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Three-wave resonant interactions and zonal flows in two-dimensional Rossby-Haurwitz wave turbulence on a rotating sphere

Kiori Obuse[1] and Michio Yamada[2]
[1]*Graduate School of Environmental and Life Science, Okayama University, Okayama, 700-8530, Japan*
[2]*Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan*

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This paper addresses three-wave resonant interactions of Rossby-Haurwitz waves in two-dimensional turbulence on a rotating sphere. Zonal modes are often omitted from the "resonant wave set" even when they satisfy the conditions for three-wave resonant interactions, as they do not transfer any energy to other modes in a resonant manner. However, the presence of zonal flows induces phase shifts in other modes, and it is not at all clear that their influence is negligible. Since it is expected that three-wave resonant interactions govern the entire dynamics of turbulence if the rotation rate of the sphere is sufficiently high, by analogy with the theorem regarding three-wave resonant interactions of Rossby waves on a β plane with sufficiently large β previously proven by Yamada and Yoneda [Physica D **245**, 1 (2013)], an appropriate definition of the resonant wave set was determined by comparing the time evolution of several wave sets on a rapidly rotating sphere. It was found that zonal waves of the form $Y_l^{m=0}\exp(i\omega t)$ with odd l, where Y_l^m are the spherical harmonics, should be considered for inclusion in the resonant wave set to ensure that the dynamics of the resonant wave set determine the overall dynamics of the turbulence on a rapidly rotating sphere. Consequently, it is suggested that the minimal resonant wave set that must be considered in the discussion of the three-wave interaction of Rossby-Haurwitz waves is the set consisting of nonzonal resonant waves and zonal waves of the form $Y_l^0\exp(i\omega t)$ with odd l.

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I. INTRODUCTION

Two-dimensional turbulence on a rotating sphere has several distinct features that set it apart from two-dimensional turbulence on a nonrotating system, though they possess many similar properties such as the occurrence of an inverse energy cascade. One of the biggest differences between these two types of turbulence is that for a rotating system, be it a sphere or a β plane, large-scale zonal flows are formed [1–4]. Another typical difference is that the rotating system has linear wave solutions, namely Rossby-Haurwitz waves on a sphere and Rossby waves on a β plane. These solutions are also nonlinear solutions to the two-dimensional Navier–Stokes equation for their respective systems, and the flow field in each system is governed by the dynamics of Rossby-Haurwitz or Rossby waves. Because of this, Rossby-Haurwitz and Rossby waves are frequently used in the discussion of characteristic features of turbulent flows on rotating spheres [4]. Specifically, three-wave resonant nonlinear interactions of the waves are considered to play an important role in the dynamics of turbulence. For example, Kartashova and L'vov defined clusters of resonant triads on a sphere in terms of the energy transfer between the resonant triads [5]. They considered the energy exchange between triads in clusters and then addressed its relation to intraseasonal oscillations in the Earth's atmosphere [6]. Additionally, Lynch [7] showed that nonlinear interactions of resonant Rossby-Haurwitz triads may result in dynamic instability of large-scale components. Sukhatm and Smith [8] considered the roles of resonant-near-resonant-nonresonant interactions of Rossby waves in the formation of zonal flows on a β plane, and Yamada and Yoneda [9] analytically showed that within a certain time range, three-wave resonant nonlinear interactions totally determine the dynamics of the flow field of two-dimensional Rossby wave turbulence on a β plane when $\beta \to \infty$.

However, although many studies have been conducted on three-wave resonant interactions of Rossby-Haurwitz and Rossby waves, there seems to be no clear definition of resonant Rossby-Haurwitz and Rossby waves that should be considered in the discussion of the effects of three-wave resonance interactions on the turbulence. Let us consider Rossby-Haurwitz waves to be defined as $Y_l^m(\phi, \mu)\exp(i\omega t)$, where $Y_l^m(\phi, \mu)$ are the spherical harmonics ($l \in \mathcal{N},\ m \in \mathcal{Z},\ -l \leqslant m \leqslant l,$). The frequency ω of a Rossby-Haurwitz wave is given as $\omega = -2\,\Omega m/[l(l+1)]$, where Ω is the rotation rate of the sphere, and the main condition for three-wave resonant interaction among waves i, j, and k is $\omega_i = \omega_j + \omega_k$. (A detailed definition is given in Sec. II.) Then for example, the impressive analysis introduced by Kartashova and L'vov [5,6] using resonant clusters is based on the assumption that zonal Rossby-Haurwitz waves, which are steady and may also be called zonal flows, can be ignored when considering resonant interactions, regardless of whether they satisfy the main condition for three-wave resonance given above. This is because they do not exchange energy with any other waves via three-wave resonant interactions [10,11] as discussed in Sec. II. This is widely known and can be easily confirmed based on the principle of detailed balance. However, it must also be taken into consideration that the existence of zonal flows shifts the phase of nonzonal Rossby-Haurwitz waves. Then the effective time-dependent part of the Rossby-Haurwitz waves becomes $\exp[i\omega t + a(t)]$, where $a(t)$ is the phase shift produced by zonal waves. The shift may not be essential to the dynamics of resonant waves when $a(t) \ll \omega t$, i.e., when the rotation rate of the sphere Ω is sufficiently high, but this assumption may not always be true.

