n_3)| \left(\int d\tau_1 |\tilde{v}(\tau_1,n_1)|^2 \right)^{1/2} \\
&\quad \times \left(\sum_{n_2} \|g_4(h_4,-n_1-n_2-n_3)\|^2_{L^2_{\tau_1}} |g_2(\tau_2,n_2)|^2 \right)^{1/2} \\
&\lesssim (1+2^{j_4-k_1^*})^{1/2} 2^{k_3/2} 2^{j_2/2} 2^{j_3/2} \prod_{i=2}^4 \|g_i\|_{L^2_\tau \ell^2_n} \\
&\lesssim 2^{-k_1^*/2} 2^{k_4^*/2} \prod_{i=2}^4 \|g_i\|_{L^2_\tau \ell^2_n}.
\end{aligned}$$

The same argument proves the claim if $k_1 \leq k_3 - 10$, $|k_2 - k_3| \leq 5$, $|k_3 - k_4| \leq 5$ by exchanging the application of the Cauchy-Schwarz inequality in n_1 and n_3 in the penultimate estimate. □

With the above two lemmas at our disposal we can find a bound on $R_4^{s,a,M}$.

Proposition 5.2.20. *Let $\alpha = 1$. Then, there are functions $c(s) \geq 0$ (with $c(s) = 0$ for $s > 1/4$) and $\varepsilon(s) > 0$ so that for any $M \in 2^{\mathbb{N}}$ we find the estimate*

$$R_4^{s,a,M}(T,u_1,\ldots,u_4) \lesssim T M^{c(s)} \prod_{i=1}^4 \|u_i\|_{F^{s,\alpha}(T)} \tag{5.82}$$

to hold whenever $s > -1/2$, $a \in S^s_\varepsilon$.

Proof. Firstly, we apply a dyadic decomposition on the spatial frequencies. We estimate $R_4^{s,a,M}(T,u_1,\ldots,u_4)$ for frequency localized functions u_i, where $P_{k_i} u = u$, $k_i \leq \log_2(M)$. For these functions we will show the estimate

$$R_4^{s,a,M}(T,u_1,\ldots,u_4) \lesssim T \prod_{i=1}^4 2^{(s-)k_i} \|u_i\|_{F^\alpha_{k_i}} \tag{5.83}$$

for $s > 1/4$.

The slightly less regularity than s on the right hand-side allows us to sum over dyadic blocks in the end. With the frequencies being smaller than M, from (5.83) for $s > 1/4$ follows already (5.82) for $s > 0$. The next reduction, independently from the interactions we consider below, is to localize time antiproportionally to the highest frequency.

Let $\gamma : \mathbb{R} \to [0,1]$ a smooth function with support in $[-1,1]$ and the property

$$\sum_{m \in \mathbb{Z}} \gamma^4(x-m) \equiv 1.$$

By means of this function write

$$
\begin{aligned}
&R_4^{s,a}(T,u_1,\ldots,u_4)\\
&=\int_0^T dt \sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*)}} \psi_{s,a}(\overline{n})\hat{u}_1(t,n_1)\hat{u}_2(t,n_2)\hat{u}_3(t,n_3)\hat{u}_4(t,n_4)\\
&=\int_0^T dt \sum_{m\in\mathbb{Z}} \sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*)}} \psi_{s,a}(\overline{n})\gamma(2^{k_1^*}t-m)\hat{u}_1(t,n_1)\\
&\quad\times \gamma(2^{\alpha k_1^*}t-m)\hat{u}_2(t,n_2)\gamma(2^{\alpha k_1^*}t-m)\hat{u}_3(t,n_3)\gamma(2^{\alpha k_1^*}t-m)\hat{u}_4(t,n_4).
\end{aligned}
$$

Already note that there are $\mathcal{O}(T2^{k_1^*})$ values of m for which the above expression does not vanish.
We localize with respect to the modulation:
With $f_{k_i}(\tau,\xi)=\mathcal{F}_{t,x}(\gamma(2^{\alpha k_1^*}t-m)u_i)(\tau,\xi)$ we set[3]

$$
f_{k_i,j_i}=\begin{cases}\eta_{\leq j_i}(\tau-n^3)f_{k_i},\ j_i=[\alpha k_1^*],\\ \eta_{j_i}(\tau-n^3)f_{k_i},\ j_i>[\alpha k_1^*].\end{cases}
$$

In the above sum, in case of non-vanishing contribution, we have to distinguish between the two cases:

$$
\begin{aligned}
\mathcal{A}&=\{m\in\mathbb{Z}|1_{[0,T]}(\cdot)\gamma(2^{\alpha k_1^*}\cdot -m)\equiv\gamma(2^{\alpha k_1^*}\cdot -m)\},\\
\mathcal{B}&=\{m\in\mathbb{Z}|1_{[0,T]}(\cdot)\gamma(2^{\alpha k_1^*}\cdot -m)\neq\gamma(2^{\alpha k_1^*}\cdot -m) \text{ and } 1_{[0,T]}(\cdot)\gamma(2^{\alpha k_1^*}\cdot -m)\neq 0\}.
\end{aligned}
$$

Note that $\#\mathcal{B}\leq 4$. Consequently, we save a factor $2^{k_1^*}$ compared to $\mathcal{A}$, and we give the necessary modifications after we have carried out the estimates in the bulk of the cases, which corresponds to $\mathcal{A}$. We have $\#\mathcal{A}\lesssim T2^{k_1^*}$.

First, we estimate $High\times High\times High\times High$-interaction: We require $|k_1-k_i|\leq 20$ for $i\in\{2,3,4\}$. From Lemma 5.2.16 we find $|\psi_{s,a}(\overline{n})|\lesssim \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}}|\Omega(\overline{n})|$. In order to be in the position to separate variables, we introduce another dyadic sum governing the size of $|\Omega(\overline{n})|$. Below we take $|\Omega(\overline{n})|\sim 2^k$ where $k_1^*-20\leq k\leq 3k_1^*+20$ and sum over k in the end.

We observe the following estimate:

$$
\begin{aligned}
\sum_{\substack{k_1^*\leq k\leq 3k_1^*,\\ |\Omega|\sim 2^k}} \sum_{m\in\mathcal{A}} \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}}|\Omega|^{1/2} &\lesssim T \sum_{\substack{k_1^*\leq k\leq 3k_1^*,\\ |\Omega|\sim 2^k}} \frac{\tilde{a}(2^{k_1^*})}{2^{k_1^*}}2^{k/2}\lesssim T\tilde{a}(2^{k_1^*})2^{k_1^*/2}\\
&\lesssim T\prod_{i=1}^4 2^{(s-)k_i}
\end{aligned}
\tag{5.84}
$$

whenever $s>1/4$.

This computation is enough to prove the claim when all frequencies are comparable. Indeed, suppose from symmetry that $j_1=j_1^*$ and together with the resonance

[3] Strictly speaking, we had to consider f_{m,k_i} or f_{m,k_i,j_i}, respectively, tracking the additional dependence on m, but with all the estimates below being uniform in m, we choose to drop dependence on m for the sake of brevity.

relation $2^{j_1^*} \gtrsim |\Omega(\overline{n})| \sim 2^k$ and three $L^6_{t,x}$-Strichartz estimate we find

$$\begin{aligned}
&\int_{\sum_i \tau_i = 0} \sum_{\substack{\sum_i n_i = 0, \\ (*)}} f_{k_1,j_1}(\tau_1,n_1) f_{k_2,j_2}(\tau_2,n_2) f_{k_3,j_3}(\tau_3,n_3) f_{k_4,j_4}(\tau_4,n_4) \\
&\lesssim 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2_\tau \ell^2_n} 2^{-k/2} \|1_{I_{k_1}} (f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4})\tilde{}\|_{L^2_\tau \ell^2_n} \\
&\lesssim 2^{-(k/2)} \prod_{i=1}^4 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n},
\end{aligned} \tag{5.85}$$

and (5.83) follows from (5.84) and (5.85) due to (2.48).

We turn to $High \times High \times Low \times Low$-interaction: Suppose that $|k_1 - k_2| \leq 5$ and $k_3, k_4 \leq k_1 - 10$. From Lemma 5.2.16 and dividing up the magnitude of the resonance function into the dyadic sum $\sum_{\substack{2k_1^* - 5 \leq k \leq 2k_1^* + k_3^* + 5, \\ |\Omega| \sim 2^k}}$, we find

$$\begin{aligned}
&\sum_{\substack{2k_1^* - 5 \leq k \leq 2k_1^* + k_3^* + 5, \\ |\Omega| \sim 2^k}} \sum_{m \in \mathcal{A}} \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} |\Omega|^{1/2} 2^{k_4^*/2} 2^{-k_1^*/2} \\
&\lesssim T \sum_{\substack{0 \leq k \leq k_3^*, \\ |\Omega| \sim 2^k}} \frac{\tilde{a}(2^{k_1^*})}{2^{k_1^*}} 2^{k/2} 2^{k_4^*/2} 2^{-k_1^*/2} \\
&\lesssim T \tilde{a}(2^{k_1^*}) 2^{k_3^*/2} 2^{k_4^*/2} 2^{-k_1^*/2} \lesssim T \prod_{i=1}^4 2^{(s-)k_i}
\end{aligned}$$

whenever $s > 1/4$.

This yields the claim because we can again suppose that $j_1 = j_1^*$, and together with the resonance identity $2^{j_1^*} \gtrsim |\Omega(\overline{n})| \sim 2^k$ and the improved Strichartz estimate from Lemma 5.2.19, it follows

$$\begin{aligned}
&\int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \sum_{\substack{n_1+n_2+n_3+n_4=0, \\ (*)}} \prod_{i=1}^4 f_{k_i,j_i}(\tau_i,n_i) \\
&\lesssim 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2_\tau \ell^2_n} 2^{-k/2} \|1_{I_{k_1}} (f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4})\tilde{}\|_{L^2_\tau \ell^2_n} \\
&\lesssim 2^{-k/2} 2^{k_4^*/2} 2^{-k_1^*/2} \prod_{i=1}^4 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n}.
\end{aligned}$$

At last, we estimate $High \times High \times High \times Low$-interaction: Suppose that $k_1 \geq \ldots \geq k_4$, $|k_1 - k_3| \leq 10$ and $k_4 \leq k_3 - 10$. Lemma 5.2.16 together with the magnitude of the resonance function $|\Omega| \sim 2^{3k_1^*}$ leads us to consider

$$\begin{aligned}
\sum_{m \in \mathcal{A}} \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} |\Omega|^{1/2} 2^{k_4^*/2} 2^{-k_1^*/2} &\lesssim T \frac{\tilde{a}(2^{k_1^*})}{2^{k_1^*}} |\Omega|^{1/2} 2^{k_4^*/2} 2^{-k_1^*/2} \lesssim T \tilde{a}(2^{k_1^*}) 2^{k_4^*/2} \\
&\lesssim T \prod_{i=1}^4 2^{(s-)k_i}
\end{aligned}$$

whenever $s > 1/4$. This proves the claim due to

$$\int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*)}} \prod_{i=1}^{4} f_{k_i,j_i}(\tau_i,n_i)$$
$$\lesssim 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2_\tau \ell^2_n} |\Omega|^{-1/2} \|1_{I_{k_1}} (f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4})\tilde{}\|_{L^2_\tau \ell^2_n}$$
$$\lesssim |\Omega|^{-1/2} 2^{k_4^*/2} 2^{-k_1^*/2} \prod_{i=1}^{4} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n}.$$

We turn to the cases from $\mathcal{B}$. We have to estimate the expression

$$\frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} \int_{\tau_1+\ldots+\tau_4=0} \sum_{\substack{n_1+\ldots+n_4=0,\\ (*)}} |\Omega(\overline{n})| f_{k_1,j_1}(\tau_1,n_1) f_{k_2,j_2}(\tau_2,n_2)$$
$$f_{k_3,j_3}(\tau_3,n_3) f_{k_4,j_4}(\tau_4,n_4),$$

where

$$f_{k_1}(\tau,n) = \mathcal{F}_{t,x}[1_{[0,T]}(t)\gamma(2^{k_1^*}t - n) u_{k_1}(t,x)].$$

The additional decomposition in modulation is given by $f_{k_1} = \sum_{j \geq 0} f_{k_1,j}$.

Suppose below that $j_1 = j_1^*$. Like above we find by three $L^6_{t,x}$-Strichartz estimates

$$\sum_{k,j_i} \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} 2^k \|f_{k_1,j_1}(\tau_1,n_1)\|_{L^2_\tau \ell^2_n} \|1_{I_{k_1}} (f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4})\tilde{}\|_{L^2_\tau \ell^2_n}$$
$$\lesssim \sum_{k,j_i} \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} 2^k 2^{j_1(1/2-\varepsilon)} 2^{-j_1(1/2-\varepsilon)} \|f_{k_1,j_1}\|_{L^2_\tau \ell^2_n} \|1_{I_{k_1}} (f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4})\tilde{}\|_{L^2_\tau \ell^2_n}$$
$$\lesssim_\varepsilon \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} 2^{(3/2+\varepsilon)k_1^*} T^\theta \prod_{i=1}^{4} \|u_i\|_{F^\alpha_{k_i}},$$

where the ultimate estimate follows from Lemma 2.5.4, Lemma 2.5.3 and (2.48). The same argument handles $j_2 = j_1^*$. □

We prove the estimate for the boundary term:

Lemma 5.2.21. *Suppose that $s \in (-1/2, 1/2)$. Then, we find the following estimate to hold:*

$$B_4^{s,a,M}(0;T) \lesssim M^{-d(s)} \|u\|^4_{F^{s,\alpha}(T)}, \tag{5.86}$$

where $d(s) > 0$.

Proof. We localize frequencies on a dyadic scale, i.e., $P_{k_i} u_i = u_i$ and suppose by symmetry that $k_1 \geq k_2 \geq k_3 \geq k_4$. Below let $m = \log_2(M)$. By virtue of Lemma 2.5.1, it is enough to derive a bound in terms of the Sobolev norms. For the evaluation at $t = 0$ we have from Lemma 5.2.16 and an application of Hölder's inequality in position space

$$\frac{\tilde{a}(2^{k_1})}{2^{2k_1}} \sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*), |n_1| \geq M}} |\hat{u}_1(0,n_1)| |\hat{u}_2(0,n_2)| |\hat{u}_3(0,n_3)| |\hat{u}_4(0,n_4)|$$
$$\lesssim \frac{\tilde{a}(2^{k_1})}{2^{2k_1}} \left(\|u_1(0)\|_{L^2_x} \|u_2(0)\|_{L^2_x} \|u_3'(0)\|_{L^\infty_x} \|u_4'(0)\|_{L^\infty_x} \right),$$

where $u_l'(0,x) = \sum |\hat{u}_l(0,k)| e^{ikx}$, $l \in \{3,4\}$.

After applying Bernstein's inequality, we sum up the dyadic pieces:

$$\sum_{\substack{k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0, \\ (*), k_1 \geq m}} \frac{\tilde{a}(2^{k_1})}{2^{2k_1}} \|u_1(0)\|_{L^2_x} \|u_2(0)\|_{L^2_x} 2^{k_3/2} \|u_3(0)\|_{L^2_x} 2^{k_4/2} \|u_4(0)\|_{L^2_x}$$

$$\lesssim \sum_{k_1 \sim k_2 \geq m} \frac{\tilde{a}(2^{k_1})}{2^{k_1}} 2^{-2sk_1} \|u_1(0)\|_{L^2_x} \|u_2(0)\|_{L^2_x} \|u(0)\|^2_{H^s_x}$$

$$\lesssim M^{-d(s)} \|u(0)\|^4_{H^s_x},$$

which we can arrange as long as $s > -1/2$ by choosing $\varepsilon = \varepsilon(s)$ sufficiently small. □

Next, we derive the crucial bound for the correction term $R_6^{s,a,M}(T, u_1, \ldots, u_6)$. Since the frequency constraint has become irrelevant, we drop it in the following.

Proposition 5.2.22. *If $s > 0$, $\alpha = 1$ and $a \in S^s_\varepsilon$ for some $\varepsilon = \varepsilon(s) > 0$, then we find the following estimate to hold:*

$$R_6^{s,a,M}(T, u_1, \ldots, u_6) \lesssim T \prod_{i=1}^{6} \|u_i\|_{F^{s,\alpha}(T)}. \tag{5.87}$$

Proof. First, we estimate the term $I(T)$.
We use the same reductions like in the proof of Proposition 5.2.20: Again, we firstly apply a decomposition into intervals in frequency space. That is we estimate $R_6^{s,a,M}(T, u_1, \ldots, u_6)$ for frequency localized functions u_i, where $P_{k_i} u = u$. For these functions we will show the estimate

$$R_6^{s,a,M}(T) \lesssim T \prod_{i=1}^{6} 2^{(s-)k_i} \|u_i\|_{F^\alpha_{k_i}} \tag{5.88}$$

and (5.87) will follow from (5.88) by summing over dyadic pieces. We localize time antiproportionally to the highest frequency.

Let $\gamma : \mathbb{R} \to [0,1]$ a smooth function with support in $[-1,1]$ and the property

$$\sum_{m \in \mathbb{Z}} \gamma^6(x - m) \equiv 1.$$

With this function we write

$$\begin{aligned} I(T,u) &= \int_0^T dt \sum_{\substack{n_1+n_2+n_3+n_4=0, \\ (*)}} \frac{\psi_{s,a}(\overline{n}) n_1}{\Omega(\overline{n})} |\hat{u}(t,n_1)|^2 \hat{u}(t,n_1)\hat{u}(t,n_2)\hat{u}(t,n_3)\hat{u}(t,n_4) \\ &= \int_0^T dt \sum_{m \in \mathbb{Z}} \sum_{\substack{n_1+n_2+n_3+n_4=0, \\ (*)}} \frac{\psi_{s,a}(\overline{n}) n_1}{\Omega(\overline{n})} \gamma(2^{\alpha k_1^*} t - m) \hat{u}_1(t,n_1) \\ &\quad \times \overline{\gamma(2^{k_1^*} t - m) \hat{u}_1(t,n_1)} \gamma(2^{k_1^*} t - m) \hat{u}_1(t,n_1) \gamma(2^{k_1^*} t - m) \hat{u}_2(t,n_2) \times \\ &\quad \times \gamma(2^{k_1^*} t - m) \hat{u}_3(t,n_3) \gamma(2^{k_1^*} t - m) \hat{u}_4(t,n_4). \end{aligned}$$

Below we carry out the argument for the majority of the cases, where the smooth cutoff function does not interact with the sharp cutoff. This contribution we denote

by $I_{\mathcal{A}}(T,u)$, so that $I(T,u) = I_{\mathcal{A}}(T,u) + I_{\mathcal{B}}(T,u)$.
We denote $f_{k_{11}}(\tau,n) = \mathcal{F}_t[\gamma(2^{k_1^*}\cdot -m)\hat{u}_1(\cdot,n)]$, $f_{k_{12}}(\tau,n) = \mathcal{F}_t[\overline{\gamma(2^{k_1^*}\cdot -m)\hat{u}_1(\cdot,n)}]$, $f_{k_{13}}(\tau,n) = \mathcal{F}_t[\gamma(2^{k_1^*}\cdot -m)\hat{u}_1(\cdot,n)]$ and $f_j(\tau,n) = \mathcal{F}_t[\gamma(2^{k_1^*}\cdot -m)\hat{u}_j(\cdot,n)]$ for $j \in \{2,3,4\}$.
We localize with respect to modulation as follows:

$$f_{j_i} = \begin{cases} \eta_{\leq j_i}(\tau - n^3)f, \ j_i = [\alpha k_1^*], \\ \eta_{j_i}(\tau - n^3) f, \ j_i > [\alpha k_1^*]. \end{cases}$$

After changing also to the Fourier side in the time variable and applying the triangle inequality, we estimate by Lemma 5.2.16 the multiplier

$$\left| \frac{\psi_{s,a}(\overline{n})}{\Omega(\overline{n})} n_1 \right| \lesssim \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} 2^{k_1}.$$

This leaves us with estimating the following expression where we assume the space-time Fourier transforms to be non-negative:

$$\sum_{j_{11},j_{12},j_{13},j_2,j_3,j_4} \sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*)}} \int_{\tau_{11}+\tau_{12}+\tau_{13}+\tau_2+\tau_3+\tau_4=0} f_{k_{11},j_{11}}(\tau_{11},n_1) \tag{5.89}$$
$$f_{k_{12},j_{12}}(\tau_{12},n_1) f_{k_{13},j_{13}}(\tau_{13},n_1) f_{k_2,j_2}(\tau_2,n_2) f_{k_3,j_3}(\tau_3,n_3) f_{k_4,j_4}(\tau_4,n_4)$$

We apply the Cauchy-Schwarz inequality in the modulation variables and n_1 to find

$$\begin{aligned}
(5.89) &= \sum_{n_1} \int_{\sum_i \tau_i = 0} \prod_{i=1}^{3} f_{k_{1i},j_{1i}}(\tau_{1i},n_1) \sum_{n_2,n_3,(*)} \prod_{i=2}^{4} f_{k_i,j_i} \\
&\leq \sum_{n_1} \left(\int |f_{k_{11},j_{11}} *_\tau f_{k_{12},j_{12}} *_\tau f_{k_{13},j_{13}}|^2 \right)^{1/2} \left(\int |f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4}|^2 \right)^{1/2} \\
&\leq \left(\sum_{n_1} \int |f_{k_{11},j_{11}} f_{k_{12},j_{12}} f_{k_{13},j_{13}}|^2 \right)^{1/2} \left(\sum_{n_1} \int \left| \sum_{\substack{n_2,n_3,\\ (*)}} f_{k_2,j_2} f_{k_3,j_3} f_{k_4,j_4} \right|^2 \right)^{1/2}.
\end{aligned}$$

The first term we estimate by Young's inequality in τ, Hölder in n_1 and by the embedding $\ell^2 \hookrightarrow \ell^3$ to find

$$\begin{aligned}
&\sum_{n_1} \left(\int_\tau |f_{k_{11},j_{11}}(n_1) *_\tau f_{k_{12},j_{12}}(n_1) *_\tau f_{k_{13},j_{13}}(n_1)|^2 \right)^{1/2} \\
&\lesssim \sum_{n_1} \| f_{k_{11},j_{11}}(n_1) \|_{L^2_\tau} 2^{j_{12}/2} \| f_{k_{12},j_{12}}(n_1) \|_{L^2_\tau} 2^{j_{13}/2} \| f_{k_{13},j_{13}} \|_{L^2_\tau} \\
&\lesssim 2^{j_{12}/2} 2^{j_{13}/2} \prod_{i=1}^{3} \| f_{k_{1i},j_{1i}} \|_{L^2_\tau \ell^2_n}.
\end{aligned}$$

The second term is amenable to the refined $L^6_{t,x}$-Strichartz estimate from Lemma 5.2.19 in case of separated frequencies and three $L^6_{t,x}$-Strichartz estimates in case of non-separated frequencies. This yields

$$(5.89) \lesssim 2^{-(k_1^*/2)-} \prod_{i=1}^{3} 2^{j_{1i}/2} \| f_{k_{1i},j_{1i}} \|_{L^2_\tau \ell^2_n} \prod_{i=2}^{4} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n}.$$

Gathering all factors, we have found the estimate

$$I_{\mathcal{A}}(T,u_1,\ldots,u_4) \lesssim T \sum_{k_1 \leq k_1^*} \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} 2^{k_1} 2^{k_1^*} 2^{-(k_1^*/2)-}$$

$$\prod_{i=1}^{3} \sum_{j_{1i} \geq k_1^*} 2^{j_{1i}/2} \|f_{k_{1i},j_{1i}}\|_{L^2_\tau \ell^2_n} \prod_{i=2}^{4} \sum_{j_i \geq k_1^*} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n},$$

and estimate (5.88) follows even for negative s due to (2.48).

Next, we turn to the exceptional cases, where

$$\gamma(2^{k_1^*} \cdot -m) 1_{[0,T]}(\cdot) \neq \gamma(2^{k_1^*} \cdot -m).$$

In the above argument replace $f_{k_{11}} = \mathcal{F}_{t,x}[1_{[0,T]}(\cdot)\gamma(2^{k_1^*} \cdot -m)]u_1$ and decompose $f_{k_{11}} = \sum_{j_{11} \geq 0} f_{k_{11},j_{11}}$.
Two applications of Cauchy-Schwarz inequality give

$$\|1_{I_{k_1}}(f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4})\|_{L^2_\tau \ell^2_n} \lesssim 2^{k_2/2} 2^{k_3/2} 2^{j_2/2} 2^{j_3/2} \prod_{i=2}^{4} \|f_{k_i,j_i}\|_{L^2_\tau \ell^2} \tag{5.90}$$

and interpolating this bound with the (refined) $L^6_{t,x}$-Strichartz estimate yields

$$(5.89) \lesssim 2^{(0+)k_1^*} \|f_{k_{11},j_{11}}\|_{L^2_\tau \ell^2_n} \prod_{i=2}^{3} 2^{j_{1i}/2} \|f_{k_{1i},j_{1i}}\|_{L^2_\tau \ell^2_n} \prod_{i=2}^{4} 2^{j_i(1/2)-} \|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n}.$$

Gathering all factors, we have found the estimate

$$I_{\mathcal{B}} \lesssim \sum_{k_1 \leq k_1^*} \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} 2^{k_1} 2^{(0+)k_1^*} \sum_{j_{11} \geq 0} \|f_{k_{11},j_{11}}\|_{L^2_\tau \ell^2_n}$$

$$\prod_{i=2}^{3} \sum_{j_{1i} \geq k_1^*} \|f_{k_{1i},j_{1i}}\|_{L^2_\tau \ell^2_n} \prod_{i=2}^{4} \sum_{j_i \geq k_1^*} 2^{j_i(1/2)-} \|f_{k_i,j_i}\|_{L^2_\tau \ell^2_n}.$$

The claim now follows from Lemma 2.5.4 and Lemma 2.5.3.

We turn to the more involved estimate of $II(T)$. With the notation and following the reductions from above, we show that

$$II(T,u_1,\ldots,u_6) = II_{\mathcal{A}} + II_{\mathcal{B}} \lesssim T^\theta \prod_{i=1}^{6} 2^{(s-)k_i} \|u_i\|_{F^\alpha_{k_i}}.$$

Also, we use an additional dyadic decomposition for n_1. That means we assume in the following $n_1 \in I_{k_1}$ and additionally sum over k_1. We denote the decreasing arrangements of k_2, k_3, k_4 by a_1^*, a_2^*, a_3^* and of k_5, k_6, k_7 by b_1^*, b_2^*, b_3^*. Note that $k_1 \leq a_1^* + 5$ due to impossible frequency interaction. We distinguish the cases $k_1^* = a_1^*$ and $k_1^* = b_1^*$.
Case A: $k_1^* = a_1^*$:

We localize time according to k_1^*. Lemma 5.2.16 yields the estimate

$$\left| \frac{|\psi_{s,a}(\overline{n})|}{|\Omega(\overline{n})|} n_1 \right| \lesssim \frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}} 2^{k_1}.$$

After introducing an additional partition in the modulation variables (although we do not write out the sum anymore) and applying the triangle inequality, we arrive at the following expression:

$$\begin{aligned}
&\sum_{0\le k_1\le k_1^*+5} 2^{k_1}\frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}}\int_{\tau_1+\ldots+\tau_7=0}\sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*),n_1\in I_{k_1}}}\prod_{i=2}^{4} f_{k_i,j_i}(\tau_i,n_i)\\
&\times \sum_{\substack{n_1+n_5+n_6+n_7=0,\\(*),n_1\in I_{k_1}}} f_{k_5,j_5}(\tau_5,n_5)f_{k_6,j_6}(\tau_6,n_6)f_{k_7,j_7}(\tau_7,n_7)\\
\lesssim &\sum_{0\le k_1\le k_1^*+5} 2^{k_1}\frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}}\|1_{I_{k_1}}(f_{k_2,j_2}*f_{k_3,j_3}*f_{k_4,j_4})\tilde{}\|_{L^2_\tau\ell^2_n}\\
&\times\|1_{I_{k_1}}(f_{k_5,j_5}*f_{k_6,j_6}*f_{k_7,j_7})\tilde{}\|_{L^2_\tau\ell^2_n}.
\end{aligned}$$

Now we apply the (refined) $L^6_{t,x}$-Strichartz estimate twice to find the following bound for the integral:

$$\begin{aligned}
&\lesssim \sum_{1\le k_1\le a_1^*} 2^{k_1}\frac{\tilde{a}(2^{k_1^*})}{2^{2k_1^*}}2^{(0+)a_1^*}2^{-a_1^*/2}2^{a_3^*/2}\prod_{i=2}^{4}2^{j_i/2}\|f_{k_i,j_i}\|_{L^2_\tau\ell^2_n}\\
&\quad\times 2^{(0+)b_1^*}2^{-b_1^*/2}2^{b_3^*/2}\prod_{i=5}^{7}2^{j_i/2}\|f_{k_i,j_i}\|_{L^2_\tau\ell^2_n}\\
&\lesssim \frac{\tilde{a}(2^{k_1^*})}{2^{k_1^*}}2^{-k_1^*/2}2^{a_3^*/2}2^{-b_1^*/2}2^{b_3^*/2}\prod_{i=2}^{7}2^{j_i/2}\|f_{k_i,j_i}\|_{L^2_\tau\ell^2_n}.
\end{aligned}$$

Taking the time localization into account gives an additional factor of $T2^{k_1^*}$, and we find (5.88) to hold for $s>0$ after summing over j_i and invoking (2.48) in case the sharp cutoff does not interact with the smooth functions.
If we are in the exceptional case that

$$1_{[0,T]}(\cdot)\gamma(2^{k_1}\cdot-m)\neq\gamma(2^{k_1}\cdot-m),$$

we interpolate the above estimate with (5.90) to find

$$\frac{\tilde{a}(2^{k_1^*})}{2^{k_1^*}}2^{(0+)k_1^*}2^{-k_1^*/2}2^{-b_1^*/2}2^{b_3^*/2}2^{k_1^*/2}\prod_{i=2}^{7}2^{(j_i/2)-}\|f_{k_i,j_i}\|_{L^2_\tau\ell^2_n},$$

where $f_{k_2}=\mathcal{F}_{t,x}[1_{[0,T]}(\cdot)\gamma(2^{k_1^*}\cdot-m)u_2]$ and $j_2\ge 0$. Now the claim follows from Lemma 2.5.3 and Lemma 2.5.4.
Case B: $k_1^*=b_1^*$:

Localize time according to a_1^*: Let γ be a smooth function supported in $[-1,1]$ such that $\sum_{m\in\mathbb{Z}}\gamma^4(\cdot-m)=1$ and write

$$\begin{aligned}
&\sum_m\sum_{0\le k_1\le a_1^*+5}\int_{\mathbb{R}}dt1_{[0,T]}(t)\sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*)}}\frac{\psi_{s,a}(\overline{n})}{\Omega(\overline{n})}n_1 1_{I_{k_1}}(n_1)\gamma(2^{a_1^*}t-m)\hat{u}_2(t,n_2)\\
&\times\gamma(2^{a_1^*}t-m)\hat{u}_3(t,n_3)\gamma(2^{a_1^*}t-m)\hat{u}_4(t,n_4)\\
&\times\sum_{\substack{n_1+n_5+n_6+n_7=0,\\(*)}}\gamma(2^{a_1^*}t-m)\hat{u}_5(t,n_5)\hat{u}_6(t,n_6)\hat{u}_7(t,n_7).
\end{aligned}$$

First, we handle the cases, where

$$1_{[0,T]}(\cdot)\gamma(2^{a_1^*}\cdot - m) = \gamma(2^{a_1^*}\cdot - m).$$

The exceptional cases is dealt with below. To lighten the notation, we omit the smooth cutoff γ below again. We estimate

$$\begin{aligned}
&\sum_{0\leq k_1\leq a_1^*+5} \int_{\mathbb{R}} dt \sum_{\substack{n_1+n_2+n_3+n_4=0,\\(*)}} \frac{\psi_{s,a}(\overline{n})}{\Omega(\overline{n})} n_1 1_{I_{k_1}}(n_1)\hat{u}_2(t,n_2)\hat{u}_3(t,n_3)\hat{u}_4(t,n_4) \\
&\quad \times \sum_{\substack{n_1+n_5+n_6+n_7=0,\\(*)}} \hat{u}_5(t,n_5)\hat{u}_6(t,n_6)\hat{u}_7(t,n_7) \\
&\lesssim \sum_{0\leq k_1\leq a_1^*+5} 2^{k_1} \int_{\mathbb{R}} dt \sum_{n_1\in I_{k_1}} \Big| \sum_{\substack{n_1+n_5+n_6+n_7=0,\\(*)}} \hat{u}_5(t,n_5)\hat{u}_6(t,n_6)\hat{u}_7(t,n_7) \Big| \\
&\quad \times \Big| \sum_{n_2,n_3,(*)} \frac{\psi_{s,a}(\overline{n})}{\Omega(\overline{n})} \hat{u}_2(t,n_2)\hat{u}_3(t,n_3)\hat{u}_4(t,-n_1-n_2-n_3) \Big| \\
&\lesssim \sum_{0\leq k_1\leq a_1^*+5} 2^{k_1} \int_{\mathbb{R}} dt \Big(\sum_{n_1} \Big| \sum_{\substack{n_1+n_5+n_6+n_7=0,\\(*)}} \hat{u}_5(t,n_5)\hat{u}_6(t,n_6)\hat{u}_7(t,n_7) \Big|^2 \Big)^{1/2} \\
&\quad \times \Big(\sum_{n_1} \Big| \sum_{n_2,n_3,(*)} \frac{\psi_{s,a}(\overline{n})}{\Omega(\overline{n})} \hat{u}_2(t,n_2)\hat{u}_3(t,n_3)\hat{u}_4(t,-n_1-n_2-n_3) \Big|^2 \Big)^{1/2}.
\end{aligned} \tag{5.91}$$

Next, we apply Hölder's inequality in time, and for $\hat{u}_2$, $\hat{u}_3$ and $\hat{u}_4$, we already insert the decomposition in the modulation variable adapted to the localization in time. This means we start with a size of the modulation variable of $2^{a_1^*}$. Further, we assume again $f_{k_i,j_i} \geq 0$. We find from applying Plancherel's theorem and the (refined) Strichartz estimate

$$\begin{aligned}
&\sum_{j_2,j_3,j_4\geq a_1^*} \Big\| \Big(\sum_{n_1} \Big| \sum_{n_2,n_3,(*)} \frac{\psi_{s,a}(\overline{n})}{\Omega(\overline{n})} \hat{u}_{2,j_2}(t,n_2)\hat{u}_{3,j_3}(t,n_3)\hat{u}_{4,j_4}(t,n_4) \Big|^2 \Big)^{1/2} \Big\|_{L^2_t} \\
&\lesssim \frac{\tilde{a}(2^{a_1^*})}{2^{2a_1^*}} \sum_{j_2,j_3,j_4\geq a_1^*} \| 1_{I_{k_1}} (f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4})\tilde{\ } \|_{L^2_\tau \ell^2_n} \\
&\lesssim \frac{\tilde{a}(2^{a_1^*})}{2^{2a_1^*}} 2^{k_1/2} 2^{-a_1^*/2} 2^{(0+)a_1^*} \prod_{i=2}^4 \sum_{j_i\geq a_1^*} 2^{j_i/2} \| f_{k_i,j_i} \|_{L^2_\tau \ell^2_n}.
\end{aligned}$$

We note that for the other convolution term in (5.91) the localization in time is not high enough to evaluate the factors in F_{k_i}. We increase localization in time to $2^{k_1^*}$, but, by L^2-almost orthogonality and the time localization up to $2^{a_1^*}$, which is already provided, this only gives a factor $2^{k_1^*/2}2^{-a_1^*/2}$. Further, we plug in the localization in modulation on scale b_1^* and suppose $f_{k_i,j_i} \geq 0$. By Plancherel's theorem and the refined Strichartz estimate, we conclude the bound

$$\begin{aligned}
&\sum_{j_5,j_6,j_7 \geq b_1^*} \left\| \left(\sum_{n_1} \left| \sum_{n_5,n_6} \hat{u}_5(t,n_5)\hat{u}_6(t,n_6)\hat{u}_7(t,n_7) \right|^2 \right)^{1/2} \right\|_{L_t^2} \\
&\lesssim \sum_{j_5,j_6,j_7 \geq b_1^*} \| 1_{I_{k_1}} (f_{k_5,j_5} * f_{k_6,j_6} * f_{k_7,j_7})\tilde{} \|_{L_\tau^2 \ell_n^2} \\
&\lesssim 2^{-k_1^*/2} 2^{k_1/2} \prod_{i=5}^{7} \sum_{j_i \geq b_1^*} 2^{j_i/2} \| f_{k_i,j_i} \|_{L_\tau^2 \ell_n^2}.
\end{aligned}$$

We gather all the factors to find:

$$\begin{aligned}
&T \sum_{1 \leq k_1 \leq a_1^*} 2^{k_1} \frac{\tilde{a}(2^{a_1^*})}{2^{2a_1^*}} 2^{a_1^*} 2^{(0+)a_1^*} 2^{b_1^*/2} 2^{-a_1^*/2} 2^{-b_1^*/2} 2^{k_1/2} \prod_{i=2}^{7} 2^{j_i/2} \| f_{ij_i} \|_{L_\tau^2 \ell_n^2} \\
&\lesssim T \tilde{a}(2^{a_1^*}) \prod_{i=2}^{7} 2^{j_i/2} \| f_{i,j_i} \|_{L_\tau^2 \ell_n^2}.
\end{aligned}$$

We find (5.88) to hold for $s > 0$ and $\varepsilon(s) > 0$ due to (2.48).
In the four exceptional cases

$$\gamma(2^{a_1^*} \cdot -m) 1_{[0,T]}(\cdot) \neq \gamma(2^{a_1^*} \cdot -m)$$

(5.91) becomes via interpolation with (5.90) like in the exceptional cases above

$$\begin{aligned}
&\sum_{j_2,j_3,j_4 \geq a_1^*} \left\| \left(\sum_{n_1} \left| \sum_{n_2,n_3,(*)} \frac{\psi_{s,a}(\overline{n})}{\Omega(\overline{n})} \hat{u}_{2,j_2}(t,n_2)\hat{u}_{3,j_3}(t,n_3)\hat{u}_{4,j_4}(t,n_4) \right|^2 \right)^{1/2} \right\|_{L_t^2} \\
&\lesssim \frac{\tilde{a}(2^{a_1^*})}{2^{2a_1^*}} \sum_{j_2,j_3,j_4 \geq a_1^*} \| 1_{I_{k_1}} (f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4}) \|_{L_\tau^2 \ell_n^2} \\
&\lesssim \frac{\tilde{a}(2^{a_1^*})}{2^{2a_1^*}} 2^{k_1/2} 2^{-a_1^*/2} 2^{a_1^*(1/2)+} \sum_{j_2 \geq 0} \| f_{k_2,j_2} \|_{L_\tau^2 \ell_n^2} \prod_{i=3}^{4} \sum_{j_i \geq a_1^*} 2^{((1/2)-)j_i} \| f_{k_i,j_i} \|_{L_\tau^2 \ell_n^2}.
\end{aligned}$$

The second factor is estimated like in the bulk of the cases. This is possible because we let the sharp cutoff only act on the first factor. The claim follows again from Lemma 2.5.3 and 2.5.4. □

Next, we see how under the hypothesis

$$\| u \|_{L_{t,x}^6(\mathbb{R} \times \mathbb{T})} \lesssim \| u \|_{X^{0+,4/9+}} \tag{5.92}$$

we can show energy estimates for negative Sobolev regularities for functions in $F^{1+\delta}$-spaces for some $\delta > 0$. We shall only prove a qualitative result because (5.92) is currently unproven.

