

Spherical CR uniformization of Dehn surgeries of the Whitehead link complement

MIGUEL ACOSTA

We apply a spherical CR Dehn surgery theorem in order to obtain infinitely many Dehn surgeries of the Whitehead link complement that carry spherical CR structures. We consider as a starting point the spherical CR uniformization of the Whitehead link complement constructed by Parker and Will, using a Ford domain in the complex hyperbolic plane $\mathbb{H}_{\mathbb{C}}^2$. We deform the Ford domain of Parker and Will in $\mathbb{H}_{\mathbb{C}}^2$ in a one-parameter family. On one side, we obtain infinitely many spherical CR uniformizations on a particular Dehn surgery on one of the cusps of the Whitehead link complement. On the other side, we obtain spherical CR uniformizations for infinitely many Dehn surgeries on the same cusp of the Whitehead link complement. These manifolds are parametrized by an integer $n \geq 4$, and the spherical CR structure obtained for $n = 4$ is the Deraux–Falbel spherical CR uniformization of the figure eight knot complement.

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Introduction

The present work takes place in the framework of the study of geometric structures on manifolds as well as in complex hyperbolic geometry. The Thurston geometrization conjecture, recently proved by Perelman, confirms that the study of the geometric structures carried by manifolds is extremely useful in order to understand their topology: any 3–dimensional manifold can be cut into pieces that carry a geometric structure. Among the 3–dimensional structures, we find the spherical CR structures: they are not on the list of the eight 3–dimensional Thurston geometries but have an interesting behavior, and there are relatively few general facts known about them. More precisely, a spherical CR structure is a (G, X) –structure, where $X = \partial_{\infty} \mathbb{H}_{\mathbb{C}}^2 \simeq S^3$ is the boundary at infinity of the complex hyperbolic plane $\mathbb{H}_{\mathbb{C}}^2$ and $G = \text{PU}(2, 1)$ is the group of holomorphic isometries of $\mathbb{H}_{\mathbb{C}}^2$. Hence, the study of discrete subgroups of $\text{PU}(2, 1)$ is closely related to the understanding of spherical CR structures. An approach to

construct such discrete subgroups is to consider triangle groups. The (p, q, r) -triangle group is the group $\Delta_{(p,q,r)}$ with presentation

$$\langle (\sigma_1, \sigma_2, \sigma_3) \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_2\sigma_3)^p = (\sigma_3\sigma_1)^q = (\sigma_1\sigma_2)^r = \text{Id} \rangle.$$

If p, q or r equals ∞ , then the corresponding relation does not appear. The representations of triangle groups into $\text{PU}(2, 1)$ have been widely studied when the images of σ_1, σ_2 and σ_3 are complex reflexions. For example, in [13], Goldman and Parker study the representations of (∞, ∞, ∞) -triangle groups, and show that they are parametrized, up to conjugation, by a real number $s \in [0, +\infty[$. They conjecture a condition on the parameter for having a discrete and faithful representation. The conjecture, proved by R Schwartz in [19] and [21], can be summarized as follows:

Theorem (Goldman–Parker; Schwartz) *A representation of the (∞, ∞, ∞) -triangle group into $\text{PU}(2, 1)$ is discrete and faithful if and only if the image of $\sigma_1\sigma_2\sigma_3$ is nonelliptic.*

A more complete picture on complex hyperbolic triangle groups can be found in the survey of Schwartz [20], where he states the following conjecture:

Conjecture (Schwartz) *Let $\Delta_{(p,q,r)}$ a triangle group with $p \leq q \leq r$. Then a representation of $\Delta_{(p,q,r)}$ into $\text{PU}(2, 1)$ where the images of the generators are complex reflexions is discrete and faithful if and only if the images of $\sigma_1\sigma_2\sigma_1\sigma_3$ and $\sigma_1\sigma_2\sigma_3$ are not elliptic. Furthermore:*

- (1) *If $p < 10$ then the representation is discrete and faithful if and only if image of $\sigma_1\sigma_2\sigma_1\sigma_3$ is nonelliptic.*
- (2) *If $p > 13$ then the representation is discrete and faithful if and only if image of $\sigma_1\sigma_2\sigma_3$ is nonelliptic.*

More recently, in [3], Deraux studies the representations of $(4, 4, 4)$ -triangle groups, and shows that the representation for which the image of $\sigma_1\sigma_2\sigma_1\sigma_3$ is of order 5 is a lattice in $\text{PU}(2, 1)$. In [17], Parker, Wang and Xie study the representations of $(3, 3, n)$ -triangle groups, and prove the conjecture of Schwartz in this case. Namely, the discrete and faithful representations are the ones for which the image of $\sigma_1\sigma_2\sigma_1\sigma_3$ is nonelliptic. These representations will appear naturally in this article.

Back to the geometric structures, determining if a manifold carries a spherical CR structure or not is a difficult question. The only negative result known to us is due to

Goldman [11], and concerns the torus bundles over the circle. However, we can obtain spherical CR structures on quotients of S^3 of the form $\Gamma \backslash \partial_\infty \mathbb{H}_\mathbb{C}^2$, containing the lens spaces. In [9], Falbel and Gusevskii construct spherical CR structures on circle bundles on hyperbolic surfaces with arbitrary Euler number $e \neq 0$.

Constructing discrete subgroups of $\mathrm{PU}(2, 1)$ can be used to construct spherical CR structures on manifolds. Among general (G, X) -structures, the structures obtained as $\Gamma \backslash X$ for $\Gamma < G$ are called *complete*, and are especially interesting since all the information is given by the group Γ . In the case of the spherical CR structures, we are interested in a more general class of structures, called *uniformizable*; we say that a spherical CR structure on a manifold M is uniformizable if it is obtained as $\Gamma \backslash \Omega_\Gamma$, where $\Omega_\Gamma \subset \partial_\infty \mathbb{H}_\mathbb{C}^2$ is the set of discontinuity of Γ .

The proofs showing that a spherical CR structure is uniformizable often extend the structure to $\mathbb{H}_\mathbb{C}^2$ and use the Poincaré polyhedron theorem, as stated for example by Parker and Will [18]. Besides the examples cited below, there are mainly three known uniformizable spherical CR structures on cusped manifolds. There are two different uniformizable spherical CR structures of the Whitehead link complement. The first one is constructed by Schwartz [22], and the second one is constructed by Parker and Will [18]. In [6], Deraux and Falbel construct a spherical CR uniformization of the figure eight knot complement. This uniformization can be deformed in a one-parameter family of uniformizations, as shown by Deraux in [5].

On the other hand, as in the real hyperbolic case treated in the notes of Thurston [23], we can expect to construct spherical CR uniformizations of other manifolds by performing a Dehn surgery on a cusp of one of the examples above. In [22], Schwartz proves a spherical CR Dehn surgery theorem, stating that, under some convergence conditions, the representations close to the holonomy representation of a uniformizable structure give spherical CR uniformizations of Dehn surgeries of the initial manifold.

This can be applied to the first uniformization of the Whitehead link complement. It leads to infinitely many uniformizable manifolds, parametrized by some rational points in an open set of a deformation space. However, the hypotheses of the Schwartz surgery theorem contain a condition on the *porosity* of the limit set of the holonomy representation, which we were unable to check in the two other cases. In [1], we show another spherical CR Dehn surgery theorem, with weaker hypotheses and weaker conclusions, giving spherical CR structures but not the uniformizability. We apply the theorem to the figure eight knot complement in [1], and we will apply it to the Parker–Will structure in Section 4.5 of this paper.

If we use Dehn surgeries to construct spherical CR uniformizations on manifolds there are two main difficult points. The first one is to apply a theorem or to prove the uniformizability of a given structure. The second one is that the two spherical CR Dehn surgery theorems give structures parametrized by the points of an open set of a space of deformations of representations that is not explicit. Two questions then naturally arise:

- (1) Can we explicitly give an open set of representations giving spherical CR structures on Dehn surgeries of the Whitehead link complement?
- (2) Are these structures uniformizable?

The aim of this article is to answer, at least partially and in a particular case, the two questions above. We will take as a starting point the Parker–Will uniformization of the Whitehead link complement. We will use as space of deformations the representations constructed by Parker and Will in [18], which factor through the group $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. This corresponds to considering a slice of the character variety $\mathcal{X}_{\text{SU}(2,1)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$ as defined in [2]. Furthermore, we will consider the representations of a whole component of the character variety, since Guilloux and Will show in [14] that a whole component of the $\text{SL}_3(\mathbb{C})$ –character variety of the fundamental group of the Whitehead link complement corresponds only to representations that factor through $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

The slice that we will consider is parametrized by a single complex number z , and the representation of the Parker–Will uniformization has parameter $z = 3$. However, we will consider the parametrization of representations used by Parker and Will in [18], given by a pair of angles $(\alpha_1, \alpha_2) \in]-\frac{\pi}{2}, \frac{\pi}{2}[^2$. With these parameters, the representation of the Parker–Will uniformization has parameter $(0, \alpha_2^{\text{lim}})$. For the deformations of the representation having parameter $(0, \alpha_2)$, we show the following two theorems:

Theorem 5.2 *Let $n \geq 4$. Let ρ_n be the representation with parameter $(0, \alpha_2)$ such that $z = 8 \cos^2(\alpha_2) = 2 \cos(\frac{2\pi}{n}) + 1$ in the Parker–Will parametrization. Then ρ_n is the holonomy representation of a spherical CR structure on the Dehn surgery of the Whitehead link complement on T_1 of type $(1, n - 3)$ (ie of slope $1/(n - 3)$)*

Theorem 5.3 *Let $\alpha_2 \in]0, \alpha_2^{\text{lim}}[$. Let ρ be the representation with parameter $(0, \alpha_2)$ in the Parker–Will parametrization. Then ρ is the holonomy representation of a spherical CR structure on the Dehn surgery of the Whitehead link complement on T_1 of type $(1, -3)$ (ie of slope $-\frac{1}{3}$).*

The corresponding representations have been studied previously by Parker and Will in [18] and by Parker, Wang and Xie in [17]. In these two articles, the authors prove

that the groups are discrete using the Poincaré polyhedron theorem in $\mathbb{H}_{\mathbb{C}}^2$, but they do not identify the topology of the manifolds at infinity. In this article, we give a new proof of these facts, but using different and more geometrical techniques, and we establish the topology of the manifolds at infinity.

On the one hand, in [18], Parker and Will study a region $\mathcal{Z} \subset]-\frac{\pi}{2}, \frac{\pi}{2}[^2$ parametrizing representations of $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ with values in $SU(2, 1)$. The region is given in Figure 3, and contains the parameters that appear in the statement of Theorem 5.3. On the other hand, in [17], Parker, Wang and Xie study the representations that appear in the statement of Theorem 5.2, since their images are index-two subgroups of a $(3, 3, n)$ -triangle group in $SU(2, 1)$. They use the Poincaré polyhedron theorem to show that the groups are discrete. The Dirichlet domain used to apply the Poincaré polyhedron theorem is very similar to the domain that we use in this article. However, they do not identify the topology of the manifold at infinity and there is no visible link between the Dirichlet domain of [17] and the Ford domain of [18], which we establish in this article.

Outline of the article This article has three main parts:

In Part I, we give the geometric background needed to state and prove the results. We will set notation and describe the complex hyperbolic plane and several objects related to this space in Section 1, and especially in the visual sphere of a point in $\mathbb{C}\mathbb{P}^2$ in Section 2. We will then focus on the definition and properties of the equidistant hypersurfaces of two points, called *bisectors*, and their continuation to $\mathbb{C}\mathbb{P}^2$, called *extors*, as well as some of their intersections in Section 3.

In Part II, we consider some spherical CR structures on the Whitehead link complement and on manifolds obtained from it by Dehn surgeries. We recall the spherical CR uniformizations of Schwartz and of Parker and Will, and describe a space of deformations of the corresponding holonomy representations. Finally, we apply the surgery theorem of [1], and identify the expected Dehn surgeries that would have a spherical CR structure if the open set of the surgery theorem is large enough.

In Part III, which is the core of the article, we give an explicit deformation of the Ford domain in $\mathbb{H}_{\mathbb{C}}^2$ constructed by Parker and Will in [18] that is bounded by bisectors. We recall the construction of Parker and Will that gives the spherical CR uniformization of the Whitehead link complement when restricted to the boundary at infinity. We consider the deformations of the holonomy representation with parameters $(0, \alpha_2)$, and deform the bisectors that border the Ford domain. By studying their intersections carefully, we show that if a particular element $[U]$ in the group is either loxodromic

or elliptic of finite order ≥ 4 , then the bisectors border a domain in $\mathbb{H}_{\mathbb{C}}^2$ with a face pairing. We identify the manifold obtained by restricting the construction to $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$ as the expected Dehn surgery of the Whitehead link complement. For the parameters for which $[U]$ is an elliptic element of finite order and for some of the parameters for which $[U]$ is loxodromic we apply the Poincaré polyhedron theorem as stated in [18], and show that the spherical CR structures obtained are uniformizable.

In Section 5, we will state the results on surgeries and uniformization and we will give the proof strategy. The rest of Part III will be devoted to proving these statements. Section 6 fixes the notation and describes the construction of Parker and Will in detail. We will prove the statements in Section 7, but admitting some technical conditions that we will prove in the three last sections. We will check the conditions on the faces of the domain, namely, a condition on the topology of the faces in Section 8 and a local combinatorics condition in Section 9, and we will show that the global combinatorics of the intersection of the faces is the expected one in Section 10.

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Part I Geometric background

1 The complex hyperbolic plane and its isometries

In this section, we set notation and recall the definition of the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^2$ and its boundary at infinity $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$. We also briefly describe some isometries and the geometric structure modeled on $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$. The main reference for these objects is the book of Goldman [12].

1.1 Notation and group of isometries

Throughout this article, we will use objects belonging to complex vector spaces and their projectivizations. For a complex vector space V and a vector $v \in V$, we will

denote by $[v]$ its image in $\mathbb{P}V$. The same notation holds for matrix groups. For example, the image of a matrix $M \in \text{SU}(2, 1)$ in the group $\text{PU}(2, 1)$ will be denoted by $[M]$.

Let V be a complex vector space of dimension 3. Let Φ be a Hermitian form of signature $(2, 1)$ on V , and define

$$V_- = \{v \in V : \Phi(v) < 0\} \quad \text{and} \quad V_0 = \{v \in V : \Phi(v) = 0\} - \{0\}.$$

The complex hyperbolic plane is the space $\mathbb{H}_{\mathbb{C}}^2 = \mathbb{P}V_-$ endowed with the Hermitian metric induced by Φ . Its boundary at infinity is the set $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2 = \mathbb{P}V_0$. We denote by $\overline{\mathbb{H}_{\mathbb{C}}^2}$ the set $\mathbb{H}_{\mathbb{C}}^2 \cup \partial_{\infty}\mathbb{H}_{\mathbb{C}}^2 \in \mathbb{C}\mathbb{P}^2$. The space $\mathbb{H}_{\mathbb{C}}^2$ is homeomorphic to a ball B^4 , and $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$ is homeomorphic to the sphere S^3 . If $V = \mathbb{C}^3$ and the Hermitian form Φ has matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

we obtain the Siegel model. In this model, we identify $\mathbb{H}_{\mathbb{C}}^2$ and $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$ as follows:

$$\mathbb{H}_{\mathbb{C}}^2 = \left\{ \left[\begin{array}{c} -\frac{1}{2}(|z|^2 + w) \\ z \\ 1 \end{array} \right] : (z, w) \in \mathbb{C}^2, \text{Re}(w) > 0 \right\},$$

$$\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2 = \left\{ \left[\begin{array}{c} -\frac{1}{2}(|z|^2 + it) \\ z \\ 1 \end{array} \right] : (z, t) \in \mathbb{C} \times \mathbb{R} \right\} \cup \left\{ \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \right\}.$$

In this case, we identify $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$ with $\mathbb{C} \times \mathbb{R} \cup \{\infty\}$.

In [12], Goldman shows that the totally geodesic subspaces of $\mathbb{H}_{\mathbb{C}}^2$ are points, real geodesics, copies of $\mathbb{H}_{\mathbb{C}}^1$, copies of $\mathbb{H}_{\mathbb{R}}^2$ and $\mathbb{H}_{\mathbb{C}}^2$ itself. The copies of $\mathbb{H}_{\mathbb{C}}^1$ are the intersections of linear subspaces of $\mathbb{P}V$ with $\mathbb{H}_{\mathbb{C}}^2$, and are called *complex geodesics*. Notice that given two distinct points of $\mathbb{H}_{\mathbb{C}}^2$ there is a unique complex geodesic containing them, as well as a unique real geodesic containing them. The boundary at infinity of a complex geodesic is called a \mathbb{C} -circle, and the boundary at infinity of a copy of $\mathbb{H}_{\mathbb{R}}^2$ is called an \mathbb{R} -circle. The group $\text{PU}(2, 1)$ acts transitively on each kind of subspace.

The group of holomorphic isometries of $\mathbb{H}_{\mathbb{C}}^2$ is the projectivization of the unitary group for the Hermitian form Φ : we denote it by $\text{PU}(2, 1)$. We will often consider matrices in the group $\text{SU}(2, 1)$ instead of elements of $\text{PU}(2, 1)$; every element of $\text{PU}(2, 1)$ admits exactly three lifts to the group $\text{SU}(2, 1)$ of unitary matrices for Φ of determinant

one. As in real hyperbolic geometry, the elements of $PU(2, 1)$ are classified by their fixed points in $\mathbb{H}_{\mathbb{C}}^2$ and $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$; and we can refine the classification by dynamical considerations. An isometry $[U] \in PU(2, 1)$ is *elliptic* if it fixes a point in $\mathbb{H}_{\mathbb{C}}^2$, *parabolic* if it is not elliptic and fixes exactly one point in $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$ and *loxodromic* if it is not elliptic and it has two fixed points in $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$.

We say that an element is *regular* if it has three different eigenvalues, and *unipotent* if it is not the identity and has three equal eigenvalues (hence equal to a cube root of 1). This last definition and the proposition above extend easily to $PU(2, 1)$. We recall some dynamical properties of regular elliptic elements, since we need to classify some of them.

A regular elliptic element $[U]$ stabilizes two complex geodesics on $\mathbb{H}_{\mathbb{C}}^2$ intersecting at the fixed point of $[U]$, and two linked \mathbb{C} -circles in $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$. In this case, $[U]$ belongs to a one-parameter subgroup of $PU(2, 1)$: the orbits of such a subgroup are the two stable \mathbb{C} -circles and torus knots turning around the two circles. We can define the *type* of a finite-order elliptic element in the following way:

Definition 1.1 Let $[U] \in PU(2, 1)$ be an elliptic of order n . Let $p, q \in \mathbb{Z} \cap]-\frac{n}{2}, \frac{n}{2}]$. We say that $[U]$ is of type $(\frac{p}{n}, \frac{q}{n})$ if it is conjugate in the ball model to

$$\begin{bmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & e^{i\gamma} \end{bmatrix}$$

with $\alpha - \gamma = 2\pi \frac{p}{n}$ and $\beta - \gamma = 2\pi \frac{q}{n}$.

Remark 1.2 Every elliptic element of finite order is of some type $(\frac{p}{n}, \frac{q}{n})$. The nonregular elements correspond to the cases when $p = 0$, $q = 0$ and $p = q$. Indeed, if $p = 0$ or $q = 0$, then the element is boundary elliptic, and if $p = q$ we recover a reflexion on a point.

Remark 1.3 If $[U]$ is elliptic of type $(\frac{p}{n}, \frac{q}{n})$, there is a one-parameter subgroup $([U_s])_{s \in \mathbb{R}}$ such that $[U_1] = [U]$. A generic orbit of the subgroup is a torus knot of type $(p/\gcd(p, q), q/\gcd(p, q))$, turning $p/\gcd(p, q)$ times around a \mathbb{C} -circle C_1 and $q/\gcd(p, q)$ times around a second \mathbb{C} -circle C_2 . The whole orbit is completed in a time $n/\gcd(p, q)$ of the flow, so the action of $[U]$ corresponds morally to $\frac{p}{n}$ turns around C_1 and $\frac{q}{n}$ turns around C_2 . Notice also that if the element is nonregular or p or q equals ± 1 , then the orbits are not knotted.

Remark 1.4 The elements of some type $(\frac{p}{n}, \frac{q}{n})$ where $\gcd(p, q) = 1$ are the ones for which the surgery theorem of [1] works, and for which a geometric structure is expected in the deformations that we consider later in this article.

1.2 Polarity and the box-product

In order to have a better understanding of the space $\mathbb{H}_{\mathbb{C}}^2$, we will sometimes use the language of polars and polarity. This language corresponds to a geometric point of view of orthogonality for the Hermitian form Φ . We state some results following immediately from linear algebra considerations and from the fact that the Hermitian form Φ is nondegenerate.

Definition 1.5 Given a point $[u] \in \mathbb{P}V$, let $[u]^{\perp} = \mathbb{P}\{v \in V - \{0\} : \langle u, v \rangle = 0\}$. It is the projectivization of the orthogonal of u for the Hermitian form Φ . Hence, it is a complex line of $\mathbb{P}V$, called *polar line* of $[u]$.

Notation 1.6 If $[u]$ and $[v]$ are distinct points of $\mathbb{P}V$, we denote by $l_{[u],[v]}$ the complex line passing through $[u]$ and $[v]$.

Definition 1.7 Given a complex line l of $\mathbb{P}V$, there is a unique point $[v] \in \mathbb{P}V$ such that $l = [v]^{\perp}$. We say that $[v]$ is the *pole* of l , and we denote it by $[v] = l^{\perp}$.

Definition 1.8 Let $[u], [v], [w] \in \mathbb{P}V$ be three nonaligned points. We say that they form an *autopolar triangle* if the poles of the lines $l_{[u],[v]}$, $l_{[v],[w]}$ and $l_{[w],[u]}$ are precisely the points $[u]$, $[v]$ and $[w]$.

We state some general remarks about the terms defined above. First, the elements of $\text{PU}(2, 1)$ preserve the polarity. Furthermore, a line is stable under some element $[U] \in \text{PU}(2, 1)$ if and only if its pole is fixed by $[U]$. Finally, if $[U] \in \text{PU}(2, 1)$ has exactly three noncollinear fixed points $[u], [v], [w] \in \mathbb{C}\mathbb{P}^2$, then they form an autopolar triangle.

We can express the polarity in an algebraic language by using the Hermitian cross product, which we define below. It is the complex version of the usual cross product on \mathbb{R}^3 . It is briefly described by Goldman in Chapter 2 of [12].

If $p, q \in \mathbb{C}^3$, we define the *Hermitian cross product* of p and q , denoted by $p \boxtimes q$, as the unique vector $s \in \mathbb{C}^3$ such that $\langle s, r \rangle = \det(p, q, r)$ for all $r \in \mathbb{C}^3$. If p and q are

collinear, then $p \boxtimes q = 0$. If not, then $[p \boxtimes q] = l_{[p],[q]}^\perp$. For the explicit computations that we make in Part III, we will need the expression of the Hermitian cross product with coordinates. We give this expression for the Siegel model in the following lemma, which we obtain immediately by checking the condition $\langle p \boxtimes q, r \rangle = \det(p, q, r)$ for r in the canonical basis of \mathbb{C}^3 .

Lemma 1.9 *In the Siegel model, we have*

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \boxtimes \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \overline{z_1}w_2 - \overline{z_2}w_1 \\ \overline{z_3}w_1 - \overline{z_1}w_3 \\ \overline{z_2}w_3 - \overline{z_3}w_2 \end{pmatrix}.$$

1.3 Spherical CR structures and horotubes

We will consider spherical CR structures on some manifolds in Parts II and III. These structures arise naturally when considering the boundary at infinity of the complex hyperbolic plane; they play the same role as the conformal structures on the boundary of the real hyperbolic space. We recall here some definitions and results about spherical CR structures as well as the definition of the horotubes, which are geometric objects that model cusps in that case.

Definition 1.10 A spherical CR structure on a manifold M is a (G, X) -structure on M for $G = \text{PU}(2, 1)$ and $X = \partial_\infty \mathbb{H}_\mathbb{C}^2$, that is, an atlas of M with charts taking values in $\partial_\infty \mathbb{H}_\mathbb{C}^2$ and transition maps given by elements of $\text{PU}(2, 1)$.

Remark 1.11 A given (G, X) -structure on a manifold M defines a developing map $\text{Dev}: \tilde{M} \rightarrow X$ and a holonomy representation $\rho: \pi_1(M) \rightarrow G$ such that for all $x \in \tilde{M}$ and $g \in \pi_1(M)$ we have $\rho(g)\text{Dev}(x) = \text{Dev}(g \cdot x)$. The holonomy representation is defined up to conjugation and the developing map up to translation by an element of G .