Hence, this paper addresses three-wave interactions of Rossby-Haurwitz waves on a rotating sphere and clarifies the definition of "resonant Rossby-Haurwitz waves" that should be considered in the discussion of the effects of three-wave resonance interactions on turbulence, specifically the treatment of zonal Rossby-Haurwitz waves. To this end, we applied the theorem that states that three-wave resonant nonlinear interactions totally determine the dynamics of the flow field of two-dimensional Rossby wave turbulence within a certain time range on a β plane when $\beta \to \infty$ [9], which also holds for a spherical geometry [12]. Numerical simulations were then conducted on a sphere with a sufficiently high rotation rate, and it was clarified which set of waves should be considered to compose the minimal "resonant wave set" that governs the entire dynamics of the turbulence. The next section gives a general definition of three-wave resonant interactions and classifies Rossby-Haurwitz waves into four sets for later discussion. The settings for the present numerical experiments are described in Sec. III, and the results are presented in Sec. IV. Then the conclusions follow in Sec. V.

II. EQUATION OF MOTION AND DEFINITION OF THREE-WAVE RESONANT INTERACTION

The model equation considered in this paper is a nondimensionalized vorticity equation for a decaying two-dimensional barotropic incompressible flow on a rotating sphere, given with respect to the longitude ϕ and sine latitude μ as[1]

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) + 2\Omega \frac{\partial \psi}{\partial \phi} = (-1)^{p+1} \nu_{2p} (\nabla^2 + 2)^p \zeta. \tag{1}$$

Here t is time; $\psi(\phi, \mu, t)$ is the stream function; $\zeta(\phi, \mu, t) \equiv \nabla^2 \psi$ is the vorticity, where ∇^2 is the horizontal Laplacian on a sphere; Ω is the dimensionless constant rotation rate of the sphere;[2] p is the hyperviscosity index; and ν_{2p} is the dimensionless kinematic hyperviscosity coefficient. $J(f, g)$

[1]Length, velocity, and time variables are nondimensionalized with respect to the radius a of the sphere, the characteristic velocity amplitude U_0 of the initial state (or the square root of the mean kinetic energy, $\sqrt{2E}$), and the advection time scale a/U_0, respectively.

[2]The dynamics of Earth's atmosphere is dominated not solely by two-dimensional mechanism, but also by several other mechanisms including thermal and moist convection, solar radiation. However, if one should

is the Jacobian operator, which is defined as $J(f, g) \equiv (\partial f/\partial \phi)(\partial g/\partial \mu) - (\partial f/\partial \mu)(\partial g/\partial \phi)$. The added constant 2 in the viscosity term is necessary for the conservation of the total angular momentum of the system, as discussed in, for example, Ref. [10].

The linear wave solutions of this system are the Rossby-Haurwitz waves $Y_l^m(\phi, \mu)\exp(i\omega t)$, where $Y_l^m(\phi, \mu)$ are the spherical harmonics ($l \in \mathcal{N}$, $m \in \mathcal{Z}$, $-l \leqslant m \leqslant l$,) and the frequency ω of the wave is given as

$$\omega = \frac{-2m\Omega}{l(l+1)}. \tag{2}$$

Hereafter, $Y_l^m(\phi, \mu)$ is sometimes referred to as Y_l^m for the sake of brevity. Additionally, in this paper, the Rossby-Haurwitz waves are sometimes called "the Rossby modes" or simply "modes."

Three-wave resonant nonlinear interactions of Rossby-Haurwitz waves are now considered. Three Rossby-Haurwitz waves $Y_{l_1}^{m_1}\exp(i\omega_1 t)$, $Y_{l_2}^{m_2}\exp(i\omega_2 t)$, and $Y_{l_3}^{m_3}\exp(i\omega_3 t)$, are said to undergo a three-wave resonant nonlinear interaction if the following conditions are satisfied:

$$m_1 = m_2 + m_3, \tag{3}$$

$$|l_2 - l_3| < l_1 < l_2 + l_3, \tag{4}$$

$$l_1 + l_2 + l_3 = \text{odd}, \tag{5}$$

$$\frac{m_1}{l_1(l_1+1)} = \frac{m_2}{l_2(l_2+1)} + \frac{m_3}{l_3(l_3+1)}. \tag{6}$$

The conditions (3)–(5) was derived in Ref. [10][3] as necessary conditions for three-wave nonlinear interactions. Note that the three-wave nonlinear interactions include both the resonant and the nonresonant nonlinear interactions in the three-wave set, i.e., a triad. When we eliminate the nonresonant interactions, the condition (6) should be taken into account together with conditions (3)–(5) [11].[4] Therefore we can take the conditions (3)–(6) as the necessary conditions for the resonant nonlinear interaction of three waves. In addition to the above resonant conditions,

$$l_1 \neq l_2,\ l_1 \neq l_3,\ l_2 \neq l_3, \tag{7}$$

are often considered when discussing resonant interactions, in order to eliminate triads which satisfy resonant conditions but which have no energy transfer between each mode in the triad. Those conditions are based on the principle of detailed balance and eliminate zonal modes $Y_l^{m=0}$ as well as the cases $l_1 = l_2 = l_3$.

