Canad. Math. Bull. Vol. **61** (3), 2018 pp. 543–552
http://dx.doi.org/10.4153/CMB-2017-049-5
© Canadian Mathematical Society 2018

Lie Derivatives and Ricci Tensor on Real Hypersurfaces in Complex Two-plane Grassmannians

Imsoon Jeong, Juan de Dios Pérez, Young Jin Suh, and Changhwa Woo

Abstract. On a real hypersurface M in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ we have the Lie derivation $\mathcal{L}$ and a differential operator of order one associated with the generalized Tanaka–Webster connection $\widehat{\mathcal{L}}^{(k)}$. We give a classification of real hypersurfaces M on $G_2(\mathbb{C}^{m+2})$ satisfying $\widehat{\mathcal{L}}^{(k)}_{\xi} S = \mathcal{L}_{\xi} S$, where ξ is the Reeb vector field on M and S the Ricci tensor of M.

1 Introduction

It is one of the most classical and interesting parts in differential geometry to find geometric properties of submanifolds on a symmetric space equipped with a Kähler structure J, *i.e.,* a Hermitian symmetric space. Among Hermitian symmetric spaces as a higher rank space of complex projective space $P_n(\mathbb{C})$, the authors have investigated the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, which consists of the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The space $G_2(\mathbb{C}^{m+2})$ is diffeomorphic to the homogeneous space $SU_{m+2}/S(U_2 \cdot U_m)$, the special unitary group SU_{m+2} acts transitively on $\mathbb{C}^{m+2}$, and $S(U_2 \cdot U_m)$ means the isotropic subgroup of SU_{m+2}. Cartan decomposition of the Lie algebra of $S(U_2 \cdot U_m)$ is expressed by $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$. We have a Kähler structure J from $\mathfrak{u}_1$, the one-dimensional center of $\mathfrak{k}$. Remarkably, we also have a quaternionic Kähler structure $\mathfrak{J}$ from $\mathfrak{su}_2$ satisfying $JJ_\nu = J_\nu J$ ($\nu = 1, 2, 3$), where $\{J_\nu\}_{\nu=1,2,3}$ is an orthonormal basis of $\mathfrak{J}$. When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper we assume $m \geq 3$.

To classify real hypersurfaces with certain geometric conditions, let us give a explanation of the geometry of real hypersurfaces on $G_2(\mathbb{C}^{m+2})$. Let us consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ and let N denote a local unit normal vector field on M

Received by the editors March 18, 2017; revised July 31, 2017.
Published electronically June 8, 2018.
This work was supported by Grant Proj. No. NRF-2015-R1A2A1A-01002459 and Proj. No. NRF-2018-R1D1A1B-05040381. Author I. J. is supported by NRF Grant funded by the Korean Government Grant Proj. No. 2017R1A2B4005317. Author J. D. P. is partially supported by MINECO-FEDER Grant MTM2013-47828-C2-1-P. Author C. W. is supported by NRF Grant funded by the Korean Government Grant Proj. No. 2017R1C1B1010265.
AMS subject classification: **53C40**, 53C15.
Keywords: real hypersurface, complex two-plane Grassmannian, Hopf hypersurface, shape operator, Ricci tensor, Lie derivation.

in $G_2(\mathbb{C}^{m+2})$. The *Reeb vector field* $\xi = -JN \in T_pM$ at $p \in M$ is induced from the Kähler structure J. Let $\mathcal{C}$ be the distribution given by the orthogonal complement of $[\xi]$ in T_pM at $p \in M$. If ξ is invariant under the shape operator A, it is said to be *Hopf*. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. It is the complex maximal subbundle of $T_pM = \mathcal{C} \oplus \mathcal{C}^\perp$. The real hypersurface M is said to be *Hopf* if $A\mathcal{C} \subset \mathcal{C}$, or equivalently, the Reeb vector field ξ is principal, where A is the shape operator of the real hypersurface M. If X is a tangent vector on M, we can put

$$JX = \phi X + \eta(X)N \quad \text{and} \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