Proposition 5.2.23. *Let $T \in (0,1]$ and suppose that (5.92) is true. There is $\delta' > 0$ and $\theta > 0$ so that for $0 < \delta < \delta'$ there is $s = s(\delta) < 0$ such that the following estimate holds:*

$$R^6_{s,a,M} \lesssim T^\theta \prod_{i=1}^{6} \|u_i\|_{F^{s,1+\delta}(T)}. \tag{5.93}$$

Proof. Like above the frequency constraint is omitted, and $R^6_{s,a,M}$ is split into dyadic blocks $R^6(K_1,K_2,K_3,K_4,K_5,K_6,K_7)$ where supp $\hat{u}_i \subseteq I_{k_i}$, $K_i = 2^{k_i}$. We may assume by symmetry that $K_2 \geq K_3 \geq K_4$, $K_5 \geq K_6 \geq K_7$. Further, let $K_1^* \geq K_2^* \ldots \geq K_6^*$ denote a decreasing rearrangement of K_i, $i = 2,\ldots,7$.
Case A: $K_2 \gtrsim K_5$. In this case $K_1^* \sim K_2$ and we add localization in time according to K_2: Let γ be a smooth function with support in $[-1,1]$ satisfying

$$\sum_m \gamma^6(t-m) \equiv 1.$$

We have to estimate

$$\sum_m \int_{\mathbb{R}} dt 1_{[0,T](t)} \sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*)}} \frac{\psi_{s,a}(\overline{n}) n_1}{\Omega(\overline{n})} \prod_{i=2}^{4} \gamma(2^{(1+\delta)k_1^*} t - m) \hat{u}_i(t,n_i)$$
$$\times \sum_{\substack{n_1+n_5+n_6+n_7=0,\\ (*)}} \prod_{i=5}^{7} \gamma(2^{(1+\delta)k_1^*} t - m) \hat{u}_i(t,n_i).$$

First, we handle the majority of cases where

$$1_{[0,T]}(\cdot)\gamma(2^{(1+\delta)k_1^*} \cdot - m) = \gamma(2^{(1+\delta)k_1^*} \cdot - m)$$

Let $f_{k_i} = \mathcal{F}_{t,x}[\gamma(2^{(1+\delta)k_1^*} \cdot - m) u_i]$.
This is further decomposed as $f_{k_i} = \sum_{j_i \geq (1+\delta)k_1^*} f_{k_i,j_i}$.
By the above, we have to estimate

$$\sum_{k_1 \leq k_2} \frac{2^{k_1}}{2^{2k_1^*}} \|f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4}\|_{L^2_\tau \ell^2_n} \|f_{k_5,j_5} * f_{k_6,j_6} * f_{k_7,j_7}\|_{L^2_\tau,\ell^2_n} \tag{5.94}$$

after which it remains to sum over $j_i \geq (1+\delta)k_1^*$ and take into account time localization, which amounts to a factor $T 2^{(1+\delta)k_1^*}$.
Above, $a \in S^s_\varepsilon$ for negative s is crudely bounded by a constant.
(5.92) yields for one factor

$$\begin{aligned} \|f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4}\|_{L^2_\tau \ell^2_n} &\lesssim \|\mathcal{F}^{-1}_{t,x}[f_{k_2,j_2}] \cdot \mathcal{F}^{-1}_{t,x}[f_{k_3,j_3}] \cdot \mathcal{F}^{-1}_{t,x}[f_{k_4,j_4}]\|_{L^2_t L^2_x} \\ &\lesssim \|\mathcal{F}^{-1}_{t,x} f_{k_2,j_2}\|_{L^6_{t,x}} \|\mathcal{F}^{-1}_{t,x} f_{k_3,j_3}\|_{L^6_{t,x}} \|\mathcal{F}^{-1}_{t,x} f_{k_4,j_4}\|_{L^6_{t,x}} \\ &\lesssim 2^{(0k_1)+} \prod_{i=2}^{4} 2^{(4j_i/9)+} \|f_{k_i,j_i}\|_2 \end{aligned}$$

and by (2.48), we find the contribution of the majority of the cases to be bounded by

$$\lesssim T 2^{(\delta k_1^*)+} \prod_{i=2}^{7} 2^{-(k_i/18)+} \|P_{k_i} u\|_{F^{1+\delta}_{k_i}}$$

with easy summation in certain Sobolev spaces with negative regularity index.

Next, suppose that

$$1_{[0,T]}(\cdot)\gamma(2^{(1+\delta)k_1^*}\cdot -m) \neq \gamma(2^{(1+\delta)k_1^*}\cdot -m).$$

Set $f_{k_2} = \mathcal{F}_{t,x}[\gamma(2^{(1+\delta)k_1^*}\cdot -m)1_{[0,T]}(\cdot)u_2]$; the further notation remains unchanged. Following along the above lines by applying six $L^6_{t,x}$-Strichartz estimates gives the estimate

$$(5.94) \lesssim \frac{1}{2^{k_1^*}} 2^{(4(1+\delta)k_1^*/9)+} \sum_{j_2 \geq 0} 2^{4j_2/9} \|f_{k_2,j_2}\|_{L^2} \prod_{i=3}^{7} \sum_{j_i \geq (1+\delta)k_1^*} 2^{(4j_i/9)+} \|f_{k_i,j_i}\|_2$$

with straight-forward summation for certain negative regularity index due to Lemma 2.5.3 and Lemma 2.5.4.

Case B: $K_5 \sim K_6 \gg K_2$.

Subcase BI: $K_5^2 \gg K_2^3$. Let

$$\Omega^{(1)}(n_1,n_2,n_3,n_4) = n_1^3 + n_2^3 + n_3^3 + n_4^3 \quad (n_1,\ldots,n_4) \in \Gamma_4$$

denote the first resonance function and

$$\Omega^{(2)}(n_1,\ldots,n_6) = \sum_{i=1}^{6} n_i^3, \quad (n_1,\ldots,n_6) \in \Gamma_6$$

denote the second resonance function.

In case $K_5^2 \gg K_2^3$ we find

$$|\Omega^{(1)}(n_1,n_2,n_3,n_4)| \ll |\Omega^{(1)}(n_1,n_5,n_6,n_7)|$$

and consequently, the second resonance function for the collected frequencies

$$\Omega^{(2)}(n_2,n_3,n_4,-n_5,-n_6,-n_7) = \Omega^{(1)}(n_1,n_2,n_3,n_4) - \Omega^{(1)}(n_1,n_5,n_6,n_7)$$

satisfies $|\Omega^{(2)}| \sim |\Omega^{(1)}(n_1,n_5,n_6,n_7)| \gtrsim K_5^2$.

Let γ be like in Case A. We add localization in time according to $K_5^{(1+\delta)}$, which leads us to estimate

$$\begin{aligned} &\int_{\mathbb{R}} dt \sum_{\substack{n_1+n_2+n_3+n_4=0,\\ (*)}} \frac{\psi_{s,a}(\overline{n})}{\Omega(\overline{n})} n_1 \prod_{i=2}^{4} \gamma(2^{(1+\delta)k_1^*}t - m)\hat{u}_2(t,n_i) \\ &\times \sum_{\substack{n_1+n_5+n_6+n_7=0,\\ (*)}} 1_{[0,T]}(t)\gamma(2^{(1+\delta)k_1^*}t - m)\hat{u}_5(t,n_5) \prod_{i=6}^{7} \gamma(2^{(1+\delta)k_1^*}t - m)\hat{u}_i(t,n_i). \end{aligned} \tag{5.95}$$

First, we deal with the majority of the cases, where

$$1_{[0,T]}(\cdot)\gamma(2^{(1+\delta)k_1^*}\cdot -m) = \gamma(2^{(1+\delta)k_1^*}\cdot -m). \tag{5.96}$$

The idea is to use two bilinear Strichartz estimates from Lemma 5.2.6 involving u_5,u_6,u_7 and the function with high modulation $j_i \geq 2k_5 - 10$. Suppose e.g. that

$j_4 \geq 2k_5 - 10$.
Up to time localization factor and summation over $j_i \geq (1+\delta)k_1^*$ we find

$$\sum_{k_1 \leq k_2} \frac{2^{k_1}}{2^{2k_2^*}} \int (f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4})(f_{k_5,j_5} * f_{k_6,j_6} * f_{k_7,j_7}) dt$$
$$\lesssim \sum_{k_1 \leq k_2} \frac{2^{k_1}}{2^{2k_2^*}} \int u_{k_2,j_2} \dots u_{k_7,j_7} dx dt,$$

where $u_{k_i,j_i} = \mathcal{F}_{t,x}^{-1}[f_{k_i,j_i}]$.
Here, we ignore the (in this case) irrelevant reflection $\tilde{f}(\tau,\xi) = f(-\tau,-\xi)$.
The contribution of (5.96) is consequently estimated by

$$\lesssim T 2^{(\delta-1/4)k_5} \frac{2^{k_3/2}}{2^{k_2/2}} \prod_{i=2}^{7} 2^{j_i/2} \|f_{k_i,j_i}\|_2$$

Hence, summation for negative regularity index is straight-forward for $\delta < 1/4$.

Next, we turn to the exceptional cases

$$1_{[0,T]}(\cdot)\gamma(2^{(1+\delta)k_1^*} \cdot -m) \neq \gamma(2^{(1+\delta)k_1^*} \cdot -m).$$

Here, we have $f_{k_5} = \mathcal{F}_{t,x}[u_5 \gamma(2^{(1+\delta)k_1^*} \cdot -m) 1_{[0,T]}(\cdot)]$ and the argument from above interpolated with (5.90) gives for $j_i \geq (1+\delta)k_1^*$, $i = 2, \dots, 7$

$$\sum_{k_1 \leq k_2+10} \int u_{k_2,j_2} \dots u_{k_7,j_7} dx dt$$
$$\lesssim \frac{2^{k_3/2}}{2^{k_2/2}} 2^{(-(5/4)+)k_5} \prod_{i=2}^{7} 2^{(j_i/2)-} \|f_{k_i,j_i}\|_2.$$

Summation over j_i gives

$$\lesssim \frac{2^{k_3/2}}{2^{k_2/2}} 2^{-(5k_5/4)+} 2^{(1+\delta)k_5/2} \left(\sum_{j_5 \geq 0} 2^{(j_5/2)-} \|f_{k_5,j_5}\|_2 \right)$$
$$\prod_{i=2, i\neq 5}^{7} \sum_{j_i \geq (1+\delta)k_5} 2^{(1/2-)j_i} \|f_{k_i,j_i}\|_2$$

and this contribution is bounded by Lemma 2.5.4 and Lemma 2.5.3.

<u>Subcase BII:</u> $K_5^2 \lesssim K_2^3$. In case $K_5 \sim K_7$ we find $|\Omega^{(1)}(n_1,n_5,n_6,n_7)| \sim K_5^3$ and consequently, $|\Omega^{(2)}| \sim K_5^3$. The argument from Subcase BI provides a sufficient estimate. Thus, suppose in the following $K_7 \ll K_5$.

<u>Subsubcase BIIa:</u> $K_3 \ll K_2$. It has to hold $K_1 \sim K_2$.
If $K_2 \ll K_7$, then $|\Omega^{(1)}(n_1,n_5,n_6,n_7)| \gtrsim K_5^2 K_7 \gg |\Omega^{(1)}(n_1,n_2,n_3,n_4)|$.
If $K_7 \ll K_2$, then $|\Omega^{(1)}(n_1,n_5,n_6,n_7)| \gtrsim K_5^2 K_2 \gg |\Omega^{(1)}(n_1,n_2,n_3,n_4)|$ because $|\Omega^{(1)}(n_1,n_2,n_3,n_4)| \lesssim K_2^2 K_3$.
In any case, $|\Omega^{(2)}| \gtrsim K_5^2$ and the argument from Subcase BI is sufficient.
It remains to check $K_2 \sim K_7$. We separate variables like in the proof of Proposition

5.1.15 (the required regularity of the multiplier is provided following Remark 5.2.18 after Lemma 5.2.16) and we have to estimate

$$\sum_{k_1 \le k_2} \frac{2^{k_1}}{2^{2k_2}} \int_0^T dt \int dx u_{k_2} \dots u_{k_7}$$

Let γ be like above and by Hölder's inequality

$$\sum_{k_1 \le k_2} \frac{2^{k_1}}{2^{2k_2}} \sum_{|m| \lesssim T 2^{(1+\delta)k_2}} \int_{\mathbb{R}} dt \int dx \gamma(2^{(1+\delta)k_2} t - m) u_{k_2} \dots \gamma(2^{(1+\delta)k_2} t - m)$$
$$1_{[0,T]}(t) u_{k_5} \gamma(\dots) u_{k_6} \gamma(\dots) u_{k_7} \gamma(\dots)$$
$$\lesssim \frac{1}{2^{k_2}} \sum_{|m| \lesssim T 2^{(1+\delta)k_2}} \|\gamma(2^{(1+\delta)k_2} t - m) u_{k_2}\|_{L^6_{t,x}} \dots$$
$$\|u_{k_5} \gamma(\dots) u_{k_6} \gamma(\dots) 1_{[0,T]}(\cdot)\|_{L^3_{t,x}} \|u_{k_7} \gamma(\dots)\|_{L^6_{t,x}}.$$

Decompose for $i \in \{2,3,4,7\}$

$$f_{k_i} = \mathcal{F}_{t,x}[\gamma(2^{(1+\delta)k_2} t - m) u_{k_i}] = \sum_{j_i \ge (1+\delta)k_2} f_{k_i,j_i}$$

and by (5.92)

$$\|\mathcal{F}^{-1}_{t,x}[f_{k_i}]\|_{L^6_{t,x}} \lesssim \sum_{j_i \ge (1+\delta)k_2} \|\mathcal{F}^{-1}_{t,x}[f_{k_i,j_i}]\|_{L^6_{t,x}} \lesssim \sum_{j_i \ge (1+\delta)k_2} 2^{((4/9)+)j_i} \|f_{k_i,j_i}\|_{L^2}$$

for $i \in \{2,3,4,7\}$. For these functions time is already localized sufficiently.

For the high frequencies we have to add localization in time, where we exploit orthogonality in time

$$\|u_{k_5} \gamma^2(2^{(1+\delta)k_2} t - m) u_{k_6} 1_{[0,T]}\|_{L^3_{t,x}}$$
$$\lesssim \Big(\sum_n \|u_{k_5} \gamma(2^{(1+\delta)k_2} t - m) \tilde{\gamma}(2^{(1+\delta)k_5} t - n)$$
$$u_{k_6} \gamma(2^{(1+\delta)k_2} t - m) \tilde{\gamma}(2^{(1+\delta)k_5} t - n)\|^3_{L^3_{t,x}} \Big)^{1/3}$$

Consequently, it is enough to estimate

$$\left(2^{(1+\delta)k_5}/2^{(1+\delta)k_2}\right)^{1/3} \|u_{k_5} \tilde{\gamma}(2^{(1+\delta)k_5} t - n)\|_{L^6_{t,x}} \|u_{k_6} \tilde{\gamma}(2^{(1+\delta)k_5} t - n)\|_{L^6_{t,x}},$$

which by $k_5 \le (3/2)k_2$, (5.92) and the above argument of splitting the modulation is achieved by

$$\lesssim 2^{(0+)k_2} 2^{(1+\delta)k_2/6} \prod_{i=5}^{6} \sum_{j_i \ge (1+\delta)k_5} 2^{(4/9+)j_i} \|f_{k_i,j_i}\|_2$$

Gathering all factors and invoking (2.48), we have derived the bound

$$R^6_{s,a}(K_2, \dots, K_7) \lesssim T \frac{2^{k_2}}{2^{2k_2}} 2^{(1+\delta)k_2} 2^{(1+\delta)k_2/6} \prod_{i=2}^{7} 2^{(-(1/18)+)k_i} \|u_i\|_{F^{1+\delta}_{k_i}}.$$

Since there are four factors with frequency higher or equal to K_2, there is enough smoothing from the $L^6_{t,x}$-estimate to sum the expression even for negative regularities choosing δ sufficiently small.

Subsubcase BIIb: $K_2 \sim K_3$.
If $K_7 \sim K_6$, then $|\Omega^{(1)}(n_1, n_5, n_6, n_7)| \gtrsim K_5^3 \gg K_2^3$ and the argument from Subcase BI applies.
Similarly, if $K_2 \ll K_7 \ll K_5$ we find

$$|\Omega^{(1)}(n_1, n_5, n_6, n_7)| \sim K_5^2 K_7 \gg K_2^3 \gtrsim |\Omega^{(1)}(n_1, n_2, n_3, n_4)|$$

Thus, we can suppose that $K_7 \lesssim K_2$. In this case the argument from Subsubcase BIIa applies because there are at least two frequencies comparable to K_2 and at most two frequencies, namely K_5 and K_6, much higher than K_2.
The proof is complete. □

Remark 5.2.24. We point out from the proofs of Propositions 5.2.20, 5.2.22 and 5.2.23 and Lemma 5.2.21 that there is some slack in the regularity. In fact, we can lower the regularity on the right hand-side depending on s (after making $\varepsilon = \varepsilon(s)$ smaller if necessary). This observation becomes important in the construction of the data-to-solution mapping.

To conclude the proof of the energy estimate, one derives a bound for the thresholds of the frequency localized energy (cmp. Lemma 5.1.16). For details we refer to Section 5.1.3.

5.2.5 Proof of new regularity results for the modified Korteweg-de Vries equation

The proof of Theorem 5.2.2 is a variant of the arguments from Section 5.1.4. The arguments follow [GO18], where low regularity periodic solutions to the NLS had been discussed.
After establishing a priori estimates on smooth solutions, a compactness argument is used to construct the solution mapping.

Lemma 5.2.25. *Let $u_0 \in H^\infty(\mathbb{T})$ and $s > 0$. There is a function $T = T(s, \|u_0\|_{H^s})$ so that we find the following estimate for the unique smooth solution to* (5.52) *to hold:*

$$\sup_{t \in [-T,T]} \|u(t)\|_{H^s(\mathbb{T})} \lesssim \|u_0\|_{H^s(\mathbb{T})}. \tag{5.97}$$

We control the $F^{s,1}(T)$-norm of the solution by a continuity argument. By Lemma 2.5.1 this is enough to prove Lemma 5.2.25.

Together with the short-time $X^{s,b}$-energy estimate, the nonlinear estimate from Proposition 5.2.14 and the energy estimate from Proposition 5.2.15 there is $\theta > 0$ and $c(s), d(s) > 0$ so that the following estimates hold true for any $M \in 2^{\mathbb{N}}$:

$$\begin{cases} \|u\|_{F^{s,1}(T)} & \lesssim \|u\|_{E^s(T)} + \|\mathfrak{N}(u)\|_{N^{s,1}(T)} \\ \|\mathfrak{N}(u)\|_{N^{s,1}(T)} & \lesssim T^\theta \|u\|^3_{F^{s,1}(T)} \\ \|u\|^2_{E^s(T)} & \lesssim \|u_0\|^2_{H^s} + T^\theta M^{c(s)} \|u\|^4_{F^{s,1}(T)} \\ & \quad + M^{-d(s)} \|u\|^4_{F^{s,1}(T)} + T^\theta \|u\|^6_{F^{s,1}(T)}. \end{cases} \tag{5.98}$$

To carry out the continuity argument, recall the continuity and limit properties of $T \mapsto \|u\|_{E^s(T)}$, $T \mapsto \|u\|_{N^s(T)}$.
We are ready to prove a priori estimates for smooth solutions.

Proof of Lemma 5.2.25. Assuming that u_0 is a smooth and real-valued initial datum, we find from the classical well-posedness theory the global existence of a smooth and real-valued solution $u \in C(\mathbb{R}, H^\infty)$ (see e.g. [Bou93a]) which satisfies the set of estimates (5.98).

We define $X(T) = \|u\|_{E^s(T)} + \|\mathfrak{N}(u)\|_{N^{s,1}(T)}$ and find the bound

$$X(T)^2 \leq C_1 \|u_0\|_{H^s}^2 + C_2((T^\theta M^{c(s)} + M^{-d(s)})X(T)^2 + T^\theta X(T)^4)X(T)^2$$

by eliminating $\|u\|_{F^{s,\alpha}(T)}$ in the above system of estimates.

Set $R = C_1^{1/2} \|u_0\|_{H^s}$ and choose $M = M(R)$ large enough so that

$$C_2 M^{-d(s)} (2R)^2 < 1/4.$$

Next, choose $T_0 = T_0(R) \leq 1$ small enough so that

$$C_2 T_0^\theta (M^{c(s)} (2R)^2 + (2R)^4) < 1/4.$$

Together with the limiting properties for $T \to 0$ a continuity argument yields $X(T) \leq 2R$ for $T \leq T_0$.
Iterating the argument gives $\sup_{t \in [0,T_0]} \|u(t)\|_{H^s(\mathbb{T})} \lesssim \|u_0\|_{H^s}$ for $T_0 = T_0(\|u_0\|_{H^s})$.
The proof is complete. □

We establish the existence of the solution mapping. For $u_0 \in H^s(\mathbb{T})$, we set $u_{0,n} = P_{\leq n} u_0$ for $n \in \mathbb{N}$. Obviously, $u_{0,n} \in H^\infty(\mathbb{T})$ and hence, the initial data give rise to smooth global solutions $u_n \in C(\mathbb{R}, H^\infty(\mathbb{T}))$. According to Lemma 5.2.25 we already have a priori estimates on a time interval $[0, T_0]$ where $T_0 = T_0(\|u_0\|_{H^s})$ independent of n. Moreover, we have the following compactness lemma (cf. Lemma 5.1.18), which is proven by the same means like above:

Lemma 5.2.26. *Let $u_0 \in H^s(\mathbb{T})$ for some $s > 0$. Let u_n be the smooth global solutions to* (5.52) *with $u_n(0) = u_{0,n}$ like above.*
Then, $(u_n)_{n \in \mathbb{N}}$ is precompact in $C([-T,T], H^s(\mathbb{T}))$ for $T \leq T_0 = T_0(\|u_0\|_{H^s})$.

Key ingredient like above is the uniform tail estimate, i.e., there is $n_0 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$

$$\|P_{\geq n_0} u_n\|_{C_T H^s} < \varepsilon. \tag{5.99}$$

We are ready to prove the main result:

Proof of Theorem 5.2.2. For $u_0 \in H^s(\mathbb{T})$ let $(u_n)_{n \in \mathbb{N}}$ denote the smooth global solutions generated from the initial data $P_{\leq n} u_0$ as described above. By Lemma 5.2.26 we find a convergent subsequence (u_{n_k}), which converges to a function $u \in C([-T,T], H^s)$. We observe that due to (5.99) the sequence also converges in $E^s(T)$. With $\|\mathfrak{N}(u_n - u)\|_{N^{s,1}(T)} \lesssim T^\theta \|u_0\|_{H^s}^2 \|u_n - u\|_{F^{s,1}(T)}$, we find for $T = T(\|u_0\|_{H^s})$ the estimate

$$\|u_n - u\|_{F^{s,1}(T)} \lesssim \|u_n - u\|_{E^s(T)}$$

to hold. The convergence in $F^{s,1}(T)$ already gives the a priori estimate for the limit. Moreover, we deduce from the multilinear estimates in Proposition 5.2.14 that $\{\mathfrak{N}(u_n)\}$ converges to $\mathfrak{N}(u)$ in $N^{s,1}(T) \hookrightarrow \mathcal{D}'$. We conclude that u satisfies (5.52) in the distributional sense with the claimed properties and the proof is complete. □

For the proof of non-existence of solutions to the unrenormalized mKdV equation, we compare smooth solutions to (5.2) and (5.52) via a gauge transform. The argument parallels [GO18].
We sketch the argument for the sake of self-containedness and for details refer to [GO18].

Proof of Theorem 5.2.3. Existence and a priori estimates of solutions to (5.52) for negative regularity conditional upon conjectured Strichartz estimates are proved like above. Here, corresponding estimates to (5.98) are utilized.

For the proof of non-existence of solutions to (5.2), we argue by contradiction. Fix $s < 0$ from the hypothesis of Theorem 5.2.3 and $u_0 \in H^s(\mathbb{T}) \backslash L^2(\mathbb{T})$. Suppose that there exists $T > 0$ and a solution $u \in C([-T,T], H^s(\mathbb{T}))$ to (5.2) in the sense of definition 5.2.1.

By defining

$$v_n(t) = e^{-2it \int_{\mathbb{T}} |u_{0,n}|^2 dx} u_n(t),$$

we find a sequence of smooth solutions to (5.52).
Further, by assumption

$$v_n(t=0) = u_n(0) \to u_0 \text{ in } H^s(\mathbb{T}).$$

By a variant of Lemma 5.2.26, there is a subsequence $(v_{n_k})_k$ converging to v in $C([-T,T], H^s)$ with $T = T(\|u_0\|_{H^s})$. The convergence of u_n implies the convergence of v_n to 0 in the sense of distributions: Let $\phi \in C_c^\infty([-T,T], C^\infty(\mathbb{T}))$. Then,

$$\langle u_n(t), \phi(t) \rangle_{L^2_x} \to F(t) := \langle u(t), \phi(t) \rangle_{L^2_x}$$

by convergence of $u_n(t)$ in $C([-T,T], H^s)$. Further, $F \in C_c(\mathbb{R})$.

It follows that

$$\begin{aligned} &| \int \int v_n \phi dx dt | = | \int e^{-2it \int_{\mathbb{T}} |u_{0,n}|^2 dx} \langle u_n(t), \phi(t) \rangle_{L^2_x} dt | \\ &\leq | \int e^{-2it \int_{\mathbb{T}} |u_{0,n}|^2 dx} F(t) dt | + \int | \langle u(t) - u_n(t), \phi(t) \rangle_{L^2_x} | dt \to 0. \end{aligned}$$

The first term vanishes according to the Riemann-Lebesgue lemma and the second term due to $u_n \to u$ in $C_T H^s$. This means that v_n converges to 0 in the distributional sense, and since $v_{n_k} \to v$ in $C_T H^s$, this implies $v \equiv 0$. This contradicts $u_0 = v(0) \neq 0$. □

Chapter 6

Local and global well-posedness for dispersion generalized Benjamin-Ono equations on the circle

6.1 Introduction to dispersion generalized Benjamin-Ono equations

In this chapter we prove new regularity results for the one-dimensional fractional Benjamin-Ono equation in the periodic case

$$\begin{cases} \partial_t u + \partial_x D_x^a u &= u \partial_x u, \ (t,x) \in \mathbb{R} \times \mathbb{T}, \\ \quad u(0) &= u_0, \end{cases} \tag{6.1}$$

where $1 < a < 2$ is considered in the following.

Previous works on dispersion generalized Benjamin-Ono equations include [HIKK10, Guo12] in the real line case and [MV15] in the periodic case. In [MV15] global well-posedness was proved in $H^s(\mathbb{T})$ for $s \geq 1 - a/2$, where $1 \leq a \leq 2$. For details on these works, we refer to the remarks on the well-posedness theory of the Benjamin-Ono equation in Chapter 1. The following results are proven via short-time analysis:

Theorem 6.1.1. *For $1 < a \leq 3/2$, (6.1) is locally well-posed in $H^s(\mathbb{T})$ provided that $s > 3/2 - a$.*
For $3/2 < a < 2$, (6.1) is globally well-posed in $L^2(\mathbb{T})$.

Remark 6.1.2. Recall that Molinet pointed out in [Mol08] that in the Benjamin-Ono case the periodic data-to-solution mapping is C^∞ on hyperplanes of initial data with fixed mean. From this, one might suspect that this is also true in the dispersion generalized case. However, Herr proved in [Her08] that (6.1) can not be solved via Picard iteration for $1 \leq a < 2$ explaining our use of short-time analysis.

The analysis extends the short-time Strichartz analysis from Chapter 3, which is further improved by considering modified energies. By this we mean correction

terms for the frequency localized energy corresponding to normal form transformations like in Chapter 5, but without symmetrization.
The improved symmetrized expression does not yield new information when analyzing differences of solutions because of reduced symmetry. Still, normal form transformations allow us to improve the energy estimates.

An early application of modified energies was given by Kwon in [Kwo08]. In this work, modified energies were combined with short-time Strichartz estimates (cf. [KT03]) in order to improve the local well-posedness theory for the fifth-order KdV equation. This was refined by Kenig-Pilod in [KP15] using short-time Fourier restriction spaces to prove global well-posedness in the energy space. In the independent work by Guo-Kwak-Kwon [GKK13] a modulation weight was used to prove the same result.
An application of modified energies in the context of short-time Fourier restriction spaces for periodic solutions was given by Kwak in [Kwa16]. In this work, the global well-posedness of the fifth-order KdV equation on the circle was proved in H^2.

On the real line, short-time analysis for dispersion generalized Benjamin-Ono equations was already carried out in [Guo12]. In [Guo12] no normal form transformations were used, which gave local well-posedness for $s \geq 2 - a$, where $1 \leq a \leq 2$. The gain from introducing modified energies is most significant for large dispersion coefficients allowing us to prove well-posedness in $L^2(\mathbb{T})$. Further, it appears as if some of the arguments can be applied in the low-dispersion case $0 < a < 1$. For these equations on the circle, which are also of physical interest, are currently no well-posedness results beyond the energy method available.
On the real line, there is the recent work by Molinet-Pilod-Vento [MPV18] refining the analysis from [MV15] by normal form transformations. Since this analysis makes use of smoothing effects unavailable on the circle, it is not clear how to extend the analysis from [MPV18] to the circle.

The local well-posedness result from Theorem 6.1.1 for $1 < a < 2$, which is globalized for $a > 3/2$ due to conservation of mass on $\mathbb{T}$ is currently the best. This improves global well-posedness for $s \geq 1 - a/2$, where $1 < a < 2$, proved in [MV15]. As argued in the previous chapters, the analysis can be transferred to the real line. This yields a possible simplification of the analysis from [HIKK10]. On the real line, the multilinear estimates relying on linear and bilinear Strichartz estimates are improved due to dispersive effects. However, the introduction of a modified energy would require additional care because the resonance

$$\Omega(\xi_1, \xi_2, \xi_3) = \xi_1 |\xi_1|^a + \xi_2 |\xi_2|^a + \xi_3 |\xi_3|^a \quad \xi_i \in \mathbb{R}, \quad \xi_1 + \xi_2 + \xi_3 = 0$$

might become arbitrarily small in modulus for non-vanishing $\xi_i \in \mathbb{R}$. To avoid this, we confine ourselves to initial data with vanishing mean. As this is a conserved quantity, there is no loss of generality in assuming

$$\int_{\mathbb{T}} u(x) dx = 0.$$

When we localize time, we do not work in Euclidean windows, but rather base the analysis on the time localization $T = T(N) = N^{a-2}$: This is explained by interpolating between Euclidean windows in the Benjamin-Ono case and the Fourier restriction norm analysis for $a = 2$, where frequency dependent time localization is no longer required. For the large data theory, it turns out to be convenient to consider the slightly shorter times $N^{a-2-\delta}$ giving an additional factor of T^θ in the

nonlinear estimates (cf. Lemma 2.5.3).
The following set of estimates will be established for the proof of Theorem 6.1.1 for a smooth solution u to (6.1) with vanishing mean. For $1 < a < 2$, $T \in (0,1]$, $M \in 2^{\mathbb{N}_0}$ and $s' \geq s \geq \max(3/2-a,0)$, there are $\delta(a,s) > 0$, $c(a,s) > 0$, $d(a,s) > 0$ and $\theta(a,s) > 0$ such that

$$\begin{cases} \|u\|_{F_a^{s',\delta}(T)} & \lesssim \|u\|_{E^{s'}(T)} + \|u\partial_x u\|_{N_a^{s',\delta}(T)} \\ \|u\partial_x u\|_{N_a^{s',\delta}(T)} & \lesssim T^\theta \|u\|_{F_a^{s',\delta}(T)} \|u\|_{F_a^{s,\delta}(T)} \\ \|u\|^2_{E^{s'}(T)} & \lesssim \|u(0)\|^2_{H^{s'}(T)} \\ & \quad + M^c T \|u\|^2_{F_a^{s',\delta}(T)} \|u\|_{F_a^{s,\delta}(T)} \\ & \quad + M^{-d} \|u\|^2_{F_a^{s',\delta}(T)} \|u\|_{F_a^{s,\delta}(T)} + T^\theta \|u\|^2_{F_a^{s',\delta}(T)} \|u\|^2_{F_a^{s,\delta}(T)}. \end{cases}$$

By the usual bootstrap arguments (cf. Section 5.2.5), the above display gives a priori estimates. In this chapter we omit the bootstrap arguments to avoid repetition.

For differences of solutions $v = u_1 - u_2$, where u_i denote smooth solutions to (6.1) with vanishing mean, we have the following set of estimates for $s > 3/2 - a$ in case $1 < a \leq 3/2$ and $s = 0$ in case $3/2 < a < 2$ and the remaining parameters like in the previous display:

$$\begin{cases} \|v\|_{F_a^{-1/2,\delta}(T)} & \lesssim \|v\|_{E^{-1/2}(T)} + \|\partial_x(v(u_1+u_2))\|_{N_a^{-1/2,\delta}(T)} \\ \|\partial_x((u_1+u_2)v)\|_{N_a^{-1/2,\delta}(T)} & \lesssim T^\theta \|v\|_{F_a^{-1/2,\delta}(T)} (\|u_1\|_{F_a^{s,\delta}(T)} + \|u_2\|_{F_a^{s,\delta}(T)}) \\ \|v\|^2_{E^{-1/2}(T)} & \lesssim \|v(0)\|^2_{H^{-1/2}} \\ & \quad + M^c T \|v\|^2_{F_a^{-1/2,\delta}(T)} (\|u_1\|_{F_a^{s,\delta}(T)} + \|u_2\|_{F_a^{s,\delta}(T)}) \\ & \quad + M^{-d} \|v\|^2_{F_a^{-1/2,\delta}(T)} (\|u_1\|_{F_a^{s,\delta}(T)} + \|u_2\|_{F_a^{s,\delta}(T)}) \\ & \quad + T^\theta \|v\|^2_{F_a^{-1/2,\delta}(T)} (\|u_1\|^2_{F_a^{s,\delta}(T)} + \|u_2\|^2_{F_a^{s,\delta}(T)}), \end{cases}$$

which yields Lipschitz-continuity in $H^{-1/2}$ for initial data in H^s.

The related set of estimates with parameters like in the previous display

$$\begin{cases} \|v\|_{F_a^{s,\delta}(T)} & \lesssim \|v\|_{E^s(T)} + \|\partial_x(v(u_1+u_2))\|_{N_a^{s,\delta}(T)} \\ \|\partial_x(v(u_1+u_2))\|_{N_a^{s,\delta}(T)} & \lesssim T^\theta \|v\|_{F_a^{s,\delta}(T)} \left(\|u_1\|_{F_a^{s,\delta}(T)} + \|u_2\|_{F_a^{s,\delta}(T)} \right) \\ \|v\|^2_{E^s(T)} & \lesssim \|v(0)\|^2_{H^s} \\ & \quad + M^c T \|v\|^2_{F_a^{s,\delta}(T)} (\|u_2\|_{F_a^{s,\delta}(T)} + \|v\|_{F_a^{s,\delta}(T)}) \\ & \quad + M^{-d} \|v\|^2_{F_a^{s,\delta}(T)} (\|u_2\|_{F_a^{s,\delta}(T)} + \|v\|_{F_a^{s,\delta}(T)}) \\ & \quad + T^\theta (\|v\|^2_{F_a^{s,\delta}(T)} (\|u_2\|^2_{F_a^{s,\delta}(T)} + \|v\|^2_{F_a^{s,\delta}(T)}) \\ & \quad + \|v\|_{F_a^{-1/2,\delta}(T)} \|v\|_{F_a^{s,\delta}(T)} \|u_2\|_{F_a^{r,\delta}(T)} \|u_2\|_{F_a^{s,\delta}(T)}), \end{cases}$$

where $r = (2-a) + s$, yields continuous dependence by a variant of the Bona-Smith approximation (cf. Section 3.5).

This chapter is structured as follows: After introducing function spaces in Section 6.2, we consider linear and bilinear estimates of functions localized in frequency and modulation in Section 6.3. In Section 6.4 the short-time bilinear estimate is carried out, and in Section 6.5 the energy estimates are proved.

6.2 Function spaces

In this section the notation from Chapter 2 is adapted to (6.1). We define a dyadically localized energy space

$$E_k = \{f \in L^2 | P_k f = f\}$$

and set

$$C_0(\mathbb{R}, E_k) = \{u_k \in C(\mathbb{R}, E_k) \,|\, \mathrm{supp}(u_k) \subseteq [-4,4] \times \mathbb{R}\}.$$

We define the short-time $X^{s,b}$-space $F_{k,a}$ for frequencies comparable to 2^k.

For $1 \leq a \leq 2$, we localize time on a scale of $2^{(a-2-\delta)k}$, where $\delta \geq 0$:

$$F^\delta_{k,a} = \{u_k \in C_0(\mathbb{R}, E_k) | \|u_k\|_{F^\delta_{k,a}} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[u_k \eta_0(2^{(2-a+\delta)k}(t-t_k))]\|_{X_{k,a}} < \infty\}. \tag{6.2}$$

Based on the observation that for $a = 1$, $T = T(N) = N^{-1}$ is a natural localization in time and that for $a = 2$, we do not need localization in time anymore to overcome the derivative loss due to sufficient dispersive effects, we choose as inbetween localization in time $T = T(N) = N^{a-2-\delta}$. It turns out that for some limiting cases small $\delta > 0$ will be useful.