Let us consider the resonant nonlinear interaction of a triad including zonal mode(s) $Y_l^{m=0}$. The nonlinear interaction of two Rossby modes is written as

$$\begin{aligned}
&J\left(\psi_{l_2}^{m_2}Y_{l_2}^{m_2}, l_3(l_3+1)\psi_{l_3}^{m_3}Y_{l_3}^{m_3}\right) + J\left(\psi_{l_3}^{m_3}Y_{l_3}^{m_3}, l_2(l_2+1)\psi_{l_2}^{m_2}Y_{l_2}^{m_2}\right)\\
&\quad = \psi_{l_2}^{m_2}\psi_{l_3}^{m_3}l_3(l_3+1)\left(\frac{\partial Y_{l_2}^{m_2}}{\partial \phi}\frac{\partial Y_{l_3}^{m_3}}{\partial \mu} - \frac{\partial Y_{l_2}^{m_2}}{\partial \mu}\frac{\partial Y_{l_3}^{m_3}}{\partial \phi}\right)\\
&\qquad + \psi_{l_2}^{m_2}\psi_{l_3}^{m_3}l_2(l_2+1)\left(\frac{\partial Y_{l_3}^{m_3}}{\partial \phi}\frac{\partial Y_{l_2}^{m_2}}{\partial \mu} - \frac{\partial Y_{l_3}^{m_3}}{\partial \mu}\frac{\partial Y_{l_2}^{m_2}}{\partial \phi}\right),
\end{aligned} \tag{8}$$

where ψ_l^m is the expansion coefficient of the stream function. Each Jacobian term in (8) clearly vanishes for the case $m_1 = m_2 = m_3 = 0$, meaning that *resonant nonlinear interaction is not*

compare Ω with the rotation rate of the Earth Ω_E, one interpretation may be $\Omega = (a_*/U_*)\Omega_E$, where a_* and U_* are the radius and representative flow velocity of the Earth.

[3]Conditions (7), (10), and (9) in Ref. [10].

[4]Condition (1) in Ref. [11].

possible among three zonal Rossby modes. According to the discussion in the Appendix, resonant nonlinear interaction is also not possible for a triad including *two zonal modes*. The Appendix further shows that resonant nonlinear interaction is possible for a triad including *only one zonal mode* $Y_l^m = 0$ *for* $l =$ odd *number*, but then we find that the energy exchange is impossible between the zonal and nonzonal modes by the resonant nonlinear interaction (8) because of the principle of detailed balance. However, we remark that the zonal mode $Y_{l=\text{odd}}^{m=0}$ in the resonant triad can change the phases of other two (nonzonal) modes. This implies that the zonal modes $Y_{l=\text{odd}}^{m=0}$ should not be neglected when we consider resonant nonlinear interactions of Rossby modes. This effect has little been discussed in previous research, where the condition (7) is applied in addition to the resonance conditions (3)–(6) [11].[5]

In the next section, we will discuss the role of the zonal Rossby modes in the dynamics of turbulence on a rotating sphere. For later convenience, all Rossby-Haurwitz modes are categorized into the following four sets:[6]

$$\text{Resonant nonzonal } (R\check{Z}) \text{ modes} = \{Y_l^m \mid \text{with (3)–(7)}\}, \tag{9}$$

$$\text{Resonant zonal } (RZ) \text{ modes} = \{Y_l^0 \mid \text{with (3)–(6)}\} = \{Y_l^0 \mid l = \text{odd integer}\}, \tag{10}$$

$$\text{Nonresonant zonal } (\check{R}Z) \text{ modes} = \{Y_l^0 \mid l = \text{even integer}\}, \tag{11}$$

$$\text{Nonresonant nonzonal } (\check{R}\check{Z}) \text{ modes} = \{\text{all Rossby-Haurwitz modes}\} \setminus \{R\check{Z}\} \setminus \{RZ\} \setminus \{\check{R}Z\}. \tag{12}$$

$R\check{Z}$ modes for $l \leqslant 31$ are shown as red dots in Fig. 1.

III. SEPARATION OF RESONANT AND NONRESONANT WAVE DYNAMICS IN FULL SIMULATIONS

According to Ref. [9], within a certain time range,[7] three-wave resonant nonlinear interactions totally determine the dynamics of the flow field of two-dimensional Rossby wave turbulence on a β plane when $\beta \to \infty$. This suggests that the same holds in two-dimensional Rossby-Haurwitz wave turbulence on a rotating sphere with an infinite rotation rate [12]. Thus, in numerical experiments on two-dimensional Rossby-Haurwitz wave turbulence on a sphere with a sufficiently high rotation rate, separation of the resonant and nonresonant wave sets in a flow field can be observed, and the dynamics of the resonant wave set can be extracted in a full numerical simulation. However, as stated in Sec. II, it is unfortunately not at all clear which definition of the resonant wave set should be adopted in these simulations. Therefore, an appropriate definition of the resonant wave set was determined in this study.

To ensure the separation of the Rossby-Haurwitz waves between the resonant and nonresonant wave sets, five different numerical time integrations of Eq. (1) that are shown in Table I with initial and numerical conditions and will be described later were considered.

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[5] Equations (11d) and (12) in Ref. [11].
[6] $\check{A}$ means negation of A. The operator $\setminus$ represents the set difference of sets: $A \setminus B = \{x \in A | x \notin B\}$.
[7] Theorems 2 and 3 in Ref. [9].