Definition 1.12 Consider a (G, X) -structure on M with holonomy ρ . Let $\Gamma = \text{Im}(\rho)$. We say that the structure is *complete* if $M \simeq \Gamma \backslash X$. We say that the structure is *uniformizable* if $M \simeq \Gamma \backslash \Omega_\Gamma$, where $\Omega_\Gamma \in X$ is the set of discontinuity of Γ .

The usual condition for a structure is to be complete, which is equivalent to being geodesically complete if X has a complete Riemannian metric. However, when considering spherical CR structures, there are very few manifolds admitting complete structures since $\partial_\infty \mathbb{H}_\mathbb{C}^2$ is compact. We will consider noncomplete structures, and

look for uniformizable ones, since they are still intrinsically related to the image of the holonomy representation.

We will consider later in this paper spherical CR uniformizations on two particular manifolds: two uniformizable structures on the Whitehead link complement, constructed by Schwartz in [22] and by Parker and Will in [18], respectively, and a uniformizable structure on the figure eight knot complement, constructed by Deraux and Falbel in [6].

For these three structures, the image of a neighborhood of a cusp by the developing map is a horotube. We recall the definition of this object, which is crucial for attempting to perform spherical CR Dehn surgeries and construct structures on other manifolds, as made in [22] or in [1].

Definition 1.13 Let $[P] \in \text{PU}(2, 1)$ be a parabolic element with fixed point $[p] \in \partial_\infty \mathbb{H}_\mathbb{C}^2$. A $[P]$ -horotube is an open set H of $\partial_\infty \mathbb{H}_\mathbb{C}^2 - \{[p]\}$, invariant under $[P]$ and such that the complement of $H/\langle [P] \rangle$ in $(\partial_\infty \mathbb{H}_\mathbb{C}^2 - \{[p]\})/\langle [P] \rangle$ is compact.

2 The visual sphere of a point in $\mathbb{C}\mathbb{P}^2$

We define here the visual sphere of a point in $\mathbb{C}\mathbb{P}^2$ and give coordinates for some charts of this object. We will use the visual sphere of a point of $\mathbb{C}\mathbb{P}^2$ in order to have a better understanding of bisectors and their topology, in Section 3, and to parametrize some intersections. We will also use this tool to control the intersections of the faces of the deformed Ford domain that we construct in Part III.

Definition 2.1 Let $[p] \in \mathbb{C}\mathbb{P}^2$. We call the set of complex lines of $\mathbb{C}\mathbb{P}^2$ passing through $[p]$ the *visual sphere* of $[p]$. We will denote it by $L_{[p]}$. In this way, $L_{[p]} = \{l_{[p],[q]} : [q] \in \mathbb{C}\mathbb{P}^2 - \{[p]\}\}$.

The space $L_{[p]}$ is isomorphic to $\mathbb{C}\mathbb{P}^1$. We can identify it in two other ways. We will often use the abuse of language corresponding to the following identifications. On the one hand, the set of lines passing through $[p]$ is the projectivization of the tangent space to $\mathbb{C}\mathbb{P}^2$ at $[p]$, hence $L_{[p]} = \mathbb{P}(T_{[p]}\mathbb{C}\mathbb{P}^2)$. On the other hand, by considering the dual space, we also have $L_{[p]} = \{[\varphi] \in \mathbb{P}((\mathbb{C}^3)^*) : \varphi(p) = 0\}$. Finally, if we have a Hermitian product, \mathbb{C}^3 is canonically identified to its dual, and we have $L_{[p]} = [p]^\perp$.

We are now going to give coordinates for the visual sphere of a point $[p]$. These coordinates will be useful for making explicit computations in this space. The following proposition gives a way to construct a chart.

Proposition 2.2 *Let $\varphi_1, \varphi_2 \in (\mathbb{C}^3)^*$ be two independent linear forms such that $\varphi_1(p) = \varphi_2(p) = 0$. Then the map*

$$f: L_{[p]} \rightarrow \mathbb{C}\mathbb{P}^1, \quad l_{[p],[q]} \mapsto \frac{\varphi_1(q)}{\varphi_2(q)},$$

is well defined and an isomorphism.

By translating this fact in terms of orthogonality, we obtain:

Proposition 2.3 *Let $[p] \in \mathbb{C}\mathbb{P}^2$. Let $[p'], [p''] \in [p]^\perp$ be two distinct points. Then the map*

$$f: L_{[p]} \rightarrow \mathbb{C}\mathbb{P}^1, \quad l_{[p],[q]} \mapsto \frac{\langle p', q \rangle}{\langle p'', q \rangle},$$

is well defined and an isomorphism.

Notation 2.4 From now on, we will denote this map by $\psi_{p',p''}: L_{[p]} \rightarrow \mathbb{C}\mathbb{P}^1$.

By choosing an autopolar triangle as a frame, the computations are easier in the corresponding chart. Indeed, if $[p], [p']$ and $[p'']$ form an autopolar triangle, then $f([p''])$ and $f([p'])$ are 0 and ∞ , the order depending on the type of the autopolar triangle.

Remark 2.5 If $[q], [q'] \in \mathbb{C}\mathbb{P}^2 - \{p\}$, then $f([q])/f([q'])$ is the cross ratio of $[p], [p'], [q]$ and $[q']$.

3 Extors, bisectors and spinal surfaces

In this section, we consider some objects appearing naturally when constructing Dirichlet or Ford domains in $\mathbb{H}_{\mathbb{C}}^2$. These objects will be the surfaces equidistant to two points, which we call metric bisectors, and some natural generalizations of them, which we simply call bisectors. In order to study them, we will also describe their analytic continuation to $\mathbb{C}\mathbb{P}^2$, called extors, and their intersection with $\partial_\infty \mathbb{H}_{\mathbb{C}}^2$, called spinal surfaces. In his book [12], Goldman dedicates Chapter 5 to the topological study of metric bisectors, Chapter 8 to extors, extending bisectors to $\mathbb{C}\mathbb{P}^2$, and Chapter 9 to some intersections of bisectors. We will use significantly this study of bisectors and extors. However, we adopt a different point of view, closer to projective geometry.

3.1 Definitions

We begin by defining the objects that we will be using, beginning with the metric bisectors, which are the equidistant surfaces of two points in $\mathbb{H}_{\mathbb{C}}^2$.

Definition 3.1 Let $[p], [q] \in \mathbb{H}_{\mathbb{C}}^2$ be two distinct points. The *metric bisector*¹ of $[p]$ and $[q]$ is the set $\mathfrak{B} = \{[z] \in \mathbb{H}_{\mathbb{C}}^2 : d([z], [p]) = d([z], [q])\}$. If $p, q \in \mathbb{C}^3$ are lifts of $[p]$ and $[q]$ such that $\langle p, p \rangle = \langle q, q \rangle$, then the bisector can be written as $\mathfrak{B} = \{[z] \in \mathbb{H}_{\mathbb{C}}^2 : |\langle z, p \rangle| = |\langle z, q \rangle|\}$. Its boundary at infinity is a *spinal sphere*.

Goldman [12, Chapter 5] shows that, topologically, a metric bisector is a 3-dimensional ball, and that a spinal sphere is a smooth sphere in $\partial_{\infty} \mathbb{H}_{\mathbb{C}}^2$. They are analytic objects, but they are not totally geodesic, since there are no totally geodesic subspaces of $\mathbb{H}_{\mathbb{C}}^2$ of dimension 3. We define the extors below. They are objects of $\mathbb{C}\mathbb{P}^2$ extending the metric bisectors. We keep the terms used by Goldman in [12] for these objects.

Definition 3.2 Let $[f] \in \mathbb{C}\mathbb{P}^2$. Let C be a real circle in $L_{[f]}$. The *extor* from $[f]$ given by C is the set $\mathfrak{E} = \{[z] \in \mathbb{C}\mathbb{P}^2 : l_{[f],[z]} \in C\}$. In this context, $[f]$ is the *focus* of \mathfrak{E} .

Note that all extors are projectively equivalent. Notice also that every metric bisector extends to an extor. If \mathfrak{B} is the metric bisector of $[p]$ and $[q]$, then it extends to an extor with focus $[p \boxtimes q]$, given by $\mathfrak{E} = \{[z] \in \mathbb{C}\mathbb{P}^2 : |\langle z, p \rangle| = |\langle z, q \rangle|\}$ if p and q are lifts of $[p]$ and $[q]$ such that $\langle p, p \rangle = \langle q, q \rangle$. The corresponding circle $C \subset L_{[p \boxtimes q]} \simeq [p \boxtimes q]^{\perp}$ is given by $C = \{[p - \alpha q] : \alpha \in S^1\}$. This facts motivate the study of extors in $\mathbb{C}\mathbb{P}^2$ and their intersections in order to understand the bisectors and their intersections.

We will consider, when deforming a Ford domain in [Part III](#), some objects defined in the same way, but with fewer restrictions on the points $[p], [q] \in \mathbb{C}\mathbb{P}^2$. Hence, we define the following generalization of the notion of metric bisector, which we study below.

Definition 3.3 Let $p, q \in \mathbb{C}^3 - \{0\}$. Then:

- The *extor* of p and q is $\mathfrak{E}(p, q) = \{[z] \in \mathbb{C}\mathbb{P}^2 : |\langle z, p \rangle| = |\langle z, q \rangle|\}$.
- The *bisector* of p and q is the intersection of the extor with $\mathbb{H}_{\mathbb{C}}^2$: $\mathfrak{B}(p, q) = \{[z] \in \mathbb{H}_{\mathbb{C}}^2 : |\langle z, p \rangle| = |\langle z, q \rangle|\}$.
- The *spinal surface* of p and q is the boundary at infinity of the bisector: $\mathfrak{S}(p, q) = \{[z] \in \partial_{\infty} \mathbb{H}_{\mathbb{C}}^2 : |\langle z, p \rangle| = |\langle z, q \rangle|\}$.

We will limit ourselves to the case where the points p and q defining an extor, a bisector or a spinal sphere have the same norm. It is always the case when they are in

¹In the literature, it is simply called the *bisector*. We will use this last term for a more general object that we will define in [Definition 3.3](#).

the same orbit for a subgroup of $SU(2, 1)$. We can recover the complex lines of the extor $\mathfrak{E}(p, q)$ in the following way:

Proposition 3.4 *Let $p, q \in \mathbb{C}^3 - \{0\}$. The extor $\mathfrak{E}(p, q)$ can be written as a union of complex lines in the following way:*

$$\mathfrak{E}(p, q) = \bigcup_{\alpha \in S^1} [q - \alpha p]^\perp.$$

Remark 3.5 By Proposition 3.4, the extor of p and q is an extor in terms of Definition 3.2. It is clear, by Definition 3.1, that a metric bisector is a bisector and that a spinal sphere is a spinal surface.

Remark 3.6 If we want to define these objects for $[p], [q] \in \mathbb{C}P^2$, different choices of lifts will lead to different objects. From now on, we will only consider the case when the lifts satisfy $\langle p, p \rangle = \langle q, q \rangle$.

3.2 Topology of bisectors and spinal surfaces

We are going to study in detail the topology of the objects that we defined above, and we will see that there are three possibilities, depending on the relative position of certain points and $\partial_\infty \mathbb{H}_\mathbb{C}^2$. We begin by defining the complex spine and the real spine of a bisector or an extor, which are a complex and a real line of $\mathbb{C}P^2$, and that will help us to understand extors and bisectors.

Definition 3.7 Let \mathfrak{E} be an extor with focus $[f]$ and given by the circle $C \subset L_{[f]}$. The *complex spine* Σ of \mathfrak{E} is the complex line $[f]^\perp$. By identifying $L_{[f]}$ with $[f]^\perp$, we define the *real spine* of \mathfrak{E} as the real circle $\sigma \subset \Sigma$ corresponding to C .

Remark 3.8 If $[f] \notin \partial_\infty \mathbb{H}_\mathbb{C}^2$, then its real spine σ is the set $\mathfrak{E} \cap \Sigma$.

Remark 3.9 In the case of metric bisectors, as described by Goldman in [12], the complex and the real spine are the intersections with $\mathbb{H}_\mathbb{C}^2$ of the ones we defined above.

Remark 3.10 Let $p, q \in \mathbb{C}^3 - \{0\}$ be two distinct points. The complex spine of the extor $\mathfrak{E}(p, q)$ is the complex line $l_{[p],[q]}$.

3.2.1 Two decompositions of extors We recall two decompositions of extors that will be useful: the slice decomposition, in complex lines, and the meridional decomposition, in real planes. For a more detailed exposition, see Goldman [12, Chapters 5 and 8].

Proposition 3.11 (slice decomposition) *Let \mathfrak{E} be an extor with focus $[f]$. Then the complex lines contained in \mathfrak{E} passing through $[f]$ form a foliation of $\mathfrak{E} - \{[f]\}$ and are the only complex lines contained in \mathfrak{E} . If, moreover, $[f] \notin \partial_\infty \mathbb{H}_\mathbb{C}^2$ and \mathfrak{E} admits Σ as complex spine and σ as real spine, they are precisely the complex lines orthogonal to Σ at the points of σ .*

Such a complex line is called a *slice* of \mathfrak{E} . This decomposition is called the *slice decomposition of the extor*. We can also consider it on the corresponding bisector. The other decomposition given by Goldman is the meridional decomposition, given in the following proposition. For a complete proof, see [12, Section 8.2.3 or Theorem 5.1.10].

Proposition 3.12 (meridional decomposition) *Let σ be a real circle in \mathbb{CP}^2 . Then the union of the real planes of \mathbb{CP}^2 containing σ form a singular foliation of the extor of real spine σ .*

Such a real plane is called a *meridian* of \mathfrak{E} ; the associated decomposition is the *meridional decomposition* of \mathfrak{E} , which we can also consider on the corresponding bisector.

3.2.2 Metric bisectors, spinal spheres and fans We begin by describing the metric bisectors, which are the usual bisectors studied in detail by Goldman in [12]. In our context, a bisector is a metric bisector if and only if its focus belongs to $\mathbb{CP}^2 - \overline{\mathbb{H}_\mathbb{C}^2}$ and its real spine is orthogonal to $\partial_\infty \mathbb{H}_\mathbb{C}^2$. Following [12], we have the next proposition about the topology of metric bisectors and spinal spheres.

Proposition 3.13 *A metric bisector is homeomorphic to a 3–dimensional ball. A spinal sphere is a smooth sphere $\partial_\infty \mathbb{H}_\mathbb{C}^2$.*

We now describe other types of bisectors that are not necessarily metric bisectors but that will be useful to construct fundamental domains on $\mathbb{H}_\mathbb{C}^2$ for the actions of some subgroups of $\text{PU}(2, 1)$. We will see the fans and the Clifford cones. We begin by defining and describing a fan.

Definition 3.14 *A fan is a bisector whose focus $[f]$ belongs to $\partial_\infty \mathbb{H}_\mathbb{C}^2$ and that does not contain $[f]^\perp$ as a slice. A spinal fan is its boundary at infinity.*

Proposition 3.15 *A fan is homeomorphic to a 3–dimensional ball. A spinal fan is a smooth sphere with a singular point in $\partial_\infty \mathbb{H}_\mathbb{C}^2$.*

Proof Let \mathfrak{E} be an extor with focus $[f] \in \partial_\infty \mathbb{H}_\mathbb{C}^2$. We work on the Siegel model and suppose, without loss of generality, that

$$[f] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Without loss of generality, suppose that the \mathbb{R} -plane of points with real coordinates is a meridian of \mathfrak{E} . The slices of the bisectors are hence of the form T_r for $r \in \mathbb{R}$, where

$$T_r = \{[f]\} \cup \left\{ \begin{bmatrix} z \\ r \\ 1 \end{bmatrix} : z \in \mathbb{C} \right\}.$$

Hence, the bisector $\mathfrak{E} \cap \mathbb{H}_\mathbb{C}^2$ is diffeomorphic to the set $\{(z, r) \in \mathbb{C} \times \mathbb{R} : 2 \operatorname{Re}(z) < r^2\}$, which is diffeomorphic to a 3-dimensional ball. Its boundary at infinity is the following set, which is a smooth sphere besides the point $[f]$, where it has a singularity:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} -\frac{1}{2}(r^2 + it) \\ r \\ 1 \end{bmatrix} : (r, t) \in \mathbb{R}^2 \right\}. \quad \square$$

3.2.3 Clifford cones and tori For the last type of bisectors, which are Clifford cones, we use partially the terms given by Goldman in his book [12]. Nevertheless, it seems preferable for us to use the term Clifford torus for the boundary at infinity of a Clifford cone, even if Goldman keeps this name for the intersections of extors.

Definition 3.16 A *Clifford cone* is a bisector whose focus belongs to $\mathbb{H}_\mathbb{C}^2$. A *Clifford torus* is its boundary at infinity.

Proposition 3.17 A *Clifford cone* is homeomorphic to $(S^1 \times D^2)/(S^1 \times \{0\})$. A *Clifford torus* is a smooth torus in $\partial_\infty \mathbb{H}_\mathbb{C}^2$.

Proof Let \mathfrak{E} be an extor with focus $[f] \in \mathbb{H}_\mathbb{C}^2$. Every complex line of $\mathbb{C}\mathbb{P}^2$ passing through $[f]$ intersects $\mathbb{H}_\mathbb{C}^2$ in a complex geodesic, which is homeomorphic to a disk D^2 , and has a \mathbb{C} -circle as boundary at infinity. By the slice decomposition of \mathfrak{E} , we know that $(\mathfrak{E} - \{[f]\}) \cap \mathbb{H}_\mathbb{C}^2$ is homeomorphic to $S^1 \times (D^2 - \{[f]\})$, and that its boundary at infinity is homeomorphic to $S^1 \times S^1$. □

Considering the relative position of two points $[p]$ and $[q]$ and of the pole $[p \boxtimes q]$ of the line $l_{[p],[q]}$, we obtain the following proposition:

Proposition 3.18 Let $p, q \in \mathbb{C}^3 - \{0\}$ be noncollinear and such that $\langle p, p \rangle = \langle q, q \rangle$. Let $r = \langle p, p \rangle \langle q, q \rangle - \langle p, q \rangle \langle q, p \rangle$. We suppose that $\mathfrak{B}(p, q) \neq \emptyset$.

- If $r < 0$, then $\mathfrak{B}(p, q)$ is a bisector and $\mathfrak{S}(p, q)$ is a spinal sphere.
- If $r = 0$, then $\mathfrak{B}(p, q)$ is a fan and $\mathfrak{S}(p, q)$ is a spinal fan.
- If $r > 0$, then $\mathfrak{B}(p, q)$ is a Clifford cone and $\mathfrak{S}(p, q)$ is a Clifford torus.

3.3 From the visual sphere

We are going to state two facts about bisectors and some particular visual spheres. We take the following notation for the natural projection on a visual sphere:

Notation 3.19 Let $[p] \in \mathbb{CP}^2$. We denote by

$$\pi_{[p]}: \mathbb{CP}^2 - \{[p]\} \rightarrow L_{[p]}, \quad [q] \mapsto l_{[p],[q]},$$

the natural projection on the visual sphere $L_{[p]}$.

The first part of the following remark follows from the definition of an extor by a focus and a circle in the visual sphere:

Remark 3.20 Let $p, q \in \mathbb{C}^3 - \{0\}$ be two distinct points such that $\langle p, p \rangle = \langle q, q \rangle$. Then $\pi_{[p \boxtimes q]}(\mathfrak{E}(p, q))$ is a circle in $L_{[p \boxtimes q]}$. Also, $\pi_{[p \boxtimes q]}(\mathfrak{B}(p, q))$ is an arc of a circle or a circle in $L_{[p \boxtimes q]}$, depending on whether or not $[p \boxtimes q] \in \overline{\mathbb{H}}_{\mathbb{C}}^2$; the set $\pi_{[p \boxtimes q]}(\mathfrak{S}(p, q))$ is its closure.

The following proposition describes the projection on the visual sphere $L_{[p]}$ of a bisector defined as $\mathfrak{B}(p, q)$ or of its corresponding spinal surface.

Proposition 3.21 Let $p, q \in \mathbb{C}^3 - \{0\}$ be noncollinear and such that $\langle p, p \rangle = \langle q, q \rangle$. Let $r = \langle p, p \rangle \langle q, q \rangle - \langle p, q \rangle \langle q, p \rangle$.

- If $r \neq 0$, then $\pi_{[p]}(\mathfrak{S}(p, q))$ is a closed disk with centers $l_{[p],[q]}$ and $l_{[p],[p \boxtimes q]}$.
- If $r = 0$, then $\pi_{[p]}(\mathfrak{S}(p, q))$ is a closed disk whose boundary contains $l_{[p],[q]}$.

In both cases, $\pi_{[p]}(\mathfrak{B}(p, q))$ is the interior of $\pi_{[p]}(\mathfrak{S}(p, q))$.

Proof First, notice that a complex line that intersects $\mathfrak{B}(p, q)$ also intersects $\mathfrak{S}(p, q)$, and that the complex lines that intersect $\mathfrak{S}(p, q)$ also intersect $\mathfrak{B}(p, q)$ unless they are tangent to $\mathfrak{S}(p, q)$. This will show the last point.

Suppose at first that $r \neq 0$. In this case, $(p, q, p \boxtimes q)$ is a basis of \mathbb{C}^3 . The stabilizer of $([p], [q])$ in $\text{PU}(2, 1)$ is then isomorphic to S^1 and its elements can be written in the basis $(p, q, p \boxtimes q)$ in the form

$$\begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{bmatrix}.$$

These elements stabilize the bisector $\mathfrak{B}(p, q)$ and, since they fix $[p]$, act in a natural way on $L_{[p]}$. In the chart $\psi_{q, p \boxtimes q}$, the action is given by a rotation centered at 0. In the chart sending $l_{[p],[q]}$ to 0 and $l_{[p],[p \boxtimes q]}$ to ∞ , the projection $\pi_{[p]}(\mathfrak{S}(p, q))$ is a compact set that is connected and invariant by the rotations of $\mathbb{C} \cup \{\infty\}$. It only remains to check that exactly one of the points 0 and ∞ belongs to the image of $\pi_{[p]}$. This corresponds to checking that only one line among $l_{[p],[q]}$ and $l_{[p],[p \boxtimes q]}$ intersects $\mathfrak{B}(p, q)$.

First case ($r < 0$) Let $\alpha \in \mathbb{C}$ be of modulus 1 such that $\langle p, \alpha q \rangle \in \mathbb{R}^-$. On the one hand, we know that $\langle p + \alpha q, p + \alpha q \rangle = 2(\langle p, p \rangle + \langle p, \alpha q \rangle) < 0$ since $r < 0$. On the other hand, $|\langle p, p + \alpha q \rangle| = |\langle p, p \rangle - |\langle p, q \rangle|| = |\langle q, q \rangle - |\langle p, q \rangle|| = |\langle q, p + \alpha q \rangle|$. Hence we know that $l_{[p],[q]}$ intersects $\mathfrak{B}(p, q)$.

Let us show that $l_{[p],[p \boxtimes q]}$ does not intersect $\mathfrak{S}(p, q)$. We know that $[p \boxtimes q] \notin \overline{\mathbb{H}^2_{\mathbb{C}}}$. Let $\lambda \in \mathbb{C}$. We have

$$|\langle p + \lambda p \boxtimes q, p \rangle| = |\langle p, p \rangle| \quad \text{and} \quad |\langle p + \lambda p \boxtimes q, q \rangle| = |\langle p, q \rangle|.$$

Since $r < 0$, the two quantities are different, hence $l_{[p],[p \boxtimes q]}$ does not intersect $\mathfrak{S}(p, q)$.

Second case ($r > 0$) In this case, we know that $l_{[p],[q]}$ does not intersect $\overline{\mathbb{H}^2_{\mathbb{C}}}$, and hence does not intersect $\mathfrak{S}(p, q)$. Furthermore, $[p \boxtimes q] \in \mathbb{H}^2_{\mathbb{C}}$ and $\langle p, p \boxtimes q \rangle = \langle p, p \boxtimes q \rangle = 0$, hence $l_{[p],[p \boxtimes q]}$ intersects $\mathfrak{B}(p, q)$.

Third case ($r = 0$) Suppose now that $r = 0$. In this case, $[p \boxtimes q] \in \partial_{\infty} \mathbb{H}^2_{\mathbb{C}}$ and $[p], [q]$ and $[p \boxtimes q]$ are collinear. The fixing subgroup of $l_{[p],[q]}$ is then a vertical unipotent subgroup isomorphic to \mathbb{R} that acts on $L_{[p]}$ as a translation. Furthermore, since $l_{[p],[q]}$ is tangent to $\partial_{\infty} \mathbb{H}^2_{\mathbb{C}}$ at $[p \boxtimes q]$, $\pi_{[p]}(\mathfrak{B}(p, q)) - \{l_{[p],[q]}\}$ is connected and invariant by one translation direction.