Correspondingly, we define the space, in which the nonlinearity is estimated, as

$$\begin{aligned} N^\delta_{k,a} &= \{u_k \in C_0(\mathbb{R}, E_k) | \\ \|u_k\|_{N^\delta_{k,a}} &= \sup_{t_k \in \mathbb{R}} \|(\tau - \omega(\xi) + i2^{(2-a+\delta)k})^{-1} \mathcal{F}[u_k \eta(2^{(2-a+\delta)k}(t-t_k))]\|_{X_{k,a}} < \infty\}. \end{aligned}$$

We localize the spaces in time for $T \in (0,1]$ as usual:

$$F^\delta_{k,a}(T) = \{u_k \in C([-T,T], E_k) | \|u_k\|_{F^\delta_{k,a}(T)} = \inf_{\tilde{u}_k = u_k \text{ in } [-T,T]} \|\tilde{u}_k\|_{F^\delta_{k,a}} < \infty\}$$

and

$$N^\delta_{k,a}(T) = \{u_k \in C([-T,T], E_k) | \|u_k\|_{N^\delta_{k,a}(T)} = \inf_{\tilde{u}_k = u_k \text{ in } [-T,T]} \|\tilde{u}_k\|_{N^\delta_{k,a}} < \infty\}.$$

The spaces E^s, $E^s(T)$, $F^{s,\delta}_a(T)$ and $N^{s,\delta}_a(T)$ are composed like in Chapter 2 via Littlewood-Paley decomposition. The dispersion relation is denoted by

$$\varphi_a(\xi) = \xi|\xi|^a.$$

The regions in Fourier space localized at frequency and modulation are denoted by

$$D^a_{k_i,j_i} = \{(\tau,\xi) \in \mathbb{R} \times \mathbb{Z} \,|\, |\xi| \sim 2^{k_i},\ |\tau - \varphi_a(\xi)| \sim 2^{j_i}\}$$

with the obvious modification for the variant $D^a_{k_i,\leq j_i}$.

6.3 Linear and bilinear estimates

In the following we derive L^2-bilinear convolution estimates for space-time functions localized in frequency and modulation. Consider k_i, j_i, $i = 1,2,3$ and $f_{k_i,j_i} \in$

$L^2_{\geq 0}(\mathbb{R} \times \mathbb{Z})$, $\mathrm{supp}(f_{k_i,j_i}) \subseteq D^a_{k_i,\leq j_i}$.
Aim is to prove estimates

$$\int\int f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) d\Gamma_3(\xi) d\Gamma_3(\tau) \lesssim \alpha(\underline{k},\underline{j}) \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \tag{6.3}$$

The following $L^4_{t,x}$-Strichartz estimate is independent of the separation of the frequencies. It is a refinement of the Strichartz estimates from decoupling for fractional dispersion relations, which was discussed in Subsection 3.2.1 in the one-dimensional case. The proof generalizes the $a = 2$-case given in [Mol12, Lemma 3.3., p. 1906].

Lemma 6.3.1. *Let $1 \leq a \leq 2$, $f_{k_i,j_i} \in L^2_{\geq 0}(\mathbb{Z} \times \mathbb{R})$, supp$f_{k_i,j_i} \subseteq D^a_{k_i,\leq j_i}$, $i = 1, 2$. Then, we find the following estimate to hold:*

$$\|f_{k_1,j_1} * f_{k_2,j_2}\|_{L^2_{\tau,\xi}} \lesssim 2^{j_{\min}/2} 2^{j_{\max}/(2(a+1))} \|f_{k_1,j_1}\|_2 \|f_{k_2,j_2}\|_2. \tag{6.4}$$

Proof. By the reflection lemma ([Tao01, Corollary 3.8.])

$$\|uv\|_2 = \|u\overline{v}\|_2,$$

we can suppose that $\mathrm{supp}_\xi f_{k_i,j_i} \subseteq \mathbb{Z}_{\geq 0}$ for $i = 1, 2$.

An application of Cauchy-Schwarz gives

$$\begin{aligned} &\int d\tau \int d\xi \left| \int d\tau_1 \int d\xi_1 f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau - \tau_1, \xi - \xi_1) \right|^2 \\ &\lesssim \sup_{\tau,\xi} \alpha(\tau,\xi) \|f_{k_1,j_1}\|_2^2 \|f_{k_2,j_2}\|_2^2, \end{aligned}$$

where

$$\begin{aligned} \alpha(\tau,\xi) &\lesssim \mathrm{mes}(\{(\tau_1,\xi_1) \in \mathbb{R} \times \mathbb{Z}_{\geq 0} | \xi - \xi_1 \in \mathbb{Z}_{\geq 0}, \langle \tau_1 - \varphi_a(\xi_1) \rangle \lesssim 2^{j_1} \\ &\qquad \text{and } \langle \tau - \tau_1 - \varphi_a(\xi - \xi_1) \rangle \lesssim 2^{j_2}\}) \\ &\lesssim 2^{j_{\min}} \# A(\tau,\xi) \end{aligned}$$

with

$$A(\tau,\xi) = \{\xi_1 \geq 0 \mid \xi - \xi_1 \geq 0 \text{ and } \langle \tau - \varphi_a(\xi_1) - \varphi_a(\xi - \xi_1) \rangle \lesssim 2^{j_{\max}}\}.$$

In the region $2^{j_{\max}} \leq \xi^{a+1}$, notice that

$$\# A(\tau,\xi) \lesssim \left(\frac{2^{j_{\max}}}{\xi^{a-1}} \right)^{1/2} + 1 \lesssim 2^{j_{\max}/(a+1)}.$$

In the region $0 \leq \xi^{a+1} \leq 2^{j_{\max}}$, use that $0 \leq \xi_1 \leq \xi$ to obtain that

$$\# A(\tau,\xi) \lesssim \#\{\xi_1 \mid 0 \leq \xi_1^{a+1} \leq 2^{j_{\max}}\} \lesssim 2^{j_{\max}/(a+1)}.$$

(6.4) follows from the above two displays. □

Lemma 6.3.2. *Let $k_2 \leq k_1 - 5$. Then, we find* (6.3) *to hold with*

$$\begin{aligned} \alpha(\underline{k},\underline{j}) = \min(&(1 + 2^{j_3 - ak_1})^{1/2} 2^{j_2/2}, (1 + 2^{j_2 - ak_1})^{1/2} 2^{j_1/2}, \\ &(1 + 2^{j_3 - (a-1)k_1 - k_2})^{1/2} 2^{j_1/2}). \end{aligned}$$

Proof. We perform a change of variables $f^{\#}_{k_i,j_i}(\tau,\xi) = f_{k_i,j_i}(\tau + \varphi_a(\xi), \xi)$ so that $\|f^{\#}_{k_i,j_i}\|_2 = \|f_{k_i,j_i}\|_2$ and $\mathrm{supp}(f^{\#}_{k_i,j_i}) \subseteq \{(\tau_i,\xi_i) \in \mathbb{R} \times \mathbb{Z} \mid |\xi_i| \sim 2^{k_i},\, |\tau_i| \lesssim 2^{j_i}\}$.[1]
The resonance function

$$\Omega^a(\xi_1,\xi_2) = (\xi_1+\xi_2)|\xi_1+\xi_2|^a - \xi_1|\xi_1|^a - \xi_2|\xi_2|^a \tag{6.5}$$

comes into play quantifying the effective support of the involved functions. Record

$$\begin{aligned} \left|\frac{\partial \Omega^a}{\partial \xi_1}\right| &= ||\xi_1+\xi_2|^a - |\xi_1|^a| \sim |\xi_1|^{a-1}|\xi_2|, \\ \left|\frac{\partial \Omega^a}{\partial \xi_2}\right| &= ||\xi_1+\xi_2|^a - |\xi_2|^a| \sim |\xi_1+\xi_2|^a. \end{aligned} \tag{6.6}$$

We prove the first estimate. An application of Cauchy-Schwarz inequality in ξ_2 yields

$$\begin{aligned} &\int\int f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) d\Gamma_3(\xi) d\Gamma_3(\tau) \\ &\lesssim \int (d\xi_1)_1 \int d\tau_1 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) \int d\tau_2 (1+2^{j_3-ak_1})^{1/2} \\ &\quad \times \left(\int (d\xi_2)_1 |f^{\#}_{k_2,j_2}(\tau_2,\xi_2)|^2 |f^{\#}_{k_3,j_3}(\tau_1+\tau_2-\Omega^a, \xi_1+\xi_2)|^2 \right)^{1/2}. \end{aligned}$$

Further applications of Cauchy-Schwarz in τ_1, ξ_1 and τ_2 yield

$$\begin{aligned} &\lesssim \int (d\xi_1)_1 \int d\tau_2 (1+2^{j_3-ak_1})^{1/2} \left(\int d\tau_1 |f^{\#}_{k_1,j_1}(\tau_1,\xi_1)|^2 \right)^{1/2} \\ &\qquad \left(\int (d\xi_2)_1 |f^{\#}_{k_2,j_2}(\tau_2,\xi_2)|^2 \int d\tau_1 |f^{\#}_{k_3,j_3}(-\tau_1-\tau_2+\Omega^a, -\xi_1-\xi_2)|^2 \right)^{1/2} \\ &\lesssim (1+2^{j_3-ak_1})^{1/2} \int d\tau_2 \|f^{\#}_{k_1,j_1}\|_2 \left(\int (d\xi_2)_1 |f^{\#}_{k_2,j_2}(\tau_2,\xi_2)|^2 \right)^{1/2} \|f^{\#}_{k_3,j_3}\|_{L^2} \\ &\lesssim 2^{j_2/2}(1+2^{j_3-ak_1})^{1/2} \prod_{i=1}^{3} \|f^{\#}_{k_i,j_i}\|_{L^2}. \end{aligned}$$

This yields the first bound.

To prove the second claim, we carry out the same computation after rearranging

$$\begin{aligned} &\int\int f_{k_3,j_3}(\tau_3,\xi_3) f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) d\Gamma_3(\xi) d\Gamma_3(\tau) \\ &= \int (d\xi_3)_1 \int d\tau_3 f^{\#}_{k_3,j_3}(\tau_3,\xi_3) \int (d\xi_1)_1 \int d\tau_1 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) \\ &\quad \times f^{\#}_{k_2,j_2}(-\tau_1-\tau_3+\Omega^a(\xi_1,\xi_3), -\xi_1-\xi_3). \end{aligned}$$

Note that

$$\left|\frac{\partial \Omega^a(\xi_1,\xi_3)}{\partial \xi_1}\right| \sim ||\xi_1+\xi_3|^a - |\xi_1|^a| \sim |\xi_1|^a.$$

[1] In the following computations, at some point we freely interchange f with $\tilde{f}(\tau,\xi) = f(-\tau,-\xi)$ as this leaves the L^2-norm and support region invariant.

Firstly, apply Cauchy-Schwarz in ξ_1 to find

$$\lesssim \int (d\xi_3)_1 \int d\tau_3 f^{\#}_{k_3,j_3}(\tau_3,\xi_3) \int d\tau_1 (1+2^{j_2-ak_1})^{1/2} \times \left(\int (d\xi_1)_1 |f^{\#}_{k_1,j_1}(\tau_1,\xi_1)|^2 |f^{\#}_{k_2,j_2}(\tau_1+\tau_3+\Omega^a,\xi_1+\xi_3)|^2 \right)^{1/2}.$$

Next, apply Cauchy-Schwarz in τ_3, ξ_3 and at last τ_1 to find the bound

$$\lesssim 2^{j_1/2}(1+2^{j_2-ak_1})^{1/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_2.$$

The third bound is established by the same argument. The difference of the group velocity is less favourable though. This leads to inferior estimates: an application of the Cauchy-Schwarz inequality in ξ_1 yields

$$\begin{aligned} &\int d\tau_2 \int (d\xi_2)_1 f^{\#}_{k_2,j_2}(\tau_2,\xi_2) \\ &\quad \int (d\xi_1)_1 \int d\tau_1 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) f^{\#}_{k_3,j_3}(\tau_1+\tau_2-\Omega^a,\xi_1+\xi_2) \\ &\lesssim \int d\tau_2 \int (d\xi_2)_1 f^{\#}_{k_2,j_2}(\tau_2,\xi_2) \int d\tau_1 (1+2^{j_3-(a-1)k_1-k_2})^{1/2} \\ &\quad \left(\int (d\xi_1)_1 |f^{\#}_{k_1,j_1}(\tau_1,\xi_1)|^2 |f^{\#}_{k_3,j_3}(\tau_1+\tau_2-\Omega^a,\xi_1+\xi_2)|^2 \right)^{1/2}. \end{aligned}$$

Now apply Cauchy-Schwarz like above in τ_2, ξ_2 and τ_1 to find

$$\lesssim 2^{j_1/2}(1+2^{j_3-(a-1)k_1-k_2})^{1/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_2.$$

This proves the third bound. □

Remark 6.3.3. Unless one introduces modulation weights like e.g. in [GPWW11], the third bound is insufficient to overcome the derivative loss in case of *High* $\times$ *Low* $\to$ *High*-interaction. Moreover, it is this estimate which complicates short-time bilinear estimates for a negative Sobolev regularity index.

Lemma 6.3.4. *Let* $1 \leq a \leq 2$. *If* $|k_i - k_j| \leq 20$, $i = 1,2,3$, *then we find* (6.3) *to hold with* $\alpha(\underline{k},\underline{j}) = 2^{j_{i_1}/2}(1+2^{j_{i_2}-(a-1)k_1})^{1/4}$ *for any* $i_1, i_2 \in \{1,2,3\}$ *provided that* $i_1 \neq i_2$.
Suppose in addition that $||\xi_{i_1}|^a - |\xi_{i_2}|^a| \sim 2^{ak_1}$ *provided that* $\xi_{i_m} \in supp_{\xi}(f_{k_{i_m},j_{i_m}})$, $i_m \in \{1,2,3\}$. *Then, we find* (6.3) *to hold with* $\alpha = 2^{j_{i_1}/2}(1+2^{j_{i_2}-ak_1})^{1/2}$.

Proof. We assume in the following that $a > 1$ because the claim is covered in Lemma

6.3.1 for $a = 1$. For the first claim, we apply Cauchy-Schwarz in ξ_2 to find

$$\begin{aligned}
&\int d\tau_1 \int (d\xi_1)_1 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) \\
&\quad \int d\tau_2 \int (d\xi_2)_1 f^{\#}_{k_2,j_2}(\tau_2,\xi_2) f^{\#}_{k_3,j_3}(\tau_1+\tau_2+\Omega^a,\xi_1+\xi_2) \\
&\lesssim \int d\tau_1 \int (d\xi_1)_1 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) \int d\tau_2 (1+2^{j_3-(a-1)k_1})^{1/4} \\
&\quad \left(\int (d\xi_2)_1 |f^{\#}_{k_2,j_2}(\tau_2,\xi_2)|^2 |f^{\#}_{k_3,j_3}(\tau_1+\tau_2+\Omega^a,\xi_1+\xi_2)|^2 \right)^{1/2}.
\end{aligned}$$

This estimate follows due to

$$\left| \frac{\partial^2 \Omega^a}{\partial \xi_2^2} \right| \sim 2^{(a-1)k_1},$$

which is derived from Case-by-Case analysis according to the signs of the involved frequencies.

Applications of Cauchy-Schwarz in τ_1, ξ_1 and τ_2 lead to

$$\lesssim 2^{j_2/2}(1+2^{j_3-(a-1)k_1})^{1/4} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_2,$$

which proves the first claim for $m_1 = 2$, $m_2 = 3$. There is no loss of generality due to the symmetry among k_i, $i = 1,2,3$.

For the second claim, we argue like in Lemma 6.3.2: Let $i_1 = 3$, $i_2 = 2$. From the proof we shall see that this is no loss of generality.
We apply the Cauchy-Schwarz inequality in ξ_2 to find

$$\begin{aligned}
&\int d\tau_1 \int (d\xi_1)_1 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) \\
&\quad \int d\tau_2 \int (d\xi_2)_1 f^{\#}_{k_2,j_2}(\tau_2,\xi_2) f^{\#}_{k_3,j_3}(\tau_1+\tau_2+\Omega^a(\xi_1,\xi_2),\xi_1+\xi_2) \\
&\lesssim \int d\tau_1 \int (d\xi_1)_1 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) \int d\tau_2 (1+2^{j_3-ak_1})^{1/2} \\
&\quad \left(\int d\xi_2 |f^{\#}_{k_2,j_2}(\tau_2,\xi_2)|^2 |f^{\#}_{k_3,j_3}(\tau_1+\tau_2+\Omega^a,\xi_1+\xi_2)|^2 \right)^{1/2}.
\end{aligned}$$

Now the claim follows from application of Cauchy-Schwarz inequality in τ_1, ξ_1 and τ_2. □

To estimate lower order terms, we use the following estimate not exploiting the dispersion relation but following from Cauchy-Schwarz inequality:

Lemma 6.3.5. *Estimate* (6.3) *holds with* $\alpha = 2^{k_{\min}/2} 2^{j_{\min}/2}$.

6.4 Short-time bilinear estimates

Purpose of this section is to prove the following proposition:

Proposition 6.4.1. *Let $T \in (0,1]$ and $u, v \in F_a^{s,\delta}(T)$, $i=1,2$.*
If $1 < a \leq 3/2$, then there are $\delta = \delta(a,s) > 0$ and $\theta = \theta(a,s) > 0$ so that we find the following estimates to hold:

$$\|\partial_x(uv)\|_{N_a^{0,\delta}(T)} \lesssim T^\theta \|u\|_{F_a^{0,\delta}(T)} \|v\|_{F_a^{0,\delta}(T)}, \tag{6.7}$$

$$\|\partial_x(uv)\|_{N_a^{-1/2,\delta}(T)} \lesssim T^\theta \|u\|_{F_a^{-1/2,\delta}(T)} \|v\|_{F_a^{s,\delta}(T)} \tag{6.8}$$

provided that $s > 3/2 - a$.
If $3/2 < a < 2$, then there are $\delta(a) > 0$ and $\theta(a) > 0$ so that we find the following estimate to hold:

$$\|\partial_x(uv)\|_{N_a^{-1/2,\delta}(T)} \lesssim T^\theta \|u\|_{F_a^{0,\delta}(T)} \|v\|_{F_a^{-1/2,\delta}(T)}. \tag{6.9}$$

We work with $\delta = 0$ in the following, which will be omitted from notation. Later we shall see how the analysis yields the estimates claimed above.
The above estimates are proved after decompositions in the frequency (cf. Subsection 4.8.2). This essentially reduces the estimates to

$$\|P_{k_3}\partial_x(u_{k_1}u_{k_2})\|_{N_{k_3,a}} \lesssim \alpha(\underline{k}) \|u_{k_1}\|_{F_{k_1,a}} \|u_{k_2}\|_{F_{k_2,a}}. \tag{6.10}$$

These estimates are proved via the L^2-bilinear estimates from the previous section. We enumerate the possible frequency interactions:

(i) $High \times Low \to High$-interaction: This case is treated in Lemma 6.4.2.

(ii) $High \times High \to High$-interaction: This case is treated in Lemma 6.4.3.

(iii) $High \times High \to Low$-interaction: This case is treated in Lemma 6.4.4.

(iv) $Low \times Low \to Low$-interaction: This case is treated in Lemma 6.4.5.

We start with $High \times Low \to High$-interaction:

Lemma 6.4.2. *Let $1 \leq a \leq 2$. Suppose that $k_3 \geq 20$, $k_2 \leq k_3 - 5$. Then, we find (6.10) to hold with $\alpha = 1$.*

Proof. By the same reductions like in Chapter 4, we find that it is enough to prove

$$2^{k_3} \sum_{j_3 \geq (2-a)k_3} 2^{-j_3/2} \|1_{D^a_{k_3,j_3}}(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim \prod_{i=1}^2 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2}, \tag{6.11}$$

where $\mathrm{supp}(f_{k_i,j_i}) \subseteq D^a_{k_i,\leq j_i}$ for $i = 2,3$, and we can suppose that $j_i \geq (2-a)k_3$. For the resonance function, we have the estimate from below

$$|\Omega^a| \gtrsim 2^{ak_3+k_2}.$$

Consequently, there is $j_i \geq ak_3 + k_2 - 10$.

Suppose that $j_3 \geq ak_3 + k_2 - 10$. Then, we apply duality and the first bound from Lemma 6.3.2 to find

$$\begin{aligned} &\sum_{j_3 \geq ak_3+k_2-10} 2^{-j_3/2} \|1_{D^a_{k_3,j_3}}(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \\ &\lesssim 2^{-(ak_3+k_2)/2} 2^{j_2/2} (1 + 2^{j_1 - ak_3})^{1/2} \prod_{i=1}^2 \|f_{k_i,j_i}\|_2. \end{aligned} \tag{6.12}$$

By the lower bound for j_1 and $a \geq 1$, it follows

$$(6.12) \lesssim 2^{-(ak_3+k_2)/2} 2^{j_2/2} 2^{j_1/2} 2^{-(2-a)k_3/2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_2 \lesssim 2^{-k_2/2} \prod_{i=1}^{2} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2}.$$

This yields (6.11).

Suppose that $j_1 \geq ak_3 + k_2 - 10$. The argument for $j_2 \geq ak_3 + k_2 - 10$ is the same. An application of the second bound from Lemma 6.4.2 yields

$$\begin{aligned}
&\sum_{j_3 \geq (2-a)k_3} 2^{-j_3/2} \|1_{D^a_{k_3,j_3}} (f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \\
&\lesssim \sum_{j_3 \geq (2-a)k_3} 2^{-j_3/2} (1 + 2^{j_1 - ak_3})^{1/2} 2^{j_2/2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_2 \\
&\lesssim 2^{-(2-a)k_3/2} 2^{-ak_3/2} \prod_{i=1}^{2} 2^{j_i/2} \|f_{k_i,j_i}\|_2 = 2^{-k_3} \prod_{i=1}^{2} 2^{j_i/2} \|f_{k_i,j_i}\|_2.
\end{aligned}$$

This completes the proof. □

We turn to *High* × *High* → *High*-interaction:

Lemma 6.4.3. *Let* $1 \leq a \leq 2$. *Suppose that* $k_3 \geq 50$, $|k_1 - k_2| \leq 10$, $|k_2 - k_3| \leq 10$. *Then, we find* (6.10) *to hold with* $\alpha = 1$.

Actually, the same argument as in *High* × *Low* → *High*-interaction is applicable since there are two frequencies with group velocity difference of size 2^{ak_1} (cf. Section 3.1). Below, we point out how to prove clearly better estimates using the resonance.

Proof. Like above it suffices to prove

$$2^{k_3} \sum_{j_3 \geq (2-a)k_3} 2^{-j_3/2} \|1_{D^a_{k_3,j_3}} (f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim \prod_{i=1}^{2} 2^{j_i/2} \|f_{k_i,j_i}\|_2 \qquad (6.13)$$

In this case we have $|\Omega^a| \gtrsim 2^{(a+1)k_3}$. Hence, due to otherwise impossible modulation interaction, there is $j_i \geq (a+1)k_3 - 20$.

If $j_3 \geq (a+1)k_3 - 20$, then we use duality and the first estimate from Lemma 6.3.4 to find

$$\begin{aligned}
&\sum_{j_3 \geq (a+1)k_3 - 10} 2^{-j_3/2} \|1_{D^a_{k_3,j_3}} (f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \\
&\lesssim 2^{-(a+1)k_3/2} 2^{j_1/2} (1 + 2^{j_2 - (a-1)k_3})^{1/4} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_2.
\end{aligned}$$

The claim follows even with extra smoothing.

If $j_1 \geq (a+1)k_3 - 10$ (or $j_2 \geq (a+1)k_3 - 10$, where the same estimate can be applied), then we use again the first estimate from Lemma 6.3.4 to derive

$$\begin{aligned}
&\sum_{j_3 \geq (2-a)k_3} 2^{-j_3/2} \| 1_{D^a_{k_3,j_3}} (f_{k_1,j_1} * f_{k_2,j_2}) \|_{L^2} \\
&\lesssim \sum_{j_3 \geq (2-a)k_3} 2^{-j_3/2} (1 + 2^{j_3-(a-1)k_3})^{1/4} 2^{j_2/2} \prod_{i=1}^{2} \| f_{k_i,j_i} \|_2 \\
&\lesssim 2^{-(1+\varepsilon(a))k_3} \prod_{i=1}^{2} 2^{j_i/2} \| f_{k_i,j_i} \|_2
\end{aligned}$$

even for some $\varepsilon = \varepsilon(a) > 0$. □

We turn to $High \times High \to Low$-interaction, which is dual to $High \times Low \to High$-interaction. We have to add localization in time in order to estimate the input frequencies in short-time spaces.

Lemma 6.4.4. *Let* $k_1 \geq 30$ *and* $k_3 \leq k_1 - 5$. *Then, we find* (6.10) *to hold with* $\alpha = (3k_1) 2^{(1-a)k_1} 2^{(a-3/2)k_3}$.

Proof. Following the definition of the $N_{a,k}$-spaces, we have to estimate

$$2^{k_3} \sum_{j_3 \geq (2-a)k_3} 2^{-j_3/2} \| 1_{D^a_{k_3,j_3}} \mathcal{F}_{t,x} (u_{k_1} v_{k_2} \eta(2^{(2-a)k_3}(t-t_k)) \|_{L^2_{\tau,\xi}}. \tag{6.14}$$

The resonance is given by $|\Omega^a| \gtrsim 2^{ak_1+k_3}$.

Suppose that $j_3 \geq ak_1 + k_3 - 10$. Then, we find

$$(6.14) \lesssim 2^{k_3} 2^{-\frac{ak_1+k_3}{2}} \| u_{k_1} v_{k_2} \eta(2^{(2-a)k_3}(t-t_k)) \|_{L^2_{t,x}}.$$

After adding localization in time (since we are estimating an L^2_t-norm at this point), it is enough to estimate

$$2^{k_3} 2^{\frac{(2-a)(k_1-k_3)}{2}} 2^{-\frac{ak_1+k_3}{2}} \| u_{k_1} v_{k_2} \eta(2^{(2-a)k_1}(t-t_\lambda)) \|_{L^2_{t,x}}. \tag{6.15}$$

Write

$$\begin{aligned}
f_{k_1,j_1} &= 1_{D^a_{k_1,(\leq)j_1}} \mathcal{F}_{t,x} [\gamma(2^{(2-a)k_1}(t-t_\lambda)) u_{k_1}], \\
f_{k_2,j_2} &= 1_{D^a_{k_2,(\leq)j_2}} \mathcal{F}_{t,x} [\gamma(2^{(2-a)k_1+10}(t-t_\mu)) v_{k_2}].
\end{aligned}$$

In the above display, the low modulations are annexed matching time localization as usual.

Then, an application of two $L^4_{t,x}$-Strichartz estimates gives

$$\begin{aligned}
(6.15) &\lesssim 2^{(1-a)k_1} 2^{\frac{a-1}{2}k_3} 2^{-\frac{k_1}{4}} \prod_{i=1}^{2} \sum_{j_i \geq (2-a)k_1} 2^{j_i/2} \| f_{k_i,j_i} \|_2 \\
&\lesssim 2^{(3/4-a)k_1} 2^{\frac{a-1}{2}k_3} \prod_{i=1}^{2} \sum_{j_i \geq (2-a)k_1} 2^{j_i/2} \| f_{k_i,j_i} \|_2.
\end{aligned}$$

This yields a first bound. Some of the above estimates are crude because the next case gives the worse bound anyway.

We turn to the sum over j_3 in (6.14), where $j_3 \leq ak_1 + k_3 - 10$.
By the reduction and notations from Chapter 5, we have to estimate

$$2^{(2-a)(k_1-k_3)} 2^{k_3} \sum_{(2-a)k_3 \leq j_3 \leq ak_1+k_3} 2^{-j_3/2} \| 1_{D^a_{k_3,\leq j_3}} (f_{k_1,j_1} * f_{k_2,j_2}) \|_{L^2_{\tau,\xi}},$$

where $j_1, j_2 \geq (2-a)k_1$.

Suppose that $j_1 \geq ak_1 + k_3 - 10$ by symmetry among f_{k_1,j_1} and f_{k_2,j_2}. An application of Lemma 6.3.2 in conjunction with duality gives

$$\begin{aligned} &\lesssim 2^{(2-a)(k_1-k_3)} 2^{k_3} \sum_{j_3 \leq ak_1+k_3} 2^{-j_3/2} 2^{j_3/2} \left(1 + 2^{j_2 - ak_1}\right)^{1/2} \|f_{k_1,j_1}\|_2 \|f_{k_2,j_2}\|_2 \\ &\lesssim (3k_1) 2^{(1-a)k_1} 2^{(a-3/2)k_3} \prod_{i=1}^{2} 2^{j_i/2} \|f_{k_i,j_i}\|_2, \end{aligned}$$

which is inferior to the first bound. The proof is complete. □

We record the estimate for *Low* × *Low* → *Low*-interaction which is immediate from Lemma 6.3.5:

Lemma 6.4.5. *Let* $k_i \leq 100$, $i = 1, 2, 3$. *Then, we find* (6.10) *to hold with* $\alpha(\underline{k}) = 1$.

Proof of Proposition 6.4.1. With the above estimates for frequency localized interactions at disposal, we can infer the claimed estimates: For $High \times Low \to High$-interaction Lemma 6.4.2 gives the estimates after square-summing

$$\begin{aligned} \|\partial_x(uv)\|_{N^0_a(T)} &\lesssim \|u\|_{F^0_a(T)} \|v\|_{F^{0+}_a(T)}, \\ \|\partial_x(uv)\|_{N^{-1/2}_a(T)} &\lesssim \|u\|_{F^{-1/2}_a(T)} \|v\|_{F^s_a(T)}, \end{aligned}$$

where $1 < a \leq 3/2$ and $s > 3/2 - a$.

Increasing time localization leads to extra smoothing (because the minimal size of the modulation regions will become larger). Together with Lemma 2.5.3, we deduce from the proof of Lemma 6.4.2

$$\begin{aligned} \|\partial_x(uv)\|_{N^{0,\delta}_a(T)} &\lesssim T^\theta \|u\|_{F^{0,\delta}_a(T)} \|v\|_{F^{0,\delta}_a(T)}, \\ \|\partial_x(uv)\|_{N^{-1/2,\delta}_a(T)} &\lesssim T^\theta \|u\|_{F^{-1/2,\delta}_a(T)} \|v\|_{F^{s,\delta}_a(T)} \end{aligned}$$

for some $\theta > 0$ for any $\delta > 0$ with a and s like in the previous display.

For $3/2 < a < 2$ the argument is analogous for $High \times Low \to High$-interaction. For $High \times High \to High$-interaction the estimates due to Lemma 6.4.3 are sufficient because of improved resonance compared to $High \times Low \to High$-interaction. For $High \times High \to Low$-interaction the short-time estimates become worse when increasing time localization. But there is room in the estimate from Lemma 6.4.4 to prove the estimates for $\delta(a) > 0$ chosen sufficiently small. □

6.5 Energy estimates

Purpose of this section is to propagate the energy norm of solutions and differences of solutions: Set

$$\|u\|_{H^m}^2 = \sum_{\xi} m(\xi)\hat{u}(\xi)\hat{u}(-\xi).$$

We consider generalized symbols $m \in S^s_\varepsilon$ like in Chapter 5. However, s can also be negative. In this case the definition is adapted following [CHT12].
The following estimates are shown:

Proposition 6.5.1. *Let $1 < a < 2$, $T \in (0,1]$, $M \in 2^{\mathbb{N}_0}$ and suppose that u is a smooth solution to (6.1) with vanishing mean. Then, there are positive $\varepsilon(s,a)$, $\theta(a,s)$, $\delta(a,s)$, $c(a,s)$, $d(a,s)$ so that we find the following estimate to hold*

$$\begin{aligned}\|u\|^2_{E^s(T)} \lesssim \|u\|^2_{H^s} &+ TM^c \|u\|^3_{F_a^{s-\varepsilon,\delta}(T)} \\ &+ M^{-d}\|u\|^3_{F_a^{s-\varepsilon,\delta}(T)} + T^\theta \|u\|^4_{F_a^{s,\delta}(T)}\end{aligned} \tag{6.16}$$

provided that $s \geq 3/2 - a$.

The following energy estimates for differences of solutions are proved.

Proposition 6.5.2. *Let $T \in (0,1]$, $1 < a < 2$ and $M \in 2^{\mathbb{N}_0}$. Suppose that $s > 3/2 - a$ and u_i, $i = 1,2$, are smooth solutions to (6.1) with vanishing mean. Then, there are positive $c(a,s), d(a,s), \theta(a,s), \delta(a,s)$ so that we find the following estimate to hold:*

$$\begin{aligned}\|v\|^2_{E^{-1/2}(T)} \lesssim \|v(0)\|^2_{H^{-1/2}} &+ TM^c \|v\|^2_{F_a^{-1/2,\delta}(T)} (\|u_1\|_{F_a^{s,\delta}(T)} + \|u_2\|_{F_a^{s,\delta}(T)}) \\ &+ M^{-d}\|v\|^2_{F_a^{-1/2,\delta}(T)} (\|u_1\|_{F_a^{s,\delta}(T)} + \|u_2\|_{F_a^{s,\delta}(T)}) \\ &+ T^\theta \|v\|^2_{F_a^{-1/2,\delta}(T)} (\|u_1\|^2_{F_a^{s,\delta}(T)} + \|u_2\|^2_{F_a^{s,\delta}(T)}).\end{aligned} \tag{6.17}$$

Furthermore, the following estimate holds:

$$\begin{aligned}\|v\|^2_{E^s(T)} \lesssim \|v(0)\|^2_{H^s} &+ M^c T \|v\|^2_{F_a^{s,\delta}(T)} (\|v\|_{F_a^{s,\delta}(T)} + \|u_2\|_{F_a^{s,\delta}(T)}) \\ &+ M^{-d}\|v\|^2_{F_a^{s,\delta}(T)} (\|v\|_{F_a^{s,\delta}(T)} + \|u_2\|_{F_a^{s,\delta}(T)}) \\ &+ T^\theta (\|v\|_{F_a^{s,\delta}(T)} \|v\|_{F_a^{-1/2,\delta}(T)} \|u_2\|_{F_a^{r,\delta}(T)} \|u_2\|_{F_a^{s,\delta}(T)} \\ &\quad + \|v\|^2_{F_a^{s,\delta}(T)} (\|u_2\|^2_{F_a^{s,\delta}(T)} + \|v\|^2_{F_a^{s,\delta}(T)})),\end{aligned} \tag{6.18}$$

where $r = s + (2-a)$.

For smooth solutions we find by the fundamental theorem of calculus and after symmetrization

$$\begin{aligned}\|u(t)\|^2_{H^m} &= \|u(0)\|^2_{H^m} \\ &+ C \int_0^t \int_{\Gamma_3} (m(\xi_1)\xi_1 + m(\xi_2)\xi_2 + m(\xi_3)\xi_3)\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\Gamma_3 ds.\end{aligned}$$

To integrate by parts like in Chapter 5, we consider the first resonance function

$$\Omega_1^a(\xi_1,\xi_2,\xi_3) = \xi_1|\xi_1|^a + \xi_2|\xi_2|^a + \xi_3|\xi_3|^a \quad (\xi_1,\xi_2,\xi_3) \in \Gamma_3. \tag{6.19}$$

A consequence of the mean value theorem is

$$|\Omega_1^a(\xi_1,\xi_2,\xi_3)| \gtrsim |\xi_{\max}|^a |\xi_{\min}|,$$

and thus, the first resonance does not vanish provided that $\xi_i \neq 0$.

Integration by parts becomes possible, and we find

$$\begin{aligned}
R_3^{s,m} &= \int_0^T dt \sum_{\substack{\xi_1+\xi_2+\xi_3=0,\\ \xi_i\neq 0}} (m(\xi_1)\xi_1 + m(\xi_2)\xi_2 + m(\xi_3)\xi_3)\hat{u}(t,\xi_1)\hat{u}(t,\xi_2)\hat{u}(t,\xi_3) \\
&= \left[\sum_{\substack{\xi_1+\xi_2+\xi_3=0,\\ \xi_i\neq 0}} \frac{(m(\xi_1)\xi_1 + m(\xi_2)\xi_2 + m(\xi_3)\xi_3)}{\Omega_1^a(\xi_1,\xi_2,\xi_3)} \hat{u}(t,\xi_1)\hat{u}(t,\xi_2)\hat{u}(t,\xi_3) \right]_{t=0}^T \\
&\quad + C \int_0^T \sum_{\substack{\xi_1+\xi_2+\xi_3=0,\\ \xi_i\neq 0}} \frac{m(\xi_1)\xi_1 + m(\xi_2)\xi_2 + m(\xi_3)\xi_3}{\xi_1|\xi_1|^a + \xi_2|\xi_2|^a + \xi_3|\xi_3|^a} \hat{u}(t,\xi_1)\hat{u}(t,\xi_2) \\
&\qquad \times \xi_3 \sum_{\substack{\xi_3=\xi_{31}+\xi_{32},\\ \xi_{3i}\neq 0}} \hat{u}(t,\xi_{31})\hat{u}(t,\xi_{32}) \\
&= B_3^{s,m}(0;T) + R_4^{s,m}(T).
\end{aligned}$$

Set

$$b_3^{s,m}(\xi_1,\xi_2,\xi_3) = \frac{m(\xi_1)\xi_1 + m(\xi_2)\xi_2 + m(\xi_3)\xi_3}{\xi_1|\xi_1|^a + \xi_2|\xi_2|^a + \xi_3|\xi_3|^a}.$$

The following estimate of the multiplier is a consequence of the mean value theorem and the lower bound for the resonance function:

Lemma 6.5.3. *Let $|\xi_1| \sim |\xi_2| \gtrsim |\xi_3| > 0$. Then, the following estimate holds:*

$$|b_3^{s,m}(\xi_1,\xi_2,\xi_3)| \lesssim \frac{\max_{i=1,2,3} |m(\xi_i)|}{|\xi_1|^a}.$$

We collect the low frequencies as

$$R_3^{s,m,M} = \int_0^T dt \sum_{\substack{\xi_1+\xi_2+\xi_3=0,\\ 1\leq|\xi_i|\leq M}} b_3^{s,m}(\xi_1,\xi_2,\xi_3)\hat{u}(t,\xi_1)\hat{u}(t,\xi_2)\hat{u}(t,\xi_3).$$

Like in Chapter 5 we differentiate by parts only $R_3^{s,m} - R_3^{s,m,M}$ such that one of the initial frequencies is higher than M.
This leads us to the boundary term $B_3^{s,m,M}$ with one of the frequencies higher than M. We have the following lemma:

Lemma 6.5.4. *Suppose that $-1/2 < s < 1/2$. Then, we find the following estimate to hold for any $1 < a < 2$, $\delta \geq 0$:*

$$B_3^{s,m,M}(0;T) \lesssim M^{-d(s,a)} \|u\|^3_{F_a^{s,\delta}(T)}. \tag{6.20}$$

Proof. Localize frequencies on a dyadic scale, i.e., $P_{k_i} u_i = u_i$ and suppose $k_1 \geq k_2 \geq k_3$ by symmetry. Let $m = \log_2(M)$. We use the embedding from Lemma 2.5.1 to reduce the bound to a bound of Sobolev norms. By Lemma 6.5.3 and Hölder in position space, we find the estimate for the evaluation at $t = 0$

$$\begin{aligned} & 2^{2\varepsilon k_1} \frac{\max(2^{2sk_1}, 2^{2sk_3})}{2^{ak_1}} \sum_{\substack{\xi_1+\xi_2+\xi_3=0, \\ \xi_i \neq 0, |\xi_1| \geq M}} |\hat{u}_1(0,\xi_1)||\hat{u}_2(0,\xi_2)||\hat{u}_3(0,\xi_3)| \\ &\lesssim 2^{2\varepsilon k_1} \frac{\max(2^{2sk_1}, 2^{2sk_3})}{2^{ak_1}} \|P_{k_1} u(0)\|_{L^2} \|P_{k_2} u(0)\|_{L^2} 2^{k_3/2} \|P_{k_3} u(0)\|_{L^2}. \end{aligned}$$

This expression sums up to the claimed estimate. □

The remainder term is symmetrized once again to find (the constraint for the initial frequencies is omitted because it is not relevant in the following)

$$\begin{aligned} R_4^{s,m} = C \int_0^T dt \int_{\Gamma_4} & d\Gamma_4 (b_3^{s,m}(\xi_1, \xi_2, \xi_{31} + \xi_{32}) - b_3^{s,m}(-\xi_{31}, -\xi_{32}, \xi_{31} + \xi_{32}))\xi_3 \\ & \times \hat{u}(t,\xi_1)\hat{u}(t,\xi_2)\hat{u}(t,\xi_{31})\hat{u}(t,\xi_{32}). \end{aligned}$$

Set

$$b_4^{s,m}(\xi_1, \xi_2, \xi_{31}, \xi_{32}) = [b_3^{s,m}(\xi_1, \xi_2, \xi_{31} + \xi_{32}) - b_3^{s,m}(-\xi_{31}, -\xi_{32}, \xi_{31} + \xi_{32})]\xi_3.$$

For the second symmetrization we record again by the mean value theorem:

Lemma 6.5.5. *With the above notation, we find the following estimate to hold:*

$$|b_4^{s,m}(\xi_1, \xi_2, \xi_{31}, \xi_{32})| \lesssim \frac{\max_{i=1,2,3} |m(\xi_i)|}{\max_{i=1,2,3} |\xi_i|^a} |\xi_3^*|,$$

where $|\xi_1^*| \geq |\xi_2^*| \geq \ldots$ *denotes a decreasing rearrangement of the* ξ_i, $i = 1, 2, 31, 32$.