where ϕX (resp. $\phi_\nu X$) is the tangential part of JX (resp. $J_\nu X$) and $\eta(X) = g(X, \xi)$ (resp. $\eta_\nu(X) = g(X, \xi_\nu)$) is the coefficient of normal part of JX (resp. $J_\nu X$). In this case, we call ϕ the structure tensor field of M. Using the Gauss and Weingarten formulas in [6, Section 1 and 2], the Kähler condition $\bar{\nabla} J = 0$ gives $\nabla_X \xi = \phi AX$ for any tangent vector field X on M, where ∇ (resp. $\bar{\nabla}$) denotes the covariant derivative on M (resp. $G_2(\mathbb{C}^{m+2})$). From this, it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

In this case, the principal curvature $\alpha = g(A\xi, \xi)$ is said to be a *Reeb curvature* of M.

From the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$, there naturally exist *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Now let us denote by $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ a 3-dimensional distribution in the tangent space T_pM at $p \in M$. In addition, $\mathcal{Q}$ stands for the orthogonal complement of $\mathcal{Q}^\perp$ in T_pM. Then it becomes a quaternionic maximal subbundle of T_pM. Thus, the tangent space of M consists of the direct sum of $\mathcal{Q}$ and $\mathcal{Q}^\perp$ as follows: $T_pM = \mathcal{Q} \oplus \mathcal{Q}^\perp$.

For two distributions $\mathcal{C}^\perp$ and $\mathcal{Q}^\perp$ defined above, we can consider two natural invariant geometric properties under the shape operator A of M, that is, $A\mathcal{C}^\perp \subset \mathcal{C}^\perp$ and $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$. The following theorem is from a paper due to Suh [13, Theorem 1.1].

Theorem A *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{Q}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In the case of (A), we want to say M is of Type (A). Similarly, in the case of (B), we say M is of Type (B).

Until now, many geometers have investigated some characterizations of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ that satisfy commuting conditions involving geometric quantities like shape operator, structure (or normal) Jacobi operator, Ricci tensor, and so on. For a tangent vector X, ϕX is the tangential part of JX; then ϕ is said to be the structure tensor field. Commuting Ricci tensor means that the Ricci tensor S and the structure tensor field ϕ commute with each other, that is, $S\phi = \phi S$. From such a point of view, Suh [12] has given a characterization of real hypersurfaces of Type (A) with commuting Ricci tensor.

On the other hand, a Jacobi field along geodesics of a given Riemannian manifold $(\overline{M}, \overline{g})$ is an important tool in the study of differential geometry. It satisfies a well-known differential equation that inspires Jacobi operators. It is defined by $(\overline{R}_X(Y))(p) = (\overline{R}(Y,X)X)(p)$, where $\overline{R}$ denotes the curvature tensor of $\overline{M}$ and X, Y denote any vector fields on $\overline{M}$. It is known to be a self-adjoint endomorphism on the tangent space $T_p\overline{M}$, $p \in \overline{M}$. Clearly, each tangent vector field X to $\overline{M}$ provides a Jacobi operator with respect to X. Thus, the Jacobi operator on a real hypersurface M of $G_2(\mathbb{C}^{m+2})$ with respect to ξ (resp. N) is said to be a *structure Jacobi operator* (resp. *normal Jacobi operator*) and will be denoted by R_ξ (resp. $\overline{R}_N$).

Among many geometric conditions, in this paper we focus on commuting conditions that have a strong relationship with hypersurfaces of tube type when the Reeb vector field ξ belongs to $\mathfrak{Q}^\perp$, that is to say, the commuting conditions between $(1,1)$ type tensor fields on real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ are used to give same results to isometric Reeb flow.

For a commuting problem concerned with structure Jacobi operator R_ξ and structure tensor ϕ of M in $G_2(\mathbb{C}^{m+2})$, that is, $R_\xi\phi = \phi R_\xi$, Suh and Yang [16] gave a characterization of a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$. Also, concerned with a commuting problem for the normal Jacobi operator $\bar{R}_N$, Pérez, Jeong, and Suh [9] gave a characterization of a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$.