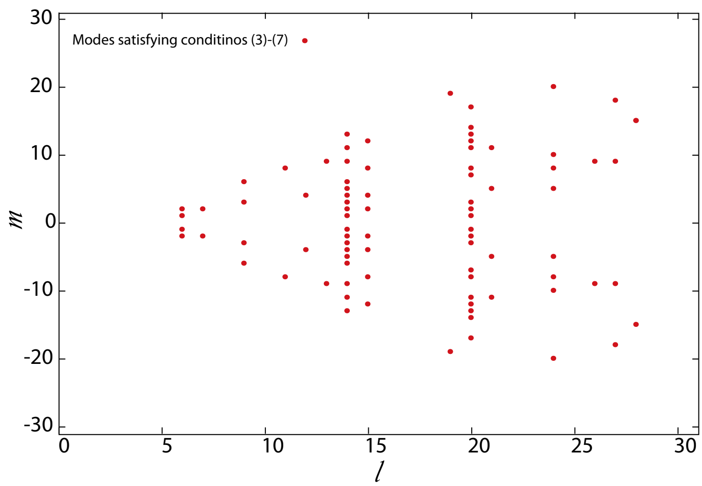

FIG. 1. Modes Y_l^m with $l \leqslant 31$ satisfying conditions (3)–(7), i.e., $R\check{Z}$ modes.

Case 1: (1,1,1,1): *Full numerical simulation of Eq. (1) from* $t = 0$ *to 10.05.* When the sphere has a sufficiently high rotation rate, the dynamics of the flow field should be governed almost exclusively by the resonant wave set for at least a certain time range, and the waves belonging to the nonresonant wave set will not experience large changes in amplitude or phase over time.

Case 2: (1,0,0,0): *Simulation with only* $R\check{Z}$ *modes retained.* RZ, $\check{R}\check{Z}$, and $\check{R}Z$ modes are not included in the numerical integration after $t = 0.05$, and the flow field until $t = 0.05$ is the same as that in Case 1: (1,1,1,1). (Although this is a short time frame, it is sufficient to get a seemingly natural initial condition from the artificial initial state at $t = 0$.) The time development is due to both resonant and nonresonant interactions among $R\check{Z}$ modes. However, when the rotation rate of the sphere is sufficiently high, nonresonant interactions can be considered to be very weak, and thus this calculation will yield *the time development of* $R\check{Z}$ *only by resonant interactions*.

Case 3: (1,1,0,0): *Simulation with only* $R\check{Z}$ *and* RZ *modes retained.* $\check{R}\check{Z}$ and $\check{R}Z$ modes are not included in the numerical integration after $t = 0.05$, and the flow field until $t = 0.05$ is the same as that in Case1: (1,1,1,1). (Although this is a short time frame, it is sufficient to get a seemingly natural initial condition from the artificial initial state at $t = 0$.) The time development is due to both resonant and nonresonant interactions among $R\check{Z}$ and RZ modes. However, when the rotation rate of the sphere is sufficiently high, nonresonant interaction can be considered to be very weak, and thus this calculation yields *the time development of* $R\check{Z} + RZ$ *by only resonant interactions*.

Case 4: (1,0,0,1): *Simulation with only* $R\check{Z}$ *and* $\check{R}Z$ *modes retained.* RZ and $\check{R}\check{Z}$ modes are not included in the numerical integration after $t = 0.05$, and the flow field until $t = 0.05$ is the same as that in Case1: (1,1,1,1). (Although this is a short time frame, it is sufficient to get a seemingly natural initial condition from the artificial initial state at $t = 0$.) The time development is due to both

TABLE I. Five different numerical time integrations.

	$R\check{Z}$	RZ	$\check{R}\check{Z}$	$\check{R}Z$
Case 1: (1,1,1,1)	✓	✓	✓	✓
Case 2: (1,0,0,0)	✓			
Case 3: (1,1,0,0)	✓	✓		
Case 4: (1,0,0,1)	✓			✓
Case 5: (1,1,0,1)	✓	✓		✓

resonant and nonresonant interactions among $R\check{Z}$ and $\check{R}Z$ modes. However, when the rotation rate of the sphere is sufficiently high, nonresonant interaction can be considered to be very weak, and thus this calculation yields *the time development of* $R\check{Z}+\check{R}Z$ *alone by only resonant interactions*.

Case 5: (1,1,0,1): *Simulation with only* $R\check{Z}$, RZ, *and* $\check{R}Z$ *modes retained.* $\check{R}\check{Z}$ modes are not included in the numerical integration after $t=0.05$, and the flow field until $t=0.05$ is the same as that in Case1: (1,1,1,1). (Although this is a short time frame, it is sufficient to get a seemingly natural initial condition from the artificial initial state at $t=0$.) The time development is due to both resonant and nonresonant interactions among $R\check{Z}$, RZ, and $\check{R}Z$ modes. However, when the rotation rate of the sphere is sufficiently high, nonresonant interaction can be considered to be very weak, and thus this calculation yields *the time development of* $R\check{Z}+RZ+\check{R}Z$ *by only resonant interactions*.