Consider coordinates in the Siegel model. By multiplying by an element of $\text{SU}(2, 1)$, we can assume that

$$p = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad p \boxtimes q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

In this case, q can be written in the following way, where $\theta \in \mathbb{R}$:

$$q = \begin{pmatrix} -1 \\ e^{i\theta} \\ 0 \end{pmatrix}.$$

We choose z_1/z_3 as the coordinate in $\mathcal{L}_{[p]}$ for the complex line passing through

$$[p] \quad \text{and} \quad \begin{bmatrix} z_1 \\ 1 \\ z_3 \end{bmatrix}.$$

With these coordinates, the fixing subgroup of $l_{[p],[q]}$ acts on $\mathcal{L}_{[p]}$ by translations of the form $z \mapsto z + it$, where $t \in \mathbb{R}$. In order to determine the image of $\mathfrak{B}(p, q)$ by $\pi_{[p]}$, it is enough to determine the set of $s \in \mathbb{R}$ such that a point of the form

$$[w_s] = \begin{bmatrix} sz_3 \\ 1 \\ z_3 \end{bmatrix}$$

belongs to $\overline{\mathfrak{B}}(p, q)$. We compute then $\langle p, w_s \rangle = 1$ and $\langle q, w_s \rangle = e^{i\theta} - z_3$. In order to have $[w_s] \in \mathfrak{B}(p, q)$, it is then necessary that z_3 could be written in the form $z_3(\phi) = e^{i\theta} + e^{i\phi}$, where $\phi \in \mathbb{R}$.

But $\langle w_s, w_s \rangle = 1 + 2s|z_3|^2$, and $0 \leq |z_3|^2 \leq 4$, where the two equalities hold when $\phi = \pi + \theta$ and $\phi = \theta$, respectively. We deduce that there exists $\phi \in \mathbb{R}$ such that $1 + 2s|z_3(\phi)|^2 \leq 0$ if and only if $s \leq -\frac{1}{8}$. The image of $\overline{\mathfrak{B}}(p, q)$ by $\pi_{[p]}$ is hence equal to

$$\{s + it : (s, t) \in \mathbb{R}^2, s \leq -\frac{1}{8}\} \cup \{\infty\}.$$

It is hence a closed disk whose boundary contains ∞ , the coordinate of $l_{[p],[q]}$. □

3.4 Pairs of extors

We now consider some intersections of bisectors in order to study some Ford domains and their deformations. We consider here, in the same way as Goldman does in [12, Chapter 8], the intersections of the corresponding extors before the intersections of bisectors. Recall that two extors \mathfrak{E}_1 and \mathfrak{E}_2 with respective foci $[f_1]$ and $[f_2]$ are called *confocal* if $[f_1] = [f_2]$, *balanced* if $[f_1] \neq [f_2]$ and $l_{[f_1],[f_2]} \subset \mathfrak{E}_1 \cap \mathfrak{E}_2$, *semibalanced* if $[f_1] \neq [f_2]$ and $l_{[f_1],[f_2]}$ is contained in exactly one of the two extors and *unbalanced* if $[f_1] \neq [f_2]$ and $l_{[f_1],[f_2]}$ is not contained in either of the two extors.

Definition 3.22 We say that a pair of extors $(\mathfrak{E}_1, \mathfrak{E}_2)$ is *coequidistant* if there exist distinct points $p, q, r \in \mathbb{C}^3 - \{0\}$ such that $\mathfrak{E}_1 = \mathfrak{E}(p, q)$ and $\mathfrak{E}_2 = \mathfrak{E}(p, r)$.

Remark 3.23 A coequidistant pair of extors is either confocal or unbalanced.

When considering Ford domains and their deformations, we will only work with coequidistant bisectors with respect to normalized lifts. The following result describes the intersection of a balanced pair of extors. See [12, Chapter 8] for a detailed proof.

Theorem 3.24 [12, Theorem 8.3.1] *Let $(\mathfrak{E}_1, \mathfrak{E}_2)$ be a balanced pair of extors. Then there exist a complex line l and a real plane P of $\mathbb{C}\mathbb{P}^2$ such that $\mathfrak{E}_1 \cap \mathfrak{E}_2 = P \cup l$.*

The following proposition is obtained by a direct computation, and describes the intersection in $\mathbb{C}\mathbb{P}^2$ of two extors of an unbalanced pair. Goldman calls this intersection a “Clifford torus”, but we chose to keep this term for the boundary at infinity of a Clifford cone.

Proposition 3.25 *Let $(\mathfrak{E}_1, \mathfrak{E}_2)$ be an unbalanced pair of extors. Then $\mathfrak{E}_1 \cap \mathfrak{E}_2$ is a torus in $\mathbb{C}\mathbb{P}^2$. If $\mathfrak{E}_1 = \mathfrak{E}(p, q)$ and $\mathfrak{E}_2 = \mathfrak{E}(p, r)$, then the intersection is parametrized by $[(q - \alpha p) \boxtimes (r - \beta p)]$, where $(\alpha, \beta) \in S^1 \times S^1$.*

3.5 Pairs of coequidistant bisectors

From now on, we will consider unbalanced pairs of bisectors defined by normalized lifts, and we will be interested in their intersections.

3.5.1 Goldman intersections In [12, Chapter 9], Goldman considers pairs of bisectors coequidistant from points of $\mathbb{H}_{\mathbb{C}}^2$ or $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$, and shows that their intersection is connected and a topological disk. We recall here some of the results, obtained by a study of tangencies of spinal spheres.

Proposition 3.26 [12, Lemma 9.1.5] *Let \mathfrak{B}_1 and \mathfrak{B}_2 be two metric bisectors with boundaries at infinity \mathfrak{S}_1 and \mathfrak{S}_2 , respectively. Then each connected component of $\mathfrak{S}_1 \cap \mathfrak{S}_2$ is a point or a circle, and each connected component of $\mathfrak{B}_1 \cap \mathfrak{B}_2$ is a disk.*

We give a particular name to these disks, which will often appear in the constructions of Ford or Dirichlet domains.

Definition 3.27 A *Giraud disk* is such a disk in the intersection of bisectors. A *Giraud circle* is its boundary at infinity.

Finally, the following theorem ensures us that the intersection of two coequidistant metric bisectors is either a point or a Giraud disk, and that the intersection of the corresponding spinal spheres is either a point or a Giraud circle.

Theorem 3.28 [12, Theorem 9.2.6] *Let $p, q, r \in \mathbb{C} - \{0\}$ such that $\langle p, p \rangle = \langle q, q \rangle = \langle r, r \rangle \leq 0$. Then $\mathfrak{S}(p, q) \cap \mathfrak{S}(p, r)$ is connected.*

3.5.2 Other intersections We will need to consider more general intersections in order to deform a Ford domain. We describe an explicit example that will be useful later. It is a very symmetrical case, where the bisectors are equidistant from points in the same real plane, and have an order 3 symmetry. We will need the following lemma for a technical point about a sign in [Proposition 3.30](#).

Lemma 3.29 *Let $p, q, r \in \mathbb{C}^3 - \{0\}$. Assume that they belong to the same \mathbb{R} -plane and that there exists $S \in \text{SU}(2, 1)$ of order 3 such that $Sp = q$ and $Sq = r$. Then $\langle p \boxtimes q, q \boxtimes r \rangle = \langle q \boxtimes r, r \boxtimes p \rangle = \langle r \boxtimes p, p \boxtimes q \rangle \in \mathbb{R}^-$.*

Proof Since p, q, r belong to the same \mathbb{R} -plane, the points $p \boxtimes q$, $q \boxtimes r$ and $r \boxtimes p$ are also in the same \mathbb{R} -plane. By the symmetry of order 3, we know that $\langle p \boxtimes q, p \boxtimes q \rangle = \langle q \boxtimes r, q \boxtimes r \rangle = \langle r \boxtimes p, r \boxtimes p \rangle$. Denote this quantity by $l \in \mathbb{R}$. On the other hand, we also know that $\langle p \boxtimes q, q \boxtimes r \rangle = \langle q \boxtimes r, r \boxtimes p \rangle = \langle r \boxtimes p, p \boxtimes q \rangle \in \mathbb{R}$. Denote this quantity by k .

Consider the generic case, where $k \neq 0$ and $(p \boxtimes q, q \boxtimes r, r \boxtimes p)$ is a basis of \mathbb{C}^3 ; the result will follow in the general case by density. In this basis, the matrix of the Hermitian form is

$$\begin{pmatrix} l & k & k \\ k & l & k \\ k & k & l \end{pmatrix}.$$

It admits a double eigenvalue equal to $l - k$ and a simple eigenvalue equal to $l + 2k$. Since the Hermitian form has signature $(2, 1)$, we deduce that $l - k > 0$ and $l + 2k < 0$, which implies $k < 0$. \square

Proposition 3.30 *Let $p, q, r \in \mathbb{C}^3 - \{0\}$. Assume that they belong to the same \mathbb{R} -plane and that there exists $S \in \text{SU}(2, 1)$ of order 3 such that $Sp = q$ and $Sq = r$. We know that $p \boxtimes q$, $q \boxtimes r$ and $r \boxtimes p$ have the same norm and belong to the same \mathbb{R} -plane. By [Lemma 3.29](#), we know that $\langle p \boxtimes q, q \boxtimes r \rangle = \langle q \boxtimes r, r \boxtimes p \rangle = \langle r \boxtimes p, p \boxtimes q \rangle \in \mathbb{R}^-$. Let $u = \langle p \boxtimes q, p \boxtimes q \rangle / \langle p \boxtimes q, q \boxtimes r \rangle$. Then:*

- If $u < \frac{2}{3}$, then $\mathfrak{B}(p, q) \cap \mathfrak{B}(p, r)$ is a disk, and its boundary at infinity is a smooth circle.
- If $u = \frac{2}{3}$, then $\mathfrak{B}(p, q) \cap \mathfrak{B}(p, r)$ is a disk, and its boundary at infinity consists of three \mathbb{C} -circles.
- If $u > \frac{2}{3}$, then $\mathfrak{B}(p, q) \cap \mathfrak{B}(p, r)$ is a torus minus two disks, and its boundary at infinity consists of two smooth circles.

Proof The intersection of the extors $\mathfrak{E}(p, q)$ and $\mathfrak{E}(p, r)$ is the torus parametrized by

$$\{(q + e^{i\theta} p) \boxtimes (r + e^{i\phi} p) : (\theta, \phi) \in [\pi, \pi]^2\}.$$

The points of the intersection $\mathfrak{B}(p, q) \cap \mathfrak{B}(p, r)$ are exactly the points of this torus with negative norm. We compute the norm of $(q + e^{i\theta} p) \boxtimes (r + e^{i\phi} p)$:

$$\begin{aligned} &\langle (q + e^{i\theta} p) \boxtimes (r + e^{i\phi} p), (q + e^{i\theta} p) \boxtimes (r + e^{i\phi} p) \rangle \\ &= \langle q \boxtimes r + e^{i\theta} p \boxtimes r + e^{i\phi} q \boxtimes p, q \boxtimes r + e^{i\theta} p \boxtimes r + e^{i\phi} q \boxtimes p \rangle \\ &= \langle q \boxtimes r, q \boxtimes r \rangle + \langle p \boxtimes r, p \boxtimes r \rangle + \langle q \boxtimes p, q \boxtimes p \rangle \\ &\quad + 2 \operatorname{Re}(e^{i\theta} \langle q \boxtimes r, p \boxtimes r \rangle + e^{i\phi} \langle q \boxtimes r, q \boxtimes p \rangle + e^{i(\phi-\theta)} \langle p \boxtimes r, q \boxtimes p \rangle) \\ &= 3 \langle q \boxtimes r, q \boxtimes r \rangle + 2 \langle q \boxtimes r, p \boxtimes r \rangle (\cos(\theta) + \cos(\phi) + \cos(\phi - \theta)) \\ &= 2 \langle q \boxtimes r, p \boxtimes r \rangle (\frac{3}{2}u + \cos(\theta) + \cos(\phi) + \cos(\phi - \theta)). \end{aligned}$$

Since $\langle q \boxtimes r, p \boxtimes r \rangle < 0$, the sign of the expression above is the opposite of the sign of $\frac{3}{2}u + \cos(\theta) + \cos(\phi) + \cos(\phi - \theta)$. The level sets of the function $(\theta, \phi) \mapsto \cos(\theta) + \cos(\phi) + \cos(\phi - \theta)$ are traced in [Figure 1](#). The set describing $\mathfrak{B}(p, q) \cap \mathfrak{B}(p, r)$ is hence given by the level sets of level $\geq -\frac{3}{2}u$, which are

- a disk with smooth boundary if $u < \frac{2}{3}$,
- a disk with boundary the circles of equations $\theta = 0$, $\phi = 0$ and $\phi - \theta = \pi \pmod{2\pi}$ if $u = \frac{2}{3}$,
- the torus minus two disks if $u > \frac{2}{3}$. □

Part II Surgeries on the Whitehead link complement

4 Surgeries on the Whitehead link complement

In this section, we will consider some spherical CR structures on the Whitehead link complement, and the Dehn surgeries of this link admitting a spherical CR structure.

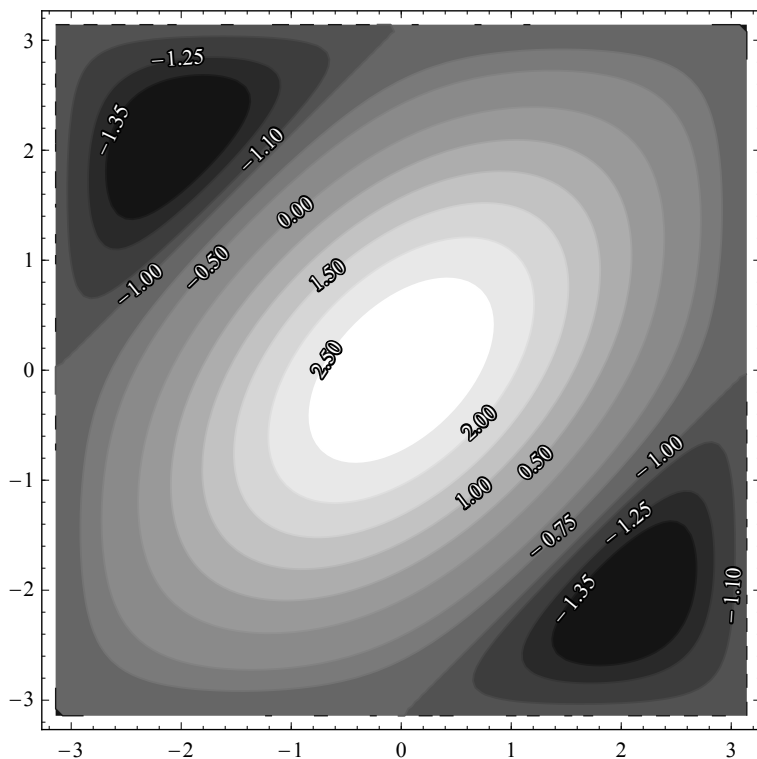


Figure 1: Level sets of the function $(\theta, \phi) \mapsto \cos(\theta) + \cos(\phi) + \cos(\phi - \theta)$ for $(\theta, \phi) \in [-\pi, \pi]^2$.

We will use the results of Schwartz, which can be found in his book [22], and of Parker and Will, given in the article [18].

4.1 The Whitehead link complement

The Whitehead link is the link given by the projection of Figure 2. It has two components and a minimal crossing number of 5. Each component is an unknotted circle.

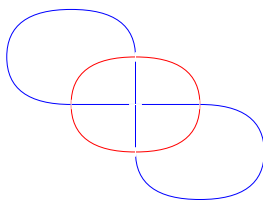


Figure 2: The Whitehead link (SnapPy).

Remark 4.1 If we denote by W the Whitehead link and W' the link obtained by exchanging its two components, then W and W' are isotopic. In other words, the two components play the same role. This fact will be reflected on the Parker–Will spherical CR structure, which we will see below.

We will denote by WLC the Whitehead link complement in S^3 . The complement of a tubular neighborhood of the link in S^3 is a compact manifold with two torus boundaries that we denote by T_1 and T_2 . Its interior is homeomorphic to WLC; we will identify $\overline{\text{WLC}}$ with $\text{WLC} \cup T_1 \cup T_2$. The fundamental group of WLC is given by the following presentation:

$$\pi_1(\text{WLC}) = \langle u, v \mid [u, v][u, v^{-1}][u^{-1}, v^{-1}][u^{-1}, v] \rangle.$$

Choosing as generators s and t satisfying $u = st$ and $v = tst$, we obtain a new presentation:

$$\pi_1(\text{WLC}) = \langle s, t \mid ts^{-1}t^{-3}s^{-2}t^{-1}st^3s^2 \rangle.$$

Remark 4.2 This presentation is the one given by SnapPy, with $t = a$ and $s^{-1} = b$.

In this presentation, the longitude–meridian pairs of the peripheral subgroups corresponding to T_1 and T_2 are given by

$$\begin{aligned} l_1 &= t^{-2}s^{-1}ts^2t^{-1}s^{-1}, & m_1 &= t^{-2}s^{-1}, \\ l_2 &= ststs^{-1}t^3s^{-1}t, & m_2 &= st. \end{aligned}$$

Remark 4.3 With this marking, the laces m_i correspond to the actual meridians of the components of WLC in S^3 and the longitudes are trivial in homology.

Notice, as Parker and Will do in [18], that by imposing $s^3 = t^3 = 1$, the group we obtain is the free product $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$; $\pi_1(\text{WLC})$ admits a surjection onto this group.

4.2 Deformation spaces

We are going to consider representations of $\pi_1(\text{WLC})$ that factor through the quotient $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, up to conjugacy. In [2], we showed that the character variety $\mathcal{X}_{\text{SL}_3(\mathbb{C})}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$ has 16 irreducible components: 15 isolated points and an irreducible component X_0 . In [14], Guilloux and Will show that the component X_0 is also an irreducible component of the character variety $\mathcal{X}_{\text{SL}_3(\mathbb{C})}(\pi_1(\text{WLC}))$. We will limit ourselves to this component X_0 , and to its intersection with the character variety $\mathcal{X}_{\text{SU}(2,1)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$. This gives us a whole component of deformations of representations of $\pi_1(\text{WLC})$ with values in $\text{SU}(2, 1)$, considered up to conjugacy.

This space can be parametrized by traces. More precisely, if

$$\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} = \langle s, t \mid s^3, t^3 \rangle,$$

the traces of $s, s^{-1}, t, t^{-1}, st, (st)^{-1}, s^{-1}t, st^{-1}$ and of the commutator $[s, t]$ determine, up to conjugacy, an irreducible representation of $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ into $SL_3(\mathbb{C})$. In [24, Section 4], Will considers the restriction to $SU(2, 1)$ or $SU(3)$. Notice that if $U \in SU(2, 1)$, then $\text{tr}(U^{-1}) = \overline{\text{tr}(U)}$; hence we will only consider the traces of $s, t, st, s^{-1}t$ and the commutator $[s, t]$. Furthermore, in the component X_0 , the images of the elements s and t are regular elliptic of order 3, and hence have trace 0. For $\rho \in \text{Hom}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}, SU(2, 1))$, write $z_\rho = \text{tr}(\rho(st))$, $w_\rho = \text{tr}(\rho(st^{-1}))$ and $x_\rho = \text{tr}(\rho([s, t]))$. As detailed in [2], the union of the two character varieties $\mathcal{X}_{SU(2,1)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$ and $\mathcal{X}_{SU(3)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$ in X_0 is described with these coordinates by

$$\{(z, w, x) \in \mathbb{C}^3 : x + \bar{x} = Q(z, w), x\bar{x} = P(z, w)\},$$

where Q and P are defined by

$$\begin{aligned} Q(z, w) &= |z|^2 + |w|^2 - 3, \\ P(z, w) &= 2 \text{Re}(z^3) + 2 \text{Re}(w^3) + |z|^2|w|^2 - 6|z|^2 - 6|w|^2 + 9. \end{aligned}$$

4.3 Parker–Will representations

In [18], Parker and Will construct a two-parameter family of representations ρ with values in $SU(2, 1)$, in such a way that $\rho(s)$ and $\rho(t)$ are regular elliptic of order 3 and $\rho(st)$ and $\rho(ts)$ are unipotent. In terms of traces, they parametrize the slice of $X_0 \cap \mathcal{X}_{SU(2,1)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$ for which one of the coordinates equals 3, ie a subset of

$$\{(3, w, x) \in \mathbb{C}^3 : x + \bar{x} = Q(3, w), x\bar{x} = P(3, w)\}.$$

In the Siegel model, this family is explicitly parametrized by $(\alpha_1, \alpha_2) \in]-\frac{\pi}{2}, \frac{\pi}{2}[^2$, in the following way:

$$\begin{aligned} \rho(s) &= e^{-\frac{1}{3}i\alpha_1} \begin{pmatrix} e^{i\alpha_1} & x_1 e^{i\alpha_1 - i\alpha_2} & -1 \\ -x_1 e^{i\alpha_2} & -e^{i\alpha_1} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ \rho(t) &= e^{\frac{1}{3}i\alpha_1} \begin{pmatrix} 0 & 0 & -1 \\ 0 & -e^{-i\alpha_1} & -x_1 e^{-i\alpha_1 - i\alpha_2} \\ -1 & x_1 e^{i\alpha_2} & e^{-i\alpha_1} \end{pmatrix}, \end{aligned}$$

where $x_1 = \sqrt{2 \cos(\alpha_1)}$. Since $\rho(s)^3 = \rho(t)^3 = \text{Id}$, we have

$$\begin{aligned} \rho(l_1) &= \rho(ts^{-1}ts^{-1}ts^{-1}), & \rho(m_1) &= \rho(ts^{-1}), \\ \rho(l_2) &= \rho(ststst), & \rho(m_2) &= \rho(st). \end{aligned}$$

Hence $\rho(l_1) = \rho(m_1)^3$ and $\rho(l_2) = \rho(m_2)^3$ in the whole family of representations. Furthermore, by construction, $\rho(m_2)$ is unipotent for all the representations parametrized by Parker and Will. For the other peripheral representation, the type of $\rho(m_1) = \rho(ts^{-1})$ is given by the curve of Figure 3. If (α_1, α_2) is on the curve, then $\rho(ts^{-1})$ is parabolic, if it is inside, $\rho(ts^{-1})$ is loxodromic, and if it is outside, elliptic. We denote these regions by \mathcal{L} and \mathcal{E} , respectively. By setting $\alpha_2^{\text{lim}} = \arccos(\sqrt{\frac{3}{8}})$, the curve has two singular points at $(0, \pm\alpha_2^{\text{lim}})$, for which $\rho(ts^{-1})$ is unipotent. Moreover, Parker and Will define the region \mathcal{Z} by the equation

$$D(4 \cos^2(\alpha_1), 4 \cos^2(\alpha_2)) > 0,$$

where $D(x, y) = x^3y^3 - 9x^2y^2 - 27xy^2 + 81xy - 27x - 27$. They show, using the Poincaré polyhedron theorem, that the image of ρ is discrete and faithful in the interior of the region \mathcal{Z} .

Remark 4.4 Considering the coordinates (z, w, x) described above for the character variety $\mathcal{X}_{\text{SU}(2,1)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$, the projection on w of the slice $z = 3$ is a double cover besides the red curve in Figure 4, for which the fibers are singletons.

With the parametrization of Parker and Will, the image of the map $\rho \mapsto z_\rho$ is the union of the three lobes of Figure 4. The type of $\rho(s^{-1}t)$ is then determined by the sign of $f(z_\rho)$, where $f(z) = |z|^4 - 8 \text{Re}(z^3) + 18|z|^2 - 27$ is the function defined by Goldman in [12, Theorem 6.2.4]. The regions \mathcal{E} and \mathcal{L} are then separated by the blue curve of Figure 4, with equation $f(z) = 0$.

4.4 Spherical CR structures with parabolic peripheral holonomy

We are going to consider spherical CR structures on the Whitehead link complement. Whenever a structure has a parabolic peripheral holonomy, we are going to deform it in order to obtain spherical CR structures either on Dehn surgeries of WLC or on manifolds obtained similarly, by gluing a torus knot complement in a lens space. At first, we will apply the surgery theorem of [1], and then give explicit bounds for deformations and study the uniformizability of the structures obtained.