Related to the Levi–Civita connection ∇, Tanno [18] introduced the generalized Tanaka–Webster connection (GTW connection) for contact metric manifolds as a generalization of the Tanaka–Webster connection. It is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold (see [17,19]). Then the GTW connection coincides with Tanaka–Webster connection if the associated CR-structure is integrable. Cho defined the GTW connection for a real hypersurface in a Kähler manifold in such a way that

$$\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + \widehat{F}^{(k)}_X Y,$$

where $k(\in \mathbb{R} \setminus \{0\})$ denotes a non-zero constant and $\widehat{F}^{(k)}_X Y$ is defined by

$$\widehat{F}^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y.$$

The skew-symmetric (1,1) type tensor $\widehat{F}^{(k)}_X$ is said to be a *Tanaka–Webster* (or *k-th-Cho*) *operator* with respect to X. In particular, if the real hypersurface satisfies $A\phi + \phi A = 2k\phi$, then the GTW connection $\widehat{\nabla}^{(k)}$ coincides with the Tanaka–Webster connection (see [1,2]).

On the other hand, we have considered real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $(\widehat{\mathcal{L}}^{(k)}_X T)Y = 0$ for any vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$, where $\widehat{\mathcal{L}}^{(k)}$ is the differential operator of order one given by

$$\widehat{\mathcal{L}}^{(k)}_X Y = \widehat{\nabla}^{(k)}_X Y - \widehat{\nabla}^{(k)}_Y X$$

for any vector fields X and Y on M, where T denotes a tensor field of type $(1,1)$.

The torsion of the GTW connection is given by

$$\widehat{\mathcal{T}}^{(k)}(X,Y) = \widehat{F}_X^{(k)}(Y) - \widehat{F}_Y^{(k)}(X).$$

The operator defined by $\widehat{\mathcal{T}}_X^{(k)}(Y) = \widehat{\mathcal{T}}^{(k)}(X,Y)$ is called the *torsion operator associated with* X.

Let S be the Ricci tensor of M. We will consider real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ satisfying

$$\text{(C-1)} \qquad \widehat{\mathcal{L}}_X^{(k)} S = \mathcal{L}_X S,$$

for any vector field X on M. This is equivalent to the fact $\widehat{\mathcal{T}}_X^{(k)} S = S\widehat{\mathcal{T}}_X^{(k)}$, for any X tangent to M.

On the other hand, Hopf hypersurfaces M are those whose Reeb vector field $\xi = -JN$ is Killing or, equivalently, a principal vector field, verifying $A\xi = \alpha\xi$, where the smooth function $\alpha = g(A\xi,\xi)$ is said to be the *Reeb curvature* of the Reeb vector field ξ. Then we can give a classification for M in $G_2(\mathbb{C}^{m+2})$ satisfying (C-1) in the particular case $X = \xi$ as follows.

Theorem 1.1 *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. The Ricci tensor S on M satisfies $\widehat{\mathcal{L}}_\xi^{(k)} S = \mathcal{L}_\xi S$ if and only if M is locally congruent to an open part of a tube of some radius $r \in (0, \frac{\pi}{2\sqrt{2}})$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

In this case, there are two kinds of focal sets in $G_2(\mathbb{C}^{m+2})$, and the distance between them is $\frac{\pi}{2\sqrt{2}}$. By virtue of this Theorem, we give another non-existence property as follows.

Corollary 1.2 *There does not exist any Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying the condition $\widehat{\mathcal{L}}_X^{(k)} S = \mathcal{L}_X S$ for any vector field X on M.*

In this paper, we refer to [6,7,11,12,14,15] for Riemannian geometric structures of a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