The temporal variation of the palinstrophy $P(t)=[\nabla\times\zeta(t)]/2$ of $R\check{Z}$ modes in Cases 1–5 was then observed.[8]

For the numerical calculations, the physical parameters in Eq. (1) were set based on those used in Ref. [13]. The hyperviscosity index is $p=8$, and the dimensionless kinematic hyperviscosity coefficient is $\nu_{2p}=10^{-34}$. The sphere was set to rotate with a very large dimensionless constant rotation rate Ω in this study, and it was set to $\Omega=4\times10^{6}$ to meet this criterion. It was confirmed that $\Omega=4\times10^{6}$ is sufficiently large for this system, as will be demonstrated later.

A spherical harmonic spectral method was used for the calculation. The stream function ψ was expanded as

$$\psi(\phi,\mu,t)=\sum_{l=0}^{N_T}\sum_{m=-l}^{l}\psi_l^m(t)Y_l^m(\phi,\mu)=\sum_{l=0}^{N_T}\sum_{m=-l}^{l}\psi_l^m(t)P_l^m(\mu)\exp(im\phi),$$

where $\psi_l^m(t)$ is the expansion coefficient. The truncation wave number was set to $N_T=31$ for the calculation,[9] and 96 and 48 spatial grid points were taken in longitudinal and latitudinal directions to eliminate aliasing errors. The linear terms in Eq. (1) were analytically treated using exponential functions. The time integration was performed using the fourth-order Runge-Kutta method with a time step of $\Delta t=0.4/\Omega$, following Ref. [13]. The initial energy distribution was set to

$$E(l,t=0)=\frac{Al^{\gamma/2}}{(l+l_0)^{\gamma}},\quad l_0=15,\quad \gamma=200, \tag{13}$$

where A is defined as

$$\sum_{l=2}^{N_T}E(l,t=0)=1.0, \tag{14}$$

essentially following Ref. [13].

IV. RESULTS OF NUMERICAL EXPERIMENTS

The time evolution of the palinstrophy $P(t)$ of the $R\check{Z}$ modes in the numerical simulations for Cases 1: (1,1,1,1) and 2: (1,0,0,0), which are shown in Fig. 2, were first compared. The time profiles for the palinstrophy of the $R\check{Z}$ modes in Cases 1: (1,1,1,1) and 2: (1,0,0,0) were significantly different at $\Omega=4\times10^{6}$, and the difference between the two cases remained almost the same even when the rotation rate of the sphere was increased 10-fold to $\Omega=4\times10^{7}$. These results show that

[8]Energy and enstrophy are conserved quantities. They are conserved in numerical calculations with an accuracy of $4.43\times10^{-8}\%$ for energy and $4.52\times10^{-8}\%$ for enstrophy.

[9]We set $\Delta t=0.4/\Omega$ to resolve time development of Rossby waves, the frequency of which is proportional to Ω. As Δt becomes very small for large Ω, it is not easy to use a large value of N_T.

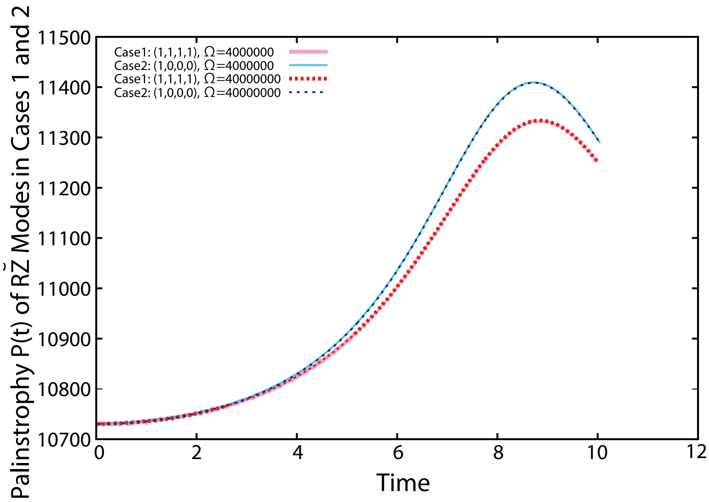

FIG. 2. Time evolution of the palinstrophy of the $R\check{Z}$ modes in the numerical simulations of Cases 1: (1,1,1,1) and 2: (1,0,0,0) (defined in Sec. II). Thick pink curve: Case 1: (1,1,1,1), $\Omega = 4 \times 10^6$; light blue curve: Case 2: (1,0,0,0), $\Omega = 4 \times 10^6$; thick red dashed curve: Case 1: (1,1,1,1), $\Omega = 4 \times 10^7$; blue dashed curve: Case 2: (1,0,0,0), $\Omega = 4 \times 10^7$. The result shows the rotation rate of the sphere Ω is high enough, and the value of Ω is not the reason of the disagreement of time variation of palinstrophy between Cases 1 and 2.

the $R\check{Z}$ modes are not dynamically separate from the $RZ + \check{R}\check{Z} + \check{R}Z$ modes despite there being no energy transfer between the $R\check{Z}$ modes and the $RZ + \check{R}\check{Z} + \check{R}Z$ modes, even when the rotation rate is sufficiently high. The $R\check{Z}$ modes therefore cannot be considered to form the resonant wave set, as the resonant wave set must evolve in time independently of other modes at a high rotation rate. To obtain a proper resonant wave set, the time evolution of the palinstrophy of the $R\check{Z}$ modes