For the more difficult remainder estimate, it is important to note that the second symmetrization cancels the second resonance

$$\Omega_2^a(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1|\xi_1|^a + \xi_2|\xi_2|^a + \xi_3|\xi_3|^a + \xi_4|\xi_4|^a, \quad (\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4. \tag{6.21}$$

Next, an estimate is derived which is effective when estimating expressions involving two high frequencies and two low frequencies provided that the second resonance is non-vanishing.

Lemma 6.5.6. *Let* $k_i, j_i \in \mathbb{N}$ *and* $f_{k_i,j_i} \in L^2_{\geq 0}(\mathbb{R} \times \mathbb{Z})$ *with supp*$(f_{k_i,j_i}) \subseteq D^a_{k_i, \leq j_i}$. *Suppose that* $k_1 \leq k_2 \leq k_3$, $k_2 \leq k_3 - 5$ *and supp*$_\xi(f_{k_m,j_m}) \subseteq I_m$, $m = 1, 2$, $|I_m| \lesssim 2^l$.
Then, we find the following estimate to hold:

$$\begin{aligned} & \int d\Gamma_4(\tau) \int d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1, \xi_1) f_{k_2,j_2}(\tau_2, \xi_2) f_{k_3,j_3}(\tau_3, \xi_3) f_{k_4,j_4}(\tau_4, \xi_4) \\ &\lesssim \min(2^{j_1/2}, 2^{j_3/2})(1 + 2^{j_4 - ak_4})^{1/2} 2^{j_2/2} 2^{l/2} \prod_{i=1}^4 \|f_{k_i,j_i}\|_2. \end{aligned} \tag{6.22}$$

Proof. Like in Section 5.2.2 we rewrite and make successive use of the Cauchy-Schwarz inequality to find

$$
\begin{aligned}
&\int d\Gamma_4(\tau)\int d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) f_{k_4,j_4}(\tau_4,\xi_4)\\
&= \int d\tau_1 \int (d\xi_1)_1 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) \int d\tau_3 \int (d\xi_3)_1 f^{\#}_{k_3,j_3}(\tau_3,\xi_3)\\
&\quad \times \int d\tau_2 \int (d\xi_2)_1 f^{\#}_{k_2,j_2}(\tau_2,\xi_2) f^{\#}_{k_4,j_4}(-\tau_1-\tau_2-\tau_3+\Omega_2^a, -\xi_1-\xi_2-\xi_3)\\
&\lesssim \int d\tau_1 \int (d\xi_1)_1 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) \int d\tau_3 \int (d\xi_3)_1 f^{\#}_{k_3,j_3}(\tau_3,\xi_3)\\
&\quad \times \int d\tau_2 (1+2^{j_4-ak_4})^{1/2} \left(\int (d\xi_2)_1 |f^{\#}_{k_2,j_2}|^2 |f^{\#}_{k_4,j_4}|^2\right)^{1/2}\\
&\lesssim (1+2^{j_4-ak_4})^{1/2} \int (d\xi_1)_1 \int d\tau_3 \int (d\xi_3)_1 f^{\#}_{k_3,j_3}(\tau_3,\xi_3)\\
&\quad \times \int d\tau_2 \left(\int d\tau_1 |f^{\#}_{k_1,j_1}(\tau_1,\xi_1)|^2\right)^{1/2}\\
&\quad \times \left(\int (d\xi_2)_1 |f^{\#}_{k_2,j_2}(\tau_2,\xi_2)|^2 \int d\tau_1 |f^{\#}_{k_4,j_4}(\tau_4,\xi_4)|^2\right)^{1/2}\\
&\lesssim 2^{l/2} 2^{j_2/2} 2^{j_3/2} (1+2^{j_4-ak_4})^{1/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_2.
\end{aligned}
$$

This yields the second estimate.

Similarly, we find the first estimate by

$$
\begin{aligned}
&\int d\tau_3 \int (d\xi_3)_1 f^{\#}_{k_3,j_3}(\tau_3,\xi_3) \int d\tau_1 \int (d\xi_2)_1 f^{\#}_{k_2,j_2}(\tau_2,\xi_2)\\
&\quad \times \int (d\xi_1)_1 \int d\tau_2 f^{\#}_{k_1,j_1}(\tau_1,\xi_1) f^{\#}_{k_4,j_4}(-\tau_1-\tau_2-\tau_3+\Omega, -\xi_1-\xi_2-\xi_3)\\
&\lesssim (1+2^{j_4-ak_4})^{1/2} \int d\tau_3 \int (d\xi_3)_1 f^{\#}_{k_3,j_3}(\tau_3,\xi_3) \int d\tau_2 \int (d\xi_2)_2 f^{\#}_{k_2,j_2}(\tau_2,\xi_2) \int d\tau_1\\
&\quad \times \left(\int (d\xi_1)_1 |f^{\#}_{k_1,j_1}(\tau_1,\xi_1)|^2 |f^{\#}_{k_4,j_4}(-\tau_1-\tau_2-\tau_3+\Omega_2^a, -\xi_1-\xi_2-\xi_3)|^2\right)^{1/2}\\
&\lesssim 2^{l/2} 2^{j_1/2} 2^{j_3/2} (1+2^{j_4-ak_4})^{1/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_2.
\end{aligned}
$$

□

Remark 6.5.7. Note that the argument is symmetric with respect to the low frequencies k_1 and k_2 above and the high frequencies k_3 and k_4. Below, we freely use the estimates obtained from such permutations.

We record the following short-time consequences (i.e., modulations large depending on the frequencies):

Lemma 6.5.8. *Let $k_i, j_i \in \mathbb{N}$, $i = 1, \ldots, 4$ and suppose that*

$$k_1 \leq k_2 \leq k_3, \quad k_2 \leq k_3 - 5, \quad j_i \geq (2-a)k_3 \text{ for } i \in \{1, \ldots, 4\} \tag{6.23}$$

Then, we find the following estimate to hold:

$$\begin{aligned} &\int d\Gamma_4(\tau) \int d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) f_{k_4,j_4}(\tau_4,\xi_4) \\ &\lesssim 2^{-k_3} \prod_{i=1}^4 2^{j_i/2} \|f_{k_i,j_i}\|_2 \end{aligned} \tag{6.24}$$

provided that $k_1 \leq k_2 - 5$.
Suppose (6.23) *and* $|k_1 - k_2| \leq 5$. *Then, we find the following estimate to hold:*

$$\begin{aligned} &\int d\Gamma_4(\tau) \int_{\xi_1+\xi_2 \neq 0} d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) f_{k_4,j_4}(\tau_4,\xi_4) \\ &\lesssim 2^{-k_4} 2^{(0+)k_2} \prod_{i=1}^4 2^{j_i/2} \|f_{k_i,j_i}\|_2. \end{aligned} \tag{6.25}$$

Proof. The first claim follows from applying Lemma 6.5.6 with $l = k_2$ and observing that $j_{\max} \geq ak_3 + k_2 - 10$.
For the second claim, we use a similar argument like in Section 5.2.2. We carry out a decomposition of the expression into $|\Omega_a^2| \sim 2^{ak_3+l}$ which is equivalent to assuming that $|\xi_1 \pm \xi_2| \sim 2^l$.
At this point, we can assume that $f_{k_1,j_1}(\tau,\cdot)$ and $f_{k_2,j_2}(\tau,\cdot)$ are supported in intervals I_m, $m = 1,2$ of length 2^l.
The decompositions $f_{k_i,j_i}^{I_i}$ are almost orthogonal, that is

$$\sum_{I_i} \|f_{k_i,j_i}^{I_i}\|_2^2 \lesssim \|f_{k_i,j_i}\|_2^2$$

and further, supposing that $|\Omega_a^2| \sim 2^{ak_3+l}$, for fixed I_1, there are only finitely many intervals I_2 such that there is a non-trivial contribution

$$\int d\Gamma_4(\tau) \int_{|\Omega_a^2| \sim 2^{ak_3+l}} d\Gamma_4(\xi) f_{k_1,j_1}^{I_1}(\tau_1,\xi_1) f_{k_2,j_2}^{I_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) f_{k_4,j_4}(\tau_4,\xi_4). \tag{6.26}$$

The localized expression is amenable to the argument yielding the first estimate. So,

$$(6.26) \lesssim 2^{-k_3} \left(\prod_{i=1}^4 2^{j_i/2} \right) \|f_{k_1,j_1}^{I_1}\|_2 \|f_{k_2,j_2}^{I_2}\|_2 \|f_{k_3,j_3}\|_2 \|f_{k_4,j_4}\|_2.$$

The claim follows from carrying out the sum over I_1 and I_2 by almost orthogonality and the sum over l, which leads to the $2^{(0+)k_2}$ loss. □

We have the following estimate due to Cauchy-Schwarz inequality to handle lower order terms:

Lemma 6.5.9. *Let* k_i, $j_i \in \mathbb{N}$ *and* $f_{k_i,j_i} \in L^2_{\geq 0}(\mathbb{R} \times \mathbb{Z})$ *with supp*$(f_{k_i,j_i}) \subseteq D^a_{k_i,\leq j_i}$ *and let* $k_1^* \geq \ldots \geq k_4^*$ *and* $j_1^* \geq \ldots \geq j_4^*$ *denote decreasing rearrangements of* k_i, j_i. *Then, we find the following estimate to hold:*

$$\begin{aligned} &\int d\Gamma_4(\tau) \int d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) f_{k_4,j_4}(\tau_4,\xi_4) \\ &\lesssim 2^{k_4^*/2} 2^{k_3^*/2} 2^{j_4^*/2} 2^{j_3^*/2} \prod_{i=1}^4 \|f_{k_i,j_i}\|_2. \end{aligned}$$

However, if $\Omega_2^a = 0$ we find $|\xi_1^*| = |\xi_2^*|$, $|\xi_3^*| = |\xi_4^*|$, where the actual frequencies have opposite signs. Thus, the sum over the frequencies collapses and two applications of Cauchy-Schwarz in the modulation variables give the following:

Lemma 6.5.10. *Let $k_i, j_i \in \mathbb{N}$ and $f_{k_i,j_i} \in L^2_{\geq 0}(\mathbb{R} \times \mathbb{Z})$ with supp$(f_{k_i,j_i}) \subseteq D^a_{k_i,\leq j_i}$. Let $|k_1 - k_2| \leq 2$, $|k_3 - k_4| \leq 2$ and $k_1 \geq k_3$ and let $j_1^* \geq \ldots \geq j_4^*$ denote a decreasing rearrangement of the j_i.*
Then, we find the following estimate to hold:

$$\int d\Gamma_4(\tau) \int_{\substack{\xi_1+\xi_2=0,\\ \xi_3+\xi_4=0}} d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) f_{k_4,j_4}(\tau_4,\xi_4)$$
$$\lesssim 2^{j_4^*/2} 2^{j_3^*/2} \prod_{i=1}^4 \|f_{k_i,j_i}\|_2.$$

In case there is one frequency clearly lower than the remaining three frequencies, the resonance is very favourable, and we make use of the following bound, which is a consequence of three $L^6_{t,x}$-Strichartz estimates:

Lemma 6.5.11. *Let $k_i, j_i \in \mathbb{N}$ and $f_{k_i,j_i} \in L^2_{\geq 0}(\mathbb{Z} \times \mathbb{R})$ with supp$(f_{k_i,j_i}) \subseteq D^a_{k_i,\leq j_i}$, and let $j_1^* \geq \ldots \geq j_4^*$ denote a decreasing rearrangement of the j_i.*
Then, we find the following estimate to hold:

$$\int d\Gamma_4(\tau) \int d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) f_{k_4,j_4}(\tau_4,\xi_4)$$
$$\lesssim 2^{-j_1^*/2} 2^{(0+)k_{\max}} \prod_{i=1}^4 2^{j_i/2} \|f_{k_i,j_i}\|_2.$$

Proof. Let $u_i = \mathcal{F}^{-1}_{t,x}[f_{k_i,j_i}]$ denote the inverse Fourier transform and to simplify the notation let $j_1 = j_1^*$.
Then, changing back to position space and applying Hölder's inequality gives

$$\begin{aligned}
&\int d\Gamma_4(\tau) \int d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1,\xi_1) \ldots f_{k_4,j_4}(\tau_4,\xi_4)\\
&= \int dt \int dx u_1(t,x) \ldots u_4(t,x)\\
&\lesssim \|u_1\|_{L^2_{t,x}} \prod_{i=2}^4 \|u_i\|_{L^6_{t,x}} \lesssim \|f_{k_1,j_1}\|_{L^2_{t,x}} \prod_{i=2}^4 2^{(0k_i)+} 2^{j_i/2} \|f_{k_i,j_i}\|_2\\
&\lesssim 2^{-j_1^*/2} 2^{(0+)k_{\max}} \prod_{i=1}^4 2^{j_i/2} \|f_{k_i,j_i}\|_2.
\end{aligned}$$

The $L^6_{t,x}$-Strichartz estimate is an instance of Proposition 3.2.1. □

Further, we have the following consequence of four $L^4_{t,x}$-Strichartz estimates:

Lemma 6.5.12. *Let $1 \leq a \leq 2$, $k_i, j_i \in \mathbb{N}$ and $f_{k_i,j_i} \in L^2_{\geq 0}(\mathbb{Z} \times \mathbb{R})$ with supp$(f_{k_i,j_i}) \subseteq D^a_{k_i,\leq j_i}$. Then, we find the following estimate to hold:*

$$\int_{\Gamma_4(\tau)} d\Gamma_4(\tau) \int_{\Gamma_4(\xi)} d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1,\xi_1) f_{k_2,j_2}(\tau_2,\xi_2) f_{k_3,j_3}(\tau_3,\xi_3) f_{k_4,j_4}(\tau_4,\xi_4)$$
$$\lesssim \prod_{i=1}^4 2^{\frac{(a+2)j_i}{4(a+1)}} \|f_{k_i,j_i}\|_2.$$

Proof. Like in Lemma 6.5.11 change to position space and apply Hölder to find

$$\begin{aligned}
&\int d\Gamma_4(\tau) \int d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1,\xi_1)\dots f_{k_4,j_4}(\tau_4,\xi_4) \\
&= \int dt \int dx u_1(t,x)\dots u_4(t,x) \\
&\lesssim \prod_{i=1}^{4} \|u_i\|_{L^4_{t,x}} \lesssim \prod_{i=1}^{4} 2^{\frac{(a+2)j_i}{4(a+1)}} \|f_{k_i,j_i}\|_2.
\end{aligned}$$

The $L^4_{t,x}$-Strichartz estimate is a consequence of Lemma 6.3.1. □

The more involved remainder estimate, for which the above multilinear estimates are deployed, is carried out in the following lemma:

Lemma 6.5.13. *Let $1 < a < 2$ and $T \in (0,1]$. Suppose that $s \geq 3/2 - a$. Then, we find the following estimate to hold:*

$$\left| \int_0^T R_4^m[u] ds \right| \lesssim T^\theta \|u\|^4_{F_a^{s-\varepsilon,\delta}(T)}$$

provided that $m \in S^s_\varepsilon$ and $\varepsilon(s,a) > 0$, $\theta(s,a) > 0$, $\delta = \delta(s,a) > 0$ are chosen sufficiently small.

Proof. In the expression

$$\int_0^T dt \int_{\Gamma_4} d\Gamma_4 b_4^{m_\varepsilon}(\xi_1,\xi_2,\xi_{31},\xi_{32}) \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_{31})\hat{u}(\xi_{32}) \tag{6.27}$$

we can suppose $|\xi_1| \gtrsim |\xi_2|$, $|\xi_{31}| \gtrsim |\xi_{32}|$ by symmetry.
Further, we break the frequencies into dyadic blocks $|\xi_1| \sim 2^{k_1}$, $|\xi_2| \sim 2^{k_2}$, $|\xi_{31}| \sim 2^{k_{31}}$, $|\xi_{32}| \sim 2^{k_{32}}$.

After dyadic frequency localization, for an estimate of (6.27), one has additionally to take into account the time localization and the multiplier bound. For this purpose, we perform a Case-by-Case analysis:

Case A. $|\xi_1| \sim |\xi_2|$

Subcase AI. $|\xi_1| \gg |\xi_3| \gtrsim |\xi_{31}| \gtrsim |\xi_{32}|$

Subcase AII. $|\xi_1| \gg |\xi_3| \ll |\xi_{31}| \sim |\xi_{32}|$

Subcase AIII. $|\xi_1| \sim |\xi_3| \gtrsim |\xi_{31}| \gtrsim |\xi_{32}|$

Subcase AIV. $|\xi_1| \sim |\xi_3| \ll |\xi_{31}| \sim |\xi_{32}|$

Case B. $|\xi_1| \gg |\xi_2|$

Subcase BI. $|\xi_1| \sim |\xi_3| \sim |\xi_{31}| \sim |\xi_{32}|$

Subcase BII. $|\xi_1| \sim |\xi_3| \ll |\xi_{31}| \sim |\xi_{32}|$

Subcase BIII. $|\xi_1| \sim |\xi_3| \sim |\xi_{31}| \gg |\xi_{32}|$

Let $\gamma : \mathbb{R} \to [0,1]$ denote a smooth function with support in $[-1,1]$ satisfying

$$\sum_{n \in \mathbb{Z}} \gamma^4(x-n) \equiv 1.$$

We have

$$\begin{aligned}
(6.27)_{|\xi_1| \sim 2^{k_1},\dots} = &\sum_{|m| \lesssim T 2^{\alpha k_{\max}}} \int_{\mathbb{R}} dt \int_{\Gamma_4, |\xi_1| \sim 2^{k_1},\dots} b_4^{m_\varepsilon}(\xi_1,\xi_2,\xi_{31},\xi_{32}) \\
&1_{[0,T]}(t) \gamma(2^{-\alpha k_{\max}} t - m) \hat{u}(\xi_1) \dots \gamma(2^{-\alpha k_{\max}} t - m) \hat{u}(\xi_{32}),
\end{aligned}$$

where $\alpha = (2 - a + \delta)$ so that the products $\gamma(2^{-\alpha k_{\max}} t - m)\hat{u}(\xi_i)$ are estimated in F^{δ}_{a,k_i}-spaces.
Here and below, we confine ourselves to the majority of the cases, where the smooth cutoff does not interact with the sharp cutoff, i.e., only the $m \in \mathbb{Z}$ are considered, for which

$$1_{[0,T]}(\cdot)\gamma(2^{-\alpha k_{\max}} \cdot - m) = \gamma(2^{-\alpha k_{\max}} \cdot - m). \tag{6.28}$$

Recall that there are at most four exceptional cases for which the above display fails. Like in Subsection 5.2.4 these can be treated by interpolation with the estimate from Lemma 6.5.9.
Thus, adapting the reductions and notations from Subsection 5.2.4, one has to estimate

$$\begin{aligned} &T 2^{(2-a+\delta)k_1^*} |b_4(2^{k_1}, 2^{k_2}, 2^{k_{31}}, 2^{k_{32}})| \int d\Gamma_4(\tau) \int_{\Omega_2^a \neq 0} d\Gamma_4(\xi) f_{k_1,j_1}(\tau_1, \xi_1) \\ &\quad f_{k_2,j_2}(\tau_2, \xi_2) f_{k_{31},j_{31}}(\tau_{31}, \xi_{31}) f_{k_{32},j_{32}}(\tau_{32}, \xi_{32}), \end{aligned} \tag{6.29}$$

where $j_i \geq (2 - a + \delta)k_1^*$, $i = 1, 2, 31, 32$ taking into account the time localization. For the sake of brevity write in the following $f_{k_3,j_3} = f_{k_{31},j_{31}}$ and $f_{k_4,j_4} = f_{k_{32},j_{32}}$. For the estimate we use Lemma 6.5.8 and 6.5.11 in case of separated frequencies and Lemma 6.3.1 whenever the frequencies are not separated. We turn to the single cases.

Subcase AI. For b_4^m we have the size estimate $|b_4^m| \lesssim \frac{\max(2^{2sk_1}, 2^{2sk_3}) 2^{2\varepsilon k_1}}{2^{ak_1}} 2^{k_3}$. The time localization yields a factor of $T 2^{(2-a+\delta)k_1}$, and an application of Lemma 6.5.8 gives

$$(6.29) \lesssim \max(2^{2sk_1}, 2^{2sk_3}) 2^{k_3 - k_1} 2^{2(1-a)k_1} 2^{\delta k_1} 2^{2\varepsilon k_1} \prod_{i=1}^{4} 2^{j_i/2} \|f_{k_i,j_i}\|_2.$$

Subcase AII. In case the frequencies are not of comparable size, one can argue like in Case AI.
Otherwise, we apply Lemma 6.3.1 to find together with the size estimate of b_4^m and the time localization

$$(6.29) \lesssim T \frac{\max(2^{2sk_1}, 2^{2sk_3})}{2^{ak_1}} 2^{k_3} 2^{(2-a+\delta)k_1} 2^{2\varepsilon k_1} 2^{-(\delta + 3\varepsilon)k_1} \prod_{i=1}^{4} 2^{j_i/2} \|f_{k_i,j_i}\|_2$$

Subcase AIII. This case can be covered following along the above lines.

Subcase AIV. The size estimate for b_4^m is $|b_4^m| \lesssim \frac{\max(2^{2sk_1}, 2^{2sk_3})}{2^{ak_1}} 2^{2\varepsilon k_1} 2^{k_1}$. The time localization yields a factor of $T 2^{(2-a+\delta)k_{31}}$ and an application of Lemma 6.5.8 gives a smoothing factor of $2^{-k_{31}} 2^{\varepsilon k_1}$, which yields

$$(6.29) \lesssim T \max(2^{2sk_1}, 2^{2sk_3}) 2^{(1-a)k_1} 2^{(1-a)k_{31}} 2^{\delta k_{31}} 2^{2\varepsilon k_1} \prod_{i=1}^{4} 2^{j_i/2} \|f_{k_i,j_i}\|_2.$$

Subcase BI. The size estimate of b_4^m is $|b_4^m| \lesssim \frac{\max(2^{2sk_1}, 2^{2sk_2}) 2^{k_1}}{2^{ak_1}} 2^{2\varepsilon k_1}$, time localization amounts to a factor of $T 2^{(2-a+\delta)k_1}$ and using the resonance $|\Omega_2^a| \gtrsim 2^{(a+1)k_1}$, hence, $j_1^* \geq (a+1)k_1/2 - 10$ in conjunction with Lemma 6.5.11 we find

$$\begin{aligned}(6.29) &\lesssim T\frac{2^{2(s+\varepsilon)k_1}}{2^{ak_1}} 2^{k_1} 2^{-(a+1)k_1/2} 2^{(2-a+\delta)k_1} 2^{3\varepsilon k_1} \prod_{i=1}^{4} 2^{j_i/2} \|f_{k_i,j_i}\|_2 \\ &\lesssim T 2^{2sk_1} 2^{(5/2)(1-a)} 2^{3\varepsilon k_1} \prod_{i=1}^{4} 2^{j_i/2} \|f_{k_i,j_i}\|_2.\end{aligned}$$

Subcase BII. The size estimate is $|b_4^m| \lesssim \frac{\max(2^{2sk_1}, 2^{2sk_2}) 2^{2\varepsilon k_1}}{2^{ak_1}}$, time localization gives a factor of $T 2^{(2-a+\delta)k_{31}}$ and by Lemma 6.5.8, we find

$$(6.29) \lesssim T \frac{\max(2^{2sk_1}, 2^{2sk_2}) 2^{k_1}}{2^{ak_1}} 2^{(2-a+\delta)k_{31}} 2^{-k_{31}} 2^{3\varepsilon k_{31}} \prod_{i=1}^{4} 2^{j_i/2} \|f_{k_i,j_i}\|_{L^2}.$$

Subcase BIII. The size of b_4^m is given by $|b_4^m| \lesssim \frac{\max(2^{2sk_1}, 2^{2sk_2})}{2^{ak_1}} 2^{(1+2\varepsilon)k_1}$. Time localization gives a factor of $T 2^{(2-a+\delta)k_1}$, and an application of Lemma 6.5.8 gives

$$\begin{aligned}(6.29) &\lesssim T \frac{\max(2^{2sk_1}, 2^{2sk_2}) 2^{k_1}}{2^{ak_1}} 2^{(2-a+\delta)k_1} 2^{3\varepsilon k_1} 2^{-k_1} \prod_{i=1}^{4} 2^{j_i/2} \|f_{k_i,j_i}\|_2 \\ &\lesssim T \max(2^{2sk_1}, 2^{2sk_2}) 2^{2(1-a)k_1} 2^{(3\varepsilon+\delta)k_1} \prod_{i=1}^{4} 2^{j_i/2} \|f_{k_i,j_i}\|_2.\end{aligned}$$

In all cases we find extra smoothing. It is straight-forward to carry out the summations. □

We turn to the proof of energy estimates for differences of solutions.

Proof of Proposition 6.5.2. We start with the proof of (6.17).
An application of the fundamental theorem of calculus gives up to irrelevant factors

$$\begin{aligned}2^{-n} \|P_n v(t)\|_{L^2}^2 &= 2^{-n} \|P_n v(0)\|_{L^2}^2 \\ &\quad + 2^{-n} \int_0^T dt \sum_{\substack{\xi_1+\xi_2+\xi_3=0, \\ \xi_i \neq 0}} \chi_n^2(\xi_1) \xi_1 \hat{v}(\xi_1) (\hat{u}_1(\xi_2) + \hat{u}_2(\xi_2)) \hat{v}(\xi_3).\end{aligned}$$

In the following we pretend that v is governed by $\partial_t v + \partial_x D_x^a v = \partial_x(v u_1)$ to lighten the notation because we can prove the same estimates replacing u_1 with u_2. This is possible due to multilinearity of the argument.

The estimate is carried out by Case-by-Case analysis, which is more involved than in the energy estimates for solutions due to reduced symmetry. For the interaction between v, u_1, v in the above display, we have to take care of the following cases:

Case I : *High* × *Low* → *High*-interaction: (v, u_1, v)

Case II : *High* × *Low* → *High*-interaction: (v, v, u_1)

Case III : *High* × *High* → *High*-interaction

Case IV : *High* × *High* → *Low*-interaction: (v, u_1, v)

We start with an analysis of Case I.

After integration by parts and switching back to position space, we find

$$2^{-n}2^k \int_0^T dt \int dx P_n u P_k u_1 P_{n'} v \quad (k \leq n-10). \tag{6.30}$$

Strictly speaking, the estimates are carried out rather for the absolute values of the space-time Fourier transform which becomes only possible after integration by parts in time first. The above notation is used in order to make the argument more readable.

Further, we omit to indicate the summation over the frequencies. One checks that the expressions sum up to the desired regularities.

Integration by parts in time is only carried out for $n \geq \log_2(M)$: This gives

$$\begin{aligned}
(6.30) &= 2^{-n}2^k 2^{-(an+k)} \left[P_n v P_k u_1 P_{n'} v\right]_{t=0}^T \\
&\quad + 2^{-n}2^k 2^{-(an+k)} \Big(\int_0^T dt \int \partial_x P_n(vu_1) P_k u_1 P_{n'} v \\
&\quad + \int_0^T dt \int P_n v \partial_x P_k(u_1^2) P_{n'} v \Big) \\
&= B_I(0;T) + I_1 + I_2, \quad k \leq n-10.
\end{aligned}$$

Like in the proof of Proposition 5.2.15 we only integrate by parts the high frequencies. The boundary term can be estimated using Hölder's inequality and Bernstein's inequality like in the estimate of the boundary term for solutions:

$$\begin{aligned}
&\sum_{n \geq m} \sum_{k \leq n-10} \sum_{|n-n'| \leq 5} 2^{-(a+1)n} \int dx P_n v(t) P_k u_1(t) P_{n'} v(t) \\
&\lesssim \sum_{n \geq m} \sum_{k \leq n-6} \sum_{|n-n'| \leq 5} 2^{-(a+1)n} \|P_n v(t)\|_{L^2} \|P_k u_1(t)\|_{L^\infty} \|P_{n'} v(t)\|_{L^2} \\
&\lesssim M^{-d} \|v\|^2_{F_a^{-1/2,\delta}(T)} \|u_1\|_{F_a^{s,\delta}(T)},
\end{aligned}$$

where the ultimate estimate follows from Lemma 2.5.1. Moreover, for the low frequencies it is straight-forward to infer by the same means that

$$\sum_{1 \leq n \leq m} \sum_{k \leq n-6} 2^{-n}2^k \int_0^T dt \int dx P_n v P_k u_1 P_{n'} v \lesssim T M^c \|v\|^2_{F_a^{-1/2,\delta}(T)} \|u_1\|_{F_a^{s,\delta}(T)}.$$

We turn to the more involved estimate of I_1 and I_2. The frequency constraint is omitted in the following. Compared to the remainder estimate for solutions the multiplier is slightly worse because we do not integrate by parts another time. Moreover, the second resonance can vanish.

We split $I_1 = I_{11} + I_{12} + I_{13}$ according to Littlewood-Paley decomposition. This means that we consider $High \times Low \to High$-interaction for I_{11}, $High \times High \to High$-interaction for I_{12} and $High \times High \to Low$-interaction for I_{13}. If the second resonance does not vanish, then Lemma 6.5.8 applies and we find

$$
\begin{aligned}
I_{11} &= 2^{-an} \left| \int_0^T dt \int (P_n v P_{k'} u_1 + P_n u_1 P_{k'} v)\, P_k u_1 P_{n'} v \right| \\
&\lesssim T 2^{(2-a+\delta)n} 2^{-an} 2^{-n} 2^{\varepsilon n} \left(\|P_n v\|_{F^\delta_{a,n}} \|P_{k'} u_1\|_{F^\delta_{a,k'}} + \|P_n u_1\|_{F^\delta_{a,n}} \|P_{k'} v\|_{F^\delta_{a,k'}} \right) \\
&\quad \|P_k u_1\|_{F^\delta_{a,k}} \|P_{n'} v\|_{F^\delta_{a,n'}}.
\end{aligned}
$$

If the second resonance vanishes, then we apply Lemma 6.5.10. This ameliorates the factor $2^{(2-a+\delta)n}$ from the time localization and gives

$$
I_{11} \lesssim T 2^{-an} \left(\|P_n v\|_{F^\delta_{a,n}} \|P_k u_1\|_{F^\delta_{a,k}} + \|P_n u_1\|_{F^\delta_{a,n}} \|P_k v\|_{F^\delta_{a,k}} \right) \|P_k u_1\|_{F_k} \|P_n v\|_{F^\delta_{a,n}}.
$$

For I_{12} we have to estimate

$$
2^{-an} \int_0^T dt \int P_n v P_{n'} u_1 P_k u_1 P_{n''} v dx, \quad k \leq n-10,\ |n-n'| \leq 5,\ |n''-n| \leq 5.
$$

The second resonance satisfies $|\Omega_2^a| \gtrsim 2^{(a+1)n}$. By Lemma 6.5.11 we find

$$
\begin{aligned}
I_{12} &\lesssim T 2^{(2-a+\delta)n} 2^{-an} 2^{-(a+1)n/2} 2^{\varepsilon n} \|P_n v\|_{F^\delta_{a,n}} \|P_{n'} u_1\|_{F^\delta_{a,n'}} \|P_k u_1\|_{F^\delta_{a,k}} \|P_{n''} v\|_{F^\delta_{a,n''}} \\
&\lesssim T 2^{(3/2-5a/2)n} 2^{(\varepsilon+\delta)n} \|P_n v\|_{F^\delta_{a,n}} \|P_{n'} u_1\|_{F^\delta_{a,n'}} \|P_k u_1\|_{F^\delta_{a,k}} \|P_{n''} v\|_{F^\delta_{a,n''}}.
\end{aligned}
$$

We turn to $High \times High \to Low$-interaction: This amounts to estimate

$$
I_{13} = 2^{-an} \int_0^T dt \int P_{m_1} v P_{m_2} u_1 P_k u_1 P_{n'} v \quad (n \leq m_1 - 5).
$$

I_{13} is amenable to Lemma 6.5.8 after adding time localization $T 2^{(2-a+\delta)m_1}$. Taking all factors together, we find

$$
I_{13} \lesssim T 2^{(1-a)m_1} 2^{(\varepsilon+\delta)m_1} 2^{-an} \|P_{m_1} v\|_{F^\delta_{a,m_1}} \|P_{m_2} u_1\|_{F^\delta_{a,m_2}} \|P_k u_1\|_{F^\delta_{a,k}} \|P_{n'} v\|_{F^\delta_{a,n'}}.
$$

For I_2 we use again Littlewood-Paley decomposition to write $I_2 = I_{21} + I_{22} + I_{23}$ like above.
Since the deployed arguments are multilinear, the estimates for I_{21} and I_{22} are carried out like above. However, in case of I_{23} we encounter the additional case of comparable frequencies

$$
2^{-n} 2^{-an} 2^k \int_0^T dt \int P_n v P_{m_1} u_1 P_{m_2} u_1 P_{n'} v \quad |m_1 - m_2| \leq 10,\ |m_1 - n| \leq 10,
$$

which is not necessarily amenable to Lemma 6.5.8.
But, after adding localization in time $T 2^{(2-a+\delta)n}$ and using Lemma 6.5.12 in the non-resonant case and Lemma 6.5.10 in the resonant case, we find the estimate

$$
I_{23} \lesssim T 2^{2(1-a)n} 2^{k-n} \|P_n v\|_{F^\delta_{a,n}} \|P_{m_1} u_1\|_{F^\delta_{a,m_1}} \|P_{m_2} u_1\|_{F^\delta_{a,m_2}} \|P_{n'} v\|_{F^\delta_{a,n'}},
$$

which is again more than enough.

In Case II we can not integrate by parts in space to put the derivative on a more favourable factor. Thus, we have to estimate the expression

$$\int_0^T dt \int P_n v P_{n'} u_1 P_k v. \tag{6.31}$$

Integration by parts in time yields

$$\begin{aligned} II &= 2^{-(an+k)} \left[P_n v P_{n'} u_1 P_k v \right]_{t=0}^T + 2^{-(an+k)} \left(\int_0^T dt \int \partial_x P_n (v u_1) P_{n'} u_1 P_k v \right. \\ &\quad \left. + \int_0^T dt \int P_n v \partial_x P_{n'} (u_1^2) P_k v + \int_0^T dt \int P_n v P_{n'} u_1 \partial_x P_k (v u_1) \right) \\ &= B_{II}(0;T) + II_1 + II_2 + II_3. \end{aligned}$$

To derive suitable estimates, we do not integrate by parts all of (6.31) but only the part with high frequencies like above. We find for the boundary term with initial frequencies $n \geq \log_2(M)$ following along the above lines of the estimate for $B_I(0;T)$:

$$B_{II,M}(0;T) \lesssim M^{-c} \|v\|^2_{F_a^{-1/2,\delta}(T)} \|u_1\|_{F_a^{s,\delta}(T)}$$

and for the low frequencies like above

$$\sum_{1 \leq n \leq m} \sum_{|n-n'| \leq 5} \sum_{k \leq n-6} \int_0^T ds \int dx P_n v P_{n'} u_1 P_k v \lesssim T M^d \|v\|^2_{F_{a,\delta}^{-1/2}(T)} \|u_1\|_{F^s_{a,\delta}(T)}.$$

We turn to the estimate of II_1. For the evaluation we plug in Littlewood-Paley decomposition of $P_n(u_1 v)$ and split like above $II_1 = II_{11} + II_{12} + II_{13}$.