2 Proof of Theorem

Let us introduce the Ricci tensor S, briefly. The curvature tensor $R(X,Y)Z$ of M in $G_2(\mathbb{C}^{m+2})$ can be derived from the curvature tensor $\overline{R}(X,Y)Z$ of $G_2(\mathbb{C}^{m+2})$. Then by contracting and using the geometric structure $JJ_\nu = J_\nu J$ $(\nu = 1,2,3)$, we can see the Ricci tensor S given by

$$g(SX,Y) = \sum_{i=1}^{4m-1} g\big(R(e_i,X)Y,e_i\big),$$

where $\{e_1,\ldots,e_{4m-1}\}$ denotes a basis of the tangent space T_pM of M, $p \in M$, in $G_2(\mathbb{C}^{m+2})$ (see [12]). From the definition of the Ricci tensor S and fundamental formulas in [12, section 2], we have

$$
\begin{aligned}
SX &= \sum_{i=1}^{4m-1} R(X, e_i)e_i \\
&= (4m+7)X - 3\eta(X)\xi + hAX - A^2X \\
&\quad + \sum_{\nu=1}^{3}\left\{-3\eta_\nu(X)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X - \eta_\nu(\phi X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\right\},
\end{aligned} \tag{2.1}
$$

where h denotes the trace of A, that is, $h = \mathrm{Tr}A$ (see [10, (1.4)]).

Using equation (2.1), we will prove that the Reeb vector field ξ of M belongs either to $\mathcal{Q}$ or $\mathcal{Q}^\perp$. Under the condition of being Hopf, we get

$$\widehat{F}_\xi^{(k)}X = -k\phi X. \tag{2.2}$$

For $X = \xi$ into (C-1), we have

$$\widehat{F}_\xi^{(k)}(SY) + \phi ASY - S\widehat{F}_\xi^{(k)}(Y) - S\phi AY = 0 \tag{2.3}$$

for any Y tangent to M. Taking the inner product of (2.3) with Z, where Z denotes a vector field tangent to M, we get

$$g(\widehat{F}_\xi^{(k)}(SY), Z) + g(\phi ASY, Z) - g(S\widehat{F}_\xi^{(k)}(Y), Z) - g(S\phi AY, Z) = 0.$$

Bearing in mind that $\widehat{F}_\xi^{(k)}$ is skew-symmetric and S is symmetric, we have

$$g\big(Y, -S\widehat{F}_\xi^{(k)}(Z) - SA\phi Z + \widehat{F}_\xi^{(k)}(SZ) + A\phi SZ\big) = 0.$$

Thus, we have $-S\widehat{F}_\xi^{(k)}(Z)SA\phi Z + \widehat{F}_\xi^{(k)}(SZ) + A\phi SZ = 0$, and, replacing Y by Z, we obtain

$$-S\widehat{F}_\xi^{(k)}(Y) - SA\phi Y + \widehat{F}_\xi^{(k)}(SY) + A\phi SY = 0. \tag{2.4}$$

Using (2.2), (2.3), and (2.4) gives us

$$
\begin{aligned}
-k\phi SY + \phi ASY + kS\phi Y - S\phi AY &= 0, \\
kS\phi Y - SA\phi Y - k\phi SY + A\phi SY &= 0,
\end{aligned} \tag{2.5}
$$

respectively.

By combining these equations, we have

$$S(\phi A - A\phi)Y = (\phi A - A\phi)SY \tag{2.6}$$

for any Y tangent to M.

Lemma 2.1 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M satisfies $\widehat{\mathcal{L}}_\xi^{(k)}S = \mathcal{L}_\xi S$, then ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.*

Proof To show this fact, we consider that the Reeb vector field ξ satisfies

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{2.7}$$

for some unit vectors $X_0 \in \mathcal{Q}$, $\xi_1 \in \mathcal{Q}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$.

Putting $Y=\xi$ in (2.5) and (2.6), by (2.7) and using basic formulas in [5, Section 2], it follows that

$$\begin{aligned}\phi AX_0 &= k\phi X_0,\\ A\phi X_0 &= k\phi X_0.\end{aligned} \tag{2.8}$$

On the other hand, to prove the lemma, we need the following equation:

(2.9)

$$\begin{aligned}\alpha A\phi X+\alpha\phi AX-2A\phi AX+2\phi X = 2\sum_{\nu=1}^{3}\Big\{&-\eta_\nu(X)\phi\xi_\nu-\eta_\nu(\phi X)\xi_\nu\\ &-\eta_\nu(\xi)\phi_\nu X+2\eta(X)\eta_\nu(\xi)\phi\xi_\nu+2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\Big\}\end{aligned}$$

([5, Lemma A]).