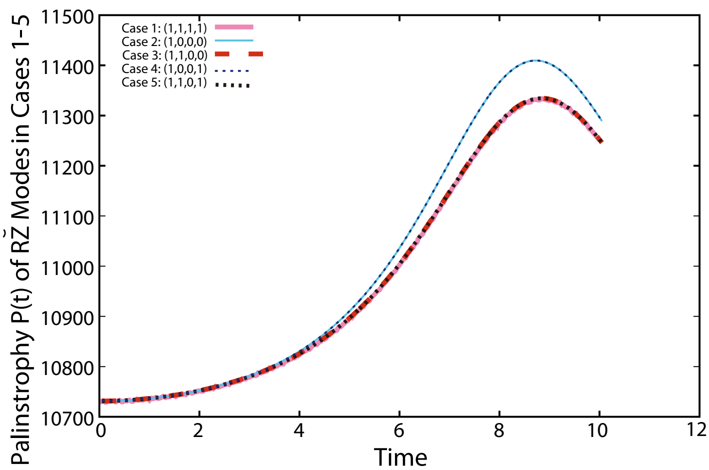

FIG. 3. Time variation in the palinstrophy of the $R\check{Z}$ modes in the numerical simulations of Cases 1–5 (defined in Sec. II). Thick pink curve: Case 1: (1,1,1,1), light blue curve: Case 2: (1,0,0,0), thick dashed red curve: Case 3: (1,1,0,0), blue dashed curve: Case 4: (1,0,0,1), thick dotted black curve: Case 5: (1,1,0,1). All simulations were conducted with $\Omega = 4 \times 10^6$. This result shows that RZ modes are not ignorable in the discussion of three-wave resonant interactions in rapidly rotating systems, while $\check{R}\check{Z}$ and $\check{R}Z$ modes are negligible.

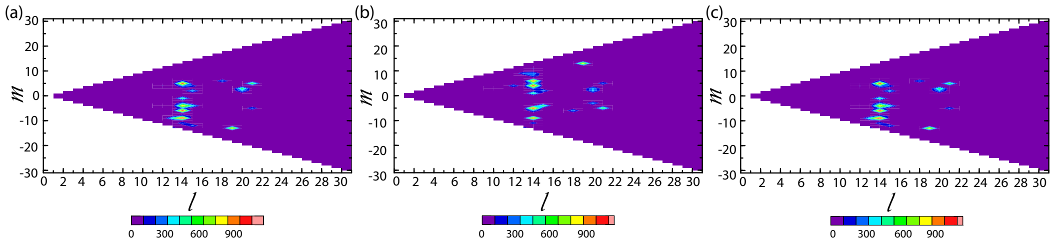

FIG. 4. Palinstrophy spectra of $R\check{Z}$ modes in l-m space at $t = 10.0$ in the numerical simulations of (a) Case 1: (1,1,1,1), (b) Case 2: (1,0,0,0), and (c) Case 3: (1,1,0,0).

in the numerical simulations for all of the cases defined in Sec. II was then compared. As shown in Fig. 3, the time variation of the palinstrophy of the $R\check{Z}$ modes in the numerical simulations for Cases 1: (1,1,1,1), 3: (1,1,0,0), and 5: (1,1,0,1) fall on a single curve, whereas those in Cases 2: (1,0,0,0) and 4: (1,0,0,1) fall on a separate curve. The agreement between Cases 1: (1,1,1,1) and 3: (1,1,0,0) and the disagreement between Cases 1: (1,1,1,1) and 2: (1,0,0,0) are also shown in the palinstrophy spectra in Fig. 4. This suggests that among all zonal waves, only the RZ modes must be added to the $R\check{Z}$ modes to form the minimal resonant wave set that gives the same $R\check{Z}$ mode dynamics as in the full numerical simulation; thus, the $R\check{Z} + RZ$ modes may be regarded as the resonant wave set in the discussion of three-wave resonant interaction and resonant wave dynamics in the flow field at high sphere rotation rates. In addition, it was verified in the same way as above that all RZ waves, i.e., not only RZ waves with low or high l (figure not shown), are necessary for the realization of the same $R\check{Z}$ dynamics as in the full numerical simulation. Additionally, it was confirmed that the result is independent of the initial conditions by taking three different initial energy spectral distributions in l-m space satisfying Eqs. (13) and (14) (figures not shown).

It should be emphasized here again that although RZ modes are important in the discussion of actual three-wave resonant-wave dynamics, they cannot undergo nonzero energy transfer in three-wave resonant-wave interactions, as stated in Sec. II, and the difference between the time profiles of the palinstrophy in Cases 1: (1,1,1,1) and 3: (1,1,0,0) and Case 2: (1,0,0,0) is not the result of energy transfer among RZ modes via three-wave resonant interactions in Cases 1: (1,1,1,1) and 3: (1,1,0,0).