We have

$$\begin{aligned} II_{11} &= 2^{-(an+k)} 2^n \left(\int_0^T dt \int P_n v P_{k'} u_1 P_{n'} u_1 P_k v + \int_0^T dt \int P_{k'} v P_n u_1 P_{n'} u_1 P_k v \right) \\ &\quad (\quad k, k' \leq n - 10). \end{aligned}$$

Time localization amounts to a factor of $T 2^{(2-a+\delta)n}$. In the non-resonant case we use Lemma 6.5.8 and in the resonant case Lemma 6.5.10 to find gathering all factors

$$\begin{aligned} II_{11} &\lesssim T 2^{-k} 2^{(1-a)n} \left(\|P_n v\|_{F^\delta_{a,n}} \|P_{k'} u_1\|_{F^\delta_{a,k'}} + \|P_{k'} u_1\|_{F^\delta_{a,k'}} \|P_n v\|_{F^\delta_{a,n}} \right) \\ &\quad \|P_{n'} u_1\|_{F^\delta_{a,n'}} \|P_k v\|_{F^\delta_{a,k}}. \end{aligned}$$

For II_{12} we have to estimate

$$2^{(1-a)n-k} \int_0^T dt \int P_{n_1} v P_{n_2} u_1 P_{n'} u_1 P_k v, \quad |n_1 - n'| \leq 5, \ |n_2 - n'| \leq 5. \tag{6.32}$$

For this we use Lemma 6.5.11 because the second resonance $|\Omega_a^2| \gtrsim 2^{(a+1)n}$ is favourable:

$$\begin{aligned} (6.32) &\lesssim T 2^{(2-a+\delta)n} 2^{(1-a)n} 2^{-k} 2^{-(a+1)n/2} \\ &\quad \|P_{n_1} v\|_{F^\delta_{a,n_1}} \|P_{n_2} u_1\|_{F^\delta_{a,n_2}} \|P_{n'} u_1\|_{F^\delta_{a,n'}} \|P_k v\|_{F^\delta_{a,k}}. \end{aligned}$$

For II_{13} estimate by Lemma 6.5.8

$$2^{(1-a)n-k}\int_0^T dt\int P_{m_1}vP_{m_2}u_2P_{n'}u_2P_kv$$
$$\lesssim T2^{(1-a+\delta)m_1}2^{(1-a)n-k}\|P_{m_1}v\|_{F^\delta_{a,m_1}}\|P_{m_2}u_2\|_{F^\delta_{a,m_2}}\|P_{n'}u_2\|_{F^\delta_{a,n'}}\|P_kv\|_{F^\delta_{a,k}},$$

where $|m_1-m_2|\leq 5$, $n'\leq m_1-6$.

Like above split $II_2=II_{21}+II_{22}+II_{23}$ and for II_{21} we have to estimate

$$2^{(1-a)n-k}\int_0^T dt\int P_nvP_{n'}u_1P_{k'}u_1P_kv \quad (k,k'\leq n-5).$$

In the non-resonant case we find by applying Lemma 6.5.6

$$II_{21}\lesssim T2^{2(1-a)n}2^{(\delta+\varepsilon)n}2^{-k}\|P_nv\|_{F^\delta_{a,n}}\|P_{n'}u_1\|_{F^\delta_{a,n'}}\|P_{k'}u_1\|_{F^\delta_{a,k'}}\|P_kv\|_{F^\delta_{a,k}}.$$

In the resonant case it follows from Lemma 6.5.10

$$II_{21}\lesssim T2^{(1-a)n-k}\|P_nv\|_{F^\delta_{a,n}}\|P_{n'}u_1\|_{F^\delta_{a,n'}}\|P_{k'}u_1\|_{F^\delta_{a,k'}}\|P_kv\|_{F^\delta_{a,k}},$$

which is still sufficient.
For II_{22} use Lemma 6.5.11 to find

$$2^{(1-a)n-k}\int_0^T dt\int P_nvP_{n_2}u_1P_{n_3}u_1P_kv$$
$$\lesssim T2^{(2-a+\delta)n}2^{(1-a)n}2^{-k}2^{-(a+1)n/2}2^{\varepsilon n}\|P_nv\|_{F^\delta_{a,n}}$$
$$\times\|P_{n_2}u_1\|_{F^\delta_{a,n_2}}\|P_{n_3}u_1\|_{F^\delta_{a,n_3}}\|P_kv\|_{F^\delta_{a,k}}$$

and for II_{23} we have to estimate

$$2^{(1-a)n-k}\int_0^T dt\int P_nvP_{m_1}u_1P_{m_2}u_1P_kv,\quad n\leq m_1-5.$$

Here, we apply Lemma 6.5.8 to find

$$II_{23}\lesssim T2^{(1-a+\delta)m_1}2^{(1-a)n-k}\|P_nv\|_{F^\delta_{a,n}}\|P_{m_1}u_1\|_{F^\delta_{a,m_1}}\|P_{m_2}u_1\|_{F^\delta_{a,m_2}}\|P_kv\|_{F^\delta_{a,k}}.$$

The estimate of II_3 is easier because the derivative hits a smaller frequency, but all frequencies can be comparable. This leads to the expression

$$2^{-an}\int_0^T dt\int P_nvP_{n'}u_1P_{m_1}vP_{m_2}u_1,$$

which can also be treated like above with Lemma 6.5.8 in the non-resonant case and Lemma 6.5.10 in the resonant case.

In Case III we have to estimate

$$\int_0^T dt\int P_{n_1}u_1P_{n_2}vP_{n_3}v \tag{6.33}$$

with $|n_i - n| \leq 10$.
The resonance is very favourable, and we find after integration by parts in time

$$\begin{aligned}(6.33) &= 2^{-(a+1)n} \left[\int P_{n_1} u_1 P_{n_2} v P_{n_3} v \right]_{t=0}^{T} + 2^{-(a+1)n} \Big(\int_0^T dt \int \partial_x P_{n_1}(u_1^2) P_{n_2} v P_{n_3} v \\ &\quad + \int_0^T dt \int P_{n_1} u_1 \partial_x P_{n_2}(v u_1) P_{n_3} v + \int_0^T dt \int P_{n_1} u_1 P_{n_2} v \partial_x P_{n_3}(v u_1) \Big) \\ &= B_{III}(0;T) + III_1 + III_2 + III_3.\end{aligned}$$

Like above integration by parts in time is only carried out for high frequencies, which gives

$$\sum_{n \geq m} \sum_{|n_i - n| \leq 10} 2^{-(a+1)n} \left[\int P_{n_1} u_1 P_{n_2} v P_{n_3} v \right]_{t=0}^{T} \lesssim M^{-d} \|v\|^2_{F_a^{-1/2,\delta}(T)} \|u_1\|_{F_a^{s,\delta}(T)}$$

and

$$\sum_{1 \leq n \leq m} \sum_{|n_i - n| \leq 10} \int_0^T dt \int P_{n_1} u_1 P_{n_2} v P_{n_3} v \lesssim M^c T \|v\|^2_{F_a^{-1/2,\delta}(T)} \|u_1\|_{F_a^{s,\delta}(T)}.$$

Due to symmetry in the frequencies and multilinearity of the applied estimates, we only estimate III_1. We split $III_1 = III_{11} + III_{12} + III_{13}$ according to Littlewood-Paley decomposition. For III_{11} we have to consider

$$2^{-an} \int_0^T ds \int P_{n_1} u_1 P_k u_1 P_{n_2} v P_{n_3} v, \quad k \leq n - 15,$$

and an application of Lemma 6.5.11 gives

$$\begin{aligned} III_{11} &\lesssim T 2^{(2-a+\delta)n} 2^{\varepsilon n} 2^{-(a+1)n/2} 2^{-an} \|P_{n_1} u_1\|_{F^\delta_{a,n_1}} \\ &\quad \|P_k u_1\|_{F^\delta_{a,k}} \|P_{n_2} v\|_{F^\delta_{a,n_2}} \|P_{n_3} v\|_{F^\delta_{a,n_3}}. \end{aligned}$$

For III_{12} we have to estimate

$$2^{-an} \int_0^T dt \int P_{n_1} u_1 P_{n_2} u_1 P_{n_3} v P_{n_4} v$$

with all frequencies comparable, i.e., $|n_i - n| \leq 15$.
In the non-resonant case use Lemma 6.5.12 and in the resonant case use Lemma 6.5.10 to find

$$III_{12} \lesssim T 2^{2(1-a)n} \|P_{n_1} u_1\|_{F^\delta_{a,n_1}} \|P_{n_2} u_1\|_{F^\delta_{a,n_2}} \|P_{n_3} v\|_{F^\delta_{a,n_3}} \|P_{n_4} v\|_{F^\delta_{a,n_4}}, \ |n_i - n| \leq 15.$$

For III_{13} we have to estimate

$$2^{-an} \int_0^T dt \int P_{m_1} u_1 P_{m_2} u_1 P_{n_2} v P_{n_3} v, \quad |m_1 - m_2| \leq 5, \quad n_2, n_3 \leq m_1 - 10.$$

An application of Lemma 6.5.8 yields

$$III_{13} \lesssim 2^{-an} T 2^{(1-a+\delta)m_1} 2^{\varepsilon n} \|P_{m_1} u_1\|_{F^\delta_{a,m_1}} \|P_{m_2} u_1\|_{F^\delta_{a,m_2}} \|P_{n'} v\|_{F^\delta_{a,n'}} \|P_{n''} v\|_{F^\delta_{a,n''}}.$$

This finishes the analysis of Case III.

In Case IV we are considering

$$\int_0^T dt \int P_n v(P_{m_1} u_1 P_{m_2} v), \quad n \leq m_1 - 5. \tag{6.34}$$

An integration by parts in time yields

$$\begin{aligned}
(6.34) &= 2^{-(am_1+n)} \left[\int P_n v P_{m_1} u_1 P_{m_2} v \right]_{t=0}^T \\
&\quad + 2^{-(am_1+n)} (\int_0^T dt \int \partial_x P_n(vu_1) P_{m_1} u_1 P_{m_2} v \\
&\quad + \int_0^T dt \int P_n v \partial_x P_{m_1}(u_1^2) P_{m_2} v + \int_0^T dt \int P_n v P_{m_1} u_1 \partial_x P_{m_2}(vu_1)) \\
&= B_{IV}(0;T) + IV_1 + IV_2 + IV_3.
\end{aligned}$$

Like above only the high frequencies are integrated by parts.

For the corresponding boundary term, we find by Hölder's inequality, Bernstein's inequality and Lemma 2.5.1 like for the previous boundary term B_I

$$\begin{aligned}
B_{IV,M}(0;T) &= \sum_{m_1 \geq m} \sum_{n \leq m_1 - 5} \sum_{|m_1 - m_2| \leq 5} 2^{-(am_1+n)} \left[\int P_n v P_{m_1} u_1 P_{m_2} v \right]_{t=0}^T \\
&\lesssim M^{-d} \|v\|^2_{F_a^{-1/2,\delta}(T)} \|u_1\|_{F_a^{s,\delta}(T)}
\end{aligned}$$

and for the low frequencies

$$\sum_{m_1 \leq m} \sum_{n \leq m_1 - 6} \sum_{|m_1 - m_2| \leq 5} \int_0^T dt \int P_n v P_{m_1} u_1 P_{m_2} v \lesssim T M^c \|v\|^2_{F_a^{-1/2,\delta}(T)} \|u_1\|_{F_a^{s,\delta}(T)}.$$

Like above we split $IV_1 = IV_{11} + IV_{12} + IV_{13}$. To estimate IV_{11}, consider

$$2^{-am_1} \int_0^T dt \int (P_n v P_k u_1 + P_k v P_n u_1) P_{m_1} u_1 P_{m_2} v, \quad k \leq n - 5.$$

Since the second resonance does not vanish, IV_{11} is amenable to Lemma 6.5.8 and we find

$$\begin{aligned}
IV_{11} &\lesssim T 2^{(1-2a)m_1} 2^{\delta m_1} (\|P_n v\|_{F^\delta_{a,n}} \|P_k u_1\|_{F^\delta_{a,k}} + \|P_k v\|_{F^\delta_{a,k}} \|P_n u_1\|_{F^\delta_{a,n}}) \\
&\quad \|P_{m_1} u_1\|_{F^\delta_{a,m_1}} \|P_{m_2} v\|_{F^\delta_{a,m_2}}.
\end{aligned}$$

For IV_{12} we can apply Lemma 6.5.8 to find

$$IV_{12} \lesssim T 2^{(1-2a)m_1} 2^{\delta m_1} 2^{\varepsilon n} \|P_n v\|_{F^\delta_{a,n}} \|P_{n'} u_1\|_{F^\delta_{a,n'}} \|P_{m_1} u_1\|_{F^\delta_{a,m_1}} \|P_{m_2} v\|_{F^\delta_{a,m_2}}$$

and for IV_{13} the only additional case arises when all frequencies are comparable in

$$2^{-am_1} \int_0^T dt \int P_{m_3} v P_{m_4} u_1 P_{m_1} u_1 P_{m_2} v, \quad \exists l : |m_i - l| \leq 10.$$

In the non-resonant case use Lemma 6.5.12 and in the resonant case Lemma 6.5.10 to find

$$IV_{13} \lesssim T 2^{2(1-a)l} 2^{\delta l} \|P_{m_1} u_1\|_{F^\delta_{a,m_1}} \|P_{m_2} v\|_{F^\delta_{a,m_2}} \|P_{m_3} v\|_{F^\delta_{a,m_3}} \|P_{m_4} u_1\|_{F^\delta_{a,m_4}}.$$

We split $IV_2 = IV_{21} + IV_{22} + IV_{23}$. In case IV_{21} we have to estimate

$$2^{(1-a)m_1} 2^{-n} \int_0^T dt \int P_n v P_{m_1} u_1 P_k u_1 P_{m_2} v \quad (k, n \le m_1 - 5).$$

In the resonant case this expression is estimated by Lemma 6.5.10 and in the non-resonant case use Lemma 6.5.8 to find

$$IV_{21} \lesssim T 2^{(1-a)m_1} 2^{-n} \|P_{m_1} u_1\|_{F^\delta_{a,n}} \|P_{m_2} v\|_{F^\delta_{a,m_2}} \|P_n v\|_{F^\delta_{a,n}} \|P_k u_1\|_{F^\delta_{a,k}}.$$

For IV_{22} consider

$$2^{(1-a)m_1 - n} \int_0^T dt \int P_n v P_{m_1} u_1 P_{m_2} u_1 P_{m_3} v, \quad \exists m' : n \le m' - 10, |m_i - m'| \le 7.$$

This we estimate by Lemma 6.5.11 to find

$$\begin{aligned} IV_{22} \lesssim\ & T 2^{(2-a+\delta)m_1} 2^{-(a+1)m_1/2} 2^{(1-a)m_1} 2^{-n} \\ & \|P_n v\|_{F^\delta_{a,n}} \|P_{m_1} u_1\|_{F^\delta_{a,m_1}} \|P_{m_2} u_1\|_{F^\delta_{a,m_2}} \|P_{m_3} v\|_{F^\delta_{a,m_3}}. \end{aligned}$$

For IV_{23} we have to estimate

$$2^{(1-a)m_1} 2^{-n} \int_0^T dt \int P_n v P_{l_1} u_1 P_{l_2} u_1 P_{m_2} v, \quad n \le m_2 - 5 \le l_1 - 10.$$

An application of Lemma 6.5.8 gives

$$IV_{23} \lesssim T 2^{(1-a)m_1 - n} 2^{(1-a)l_1} 2^{\delta l_1} \|P_n v\|_{F^\delta_{a,n}} \|P_{l_1} u_1\|_{F^\delta_{a,l_1}} \|P_{l_2} u_1\|_{F^\delta_{a,l_2}} \|P_{m_2} v\|_{F^\delta_{a,m_2}}.$$

IV_3 is estimated like IV_2. This completes the proof of (6.17).

In order to prove (6.18), we write by the fundamental theorem of calculus up to irrelevant factors

$$\begin{aligned} 2^{2ns} \|P_n v(T)\|^2_{L^2} &= 2^{2ns} \|P_n v(0)\|^2_{L^2} + 2^{2ns} \int_0^T dt \int P_n v \partial_x P_n(v^2) \\ &\quad + 2^{2ns} \int_0^T dt \int P_n v \partial_x P_n(vu) \\ &= 2^{2ns} \|P_n v(0)\|^2_{L^2} + 2^{2ns}(A + B), \end{aligned}$$

where

$$\begin{aligned} A &= 2^{2ns} \int_0^T dt \int_{\Gamma_3} \chi_n^2(\xi_1) \hat{v}(\xi_1)(i\xi_1) \hat{v}(\xi_2) \hat{v}(\xi_3) d\Gamma_3, \\ &= C 2^{2ns} \int_0^T dt \int_{\Gamma_3} d\Gamma_3 (\chi_n^2(\xi_1)\xi_1 + \chi_n^2(\xi_2)\xi_2 + \chi_n^2(\xi_3)\xi_3) \hat{v}(\xi_1) \hat{v}(\xi_2) \hat{v}(\xi_3). \end{aligned}$$

After integration by parts in time we find

$$
\begin{aligned}
A = & \left[\sum_{\substack{\xi_1+\xi_2+\xi_3=0,\\ \xi_i\neq 0}} \frac{(\chi_n^2(\xi_1)\xi_1 + \chi_n^2(\xi_2)\xi_2 + \chi_n^2(\xi_3)\xi_3)}{\xi_1|\xi_1|^a + \xi_2|\xi_2|^a + \xi_3|\xi_3|^a} \hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_3) \right]_{t=0}^{T} \\
& + \int_0^T dt \sum_{\substack{\xi_1+\xi_2+\xi_3=0,\\ \xi_i\neq 0}} \frac{(\chi_n^2(\xi_1)\xi_1 + \chi_n^2(\xi_2)\xi_2 + \chi_n^2(\xi_3)\xi_3)}{\xi_1|\xi_1|^a + \xi_2|\xi_2|^a + \xi_3|\xi_3|^a} \hat{v}(\xi_1)\hat{v}(\xi_2) \\
& \xi_3 \sum_{\substack{\xi_3=\xi_{31}+\xi_{32},\\ \xi_{3i}\neq 0}} \hat{v}(\xi_{31})\hat{v}(\xi_{32}) \\
& + \int_0^T dt \sum_{\substack{\xi_1+\xi_2+\xi_3=0,\\ \xi_i\neq 0}} \frac{(\chi_n^2(\xi_1)\xi_1 + \chi_n^2(\xi_2)\xi_2 + \chi_n^2(\xi_3)\xi_3)}{\xi_1|\xi_1|^a + \xi_2|\xi_2|^a + \xi_3|\xi_3|^a} \hat{v}(\xi_1)\hat{v}(\xi_2) \\
& \xi_3 \sum_{\substack{\xi_3=\xi_{31}+\xi_{32},\\ \xi_{3i}\neq 0}} \hat{v}(\xi_{31})\hat{u}_2(\xi_{32}) \\
= & B_A(0;T) + A_1 + A_2.
\end{aligned}
$$

Set

$$
b_3(\xi_1,\xi_2,\xi_3) = \frac{\chi_n^2(\xi_1)\xi_1 + \chi_n^2(\xi_2)\xi_2 + \chi_n^2(\xi_3)\xi_3}{\xi_1|\xi_1|^a + \xi_2|\xi_2|^a + \xi_3|\xi_3|^a}.
$$

A second symmetrization like in the proof of the energy estimates for solutions gives

$$
\begin{aligned}
A_1 &= C\int_0^T dt \int_{\Gamma_4} d\Gamma_4 b_3(\xi_1,\xi_2,\xi_{31}+\xi_{32})\xi_3\hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_{31})\hat{v}(\xi_{32}) \\
&= C\int_0^T dt \int_{\Gamma_4} d\Gamma_4 [b_3(\xi_1,\xi_2,\xi_{31}+\xi_{32}) - b_3(-\xi_{31},-\xi_{32},\xi_{31}+\xi_{32})] \\
&\quad \xi_3\hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_{31})\hat{v}(\xi_{32}) \\
&= C\int_0^T dt \int_{\Gamma_4} d\Gamma_4 b_4(\xi_1,\xi_2,\xi_{31},\xi_{32})\hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_{31})\hat{v}(\xi_{32}),
\end{aligned}
$$

and the expression is estimated like in Lemma 6.5.13.

To estimate

$$
A_2 = \int_0^T dt \int_{\Gamma_4} d\Gamma_4 b_3(\xi_1,\xi_2,\xi_3)\xi_3\hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_{31})\hat{u}_2(\xi_{32}),
$$

we conduct a Case-by-Case analysis plugging in Littlewood-Paley decomposition. For the interaction of (v,v,v) before integration by parts in time, we have to take into account the following cases:

Case I: $High \times Low \to High$ $(|\xi_1| \sim |\xi_3| \gg |\xi_2|)$,

Case II: $High \times High \to High$ $(|\xi_1| \sim |\xi_2| \sim |\xi_3|)$,

Case III: $High \times High \to Low$ $(|\xi_3| \ll |\xi_1| \sim |\xi_2|)$.

Here, we additionally plug in the possible frequency interactions for $(\xi_3, \xi_{31}, \xi_{32})$ like $I = I_1 + I_2 + I_3$. For I_1 we have to estimate

$$I_1 = 2^{2ns} 2^{(1-a)n} \left(\int_0^T dt \int P_n v P_k v \left(P_{n'} v P_{k'} u_2 + P_{k'} v P_{n'} u_2 \right) \right),$$
$$(k, k' \leq n - 5).$$

In the non-resonant case both expressions can be handled with Lemma 6.5.8 and in the resonant case Lemma 6.5.10 yields

$$I_1 \lesssim 2^{2ns} T 2^{(1-a)n} \|P_n v\|_{F^\delta_{a,n}} \|P_k v\|_{F^\delta_{a,k}}$$
$$(\|P_{n'} v\|_{F^\delta_{a,n'}} \|P_{k'} u_2\|_{F^\delta_{a,k'}} + \|P_{k'} v\|_{F^\delta_{a,k'}} \|P_{n'} u_2\|_{F^\delta_{a,n'}}).$$

I_2 is amenable to Lemma 6.5.11 which gives

$$I_2 \lesssim 2^{2ns} T 2^{(2-a+\delta)n} 2^{(1-a)n} 2^{\varepsilon n} 2^{-(a+1)n/2}$$
$$\|P_n v\|_{F^\delta_{a,n}} \|P_k v\|_{F^\delta_{a,k}} \|P_{n_2} v\|_{F^\delta_{a,n_2}} \|P_{n_3} u_2\|_{F^\delta_{a,n_3}},$$

where $|n - n_i| \leq 5$, $k \leq n - 10$.
For I_3 consider

$$2^{2ns} 2^{(1-a)n} \int_0^T dt \int P_n v P_k v P_{l_1} v P_{l_2} u_2, \quad n \leq l_1 - 5, \ k \leq n - 5.$$

Lemma 6.5.8 gives

$$I_3 \lesssim 2^{2ns} T 2^{(1-a+\delta)l_1} 2^{(1-a)n} \|P_n v\|_{F^\delta_{a,n}} \|P_k v\|_{F^\delta_{a,k}} \|P_{l_1} v\|_{F^\delta_{a,l_1}} \|P_{l_2} u_2\|_{F^\delta_{a,l_2}}.$$

Consider Case II next. Split $II = II_1 + II_2 + II_3$. For II_1 we have to consider

$$2^{2ns} 2^{(1-a)n} \left(\int_0^T dt \int P_{n_1} v P_{n_2} v P_{n_3} v P_k u_2 + \int_0^T dt \int P_{n_1} v P_{n_2} v P_k v P_{n_3} u_2 \right),$$
$$|n_1 - n_2| \leq 3, |n_1 - n_3| \leq 3, k \leq n_1 - 6.$$

This we estimate by Lemma 6.5.11 to find

$$II_1 \lesssim 2^{2ns} T 2^{(2-a+\delta)n} 2^{(1-a)n} 2^{-(a+1)n/2} \|P_{n_1} v\|_{F^\delta_{a,n_1}} \|P_{n_2} v\|_{F^\delta_{a,n_2}}$$
$$(\|P_{n_3} v\|_{F^\delta_{a,n_3}} \|P_k u_2\|_{F^\delta_{a,k}} + \|P_k v\|_{F^\delta_{a,k}} \|P_{n_3} u_2\|_{F^\delta_{a,n_3}}).$$

For II_2 consider

$$2^{2ns} 2^{(1-a)n} \int_0^T dt \int P_{n_1} v P_{n_2} v P_{n_3} v P_{n_4} u_2, \quad |n_1 - n_i| \leq 10, \ i = 2, 3, 4.$$

This we estimate by Lemma 6.5.8 in the non-resonant case and by Lemma 6.5.10 in the resonant case to find

$$II_2 \lesssim 2^{2ns} T 2^{(3-2a)n} \|P_{n_1} v\|_{F^\delta_{a,n_1}} \|P_{n_2} v\|_{F^\delta_{a,n_2}} \|P_{n_3} v\|_{F^\delta_{a,n_3}} \|P_{n_4} u_2\|_{F^\delta_{a,n_4}}.$$

For II_3 we have to consider

$$2^{2ns} 2^{(1-a)n_1} \int_0^T dt \int P_{n_1} v P_{n_2} v P_{l_1} v P_{l_2} u_2, \quad n_1 \leq l_1 - 10, \ |n_1 - n_2| \leq 5.$$

This is amenable to Lemma 6.5.8 which yields the estimate

$$II_3 \lesssim 2^{2ns} T 2^{(1-a+\delta)l_1} 2^{(1-a)n_1} \|P_{n_1} v\|_{F^\delta_{a,n_1}} \|P_{n_2} v\|_{F^\delta_{a,n_2}} \|P_{l_1} v\|_{F^\delta_{a,l_1}} \|P_{l_2} u_2\|_{F^\delta_{a,l_2}}.$$

We estimate $III = III_1 + III_2 + III_3$. For III_1 consider

$$2^{2ns} 2^{(1-a)n} \int_0^T dt \int P_{n_1} v P_{n_2} v (P_k v P_{k'} u_2 + P_{k'} v P_k u_2),$$

where $k \leq n_1 - 5$, $k' \leq k - 5$.
The expressions are amenable to Lemma 6.5.8 and we find

$$\begin{aligned} III_1 \lesssim & 2^{2ns} T 2^{2(1-a)n} 2^{\delta n} \|P_{n_1} v\|_{F^\delta_{a,n_1}} \|P_{n_2} v\|_{F^\delta_{a,n_2}} \\ & (\|P_k v\|_{F^\delta_{a,k}} \|P_{k'} u_2\|_{F^\delta_{a,k'}} + \|P_{k'} v\|_{F^\delta_{a,k'}} \|P_k u_2\|_{F^\delta_{a,k}}). \end{aligned}$$

The same argument applies to III_2 because there can not be a resonant case, which gives

$$\begin{aligned} III_2 \lesssim & 2^{2ns} T 2^{(1-a)n_1} 2^{\delta n} \|P_{n_1} v\|_{F^\delta_{a,n_1}} \|P_{n_2} v\|_{F^\delta_{a,n_2}} \|P_{l_1} v\|_{F^\delta_{a,l_1}} \|P_{l_2} u_2\|_{F^\delta_{a,l_2}} \\ & |l_1 - l_2| \leq 5, \ l_1 \leq n_1 - 10. \end{aligned}$$

For III_3 we have to consider

$$2^{2ns} \int_0^T dt \int P_{n_1} v P_{n_2} v P_k (P_{l_1} v P_{l_2} u_2), \quad k \leq l_1 - 10, k \leq n_1 - 10.$$

If $|n_1 - l_1| \geq 15$, we can argue like above. Otherwise, all frequencies are comparable and applying Lemma 6.5.12 in the non-resonant case and Lemma 6.5.10 in the resonant case to find

$$III_3 \lesssim 2^{2ns} T 2^{(3-2a)n} \|P_{n_1} v\|_{F^\delta_{a,n_1}} \|P_{n_2} v\|_{F^\delta_{a,n_2}} \|P_{l_1} v\|_{F^\delta_{a,l_1}} \|P_{l_2} u_2\|_{F^\delta_{a,l_2}}, |n_1 - l_1| \leq 5.$$

For the estimate of B, we are again in the situation from the proof of (6.17). The only difference is that we do not have the extra smoothing from the $H^{-1/2}$-input regularity, which leads to the shift in regularity.
We have the following cases:

Case I: $High \times Low \to High(v, u_2, v)$,

Case II: $High \times Low \to High(v, v, u_2)$,

Case III: $High \times High \to High$,

Case IV: $High \times High \to Low(v, u_2, v)$.

To estimate the individual contributions, we use exactly the same arguments from above. Hence, we shall be brief.

In Case I we integrate by parts to put the derivative on the lowest frequency from above to arrive at the expression

$$2^{2ns} 2^k \int_0^T dt \int dx P_n v P_k u_2 P_{n'} v \quad (k \leq n - 5). \tag{6.35}$$

Integration by parts in time gives modulo boundary terms and irrelevant factors

$$\begin{aligned}(6.35) - B_I(0;T) &= 2^{2ns}2^{-an}\Big(\int_0^T dt \int \partial_x P_n(v(v+u_2))P_k u_1 P_{n'} v \\ &\quad + \int_0^T dt \int P_n v P_k \partial_x (u_2^2) P_{n'} v\Big) \\ &= I_1 + I_2, \quad (k, k' \leq n-5).\end{aligned}$$

The boundary terms are handled like in the proof of (6.17). We omit the estimates of the boundary terms in the following. Split $I_1 = I_{11} + I_{12} + I_{13}$. Using Lemma 6.5.8 in case of non-vanishing resonance and Lemma 6.5.10 in case of vanishing second resonance, we find

$$\begin{aligned}I_{11} &\lesssim 2^{2ns}T2^{(1-a)n}\|P_k u_1\|_{F^\delta_{a,k}}\|P_n v\|_{F^\delta_{a,n}}(\|P_{n_1}v\|_{F^\delta_{a,n_1}}\|P_{k_1}u_2\|_{F^\delta_{a,k}} \\ &\quad + \|P_{n_1}v\|_{F^\delta_{a,k_1}}\|P_{n_1}u_2\|_{F^\delta_{a,n_1}} + \|P_{n_1}v\|_{F^\delta_{a,n_1}}\|P_{k_1}v\|_{F^\delta_{a,k_1}}).\end{aligned}$$

For I_{12} we find by the above argument

$$I_{12} \lesssim 2^{2ns}T2^{(1-a)n}\|P_{n_1}v\|_{F^\delta_{a,n_1}}(\|P_{n_2}u_2\|_{F^\delta_{a,n_2}} + \|P_{n_2}v\|_{F^\delta_{a,n_2}})\|P_k u_2\|_{F^\delta_{a,k}}\|P_{n_3}v\|_{F^\delta_{a,n_3}}$$

with $|n_i - n| \leq 5$, $k \leq n-10$.
Further,

$$\begin{aligned}I_{13} &\lesssim 2^{2ns}T2^{(1-a)n}2^{(1-a+\delta)m_1}\|P_{m_1}v\|_{F^\delta_{a,m_1}}\|P_k u_2\|_{F^\delta_{a,k}}\|P_{k'}v\|_{F^\delta_{a,k'}} \\ &\quad (\|P_{m_2}u_2\|_{F^\delta_{a,m_2}} + \|P_{m_2}v\|_{F^\delta_{a,m_2}}).\end{aligned}$$

In case of I_{23} the additional case of comparable frequency occurs

$$2^{2ns}2^{-an}2^k \int_0^T dt \int P_n v P_{m_1}u_2 P_{m_2}u_2 P_{n'}v, \quad |m_1 - m_2| \leq 10, \ |m_1 - n| \leq 10,$$

and we find by Lemma 6.5.12 or Lemma 6.5.10, respectively,

$$I_{23} \lesssim 2^{2ns}T2^{2(1-a)n}\|P_n v\|_{F^\delta_{a,n}}\|P_{m_1}u_2\|_{F^\delta_{a,m_1}}\|P_{m_2}u_2\|_{F^\delta_{a,m_2}}\|P_{n'}v\|_{F^\delta_{a,n'}}.$$

In Case II we have to estimate the expression

$$2^{2ns}2^n \int_0^T dt \int P_n v P_{n'} u_2 P_k v \quad k \leq n-5.$$

This we integrate by parts in time to find

$$\begin{aligned}II - B_{II}(0;T) &= 2^{2ns}2^{(1-a)n-k}\Big(\int_0^T dt \int \partial_x P_n(v(v+u_2))P_{n'}u_2 P_k v \\ &\quad + \int_0^T dt \int P_n v \partial_x P_{n'}(u_2^2) P_k v \\ &\quad + \int_0^T dt \int P_n v P_{n'} u_2 \partial_x P_k(v(v+u_2))\Big) \\ &= II_1 + II_2 + II_3.\end{aligned}$$

By the above notation and arguments we find

$$II_{11} \lesssim 2^{2ns} T 2^{(2-a)n-k} \|P_{n'} u_2\|_{F^\delta_{a,n'}} \|P_k v\|_{F^\delta_{a,k}} (\|P_n v\|_{F^\delta_{a,n}} \|P_{k'} u_2\|_{F^\delta_{a,k'}} + \|P_{k'} v\|_{F^\delta_{a,k'}} \|P_n u_2\|_{F^\delta_{a,n}} + \|P_n v\|_{F^\delta_{a,n}} \|P_{k'} v\|_{F^\delta_{a,k'}}) \quad k, k' \leq n - 10$$

with an improved estimate for $k \neq k'$.
For II_{12} estimate by Lemma 6.5.11

$$\begin{aligned} & 2^{2ns} 2^{(2-a)n-k} \int_0^T dt \int P_{n_1} v P_{n_2} u_2 P_{n'} u_2 P_k v \\ & \lesssim 2^{2ns} T 2^{(2-a)n-k} 2^{-(a+1)/2} 2^{(\varepsilon+\delta)n} \|P_{n_1} v\|_{F^\delta_{a,n_1}} (\|P_{n_2} u_2\|_{F^\delta_{a,n_2}} + \|P_{n_2} v\|_{F^\delta_{a,n_2}}) \\ & \|P_{n_3} u_2\|_{F^\delta_{a,n'}} \|P_k v\|_{F^\delta_{a,k}} \quad |n_i - n'| \leq 5, \ k \leq n - 10. \end{aligned}$$

For II_{13} estimate by Lemma 6.5.8

$$\begin{aligned} & 2^{2ns} 2^{(2-a)n-k} \int_0^T dt \int P_{m_1} v (P_{m_2} u_2 + P_{m_2} v) P_{n'} u_2 P_k v \\ & \lesssim 2^{2ns} T 2^{(1-a+\delta)m_1} 2^{(2-a)n-k} \|P_{m_1} v\|_{F^\delta_{a,m_1}} \|P_{n'} u_2\|_{F^\delta_{a,n'}} \|P_k v\|_{F^\delta_{a,k}} \\ & (\|P_{m_2} u_2\|_{F^\delta_{a,m_2}} + \|P_{m_2} v\|_{F^\delta_{a,m_2}}), \end{aligned}$$

where $n \leq m_1 - 5$.
For II_{21} estimate

$$2^{2ns} 2^{(2-a)n-k} \int_0^T dt \int P_n v P_{n'} u_2 P_{k'} u_2 P_k v, \quad k, k' \leq n - 10,$$

and it follows like above

$$II_{21} \lesssim 2^{2ns} T 2^{(2-a)n-k} \|P_n v\|_{F^\delta_{a,n}} \|P_{n'} u_2\|_{F^\delta_{a,n'}} \|P_{k'} u_2\|_{F^\delta_{a,k'}} \|P_k v\|_{F^\delta_{a,k}}$$

with an improved estimate for $k \neq k'$.
For II_{22} we find by Lemma 6.5.11

$$\begin{aligned} II_{22} \lesssim\ & 2^{2ns} T 2^{(2-a+\delta)n} 2^{\varepsilon n} 2^{(2-a)n-k} 2^{-(a+1)n/2} \\ & \|P_n v\|_{F^\delta_{a,n}} \|P_{n_2} u_2\|_{F^\delta_{a,n_2}} \|P_{n_3} u_2\|_{F^\delta_{a,n_3}} \|P_k v\|_{F^\delta_{a,k}}, \end{aligned}$$

where $|n - n_i| \leq 5$, $k \leq n - 10$.
For II_{23} we find by Lemma 6.5.8

$$II_{23} \lesssim 2^{2ns} T 2^{(2-a)n-k} 2^{(1-a+\delta)m_1} \|P_n v\|_{F^\delta_{a,n}} \|P_{m_1} u_2\|_{F^\delta_{a,m_1}} \|P_{m_2} u_2\|_{F^\delta_{a,m_2}} \|P_k v\|_{F^\delta_{a,k}}$$

For II_3 we can argue in case of separated frequencies like in II_1 or II_2 and the conclusion is easier because the derivative hits a low frequency.
However, in case of comparable frequencies there is the additional case

$$2^{2ns} 2^{(1-a)n} \int_0^T dt \int P_{n_1} v P_{n_2} u_2 P_{n_3} v (P_{n_4} u_2 + P_{n_4} v) \quad \exists n : |n_i - n| \leq 10. \quad (6.36)$$

This is estimated by Lemma 6.5.12 in case of non-vanishing resonance and 6.5.10 otherwise to find

$$\begin{aligned}(6.36) \lesssim\ & T 2^{2ns} 2^{(1-a)n} 2^{(2-a)n} \|P_{n_1} v\|_{F^{\delta}_{a,n_1}} \|P_{n_2} u_2\|_{F^{\delta}_{a,n_2}} \|P_{n_3} v\|_{F^{\delta}_{a,n_3}} \\ & (\|P_{n_4} u_2\|_{F^{\delta}_{a,n_4}} + \|P_{n_4} v\|_{F^{\delta}_{a,n_4}}).\end{aligned}$$

In Case III we find via the above arguments

$$\begin{aligned}III_{11} \lesssim\ & 2^{2ns} T 2^{(2-a+\delta)n} 2^{\varepsilon n} 2^{-(a+1)n/2} 2^{(1-a)n} \\ & \|P_{n_1} u_2\|_{F^{\delta}_{a,n_1}} \|P_k u_2\|_{F^{\delta}_{a,k}} \|P_{n_2} v\|_{F^{\delta}_{a,n_2}} \|P_{n_3} v\|_{F^{\delta}_{a,n_3}} \\ & (|n - n_i| \leq 10,\ k \leq n - 15) \\ III_{12} \lesssim\ & 2^{2ns} T 2^{(3-2a)n} \|P_{n_1} u_2\|_{F^{\delta}_{a,n_1}} \|P_{n_2} u_2\|_{F^{\delta}_{a,n_2}} \|P_{n_3} v\|_{F^{\delta}_{a,n_3}} \|P_{n_4} v\|_{F^{\delta}_{a,n_4}} \\ & (|n_i - n| \leq 15) \\ III_{13} \lesssim\ & T 2^{(1-a+\delta)m_1} 2^{(1-a)n} 2^{\varepsilon n} \\ & \|P_{m_1} u_2\|_{F^{\delta}_{a,m_1}} \|P_{m_2} u_2\|_{F^{\delta}_{a,m_2}} \|P_{n_1} v\|_{F^{\delta}_{a,n_1}} \|P_{n_2} v\|_{F^{\delta}_{a,n_2}} \quad (n \leq m_1 - 10)\end{aligned}$$

and due to symmetry and multilinearity the remaining cases are omitted.