Putting $X=X_0$ into (2.9), we have $\alpha k-k^2=\eta^2(X_0)$.

Since k is non-zero constant, differentiating this with respect to ξ, we have

$$\begin{aligned}\xi\alpha &= -\frac{4}{k}\eta(X_0)\big\{g(\nabla_\xi X_0,\xi)+g(X_0,\nabla_\xi\xi)\big\} = -\frac{4}{k}\eta(X_0)g(\nabla_\xi X_0,\xi_1)\\ &= -\frac{4}{k}\eta(X_0)g(X_0,\phi_1 A\xi) = \frac{4}{k}\eta(X_0)\alpha g(X_0,\phi_1\xi)=0\end{aligned}$$

where we have used $\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX$.

This gives $\xi\alpha=0$.

Due to [4, Equation (2.10)], $A\xi_1=\alpha\xi_1$ is derived from $\xi\alpha=0$. Equation (2.8) becomes

$$(\alpha-k)\phi\xi_1=0.$$

As k is nonzero constant and ϕX_0 never vanishes, we have $\alpha=k$. Then by the equation $Y\alpha=(\xi\alpha)\eta(Y)-4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$ in [5, Lemma A], we easily obtain that ξ belongs either to $\mathfrak{Q}$ or to $\mathfrak{Q}^\perp$ (see [10]). ■

Then by Lemma 2.1, we can divide our consideration into two cases being that ξ belongs to either $\mathfrak{Q}^\perp$ or $\mathfrak{Q}$, respectively. Then first we consider the case $\xi\in\mathfrak{Q}^\perp$. We can put $\xi=\xi_1\in\mathfrak{Q}^\perp$ for our convenience sake.

Then [8, lemma 1.2] tells us Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ and $\xi\in\mathfrak{Q}^\perp$ gives $AS=SA$. Thus, (2.6) is changed into

$$\begin{aligned}0 &= S(\phi A-A\phi)Y-(\phi A-A\phi)SY = S\phi AY-SA\phi Y-\phi ASY+A\phi SY\\ &= S\phi AY-AS\phi Y-\phi SAY+A\phi SY=(S\phi-\phi S)AY-A(S\phi-\phi S)Y\end{aligned}$$

By virtue of Lemma 2.1 and the above equations, we assert the following:

Lemma 2.2 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If M satisfies $A(\phi S-S\phi)=(\phi S-S\phi)A$ and $\xi\in\mathfrak{Q}^\perp$, then we obtain $S\phi=\phi S$.*

Proof Since the shape operator A and the tensor $\phi S-S\phi$ are both symmetric operators and commute with each other, by using the method due to Horn and Johnson [3], there exists a common basis $\{E_i\}_{i=1,\dots,4m-1}$ that gives a simultaneous diagonalization. Since $A\xi=\alpha\xi$ and $(\phi S-S\phi)\xi=0$, ξ is principal for A and $\phi S-S\phi$. We write

$AE_i = \lambda_i E_i$ and $(\phi S - S\phi)E_i = \beta_i E_i$, where eigenvalues λ_i and β_i are real valued functions for all $i \in \{1, 2, \dots 4m-1\}$.

Bearing in mind that $\xi = \xi_1 \in \mathfrak{Q}^\perp$, (2.1) is simplified:

$$SX = (4m+7)X - 7\eta(X)\xi - 2\eta_2(X)\xi_2 - 2\eta_3(X)\xi_3 + \phi_1\phi X + hAX - A^2X. \tag{2.10}$$

As ξ is principal for both A and $\phi S - S\phi$, we get

Case 1. We can restrict $X \in [\xi]^\perp$. Here replacing X by ϕX in (2.10) (resp. applying ϕ to (2.10)), we have

$$\begin{aligned} S\phi X &= (4m+7)\phi X - \phi_1 X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + hA\phi X - A^2\phi X, \\ \phi SX &= (4m+7)\phi X - \phi_1 X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + h\phi AX - \phi A^2 X. \end{aligned} \tag{2.11}$$