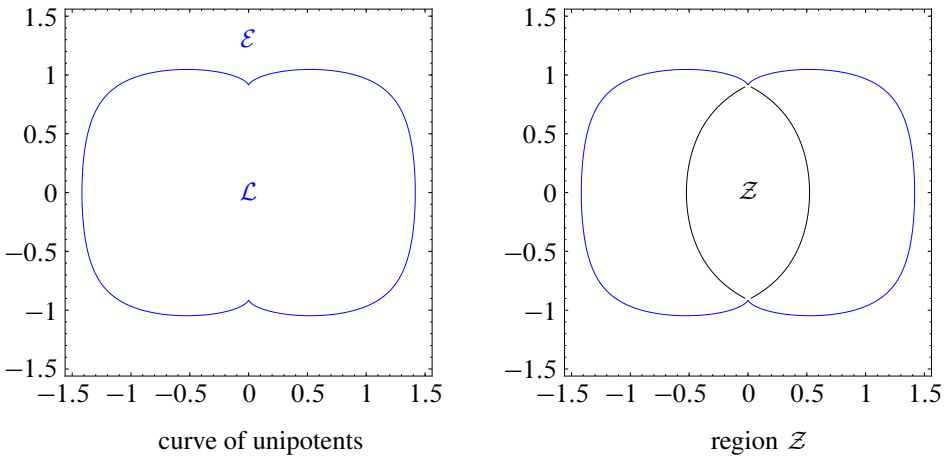


Figure 3: Curve of unipotents and the region \mathcal{Z} in the Parker–Will slice.

We will then consider in detail the structure constructed by Parker and Will in [18] which admits as holonomy representation the representation ρ with coordinates $(0, \alpha_2^{\text{lim}})$. We will also briefly cite the structure constructed by Schwartz in his book [22], corresponding to the point $(\alpha_1^{\text{lim}}, 0)$, where $\alpha_1^{\text{lim}} = \arccos(\frac{\sqrt{3}}{4})$.

4.4.1 The Parker–Will structure In [18], Parker and Will study the groups inside the region \mathcal{Z} . Using the Poincaré polyhedron theorem, they show that in this case they are faithful and discrete representations of $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, and that they are holonomy representations for open orbifolds of dimension 4 with a $(\text{PU}(2, 1), \mathbb{H}_{\mathbb{C}}^2)$ –structure.

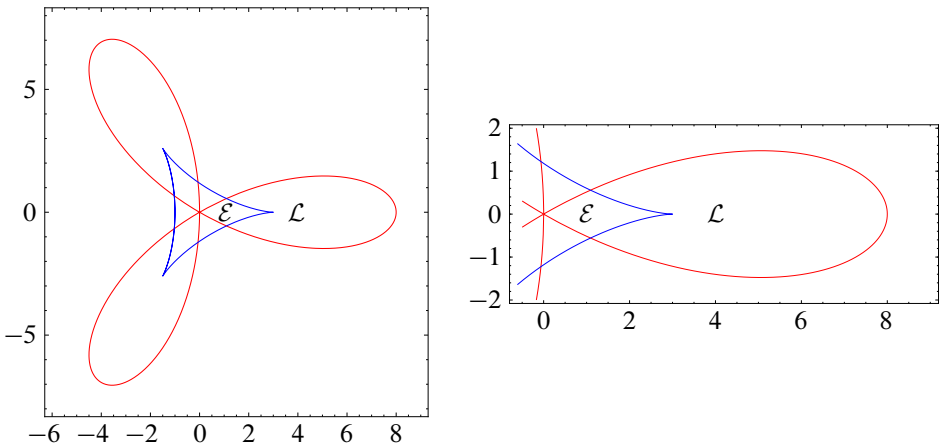


Figure 4: The Parker–Will slice in trace coordinates.

In [18, Section 6], they study the group of parameter $(0, \alpha_2^{\text{lim}})$ and show that it is the holonomy representation of a spherical CR uniformization of the Whitehead link complement. In this case, they compute the images of s and t :

$$\rho(s) = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} - i\frac{\sqrt{5}}{2} & -1 \\ -\frac{\sqrt{3}}{2} - i\frac{\sqrt{5}}{2} & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \rho(t) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -\frac{\sqrt{3}}{2} + i\frac{\sqrt{5}}{2} \\ -1 & \frac{\sqrt{3}}{2} + i\frac{\sqrt{5}}{2} & 1 \end{bmatrix}.$$

They construct then a Ford domain invariant by the holonomy $\rho(m_2) = \rho(st)$ of the second cusp. This domain is a horotube; see [18, Figure 11 and Proposition 6.8].

Remark 4.5 For the holonomy representation of the uniformization, the traces of the images of s , t , st , $s^{-1}t$ and $[s, t]$ are, respectively, 0 , 0 , 3 , 3 and x_0 , where $x_0 = \frac{15}{2} - \frac{3}{2}i\sqrt{15}$.

Proposition 4.6 Consider WLC endowed with the uniformizable spherical CR structure of Parker and Will. Then there is an antiholomorphic involution ι of WLC exchanging the two cusps.

Proof Consider the automorphism φ of $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ given by $\varphi(s) = s^{-1}$ and $\varphi(t) = t$ and the representation $\rho' = \rho \circ \varphi$. Since $[s^{-1}, t]$ is conjugate to $[s, t]^{-1}$, the character χ_ρ has coordinates $(0, 0, 3, 3, x_0)$ and $\chi_{\rho'}$ has coordinates $(0, 0, 3, 3, \bar{x}_0)$. By the Lubotzky–Magid theorem [15, Theorem 1.28] on characters of semisimple representations, the representations ρ and $\bar{\rho}'$ are conjugate in $\text{SL}_n(\mathbb{C})$. Since they are irreducible and with values in $\text{SU}(2, 1)$, they are conjugate in $\text{SU}(2, 1)$. Hence there exist antiholomorphic involutions ι and η of $\mathbb{C}\mathbb{P}^2$ and $\text{SU}(2, 1)$, respectively, such that:

- (1) ι stabilizes $\mathbb{H}_{\mathbb{C}}^2$ and $\partial_\infty \mathbb{H}_{\mathbb{C}}^2$.
- (2) For all $U \in \text{SU}(2, 1)$ and $[Z] \in \mathbb{C}\mathbb{P}^2$ we have $[\eta(U)][\iota(Z)] = \iota([U][Z])$.
- (3) $\eta(\rho(s)) = \rho'(s) = \rho(s)^{-1}$.
- (4) $\eta(\rho(t)) = \rho'(t) = \rho(t)$.

Since $\text{Im}(\rho) = \text{Im}(\rho') = \Gamma$, the domain of discontinuity Ω_Γ is stabilized by ι . Hence, the involution ι induces an antiholomorphic involution of $\text{WLC} = \Gamma \backslash \Omega_\Gamma$. The holonomy representation of the structure given by $\iota(\Gamma \backslash \Omega_\Gamma)$ is then ρ' . Thus, $\rho'(st) = \rho(s^{-1}t) = \rho(t^{-1})\rho(ts^{-1})\rho(t)$ and $\rho'(s^{-1}t) = \rho(st)$. Hence, the involution η exchanges the peripheral holonomies of the two cusps: we deduce that ι exchanges the two cusps of WLC. □

In particular, we deduce that there exists a neighborhood of the first cusp whose image by the developing map is a horotube invariant by $\rho(m_1) = \rho(ts^{-1})$. It is, indeed, the image by ι of a neighborhood of the second cusp, whose image by the developing map is a horotube invariant by $\rho(m_2) = \eta(\rho(s)^{-1}\rho(m_1)\rho(s))$.

4.4.2 The Schwartz structure In his book [22], Schwartz had already studied the groups corresponding to the real axis of the Parker–Will parametrization, constructing them as subgroups of triangle groups. He shows, in particular, that the representations are discrete in the segment $[-\alpha_1^{\text{lim}}, \alpha_1^{\text{lim}}] \times \{0\}$, and that the representation with coordinates $(\alpha_1^{\text{lim}}, 0)$ is the holonomy representation of a spherical CR uniformization of the Whitehead link complement. Furthermore, his results can be reformulated to establish that the image of a neighborhood of each cusp by the developing map is a horotube. Schwartz also describes the two peripheral holonomies: the first one is horizontal unipotent and the second is generated by an ellipto-parabolic element $[P]$. By explicitly computing $[P]$ from the data in Chapter 4 of [22], we obtain that the matrix P is conjugate in $SU(2, 1)$ to

$$e^{i\theta} \begin{pmatrix} 1 & 0 & -\frac{1}{2}i \\ 0 & e^{-3i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\theta = \frac{1}{3} \arccos(-\frac{7}{8})$. In the trace coordinates of $\mathcal{X}_{SU(2,1)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$, this representation is at the point $(3, w_{\text{sch}})$, where $w_{\text{sch}} = 2e^{i\theta} + e^{-2i\theta} \approx 1.09062813494126 + 0.557252430478823i$. It is the intersection point of the red and blue curves of the Parker–Will slice, as in Figure 4.

4.5 Spherical CR surgeries

Now, we apply the spherical CR surgery theorem of [1] to the Parker–Will uniformization of the Whitehead link complement. We keep the notation of [1] for the boundary thickening in order to state the result in a simpler way. We denote by (Dev_0, ρ_0) the spherical CR structure on WLC given by the Parker–Will uniformization. We use the abuse of notation of identifying $\rho: \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \rightarrow SU(2, 1)$ with representations $\rho: \pi_1(\text{WLC}) \rightarrow PU(2, 1)$.

4.5.1 Applying the surgery theorem We have seen below that the hypotheses of the surgery theorem of [1] are satisfied: the images of the two peripheral holonomies are generated by the unipotent elements $\rho(m_1) = \rho(st)$ and $\rho(m_2) = \rho(ts^{-1})$, and

there exists $s \in [0, 1[$ such that $\text{Dev}_0(\widetilde{T}_{1[s,1[})$ and $\text{Dev}_0(\widetilde{T}_{2[s,1[})$ are horotubes invariant under $\rho_0(m_1)$ and $\rho_0(m_2)$, respectively.

Remark 4.7 For the representations of $\pi_1(\text{WLC})$ coming from representations of $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, the relations $\rho(l_1) = \rho(m_1)^3$ and $\rho(l_2) = \rho(m_2)^3$ hold. Hence, the space parametrized by Parker and Will is contained in $\mathcal{R}_1(\pi_1(\text{WLC}), \text{PU}(2, 1))$. These relations are rigid: they are satisfied in the whole component of the $\text{SL}_n(\mathbb{C})$ -character variety of WLC. This fact is proven, with other techniques, by Guilloux and Will in [14].

Applying the surgery theorem of [1], we obtain:

Proposition 4.8 *There is an open neighborhood Ω of ρ_0 in $\mathcal{R}_1(\pi_1(\text{WLC}), \text{PU}(2, 1))$ such that for all $\rho \in \Omega$, there exists a spherical CR structure (Dev_ρ, ρ) on WLC and for $i = 1, 2$:*

- (1) *If $\rho(m_i)$ is loxodromic, then the structure (Dev_ρ, ρ) extends to a structure on the Dehn surgery of WLC of type $(-1, 3)$ on T_i .*
- (2) *If $\rho(m_i)$ is elliptic of type $(\frac{p}{n}, \frac{1}{n})$, then the structure (Dev_ρ, ρ) extends to a structure on the Dehn surgery of WLC of type $(-p, n + 3p)$ on T_i .*
- (3) *If $\rho(m_i)$ is elliptic of type $(\frac{p}{n}, \frac{q}{n})$ with p, q and n relatively prime, then the structure (Dev_ρ, ρ) extends to a structure on the gluing of WLC with the manifold $V(p, q, n)$ along T_i .*

Remark 4.9 The marking (l_0, m_0) of the surgery theorem of [1] does not correspond to the marking (l_1, m_1) that we consider here. We know that $3m_1 - l_1 \in \ker(\rho)$ and that the image of each peripheral holonomy of T_1 is generated by $\rho(m_1)$. Hence the conclusions of the surgery theorem apply to the marking $l_0 = m_1$ and $m_0 = 3m_1 - l_1$. In particular, $nl_0 + pm_0 = nm_1 + p(3m_1 - l_1) = -3l_1 + (n + 3p)m_1$. The same relation holds for the peripheral holonomy of T_2 .

In particular, in the region parametrized by Parker and Will, the peripheral holonomy of T_2 is always unipotent, and in a neighborhood of ρ_0 , there exist open sets for which the peripheral holonomy of T_1 is loxodromic and elliptic, respectively. Hence we obtain:

Corollary 4.10 *There are infinitely many spherical CR structures on the Dehn surgery of WLC of type $(-1, 3)$ on T_1 .*

Remark 4.11 Using SnapPy, we know that the Dehn surgery of WLC of type $(-1, 3)$ on T_1 is a manifold with a torus boundary and fundamental group $\langle a, b \mid a^3b^3 \rangle$ which

is not hyperbolic. The Dehn surgery on T_2 of type $(0, 1)$ of this last manifold is the lens space $L(3, 1)$.

4.5.2 Expected Dehn surgeries To make explicit the third point of Proposition 4.8 and determine the Dehn surgeries on T_1 admitting spherical CR structures extending (Dev_ρ, ρ) , we need to identify the type of the elliptic element which generates the peripheral holonomy through the deformation. Outside the curve of unipotents of the region parametrized by Parker and Will, $\rho(m_1)$ is elliptic. First, consider the point ρ_1 with coordinates $(\alpha_1, \alpha_2) = (0, \frac{2\pi}{3})$. Writing $\omega = e^{i\frac{2\pi}{3}}$, the element $\rho_1(m_1)$ is conjugate to

$$\rho_1(s^{-1}t) = \begin{pmatrix} 1 & -\sqrt{2}\omega & -1 \\ -\sqrt{2}\omega & 1 + 2\omega^2 & -\sqrt{2} \\ -1 & -\sqrt{2} & -2\omega^2 \end{pmatrix}.$$

The eigenvalues of this matrix are $-\omega^2$, $-\omega$ and 1 , of respective eigenvectors

$$V_1 = \begin{pmatrix} 1 \\ -\sqrt{2}\omega \\ \omega^2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \\ 0 \\ -\omega \end{pmatrix} \quad \text{and} \quad V_3 = \begin{pmatrix} \sqrt{2} \\ -\omega \\ \sqrt{2}\omega^2 \end{pmatrix}.$$

Moreover, these vectors have norms $\Phi(V_1) = \Phi(V_2) = 1$ and $\Phi(V_3) = -1$. Hence the fixed point of $\rho(s^{-1}t)$ in $\mathbb{H}_{\mathbb{C}}^2$ is $[V_3]$. Since the eigenvalues of $\rho(s^{-1}t)$ are $-\omega^2 = e^{\frac{2\pi}{6}}$, $-\omega = e^{-\frac{2\pi}{6}}$ and 1 , it is of type $(\frac{1}{6}, -\frac{1}{6})$. If a regular elliptic element admits as eigenvalues of its positive eigenvectors $e^{2i\pi\alpha}$ and $e^{2i\pi\beta}$, then $2\alpha + \beta$ and $2\beta + \alpha$ are nonzero. If, in addition, the element is of type $(\frac{p}{n}, \frac{q}{n})$, up to exchanging α and β , we have $2\alpha + \beta = \frac{p}{n}$ and $2\beta + \alpha = \frac{q}{n}$. Since \mathcal{E} is connected and the eigenvectors and eigenvalues of a matrix are continuous, if $\rho \in \mathcal{E}$ has type $(\frac{p}{n}, \frac{q}{n})$ with $p \geq q$, then $p > 0 > q$.

Since $\text{tr}(m_1)$ is a local parameter of the region parametrized by Parker and Will, there exists an open neighborhood of $3 \in \mathbb{C}$ of traces reached by $\text{tr}(m_1)$. But if $\rho(m_1)$ is in \mathcal{E} and its trace is of the form $e^{2i\pi(2p-1)/(3n)} + e^{2i\pi(2-p)/(3n)} + e^{2i\pi(-p-1)/(3n)}$ with $1 > 0 > p > -n$, then $\rho(m_1)$ is elliptic of type $(\frac{p}{n}, \frac{1}{n})$. By exchanging p with $-p$ to have a clearer statement, we obtain the following proposition:

Proposition 4.12 *There exists $\delta > 0$ such that for all relatively prime integers p and n satisfying $0 < p < n$ and $\frac{p}{n} < \delta$, there exists a deformation of the structure (Dev_0, ρ_0) which extends into a spherical CR structure on the Dehn surgery of WLC of type $(p, n - 3p)$ on T_1 .*

Remark 4.13 If the open set for which Proposition 4.12 holds is large enough, we will recognize some spherical CR structures on manifolds that have already been studied. Indeed, if $(p, n) = (1, n)$, the eigenvalues of $\rho(s^{-1}t)$ are $e^{\frac{2\pi}{n}}$, $e^{-i\frac{2\pi}{n}}$ and 1, and its type will be $(\frac{1}{n}, -\frac{1}{n})$. This corresponds to some parameter $(0, \alpha_2)$ satisfying

$$\text{tr}(\rho(s^{-1}t)) = 8 \cos^2(\alpha_2) = 2 \cos\left(\frac{2\pi}{n}\right) + 1.$$

We can notice, as Parker and Will do in [18, page 3999], that for the parameters $(p, n) = (1, 4)$, the obtained surgery is the figure eight knot complement, and the corresponding representation is the holonomy representation of the Deraux–Falbel structure constructed in [6]. Moreover, when $(p, n) = (1, 5)$, the obtained Dehn surgery is the manifold m009 of the census of Falbel, Koseleff and Rouillier of [10], and the representation is the one studied by Deraux in [4], where he shows that it gives a spherical CR uniformization of the manifold m009.

Hence, we expect that these structures can be obtained as spherical CR Dehn surgeries of the Parker–Will structure on WLC. We will prove this fact in Part III; the corresponding statement is Theorem 5.2.

4.5.3 Surgeries on the Schwartz structure We can also apply the spherical CR surgery theorem of [1] to the Schwartz uniformization of WLC that we briefly described in Section 4.4.2. In the rest of this article we will not consider this case any more, and we will take the Parker–Will uniformization as a starting point.

The Schwartz structure satisfies the hypotheses of the surgery theorem of [1] for the cusp of unipotent peripheral holonomy; we only need to describe a space of representations where the nonunipotent cusp has constant peripheral holonomy and the image of the holonomy of the unipotent cusp becomes elliptic or loxodromic. This is equivalent to studying the points of $\mathcal{X}_{\text{SU}(2,1)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$ with coordinates (z, w_{sch}) , taking as a starting point the coordinates $(3, w_{\text{sch}})$ of the Schwartz representation described in Section 4.4.2.

These points correspond to the interior of the red lobes of Figure 5, in which there is also traced the curve of nonregular elements. We remark that in a neighborhood of the point with coordinate 3, there are representations with loxodromic peripheral holonomy as well as elliptic. For determining the type of the elliptic elements when deforming, the continuity argument in Section 4.5.2 still holds. Hence, the elliptic elements that appear are of type $(\frac{p}{n}, \frac{q}{n})$, where $p > 0 > q$. By applying the surgery theorem of [1], and denoting by ρ_{sch} the holonomy representation of the Schwartz structure on WLC we obtain:

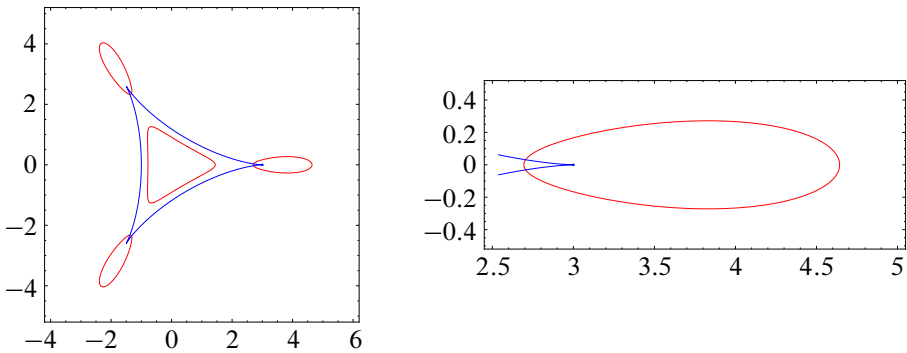


Figure 5: The Schwartz slice, $w = w_{\text{sch}}$, of $\mathcal{X}_{\text{SU}(2,1)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$.

Proposition 4.14 *There is an open neighborhood Ω of ρ_{sch} in $\mathcal{R}_1(\pi_1(\text{WLC}), \text{PU}(2, 1))$ such that for all $\rho \in \Omega$, there exists a spherical CR structure (Dev_ρ, ρ) on WLC, close to the Schwartz uniformization, such that:*

- (1) *If $\rho(m_1)$ is loxodromic, then the structure (Dev_ρ, ρ) extends to a structure on the Dehn surgery of WLC of type $(-1, 3)$ on T_1 .*
- (2) *If $\rho(m_1)$ is elliptic of type $(\frac{p}{n}, \frac{1}{n})$, then the structure (Dev_ρ, ρ) extends to a structure on the Dehn surgery of WLC of type $(-p, n + 3p)$ on T_1 .*
- (3) *If $\rho(m_1)$ is elliptic of type $(\frac{p}{n}, \frac{q}{n})$ with p, q and n relatively prime, then the structure (Dev_ρ, ρ) extends to a structure on the gluing of WLC with the manifold $V(p, q, n)$ along T_1 .*

Notice that all three cases happen in this framework. For the Dehn surgeries, we obtain the following two propositions:

Proposition 4.15 *There is an open set of real dimension 2 parametrizing spherical CR structures on the Dehn surgery of WLC of type $(-1, 3)$ on T_1 , which are obtained by deforming the Schwartz uniformization.*

Proposition 4.16 *There exists $\delta > 0$ such that for all relatively prime integers p and n satisfying $0 < p < n$ and $\frac{p}{n} < \delta$, there exists a deformation of the Schwartz uniformization which extends to a spherical CR structure on the Dehn surgery of WLC of type $(p, n - 3p)$ on T_1 .*

4.5.4 The figure eight knot complement Finally, we make some remarks about the following observation, made by Parker and Will in [18]:

Remark 4.17 At the point with coordinates $(\alpha_1, \alpha_2) = (0, \arctan(\sqrt{7}))$, by setting $G_1 = \rho(st)$, $G_2 = \rho((tst)^3)$ and $G_3 = \rho(ts)$, we obtain the representation ρ_2 of the fundamental group of the figure eight knot complement found by Falbel in [7]. This representation is also the holonomy representation of the uniformization of Deraux and Falbel, constructed in [6].

Remark 4.18 At the point with coordinates $(\alpha_1, \alpha_2) = (0, \arctan(\sqrt{7}))$, the element $\rho(m_1) = \rho(sts^{-1})$ is elliptic of type $(-\frac{1}{4}, \frac{1}{4})$. If the open set Ω given by Proposition 4.8 contains the point $(0, \arctan(\sqrt{7}))$, then the expected Dehn surgery is of type $(1, -1)$ on T_1 . It is, indeed, the figure eight knot complement.

By noticing that tst is conjugate to st^{-1} , we obtain the following proposition:

Proposition 4.19 *Let $\rho \in \text{Hom}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}, \text{SU}(2, 1))$ be a representation. The following assertions are equivalent:*

- (1) $\rho(tst)$ has order 4.
- (2) $\text{tr}(\rho(st^{-1})) = 1$.
- (3) χ_ρ has coordinates $(0, 0, z, 1, x)$ in $\mathcal{X}_{\text{SU}(2,1)}(\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$ with $z, x \in \mathbb{C}$.
- (4) By setting $G_1 = \rho(st)$, $G_2 = \rho((tst)^3)$ and $G_3 = \rho(ts)$, we obtain a representation of $\pi_1(M_8)$.

Remark 4.20 The character χ_ρ has trace coordinates of the form $(0, 0, z_\rho, 1, x_\rho)$ in this case. The map $[\rho] \mapsto z_\rho$ is a double cover over its image, besides the boundary curve, where the fibers are singletons. We have then the parametrization of the component of the character variety of the figure eight knot given by Falbel, Guilloux, Koseleff, Rouillier and Thistlethwaite in [8] and studied in [2].

Part III Effective deformation of a Ford domain

5 Statements and strategy of proof

We will give an explicit bound, at least in one direction, to Proposition 4.8, by deforming the Ford domain of the Parker–Will uniformization. We are going to consider the representations parametrized by Parker and Will in [18] with parameter $(0, \alpha_2)$, where $\alpha_2 \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. We take as a starting point the point with parameter $(0, \alpha_2^{\text{lim}})$, corresponding to the holonomy representation of the uniformization.

5.1 Spherical CR structures: statements

Notation 5.1 If ρ is a representation with parameter $(0, \alpha_2)$ in the Parker–Will parametrization, we denote by $\Gamma(\alpha_2) = \text{Im}(\rho)$ its image. In order to avoid heavy notation, we will often use Γ instead of $\Gamma(\alpha_2)$ if there is not ambiguity for the parameter. For α_2^{lim} , we denote $\rho_\infty = \rho(\alpha_2^{\text{lim}})$ and $\Gamma_\infty = \Gamma(\alpha_2^{\text{lim}})$. We finally denote by ρ_n , for $n \geq 4$, the representation $\rho(\alpha_2)$ such that $8 \cos^2(\alpha_2) = 2 \cos\left(\frac{2\pi}{n}\right) + 1$.