In Case IV consider

$$2^{2ns} 2^n \int_0^T dt \int P_n v (P_{m_1} u_2 P_{m_2} v), \quad n \leq m_1 - 5.$$

With the notation from above, we find

$$\begin{aligned}IV_{11} \lesssim\ & 2^{2ns} T 2^n 2^{(1-2a)m_1} 2^{\delta m_1} \|P_{m_1} u_2\|_{F^{\delta}_{a,m_1}} \|P_{m_2} v\|_{F^{\delta}_{a,m_2}} \\ & (\|P_n v\|_{F^{\delta}_{a,n}} (\|P_k u_2\|_{F^{\delta}_{a,k}} + \|P_k v\|_{F^{\delta}_{a,k}}) + \|P_k v\|_{F^{\delta}_{a,k}} \|P_n u_2\|_{F^{\delta}_{a,n}}), \\ IV_{12} \lesssim\ & 2^{2ns} T 2^n 2^{(1-2a)m_1} 2^{\delta m_1} \|P_n v\|_{F^{\delta}_{a,n}} (\|P_{n'} u_2\|_{F^{\delta}_{a,n'}} + \|P_{n'} v\|_{F^{\delta}_{a,n'}}) \\ & \|P_{m_1} u_2\|_{F^{\delta}_{a,m_1}} \|P_{m_2} v\|_{F^{\delta}_{a,m_2}}, \\ IV_{13} \lesssim\ & 2^{2ns} T 2^{(2-3a)m_1} (\|P_{m_1} u_2\|_{F^{\delta}_{a,m_1}} + \|P_{m_1} v\|_{F^{\delta}_{a,m_1}}) \\ & \|P_{m_2} v\|_{F^{\delta}_{a,m_2}} \|P_{m_3} v\|_{F^{\delta}_{a,m_3}} \|P_{m_4} u_2\|_{F^{\delta}_{a,m_4}}.\end{aligned}$$

For the other cases record

$$\begin{aligned}IV_{21} &\lesssim 2^{2ns} T 2^{(1-a+\delta)m_1} \|P_n v\|_{F^{\delta}_{a,n}} \|P_{m_1} u_2\|_{F^{\delta}_{a,m_1}} \|P_k u_2\|_{F^{\delta}_{a,k}} \|P_{m_2} v\|_{F^{\delta}_{a,m_2}}, \\ IV_{22} &\lesssim 2^{2ns} T 2^{(2-a+\delta)m_1} \|P_n v\|_{F^{\delta}_{a,n}} \|P_{m_1} u_2\|_{F^{\delta}_{a,m_1}} \|P_{m_2} u_2\|_{F^{\delta}_{a,m_2}} \|P_{m_3} v\|_{F^{\delta}_{a,m_3}}, \\ IV_{23} &\lesssim 2^{2ns} T 2^{(1-a)m_1} 2^{(1-a)l_1/2} 2^{\delta l_1} \|P_n v\|_{F^{\delta}_{a,n}} \|P_{l_1} u_2\|_{F^{\delta}_{a,l_1}} \|P_{l_2} u_2\|_{F^{\delta}_{a,l_2}} \|P_{m_2} v\|_{F^{\delta}_{a,m_2}}.\end{aligned}$$

Case IV_3 is omitted due to multilinearity and symmetry.
All frequency localized estimates sum up to one of the below expressions choosing δ sufficiently small

$$\begin{aligned}& T \|v\|^3_{F^{s,\delta}_a(T)} \|u_2\|_{F^{s,\delta}_a(T)}, \\ & T \|v\|^2_{F^{s,\delta}_a(T)} \|u_2\|^2_{F^{s,\delta}_a(T)}, \\ & T \|v\|_{F^{s,\delta}_a(T)} \|v\|_{F^{-1/2,\delta}_a(T)} \|u_2\|_{F^{s+(2-a),\delta}_a(T)} \|u_2\|_{F^{s,\delta}_a(T)}.\end{aligned}$$

This finishes the proof of (6.18). □

Chapter 7

Variable-coefficient decoupling and smoothing estimates for elliptic and hyperbolic phase functions

In this chapter new regularity results for variable-coefficient elliptic and hyperbolic phase functions are discussed. This relates to short-time estimates for the oscillatory integral operators which arise when analyzing dispersive equations on compact manifolds.

The linear and bilinear short-time Strichartz estimates on arbitrary compact manifolds from Chapter 3 are inferred from results for the respective oscillatory integral operators. Unfortunately, the established results in this chapter do not directly yield a regularity result for dispersive PDE on compact manifolds on times, which do not depend on the frequency. We hope that the connection can be established in the future.

Below, variable-coefficient versions of ℓ^2-decoupling inequalities for elliptic and hyperbolic phase functions are proved and applications are given. Further, we consider frequency localized L^p-smoothing estimates for variable coefficients. Here, the bilinear approach yields the same estimates like in the constant-coefficient case.

7.1 Introduction to variable-coefficient oscillatory integral operators

We consider smooth functions $a \in C_c^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^n)$, $a = a_1 \otimes a_2$, $0 \leq a_1, a_2 \leq 1$ and $\phi : B^{n+1}(0,1) \times B^n(0,1) \to \mathbb{R}$, which we shall refer to as amplitude and phase function, respectively.

We associate the oscillatory integral operator

$$Tf(t,x) = \int_{\mathbb{R}^n} e^{i\phi(t,x,\xi)} a(t,x,\xi) f(\xi) d\xi \tag{7.1}$$

and the rescaled versions

$$T^\lambda f(t,x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(t/\lambda,x/\lambda,\xi)} a(t/\lambda,x/\lambda,\xi) f(\xi) d\xi \tag{7.2}$$

for different classes of phase functions.

Subject of discussion are variable-coefficient generalizations of the phase function

$$\phi_{hyp}(t,x;\xi) = \langle x,\xi\rangle + \frac{t\langle \xi, I_n^k \xi\rangle}{2}, \quad I_n^k = diag(1,\dots,1,\underbrace{-1,\dots,-1}_{k}), \quad 0 \leq k \leq n/2.$$

Set also $I_n = diag(1,\dots,1) \in \mathbb{R}^{n\times n}$.
In the following we shall always assume that there are at most as many negative eigenvalues as positive eigenvalues, which is no loss of generality since time reversal $t \to -t$ flips signs.

We define the Gauss map by

$$G : B^{n+1} \times B^n \to \mathbb{S}^n, \quad G(z;\xi) = \frac{G_0(z;\xi)}{|G_0(z;\xi)|}; \quad z = (t,x), \tag{7.3}$$

where $m \in \mathbb{N}$ and B_m denotes the unit ball in $\mathbb{R}^m$ and

$$G_0(z;\omega) = \bigwedge_{j=1}^{n} \partial_{\xi_j} \partial_z \phi(z;\xi) \tag{7.4}$$

with the standard identification $\bigwedge^n \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.

We impose the following conditions on the phase function:

$$\begin{aligned} &H1) \quad \text{rank}\, \partial^2_{\xi x}\phi(z;\xi) = n \quad \forall\, (z,\xi) \in B^{n+1} \times B^n, \\ &H2) \quad \partial^2_{\xi\xi} \left.\langle \partial_z \phi(z;\xi), G(z;\xi_0)\rangle\right|_{\xi=\xi_0} \text{ is non-degenerate.} \end{aligned}$$

$H1$) is a non-degeneracy condition, and $H2$) implies that the constant coefficient approximation of ϕ is the adjoint Fourier restriction operator (i.e. extension operator) associated to a non-degenerate surface. These assumptions are known as Carleson-Sjölin conditions (cf. [CS72]). For an exposition, see e.g. [Sog17, Section 2.2].

Contrary to the constant-coefficient case ϕ_{hyp}, rescaling $(t,x) \to (\lambda^2 t, \lambda x)$, $\xi \to \xi/\lambda$ yields no exact symmetry. Therefore, it is useful to quantify the conditions $H1$) and $H2$). Before doing so, we point out the following more precise versions of $H1$) and $H2$), which one may assume without loss of generality:

$$\begin{aligned} &H1') \quad \det \partial^2_{\xi x}\phi(z;\xi) \neq 0 \text{ for all } (z;\xi) \in T \times X \times \Xi = Z \times \Xi; \\ &H2'_{[k]}) \quad \partial_t \partial^2_{\xi\xi}\phi(z;\xi) \text{ is non-degenerate for all } (z;\xi) \in Z \times \Xi \\ &\qquad \text{and has exactly } k \text{ negative eigenvalues.} \end{aligned}$$

Here, T, X, Ξ denote balls of radius less or equal to one around the origin. To reduce from $H1$) and $H2$) to the conditions in the above display, one applies a rotation in space-time. This gives $G(0;0) = e_{n+1}$, and then one uses a partition of unity to suitably localize the support. Moreover, the implicit function theorem implies the existence of smooth functions Φ and Ψ taking values in X and Ω, respectively, such that

$$\partial_x \phi(z; \Psi(z;\xi)) = \xi \tag{7.5}$$

and

$$\partial_\xi \phi(t, \Phi(t,x;\xi);\xi) = x. \tag{7.6}$$

The first identity allows us to find a graph parametrization

$$\xi \mapsto (\partial_z \phi(z; \Psi(z;\xi))) = (\xi, (\partial_t \phi)(z; \Psi(z;\xi)))$$

for a hypersurface Σ_z with non-vanishing curvature. From differentiating the second identity we find $\partial_x \Phi(0;0) = \partial^2_{x\xi}\phi(0;0)^{-1}$.

Later on, $H1'$) and $H2'$) are quantified. It turns out that one can perceive any phase function satisfying $H1'$) and $H2'$) after introducing a partition of unity and rescaling as small smooth perturbations of $\phi_{hyp} = \langle x, \xi \rangle + \frac{t\langle \xi, I^n_k \xi \rangle}{2}$.

For $h \in C^2(B^n(0,1), \mathbb{R})$ let the extension operator E_h be given by

$$E_h f(t,x) = \int_{B^n(0,1)} e^{i(x\xi + th(\xi))} f(\xi) d\xi,$$

where $f \in L^2$, $\operatorname{supp}(f) \subseteq B^n(0,1)$ and define a smooth weight function, which is essentially a characteristic function on some ball $B^{n+1}(\overline{z}, R)$, $\overline{z} = (\overline{t}, \overline{x})$:

$$w_{B(\overline{z},R)}(t,x) = (1 + R^{-1}|x - \overline{x}| + R^{-1}|t - \overline{t}|)^{-N}$$

for some large integer $N \in \mathbb{N}$, which is fixed later.

We define the decoupled L^p-norm for variable coefficient operators for $1 \leq R \leq \lambda$. Let $\mathcal{T}_R$ denotes a finitely overlapping family of $R^{-1/2}$ balls covering $B^n(0,1)$. Set

$$\|T^\lambda f\|_{L^{p,R}_{dec}(S)} = \left(\sum_{\tau \in \mathcal{T}_R} \|T^\lambda f_\tau\|^2_{L^p(S)} \right)^{1/2}$$

for S measurable and

$$\alpha(p,k) = \begin{cases} k\left(\frac{1}{4} - \frac{1}{2p}\right), & 2 \leq p \leq \frac{2(n+2-k)}{n-k}, \\ \frac{n}{4} - \frac{n+2}{2p}, & \frac{2(n+2-k)}{n-k} \leq p < \infty. \end{cases} \tag{7.7}$$

We recall the constant-coefficient ℓ^2-decoupling theorem:

Theorem 7.1.1. *[BD17a, Theorem 1.2, p. 280] Let $R \geq 1$, $N \geq 10$, $2 \leq p < \infty, 0 \leq k \leq n/2$, $\alpha(p,k)$ as in (7.7) and $h : B^n(0,1) \to \mathbb{R}$ be a C^2-function with Hessian $\partial^2_{\xi\xi} h$ having modulus of eigenvalues in $[C^{-1}, C]$ for some $C > 0$. Then, we find for f with supp$(f) \subseteq B(0,1)$ the following estimate to hold:*

$$\|E_h f\|_{L^p(w_{B_R})} \lesssim_{C,N,\varepsilon} R^{\alpha(p,k)+\varepsilon} \|E_h f\|_{L^{p,R}_{dec}(w_{B_R})}$$

provided that $N \geq N(n,p)$.

Strictly speaking, this result was proved in [BD17a] only for the hyperboloid $h(\xi) = \sum_{i=1}^n \alpha_i \xi_i^2$. However, the arguments from [PS07], which are illustrated in the context of elliptic surfaces in [BD15, Section 7], yield the more general translation invariant case in a straight-forward manner. See also the discussion below.

Originally, decoupling inequalities were studied for the cone by Wolff in [LaW02, Wol00] to make progress on L^p-smoothing estimates (cf. [MSS92, MSS93]) for the wave equation. These estimates were refined (cf. [GS09, Bou13]) until the breakthrough result of Bourgain-Demeter (cf. [BD15, BD17b]) where sharp decoupling inequalities for the paraboloid were proved. Subsequently, the result was generalized to hyperboloids (cf. [BD17a]). These results also give estimates for exponential sums, in particular essentially sharp Strichartz estimates on irrational tori.

The theory was also extended to non-degenerate curves (cf. [BDG16]). As already pointed out in Beltran-Hickman-Sogge [BHS18], the decoupling theory seems to extend to the variable coefficient case sharply divergent from the $L^p - L^q$-estimates for oscillatory integral operators. In fact, it is well known that there are strictly less estimates admissible in the constant coefficient case due to Kakeya compression (cf. [Bou91, Bou95, Wis05]). For recently proved sharp $L^p - L^p$-estimates for variable-coefficient oscillatory integral operators, we refer to Guth-Hickman-Iliopoulou [GHI17].

Our first result is the following extension of Theorem 7.1.1:

Theorem 7.1.2. *Let $2 \leq p < \infty$, $n, M \in \mathbb{N}$, $0 \leq k \leq n/2$ and $\alpha(p,k)$ like in (7.7). Suppose that (ϕ, a) satisfies $H1'$) and $H2'_{[k]}$). Then, we find the following estimate to hold:*

$$\|T^\lambda f\|_{L^p(\mathbb{R}^{n+1})} \lesssim_{\varepsilon,\phi,M,a} \lambda^{\alpha(p,k)+\varepsilon} \|T^\lambda f\|_{L^{p,\lambda}_{dec}(\mathbb{R}^{n+1})} + \lambda^{-M} \|f\|_2. \qquad (7.8)$$

For variable-coefficient generalizations of the phase function $\phi_{cone}(t,x;\xi) = \langle x, \xi \rangle + t|\xi|$ associated to the adjoint Fourier restriction problem of the cone this was carried out in [BHS18]. The proof of Theorem 7.1.2 adapts this general strategy from [BHS18] to prove variable-coefficient decoupling from constant-coefficient decoupling: on small spatial scales the variable-coefficient oscillatory integral operator is well-approximated by a constant-coefficient operator. It is enough to make progress on this small scale because it extends to any scale by means of parabolic rescaling.

Already in the context of constant coefficients, approximating one surface by another on small scales and recovering arbitrary scales by rescaling was used to derive decoupling estimates for more general elliptic surfaces or the cone (cf. [BD15, Section 7, 8]), see also [PS07, GO16].

It seems plausible that a similar approximation derives the variable-coefficient cone decoupling from the variable-coefficient paraboloid decoupling. Recently, in [Har18] was shown by the same approximation that broad-narrow considerations are also valid for the cone. We do not pursue this line of argument.

The following different consequences of Theorem 7.1.2 are given in Section 7.3: The variable-coefficient ℓ^2-decoupling implies a stability theorem for exponential sums which is proved using the argument in [BD15] for the constant-coefficient case. Moreover, on small spatial scales the broad-narrow considerations from the constant-coefficient case extends to the variable- coefficient case. Further, we point out how the decoupling theorem implies Strichartz and smoothing estimates without further arguments (e.g. dispersive estimates for the propagator). Here, we use a localization property of the kernel which was used in the constant-coefficient case in [RS10] and in the context of variable coefficients in [Lee06b]. The found Strichartz estimates are inferior to the known Strichartz estimates for Schrödinger equations with variable coefficients (cf. [BGT04, ST02]). We rather point out this application due to its simplicity.

In Rogers-Seeger [RS10] L^p-smoothing estimates of the following Schrödinger-like equations were considered:

$$\|e^{itD^\alpha} f\|_{L^p(\mathbb{R}^n \times I)} \lesssim_{p,I,\alpha,n} \|f\|_{L^p_\beta(\mathbb{R}^n)}, \quad \alpha > 1, \quad p \in (2 + \frac{4}{n+1}, \infty),$$
$$\frac{\beta}{\alpha} = n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p}. \tag{7.9}$$

These estimates are the analog of local smoothing estimates for the wave equation, which is a deeper question. Further, estimates of this kind improve the sharp fixed time estimates of Miyachi [Miy81] and Fefferman-Stein [FS72].

In the context of Schrödinger's equation these estimates were previously discussed in Rogers [Rog08]. In this work a conditional result was proved according to which an L^p-restriction estimate implies a smoothing estimate. According to the restriction conjecture, (7.9) should hold for $p > \frac{2(n+1)}{n}$. However, the Sobolev regularity was shown to be sharp. These considerations were extended in Lee et al. [LRV11, LRS13] where more general space-time estimates

$$\|e^{it\Delta} f\|_{L^q_x(\mathbb{R}^n, L^r_t(I))} \lesssim_{I,r,q,p,\alpha} \|f\|_{L^p_\alpha(\mathbb{R}^n)} \tag{7.10}$$

were taken into account. See also the previous work by Rogers [Rog09]. In [LRS13] was also shown a more precise equivalence between extension estimates and smoothing estimates.

In [RS10] (7.9) was proved for $p > 2 + \frac{4}{n+1}$ relying on the bilinear adjoint Fourier restriction theorem for elliptic surfaces by Tao (cf. [Tao03], see also [TVV98]) after frequency localization. In the last section we improve the L^p-smoothing results for variable-coefficient elliptic phase functions derived from decoupling by using the bilinear argument from [RS10]. Here, we use the variable-coefficient generalization by Lee (cf. [Lee06b]) of Tao's bilinear adjoint Fourier restriction theorem for the paraboloid. Again, we use the localization property of the kernel. The following estimate is proved:

Theorem 7.1.3. *Let $p > 2 + \frac{4}{n+1}$ and (ϕ, a) be data giving rise to the Hörmander-operator T^λ which satisfies $H1')$ and $H2'_{[0]})$. Then, for any $\varepsilon > 0$ we find the following estimate to hold:*

$$\|T^\lambda \hat{f}\|_{L^p(\mathbb{R}^{n+1})} \lesssim_{\phi,\varepsilon} \lambda^{n(1/2-1/p)+\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}. \tag{7.11}$$

It is well-known that for constant coefficients the frequency localized estimates admit globalization (cf. [See90, RS10]) to

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^n \times [0,1])} \lesssim \|f\|_{L^p_\beta(\mathbb{R}^n)}.$$

Once one can perceive the operators T^λ as suitable localization of some (say) generalized Schrödinger evolution, it seems feasible to globalize (7.11) in a similar spirit. We do not pursue this question.

7.2 Variable-coefficient decoupling for hyperbolic phase functions

7.2.1 Basic reductions

Before we begin the proof of Theorem 7.1.2 in earnest, we carry out several reductions. Most importantly, we quantify the conditions $H1'$) and $H2'_{[k]}$). In dependence of ε, M and p from Theorem 7.1.2, we choose a small constant $0 < c_{par} \ll 1$ and a large integer $N = N_{\varepsilon,M,p}$ and define the following conditions which we will impose on the phase function for $A \geq 1$:

$H1_1$) $|\partial^2_{x\xi}\phi(z;\xi) - I_n| \leq c_{par}$ for all $(z,\xi) \in Z \times \Xi$

$H2^1_{[k]}$) $|\partial_t \partial^2_\xi \phi - I_n| \leq c_{par}$ for all $(z,\xi) \in Z \times \Xi$

$D1^1_1$) $\|\partial_{x_k}\partial^\beta_\xi \phi\|_{L^\infty(Z\times\Xi)} \leq c_{par}$ for $2 \leq |\beta| \leq N$

$D1^2_1$) $\|\partial_t \partial^\beta_\xi \phi\|_{L^\infty(Z\times\Xi)} \leq c_{par}$ for $3 \leq |\beta| \leq N$

$D2_A$) $\|\partial^2_z \partial^\beta_\xi \phi\|_{L^\infty} \leq \frac{c_{par}A}{100n}$ for $1 \leq |\beta| \leq 2N$

For technical reasons we also impose the following margin condition on the positional part a_1 of the amplitude a:

M_A) $\mathrm{dist}(\mathrm{supp} a_1, \mathbb{R}^{n+1}\backslash Z) \geq 1/(4A)$

We already note the following consequence of $H2^1_{[k]}$):

$$|\partial_t \nabla_\xi \phi| \leq 2|\xi|. \tag{7.12}$$

In [GHI17] it was shown that after introducing suitable partition of unities and performing changes of variables an elliptic phase function satisfying $H1'$) and $H2'_{[0]}$) reduces to the following normal form:

$$\phi(t,x;\xi) = \langle x, \xi \rangle + \frac{t|\xi|^2}{2} + \mathcal{E}(x,t;\xi) \tag{7.13}$$

with $\mathcal{E}$ being quadratic in (t,x) and ξ, to say

$$\partial^\alpha_{(x,t)} \partial^\beta_\xi \mathcal{E}(0;\xi) = 0 \quad \forall |\alpha| \leq 1,\ \beta \in \mathbb{N}^n_0.$$

The explicit representation (7.13) is not required for the following arguments. However, it is useful to keep it in mind stressing the nature of a small smooth perturbation to ϕ_{hyp}. We refer to data (ϕ, a) satisfying the above conditions for some $A \geq 1$ and $0 \leq k \leq n/2$ as type (A,k)-data. The notation and nomenclature is analogous to [BHS18] to point out the similarity to the case of homogeneous variable-coefficient phase functions.

It turns out that these conditions are invariant under parabolic rescaling in a uniform sense, and this allows us to run the induction argument for normalized data. However, to reduce arbitrary hyperbolic phase functions, we have to do one rescaling which depends on the phase function. This gives rise to the dependence on ϕ in (7.8). If we confine ourselves in (7.8) to normalized data, there will be no explicit dependence on ϕ. We define the relevant constant as follows, where ε, M

and p were fixed above and c_{par} and $N = N_{\varepsilon,M,p}$ in the definition of normalized data are chosen in dependence.

We denote by $\mathfrak{D}^{\varepsilon}_{A,k}(\lambda; R)$ the infimum over all $D \geq 0$ so that the estimate

$$\|T^\lambda f\|_{L^p(B_R)} \leq D R^{\alpha(p,k)+\varepsilon} \|T^\lambda f\|_{L^{p,R}_{dec}(w_{B_R})} + R^{2n}(\lambda/R)^{-N/8}\|f\|_{L^2}$$

holds true for all data (ϕ, a) of type (A, k), balls B_R of radius R contained in $B(0,\lambda)$ and $f \in L^2(B^n(0,1))$. For the weight function we take N as in $D2_A$).

The estimate

$$\mathfrak{D}^{\varepsilon}_{1,k}(\lambda; R) \leq C_\varepsilon \tag{7.14}$$

implies Theorem 7.1.2 since we can reduce to normal data. It turns out that it is enough to prove the following proposition:

Proposition 7.2.1. *Let $1 \leq R \leq \lambda^{1-\varepsilon/n}$. Then, we find the estimate* (7.14) *to hold true.*

In fact, we observe that for any $1 \leq \rho \leq R$ and $\rho^{-1/2}$-ball θ one may write

$$T^\lambda f_\theta = \sum_{\substack{\sigma \cap \tilde{\theta} \neq \emptyset, \\ \sigma: R^{-1/2}-ball}} T^\lambda f_\sigma,$$

where $\tilde{\theta}$ denotes the intersection of supp(f) and θ. We compute using Minkowski's and Cauchy-Schwarz inequality that for any weight w one has

$$\begin{aligned}
\|T^\lambda f\|_{L^{p,\rho}_{dec}(w)} &= \Big(\sum_{\theta: \rho^{-1/2}-ball} \|T^\lambda f_\theta\|^2_{L^p(w)} \Big)^{1/2} \\
&= \Big(\sum_{\theta: \rho^{-1/2}-ball} \Big(\sum_{\substack{\sigma: R^{-1/2}-ball, \\ \sigma \cap \tilde{\theta} \neq \emptyset}} \|T^\lambda f_\theta\| \Big)^2 \Big)^{1/2} \\
&\leq \Big(\sum_{\theta: \rho^{-1/2}-ball} (R/\rho)^{n/2} \sum_{\substack{\sigma: R^{-1/2}-ball, \\ \sigma \cap \tilde{\theta} \neq \emptyset}} \|T^\lambda f_\sigma\|^2_{L^p} \Big)^{1/2} \\
&\lesssim (R/\rho)^{\frac{n}{4}} \|T^\lambda f\|_{L^{p,R}_{dec}(w)}.
\end{aligned} \tag{7.15}$$

Since $\|T^\lambda f\|_{L^p(B_R)} \lesssim \|T^\lambda f\|_{L^{p,1}_{dec}(B_R)}$, from taking $\rho = 1$ in the above display it follows that

$$\mathfrak{D}^{\varepsilon}_{A,k}(\lambda; R) \lesssim R^{\frac{n}{4} - \alpha(p,k) - \varepsilon}, \tag{7.16}$$

which yields finiteness of $\mathfrak{D}^\varepsilon$.

Moreover, we can reduce to

$$\mathfrak{D}^{\varepsilon}_{A,k}(\lambda; \lambda^{1-\frac{\varepsilon}{n}}) \leq C_\varepsilon. \tag{7.17}$$

Indeed, the support conditions of the amplitude a imply that the support of $T^\lambda f$ is always contained in $B(0,\lambda)$. We cover $B(0,\lambda)$ by an essentially disjoint family of $\lambda^{1-\frac{\varepsilon}{n}}$-balls

$$\|T^\lambda f\|^p_{L^p(B(0,\lambda))} \leq \sum_{B: \lambda^{1-\frac{\varepsilon}{n}}-balls} \|T^\lambda f\|^p_{L^p(B)},$$

and using Minkowski's inequality we find

$$
\begin{aligned}
\|T^\lambda f\|_{L^p(B)} &\lesssim \mathfrak{D}^\varepsilon_{A,k}(\lambda;\lambda^{1-\frac{\varepsilon}{n}})\lambda^{\frac{\varepsilon}{4}}(\lambda^{1-\frac{\varepsilon}{n}})^{\alpha(p,k)+\varepsilon} \\
&\qquad \Big(\sum_{\theta:(\lambda^{1-\frac{\varepsilon}{n}})^{-1/2}-balls} \|T^\lambda f_\theta\|^2_{L^p(w_B)} \Big)^{1/2} + (\lambda^{1-\varepsilon/n})^{2(n+1)} \lambda^{-\varepsilon N/8} \|f\|_2 \\
&\lesssim \mathfrak{D}^\varepsilon_{A,k}(\lambda;\lambda^{1-\frac{\varepsilon}{n}})(\lambda^{1-\frac{\varepsilon}{n}})^{\alpha(p,k)+\varepsilon}\lambda^{\frac{\varepsilon}{4}} \\
&\qquad \Big(\sum_{\theta:\lambda^{-1/2}-balls} \|T^\lambda f_\theta\|^2_{L^p(w_{B(0,\lambda)})} \Big)^{1/2} + \lambda^{2(n+1)-\frac{2\varepsilon(n+1)}{n}} \lambda^{-\varepsilon N/8} \|f\|_2.
\end{aligned}
$$

For N large enough in dependence of ε, n and M we find (7.8) to hold from (7.15) for normalized data.

7.2.2 Rescaling of variable-coefficient phase functions

We record the following trivial rescaling allowing us to reduce data of type A to data of type 1:

Lemma 7.2.2. *For any $A \geq 1$ we find the following estimate to hold:*

$$\mathfrak{D}^\varepsilon_{A,k}(\lambda;R) \lesssim_A \mathfrak{D}^\varepsilon_{1,k}(\lambda/A;R/A). \tag{7.18}$$

Proof. Let (ϕ,a) be a datum in A-normal form. We define $\tilde{\phi}(z;\xi) = A\phi(z/A;\xi)$ and amplitude $\tilde{a}(z;\xi) = a(z/A;\xi)$ and observe that $T^\lambda f = \tilde{T}^{\lambda/A} f$. Note the equivalent behaviour of ϕ and $\tilde{\phi}$ under one positional derivative. Hence, we find $(\tilde{\phi},\tilde{a})$ to satisfy $H1_1$), $H2^1_{[k]}$), $D1^1_1$), $D1^2_1$), and the second derivative amounts to an additional factor of $1/A$. Hence, we find $D2_1$) to hold. The new margin of the new amplitude $\tilde{a}$ has been increased to size 1/4 and we find M_1) to hold. This step might require the additional argument of decomposing the amplitude function through a partition of unity and translating each piece, if necessary, to adjust to the enlarged support $A\text{supp}(a)$. This involves a sum over $\mathcal{O}(A^{n+1})$ operators where each is associated to type 1-data.

Covering $B(0,R)$ with R/A-balls yields another factor of $\mathcal{O}(A^{n+1})$, but these pieces can be bounded by $\mathfrak{D}^\varepsilon_{1,k}(\lambda/A;R/A)$, and the proof is complete. Moreover, the form of the error term allows us to summarize the sum over $\mathcal{O}(A^{n+1})$ error terms again as error term. □

Next, we show the following stability result for normalized phase functions under parabolic rescaling[1]. This allows us to properly run an induction argument.

Lemma 7.2.3. *[Parabolic rescaling for hyperbolic phase functions] Let $2 \leq p < \infty$, $1 \leq \rho \leq R \leq \lambda$, $0 \leq k \leq n/2$ and $\alpha(p,k)$ like in (7.7). Suppose that (ϕ,a) satisfies $H1'$) and $H2'_{[k]}$) and let T^λ be the associated oscillatory integral operator. If g is supported in a ρ^{-1}-ball and ρ is sufficiently large, then there exists a constant $\overline{C}(\phi) \geq 1$ such that*

$$
\begin{aligned}
\|T^\lambda g\|_{L^p(w_{B_R})} \lesssim_{\varepsilon,N,\phi} &\ \mathfrak{D}^\varepsilon_{1,k}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon} \|T^\lambda g\|_{L^{p,R}_{dec}(w_{B_R})} \\
&+ R^{2(n+1)}(\lambda/R)^{-N/8}\|g\|_2.
\end{aligned}
$$

[1]Here, the term parabolic refers to the rescaling of time by a quadratic factor compared to space and is not restricted to phase functions related to elliptic (parabolic) surfaces.

Proof of Lemma 7.2.3 for phase functions of type 1. Let $\xi_0 \in B^n(0,1)$ be the centre of ρ^{-1}-ball where g is supported. We perform the change of variables $\xi' = \rho(\xi - \xi_0)$ and we compute

$$\begin{aligned} T^\lambda g(z) &= \int_{\mathbb{R}^n} e^{i\phi^\lambda(z;\xi)} a^\lambda(z;\xi) g(\xi) d\xi \\ &= \int_{\mathbb{R}^n} e^{i\phi^\lambda(z;\xi_0+\rho^{-1}\xi')} a^\lambda(z;\xi_0+\rho^{-1}\xi') \underbrace{\rho^{-n} g(\xi_0+\rho^{-1}\xi')}_{\tilde{g}(\xi')} d\xi'. \end{aligned}$$

We expand ϕ to find

$$\phi(z;\xi_0+\xi'/\rho) = \phi(z;\xi_0)+[\nabla_\xi \phi(z;\xi_0)]\frac{\xi'}{\rho}+\rho^{-2}\int_0^1 (1-r)\langle \partial^2_{\xi\xi}\phi(z;\xi_0+r\xi'/\rho)\xi',\xi'\rangle dr.$$

Let $\Phi_{\xi_0}(t,x) = (t, \Phi(t,x;\xi_0))$; $\quad \Phi^\lambda(t,x) = \lambda \Phi_{\xi_0}(t/\lambda, x/\lambda)$ and we introduce the dilations $D_\rho(t,x) = (\rho^2 t, \rho x)$ and $D'_{\rho^{-1}}(x) = \rho^{-1}x$. We find

$$e^{i\lambda\phi(\Phi_{\xi_0}(\rho^2 t/\lambda,\rho x/\lambda);\xi_0)} T^\lambda g \circ \Phi^\lambda_{\xi_0} \circ D_\rho = \tilde{T}^{\lambda/\rho^2}\tilde{g}, \tag{7.19}$$

where

$$\tilde{T}^{\lambda/\rho^2}\tilde{g}(t,x) = \int_{\mathbb{R}^n} e^{i\tilde{\phi}^{\lambda/\rho^2}(t,x;\xi)} \tilde{a}^{\lambda/\rho^2}(x;\xi)\tilde{g}(\xi) d\xi,$$

and the phase $\tilde{\phi}(t,x;\xi)$ is given by

$$\langle x,\xi\rangle + \int_0^1 (1-r)\langle \partial^2_{\xi\xi}\phi(\Phi_{\xi_0}(t, D'_{\rho^{-1}}x);\xi_0+r\xi/\rho)\xi,\xi\rangle dr,$$

and the amplitude $\tilde{a}(y,t;\xi) = a(\Phi_{\xi_0}(t;D'_{\rho^{-1}}y);\xi_0+\xi/\rho)$.

We verify (7.19): From the definition

$$(\Phi^\lambda_{\xi_0} \circ D_\rho)(t,x) = \lambda\Phi_{\xi_0}(\rho^2 t/\lambda, \rho x/\lambda)$$

and

$$\begin{aligned} &\phi^\lambda(\Phi^\lambda_{\xi_0}(D_\rho(t,x)), \xi_0+\xi/\rho) \\ &= \lambda\phi(\Phi_{\xi_0}(\rho^2 t/\lambda, \rho x/\lambda);\xi_0+\xi/\rho) \\ &\to \lambda\phi(\Phi_{\xi_0}(\rho^2 t/\lambda,\rho x/\lambda);\xi_0) + \lambda[\nabla_\xi\phi(\Phi_{\xi_0}(\rho^2 t/\lambda,\rho x/\lambda);\xi_0)]\frac{\xi}{\rho} \\ &\quad + \rho^{-2}\lambda\int_0^1 (1-r)\langle\partial^2_{\xi\xi}\phi(\Phi_{\xi_0}(\rho^2 t/\lambda,\rho x/\lambda);\xi_0+r\rho^{-1}\xi)\xi,\xi\rangle dr, \end{aligned}$$

which proves (7.19).

If ϕ is in normal form, then we can also write

$$\tilde{\phi}(t,x;\xi) = \langle x,\xi\rangle + \frac{t|\xi|^2}{2} + \int_0^1 (1-r)\langle \partial^2_{\xi\xi}\mathcal{E}(\Phi_{\xi_0}(t,D'_{\rho^{-1}}x),\xi_0+r\xi/\rho)\xi,\xi\rangle, \tag{7.20}$$

and with $\tilde{g}$ being supported in $B^n(0,1)$ we can assume that $|\xi| \leq 1$.

A change of spatial variables gives

$$\|T^\lambda g\|_{L^p(B_R)} \lesssim_\phi \rho^{\frac{n+2}{p}} \|\tilde{T}^{\lambda/\rho^2}\tilde{g}\|_{L^p((\Phi^\lambda_{\xi_0}\circ D_\rho)^{-1}(B_R))},$$

where the implicit constant stems from the Jacobian of Φ_{ξ_0}, which is controlled by property $D1_1$). Note that the implicit constant can be chosen constant for data of type 1 provided that $c_{par} > 0$ is chosen small enough. We cover $(\Phi^{\lambda}_{\xi_0} \circ D_\rho)^{-1}(B_R)$ with essentially disjoint R/ρ^2-balls, $B_{R/\rho^2} \in \mathcal{B}_{R/\rho^2}$ and find

$$\|T^\lambda g\|_{L^p(B_R)} \lesssim_\phi \rho^{(n+2)/p} \left(\sum_{B_{R/\rho^2} \in \mathcal{B}_{R/\rho^2}} \|\tilde{T}^{\lambda/\rho^2} \tilde{g}\|^p_{L^p(B_{R/\rho^2})} \right)^{1/p}.$$

We argue below that

$$\begin{aligned} \|\tilde{T}^{\lambda/\rho^2} \tilde{g}\|_{L^p(B_{R/\rho^2})} \lesssim_{\varepsilon,N} \mathfrak{D}^\varepsilon_{1,k}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon} \|\tilde{T}^{\lambda/\rho^2} \tilde{g}\|_{L^{p,R/\rho^2}_{dec}(w_{B_{R/\rho^2}})} \\ + (R/\rho^2)^{2(n+1)}(\lambda/R)^{-N/8}\|g\|_{L^2(\mathbb{R}^n)} \end{aligned} \tag{7.21}$$

holds for each $B_{R/\rho^2} \in \mathcal{B}_{R/\rho^2}$ and some $\overline{C} \geq 1$.

If $(\tilde{\phi}, \tilde{a})$ was a type-1 datum, this would be a consequence of the definitions. First, we show how to conclude the proof with (7.21): we can write

$$\cup_{B_{R/\rho^2} \in \mathcal{B}_{R/\rho^2}} B_{R/\rho^2} \subseteq (\Phi^\lambda_{\xi_0} \circ D_\rho)^{-1}(B_{C_\phi R}) = C_{R'},$$

where $B_{C_\phi R}$ is a ball concentric to B_R, but with enlarged radius $C_\phi R$ for some $C_\phi \geq 1$ because Φ_{ξ_0} is a diffeomorphism.
Hence, we find from summing the pth power on both sides over R/ρ^2 balls and inverting the change of variables

$$\begin{aligned} &\mathfrak{D}^\varepsilon_{1,k}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon} \left(\sum_{B_{R/\rho^2} \in \mathcal{B}} \|\tilde{T}^{\lambda/\rho^2} \tilde{g}\|^p_{L^{p,R/\rho^2}_{dec}(w_{B_{R/\rho^2}})} \right)^{1/p} \\ &\leq \mathfrak{D}^\varepsilon_{1,k}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon} \|\tilde{T}^{\lambda/\rho^2} \tilde{g}\|_{L^{p,R/\rho^2}_{dec}(w_{C_{R'}})}. \end{aligned}$$

Inverting the change of coordinates yields

$$\begin{aligned} \|T^\lambda g\|_{L^p(B_R)} &\lesssim_{\varepsilon,N,\phi} \mathfrak{D}^\varepsilon_{1,k}(\lambda/\overline{C}\rho^2, R/\overline{C}\rho^2)(R/\rho^2)^{\alpha(p,k)+\varepsilon} \\ &\quad (\sum_{\tilde{\theta}:(R/\rho^2)^{-1/2}-ball} \|T^\lambda g_\theta\|^p_{L^p(w_{B_R})})^{1/p} + R^{2(n+1)}(\lambda/R)^{-N/8}\|g\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

It is straight-forward to check that the θ, which are the images of $\tilde{\theta}$ under the mapping $\xi \mapsto \rho(\xi - \xi_0)$, which inverts the change of variables in frequency space, form a cover of the supp$g$ with $R^{-1/2}$-balls. Note how the error term compensates the decomposition into R/ρ^2 balls. In fact, any R/ρ^2-ball contributes with $(R/\rho^2)^{2n}$ and there are roughly $\rho^{2(n+1)}$ R/ρ^2-balls.