Combining equations in (2.11), we get

$$S\phi X - \phi SX = hA\phi X - A^2\phi X - h\phi AX + \phi A^2 X. \tag{2.12}$$

Putting $X = E_i$ into (2.12) and using $AE_i = \lambda_i E_i$, we obtain

$$(S\phi - \phi S)E_i = hA\phi E_i - A^2\phi E_i - h\lambda_i\phi E_i + \lambda_i^2\phi E_i. \tag{2.13}$$

Taking the inner product with E_i into (2.13), we have

$$\beta_i g(E_i, E_i) = h\lambda_i g(\phi E_i, E_i) - \lambda_i^2 g(\phi E_i, E_i) = 0.$$

Since $g(E_i, E_i) \neq 0$, $\beta_i = 0$ for all $i \in 1, 2, \dots, 4m-2$. This is equivalent to $(S\phi - \phi S)E_i = 0$ for all $i \in 1, 2, \dots, 4m-2$.

Case 2. For $X \in [\xi]$. This gives $(S\phi - \phi S)\xi = 0$. It follows that $S\phi X = \phi SX$ for any tangent vector field X on M. ∎

Summing up Lemmas 2.1, 2.2 and [12, Theorem], we conclude that if M is a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ satisfying (C-1) for $X = \xi$ and $\xi\alpha = 0$, then M satisfies the condition of type (A) real hypersurfaces. Hereafter, let us check whether the Ricci tensor of a model space of type (A) satisfies the given condition (C-1) for $X = \xi$.

First let us consider $X = \xi$; then (C-1) becomes

$$(\widehat{\mathcal{L}}_\xi^{(k)} S)Y = (\mathcal{L}_\xi S)Y, \tag{2.14}$$

which is equivalent to

$$-k\phi SY - \phi ASY + kS\phi Y - S\phi AY = 0. \tag{2.15}$$

When ξ is Hopf vector field and $\xi \in \mathfrak{Q}^\perp$, the Ricci tensor S commutes with the structure tensor ϕ and by [8, lemma 1.2], M_A satisfies (2.15).

If the Reeb vector field ξ belongs to the maximal quaternionic subbunble $\mathfrak{Q}$, then a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ is locally congruent to one of type (B) by virtue of [6, Main Theorem].

For M_B, (2.14) is also equivalent to (2.15). So we assume M_B satisfies (2.15). For each eigenspace, we have

$$SX = \begin{cases} (4m+4+h\alpha-\alpha^2)\xi & \text{if } X = \xi \in T_\alpha, \\ (4m+4+h\beta-\beta^2)\xi_\ell & \text{if } X = \xi_\ell \in T_\beta, \\ (4m+8)\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma, \\ (4m+7+h\lambda-\lambda^2)X & \text{if } X \in T_\lambda, \\ (4m+7+h\mu-\mu^2)X & \text{if } X \in T_\mu. \end{cases}$$

From [13], we obtain the following equations:

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r),$$
$$\lambda + \mu = \beta \quad \text{and} \quad h = \alpha + 3\beta + (4n-4)(\lambda+\mu) = \alpha + (4n-1)\beta. \tag{2.16}$$

Thus, we get

$$(\widehat{\mathcal{L}}_\xi^{(k)}S)Y - (\mathcal{L}_\xi S)Y = \begin{cases} 0 & \text{if } Y = \xi \in T_\alpha, \\ (4-h\beta+\beta^2)(\beta-k)\xi_\ell & \text{if } Y = \xi_\nu \in T_\beta, \\ (4-h\beta+\beta^2)\phi\xi_\ell & \text{if } Y = \phi\xi_\nu \in T_\gamma, \\ (-k+\lambda)(\lambda-\mu)(h-\lambda-\mu)\phi Y & \text{if } Y \in T_\lambda, \\ (-k+\mu)(\mu-\lambda)(h-\mu-\lambda)\phi Y & \text{if } Y \in T_\mu. \end{cases}$$

From the fourth equation of above (resp., fifth), since $\mu \neq \lambda$, due to (2.16), we have $k = \mu$ or $h = \beta$ (resp., $k = \lambda$ or $h = \beta$). However, if $h = \beta$, the third one cannot happen. So we have $k = \mu = \lambda$. This gives a contradiction.