We will prove the following two theorems:

Theorem 5.2 *Let $n \geq 4$. Let ρ_n be the representation with parameter $(0, \alpha_2)$ such that $8 \cos^2(\alpha_2) = 2 \cos\left(\frac{2\pi}{n}\right) + 1$ in the Parker–Will parametrization. Then ρ_n is the holonomy representation of a spherical CR structure on the Dehn surgery of the Whitehead link complement on T_1 of type $(1, n - 3)$ (ie of slope $1/(n - 3)$).*

Theorem 5.3 *Let $\alpha_2 \in]0, \alpha_2^{\text{lim}}[$. Let ρ be the representation with parameter $(0, \alpha_2)$ in the Parker–Will parametrization. Then ρ is the holonomy representation of a spherical CR structure on the Dehn surgery of the Whitehead link complement on T_1 of type $(1, -3)$ (ie of slope $-\frac{1}{3}$).*

5.2 Strategy of proof

By studying the image Γ_∞ of the holonomy representation $\rho_\infty: \pi_1(\text{WLC}) \rightarrow \text{PU}(2, 1)$, Parker and Will construct a Ford domain for the set of left cosets $[A] \backslash \Gamma_\infty$ for some unipotent element $[A]$, which generates the image of one of the peripheral holonomies. This structure has a symmetry exchanging the two cusps, seen in [Proposition 4.6](#); we denote by U the image of A by the corresponding involution η .

We will consider some deformations of the holonomy representation ρ given by Parker and Will and deform the Ford domain for $[U] \backslash \Gamma$ into a domain centered at a fixed point of $[U]$ and invariant by $[U]$, with face identifications given by elements of Γ and with the same local combinatorics as the Parker–Will domain. If $[U]$ is elliptic, it will be a Dirichlet domain, like the one given by Deraux and Falbel in [\[6\]](#) for the figure eight knot complement. If $[U]$ is loxodromic, the domain will be centered *outside* $\mathbb{H}_{\mathbb{C}}^2$.

Considering separately a neighborhood of the cusp and the rest of the structure, we identify the obtained structures as Dehn surgeries, like in the surgery theorem of [\[1\]](#).

In order to deform the Ford domain of Parker and Will, we will deform their construction by defining a one-parameter family of bisectors $(\mathcal{J}_k^\pm)_{k \in \mathbb{Z}} \subset \mathbb{H}_{\mathbb{C}}^2$ invariant by the action

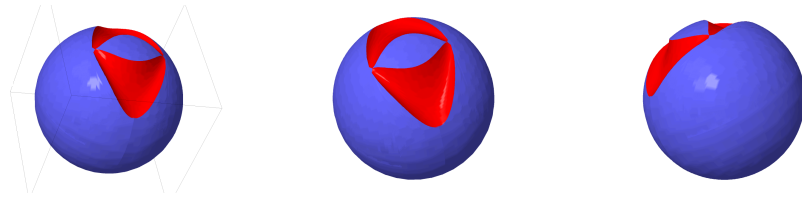


Figure 6: Simplified picture of a face \mathcal{F}_k^\pm . The blue region is in $\partial_\infty \mathbb{H}_\mathbb{C}^2$; the two red regions are Giraud disks in $\mathbb{H}_\mathbb{C}^2$. The blue region consists of a bigon B_k^\pm and a quadrilateral Q_k^\pm .

of $[U]$. We will define the 3-faces of our domain $\mathcal{F}_k^\pm \subset \mathcal{J}_k^\pm$ by cutting off a part of the bisector \mathcal{J}_k^+ by the bisectors \mathcal{J}_k^- and \mathcal{J}_{k+1}^- to define the face \mathcal{F}_k^+ and by cutting off a part of the bisector \mathcal{J}_k^- by the bisectors \mathcal{J}_k^+ and \mathcal{J}_{k-1}^+ to define the face \mathcal{F}_k^- . See Figure 6 in order to have in mind the shape of these 3-faces. We will give a more precise definition in Section 6, as well as the notation that we will use from that point on.

We will need to check three conditions to establish our results: a condition on the topology of faces, which we will denote by (TF), a condition on the local combinatorics of intersections, which we will denote by (LC), and a condition on the global combinatorics of intersections, which we will denote by (GC). More precisely, we can state them in the following way:

Notation 5.4 Conditions (TF), (LC) and (GC) are as follows:

- (TF) The intersections of the form $\mathcal{F}_k^+ \cap \mathcal{F}_k^-$ and $\mathcal{F}_k^+ \cap \mathcal{F}_{k-1}^-$ are bitangent Giraud disks. In particular, $\partial_\infty \mathcal{F}_k^+$ is bounded by two bitangent circles, defining a “bigon” and a “quadrilateral”. The same holds for intersections of the form $\mathcal{F}_k^- \cap \mathcal{F}_k^+$ and $\mathcal{F}_k^- \cap \mathcal{F}_{k+1}^+$.
- (LC) The intersections of the form $\mathcal{F}_k^+ \cap \mathcal{F}_{k+1}^+$ and $\mathcal{F}_k^- \cap \mathcal{F}_{k+1}^-$ contain exactly two points; the ones of the form $\mathcal{F}_k^+ \cap \mathcal{F}_{k+1}^-$ contain exactly one point; all these points are in $\partial_\infty \mathbb{H}_\mathbb{C}^2$.
- (GC) The face \mathcal{F}_k^\pm intersects \mathcal{F}_{k+l}^\pm if and only if $l \in \{-1, 0, 1\}$. The face \mathcal{F}_k^+ intersects \mathcal{F}_{k+l}^- if and only if $l \in \{-2, -1, 0, 1\}$. The indices are modulo n whenever $[U]$ is elliptic of order n .

If these three conditions are satisfied, then the faces \mathcal{F}_k^\pm are well defined and border a domain in $\mathbb{H}_\mathbb{C}^2$, which has the same side pairing as the one given by Parker and Will in [18]. The same holds for the boundary at infinity of the domain, which is bordered by the bigons and quadrilaterals of $\partial_\infty \mathcal{F}_k^\pm$. A simplified picture of the topology of

the faces is given in [Figure 6](#). If the three conditions are satisfied, then the domain defined in $\partial_\infty \mathbb{H}_\mathbb{C}^2$ together with the side pairings determines a spherical CR structure on WLC, which extends to the surgery expected by [Proposition 4.8](#). More precisely:

- (1) If $[U]$ is loxodromic, then the structure (Dev_ρ, ρ) extends to a structure on the Dehn surgery on WLC of type $(-1, 3)$ on T_1 .
- (2) If $[U]$ is elliptic of type $(\frac{p}{n}, \frac{1}{n})$, then the structure (Dev_ρ, ρ) extends to a structure on the Dehn surgery on WLC of type $(-p, n + 3p)$ on T_1 .
- (3) If $\rho(m_1)$ is elliptic of type $(\frac{p}{n}, \frac{q}{n})$ with p, q and n relatively prime, then the structure (Dev_ρ, ρ) extends to a structure on the gluing of WLC with the manifold $V(p, q, n)$ along T_1 .

We are going to show conditions [\(TF\)](#), [\(LC\)](#) and [\(GC\)](#) in the particular case of deformations with parameter $(0, \alpha_2)$ for $\alpha_2 \in]0, \frac{\pi}{2}[$ in the Parker–Will parametrization. This will give [Theorems 5.2](#) and [5.3](#). We will begin by setting notation and recalling the initial combinatorics in [Section 6](#). Then, assuming conditions [\(TF\)](#), [\(LC\)](#) and [\(GC\)](#), we will prove the statements on spherical CR structures in [Section 7](#). Thereafter, we will check the three conditions, which is mostly technical work. We will show the condition [\(TF\)](#) of the topology of the faces in [Section 8](#), then the condition [\(LC\)](#) of local combinatorics in [Section 9](#) and finally we will consider the condition [\(GC\)](#) of global combinatorics in two steps, beginning in [Section 10](#) with a global strategy of proof, and finishing in [Sections 10.4](#) and [10.5](#) with the cases of $[U]$ loxodromic and $[U]$ elliptic.

5.3 Results involving the Poincaré polyhedron theorem

We can wonder if the spherical CR structures given by [Theorems 5.2](#) and [5.3](#) are uniformizable. In order to prove such a result, we will need to apply a Poincaré polyhedron theorem in $\mathbb{H}_\mathbb{C}^2$, as stated for example in [\[18\]](#). A complete proof of this theorem will appear in the book of Parker [\[16\]](#). By applying the Poincaré polyhedron theorem in $\mathbb{H}_\mathbb{C}^2$ from [Theorem 5.2](#), we obtain the following theorem:

Theorem 5.5 *Let $n \geq 4$. Then the Dehn surgery of the Whitehead link complement on T_1 of slope $1/(n - 3)$ admits a spherical CR uniformization, given by the group Γ_n .*

The use of the Poincaré polyhedron theorem is the same as that of Parker, Wang and Xie in [\[17\]](#). For $\alpha_2 \in [\frac{\pi}{6}, \alpha_2^{\text{lim}}[$, the Poincaré polyhedron theorem can still be applied, and apart from the condition of being a *polyhedron* (where all faces are homeomorphic to balls), we check the hypothesis for $\alpha_2 \in]0, \frac{\pi}{6}[$. See [Lemma 8.1](#) for more details. This allows us to conjecture the following result:

Conjecture 5.6 Let $\alpha_2 \in]0, \alpha_2^{\text{lim}}[$. Then the group $\Gamma(\alpha_2)$ gives a spherical CR uniformization on the Dehn surgery of the Whitehead link complement on T_1 of slope $-\frac{1}{3}$.

6 Notation and initial combinatorics

With the notation and the tools related to bisectors, extors and the intersections that we studied in Section 3.5, we will describe a deformation of the Ford domain constructed by Parker and Will in [18]. Let us begin by recalling the combinatorics of the domain and fix notation for some notable elements of the group as well as some points of \mathbb{CP}^2 .

6.1 Notation

First, we set notation for the elements of the group and some notable points that we will use below.

Notation 6.1 (elements) In the family of representations ρ of $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} = \langle s, t \rangle$ with values in $\text{SU}(2, 1)$ parametrized by Parker and Will in [18], we denote by $S = \rho(s)$ and $T = \rho(t)$. They are two regular elliptic elements of order 3.

In the same way as Parker and Will, we write $A = ST$ and $B = TS$. These two elements, conjugate by S , are unipotent in the family of representations, parametrized by $(\alpha_1, \alpha_2) \in]-\frac{\pi}{2}, \frac{\pi}{2}[^2$. Finally, we write $U = S^{-1}T$, $V = SUS^{-1} = TS^{-1}$ and $W = SVS^{-1} = STS$.

Notation 6.2 (fixed points) If $[G] \in \text{PU}(2, 1)$ is unipotent or regular, we will denote by $[p_G] \in \mathbb{CP}^2$ a particular fixed point.

- If $[G]$ is parabolic, denote by $[p_G]$ its unique fixed point in $\partial_\infty \mathbb{H}_\mathbb{C}^2$.
- If $[G]$ is regular elliptic, denote by $[p_G]$ its unique fixed point in $\mathbb{H}_\mathbb{C}^2$.
- If $[G]$ is loxodromic, denote by $[p_G]$ its unique fixed point in $\mathbb{CP}^2 - \overline{\mathbb{H}_\mathbb{C}^2}$.

Remark 6.3 When $[G]$ is unipotent or regular, the map $[G] \mapsto [p_G]$ is continuous.

In this way, in the region parametrized by Parker and Will, we have

$$[p_A] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad [p_B] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We will limit ourselves to the deformations of the Parker–Will structure with coordinate $\alpha_1 = 0$. In this case, we have

$$S = \begin{pmatrix} 1 & \sqrt{2}e^{-i\alpha_2} & -1 \\ -\sqrt{2}e^{i\alpha_2} & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -\sqrt{2}e^{-i\alpha_2} \\ -1 & \sqrt{2}e^{i\alpha_2} & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & -\sqrt{2}e^{i\alpha_2} & -1 \\ -\sqrt{2}e^{i\alpha_2} & 1 + 2e^{2i\alpha_2} & 2\sqrt{2} \cos(\alpha_2) \\ -1 & 2\sqrt{2} \cos(\alpha_2) & 2 + 2e^{-2i\alpha_2} \end{pmatrix}.$$

Then

$$[p_U] = \begin{bmatrix} 1 \\ -\frac{\sqrt{2}}{2}e^{i\alpha_2} \\ e^{2i\alpha_2} \end{bmatrix}, \quad [p_V] = [Sp_U] = \begin{bmatrix} -e^{2i\alpha_2} \\ -\frac{\sqrt{2}}{2}e^{i\alpha_2} \\ -1 \end{bmatrix},$$

$$[p_W] = [Sp_V] = \begin{bmatrix} -e^{2i\alpha_2} \\ \sqrt{2}e^{3i\alpha_2} + \frac{\sqrt{2}}{2}e^{i\alpha_2} \\ e^{2i\alpha_2} \end{bmatrix}.$$

Notation 6.4 If U is not unipotent, it has three different eigenvalues. We will denote by p'_U and p''_U two eigenvectors associated to the eigenvalues different from 1. Denoting by δ a square root of $(8 \cos^2(\alpha_2) - 3)(8 \cos^2(\alpha_2) + 1)$, we have

$$[p'_U] = \begin{bmatrix} 2(2e^{2i\alpha_2} + 1) \\ -\sqrt{2}e^{i\alpha_2}(2e^{2i\alpha_2} + 1 + \delta) \\ -(8 \cos^2(\alpha_2) + 1) - \delta \end{bmatrix}, \quad [p''_U] = \begin{bmatrix} 2(2e^{2i\alpha_2} + 1) \\ -\sqrt{2}e^{i\alpha_2}(2e^{2i\alpha_2} + 1 - \delta) \\ -(8 \cos^2(\alpha_2) + 1) + \delta \end{bmatrix}.$$

Remark 6.5 If $\alpha_2 > \alpha_2^{\text{lim}}$, then U is a regular elliptic element with eigenvalues $1, e^{i\beta}$ and $e^{-i\beta}$ for some $\beta \in]0, \frac{\pi}{2}[$. The respective eigenvectors are then p_U, p'_U and p''_U . In this case, $\text{tr}(U) = 2 \cos(\beta) + 1$ and $(8 \cos^2(\alpha_2) - 3)(8 \cos^2(\alpha_2) + 1) = (\text{tr}(U) - 3)(\text{tr}(U) + 1) = -4 \sin^2(\beta)$. We will take $\delta = 2i \sin(\beta)$.

Remark 6.6 If $\alpha_2 < \alpha_2^{\text{lim}}$, then U is a loxodromic element with eigenvalues $1, e^l$ and e^{-l} for some $l \in \mathbb{R}^+$. The respective eigenvectors are p_U, p'_U and p''_U . Then $\text{tr}(U) = 2 \cosh(l) + 1$ and $(8 \cos^2(\alpha_2) - 3)(8 \cos^2(\alpha_2) + 1) = (\text{tr}(U) - 3)(\text{tr}(U) + 1) = 4 \sinh^2(l)$. We will take then $\delta = 2 \sinh(l)$.

6.2 Combinatorics of the Parker–Will uniformization

Notation 6.7 Following the article of Parker and Will [18], let $\mathcal{I}_0^+ = \overline{\mathfrak{B}}(p_A, S^{-1} p_A)$, and for $k \in \mathbb{Z}$, let $\mathcal{I}_k^+ = A^k \mathcal{I}_0^+ = \overline{\mathfrak{B}}(p_A, A^k S^{-1} p_A)$. Similarly, let $\mathcal{I}_0^- = \overline{\mathfrak{B}}(p_A, S p_A)$ and $\mathcal{I}_k^- = A^k \mathcal{I}_0^- = \overline{\mathfrak{B}}(p_A, A^k S p_A)$. These sets are closed bisectors in $\overline{\mathbb{H}}_{\mathbb{C}}^2$.

Parker and Will show that for the representation with parameter $(0, \alpha_2^{\text{lim}})$, the bisectors \mathcal{I}_k^{\pm} bound an infinite polyhedron in $\mathbb{H}_{\mathbb{C}}^2$, locally finite and invariant by $[A]$, which is endowed with a side pairing. Its boundary at infinity is the region of $\partial_{\infty} \mathbb{H}_{\mathbb{C}}^2$ containing $[p_A]$ and limited by the spinal spheres $\partial_{\infty} \mathcal{I}_k^+$ and $\partial_{\infty} \mathcal{I}_k^-$. Parker and Will show that this region is a Ford domain invariant by $[A]$ for the spherical CR uniformization of the Whitehead link complement.

The domain in $\partial_{\infty} \mathbb{H}_{\mathbb{C}}^2$ has four classes of faces: quadrilaterals Q_k^+ and Q_k^- , contained in $\partial_{\infty} \mathcal{I}_k^+$ and $\partial_{\infty} \mathcal{I}_k^-$, respectively, and bigons B_k^+ and B_k^- , contained in $\partial_{\infty} \mathcal{I}_k^+$ and $\partial_{\infty} \mathcal{I}_k^-$, respectively. We can see the incidences, combinatorially, in Figure 7.

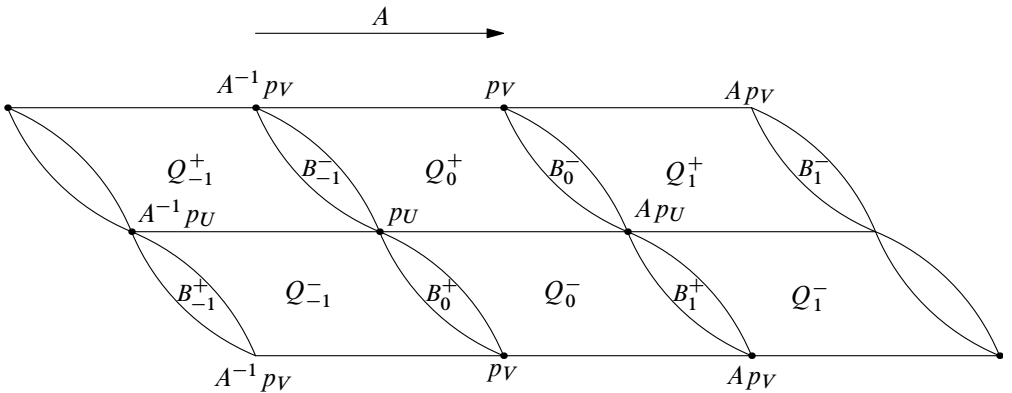


Figure 7: Combinatorics of the boundary at infinity of the Ford domain for Γ_{∞} constructed by Parker and Will. The two horizontal boundaries are identified by a vertical translation.

In this way, the quadrilateral Q_0^+ has vertices $[pU]$, $[A^{-1}pv]$, $[pv]$ and $[ApU]$ and the bigon B_0^+ has vertices $[pU]$ and $[pv]$. The sides of Q_0^+ and B_0^+ are arcs of the Giraud circles $\partial_{\infty} \mathcal{I}_0^+ \cap \partial_{\infty} \mathcal{I}_{-1}^+$ and $\partial_{\infty} \mathcal{I}_0^+ \cap \partial_{\infty} \mathcal{I}_0^-$, which contain $\{[A^{-1}pv], [pU], [pv]\}$ and $\{[Apv], [pU], [pv]\}$, respectively.

Remark 6.8 This configuration gives indeed a bigon and a quadrilateral since the two Giraud circles are tangent at $[pU]$ and $[pv]$. The reader can find a detailed proof in [18], and we will show this fact again when deforming the domains.

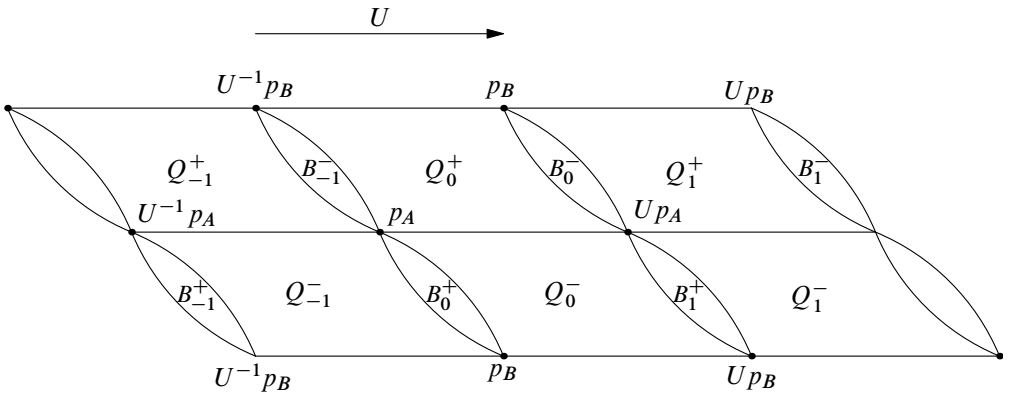


Figure 8: Combinatorics of the boundary at infinity of the Ford domain after the involution. The two horizontal boundaries are identified by a vertical translation.

However, we will not use this domain for the deformation, but its image by the involution ι defined in Proposition 4.6. Recall that ι is compatible with the involution $\eta: \text{Im}(\rho) \rightarrow \text{Im}(\rho)$ given by $\eta(T) = T$ and $\eta(S) = S^{-1}$. Hence, $\eta(A) = U$ and $\eta(B) = V$.

Notation 6.9 We will write $\mathcal{J}_0^\pm = \iota(\mathcal{I}_0^\pm)$. Therefore, $\mathcal{J}_k^+ = U^k \overline{\mathfrak{B}}(p_U, p_V)$ and $\mathcal{J}_k^- = U^k \overline{\mathfrak{B}}(p_U, p_W)$. Furthermore, by abuse of language we will still denote by Q_k^\pm and B_k^\pm the images of the quadrilaterals and the bigons by the involution ι .

The incidences and the combinatorics of the boundary of the new Ford domain in $\partial_\infty \mathbb{H}_\mathbb{C}^2$ that we consider are given in Figure 8. The quadrilateral Q_0^+ has vertices $[p_A]$, $[U^{-1}p_B]$, $[p_B]$ and $[Up_A]$ and the bigon B_0^+ has vertices $[p_A]$ and $[p_B]$. The sides of Q_0^+ and B_0^+ are arcs of the Giraud circles $\partial_\infty \mathcal{J}_0^+ \cap \partial_\infty \mathcal{J}_{-1}^+$ and $\partial_\infty \mathcal{J}_0^+ \cap \partial_\infty \mathcal{J}_0^-$, which contain $\{[U^{-1}p_B], [p_A], [p_B]\}$ and $\{[Up_A], [p_A], [p_B]\}$, respectively. Furthermore, we denote the faces of the domain contained in $\mathbb{H}_\mathbb{C}^2$ in the following way:

Notation 6.10 For $k \in \mathbb{Z}$, we define the 3-face \mathcal{F}_k^+ as

$$\mathcal{F}_k^+ = \{[z] \in \mathcal{J}_k^+ : |\langle z, p_U \rangle| \leq \min(|\langle z, U^k p_W \rangle|, |\langle z, U^{k-1} p_W \rangle|)\}.$$

Its boundary in \mathcal{J}_k^+ is then given by $\mathcal{J}_k^+ \cap (\mathcal{J}_k^- \cup \mathcal{J}_{k-1}^-)$. We also define the face \mathcal{F}_k^- as

$$\mathcal{F}_k^- = \{[z] \in \mathcal{J}_k^- : |\langle z, p_U \rangle| \leq \min(|\langle z, U^k p_V \rangle|, |\langle z, U^{k+1} p_V \rangle|)\}.$$

Its boundary in \mathcal{J}_k^- is then given by $\mathcal{J}_k^- \cap (\mathcal{J}_k^+ \cup \mathcal{J}_{k+1}^+)$.

Remark 6.11 The boundary of the face \mathcal{F}_k^+ is, a priori, the union of the two Giraud disks $\mathcal{J}_k^+ \cap \mathcal{J}_k^-$ and $\mathcal{J}_k^+ \cap \mathcal{J}_{k-1}^-$. We are going to show, in [Section 8](#), that it is the case, and that these two disks are bitangent at infinity during the entire deformation.

We are going to deform the Ford domain of Parker and Will into a domain with the same local combinatorics. It will be either a Dirichlet domain, or a Ford domain centered *outside* $\mathbb{H}_{\mathbb{C}}^2$ depending on whether $[U]$ is elliptic or loxodromic. From now on, all points and elements of the representation of $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ with values in $SU(2, 1)$ depend on the parameter α_2 , which will vary.