It remains to prove (7.21) for each $B_{R/\rho^2} \in \mathcal{B}_{R/\rho^2}$. For this purpose record the following representations of $\tilde{\phi}_L = \tilde{\phi}(t, (L^{-1})^t x; L\xi)$:

$$\tilde{\phi}_L(t,x;\xi) = \langle x, \xi \rangle + \int_0^1 (1-r)\langle \partial^2_{\xi\xi}\phi(\Phi_{\xi_0}(t, D'_{\rho^{-1}} \circ L^{-1}x), \xi_0 + Lr\xi/\rho)L\xi, L\xi\rangle dr, \tag{7.22}$$

and from Taylor expansion we find (up to an irrelevant phase factor)

$$\tilde{\phi}_L(t,x;\xi) = \rho^2 \phi(t, \Phi_{\xi_0}(t, D'_{\rho^{-1}} \circ L^{-1}x); \xi_0 + L\xi/\rho). \tag{7.23}$$

$\tilde{\phi}_L$ is an affinely changed version of $\tilde{\phi}$ for some invertible L, so that $\partial_t \partial^2_{\xi\xi} \tilde{\phi}_L(0,0;0) = I^k_n$. We perceive $L = diag(\sqrt{\mu_1}, \dots, \sqrt{\mu_n}) \cdot R$, where R is a rotation and $\mu_1, \dots, \mu_n$ are the eigenvalues of $\partial_t \partial^2_{\xi\xi} \tilde{\phi}$ which is already close to I^k_n quantified by property $H2^{[k]}_1$).

We verify $H1_1$) for $\tilde{\phi}$: Taking an x derivative of the integral term leads to an expression of the kind $\partial_x \partial^2_{\xi\xi} \phi \cdot \partial_x \Phi_{\xi_0} \cdot \rho^{-1}$.
$\partial_x \partial^2_{\xi\xi} \phi$ is controlled by property D^1_1) of ϕ. From the definition of Φ_{ξ_0} and the chain rule we find $\partial_x \Phi_{\xi_0} = (\partial_x \partial_\xi \phi)^{-1}$. Since $|\partial^2_{x\xi} \phi - I^k_n| \leq c_{par}$, we have $|\partial_x \Phi_{\xi_0}| \leq 2$ and we find the total expression to be of order c_{par}/ρ. Note that taking a frequency derivative does not magnify the size.
Likewise we verify $D1^1_1$) for $|\beta| = 2$. For higher derivatives in ξ we can argue with the representation (7.23) and observe that the bounds for large ρ become smaller and smaller since any derivative in ξ gives rise to a factor of ρ^{-1}. In this way one checks the validity of $D2_1$).

We check $H2^{[k]}_1$): For this purpose we write

$$\partial_t \partial^2_{\xi} \tilde{\phi}_L(t,x;\xi) - I^k_n = \partial_t \partial^2_{\xi} \tilde{\phi}_L(t,x;\xi) - \partial_t \partial^2_{\xi\xi} \tilde{\phi}_L(0,0;0)$$

and use the fundamental theorem of calculus. For an additional ξ derivative we find the contribution to be of size $\mathcal{O}(c_{par}\rho^{-1})$. For positional derivatives we use property $D2_1$) of ϕ to find this contribution to be also much smaller than c_{par}, and thus the claim follows.

The only cases of $D2_1$) which require additional reasoning to the above arguments are when there are two time derivatives and only one or two frequency derivatives. Else, the smallness is immediate from (7.23). In the case of two time derivatives and few frequency derivatives we have to consider combinations $\partial^2_{tt} \partial^2_{\xi\xi} \mathcal{E}$, $\partial_t \Phi_{\xi_0}$ and $\partial^2_{tt} \Phi_{\xi_0}$. $\partial^2_{tt} \partial^2_{\xi\xi} \phi$ and higher frequency derivatives are controlled by property $D2_1$) of ϕ and above we have seen that $\partial_t \Phi_{\xi_0}$ is controlled quantitatively through (7.12) of ϕ. The control over $\partial^2_{tt} \Phi_{\xi_0}$ follows from considering one further time derivative:

$$\begin{aligned} &\partial_{tt} \partial_\xi \phi^\lambda(\Phi^\lambda(t,x;\xi_0);\xi_0) + \partial_t \partial^2_{x\xi} \phi^\lambda(\Phi^\lambda(t,x;\xi_0)) \partial_t \Phi^\lambda \\ &+ \partial_t \partial^2_{x\xi} \phi^\lambda(\Phi^\lambda(t,x;\xi_0);\xi_0) \partial_t \Phi^\lambda(t,x;\xi_0) + \partial^2_{xx} \partial_\xi \phi^\lambda(\Phi^\lambda(t,x;\xi_0);\xi_0)(\partial_t \Phi^\lambda)^2 \\ &+ \partial^2_{x\xi} \phi^\lambda \partial^2_{tt} \Phi^\lambda = 0. \end{aligned} \tag{7.24}$$

Hence, we find $|\partial^2_{tt} \partial_\xi \tilde{\phi}_L|, |\partial^2_{tt} \partial^2_{\xi\xi} \tilde{\phi}_L| \leq C$ independent of ϕ with dependence only on the parameters in the definition of type 1 data. After invoking Lemma 7.2.2 with some constant independent of ϕ provided that ϕ is a datum of type 1, the proof is complete. □

Finally, we deal with the case of a general phase function. The proof is essentially a reprise of the proof of Lemma 7.2.3. However, the implicit constants are now allowed to depend on ϕ, and since we are not dealing with a normalized datum from the beginning, the constants may become arbitrarily large.

Proof. First, we use the trivial rescaling from Lemma 7.2.2 $\phi \to \phi^A = A\phi(z/A, \xi)$ to ensure that

$$\|\partial_z^2 \partial_\xi^\beta \phi\|_{L^\infty} \leq \frac{c_{par}}{100nA} \quad \text{for } |\beta| = 1, 2.$$

Later, we shall see how to choose $A = A(\phi)$. Next, we break the support of g into ρ^{-1}-balls, and again we will choose $\rho \geq 1$ later in dependence of ϕ.

We carry out the changes of coordinates from the proof of Lemma 7.2.3 and again arrive at the representations

$$\tilde{\phi}^A(t, x; \xi) = \langle x, \xi \rangle + \int_0^1 (1 - r)\langle \partial^2_{\xi\xi} \phi^A(\Phi^A_{\xi_0}(t, D'_{\rho^{-1}} x); \xi_0 + r\xi/\rho)\xi, \xi\rangle dr, \tag{7.25}$$

$$\tilde{\phi}^A(t, x; \xi) = \rho^2 \phi^A(\Phi^A_{\xi_0}(t, D'_{\rho^{-1}} x); \xi_0 + \xi/\rho), \tag{7.26}$$

and we define $\tilde{\phi}^A_L$ analogous to the proof of Lemma 7.2.3. We check $H1_1$) from (7.25) which shows that

$$\partial^2_{x\xi} \tilde{\phi}^A_L = I_n + \mathcal{O}_\varphi(\rho^{-1}). \tag{7.27}$$

We also find $\|\partial_{x_k} \partial^2_{\xi\xi} \phi\|_{L^\infty} = \mathcal{O}_\phi(\rho^{-1})$ also follows from (7.26). Moreover, for higher order derivatives in ξ we get additional factors of ρ^{-1} which proves property $D1^1_1$). Likewise, we verify $D1^2_1$) for sufficiently large ρ.

For the proof of $H2^{[k]}_1$) we write again

$$\partial_t \partial^2_{\xi\xi} \tilde{\phi}^A_L - I^k_n = \partial_t \partial^2_{\xi\xi} \tilde{\phi}^A_L(t, x; \xi) - \partial_t \partial^2_{\xi\xi} \tilde{\phi}^A_L(0, 0; 0) \tag{7.28}$$

and estimate the difference deploying the fundamental theorem of calculus. The above arguments already yield $\partial_t \partial^3_{\xi\xi\xi} \tilde{\phi}^A_L = \mathcal{O}_\phi(\rho^{-1})$, for positional derivatives we choose $A = A(\phi)$ large enough, so that $\partial_t \partial_z \partial^2_{\xi\xi} \tilde{\phi}^A_L \leq \frac{c_{par}}{100n}$ and we can also control this contribution. Note that here we also need $|\partial^2_{tt} \Phi^\lambda| = \mathcal{O}_\phi(A^{-1})$ which follows from (7.24).

We check $D2_1$) like in the proof of Lemma 7.2.3 after choosing $A = A(\phi)$ sufficiently large. ☐

7.2.3 Approximation by extension operators

Let (ϕ, a) be a datum of type 1 giving rise to the oscillatory integral operator T^λ and recall that we assume the amplitude function to be of product type: $a(z; \xi) = a_1(z)a_2(\xi)$. Further, recall that

$$\xi \mapsto (\nabla_{x,t} \phi^\lambda)(\overline{z}; \Psi^\lambda(\overline{z}; \xi))$$

is a graph parametrisation of a hypersurface $\Sigma_{\overline{z}}$. Thus, we have

$$\langle z, (\nabla_{x,t} \phi^\lambda(\overline{z}; \Psi^\lambda(\overline{z}; \xi))\rangle = \langle x, \xi \rangle + t h_{\overline{z}}(\xi) \tag{7.29}$$

for all $z = (x, t) \in \mathbb{R}^{n+1}$ with $z/\lambda \in Z$ where $h_{\overline{z}}(\xi) = (\partial_t \phi^\lambda(\overline{z}; \Psi^\lambda(\overline{z}; \xi)))$.

Moreover, from the definition of Ψ^λ we have

$$\begin{aligned} \xi &= \partial_x \phi^\lambda(\overline{z}; \Psi^\lambda(\overline{z}; \xi)), \\ I_n &= \partial^2_{x\xi} \phi^\lambda(\overline{z}; \Psi^\lambda(\overline{z}; \xi))(\partial_\xi \Psi^\lambda(\overline{z}; \xi)), \\ 0 &= \partial^3_{x\xi\xi} \phi(\overline{z}; \Psi^\lambda(\xi))(\partial_\xi \Psi^\lambda(\overline{z}; \xi))^2 + \partial^2_{x\xi} \phi^\lambda(\overline{z}; \Psi^\lambda(\xi))\partial^2_{\xi\xi} \Psi^\lambda(\overline{z}; \xi). \end{aligned} \tag{7.30}$$

And consequently, we find for 1-normalized data

$$\begin{aligned} |\partial_\xi \Psi^\lambda(\overline{z};\xi) - I_n| &\ll 1, \\ |\partial^2_{\xi\xi} \Psi^\lambda(\overline{z};\xi)| &\ll 1. \end{aligned} \tag{7.31}$$

Let $E_{\overline{z}}$ denote the extension operator associated to $\Sigma_{\overline{z}}$ given by

$$E_{\overline{z}} g(x,t) = \int_{\mathbb{R}^n} e^{i(\langle x,\xi\rangle + t h_{\overline{z}}(\xi))} a_{\overline{z}}(\xi) g(\xi) d\xi,$$

where $a_{\overline{z}}(\xi) = a_2 \circ \Psi^\lambda(\overline{z};\xi)|\det \partial_\xi \Psi^\lambda(\overline{z};\xi)|$.

We shall see that on small spatial scales T^λ is effectively approximated by $E_{\overline{z}}$ and vice versa. We record the following consequence of dealing with 1-normalized data:

Lemma 7.2.4. *Let (ϕ, a) be a type 1 datum. Each eigenvalue μ of $\partial^2_{\xi\xi} h_{\overline{z}}$ satisfies $|\mu| \sim 1$ on supp$(a_{\overline{z}})$. For elliptic phase functions of type 1 we have $\mu \sim 1$ on supp$(a_{\overline{z}})$.*

Proof. From the definition of $h_{\overline{z}}$ we find

$$\begin{aligned} \partial_\xi h_{\overline{z}}(\xi) &= (\partial_t \partial_\xi \phi^\lambda(\overline{z}; \Psi^\lambda(\overline{z};\xi)) \partial_\xi \Psi^\lambda(\overline{z};\xi), \\ \partial^2_{\xi\xi} h_{\overline{z}}(\xi) &= (\partial_t \partial^2_{\xi\xi} \phi^\lambda(\overline{z}; \Psi^\lambda(\overline{z};\xi))(\partial_\xi \Psi^\lambda(\overline{z};\xi))^2 + \partial_t \partial_\xi \phi^\lambda(\overline{z}; \Psi^\lambda(\overline{z};\xi)) \partial^2_{\xi\xi} \Psi^\lambda(\overline{z};\xi), \end{aligned}$$

and the claim follows from (7.31). □

This becomes useful when it comes to applying the constant-coefficient ℓ^2-decoupling theorem, which we repeated in Theorem 7.1.1, because Lemma 7.2.4 ensures uniformity of the constant from the decoupling inequality.

In the following we analyze $T^\lambda f(z)$ for $z \in B^{n+1}(\overline{z}; K) \subseteq B(0, 3\lambda/4)$ and $1 \leq K \leq \lambda^{1/2}$. The containment property can be assumed due to the margin condition. We see that the desired approximation identity holds on this spatial scale: we perform a change of variables $\xi = \Psi^\lambda(\overline{z};\tilde{\xi})$ and expand ϕ^λ around $\overline{z}$ to find

$$T^\lambda f(z) = \int_{\mathbb{R}^n} e^{i(\langle z-\overline{z}, \nabla_{x,t}\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\tilde{\xi}))\rangle + \mathcal{E}^\lambda_{\overline{z}}(z-\overline{z};\tilde{\xi}))} a_1^\lambda(z) a_{\overline{z}}(\tilde{\xi}) f_{\overline{z}}(\tilde{\xi}) d\tilde{\xi},$$

where $f_{\overline{z}} = e^{i\phi^\lambda(\overline{z};\Psi^\lambda(\overline{z};\cdot))} f \circ \Psi^\lambda(\overline{z};\cdot)$ and

$$\mathcal{E}^\lambda_{\overline{z}}(v;\xi) = \frac{1}{\lambda}\int_0^1 (1-r)\langle(\partial^2_{zz}\phi)((\overline{z}+rv)/\lambda; \Psi^\lambda(\overline{z};\xi))v; v\rangle dr.$$

Lemma 7.2.5. *Let T^λ be an operator associated to a 1-normalized datum (ϕ, a), $0 < \delta \leq 1/2$, $1 \leq K \leq \lambda^{1/2-\delta}$ and $\overline{z}/\lambda \in Z$ so that $B(\overline{z}; K) \subseteq B(0, 3\lambda/4)$.*
Then, we find the estimates

$$\|T^\lambda f\|_{L^p(w_{B(\overline{z};K)})} \lesssim_N \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0;K)})} + \lambda^{-\delta N/2}\|f\|_2, \tag{7.32}$$

$$\|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0;K)})} \lesssim_N \|T^\lambda f\|_{L^p(w_{B(\overline{z};K)})} + \lambda^{-\delta N/2}\|f\|_2. \tag{7.33}$$

to hold provided that N is chosen sufficiently large depending on n, δ and p. Here, the constant N is the same for the weight functions, the conditions on the derivatives $D1^1_1$), $D1^2_1$), $D2_1$) and in the exponent of λ in the above estimates. Moreover, in case of sharp cutoff (7.32) becomes

$$\|T^\lambda f\|_{L^p(B(\overline{z};K))} \lesssim_N \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0;K)})} + \lambda^{-\delta N/2}\|f\|_2. \tag{7.34}$$

Proof. We can replace f by $f\varphi$, where in view of the definition $a_{\overline{z}}$ and the fact that we are dealing with a datum of type 1, we can assume that φ is supported in $[0,2\pi]^n$. After performing a Fourier series decomposition of $e^{i\mathcal{E}^{\lambda}_{\overline{z}}(v,\xi)}\varphi(\xi)$, one may write

$$e^{i\mathcal{E}^{\lambda}_{\overline{z}}(v,\xi)}\varphi(\xi)=\sum_{k\in\mathbb{Z}^n}a_k(v)e^{i\langle k,\xi\rangle}, \tag{7.35}$$

where $a_k(v)=\int_{[0,2\pi]^n}e^{-i\langle k,\xi\rangle}e^{i\mathcal{E}^{\lambda}_{\overline{z}}(v;\xi)}\varphi(\xi)d\xi$.

Since $K\le\lambda^{1/2}$ we find the favourable bound

$$\sup_{(v;\xi)\in B(0,K)\times\mathrm{supp}a_{\overline{z}}}|\partial^{\beta}_{\xi}\mathcal{E}^{\lambda}_{\overline{z}}(v;\xi)|\lesssim_N\frac{|v|^2}{\lambda}$$

as long as $\beta\in\mathbb{N}_0^n$ with $1\le|\beta|\le 2N$ by virtue of property $D2_1$) and the computation in (7.30) showing that $|\partial^{\beta}_{\xi}\Psi^{\lambda}(\overline{z};\xi)|\lesssim 1$ as long as $1\le|\beta|\le 2N$.

Consequently, integration by parts yields

$$|a_k(v)|\lesssim_N(1+|k|)^{-N},$$

whenever $|v|\le 2\lambda^{1/2}$. We derive the following pointwise identity from (7.35):

$$|T^{\lambda}f(\overline{z}+v)|\le\sum_{k\in\mathbb{Z}^n}|a_k(v)||E_{\overline{z}}(f_{\overline{z}}e^{i\langle k,\cdot\rangle})(v)|\lesssim\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-N}|E_{\overline{z}}(f_{\overline{z}}e^{i\langle k,\cdot\rangle}(v)|.$$

We decompose further:

$$\begin{aligned}\|T^{\lambda}f\|_{L^p(w_{B(\overline{z};K)})}\le\ &\|(T^{\lambda}f)1_{B(\overline{z};2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})}\\&+\|(T^{\lambda}f)1_{\mathbb{R}^{n+1}\setminus B(\overline{z};2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})}.\end{aligned}$$

The second term leads to the error term, that is

$$\|(T^{\lambda}f)\chi_{\mathbb{R}^n\setminus B(\overline{z};2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})}\lesssim\lambda^{\frac{n}{2p}-\delta(N-(n+2))}\|f\|_{L^2(\mathbb{R}^n)}. \tag{7.36}$$

In fact, we have $\|T^{\lambda}f\|_{L^\infty}\lesssim\|f\|_2$, and consequently,

$$\left(\int_{\mathbb{R}^{n+1}}(1+K^{-1}|x|)^{-(n+2)}|T^{\lambda}f|^p\right)^{1/p}\lesssim K^{n/p}\|f\|_{L^2}\lesssim\lambda^{\frac{1}{2p}}\|f\|_2,$$

and the factor $\lambda^{-\delta(N-(n+2))}$ stems from the additional decay of the weight $(1+K^{-1}|x|)^{-N}$ we are actually considering.
This gives (7.36), and since the operator $E_{\overline{z}}$ is translation-invariant,

$$E_{\overline{z}}[e^{i\langle k,\cdot\rangle}g](t,x)=E_{\overline{z}}g(t,x+k)\quad\forall(t,x)\in\mathbb{R}^{n+1}\text{ and }k\in\mathbb{R}^n. \tag{7.37}$$

Minkowski's inequality yields

$$\|T^{\lambda}f1_{B(\overline{z};2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})}\lesssim_N\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-N}\|E_{\overline{z}}f_{\overline{z}}\|_{L^p(w_{B((k,0),K)})}. \tag{7.38}$$

Next, observe that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-N}\|E_{\overline{z}}f_{\overline{z}}\|_{L^p(w_{B((k,0),K)})}\\
&=\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-\frac{N}{p}}(1+|k|)^{N(\frac{1}{p}-1)}\|E_{\overline{z}}f_{\overline{z}}\|_{L^p(w_{B((k,0),K)}}\\
&\leq\left[\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-N}\|E_{\overline{z}}f_{\overline{z}}\|^p_{L^p(w_{B((k,0),K)}}\right]^{1/p}\left(\sum_{k\in\mathbb{Z}^n}(1+|k|)^{N(\frac{1}{p}-1)p'}\right)^{1/p'} \qquad (7.39)\\
&=C(n,p,N)\left(\int|E_{\overline{z}}f_{\overline{z}}|^p\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-N}w_{B((k,0),K)}\right)^{1/p}\\
&\lesssim_{n,p}\|E_{\overline{z}}f_{\overline{z}}\|_{L^p(w_{B(0,K)})}.
\end{aligned}$$

For the ultimate estimate one observes

$$\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-N}w_{B((k,0),K)}\lesssim w_{B(0,K)}.$$

In order to prove (7.33), we write

$$E_{\overline{z}}f_{\overline{z}}(v)=\int_{\mathbb{R}^n}e^{i\phi^\lambda(\overline{z}+v;\Psi^\lambda(\overline{z};\xi)}e^{-\mathcal{E}^\lambda_{\overline{z}}(v;\xi)}a_{\overline{z}}(\xi)f\circ\Psi^\lambda(\overline{z};\xi)d\xi.$$

Again, we insert a smooth cutoff $\varphi(\xi)$ supported in $[0,2\pi]^n$ so that

$$e^{-i\mathcal{E}^\lambda_{\overline{z}}(v;\xi)}\varphi(\xi)=\sum_{k\in\mathbb{Z}^n}e^{i\langle k,\xi\rangle}b_k(v),$$

where $b_k(v)=\int_{[0,2\pi]^n}e^{-i\langle k,\xi\rangle}e^{-i\mathcal{E}^\lambda_{\overline{z}}(v;\xi)}\varphi(\xi)d\xi$.

Once more, integration by parts yields the pointwise bound

$$|b_k(v)|\lesssim_N(1+|k|)^{-2N},$$

and inverting the change of variables gives

$$|E_{\overline{z}}f_{\overline{z}}(v)|\lesssim_N\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-2N}|T^\lambda\underbrace{[e^{i\langle k,\partial_z\phi^\lambda(\overline{z},\cdot)\rangle}f]}_{\tilde{f}_k}(\overline{z},v)|.$$

From a similar argument to the one from the proof of (7.33), we have

$$\|E_{\overline{z}}f_{\overline{z}}\|_{L^p(w_{B(0,K)})}\lesssim_N\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-2N}\|(T^\lambda\tilde{f}_k)\chi_{B(\overline{z},2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z},K)})}+\lambda^{-\delta N/2}\|f\|_2. \qquad (7.40)$$

The $k=0$ term is alright because it yields the desired quantity. For the higher order terms we use the estimate (7.32) and (7.39) to conclude

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n,k\neq0}(1+|k|)^{-2N}\|(T^\lambda\tilde{f}_k)\chi_{B(\overline{z},2\lambda^{1/2})}\|_{L^p(w_{B(\overline{z};K)})}\\
&\lesssim_N 2^{-N}\sum_{k\in\mathbb{Z}^N,k\neq0}(1+|k|)^{-N}\|E_{\overline{z}}f_{\overline{z}}\|_{L^p(w_{B((k,0),K)})}\\
&\lesssim_N 2^{-N}\|E_{\overline{z}}f_{\overline{z}}\|_{L^p(w_{B(0,K)})}.
\end{aligned}$$

Choosing N large enough depending on n and p this quantity can be absorbed into the lefthandside of (7.40) which yields the claim. □

7.2.4 Conclusion of the proof

Proof of Proposition 7.2.1. To show Proposition 7.2.1 for fixed parameters n, ε and $N = N(n,\varepsilon)$, it is enough to prove that

$$\mathfrak{D}^{\varepsilon}_{1,k}(\lambda;R) \lesssim_{\varepsilon} 1 \text{ for all } 1 \le R \le \lambda^{1-\varepsilon/n}. \tag{7.41}$$

We perform an induction on the radius, and with the base case (small R) readily settled, we contend the following induction hypothesis:

There is a constant $\overline{C}_{\varepsilon} \ge 1$ such that $\mathfrak{D}^{\varepsilon}_{1,k}(\lambda';R') \le \overline{C}_{\varepsilon}$ holds for all $1 \le R' \le R/2$ and all λ' satisfying $R' \le (\lambda')^{1-\varepsilon/n}$.

We use the approximation lemma on a small spatial scale and lift the resulting estimates to the correct spatial scales through parabolic rescaling: Let $\mathcal{B}_K$ denote a family of finitely-overlapping K-balls covering B_R for some $2 \le K \le \lambda^{1/4}$. After breaking B_R into $B(\overline{z};K)$-balls the estimate from Lemma 7.2.5 implies

$$\begin{aligned} \|T^{\lambda} f\|_{L^p(B_R)} &\lesssim \Big(\sum_{B(\overline{z};K)\in\mathcal{B}_K} \|T^{\lambda} f\|^p_{L^p(B(\overline{z};K))} \Big)^{1/p} \\ &\lesssim \Big(\sum_{B(\overline{z};K)\in\mathcal{B}_K} \|E_{\overline{z}} f_{\overline{z}}\|^p_{L^p(w_{B(0;K)})} \Big)^{1/p}. \end{aligned} \tag{7.42}$$

We apply the constant-coefficient decoupling theorem (Theorem 7.1.1) to each small scale and find after reverting the change of coordinates (again using that we are dealing with 1-normalized data):

$$\begin{aligned} \|E_{\overline{z}} f_{\overline{z}}\|_{L^p(w_{B(0,K)})} &\lesssim_{\varepsilon} K^{\varepsilon/2+\alpha(p,k)} \|E_{\overline{z}} f_{\overline{z}}\|_{L^{p,K}_{dec}(w_{B(0,K)})} \\ &\lesssim K^{\alpha(p,k)+\varepsilon/2} \Big(\sum_{\sigma:K^{-1/2}-ball} \|T^{\lambda} f_{\sigma}\|^2_{L^p(w_{B(\overline{z};K)})} \Big)^{1/2} \\ &\quad + \lambda^{-N/8} K^{2n} \|f\|_2. \end{aligned} \tag{7.43}$$

Moreover, this estimate holds uniformly in $\overline{z}$ by virtue of the uniform estimates on the Hessian of $h_{\overline{z}}$ derived in Lemma 7.2.4.

We plug (7.43) into (7.42) to find after using Minkowski's inequality:

$$\|T^{\lambda} f\|_{L^p(B_R)} \lesssim K^{\alpha(p,k)+\varepsilon/2} \Big(\sum_{\sigma:K^{-1/2}-ball} \|T^{\lambda} f_{\sigma}\|^2_{L^p(w_{B_R})} \Big)^{1/2} + \lambda^{-N/8} K^{2n} R^n \|f\|_{L^2}. \tag{7.44}$$

Next, apply Lemma 7.2.3 to each $T^{\lambda} f_{\sigma}$ which gives the estimate

$$\begin{aligned} \|T^{\lambda} f_{\sigma}\|_{L^p(w_{B_R})} &\le \mathfrak{D}^{\varepsilon}_{1,k}(\lambda/(\overline{C}K^2), R/(\overline{C}K^2))(R/K^2)^{\alpha(p,k)+\varepsilon} \|T^{\lambda} f_{\sigma}\|_{L^{p,R}_{dec}(w_{B_R})} \\ &\quad + R^{2(n+1)}(\lambda/R)^{-N/8} \|f_{\sigma}\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{7.45}$$

We note that $\mathfrak{D}^{\varepsilon}_{1,k}(\lambda/(\overline{C}K^2), R/(\overline{C}K^2)) \lesssim_\varepsilon 1$ according to our induction hypothesis. Plugging (7.45) into (7.44) gives after applying orthogonality

$$\begin{aligned}\|T^\lambda f\|_{L^p(B_R)} &\leq C_\varepsilon \overline{C}_\varepsilon K^{\varepsilon/2} (R/K^2)^{\alpha(p,k)+\varepsilon} \Big(\sum_{\sigma: K^{-1/2}-ball} \|T^\lambda f_\sigma\|^2_{L^{p,R}_{dec}(w_{B_R})} \Big)^{1/2} \\ &\quad + R^{2(n+1)} (\lambda/R)^{-N/8} \|f\|_2 \\ &\leq C_\varepsilon \overline{C}_\varepsilon K^{-\varepsilon/2} R^{\alpha(p,k)+\varepsilon} \|T^\lambda f\|_{L^{p,R}_{dec}(w_{B_R})} \\ &\quad + R^{2(n+1)} (\lambda/R)^{-N/8} \|f\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

and we see that induction closes. □

Proof of Theorem 7.1.2. To finish the proof of Theorem 7.1.2, we break the support of $f \in L^2(B^n(0,1))$ into ρ^{-1}-balls, $\rho = \rho(\phi)$, so that after parabolic rescaling we are dealing with a normalized phase function $\tilde{\phi}$. We can apply Proposition 7.2.1 to $\tilde{\phi}$, and the proof is completed using Lemma 7.2.3. □

7.3 Applications of variable-coefficient decoupling

Below, we state consequences of the variable-coefficient decoupling. Mainly, we argue how phenomena from the constant- coefficient case lift to variable coefficients. In the following we shall occasionally refer to normalized initial data. This means that after we fix the parameters c_{par} small and N large enough depending on n, we find the stated estimates to hold uniformly. The claims extend to unnormalized phase functions by parabolic rescaling.

7.3.1 Discrete L^2-restriction theorem

We generalize the following discrete L^2-restriction theorem proved in [BD15]:

Theorem 7.3.1. *[BD15, Theorem 2.1] Let S be a compact C^2-hypersurface in $\mathbb{R}^{n+1}$ with positive definite second fundamental form. Let $\Lambda \subseteq S$ be a $\delta^{-1/2}$-separated set, and let $R \gtrsim \delta^{-1}$. Then, for each $\varepsilon > 0$, we find the following estimate to hold*

$$\left(\frac{1}{|B^{n+1}(0,R)|} \int_{B^{n+1}(0,R)} \Big| \sum_{\xi \in \Lambda} a_\xi e^{i\langle x, \xi \rangle} \Big|^p \right)^{1/p} \lesssim \delta^{-\alpha(p)-\varepsilon} \|a_\xi\|_2 \qquad (7.46)$$

provided that $p \geq \frac{2(n+2)}{n}$ and $\alpha(p) = \alpha(p,0)$ from (7.7).

The proof relies on the ℓ^2-decoupling theorem for elliptic surfaces and approximating delta-functions on small caps with bump functions. Also recall the estimate for $R \gtrsim \delta^{-1/2}$

$$\Big\| \sum_{\xi \in \Lambda} a_\xi e^{i\langle x, \xi \rangle} \Big\|_{L^2(B_R^{n+1})} \sim |B_R^{n+1}|^{1/2} \|a_\xi\|_2, \qquad (7.47)$$

which is a consequence of Plancherel's theorem. Indeed, take $\psi(x)$ to be a bump function $\psi(x) = 1$ on $B_1(0)$ and *supp*$(\psi) \subseteq B_2(0)$, and $\psi_R(x) = \psi(x/R)$. Then, we compute

$$\Big\| \sum_{\xi \in \Lambda} a_\xi e^{i\langle x,\xi\rangle} \Big\|_{L^2(B^{n+1}(0,R))} \sim \Big\| \sum_{\xi \in \Lambda} a_\xi e^{i\langle x,\xi\rangle} \psi_R(x) \Big\|_{L^2(\mathbb{R}^n)} .$$

Now we use Plancherel's theorem to find that $\mathcal{F}(a_\xi e^{i\langle x,\xi\rangle}\psi_R)$ is essentially a bump function on scale R^{-1} centered at ξ. The separation condition lets the bumps behave decoupled so that

$$\begin{aligned}\Big\| \sum_{\xi \in \Lambda} a_\xi e^{i\langle x,\xi\rangle} \psi_R(x) \Big\|^2_{L^2(\mathbb{R}^n)} &\sim \sum_{\xi \in \Lambda} |a_\xi|^2 \int |\tilde{\varphi_R}(\eta)|^2 d\eta \\ &\sim |B_R| \sum_{\xi \in \Lambda} |a_\xi|^2,\end{aligned}$$

which yields the claim.

The estimate in the above display generalizes as follows. The below lemma is a special case of [GHI17, Lemma 11.5].

Lemma 7.3.2. *Let $A \subseteq B_1^n(0)$ be a λ^{-1}-separated set and φ be a 1-normalized elliptic phase function. Then, we find the estimate*

$$\Big\| \sum_{\xi \in A} a_\xi e^{i\varphi^\lambda(t,x;\xi)} \Big\|_{L^2(B_\lambda^{n+1})} \lesssim |B_\lambda^{n+1}|^{1/2} \lambda^\varepsilon \|a_\xi\|_{\ell^2}$$

to hold.

The discrete L^2-restriction theorem generalizes as follows:

Theorem 7.3.3. *Let $n \geq 1$, $0 \leq k \leq n/2$ and suppose that ϕ is a phase function satisfying $H1')$ and $H2'_{[k]})$, $A \subseteq B^n(0,1)$ be a $\lambda^{-1/2}$-separated set, $p \geq \frac{2(n+2-k)}{n-k}$ and $\alpha(p,k)$ like in (7.7). Then, we find the following estimate to hold:*

$$\left(\frac{1}{|B_\lambda|} \int_{B_\lambda} \Big| \sum_{\xi \in A} a_\xi e^{i\phi^\lambda(t,x;\xi)} \Big|^p \right)^{1/p} \lesssim_{\phi,\varepsilon} \lambda^{\alpha(p,k)+\varepsilon} \left(\sum_{\xi \in A} |a_\xi|^2 \right)^{1/2} .$$

Proof. As initial data choose

$$f_\tau = \sum_{\xi \in A} a_\xi |B^{n-1}(\tau,\xi)| 1_{B^{n-1}(\tau,\xi)}.$$

It follows that

$$\begin{aligned}T^\lambda f_\tau(t,x) = F_\tau(t,x) &= \int_\Omega \sum_{\xi \in A} a_\xi \frac{1_{B(\xi,\tau)}}{|B(\xi,\tau)|} e^{i\phi^\lambda(t,x;\xi')} a^\lambda(t,x;\xi') d\xi' \\ &= \sum_{\xi \in A} a_\xi \frac{1}{|B(\xi,\tau)|} \int_{B(\xi,\tau)} e^{i\phi^\lambda(t,x;\xi')} a^\lambda(t,x;\xi') d\xi' \\ &\to \sum_{\xi \in A} a_\xi a^\lambda(t,x;\xi) e^{i\phi^\lambda(t,x;\xi)}\end{aligned}$$

and for the frequency localized pieces

$$\begin{aligned}T^\lambda f_\theta = F_{\theta,\tau}(t,x) &= \int_\Omega \sum_{\xi \in A} a_\xi \frac{1_{B(\xi,\tau)}}{|B(\xi,\tau)|} e^{i\phi^\lambda(t,x;\xi')} a^\lambda(t,x;\xi') 1_\theta(\xi') d\xi' \\ &= \frac{1}{|B(\theta,\tau)|} \int_{B(\theta,\tau)} a_\theta e^{i\phi^\lambda(t,x;\xi')} a^\lambda(t,x;\xi') d\xi' \\ &\to a_\theta e^{i\phi^\lambda(t,x;\theta)} a^\lambda(t,x;\theta)\end{aligned}$$

and

$$\|F_{\theta,\tau}\|_{L^p(w_{B_\lambda})} \to |a_\theta| \|1\|_{L^p(w_{B_\lambda})} \sim |a_\theta| |B_\lambda|^{1/p},$$

which yields the claim by applying Theorem 7.1.2. □

7.3.2 Decoupling inequalities imply Strichartz inequalities and smoothing inequalities for variable coefficients

Already in [Wol00] was pointed out how decoupling inequalities for the cone imply L^p-smoothing estimates; in [BHS18] this was extended to the variable-coefficient case.

Purpose of this section is to point out how a localization property of the kernel ([RS10, Lee06b]) implies L^p-smoothing and Strichartz estimates without further arguments (e.g. dispersive estimates for the propagator or multilinear considerations) for elliptic and hyperbolic phase functions from decoupling estimates. We detail the argument in the next section to avoid repetition. The obtained estimates do not recover the classical range, but the Tomas-Stein restriction theorem in the elliptic case. We phrase this as a conditional result.

Proposition 7.3.4. *Let $n \in \mathbb{N}$, $2 \leq p < \infty$ and suppose that (ϕ, a) satisfies $H1')$ and $H2'_{[k]})$ and the decoupling inequality*

$$\|T^\lambda f\|_{L^p(B_\lambda)} \lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \left(\sum_{\theta:\lambda^{-1/2}-balls} \|T^\lambda f_\theta\|^2_{L^p(w_{B_\lambda})} \right)^{1/2} + \lambda^{-M} \|f\|_2. \tag{7.48}$$

Then, we find the Strichartz estimate

$$\|T^\lambda f\|_{L^p(B_\lambda)} \lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \lambda^{\frac{n+2}{2p}} \lambda^{-\frac{n}{4}} \|f\|_2 \tag{7.49}$$

to hold for any $f \in L^2$.

Furthermore, we find the local smoothing estimate

$$\|T^\lambda \hat{f}\|_{L^p(B_\lambda)} \lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \lambda^{\frac{n}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^p} \tag{7.50}$$

to hold for any $f \in L^p$.

Proof. In both cases, after applying the decoupling inequality one uses parabolic rescaling on $T^\lambda f_\theta$ (cf. the proof of Lemma 7.2.3) which deforms the B_λ-ball into a $\lambda^{1/2} \times \ldots \times \lambda^{1/2} \times 1$-ellipse.

Following the computation of the proof in Lemma 7.2.3, we find

$$\|T^\lambda f_\theta\|_{L^p(B_\lambda)} \lesssim_\phi \lambda^{\frac{n+2}{2p}} \|\tilde{T}^1 g_\theta\|_{L^p(C_{\lambda^{1/2}})},$$

where $g_\theta(\xi') = \lambda^{-n/2} f(\xi_\theta + \lambda^{-1/2}\xi')$, ξ_θ denoting the center of θ. Without loss of generality we can suppose that the phase function $\tilde{\phi}$ under rescaling is of type 1.