Remark 2.3 Let M be a real hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$; then M_B does not satisfy the given condition $(\widehat{\mathcal{L}}_\xi^{(k)}S)Y = (\mathcal{L}_\xi S)Y$, for any Y tangent to M.

Thus, we have asserted Theorem 1.1 in the introduction.

Secondly, we assume that M_A satisfies (C-1). Putting $Y = \xi$ into (C-1), we obtain

$$-\sigma\phi AX + k\sigma\phi X + S\phi AX - kS\phi X = 0,$$

where $S\xi = \sigma\xi = (4m + h\alpha - \alpha^2)\xi$.

From [13], we obtain the following equation:

$$SX = \begin{cases} (4m+h\alpha-\alpha^2)\xi & \text{if } X = \xi \in T_\alpha, \\ (4m+6+h\beta-\beta^2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta, \\ (4m+6+h\lambda-\lambda^2)X & \text{if } X \in T_\lambda, \\ (4m+8)X & \text{if } X \in T_\mu. \end{cases}$$

For $Y = \xi \in T_\alpha$, we get

$$(2.17)\quad (\widehat{\mathcal{L}}^{(k)}_X S)\xi - (\mathcal{L}_X S)\xi = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha, \\ (k-\beta)(-h\alpha + \alpha^2 + 6 + h\beta - \beta^2)\xi_3 & \text{if } X = \xi_2 \in T_\beta, \\ (k-\beta)(-h\alpha + \alpha^2 + 6 + h\beta - \beta^2)\xi_2 & \text{if } X = \xi_3 \in T_\beta, \\ (k-\lambda)(h\alpha - \alpha^2 - 6 - h\lambda + \lambda^2)\phi X & \text{if } X \in T_\lambda, \\ (h\alpha - \alpha^2 - 8)\phi X & \text{if } X \in T_\mu. \end{cases}$$

From the fifth equation in (2.17), we obtain

$$(2.18)\qquad h\alpha - \alpha^2 - 8 = 0,$$

and from the definition of h, we obtain $h = \alpha + 2\beta + (2m-2)(\lambda + \mu)$.

Summing these up, by [13] we have

$$(2.19)\qquad (m-1)t^2 - (m+2)t + 4 = 0,$$

where $t = tan^2(\sqrt{2}r)$.

From the second equation of (2.17) and (2.18), we obtain

$$(2.20)\qquad (k-\beta)(h\beta - \beta^2 - 2) = 0.$$

If we assume that $h\beta - \beta^2 - 2 = 0$, then by summing up (2.19) with (2.20), we have $m = -1$, which gives us a contradiction. Thus, $k = \beta$, so from the fourth equation of (2.17) and (2.18), we get

$$h\lambda - \lambda^2 - 2 = 0,$$

which becomes

$$(2.21)\qquad (2m-3)t^2 - 4t + 1 = 0.$$

Combining (2.19) and (2.21) implies

$$t = \frac{-7m + 11}{(m-2)(2m+1)}.$$

Since $m \geq 3$ and $t \geq 0$, this gives us a contradiction.

By virtue of Remark 2.3, we also get the fact that M_B does not satisfy the given condition $(\widehat{\mathcal{L}}^{(k)}_X S)Y = (\mathcal{L}_X S)Y$. Thus, we assert Corollary 1.2.

Acknowledgment The authors would like to express their deep gratitude to Dr. Lee working at RIRCM for giving us nice comments for this paper.

References

[1] J. T. Cho, *CR structures on real hypersurfaces of a complex space form*. Publ. Math. Debrecen **54**(1999), no. 3–4, 473–487.