Recall that the representations with parameter $\alpha_2 \in [-\alpha_2^{\text{lim}}, \alpha_2^{\text{lim}}]$ have been studied by Parker and Will in [\[18\]](#) by considering a Ford domain in $\mathbb{H}_{\mathbb{C}}^2$ centered at the point $[p_A]$. They show, using the Poincaré polyhedron theorem, that the representation of $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ is then discrete and faithful, and that the quotient of $\mathbb{H}_{\mathbb{C}}^2$ by its image is a manifold. Furthermore, when $\alpha_2 = \alpha_2^{\text{lim}}$, they show that the manifold at infinity is the Whitehead link complement, as described above.

On the other hand, Parker, Wang and Xie study in [\[17\]](#) the representations with parameter $\alpha_2 \in]\alpha_2^{\text{lim}}, \frac{\pi}{2}[$ for which $[U]$ is of finite order ≥ 4 . They obtain these groups as the index-two subgroup of some $(3, 3, n)$ -triangle group generated by involutions I_1, I_2 and I_3 . With our notation, we have $S = I_3 I_1, T = I_3 I_2$ and $U = I_1 I_2$. They construct a Dirichlet domain in $\mathbb{H}_{\mathbb{C}}^2$ for these groups, and show, using the Poincaré polyhedron theorem, that they are discrete, and that the quotients of $\mathbb{H}_{\mathbb{C}}^2$ by these groups are manifolds. These Dirichlet domains are the same as the ones that we will construct below, but we will study the combinatorics of the intersections with different techniques, hoping for them to be also useful for representations not coming from triangle groups. We will discuss this in more detail in [Section 10.5](#).

7 Effective deformation: proof

We will assume in this section that the conditions [\(TF\)](#) on the topology of the faces, [\(LC\)](#) of local combinatorics and [\(GC\)](#) of global combinatorics are satisfied, and we will show [Theorems 5.2](#) and [5.3](#). Then, considering the Poincaré polyhedron theorem, we will discuss [Theorem 5.5](#) and [Conjecture 5.6](#). We will establish thereafter the conditions in [Sections 8, 9](#) and [10](#), respectively. From now on, we will assume that the combinatorics of the incidence of the faces \mathcal{F}_k^{\pm} , as well as their boundaries at infinity, is the one expected for $\alpha_2 \in]0, \alpha_2^{\text{lim}}]$ when $[U]$ is loxodromic or unipotent, and for $\alpha_2 \in]\alpha_2^{\text{lim}}, \frac{\pi}{2}[$ when $[U]$ is elliptic of finite order ≥ 4 .

Notation 7.1 We say a parameter α_2 is *admissible* if $\alpha_2 \in]0, \alpha_2^{\text{lim}}[$ or if $\alpha_2 \in]\alpha_2^{\text{lim}}, \frac{\pi}{2}[$ and $[U]$ is elliptic of type $(\frac{1}{n}, -\frac{1}{n})$ with $n \geq 4$.

We begin by fixing some notation and recalling the uniformization result of Parker and Will, which gives a spherical CR structure on WLC for the parameter $\alpha_2 = \alpha_2^{\text{lim}}$.

Notation 7.2 Denote by $D_0 \subset \mathbb{H}_{\mathbb{C}}^2$ the Ford domain given by Parker and Will in [18]. It corresponds to the parameter $\alpha_2 = \alpha_2^{\text{lim}}$. Its boundary is given by the faces \mathcal{F}_k^{\pm} . Let $\partial_{\infty} D_0 \subset \partial_{\infty} \mathbb{H}_{\mathbb{C}}^2$ be its boundary at infinity. The boundary of this last set in $\partial_{\infty} \mathbb{H}_{\mathbb{C}}^2$ consists of the bigons B_k^{\pm} and the quadrilaterals Q_k^{\pm} .

In [18], Parker and Will show that the side pairings of the faces \mathcal{F}_k^{\pm} of D_0 are given by the group Γ_{∞} modulo the action of $[U]$: using the Poincaré polyhedron theorem, they show that in that case, the group Γ_{∞} is discrete and that the manifold at infinity, which is homeomorphic to WLC, is uniformizable. Hence, we know that $\Gamma_{\infty} \backslash \partial_{\infty} D_0$ is homeomorphic to the Whitehead link complement. We have seen, in Section 4.5, that the image by the developing map of a neighborhood of the cusp corresponding to T_1 is a horotube for the action of $[U]$. If we only consider a thickening of the quadrilaterals and the bigons before taking the quotient of $\partial_{\infty} D_0$ by Γ_{∞} , we obtain the structure on WLC apart from a neighborhood of the first cusp.

Using the conditions (LC) and (GC) about the combinatorics, we prove the following lemma, which ensures that the domain D_0 can be deformed into a domain in $\mathbb{H}_{\mathbb{C}}^2$ with boundary the faces \mathcal{F}_k^{\pm} .

Lemma 7.3 *If α_2 is an admissible parameter, then the faces \mathcal{F}_k^{\pm} border a domain $D(\alpha_2)$ in $\mathbb{H}_{\mathbb{C}}^2$, obtained as a deformation of D_0 .*

Proof Let

$$D(\alpha_2) = \{[z] \in \mathbb{H}_{\mathbb{C}}^2 : |\langle z, p_U \rangle| \leq \min(|\langle z, U^k p_V \rangle|, |\langle z, U^k p_W \rangle|) \text{ for all } k \in \mathbb{Z}\}.$$

It is the set of points of $\mathbb{H}_{\mathbb{C}}^2$ which are “nearer to” $[p_U]$ than to the orbits by $[U]$ of $[p_V]$ and $[p_W]$. When $\alpha_2 = \alpha_2^{\text{lim}}$, this domain is exactly the Ford domain D_0 of Parker and Will. It is then bordered by the faces \mathcal{F}_k^{\pm} which are contained in the bisectors \mathcal{J}_k^{\pm} for $k \in \mathbb{Z}$. If $\alpha_2 \in]0, \frac{\pi}{2}[$ is an admissible parameter, the element $[U]$ generates a discrete subgroup and the boundary of the domain $D(\alpha_2)$ is contained in the bisectors \mathcal{J}_k^{\pm} . By the global combinatorics condition (GC), the bisectors only intersect their neighbors in the local combinatorics, and they determine the faces \mathcal{F}_k^{\pm} . Hence, these faces form the boundary of a domain $D(\alpha_2)$, which is obtained as a deformation of D_0 . \square

The same holds for the boundary at infinity: the quadrilaterals and the bigons border a domain of $\partial_\infty \mathbb{H}_\mathbb{C}^2$ which is obtained by deforming $\partial_\infty D_0$.

Lemma 7.4 *If α_2 is an admissible parameter, then the bigons and quadrilaterals B_k^\pm and Q_k^\pm border a domain in $\partial_\infty \mathbb{H}_\mathbb{C}^2$, obtained as a deformation of $\partial_\infty D'_0$. It is the boundary at infinity of $D(\alpha_2)$; we will denote it by $\partial_\infty D(\alpha_2)$.*

We are now going to study the topology of the manifold $\Gamma(\alpha_2) \setminus \partial_\infty D(\alpha_2)$ for the admissible parameters, and we are going to show that it corresponds to the one expected by the surgery theorem of [1], which we identified in Section 4.5.2. In order to do so, we cut the domain $\partial_\infty D(\alpha_2)$ into two pieces: the first “near” the faces, which will give the structure on WLC apart from the cusp associated to T_1 , and the second “far” from the faces, which will give the solid torus glued to T_1 in order to obtain a Dehn surgery.

Notation 7.5 Let \mathcal{V}_0 be a thickening of $\bigcup_{k \in \mathbb{Z}} (Q_k^+ \cup B_k^+ \cup Q_k^- \cup B_k^-)$ in $\partial_\infty D_0$, and N_0 be its complement in $\partial_\infty D_0$.

Lemma 7.6 *If α_2 is an admissible parameter, then \mathcal{V}_0 deforms into a thickening $\mathcal{V}(\alpha_2)$ of $\bigcup_{k \in \mathbb{Z}} (Q_k^+ \cup B_k^+ \cup Q_k^- \cup B_k^-)$ in $\partial_\infty D(\alpha_2)$. The quotient $\Gamma(\alpha_2) \setminus \mathcal{V}(\alpha_2)$ is homeomorphic to WLC minus the cusp corresponding to T_1 .*

Proof By the conditions on local and global combinatorics (LC) and (GC), the bigons and quadrilaterals B_k^\pm and Q_k^\pm intersect with the same local combinatorics as for the parameter α_2^{lim} , and form a surface in $\partial_\infty \mathbb{H}_\mathbb{C}^2$. Hence, we can consider a thickening $\mathcal{V}(\alpha_2)$ of this surface, which is a deformation of \mathcal{V}_0 . A fundamental domain in $\mathcal{V}(\alpha_2)$ for the action of $\langle [U] \rangle$ is then given by a thickening of $B_0^+ \cup Q_0^+ \cup B_0^- \cup Q_0^-$. Since the side pairings are given by the same elements of the group, the topology of the quotient $\Gamma(\alpha_2) \setminus \mathcal{V}(\alpha_2)$ is the same as the one of $\Gamma_\infty \setminus \mathcal{V}_0$, which is precisely the Parker–Will structure on WLC minus the cusp corresponding to T_1 . □

We will focus now on the other part of the structure, which allows us to identify the manifold on which the spherical CR structure on WLC minus a cusp can be extended by Lemma 7.6. It will always be the expected Dehn surgery, as described in Section 4.5.2. Let $N(\alpha_2) = \partial_\infty D(\alpha_2) - \mathcal{V}(\alpha_2)$. The following two propositions complete the proof of Theorems 5.2 and 5.3, by identifying the topology of the manifold $\langle [U] \rangle \setminus N(\alpha_2)$, glued to the spherical CR structure on WLC given by $\langle [U] \rangle \setminus D(\alpha_2)$.

Proposition 7.7 *If $\alpha_2 \in]0, \alpha_2^{\text{lim}}[$, then the quotient $\langle [U] \rangle \setminus N(\alpha_2)$ is a solid torus, in which the curve $l_1^{-1} m_1^3$ is homotopically trivial.*

Proof The domain $\partial_\infty D_0$ is a horotube, bordered by the quadrilaterals Q_k^\pm and the bigons B_k^\pm when $\alpha_2 = \alpha_2^{\text{lim}}$. If $\alpha_2 < \alpha_2^{\text{lim}}$, the element $[U]$ is loxodromic. In this case, $N(\alpha_2)$ becomes homeomorphic to a cylinder in which the curve $l_1^{-1}m_1^3$ is homotopically trivial. This verification is analogous to the one made in the proof of the surgery theorem of [1].

If $\alpha_2 < \frac{\pi}{6}$, then the cylinder becomes a surface of infinite genus, but by the continuity of the deformation, the same curve is homotopically trivial, since it stays in a fixed compact set. \square

Proposition 7.8 *Let $\alpha_2 \in]\alpha_2^{\text{lim}}, \frac{\pi}{2}[$ be an admissible parameter. Then $[U]$ is elliptic of order $n \geq 4$. The quotient $\langle [U] \rangle \backslash N(\alpha_2)$ is then a solid torus, in which the curve $l_1 m_1^{n-3}$ is homotopically trivial.*

Proof The domain $\partial_\infty D_0$ is a horotube, bordered by the quadrilaterals Q_k^\pm and the bigons B_k^\pm when $\alpha_2 = \alpha_2^{\text{lim}}$. If $\alpha_2 > \alpha_2^{\text{lim}}$ is an admissible parameter, the element $[U]$ is elliptic and $\partial_\infty D(\alpha_2)$ is a torus invariant by $[U]$. Since $[U] = \rho(m_1)$ has type $(\frac{1}{n}, -\frac{1}{n})$, this torus is not knotted, and $N(\alpha_2)$ is a solid torus invariant by $[U]$ in which the curve $l_1 m_1^{n-3}$ is homotopically trivial. Indeed, with the notation of Remark 4.9, the curves $l_0^n = m_1^n$ and $m_0 = l_1^{-1}m_1^3$ are homotopic in $N(\alpha_2)$ to one of the two \mathbb{C} -circles invariant by $[U]$: the curve $l_1 m_1^{n-3}$ is homotopically trivial in $N(\alpha_2)$. We deduce that the quotient $\langle [U] \rangle \backslash N(\alpha_2)$ is also a solid torus in which the curve $l_1 m_1^{n-3}$ is homotopically trivial. This verification is analogous to the one made in the proof of the surgery theorem of [1]. \square

The last two propositions complete the proof of Theorems 5.2 and 5.3. It remains to discuss the use of the Poincaré polyhedron theorem for $\mathbb{H}_{\mathbb{C}}^2$ to obtain Theorem 5.5 and a part of Conjecture 5.6. If $\alpha_2 \in]\frac{\pi}{6}, \frac{\pi}{2}[$ is an admissible parameter, the domain $D(\alpha_2)$ is a polyhedron invariant by the action of $[U]$, and has a side pairing given by S and its conjugates by $[U]$. The hypotheses of the Poincaré polyhedron theorem for $\mathbb{H}_{\mathbb{C}}^2$, as stated in [18], are satisfied. A complete proof of the theorem will appear in the book of Parker [16]. By applying this theorem to the domain $D(\alpha_2)$ and the side pairing between \mathcal{F}_k^\pm in $\mathbb{H}_{\mathbb{C}}^2$, we deduce that for the admissible parameters α_2 , the group $\Gamma(\alpha_2)$ is discrete, and that the structure on the manifold at infinity $\Gamma(\alpha_2) \backslash D(\alpha_2)$ is uniformizable. This proves Theorem 5.5 and a part of Conjecture 5.6.

When $\alpha_2 \in]0, \frac{\pi}{6}[$, the domain $D(\alpha_2)$ has the same side pairings, but the faces \mathcal{F}_k^\pm are no longer homeomorphic to a 3-dimensional ball, so we no longer have the conditions

to apply the Poincaré polyhedron theorem for $\mathbb{H}_{\mathbb{C}}^2$. However, we can expect that a similar statement can be applied for faces that are not balls, but we won't go in that direction, and limit ourselves to stating [Conjecture 5.6](#).

In order to complete the proofs above, it remains to establish the conditions on the topology of the faces (TF), local combinatorics (LC) and global combinatorics (GC). The rest of the article will consist of a rather technical proof of these conditions. Indeed, it is not trivial to reduce the global considerations to a finite number of verifications, and we will need to use the techniques involving visual spheres described in [Part I](#).

8 Topology of faces during the deformation (TF)

In this section, we show the condition (TF) on the topology of the faces and give almost all the tools for showing the local combinatorics condition (LC) given in the strategy of proof of [Section 5](#). First of all, note that there is a change in the topology of bisectors at the point $\alpha_2 = \frac{\pi}{6}$.

Lemma 8.1 *If $\alpha_2 \in]\frac{\pi}{6}, \frac{\pi}{2}[$, then \mathcal{J}_k^+ and \mathcal{J}_k^- are metric bisectors. If $\alpha_2 = \frac{\pi}{6}$, then \mathcal{J}_k^+ and \mathcal{J}_k^- are fans. If $\alpha_2 \in]0, \frac{\pi}{6}[$, then \mathcal{J}_k^+ and \mathcal{J}_k^- are Clifford cones.*

Proof By [Proposition 3.18](#), we only need to consider the signature of the restriction of the Hermitian form to the plane generated by p_U and p_V . We have

- $\langle p_U, p_U \rangle = \langle p_V, p_V \rangle = 4 \cos^2(\alpha_2) - \frac{3}{2}$,
- $\langle p_U, p_V \rangle = -\frac{3}{2}$.

In the basis (p_U, p_V) , the determinant of the Hermitian form equals

$$(4 \cos^2(\alpha_2) - \frac{3}{2})^2 - (\frac{3}{2})^2 = 4 \cos^2(\alpha_2)(4 \cos^2(\alpha_2) - 3).$$

Hence it is positive if $\alpha_2 \in [0, \frac{\pi}{6}[$, zero if $\alpha_2 = \frac{\pi}{6}$ and negative if $\alpha_2 \in]\frac{\pi}{6}, \frac{\pi}{2}[$. \square

8.1 Incidence of points and bisectors

We begin by checking the incidence of the points and the bisectors: the points will be the vertices of the faces which lie on spinal surfaces.

Lemma 8.2 *The point $[p_A]$ is contained in the bisectors $\mathcal{J}_0^+, \mathcal{J}_0^-, \mathcal{J}_{-1}^+$ and \mathcal{J}_{-1}^- .*

Proof We only need to compute the Hermitian products of p_A with $p_U, p_V, p_W, U^{-1}p_V$ and $U^{-1}p_W$ and check that they have the same modulus. We compute

that $\langle p_A, p_U \rangle = e^{2i\alpha_2}$, $\langle p_A, p_V \rangle = -1$ and $\langle p_A, p_W \rangle = e^{2i\alpha_2}$. Furthermore, we find $\langle p_A, U^{-1}p_V \rangle = \langle Up_A, p_V \rangle = e^{2i\alpha_2}$ and $\langle p_A, U^{-1}p_W \rangle = \langle Up_A, p_W \rangle = -1$. All these products have modulus 1, which completes the proof. \square

Lemma 8.3 *The point $[p_B]$ is contained in the bisectors \mathcal{J}_0^+ , \mathcal{J}_0^- , \mathcal{J}_1^+ and \mathcal{J}_{-1}^- .*

Proof We only need to compute the Hermitian products of p_B with p_U , p_V , p_W , Up_V and $U^{-1}p_W$ and check that they have the same modulus. We compute $\langle p_B, p_U \rangle = 1$, $\langle p_B, p_V \rangle = -e^{-2i\alpha_2}$ and $\langle p_B, p_W \rangle = 1$. Furthermore, $\langle p_B, Up_V \rangle = 1$ and $\langle p_B, U^{-1}p_W \rangle = \langle Up_B, p_W \rangle = 1$. All these products have modulus 1, which completes the proof. \square

Considering the translation by U^k , we obtain the following corollary:

Corollary 8.4 *For $k \in \mathbb{Z}$ we have the following incidences:*

- The bisector \mathcal{J}_k^+ contains the points $U^k p_A$, $U^{k+1} p_A$, $U^k p_B$ and $U^{k-1} p_B$.
- The bisector \mathcal{J}_k^- contains the points $U^k p_A$, $U^{k+1} p_A$, $U^k p_B$ and $U^{k+1} p_B$.

Lemma 8.5 *The intersections $\mathcal{J}_k^+ \cap \mathcal{J}_k^-$ and $\mathcal{J}_k^- \cap \mathcal{J}_{k+1}^+$ are Giraud disks. If $\alpha_2 \neq 0$, their boundary is a smooth circle in $\partial_\infty \mathbb{H}_\mathbb{C}^2$.*

Proof The hypotheses of Proposition 3.30 are satisfied for the intersection $\mathcal{J}_k^+ \cap \mathcal{J}_k^-$, since $p_V = Sp_U$ and $p_W = S^2 p_U$. It is the same for $\mathcal{J}_k^- \cap \mathcal{J}_{k+1}^+$ since $p_V = T p_U$ and $p_W = T^2 p_U$. Hence, it is enough to compute

$$\frac{\langle p_U \boxtimes p_V, p_U \boxtimes p_V \rangle}{\langle p_U \boxtimes p_V, p_W \boxtimes p_U \rangle}$$

and compare it to $\frac{2}{3}$; the other case is analogous. We compute

$$\begin{aligned} \langle p_U \boxtimes p_V, p_U \boxtimes p_V \rangle &= -4 \cos^2(\alpha_2)(4 \cos^2(\alpha_2) - 3), \\ \langle p_U \boxtimes p_V, p_W \boxtimes p_U \rangle &= -6 \cos^2(\alpha_2). \end{aligned}$$

The quotient is then equal to $\frac{2}{3}(4 \cos^2(\alpha_2) - 3)$, which is $\leq \frac{2}{3}$, with equality if and only if $\alpha_2 = 0$. \square

Corollary 8.6 *For all $k \in \mathbb{Z}$ we have:*

- $\mathcal{J}_k^+ \cap \mathcal{J}_k^-$ is a Giraud disk containing the points $U^k p_A$, $U^k p_B$ and $U^{k+1} p_A$.
- $\mathcal{J}_k^- \cap \mathcal{J}_{k+1}^+$ is a Giraud disk containing the points $U^k p_B$, $U^{k+1} p_A$ and $U^{k+1} p_B$.

Remark 8.7 If $\alpha_2 = 0$, then $\frac{2}{3}(4 \cos^2(\alpha_2) - 3) = \frac{2}{3}$, and the intersections $\mathcal{J}_k^+ \cap \mathcal{J}_k^-$ and $\mathcal{J}_k^- \cap \mathcal{J}_{k+1}^+$ are hexagons with their opposite vertices identified. It is the limit case of Proposition 3.30; the boundary of each hexagon consists of three \mathbb{C} -circles.

Recall now the definition of the 3-dimensional faces of the Ford domain in $\mathbb{H}_{\mathbb{C}}^2$. They are contained in the bisectors \mathcal{J}_k^{\pm} and will be deformed. Their boundary at infinity will consist of bigons and quadrilaterals, which will border the domain $\partial_{\infty} D(\alpha_2)$ of $\partial_{\infty} \mathbb{H}_{\mathbb{C}}^2$. This domain, endowed with the side pairing given by the group Γ , gives spherical CR structures on some Dehn surgeries on the Whitehead link complement.

Definition 8.8 For $k \in \mathbb{Z}$, we define the 3-face \mathcal{F}_k^+ as

$$\mathcal{F}_k^+ = \{[z] \in \mathcal{J}_k^+ : |\langle z, p_U \rangle| \leq \min(|\langle z, U^k p_W \rangle|, |\langle z, U^{k-1} p_W \rangle|)\}.$$

Its boundary in \mathcal{J}_k^+ is then given by $\mathcal{J}_k^+ \cap (\mathcal{J}_k^- \cup \mathcal{J}_{k-1}^-)$. We define the 3-face \mathcal{F}_k^- as

$$\mathcal{F}_k^- = \{[z] \in \mathcal{J}_k^- : |\langle z, p_U \rangle| \leq \min(|\langle z, U^k p_V \rangle|, |\langle z, U^{k+1} p_V \rangle|)\}.$$

Its boundary in \mathcal{J}_k^- is then given by $\mathcal{J}_k^- \cap (\mathcal{J}_k^+ \cup \mathcal{J}_{k+1}^+)$.

Remark 8.9 The boundary of the face \mathcal{F}_k^+ is, a priori, the union of the two Giraud disks $\mathcal{J}_k^+ \cap \mathcal{J}_k^-$ and $\mathcal{J}_k^+ \cap \mathcal{J}_{k-1}^-$. We are going to show, in the following section, that it is indeed the case, and that, during the whole deformation, these two disks are bitangent at infinity.

8.2 Symmetry

The domain that we are going to deform admits a symmetry exchanging the faces \mathcal{F}_k^+ and \mathcal{F}_k^- . Thanks to this symmetry and to the invariance of the domain by $[U]$, it will be enough to check most of the statements only for the bisector \mathcal{J}_0^+ or \mathcal{J}_0^- to prove them for the whole family. Consider the involution I_1 of $U(2, 1)$ given by²

$$I_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{2}e^{i\alpha_2} & -1 & 0 \\ -1 & -\sqrt{2}e^{-i\alpha_2} & 1 \end{pmatrix}.$$

It satisfies $I_1 p_U = p_U$, $I_1 p_V = p_W$ and $I_1 p_W = p_V$. Furthermore, we have

²The involution I_1 is the same as the one considered in [17]. See the discussion at the end of Section 6.2.

that $I_1UI_1 = U^{-1}$; we deduce that $I_1U^k p_V = I_1U^k I_1 p_W = U^{-k} p_W$ for all $k \in \mathbb{Z}$, and hence

$$I_1\mathcal{J}_k^+ = \mathcal{J}_{-k}^- \quad \text{for all } k \in \mathbb{Z}.$$

In particular, the action of $[I_1]$ on \mathbb{CP}^2 exchanges the bisectors \mathcal{J}_0^+ and \mathcal{J}_0^- .

8.3 The intersection $\mathcal{F}_0^- \cap \mathcal{F}_{-1}^-$

We begin by studying the intersection of 3-faces of the form $\mathcal{F}_k^\pm \cap \mathcal{F}_{k+1}^\pm$ in $\overline{\mathbb{H}_\mathbb{C}^2}$. We are going to show that these two faces are bitangent in $\partial_\infty \mathbb{H}_\mathbb{C}^2$, and that they don't intersect elsewhere. This will show the condition on the topology of faces (TF) of Section 5. By symmetry, it is enough to study the intersection $\mathcal{F}_0^- \cap \mathcal{F}_{-1}^-$. We will use the following lemma, which determines the intersection $\mathfrak{E}(p_U, p_V) \cap \mathfrak{E}(p_W, U^{-1} p_W)$, in order to study the intersection of the bisectors \mathcal{J}_0^- and \mathcal{J}_{-1}^- .