Figure 5(a) shows $\int_0^{10.05} |dE(l, m = 0)/dt| dt$ plotted against the total wave number l for Cases 1: (1,1,1,1) and 3: (1,1,0,0).[10] As shown in Fig. 5(a), the value of this integral for the RZ ($l =$ odd integer) and $\check{R}Z$ ($l =$ even integer) modes was very small, and approximately two orders of magnitude smaller than that for the $R\check{Z}$ waves (not shown), and this verifies that there is no energy transfer among RZ modes via three-wave resonant interactions.[11] Then of course, the palinstrophy spectrum of the RZ ($l =$ odd integer) and $\check{R}Z$ ($l =$ even integer) modes remained constant over time [Fig. 5(b) for Case 3: (1,1,0,0)].

The clear differences among the three palinstrophy spectra in Fig. 4 arose from the time development of the $R\check{Z}$ waves in Cases 1: (1,1,1,1) and 3: (1,1,0,0) differing from that in Case 2: (1,0,0,0). That is, the effect of the phase shift of the $R\check{Z}$ modes by RZ modes, i.e., the equatorially symmetric zonal flow, changed the time development of the $R\check{Z}$ modes. The time-dependent part of $R\check{Z}$ Rossby-Haurwitz modes including the shift $a(t)$ induced by RZ waves is $\exp[i(\omega t + a(t))]$, where $a(t) \ll \omega t$ at a high sphere rotation rate. It is noteworthy that although the timescale of the change in the energy of $R\check{Z}$ modes caused by three-wave resonant interaction is much larger than that caused by the phase shift, the phase shift still has a non-negligible effect on the $R\check{Z}$ dynamics.

[10] Case 2: (1,0,0,0) has no energy transfer to zonal modes by its definition.

[11] This nonzero value may be a result of very weak nonresonant three-wave interactions (as Ω is always finite in numerical simulations) in addition to numerical error.

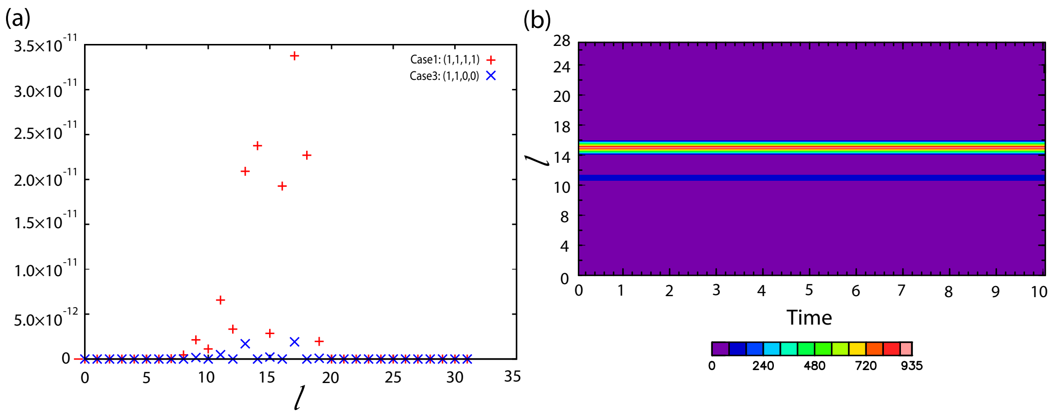

FIG. 5. (a) $\int_0^{10.05} |dE(l, m = 0)/dt|dt$ in Cases 1: (1,1,1,1) and 3: (1,1,0,0). Absolute value is employed to verify that there is no short-time cancellation of high-frequency oscillation. (b) Time variation in the palinstrophy spectrum of the RZ and $\check{R}Z$ modes in Case 3: (1,1,0,0).

It should also be noted that near-resonant interactions are not an important influence on the disagreement between Cases 1: (1,1,1,1) and 2: (1,0,0,0) in this system. Figure 6 shows the time variation of the palinstrophy in Case 2: (1,0,0,0) and in the full numerical simulation but with the energy of the RZ modes set to zero at $t = 0.05$. A weak three-wave nonresonant interaction is active because Ω is finite despite Ω being very large, and the nonresonant interaction most effective here should be a near-resonant interaction. However, the two curves in Fig. 6 show good agreement, which means that any near-resonant interactions among $R\check{Z} + \check{R}\check{Z}$ modes are negligible. However, the time variation of the palinstrophy in Case 2: (1,0,0,0) does not agree with that in the full numerical simulation with the energy of $\check{R}Z$ or $\check{R}\check{Z}$ set to zero at $t = 0.05$ (not shown). This suggests that the existence of the RZ modes is much more important than the effect of near-resonant interactions in this system, unlike in other systems, such as water wave fields [14].

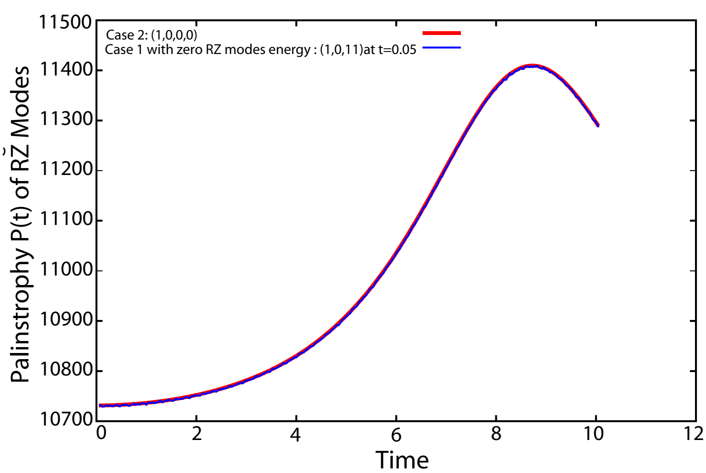

FIG. 6. Time variation of the palinstrophy in Case 2: (1,0,0,0) and in the full numerical with the energy of the RZ modes set to zero at $t = 0.05$. The good agreement of two graphs shows that near-resonant interactions among $R\check{Z} + \check{R}\check{Z}$ modes are negligible.