[2] ———, *Levi parallel hypersurfaces in a complex space form*. Tsukuba J. Math. **30**(2006), no. 2, 329–343. http://dx.doi.org/10.21099/tkbjm/1496165066

[3] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge University Press, Cambridge, 1985. http://dx.doi.org/10.1017/CBO9780511810817

[4] I. Jeong, C. Machado, J. D. Pérez, and Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with $\mathcal{D}^\perp$-parallel structure Jacobi operator*. Internat. J. Math. **22**(2011), no. 5, 655–673. http://dx.doi.org/10.1142/S0129167X11006957

[5] H. Lee, Y. S. Choi, and C. Woo, *Hopf hypersurfaces in complex two-plane Grassmannians with Reeb parallel shape operator*. Bull. Malays. Math. Sci. Soc. **38**(2015), no. 2, 617–634. http://dx.doi.org/10.1007/s40840-014-0039-3

[6] H. Lee and Y. J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector*. Bull. Korean Math. Soc. **47**(2010), no. 3, 551–561. http://dx.doi.org/10.4134/BKMS.2010.47.3.551

[7] T. H. Loo, *Semi-parallel real hypersurfaces in complex two-plane Grassmannians*. Differential Geom. Appl. **34**(2014), 87–102. http://dx.doi.org/10.1016/j.difgeo.2014.03.011

[8] E. Pak, Y. J. Suh, and C. Woo, *Restricted Ricci conditions for real hypersurfaces in complex two-plane Grassmannians*. Houston J. Math. **41**(2015), no. 3, 767–783.

[9] J. D. Pérez, I. Jeong, and Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with commuting normal Jacobi operator*. Acta Math. Hungar. **117**(2007), 201–217. http://dx.doi.org/10.1007/s10474-007-6091-9

[10] J. D. Pérez and Y. J. Suh, *The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians*. J. Korean Math. Soc. **44**(2007), 211–235. http://dx.doi.org/10.4134/JKMS.2007.44.1.211

[11] J. D. Pérez, Y. J. Suh, and Y. Watanabe, *Generalized Einstein real hypersurfaces in complex two-plane Grassmannians*. J. Geom. Phys. **60**(2010), 1806–1818. http://dx.doi.org/10.1016/j.geomphys.2010.06.017

[12] Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with commuting Ricci tensor*. J. Geom. Phys. **60**(2010), 1792–1805. http://dx.doi.org/10.1016/j.geomphys.2010.06.007

[13] ———, *Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor*. Proc. Roy. Soc. Edinburgh Sect. A. **142**(2012), 1309–1324. http://dx.doi.org/10.1017/S0308210510001472

[14] ———, *Real hypersurfaces in the complex quadric with parallel Ricci tensor*. Adv. Math. **281**(2015), 886–905. http://dx.doi.org/10.1016/j.aim.2015.05.012

[15] ———, *Real hypersurfaces in the complex quadric with harmonic curvature*. J. Math. Pures Appl. **106**(2016), 393–410. http://dx.doi.org/10.1016/j.matpur.2016.02.015

[16] Y. J. Suh and H. Y. Yang, *Real hypersurfaces in complex two-plane Grassmannians with commuting structure Jacobi operator*. Bull. Korean Math. Soc. **45**(2008), no. 3, 495–507. http://dx.doi.org/10.4134/BKMS.2008.45.3.495

[17] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*. Japan. J. Math. (N.S.) **2**(1976), 131–190.

[18] S. Tanno, *Variational problems on contact Riemannian manifolds*. Trans. Amer. Math. Soc. **314**(1989), 349–379. http://dx.doi.org/10.1090/S0002-9947-1989-1000553-9

[19] S. M. Webster, *Peudo-Hermitian structures on a real hypersurface*. J. Differential Geom. **13**(1978), 25–41. http://dx.doi.org/10.4310/jdg/1214434345

Division of Future Capability Education, Ju Si-Gyeong College, Pai Chai University, 155-40 Baejae-ro, Seo-gu, Daejeon, 35345, Republic of Korea
e-mail: imsoon.jeong@gmail.com

Departamento de Geometria y Topologia, Universidad de Granada, 18071-Granada, Spain
e-mail: jdperez@ugr.es

Department of Mathematics and Research Institute of Real and Complex Manifold, Kyungpook National University, Daegu 41566, Republic of Korea
e-mail: yjsuh@knu.ac.kr

Department of Mathematics Education, Woosuk University, 565-701 Wanju, Jeonbuk, Republic Of Korea
e-mail: legalgwch@naver.com