Lemma 8.10 *The extors $\mathfrak{E}(p_U, p_V)$ and $\mathfrak{E}(p_W, U^{-1} p_W)$ form a balanced pair. Their intersection is the union of a real plane \mathfrak{m} and a complex line l given by*

$$\mathfrak{m} = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \in \mathbb{CP}^2 : z_1 \in \mathbb{R}, z_2 \in i\mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 1 \\ z_2 \\ 0 \end{bmatrix} \in \mathbb{CP}^2 : z_2 \in i\mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$l = \begin{bmatrix} \sin(\alpha_2) \\ -i \frac{\sqrt{2}}{2} \\ -\sin(\alpha_2) \end{bmatrix}^\perp.$$

Proof Consider the vectors

$$f = \begin{pmatrix} -1 \\ -i \frac{\sqrt{2}}{2} \sin(\alpha_2) \\ 1 \end{pmatrix} \quad \text{and} \quad f' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We have $\langle f, p_U \rangle = \langle f, p_V \rangle = \langle f', p_W \rangle = \langle f', U^{-1} p_W \rangle = 0$, hence $[f]$ is the focus of $\mathfrak{E}(p_U, p_V)$ and $[f']$ is the focus of $\mathfrak{E}(p_W, U^{-1} p_W)$. We compute

$$\begin{aligned} \langle p_U, f' \rangle &= 2 \cos(\alpha_2) e^{-i\alpha_2}, \\ \langle p_V, f' \rangle &= -2 \cos(\alpha_2) e^{-i\alpha_2}, \\ \langle p_W, f \rangle &= (i \sin(\alpha_2)(4e^{-2i\alpha_2} + 2) - 2e^{-i\alpha_2}) e^{-i\alpha_2}, \\ \langle U^{-1} p_W, f \rangle &= (i \sin(\alpha_2)(4e^{2i\alpha_2} + 2) + 2e^{i\alpha_2}) e^{-i\alpha_2} = -e^{-2i\alpha_2} \overline{\langle p_W, f \rangle}. \end{aligned}$$

Consequently, $[f'] \in \mathfrak{E}(p_U, p_V)$ and $[f] \in \mathfrak{E}(p_W, U^{-1}p_W)$. Hence it is a balanced pair. By [Theorem 3.24](#), the intersection of the two extors is given by a complex line l and an \mathbb{R} -plane. The complex line is given by $l = l_{[f],[f']}$; we easily check that we have

$$l = l_{[f],[f']} = \begin{bmatrix} \sin(\alpha_2) \\ -i\frac{\sqrt{2}}{2} \\ -\sin(\alpha_2) \end{bmatrix}^\perp.$$

We know that the points $[f]$, $[f']$ and $[p_A]$ lie in the \mathbb{R} -plane \mathfrak{m} given in the statement. Since these points are not aligned, there is at most one \mathbb{R} -plane containing them. By [Corollary 8.4](#), we know that $[p_A] \in \mathfrak{E}(p_U, p_V) \cap \mathfrak{E}(p_U, p_W) \cap \mathfrak{E}(p_U, U^{-1}p_W)$, and hence that $[p_A]$ is in $\mathfrak{E}(p_U, p_V) \cap \mathfrak{E}(p_W, U^{-1}p_W)$. Since $[p_A] \notin l$, the \mathbb{R} -plane \mathfrak{m} passing through $[f]$, $[f']$ and $[p_A]$ is the \mathbb{R} -plane of the intersection of the balanced pair of extors. □

In order to understand the intersection of the faces \mathcal{F}_0^- and \mathcal{F}_{-1}^- , we consider the triple intersection of extors $\mathfrak{E}_0^+ \cap \mathfrak{E}_0^- \cap \mathfrak{E}_{-1}^-$. We will use in a crucial way the following lemma:

Lemma 8.11 *The triple intersection of extors $\mathfrak{E}_0^+ \cap \mathfrak{E}_0^- \cap \mathfrak{E}_{-1}^-$ consists of two topological circles, one contained in the \mathbb{R} -plane \mathfrak{m} and the other contained in the complex line l . The triple intersection of bisectors $\mathcal{J}_0^+ \cap \mathcal{J}_0^- \cap \mathcal{J}_{-1}^-$ is the set $\{[p_A], [p_B]\}$.*

Proof By [Corollary 8.4](#), we know that $[p_A]$ and $[p_B]$ are in the triple intersection. Consider the intersection $\mathfrak{E}_0^+ \cap \mathfrak{E}_0^- \cap \mathfrak{E}_{-1}^-$. We have

$$\begin{aligned} \mathfrak{E}_0^+ \cap \mathfrak{E}_0^- \cap \mathfrak{E}_{-1}^- &= \mathfrak{E}(p_U, p_V) \cap \mathfrak{E}(p_U, p_W) \cap \mathfrak{E}(p_U, U^{-1}p_W) \\ &= \mathfrak{E}(p_U, p_V) \cap \mathfrak{E}(p_U, p_W) \cap \mathfrak{E}(p_W, U^{-1}p_W) \\ &= \mathfrak{E}(p_U, p_W) \cap (\mathfrak{m} \cup l), \end{aligned}$$

where \mathfrak{m} and l are the real plane and the complex line of [Lemma 8.10](#). We consider first the intersection $\mathfrak{E}(p_U, p_W) \cap \mathfrak{m}$ with the real plane. For $r, s \in \mathbb{R}$, let

$$q_{r,s} = \begin{pmatrix} r \\ i\sqrt{2}s \\ 1 \end{pmatrix} \quad \text{and} \quad q_r = \begin{pmatrix} r \\ i\sqrt{2} \\ 0 \end{pmatrix}.$$

The \mathbb{R} -plane \mathfrak{m} is then equal to the set $\{[p_A]\} \cup \{[q_r] : r \in \mathbb{R}\} \cup \{[q_{r,s}] : r, s \in \mathbb{R}\}$. Let us show that $\mathfrak{m} \cap \mathcal{J}_0^- = \{[p_A], [p_B]\}$. Let $r, s \in \mathbb{R}$. We have

$$\begin{aligned} |\langle p_U, q_r \rangle|^2 &= r^2 + 2r \sin(\alpha_2) + 1, \\ |\langle p_W, q_r \rangle|^2 &= r^2 + 2r \sin(\alpha_2) + 1 + 8 \cos^2(\alpha_2). \end{aligned}$$

Hence, the points of the form $[q_r]$ are not in the triple intersection. We also compute

$$|\langle p_U, q_{r,s} \rangle|^2 = r^2 + s^2 + 1 + 2r(2 \cos(\alpha_2)^2 - 1) + 2(r - 1)s \sin(\alpha_2),$$

$$|\langle p_W, q_{r,s} \rangle|^2 = (8 \cos(\alpha_2)^2 + 1)s^2 + r^2 + 2(r - 1)s \sin(\alpha_2) - 2r + 1.$$

We deduce that $|\langle p_W, q_{r,s} \rangle|^2 = |\langle p_U, q_{r,s} \rangle|^2$ if and only if $4 \cos^2(\alpha_2)r = 8 \cos^2(\alpha_2)s^2$. Since $\alpha_2 \neq \pm \frac{\pi}{2}$, this condition is equivalent to $2s^2 - r = 0$. But we know that $\langle q_{r,s}, q_{r,s} \rangle = 2s^2 + 2r$. If this point lies in \mathcal{J}_0^- , then $s^2 + r \leq 0$ and $2s^2 - r \geq 3s^2 \geq 0$, with equality if and only if $r = s = 0$. Hence $\mathfrak{m} \cap \mathcal{J}_0^- = \{[p_A], [p_B]\}$.

Now consider the intersection with the complex line. The vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 2\sqrt{2}i \sin(\alpha_2) \\ 1 \end{bmatrix}$$

form a basis for

$$\begin{bmatrix} \sin(\alpha_2) \\ -i \frac{\sqrt{2}}{2} \\ -\sin(\alpha_2) \end{bmatrix}^\perp.$$

For $\mu \in \mathbb{C}$, let

$$q_\mu = \begin{pmatrix} -1 + \mu \\ 2\sqrt{2}i \sin(\alpha_2) \\ 1 + \mu \end{pmatrix}.$$

We have

$$|\langle p_U, q_\mu \rangle|^2 = 4 \cos^2(\alpha_2)|\mu|^2 \quad \text{and} \quad |\langle p_W, q_\mu \rangle|^2 = 4(9 - 8 \cos^2(\alpha_2)) \cos^2(\alpha_2).$$

The point $[q_\mu]$ is hence in the triple intersection if and only if $|\mu|^2 = 9 - 8 \cos^2(\alpha_2)$. But $\langle q_\mu, q_\mu \rangle = 2(|\mu|^2 - 4 \cos^2(\alpha_2) + 3)$, so for the points in the triple intersection this quantity equals $24 \sin^2(\alpha_2)$. Hence, these points never lie in $\overline{\mathbb{H}}_{\mathbb{C}}^2$.

We deduce that the triple intersection $\mathcal{J}_0^+ \cap \mathcal{J}_0^- \cap \mathcal{J}_{-1}^-$ is the set $\{[p_A], [p_B]\}$. □

By studying the torus $\mathfrak{T} = \mathfrak{E}_0^- \cap \mathfrak{E}_{-1}^- \subset \mathbb{C}\mathbb{P}^2$, and its intersections with \mathfrak{E}_0^+ and $\overline{\mathbb{H}}_{\mathbb{C}}^2$, we obtain the following proposition. The combinatorics of these intersections is given by [Figure 9](#).

Proposition 8.12 *The intersection $\mathcal{F}_0^- \cap \mathcal{F}_{-1}^-$ is reduced to $\{[p_A], [p_B]\}$.*

Proof We are going to study the torus $\mathfrak{T} = \mathfrak{E}_0^- \cap \mathfrak{E}_{-1}^- \subset \mathbb{C}\mathbb{P}^2$, and its intersections with \mathfrak{E}_0^+ and $\overline{\mathbb{H}}_{\mathbb{C}}^2$. We know, by [Lemma 8.11](#), that $\mathfrak{T} \cap \mathfrak{E}_0^+$ is the union of two circles cutting \mathfrak{T} into two pieces. We are going to show that $\mathfrak{T} \cap \overline{\mathbb{H}}_{\mathbb{C}}^2$ is always in the same

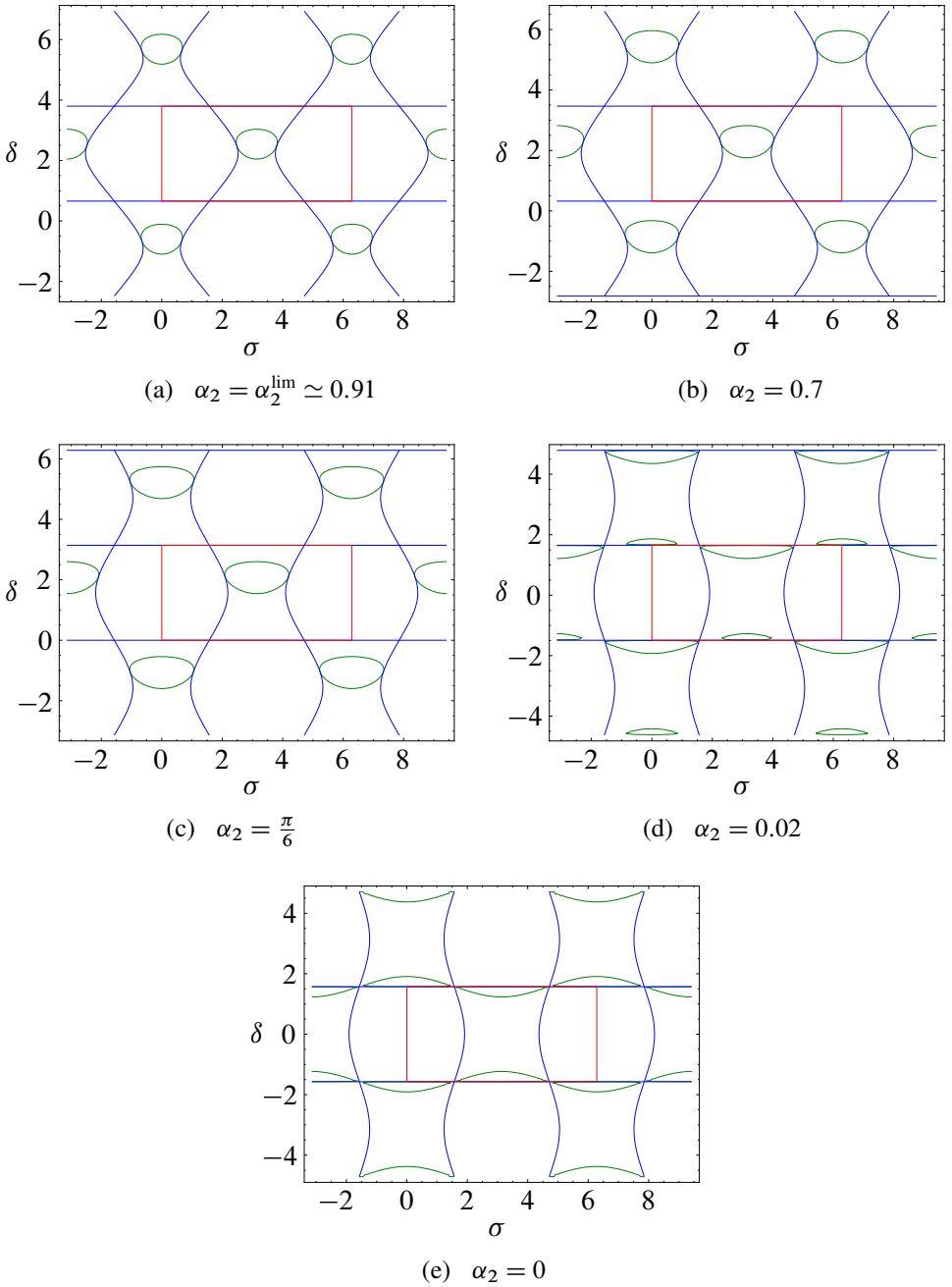


Figure 9: The curves of equations (1) and (2) (blue), the curve of the intersection with $\partial_\infty \mathbb{H}_\mathbb{C}^2$ (green) and the rectangle of study (red) for $\alpha_2 \in \{0, 0.02, \frac{\pi}{6}, 0.7, \alpha_2^{\text{lim}}\}$.

side of these two circles, excepting the points $[p_A]$ and $[p_B]$; we will conclude the proof by a continuity argument.

We begin by parametrizing the torus \mathfrak{T} . We know $\mathfrak{T} = \mathfrak{E}(p_U, p_W) \cap \mathfrak{E}(p_U, U^{-1}p_W)$, hence we can parametrize it, by Proposition 3.25, by

$$\mathfrak{T} = \{[(p_W - e^{i\theta} p_U) \boxtimes (U^{-1} p_W - e^{i\phi} p_U)] : (\theta, \phi) \in (\mathbb{R}/2\pi\mathbb{Z})^2\}.$$

Let us compute the coordinates of these points. We have

$$\begin{aligned} (p_W - e^{i\theta} p_U) \boxtimes (U^{-1} p_W - e^{i\phi} p_U) \\ = p_W \boxtimes U^{-1} p_W - e^{-i\theta} p_U \boxtimes U^{-1} p_W - e^{-i\phi} p_W \boxtimes p_U. \end{aligned}$$

We compute

$$\begin{aligned} p_W \boxtimes U^{-1} p_W &= \sqrt{2} \cos(\alpha_2) e^{-2i\alpha_2} \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, \\ p_U \boxtimes U^{-1} p_W &= \sqrt{2} \cos(\alpha_2) e^{-2i\alpha_2} \begin{pmatrix} 2e^{2i\alpha_2} \\ \sqrt{2}e^{i\alpha_2} \\ -1 \end{pmatrix}, \\ p_W \boxtimes p_U &= \sqrt{2} \cos(\alpha_2) e^{-2i\alpha_2} \begin{pmatrix} -1 \\ \sqrt{2}e^{-i\alpha_2} \\ 2e^{-2i\alpha_2} \end{pmatrix}. \end{aligned}$$

Let, for $(\theta, \phi) \in \mathbb{R}^2/\mathbb{Z}^2$,

$$\begin{aligned} v(\theta, \phi) &= \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} - e^{-i\theta} \begin{pmatrix} 2e^{2i\alpha_2} \\ \sqrt{2}e^{i\alpha_2} \\ -1 \end{pmatrix} - e^{-i\phi} \begin{pmatrix} -1 \\ \sqrt{2}e^{-i\alpha_2} \\ 2e^{-2i\alpha_2} \end{pmatrix} \\ &= \begin{pmatrix} -3 - 2e^{i(2\alpha_2 - \theta)} + e^{-i\phi} \\ -\sqrt{2}(e^{i(\alpha_2 - \theta)} + e^{i(-\alpha_2 - \phi)}) \\ -3 + e^{-i\theta} - 2e^{i(-2\alpha_2 - \phi)} \end{pmatrix}. \end{aligned}$$

The vector $v(\theta, \phi)$ is a multiple of $(p_W - e^{i\theta} p_U) \boxtimes (U^{-1} p_W - e^{i\phi} p_U)$. Hence, we have $\mathfrak{T} = \{[v(\theta, \phi)] : (\theta, \phi) \in \mathbb{R}^2/\mathbb{Z}^2\}$.

To simplify the following computations, we change the variables. Let $\sigma = \frac{1}{2}(\theta + \phi)$ and $\delta = \frac{1}{2}(\theta - \phi)$. With these new variables, we have

$$v(\theta, \phi) = e^{-i\sigma} \begin{pmatrix} -3e^{i\sigma} - 2e^{i(2\alpha_2 - \delta)} + e^{i\delta} \\ -2\sqrt{2} \cos(\alpha_2 - \delta) \\ -3e^{i\sigma} - 2e^{i(-2\alpha_2 + \delta)} + e^{-i\delta} \end{pmatrix}.$$

We study now the intersection of \mathfrak{T} with \mathfrak{E}_0^+ . This intersection is given by the intersection of \mathfrak{T} with the real plane \mathfrak{m} and the complex line l of Lemma 8.10.

Consider first the intersection with \mathfrak{m} . The point $[v(\theta, \phi)]$ is in the \mathbb{R} -plane \mathfrak{m} if and only if the quotients of the first and third coefficients of the vector by the second one are imaginary, which is equivalent to

$$(1) \quad -3 \cos(\sigma) - 2 \cos(2\alpha_2 - \delta) + \cos(\delta) = 0.$$

Notice that if $\alpha_2 \in]0, \frac{\pi}{2}[$, this equation has no solutions in δ if $\sigma = 0$. Consider now the intersection with the complex line l . The point $[v(\theta, \phi)]$ is in the complex line l if and only if

$$\left\langle v, \begin{bmatrix} \sin(\alpha_2) \\ -i \frac{\sqrt{2}}{2} \\ -\sin(\alpha_2) \end{bmatrix} \right\rangle = 0,$$

which can be written in coordinates as

$$2i (\cos(\alpha_2 - \delta) - \sin(\alpha_2)(2 \sin(2\alpha_2 - \delta) - \sin(\delta))) = 0,$$

but

$$\begin{aligned} &\cos(\alpha_2 - \delta) - \sin(\alpha_2)(2 \sin(2\alpha_2 - \delta) - \sin(\delta)) \\ &= \cos(\alpha_2) \cos(\delta) + 2 \sin(\alpha_2) \sin(\delta) - 2 \sin(\alpha_2)(\sin(2\alpha_2) \cos(\delta) - \cos(2\alpha_2) \sin(\delta)) \\ &= \cos(\alpha_2) \cos(\delta)(1 - 4 \sin^2(\alpha_2)) + 4 \cos^2(\alpha_2) \sin(\alpha_2) \sin(\delta) \\ &= \cos(\alpha_2)(-\cos(\delta) + 2 \cos(\delta) \cos(2\alpha_2) + 2 \sin(2\alpha_2) \sin(\delta)) \\ &= \cos(\alpha_2)(-\cos(\delta) + 2 \cos(2\alpha_2 - \delta)). \end{aligned}$$

Hence the point $v(\theta, \phi)$ is in the complex line l if and only if

$$(2) \quad 2 \cos(2\alpha_2 - \delta) - \cos(\delta) = 0.$$

If $\alpha_2 \neq 0$, we can rewrite this condition as

$$(3) \quad \tan(\delta) = \frac{1 - 2 \cos(2\alpha_2)}{2 \sin(2\alpha_2)}.$$

Writing $\delta_0 = \arctan((1 - 2 \cos(2\alpha_2))/(2 \sin(2\alpha_2)))$, a point of \mathfrak{T} is in the complex line l if and only if $\delta = \delta_0 \bmod \pi$. From now on, we will consider the domain $\{(\delta, \sigma) \in \mathbb{R}^2 : \delta_0 \leq \delta \leq \delta_0 + \pi, 0 \leq \sigma \leq 2\pi\}$ as a chart to study \mathfrak{T} , and we will denote by $v(\sigma, \delta)$ the parametrization. In order to understand the intersection of $\overline{\mathbb{H}}_{\mathbb{C}}^2$ with \mathfrak{T} , we study the function

$$h: (\sigma, \delta) \mapsto \langle v(\sigma, \delta), v(\sigma, \delta) \rangle.$$

We have

$$\begin{aligned} \frac{\partial h}{\partial \sigma}(\sigma, \delta) &= 2 \operatorname{Re} \left(\left\langle \frac{\partial(e^{i\sigma} v(\sigma, \delta))}{\partial \sigma}, e^{i\sigma} v(\sigma, \delta) \right\rangle \right) \\ &= 2 \operatorname{Re} \left(\left\langle \begin{pmatrix} -3i e^{i\sigma} \\ 0 \\ -3i e^{i\sigma} \end{pmatrix}, \begin{pmatrix} -3e^{i\sigma} - 2e^{i(2\alpha_2 - \delta)} + e^{i\delta} \\ -2\sqrt{2} \cos(\alpha_2 - \delta) \\ -3e^{i\sigma} - 2e^{i(-2\alpha_2 + \delta)} + e^{-i\delta} \end{pmatrix} \right\rangle \right) \\ &= 6(2(\sin(2\alpha_2 - \delta - \sigma) + \sin(-2\alpha_2 + \delta - \sigma)) + \sin(\sigma - \delta) + \sin(\delta + \sigma)) \\ &= -12 \sin(\sigma)(2 \cos(2\alpha_2 - \delta) - \cos(\delta)). \end{aligned}$$

Hence, this partial derivative is zero if and only if $\delta \in \{\delta_0, \delta_0 + \pi\}$ or $\sigma \in \{0, \pm\pi\}$.

Now, fix $\delta_1 \in]\delta_0, \delta_0 + \pi[$. We have, in this case, $2 \cos(2\alpha_2 - \delta_1) - \cos(\delta_1) > 0$. Consider the function

$$h_{\delta_1}: [-\pi, \pi] \rightarrow \mathbb{R}, \quad \sigma \mapsto h(\sigma, \delta_1).$$

Since $h'_{\delta_1}(\sigma) = (\partial h / \partial \sigma)(\sigma, \delta_1)$ has the same sign as $-\sin(\sigma)$, the function h_{δ_1} is decreasing on $[0, \pi]$ and increasing on $[\pi, 2\pi]$. Hence the values of σ for which $h_{\delta_1}(\sigma) \leq 0$ form an interval (possibly empty) centered at π . By Lemma 8.11, we know that $\mathfrak{T} \cap \mathfrak{E}_0^+ \cap \overline{\mathbb{H}_{\mathbb{C}}^2}$ is reduced to $\{[p_A], [p_B]\}$, which are in the \mathbb{R} -plane \mathfrak{m} . Hence the set $\mathfrak{T} \cap \overline{\mathbb{H}_{\mathbb{C}}^2} - \{[p_A], [p_B]\}$ is in the same connected component of $\mathfrak{T} - \mathfrak{E}_0^+$ as the interval of points $v(\pi, \delta)$ for $\delta_0 < \delta < \delta_0 + \pi$.

When $\alpha_2 = \alpha_2^{\lim}$, Parker and Will show in [18] that $\mathcal{F}_0^- \cap \mathcal{F}_{-1}^-$ is reduced to $\{[p_A], [p_B]\}$. Hence, we know that for this parameter α_2 the set $\mathfrak{T} \cap \overline{\mathbb{H}_{\mathbb{C}}^2} - \{[p_A], [p_B]\}$ is in the connected component of $\mathfrak{T} - \mathfrak{E}_0^+$ that is not contained in \mathcal{F}_0^- . By continuity of the deformation, it is also true for all the parameters $\alpha_2 \in]0, \frac{\pi}{2}[$.