V. CONCLUSIONS

This paper presented an appropriate definition of the resonant wave set for the discussion of three-wave resonant interaction and resonant wave dynamics in a flow field at a high sphere rotation rate, specifically addressing whether zonal modes should be added to $R\check{Z}$ modes or not. A comparison of the results of a full numerical simulation and four numerical simulations only allowing nonzero energy transfer among $R\check{Z}$ modes and select zonal modes revealed that RZ modes must be considered in addition to $R\check{Z}$ modes to yield $R\check{Z}$ dynamics identical to those in the full numerical simulation. This means that the $R\check{Z} + RZ$ modes comprise the minimal resonant wave set, which evolves in time independently of other modes at high rotation rates.

This is because that the phase shift of the $R\check{Z}$ modes caused by the existence of nonzero RZ modes, i.e., the equatorially symmetric zonal flow, changes the time development of the $R\check{Z}$ modes. The time-dependent part of $R\check{Z}$ Rossby-Haurwitz modes including the shift $a(t)$ induced by RZ waves is $\exp[i(\omega t + a(t)]$, where $a(t) \ll \omega t$ at a high sphere rotation rate. It is noteworthy that although the timescale of the change in the energy of $R\check{Z}$ modes caused by three-wave resonant interaction is much larger than that caused by the phase shift, the phase shift still has a non-negligible effect on the $R\check{Z}$ dynamics.

The results of this study suggest that the effect of the phase shift of the $R\check{Z}$ modes caused by the RZ modes is not negligible when considering the actual dynamics of $R\check{Z}$ modes in a full simulation. Thus, a suitable definition of the resonant wave set is $R\check{Z}$ modes accompanied by RZ modes.

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APPENDIX: THREE-WAVE RESONANT CONDITION AND ZONAL MODES

When $m_1 = 0$, $m_2 \neq 0$, and $m_3 \neq 0$:

From the condition (6)

$$\frac{m_2}{l_2(l_2+1)} = -\frac{m_3}{l_3(l_3+1)}, \tag{A1}$$

is required. This is satisfied when $l_2 = l_3$, and $m_2 = -m_2$. Then the three modes $(l_1, 0), (l_2, m_2), (l_2, -m_2)$ with $0 < l_1 < 2l_2$ and $l_1 =$ odd number satisfy necessary conditions for the resonant nonlinear interaction.

When $m_1 \neq 0$, $m_2 = 0$, and $m_3 \neq 0$:

From the condition (6)

$$\frac{m_1}{l_1(l_1+1)} = \frac{m_3}{l_3(l_3+1)}, \tag{A2}$$

is required. This is satisfied when $l_1 = l_3$, and $m_1 = m_3$. Then the three modes $(l_1, m_1), (l_2, 0), (l_1, m_1)$ with $|l_2 - l_1| < l_1 < l_1 + l_2$ and $l_2 =$ odd number satisfy necessary conditions for the resonant nonlinear interaction.[12]

[12]We treat this as a three-wave nonlinear resonant interaction.

When more than two zonal modes are involved:

Only cases $(l_1, 0)$, $(l_2, 0)$, $(l_3, 0)$ with $|l_2 - l_3| < l_1 < l_2 + l_3$ and $l_1 + l_2 + l_3 =$ odd number satisfy the necessary conditions for the resonant nonlinear interaction of three waves. However, each Jacobian term (8) vanishes, meaning that resonant nonlinear interaction among these three zonal modes is not possible.

The above discussion suggests that three-mode combinations $(l, m), (l, \pm m)$, $(l', 0)$ with a condition $l' =$ odd number satisfy the necessary conditions for the resonant nonlinear interaction. Now, let us consider the principle of detailed balance

$$0 = \frac{d}{dt}\left(E_{l_1}^{m_1} + E_{l_2}^{m_2} + E_{l_3}^{m_3}\right), \tag{A3}$$

where $E_l^m = l(l+1)|\psi_l^m|^2$, ψ_l^m is the expansion coefficient of the stream function. For the three modes (l, m), $(l, \pm m)$, and $(l', 0)$, this becomes

$$0 = \frac{d}{dt}\left(E_{l'}^0 + E_l^m + E_l^{\pm m}\right) = \frac{d}{dt}\left(E_{l'}^0 + 2E_l^m\right), \tag{A4}$$

and as $dE_{l'}^0/dt$ results from the nonlinear interaction (8) of (l, m) and $(l, -m)$ modes is zero, we obtain

$$\frac{d}{dt}E_l^m = 0 \tag{A5}$$

meaning that zonal mode does not exchange energy with nonzonal modes by resonant interaction.

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