Next, cover $Q(\lambda^{1/2})$ with cubes of sidelength 1 and as detailed in the next section, we find from a kernel estimate

$$\|\tilde{T}^1 g_\theta\|_{L^p(C_{\lambda^{1/2}})} \lesssim \left(\sum_{l,k:\lambda^\varepsilon Q_k \cap Q_l \neq \emptyset} \|\tilde{T}^1(g_\theta \chi_{Q_k})\|^p_{L^p(Q_l \times [-1,1])} \right)^{1/p} + \lambda^{-N} \|g_\theta\|_2$$

and the estimate

$$\|\tilde{T}^1(g_\theta \chi_{Q_l})\|_{L^p(Q_l \times [-1,1])} \lesssim \|g_\theta \chi_{Q_l}\|_{L^q} \text{ for } q \in \{2; p\}$$

is trivial due to the normalized domain of integration. Due to $q \leq p$, we find

$$\begin{aligned} \|\tilde{T}^1 g_\theta\|_{L^p(C_{\lambda^{1/2}})} &\lesssim_\varepsilon \lambda^\varepsilon \left(\sum_l \|g_\theta \chi_{Q_l}\|^p_{L^q} \right)^{1/p} + \lambda^{-N} \|g_\theta\|_2 \\ &\lesssim_\varepsilon \lambda^\varepsilon \|g_\theta\|_{L^q} + \lambda^{-N} \|g_\theta\|_2. \end{aligned} \tag{7.51}$$

We find from inverting the change of coordinates

$$\|g_\theta\|_{L^p} = \lambda^{-\frac{n}{2p}} \|f_\theta\|_{L^p}.$$

To prove (7.49) after using (7.48) and (7.51) with $q = 2$, we find

$$\begin{aligned} \|T^\lambda f\|_{L^p(B_\lambda)} &\lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \left(\sum_{\theta:\lambda^{-1/2}-balls} \|T^\lambda f_\theta\|^2_{L^p(B_\lambda)} \right)^{1/2} + \lambda^{-M} \|f\|_2 \\ &\lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \lambda^{\frac{n+2}{2p}} \left(\sum_{\theta:\lambda^{-1/2}-balls} \|\tilde{T}^1 g_\theta\|^2_{L^p(C_{\lambda^{1/2}})} \right)^{1/2} \\ &\quad + \lambda^{-M} \|f\|_2 \\ &\lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \lambda^{\frac{n+2}{2p}} \lambda^{-n/4} \left(\sum_{\theta:\lambda^{-1/2}-balls} \|f_\theta\|^2_{L^2} \right)^{1/2} + \lambda^{-M} \|f\|_2 \\ &\lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \lambda^{\frac{n+2}{2p}} \lambda^{-n/4} \|f\|_2, \end{aligned}$$

where in the ultimate estimate almost orthogonality is applied. For the proof of (7.50) let

$$(P_\theta f)\hat{}(\xi) = \chi_\theta \hat{f}(\xi),$$

where χ is a smooth version of the characteristic function on θ. Following along the above lines gives together with Hölder's inequality

$$\begin{aligned} \|T^\lambda \hat{f}\|_{L^p(B_\lambda)} &\lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \lambda^{1/p} \left(\sum_{\theta:\lambda^{-1/2}-balls} \|P_\theta f\|^2_{L^p} \right)^{1/p} + \lambda^{-M} \|f\|_2 \\ &\lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \lambda^{1/p} \lambda^{\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \left(\sum_\theta \|P_\theta f\|^p_{L^p} \right)^{1/p} + \lambda^{-M} \|f\|_2 \\ &\lesssim_{\varepsilon,\phi,M,N} \lambda^{\beta(p,k)+\varepsilon} \lambda^{1/p} \lambda^{\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^p} + \lambda^{-M} \|f\|_2, \end{aligned}$$

where the estimate was concluded by the following estimate

$$\left(\sum_\theta \|P_\theta f\|_{L^p}^p \right)^{1/p} \lesssim_p \|f\|_{L^p},$$

which holds for $2 \leq p < \infty$. This can be inferred from Plancherel's theorem for $p = 2$ and from Young's inequality for $p = \infty$. The remaining cases follow from interpolation. □

7.4 L^p-smoothing estimates for elliptic phase functions with variable coefficients

Purpose of this section is to point out how the frequency localized estimates from [RS10], which was proved via bilinear methods, generalize to the variable-coefficient case.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth elliptic phase function and χ be a smooth function supported in $\mathcal{U} \subseteq Q(2)$, which denotes the cube with side-length 2 centered at the origin. Let S denote the operator

$$Sf(t,x) = S_\chi^\varphi f(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \chi(\xi) e^{it\varphi(\xi)} \hat{f}(\xi) e^{i\langle x, \xi \rangle} d\xi$$

so that $u = Sf$ describes the solution to the dispersive PDE

$$\begin{cases} i\partial_t u + \varphi(\nabla/i)u &= 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0) &= f, \end{cases}$$

where f has Fourier support in $\mathcal{U}$.

In [RS10] was shown the following estimate:

Proposition 7.4.1 ([RS10, Proposition 3.1., p. 51])**.** *Let $p > 2 + \frac{4}{n+1}$, $\chi \in C_0^\infty(\mathcal{U})$, and let φ be an elliptic phase function on $\mathcal{U}$. Then, we find the following estimate to hold:*

$$\|Sf\|_{[-\lambda,\lambda] \times L^p(\mathbb{R}^n)} \lesssim_\varphi \lambda^{n(1/2-1/p)} \|f\|_{L^p(\mathbb{R}^n)}.$$

By rescaling and interpolation this estimate implies L^p-smoothing estimates for Schrödinger-like equations, which are sharp with respect to the Sobolev regularity of the initial data (cf. [RS10, Section 2]):

Theorem 7.4.2. *Let $p \in (2 + 4/(n+1), \infty)$ and $\alpha > 1$. Then, for any compact time interval I,*

$$\left(\int_I \|e^{itD^\alpha} f\|_{L^p}^p dt \right)^{1/p} \leq C_{I,p,\alpha} \|f\|_{L^p_\beta(\mathbb{R}^n)}, \quad \frac{\beta}{\alpha} = n \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{1}{p}.$$

This section is devoted to the proof of the variable-coefficient generalization of Proposition 7.4.1.

In [RS10] Proposition 7.4.1 was proven by the bilinear adjoint Fourier restricition theorem for elliptic surfaces by Tao (cf. [Tao03]) and orthogonality considerations. Purpose of this section is to prove the same result for variable coefficients as stated in Theorem 7.1.3.

For the proof we reduce to normalized phase functions by parabolic rescaling. It is enough to prove the following proposition:

Proposition 7.4.3. *Let $p > 2 + \frac{4}{n+1}$ and ϕ be a normalized elliptic phase function giving rise to the Hörmander operator T^λ. Then, for any $\varepsilon > 0$, we find the following estimate to hold:*

$$\|T^\lambda \hat{f}\|_{L^p(\mathbb{R}^{n+1})} \lesssim_\varepsilon \lambda^{n(1/2-1/p)+\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}. \tag{7.52}$$

For a phase function ϕ satisfying $H1)$ let $q(t,x;\xi) = \partial_t \phi(t,x;[\partial_x \phi(t,x;\cdot)]^{-1}(\xi))$ denote the local parametrization of the surface associated to the adjoint Fourier restriction operator. To prove the L^p-smoothing estimate for variable-coefficient phase functions, crucial use is made of the following bilinear estimate using transversality, which is a generalization of the theorem from [Tao03]:

Theorem 7.4.4 ([Lee06b, Theorem 1.1, p. 58])**.** *For $i = 1, 2$, let ϕ_i be smooth functions satisfying $H1)$. Suppose that the Hessian $\partial^2_{\xi\xi} q_i$ satisfies*

$$\det \partial^2_{\xi\xi} q_i(t,x;\partial_x \phi_i(t,x;\xi_i)) \neq 0$$

on the support of a_i and for $(t,x;\xi_i) \in supp(a_i)$

$$\begin{aligned} &|\langle \partial^2_{x\xi}\phi_i(t,x;\xi_i)\delta(t,x;\xi_1,\xi_2), \\ &\quad [\partial^2_{x\xi}\phi_i(t,x;\xi_i)]^{-1}[\partial^2_{\xi\xi}q_i(t,x;u_i)]^{-1}\delta(t,x;\xi_1,\xi_2)\rangle| \geq c > 0 \end{aligned}$$

for $i = 1, 2$, where $u_i = \partial_x \phi_i(t,x;\xi_i)$ and

$$\delta(t,x;\xi_1,\xi_2) = \partial_\xi q_1(t,x;u_1) - \partial_\xi q_2(t,x;u_2).$$

Then, for any $\varepsilon > 0$, there is a constant $C = C(\varepsilon)$ such that for $p \geq (n+3)/(n+1)$,

$$\|T_1^\lambda f T_2^\lambda g\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \leq C_\varepsilon \lambda^\varepsilon \|f\|_2 \|g\|_2,$$

where T_i^λ denotes the Hörmander-operator associated to the data (ϕ_i, a_i).

For normalized elliptic phase functions this yields the following corollary:

Corollary 7.4.5. *Let ϕ be a normalized elliptic phase function and suppose for $f_1, f_2 \in L^2(\mathbb{R}^n)$ with $\operatorname{supp}(f_i) \subseteq Q(2)$. Then, for any $\varepsilon > 0$ there is C_ε such that for $p \geq (n+3)/(n+1)$ we find the following estimate to hold:*

$$\|T^\lambda f_1 T^\lambda f_2\|_{L^p(\mathbb{R}^n\times\mathbb{R})} \leq C_\varepsilon \lambda^\varepsilon \|f_1\|_2 \|f_2\|_2.$$

Proof. We check that the assumptions of the more general Theorem 7.4.4 are satisfied: Indeed, from normalization we infer $\partial^2_{x\xi}\phi \sim I_n$, $\delta(x,t;\xi_1,\xi_2) \sim \xi_1 - \xi_2$, $\partial^2_{\xi\xi} q_i \sim I_n$ and consequently,

$$\begin{aligned} &|\langle \partial^2_{x\xi}\phi(x,t;\xi_i)\delta(x,t;\xi_1,\xi_2), [\partial^2_{x\xi}\phi(x,t;\xi_i)]^{-1}[\partial^2_{\xi\xi}q_i(x,t;u_i)]^{-1}\delta(x,t;\xi_1,\xi_2)\rangle| \\ &\sim |\xi_1 - \xi_2|^2 \geq c^2. \end{aligned}$$

This yields the claim. □

Due to a localization property of the kernel, this yields the following L^p-estimates.

Lemma 7.4.6. *Let $p > \frac{2(n+3)}{n+1}$, $B_1, B_2 \subseteq Q(1)$ balls with dist$(B_1, B_2) \geq c > 0$ and ϕ be a normalized elliptic phase function. Then, for f_1, f_2 with supp$(\hat{f}_i) \subseteq B_i$, $i = 1, 2$, we find the following estimate to hold:*

$$\|T^\lambda \hat{f}_1 T^\lambda \hat{f}_2\|_{L^{p/2}(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{p,\varepsilon} \lambda^{n(1-2/p)+\varepsilon} \|f_1\|_{L^p(\mathbb{R}^n)} \|f_2\|_{L^p(\mathbb{R}^n)}.$$

Proof. First, we note the localization property for

$$K(t, x; y) = a_0^\lambda(t, x) \int_{\mathbb{R}^n} e^{i(\phi^\lambda(t,x;\xi) - \langle y,\xi\rangle)} \beta(\xi) d\xi.$$

In fact,

$$\begin{aligned}
\partial_\xi[\phi^\lambda(t, x; \xi) - \langle y, \xi\rangle] &= \partial_\xi \phi^\lambda(0, x; \xi) + \int_0^{t/\lambda} \partial_t \partial_\xi \phi(t', x/\lambda; \xi) dt' - y \\
&= x - y + \lambda \int_0^{t/\lambda} dt' \int_0^1 ds \partial_t \partial^2_{\xi\xi} \phi(t', x/\lambda; s\xi) \cdot \xi \\
&= x - y + O(\lambda).
\end{aligned}$$

Consequently, integration by parts yields that there is a constant C which depends only on the constants in the definition of normalized elliptic phase functions so that for $|y| \geq C\lambda$ we find $|K(t, x; y)| \leq C_N \lambda^{-N} |x - y|^{-N}$ from integration by parts. Let χ_Q be a smooth function which is essentially supported on $Q(C\lambda)$ with Fourier support in $B(0, c/10)$ and decompose $f_i = \chi_Q f_i + f_{i2} = f_{i1} + f_{i2}$.

From trivial kernel estimates we find

$$\|T^\lambda \hat{f}_{i1}\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \leq \lambda^{C'} \|f_i\|_{L^p(\mathbb{R}^n)},$$

where $C' = C(n, p)$ and from the kernel estimate and Young's inequality

$$\begin{aligned}
\|T^\lambda \hat{f}_{i2}\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} &= \left\| \int K(t, x; y) f_{i2}(y) dy \right\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \\
&\leq C_N \lambda^{-N} \|f_{i2}\|_{L^p(\mathbb{R}^n)} \leq C_N \lambda^{-N} \|f_i\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

We split

$$\begin{aligned}
\|T^\lambda \hat{f}_1 T^\lambda \hat{f}_2\|_{L^{p/2}(\mathbb{R}\times\mathbb{R}^n)} &\leq \|T^\lambda \hat{f}_{11} T^\lambda \hat{f}_{21}\|_{L^{p/2}(\mathbb{R}\times\mathbb{R}^n)} + \|T^\lambda \hat{f}_{12} T^\lambda \hat{f}_{21}\|_{L^{p/2}(\mathbb{R}\times\mathbb{R}^n)} \\
&\quad + \|T^\lambda \hat{f}_{11} T^\lambda \hat{f}_{22}\|_{L^{p/2}(\mathbb{R}\times\mathbb{R}^n)} + \|T^\lambda \hat{f}_{12} T^\lambda \hat{f}_{22}\|_{L^{p/2}(\mathbb{R}\times\mathbb{R}^n)} \\
&\leq I + II + III + IV.
\end{aligned}$$

II, III, IV are estimated by the above elementary estimates. For instance,

$$II \leq \|T^\lambda \hat{f}_{12}\|_{L^p} \|T^\lambda \hat{f}_{21}\|_{L^p} \leq C_N \lambda^{-N} \|f_1\|_{L^p} \lambda^{C'} \|f_2\|_{L^p}.$$

Only for I we have to invoke the bilinear estimate from above. Due to the support properties of χ_Q we can still use Corollary 7.4.5

$$\begin{aligned}
\|T^\lambda \hat{f}_{11} T^\lambda \hat{f}_{21}\|_{L^{p/2}(\mathbb{R}^n\times\mathbb{R})} &\lesssim_\varepsilon \lambda^\varepsilon \|\widehat{\chi_Q f_1}\|_{L^2} \|\widehat{\chi_Q f_2}\|_{L^2} \\
&\leq C_\varepsilon \lambda^\varepsilon \|\chi_Q f_1\|_{L^2} \|\chi_Q f_2\|_{L^2} \\
&\leq C_\varepsilon \lambda^\varepsilon \lambda^{n(1-2/p)} \|f_1\|_{L^p(\mathbb{R}^n)} \|f_2\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

and the proof is complete. ∎

Lemma 7.4.6 is the most important building block for further arguments. To utilize it efficiently, we carry out a Whitney decomposition of $T^\lambda \hat{f}_1 T^\lambda \hat{f}_2$ in terms of frequency support and separation.

Precisely, let

$$B(1) \times B(1) \backslash D = \bigcup_{0 \leq j \leq j'} \bigcup_{\mathrm{dist}(Q^j_{k_1}, Q^j_{k_2}) \sim 2^{-j}} Q^j_{k_1} \times Q^j_{k_2},$$

where for $0 \leq j < j' = \frac{1}{2} \log_2 \lambda^{1/2}$ the cubes $Q^j_{k_1}, Q^j_{k_2}$ have side length comparable to 2^{-j} and are separated with a distance comparable to 2^{-j}. For $j = j'$ the separation is less or equal to 2^{-j}.

Adapted to the Whitney-decomposition, we write

$$\begin{aligned}(T^\lambda \hat{f})^2 &= \sum_{0 \leq j \leq j'} \sum_{k_1, k_2} T^\lambda(\beta^j_{k_1} \hat{f}) T^\lambda(\beta^j_{k_2} \hat{f}) \\ &=: \sum_{0 \leq j \leq j'} \tilde{\mathcal{B}}_j[f, f],\end{aligned}$$

where the β^j_k denote a smooth partition of unity adapted to the Whitney-decomposition.

For $j < j'$ we prove the following estimate by scaling:

Lemma 7.4.7. *Let ϕ be a normalized elliptic phase function and f_i have Fourier support in cubes of side length 2^{-j} which are also separated with a distance of 2^{-j}. Then, we find the following estimate to hold provided that $1 \leq 2^{2j} \leq \lambda$:*

$$\| T^\lambda \hat{f}_1 T^\lambda \hat{f}_2 \|_{L^{p/2}} \lesssim_\varepsilon \lambda^\varepsilon 2^{\frac{4j}{p}} (\lambda / 2^{2j})^{n(1-2/p)} \| f_1 \|_{L^p(\mathbb{R}^n)} \| f_2 \|_{L^p(\mathbb{R}^n)}. \tag{7.53}$$

Proof. We use parabolic rescaling to write

$$\begin{aligned}&\| T^\lambda \hat{f}_1 T^\lambda \hat{f}_2 \|_{L^{p/2}(\mathbb{R}^{n+1})} \\ &= 2^{\frac{2jn}{p}} 2^{-2jn} 2^{\frac{4j}{p}} \| \int e^{i\phi^\lambda(2^{2j}t', 2^j y, \xi_0 + 2^{-j}\xi')} a^\lambda(2^{2j}t', 2^j y; \xi_0 + 2^{-j}\xi') \hat{f}_1(\xi_0 + 2^{-j}\xi') d\xi' \\ &\quad \int e^{i\phi^\lambda(2^{2j}t, 2^j y; \xi_0 + 2^{-j}\xi')} a^\lambda(2^{2j}t', 2^j y; \xi_0 + 2^{-j}\xi') \hat{f}_2(\xi_0 + 2^{-j}\xi') d\xi' \|_{L^{p/2}(\mathbb{R}^{n+1})} \\ &= 2^{-2jn} 2^{\frac{2jn}{p}} 2^{\frac{4j}{p}} \| \int e^{i\tilde{\phi}^{\lambda/2^{2j}}(t', y; \xi')} \tilde{a}^{\lambda/2^{2j}}(t', y; \xi') \hat{g}_1(\xi') d\xi' \\ &\quad \int e^{i\tilde{\phi}^{\lambda/2^{2j}}(t', y; \xi')} \tilde{a}^{\lambda/2^{2j}}(t', y; \xi') \hat{g}_2(\xi') d\xi' \|_{L^{p/2}(\mathbb{R}^n \times \mathbb{R})}.\end{aligned} \tag{7.54}$$

Due to $\hat{g}_i(\xi) = \hat{f}_i(\xi_0 + 2^{-j}\xi)$, we have $\|g_i\|_{L^p} = 2^{nj} 2^{-\frac{nj}{p}} \|f_i\|_{L^p}$. We argue that the latter expression is amenable to Lemma 7.4.6: Note that

$$\mathrm{supp}(\tilde{a}^{\lambda/2^{2j}}(t, \cdot; \xi)) \subseteq Q(\lambda/2^j).$$

However, we want to apply Lemma 7.4.6 with parameter $\lambda/2^{2j}$. For this purpose we use again the localization property of the kernel. Note that this holds for $\tilde{\phi}^{\lambda/2^{2j}}$ independently of x and in particular, there is no renormalization with an affine change of coordinates necessary for this property to hold.

Cover $Q(\lambda/2^j)$ by $\lambda/2^{2j}$ cubes Q_l and let $g = \sum_l \chi_{Q_l} g$, where χ_{Q_l} are smooth functions essentially supported on Q_l with $\sum \chi_{Q_l} = 1$ and

$$\mathrm{supp}(\widehat{\chi_{Q_l}}) \subseteq B(0,c) \tag{7.55}$$

for some $c \ll 1$. This is possible due to the Poisson summation formula and $\lambda/2^{2j} \geq 1$.

Effectively, we work with the decomposition

$$g = \sum^{N} \chi_{Q_l} g + \chi_{Q^*} g, \quad \chi_{Q^*} g = \sum_{Q_l \cap Q(\lambda^{1+\varepsilon}/2^j) = \emptyset} \chi_{Q_l} g.$$

Like in the proof of Lemma 7.4.6, we find that the terms containing $\chi_{Q^*} g$ can safely be neglected.

In fact, we find the main contribution of (7.54) raised to $p/2$ to be given by

$$\sum_{\substack{Q_k, l, m: \\ Q_l \cap \lambda^\varepsilon Q_k \neq \emptyset, \\ Q_m \cap \lambda^\varepsilon Q_k \neq \emptyset}} \left\| \tilde{T}^{\lambda/2^{2j}}(\widehat{\chi_{Q_l} g_1}) \tilde{T}^{\lambda/2^{2j}}(\widehat{\chi_{Q_m} g_2}) \right\|^{p/2}_{L^{p/2}([-\lambda/2^{2j}, \lambda/2^{2j}] \times Q_k)} \tag{7.56}$$

The conditions on Q_k, l, m in the above display are denoted by $Q_k, l, m : (*)$ in the following.

From (7.55) the supports of $\widehat{\chi_{Q_l} g_1}$ and $\widehat{\chi_{Q_m} g_2}$ are still separated of unit order, and after the change of variables $x \to x_0 + x$, we find

$$\begin{aligned} &\|\tilde{T}^{\lambda/2^{2j}}(\widehat{\chi_{Q_l} g_1}) \tilde{T}^{\lambda/2^{2j}}(\widehat{\chi_{Q_m} g_2})\|_{L^{p/2}([-\lambda/2^{2j}, \lambda/2^{2j}] \times Q_k)} \\ &= \|\tilde{T}_k^{\lambda/2^{2j}}(e^{i\tilde{\phi}} \widehat{\chi_{Q_l} g_1}) \tilde{T}_k^{\lambda/2^{2j}}(e^{i\tilde{\phi}} \widehat{\chi_{Q_m} g_2})\|_{L^{p/2}([-\lambda/2^{2j}, \lambda/2^{2j}] \times Q_0)}, \end{aligned} \tag{7.57}$$

where $\phi_k = \tilde{\phi}(t, x + x_0; \xi) - \tilde{\phi}(0, x_0; \xi)$.

Note that the ϕ_k are normalized elliptic phase functions after an additional affine transformation, which is close to the identity mapping. Hence, (7.57) is amenable to Lemma 7.4.6 with parameter $\lambda/2^{2j}$, and we conclude the bound by means of Cauchy-Schwarz:

$$\begin{aligned} &\sum_{Q_k, l, m:(*)} \|\tilde{T}^{\lambda/2^{2j}}(\widehat{\chi_{Q_l} g_1}) \tilde{T}^{\lambda/2^{2j}}(\widehat{\chi_{Q_m} g_2})\|^{p/2}_{L^{p/2}([-\lambda/2^{2j}, \lambda/2^{2j}] \times Q_k)} \\ &\lesssim_\varepsilon \sum_{Q_k, l, m:(*)} (\lambda/2^{2j})^{[n(1-2/p)+\varepsilon](p/2)} \|\chi_{Q_l} g_1\|^{p/2}_{L^p} \|\chi_{Q_m} g_2\|^{p/2}_{L^p} \\ &\lesssim_\varepsilon (\lambda/2^{2j})^{[n(1-2/p)+2\varepsilon](p/2)} \left(\sum_l \|\chi_{Q_l} g_1\|^p_{L^p} \right)^{1/2} \left(\sum_m \|\chi_{Q_m} g_2\|^p_{L^p} \right)^{1/2} \\ &\lesssim_\varepsilon (\lambda/2^{2j})^{[n(1-2/p)+2\varepsilon](p/2)} \|g_1\|^{p/2}_{L^p} \|g_2\|^{p/2}_{L^p}, \end{aligned}$$

and the proof is complete. □

For possibly vanishing separation for cubes with side length $\lambda^{-1/2}$ we have the following lemma:

Lemma 7.4.8. *Let supp($\hat{f}$) $\subseteq B$, where B is a ball of radius $\lambda^{-1/2}$. Then, we find the following estimate to hold:*

$$\|T^\lambda \hat{f}\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \lesssim_\varepsilon \lambda^\varepsilon \lambda^{1/p} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We use parabolic rescaling to write like in the proof of Lemma 7.4.7

$$\|T^\lambda \hat{f}\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \lesssim \lambda^{-n/2}\lambda^{n/2p}\lambda^{1/p}\|\tilde{T}^1\hat{g}\|_{L^p(\mathbb{R}\times\mathbb{R}^n)},$$

where $\|g\|_{L^p} = \lambda^{n/2}\lambda^{-n/2p}\|f\|_{L^p}$. It is enough to prove

$$\|\tilde{T}^1\hat{g}\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \lesssim_\varepsilon \lambda^\varepsilon \|g\|_{L^p(\mathbb{R}^n)}.$$

Again, we use the localization property of the kernel to consider cubes of unit size and adapted functions with localized Fourier transform χ_Q. Decompose like in the proof of Lemma 7.4.7

$$g = \sum_{l: Q_l \cap Q(\lambda^{1/2+\varepsilon}) \neq \emptyset} \chi_{Q_l} g + \chi_{Q^*} g.$$

Like in the proof of Lemma 7.4.7 one finds

$$\|\tilde{T}^1\hat{g}\|^p_{L^p([-1,1]\times Q)} \lesssim \sum_{Q,l} \|\tilde{T}^1\widehat{\chi_{Q_l} g}\|^p_{L^p([-1,1]\times Q)} + C_N \lambda^{-N}\|g\|_{L^p}.$$

The proof is concluded following along the lines of the proof of Lemma 7.4.7. □

To separate the contribution of the different k_i, we use a natural orthogonality of the oscillatory integrals, which was used in a similar context in [Lee06b].

Lemma 7.4.9 ([Lee06b, Lemma 3.3, p. 82])**.** *Let ϕ^λ be a normalized elliptic phase function, $1 \leq p/2 \leq 2$ and $j \geq \frac{1}{2}\log_2 \lambda$. Then, we find the following estimate to hold:*

$$\begin{aligned}
&\Big\| \sum_{\mathrm{dist}(Q^j_{k_1}, Q^j_{k_2}) \sim 2^{-j}} T^\lambda(\beta^j_{k_1}\hat{f}) T^\lambda(\beta^j_{k_2}\hat{f}) \Big\|_{p/2} \\
&\leq C_\varepsilon \lambda^\varepsilon \Big(\sum_{k_1,k_2} \|T^\lambda(\beta^j_{k_1}\hat{f}) T^\lambda(\beta^j_{k_2}\hat{f})\|^{p/2}_{p/2} \Big)^{2/p} \\
&\quad + C_N \lambda^{-N} \|f\|_p^2.
\end{aligned}$$

For the error term recall that we can always suppose that f in position space is localized to a cube of length λ from the kernel estimate. Hence, using the estimate from [Lee06b, Lemma 3.3] with $r = 2$ yields the claim after applications of Hölder's inequality and Plancherel's theorem.

We prove the following bound for $\tilde{\mathcal{B}}_j$.

Proposition 7.4.10. *Let $0 \leq j \leq \frac{1}{2}\log_2(\lambda)$ and $\tilde{B}$ like above. Then, we find the following estimate to hold:*

$$\|\tilde{\mathcal{B}}_j[f,f]\|_{L^{p/2}} \lesssim_\varepsilon \begin{cases} \lambda^\varepsilon 2^{4j/p}(\lambda/2^{2j})^{n(1-2/p)}\|f\|^2_{L^p}, & 2 + \frac{n+4}{n+1} \leq p \leq 4, \\ \lambda^\varepsilon 2^{4j/p}(\lambda/2^{2j})^{n(1-2/p)}\|f\|^2_{L^p}, & 4 < p < \infty. \end{cases} \tag{7.58}$$

Proof. We have to distinguish between $1 \le p/2 \le 2$ and $p > 4$. In fact, for $1 \le p/2 \le 2$ we can apply Lemma 7.4.9, but for $p > 4$ we use instead the trivial observation

$$\left\| \sum_{\mathrm{dist}(Q^j_{k_1}, Q^j_{k_2}) \sim 2^{-j}} T^\lambda(\beta^j_{k_1} \hat{f}) T^\lambda(\beta^j_{k_2} \hat{f}) \right\|_{L^\infty} \lesssim 2^{jn} \sup_{k_1,k_2} \| T^\lambda(\beta^j_{k_1} \hat{f}) T^\lambda(\beta^j_{k_2} \hat{f}) \|_{L^\infty}$$

and interpolation yields

$$\begin{aligned} \| \sum T^\lambda(\beta^j_{k_1} \hat{f}) T^\lambda(\beta^j_{k_2} \hat{f}) \|_{L^{p/2}} &\le C_\varepsilon \lambda^\varepsilon 2^{jn(1-4/p)} \\ &\qquad \left(\sum_{k_1,k_2} \| T^\lambda(\beta^j_{k_1} \hat{f}) T^\lambda(\beta^j_{k_2} \hat{f}) \|^{p/2}_{L^{p/2}} \right)^{2/p}. \end{aligned} \tag{7.59}$$

Consequently, for $1 \le p/2 \le 2$:

$$\begin{aligned} &\| \sum_{\mathrm{dist}(Q^j_{k_1}, Q^j_{k_2}) \sim 2^{-j}} T^\lambda(\beta^j_{k_1} \hat{f}) T^\lambda(\beta^j_{k_2} \hat{f}) \|_{L^{p/2}} \\ &\lesssim_\varepsilon \lambda^\varepsilon \left(\sum_{dist \sim 2^{-j}} \| T^\lambda(\beta^j_{k_1} \hat{f}) T^\lambda(\beta^j_{k_2} \hat{f}) \|^{p/2}_{L^{p/2}} \right)^{2/p} \\ &\lesssim_\varepsilon \lambda^{2\varepsilon} 2^{4j/p} (\lambda/2^{2j})^{n(1-2/p)} \left(\sum_{k_1,k_2} \| P^j_{k_1} f \|^{p/2}_{L^p} \| P^j_{k_2} f \|^{p/2}_{L^p} \right)^{2/p} \\ &\lesssim_\varepsilon \lambda^{2\varepsilon} 2^{4j/p} (\lambda/2^{2j})^{n(1-2/p)} \left(\sum_{k_1} \| P^j_{k_1} f \|^p_{L^p} \right)^{1/p} \left(\sum_{k_2} \| P^j_{k_2} f \|^p_{L^p} \right)^{1/p} \\ &\le C_\varepsilon \lambda^{2\varepsilon} 2^{4j/p} (\lambda/2^{2j})^{n(1-2/p)} \| f \|_{L^p} \| f \|_{L^p} \end{aligned}$$

and for $p > 4$ by the above means, but appealing to (7.59) than Lemma 7.4.9,

$$\begin{aligned} \left\| \sum_{\mathrm{dist}(Q^j_{k_1}, Q^j_{k_2}) \sim 2^{-j}} T^\lambda(\beta^j_{k_1} \hat{f}) T^\lambda(\beta^j_{k_2} \hat{f}) \right\|_{L^{p/2}} &\lesssim_\varepsilon \lambda^{2\varepsilon} 2^{jn(1-4/p)} \\ &\qquad 2^{4j/p} (\lambda/2^{2j})^{(1-2/p)} \| f \|^2_{L^p}. \end{aligned}$$

□

Proof of Proposition 7.4.3. We conclude that for small p the main contribution comes from $j = 0$ and summing a geometric series yields the bound

$$\begin{aligned} \| T^\lambda \hat{f} \|_{L^p} = \| (T^\lambda \hat{f})^2 \|^{1/2}_{L^{p/2}} &\le \left(\sum_{0 \le j \le j'} \| \tilde{B}_j[f,f] \|_{L^{p/2}} \right)^{1/2} \\ &\lesssim_\varepsilon \left(\sum_{0 \le j \le j'} \lambda^\varepsilon \lambda^{n(1-2/p)} 2^{4j/p} 2^{-2j(n(1-2/p))} \| f \|^2_{L^p} \right)^{1/2} \\ &\lesssim \lambda^\varepsilon \lambda^{n(1/2-1/p)} \| f \|_{L^p}. \end{aligned}$$

The same argument works for $p \to \infty$ and interpolation between the bounds for small p and large p proves the claim. $\square$

Conclusively, we argue how Proposition 7.4.3 implies Theorem 7.1.3.

Proposition 7.4.3 ⇒ Theorem 7.1.3. Let $\hat{f}$ be supported in a ρ^{-1}-ball centered at ρ_0. It follows from parabolic rescaling (cf. Section 7.2)

$$e^{i\lambda\phi(\Phi_{\xi_0}(\rho^2 t/\lambda, \rho x/\lambda);\xi_0)} T^\lambda \hat{f} \circ \Phi^\lambda_{\xi_0} \circ D_\rho = \tilde{T}^{\lambda/\rho^2} \hat{g},$$

where

$$\tilde{T}^{\lambda/\rho^2} \hat{g}(t,x) = \int_{\mathbb{R}^n} e^{i\tilde{\phi}^{\lambda/\rho^2}(t,x;\xi)} \tilde{a}^{\lambda/\rho^2}(t,x;\xi) \hat{g}(\xi) d\xi,$$

and the phase $\tilde{\phi}$ is given by

$$\langle x, \xi \rangle + \int_0^1 (1-r) \langle \partial^2_{\xi\xi} \phi(\Phi_{\xi_0}(t, D'_{\rho^{-1}} x), \xi_0 + r\xi/\rho)\xi, \xi \rangle d\xi.$$

A change of spatial variables gives

$$\| T^\lambda \hat{f} \|_{L^p(\mathbb{R} \times \mathbb{R}^n)} \lesssim_\phi \rho^{\frac{n+2}{p}} \rho^{-n} \| \tilde{T}^{\lambda/\rho^2} \hat{g} \|_{L^p((\Phi^\lambda_{\xi_0} \circ D_\rho)^{-1}(\mathbb{R} \times \mathbb{R}^n))},$$

where like above $\|g\|_{L^p} = \rho^n \rho^{-n/p} \|f\|_{L^p}$.

The support of $\tilde{T}^{\lambda/\rho^2} \hat{g}$ is essentially a $\lambda/\rho \times \ldots \times \lambda/\rho \times \lambda/\rho^2$-ellipse. As argued in Section 7.2, $\tilde{\phi}$ is up to an affine transformation a normalized elliptic phase function. Note that in the current context, we choose ρ depending on ϕ so that the phase function, we arrive at after rescaling and an additional affine change of coordinates, is actually normalized. Moreover, the magnitude of the Jacobian of Φ_{ξ_0} and L (see Section 7.2 for notation) depend on ϕ. Thus, the implicit constant depends on ϕ contrary to the applications of parabolic rescaling for normalized elliptic phase functions. To utilize Proposition 7.4.3, we perform an additional decomposition of the ellipse into λ/ρ^2-cubes and use again the orthogonality property which follows from the localization property of the kernel. The argument is concluded like in the proofs of Lemma 7.4.7 and 7.4.8. In order to avoid repetition, the details are omitted. $\square$

In the case of constant coefficients it was shown in [RS10] how a localization property of the kernel and an interpolation argument yield globalization.

We point out that the bilinear approach depicted above for variable coefficients also applies to the hyperbolic Schrödinger equation (cf. [Lee06a, Var05]) in two dimensions:

$$\begin{cases} i\partial_t u(t,x) + (\partial_{xx} - \partial_{yy}) u(t,x) &= 0, \ (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0,x) &= u_0(x) \in L^p_\beta(\mathbb{R}^2), \end{cases} \tag{7.60}$$

and using the localization and interpolation argument from [RS10, Section 4] which applies mutatis mutandis yields the following result:

Theorem 7.4.11. *For $p > 10/3$ we find the following estimate to hold:*

$$\| e^{it(\partial_{xx} - \partial_{yy})} u_0 \|_{L^p([0,1] \times \mathbb{R}^2)} \lesssim \| u_0 \|_{L^p_\beta(\mathbb{R}^2)}, \quad \beta \geq 4(\frac{1}{2} - \frac{1}{p}) - \frac{2}{p} \tag{7.61}$$

This covers the endpoint Sobolev regularity, which is not covered in [Rog08].

Summary

A detailed survey and motivation for our work on quasilinear dispersive equations is provided in the introductory Chapter 1.

In Chapter 2 notation is fixed. Function spaces, which are suitable for the analysis in the following chapters, are revisited. References for the function spaces are [IKT08, KT07, HHK09, CHT12].

Chapter 3 starts with a discussion of linear and bilinear Strichartz estimates. New linear Strichartz estimates on tori are derived via the decoupling inequalities from [BD15, BD17a].

We discuss how frequency dependent time localization allows one to recover linear and bilinear Strichartz estimates on tori. Here, we elaborate on previous works (cf. [BGT04, MV08, Han12, Din17]). Results from these works are recast and modestly generalized in a form, which is useful for our purposes.

Next, we point out how to overcome derivative loss and to improve the energy method (cf. [BS75]) via a novel combination of short-time bilinear Strichartz estimates with a well-known iteration of perturbative and energy arguments (cf. [IKT08]). The results from Chapter 3 were made publicly available in [Sch19b] and [Sch18].

In Chapter 4 the argument to improve the energy method from the previous chapter is applied to higher-dimensional Benjamin-Ono equations and fractional variants, which relate to the Zakharov-Kuznetsov equation. Previous local well-posedness results (cf. [BJUM19, LRRW19, LPRT19]) are improved and unified using transversality. The results were made publicly available in [Sch19c].

In Chapter 5 dispersive equations with cubic derivative nonlinearities on the circle are analyzed with short-time analysis. New a priori estimates and existence results are proved for solutions to the modified Benjamin-Ono equation, the derivative nonlinear Schrödinger equation and the modified Korteweg-de Vries equation. Here, the arguments from Chapters 3 and 4 are combined with the introduction of correction terms for the Sobolev energies of the solutions. This argument is closely related to normal form transformations and the I-method (cf. [CKS$^+$02, CKS$^+$03]).

In case of quadratic dispersion relations the regularity results previously known on the real line (cf. [Guo11]) are extended to the circle. In addition, for the focusing modified Korteweg-de Vries equation new existence results are shown in Sobolev spaces with positive regularity index. In the defocusing case this was previously known making use of complete integrability (cf. [KT05a, Mol12]).

Conditional upon conjectured Strichartz estimates we prove existence of solutions to a certain renormalization of the modified Korteweg-de Vries equation for negative Sobolev regularity. This implies non-existence of solutions to the unrenormalized modified Korteweg-de Vries equation for negative Sobolev regularity. The

results from this chapter were made available in [Sch17a] and [Sch17b].

In Chapter 6 dispersion generalized Benjamin-Ono equations on the circle (cf. [MV15]) are considered. Here, normal form transformations are also utilized to analyze differences of solutions. New local and global well-posedness results are proved. The analysis was made available in [Sch19a].

In Chapter 7 oscillatory integral operators related to the short-time linear evolution of dispersive equations on compact manifolds (cf. [ST02, BGT04]) are discussed. We prove a variable-coefficient decoupling theorem and new L^p-smoothing estimates. The results extend previous theorems for constant coefficients (cf. [BD17a, RS10]) to variable coefficients.

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