We deduce that the intersection $\mathcal{F}_0^- \cap \mathcal{F}_{-1}^-$ is reduced to $\{[p_A], [p_B]\}$. □

Remark 8.13 The intersection of the bisectors \mathcal{J}_0^- and \mathcal{J}_{-1}^- is not always connected. For α_2 close to 0, it has two connected components, as shown in Figure 9d.

8.4 The faces in $\partial_{\infty} \mathbb{H}_{\mathbb{C}}^2$ are well defined

Now, we show that the 2-faces in $\partial_{\infty} \mathbb{H}_{\mathbb{C}}^2$ are well defined and that the local incidences of the bisectors are the same as the ones of the Parker–Will structure for $\alpha_2 \in]0, \frac{\pi}{2}[$. This fact will almost show the local combinatorics condition (LC) stated in Section 5. We need to show that each spinal surface of the form $\partial_{\infty} \mathcal{F}_k^{\pm}$ is cut into a quadrilateral and a bigon with vertices in the orbits of $[p_A]$ and $[p_B]$ by powers of $[U]$. We take as

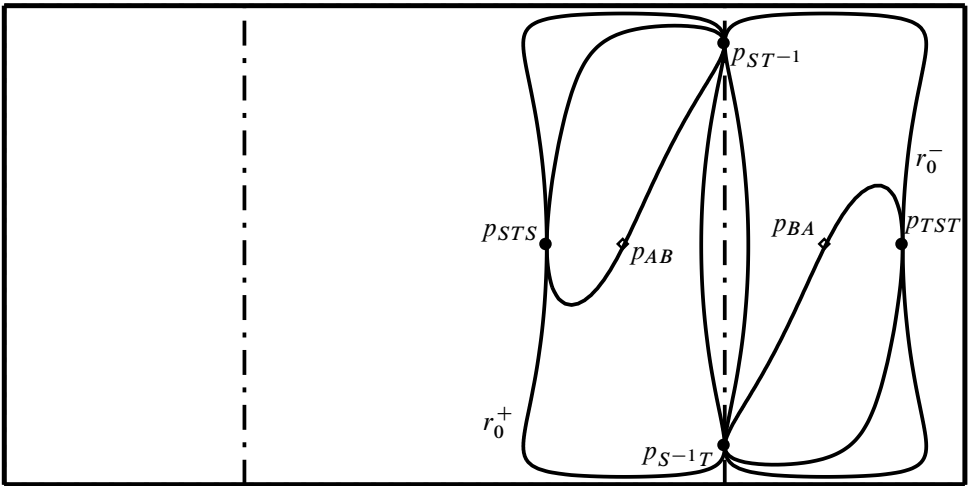


Figure 10: The traces of the bisectors near \mathcal{I}_0^+ on $\partial_\infty \mathcal{I}_0^+$, in geographical coordinates on the sphere $\partial_\infty \mathcal{I}_0^+$. We see a bigon with vertices p_{ST-1} and p_{S-1T} and a quadrilateral with vertices p_{ST-1} , p_{ST} , p_{S-1T} and p_{ST} . (From [18].)

a starting point and inspiration the proof of Parker and Will in [18]. Proposition 8.12 gives immediately the following lemma:

Lemma 8.14 *The Giraud disks $\mathcal{J}_0^+ \cap \mathcal{J}_0^-$ and $\mathcal{J}_0^+ \cap \mathcal{J}_{-1}^-$ are tangent at $[p_A]$ and at $[p_B]$.*

Proposition 8.15 *For all $\alpha_2 \in]0, \frac{\pi}{2}[$ and for all $k \in \mathbb{Z}$, the bigons B_k^+ and B_k^- , as well as the quadrilaterals Q_k^+ and Q_k^- , are well defined. If $\alpha_2 \in]0, \frac{\pi}{6}[$, then the quadrilateral Q_k^\pm is of genus 1, ie it is diffeomorphic to a torus minus a disk.*

Proof By symmetry, we only need to prove that the bigon B_0^+ and the quadrilateral Q_0^+ are well defined through the deformation. By Lemma 8.14, the Giraud circles $\partial_\infty \mathcal{J}_0^+ \cap \partial_\infty \mathcal{J}_0^-$ and $\partial_\infty \mathcal{J}_0^+ \cap \partial_\infty \mathcal{J}_{-1}^-$ are bitangent and cut the spinal surface $\partial_\infty \mathcal{J}_0^+$ in four connected components, as in Figure 10, which is taken from [18] and traced for the parameter $\alpha_2 = \alpha_2^{\text{lim}}$. The points $[p_A]$ and $[p_B]$ cut those Giraud circles into two arcs each. The arc of $\partial_\infty \mathcal{J}_0^+ \cap \partial_\infty \mathcal{J}_0^-$ containing $[Up_A]$ and the arc of $\partial_\infty \mathcal{J}_0^+ \cap \partial_\infty \mathcal{J}_{-1}^-$ containing $[U^{-1}p_B]$ border a “quadrilateral” with vertices $[p_A]$, $[Up_A]$, $[p_B]$ and $[U^{-1}p_B]$. The two other arcs border a “bigon” with vertices $[p_A]$ and $[p_B]$.

It remains to do a topological verification. Indeed, if $\alpha_2 \in]\frac{\pi}{6}, \frac{\pi}{2}[$, then the spinal surface $\partial_\infty J_0^+$ is a smooth sphere; if $\alpha_2 = \frac{\pi}{6}$, then $\partial_\infty J_0^+$ is a sphere with a singular point (which is the focus of the bisector); and if $\alpha_2 \in]0, \frac{\pi}{6}[$, then $\partial_\infty J_0^+$ is a torus. In the first two cases, the two bitangent Giraud circles cut $\partial_\infty J_0^+$ into four topological disks, but in the last case we obtain three disks and a torus minus a disk. An ideal picture is given in Figures 6 and 11. In order to identify the component that becomes of genus 1 while deforming, it is enough to check that when $\alpha_2 = \frac{\pi}{6}$, the singular point is in the interior of the quadrilateral Q_0^+ .

When $\alpha_2 = \frac{\pi}{6}$, we have

$$\begin{aligned}
 [p_U] &= \begin{bmatrix} 1 \\ -\frac{\sqrt{2}}{2}e^{i\frac{\pi}{6}} \\ e^{i\frac{\pi}{3}} \end{bmatrix}, & [p_V] &= \begin{bmatrix} -e^{i\frac{\pi}{3}} \\ -\frac{\sqrt{2}}{2}e^{i\frac{\pi}{6}} \\ -1 \end{bmatrix}, \\
 [Up_A] &= \begin{bmatrix} 1 \\ -\sqrt{2}e^{i\frac{\pi}{6}} \\ -1 \end{bmatrix}, & [U^{-1}p_B] &= \begin{bmatrix} 1 \\ \sqrt{2}e^{-i\frac{\pi}{6}} \\ -1 \end{bmatrix}.
 \end{aligned}$$

Hence, the focus of $\mathfrak{E}_0^+ = \mathfrak{E}(p_U, p_V)$ is the point

$$[f] = \begin{bmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{bmatrix}.$$

The points $[f]$, $[Up_A]$ and $[U^{-1}p_B]$ are hence aligned, in the same slice of the extor \mathfrak{E}_0^+ . Consider the intersection of this complex line with $\partial_\infty \mathbb{H}_\mathbb{C}^2$. It is the \mathbb{C} -circle $\{[q_\theta] : \theta \in [0, 2\pi]\}$, where

$$q_\theta = \begin{pmatrix} 1 \\ \sqrt{2}e^{i\theta} \\ -1 \end{pmatrix}.$$

We have

$$\begin{aligned}
 \langle p_U, q_\theta \rangle &= \langle p_V, q_\theta \rangle = -1 - e^{i(\theta - \frac{\pi}{6})} + e^{-i\frac{\pi}{3}} \\
 &= -e^{i\frac{\pi}{3}} - e^{i(\theta - \frac{\pi}{6})} \\
 &= -2 \cos\left(\frac{\pi}{4} + \frac{1}{2}\theta\right) e^{i(\frac{\pi}{12} + \frac{1}{2}\theta)},
 \end{aligned}$$

and hence

$$|\langle p_U, q_\theta \rangle|^2 = |\langle p_V, q_\theta \rangle|^2 = 4 \cos^2\left(\frac{\pi}{4} + \frac{1}{2}\theta\right) = 2(1 + \sin(\theta)).$$

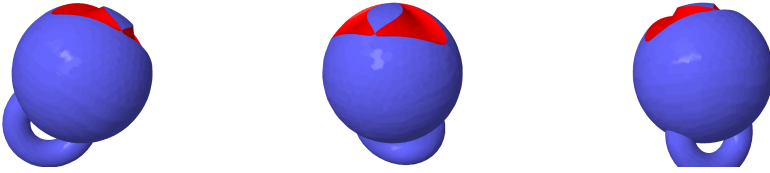


Figure 11: Ideal pictures of a face \mathcal{F}_k^\pm for $\alpha_2 \in]0, \frac{\pi}{6}[$. The blue region is in $\partial_\infty \mathbb{H}_\mathbb{C}^2$; the two red regions are Giraud disks in $\mathbb{H}_\mathbb{C}^2$. This time, the boundary at infinity of the face is on a torus, and we have a singular point in $\mathbb{H}_\mathbb{C}^2$ which is not in the picture. The quadrilateral Q_k^\pm has now a handle.

Furthermore, we have

$$p_W = e^{i\frac{\pi}{3}} \begin{pmatrix} -1 \\ \sqrt{2}e^{i\frac{\pi}{6}} + \frac{\sqrt{2}}{2}e^{-i\frac{\pi}{6}} \\ 1 \end{pmatrix} \quad \text{and} \quad U^{-1}p_W = \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2}e^{i\frac{\pi}{6}} + \sqrt{2}e^{-i\frac{\pi}{6}} \\ -1 \end{pmatrix}.$$

We compute then

$$|\langle p_W, q_\theta \rangle|^2 = |2 + 2e^{i(\theta-\frac{\pi}{6})} + e^{i(\theta+\frac{\pi}{6})}|^2 = 6\sqrt{3}\cos(\theta) + 2\sin(\theta) + 11,$$

$$|\langle U^{-1}p_W, q_\theta \rangle|^2 = |-2 + e^{i(\theta-\frac{\pi}{6})} + 2e^{i(\theta+\frac{\pi}{6})}|^2 = -6\sqrt{3}\cos(\theta) + 2\sin(\theta) + 11.$$

We deduce that $[q_\theta] \in \mathcal{J}_0^+$ if and only if $\theta \in [\frac{\pi}{6}, \frac{2\pi}{3}] \cup [\frac{4\pi}{3}, \frac{11\pi}{6}]$. But $Up_A = q_{\frac{4\pi}{3}}$, $U^{-1}p_B = q_{\frac{11\pi}{6}}$ and $f = q_{\frac{3\pi}{2}}$. Hence $[f]$ is in the interior of Q_0^+ . \square

9 Local combinatorics (LC)

Combining the results of the last section, we can now prove the local combinatorics condition (LC) stated in Section 5. We are going to show that the local combinatorics of the faces stays constant through the deformation. More precisely, we have the following two propositions. The first is about the 3-dimensional faces $\mathcal{F}_k^\pm \subset \mathbb{H}_\mathbb{C}^2$; the second about their boundary at infinity, composed by bigons and quadrilaterals in $\partial_\infty \mathbb{H}_\mathbb{C}^2$.

Proposition 9.1 *We have the following intersections of the 3-faces:*

- (1) \mathcal{F}_0^+ intersects \mathcal{F}_0^- and \mathcal{F}_{-1}^- along Giraud disks.
- (2) \mathcal{F}_0^- intersects \mathcal{F}_0^+ and \mathcal{F}_1^+ along Giraud disks.
- (3) $\mathcal{F}_0^+ \cap \mathcal{F}_1^+$ is reduced to $\{[p_B], [Up_A]\}$.
- (4) $\mathcal{F}_0^- \cap \mathcal{F}_{-1}^-$ is reduced to $\{[p_B], [p_A]\}$.

Proof The first two points are given by [Lemma 8.5](#). The third and the fourth are symmetrical, and given by [Proposition 8.12](#). \square

By considering the boundary at infinity, the next proposition follows:

Proposition 9.2 *The faces Q_0^+ and B_0^+ intersect the faces contained in \mathcal{J}_0^- , \mathcal{J}_{-1}^- , \mathcal{J}_{-1}^+ and \mathcal{J}_1^+ exactly as in [Figure 8](#). More precisely:*

- (1) B_0^+ intersects Q_0^- and Q_{-1}^- in the two arcs of its boundary.
- (2) Q_0^+ intersects Q_0^- , B_0^- , Q_{-1}^- and B_{-1}^- in the four arcs of its boundary.
- (3) The intersection of the faces B_0^+ and Q_0^+ with the faces B_{-1}^+ and Q_{-1}^+ is reduced to $\{[U^{-1}p_B], [p_A]\}$.
- (4) The intersection of B_0^+ and Q_0^+ with B_1^+ and Q_1^+ is reduced to $\{[p_B], [Up_A]\}$.

By symmetry, we obtain the local combinatorics for the faces contained in \mathcal{J}_0^- :

Proposition 9.3 *The faces Q_0^- and B_0^- intersect the faces contained in \mathcal{J}_0^+ , \mathcal{J}_1^+ , \mathcal{J}_{-1}^- and \mathcal{J}_1^- exactly as in [Figure 8](#). More precisely:*

- (1) B_0^- intersects Q_0^+ and Q_1^+ in the two arcs of its boundary.
- (2) Q_0^- intersects Q_0^+ , B_0^+ , Q_1^+ and B_1^+ in the four arcs of its boundary.
- (3) The intersection of B_0^- and Q_0^- with B_{-1}^- and Q_{-1}^- is reduced to $\{[p_A], [p_B]\}$.
- (4) The intersection of B_0^- and Q_0^- with B_1^- and Q_1^- is reduced to $\{[Up_A], [Up_B]\}$.

10 Global combinatorics (GC)

Finally, it remains to check the global combinatorics condition (GC) of the strategy of proof of [Section 5](#). This point is more technical than the two preceding ones: when $[U]$ is loxodromic, we are going to use explicit projections on visual spheres in order to prove it. When $[U]$ is elliptic, the same techniques can prove the results but fail whenever $[U]$ has small order (namely 4, 5, 6, 7 or 8). We prefer to use the results of Parker, Wang and Xie in [\[17\]](#) in order to establish the global combinatorics for the elliptic side. We begin by setting a strategy of proof for the condition (GC).

10.1 Strategy

We want to show that, through the deformation, if $[U]$ is loxodromic or elliptic of type $(\frac{1}{n}, -\frac{1}{n})$ with $n \geq 4$, then the intersections of the faces \mathcal{F}_k^+ and \mathcal{F}_k^- and of their

boundaries at infinity are exactly the ones described by the local combinatorics. Since the domain is invariant by $[U]$ and since we have described the combinatorics of the intersections of the faces \mathcal{F}_k^\pm as well as of B_0^\pm and Q_0^\pm with the faces contained in the neighboring bisectors, it is enough to show the following two propositions to obtain the global combinatorics of the intersections of the faces.

Proposition 10.1 *If $[U]$ is loxodromic, then:*

- \mathcal{J}_0^+ intersects \mathcal{J}_k^+ if and only if $k \in \{-1, 0, 1\}$.
- \mathcal{J}_0^- intersects \mathcal{J}_k^- if and only if $k \in \{-1, 0, 1\}$.
- \mathcal{J}_0^+ intersects \mathcal{J}_k^- if and only if $k \in \{-2, -1, 0, 1\}$. Further, $\mathcal{J}_0^+ \cap \mathcal{J}_1^- = \{[Up_A]\}$ and $\mathcal{J}_0^+ \cap \mathcal{J}_{-2}^- = \{[U^{-1}p_B]\}$.

Proposition 10.2 *If $[U]$ is elliptic of order ≥ 4 , then:*

- \mathcal{J}_0^+ intersects \mathcal{J}_k^+ if and only if $k \equiv -1, 0, 1 \pmod n$.
- \mathcal{J}_0^- intersects \mathcal{J}_k^- if and only if $k \equiv -1, 0, 1 \pmod n$.
- \mathcal{J}_0^+ intersects \mathcal{J}_k^- if and only if $k \equiv -2, -1, 0, 1 \pmod n$. Further, $\mathcal{J}_0^+ \cap \mathcal{J}_1^- = \{[Up_A]\}$ and $\mathcal{J}_0^+ \cap \mathcal{J}_{-2}^- = \{[U^{-1}p_B]\}$.

In order to prove [Proposition 10.1](#), we will project the bisectors \mathcal{J}_k^\pm on the visual sphere of $[p_U]$. Then, we will obtain a family of disks invariant by the action of $[U]$ on the visual sphere.

Parker, Wang and Xie give a proof of [Proposition 10.2](#) in [[17](#), Theorem 4.3]. The correspondence between their notation and ours is given by $\mathcal{J}_k^+ = \mathcal{B}_{-2k}$ and $\mathcal{J}_k^- = \mathcal{B}_{-2k-1}$. The projection techniques that we will use when $[U]$ is loxodromic can also be used in this case, but have to be refined by considering a projection into the real visual sphere and fail to prove the statement for small values of n , namely $n = 4, 5, 6, 7$ or 8 . We discuss this point in [Section 10.5](#).

10.2 First data

We establish first some results on the projection of the bisectors \mathcal{J}_k^\pm on the visual sphere $L_{[p_U]}$ of $[p_U]$. We identify those projections as disks with boundaries passing through the images of some notable points. Recall that we studied the visual spheres of points of $\mathbb{C}\mathbb{P}^2$ in [Section 2](#) of [Part I](#). First, we establish a criterion for the tangency of a complex line with a spinal surface.

Lemma 10.3 (tangency criterion) *Let $p, q \in \mathbb{C}^3 - \{0\}$ be such that $\langle p, p \rangle = \langle q, q \rangle \neq 0$ and $\langle p, q \rangle \in \mathbb{R} - \{0\}$. Let $[r] \in \mathfrak{S}(p, q)$. Then the complex line $l_{[p],[r]}$ is tangent to $\mathfrak{S}(p, q)$ at $[r]$ if and only if there exists $\epsilon \in \{\pm 1\}$ such that $\langle p, r \rangle = \epsilon \langle q, r \rangle$ and $\langle q, p \rangle \neq \epsilon \langle p, p \rangle$.*

Proof We are going to show that $l_{[p],[r]}$ intersects $\mathfrak{S}(p, q)$ only at $[r]$ if and only if the conditions of the statement are satisfied. Another point of $l_{[p],[r]} \cap \mathfrak{S}(p, q)$ can be written as $[p + \lambda r]$ with $\lambda \in \mathbb{C}$, and satisfies

$$\langle p + \lambda r, p + \lambda r \rangle = 0 \quad \text{and} \quad |\langle p, p + \lambda r \rangle|^2 = |\langle q, p + \lambda r \rangle|^2.$$

By expanding and using that $\langle r, r \rangle = 0$, that $\langle p, p \rangle = \langle q, q \rangle \in \mathbb{R}$ and that $|\langle p, r \rangle| = |\langle q, r \rangle|$, we obtain

$$(4) \quad \langle p, p \rangle + 2 \operatorname{Re}(\lambda \langle p, r \rangle) = 0,$$

$$(5) \quad \langle p, p \rangle \operatorname{Re}(\lambda \langle p, r \rangle) = \operatorname{Re}(\lambda \langle q, p \rangle \langle q, r \rangle).$$

Replacing $\operatorname{Re}(\lambda \langle p, r \rangle)$ in the second equation and noticing $\langle q, p \rangle \in \mathbb{R} - \{0\}$, we have

$$(6) \quad -\frac{1}{2} \langle p, p \rangle = \operatorname{Re}(\lambda \langle p, r \rangle),$$

$$(7) \quad -\frac{1}{2} \frac{\langle p, p \rangle^2}{\langle q, p \rangle} = \operatorname{Re}(\lambda \langle q, r \rangle).$$

We obtain two equations of real lines in \mathbb{C} for λ . The intersection of the lines is empty if and only if the lines are different and parallel. Since $|\langle p, r \rangle| = |\langle q, r \rangle|$, this is equivalent to the existence of $\epsilon \in \{\pm 1\}$ such that $\langle p, r \rangle = \epsilon \langle q, r \rangle$ and $-\frac{1}{2} \langle p, p \rangle \neq \epsilon (-\frac{1}{2} \langle p, p \rangle^2 / \langle q, p \rangle)$, ie $\langle q, p \rangle \neq \epsilon \langle p, p \rangle$. □

Lemma 10.4 *For $\alpha_2 \neq \pm \alpha_2^{\lim}, \pm \frac{\pi}{2}$, the complex lines $l_{[p_U],[U p_A]}$ and $l_{[p_U],[U^{-1} p_B]}$ are tangent to the sphere $\mathfrak{S}(p_U, p_V)$ at $[U p_A]$ and $[U^{-1} p_B]$, respectively.*

Proof By Lemmas 8.2 and 8.3, we know $[U p_A]$ and $[U^{-1} p_B]$ belong to $\mathfrak{S}(p_U, p_V)$. Furthermore, we have

$$\begin{aligned} \langle p_U, p_U \rangle &= \langle p_V, p_V \rangle = 4 \cos^2(\alpha_2) - \frac{3}{2}, \\ \langle p_U, U p_A \rangle &= \langle p_V, U p_A \rangle = e^{-2i\alpha_2}, \\ \langle p_U, U^{-1} p_B \rangle &= \langle p_V, U^{-1} p_B \rangle = 1, \\ \langle p_V, p_U \rangle &= -\frac{3}{2}. \end{aligned}$$

By Lemma 10.3, we have the tangencies of the statement. □

Since, by Proposition 3.21, we know that the projection of a bisector of the form $\mathfrak{B}(p, q)$ on the visual sphere $L_{[p]}$ is a disk, we deduce the following corollary:

Corollary 10.5 *The set $\pi_{[p_U]}(\mathcal{J}_0^+)$ is a closed disk whose boundary contains $\pi_{[p_U]}(l_{[p_U],[U p_A]})$ and $\pi_{[p_U]}(l_{[p_U],[U^{-1} p_B]})$.*

Notation 10.6 We will denote those disks by D_k^\pm , in order to have $D_k^\pm = \pi_{[p_U]}(\mathcal{J}_0^\pm)$.

Recall, from Section 8.2, that we also have a symmetry associated to the involution $I_1 \in U(2, 1)$ given by

$$I_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{2}e^{i\alpha_2} & -1 & 0 \\ -1 & -\sqrt{2}e^{-i\alpha_2} & 1 \end{pmatrix}.$$

As seen in Section 8.2, I_1 satisfies $I_1 p_U = p_U$, $I_1 p_V = p_W$ and $I_1 p_W = p_V$. The action of $[I_1]$ on $\mathbb{C}\mathbb{P}^2$ fixes $[p_U]$ and exchanges the bisectors \mathcal{J}_0^+ and \mathcal{J}_0^- .

10.3 The chart $\psi_{p'_U, p''_U}$ of $L_{[p_U]}$

Recall that, given $[p] \in \mathbb{C}\mathbb{P}^2$ and two points $[p'], [p''] \in [p]^\perp$, we defined a map $\psi_{p', p''}: L_{[p]} \rightarrow \mathbb{C}\mathbb{P}^1$ in Notation 2.4. This map gives coordinates for the visual sphere $L_{[p]}$. We make some computations in the chart $\psi_{p'_U, p''_U}$ of $L_{[p_U]}$ in order to identify the intersections of the disks D_k^\pm and deduce the global combinatorics of the intersections of the bisectors \mathcal{J}_k^\pm .

Notation 10.7 In order to avoid heavy notation, if $[q] \in \mathbb{C}\mathbb{P}^2 - \{[p_U]\}$ we write $\psi(q)$ instead of $\psi_{p'_U, p''_U}(l_{[p_U],[q]})$.

Remark 10.8 The image by ψ of several notable points is easy to compute:

- If $[U]$ is elliptic, then $\psi(p'_U) = \infty$ and $\psi(p''_U) = 0$.
- If $[U]$ is loxodromic, then $\psi(p'_U) = 0$ and $\psi(p''_U) = \infty$.
- $\psi(p_B) = 1$.
- $\psi(p_A) = \frac{-(8 \cos^2(\alpha_2) + 1) + \delta}{-(8 \cos^2(\alpha_2) + 1) - \delta} = \frac{\text{tr}(U) + 1 - \delta}{\text{tr}(U) + 1 + \delta}$.

The elements U and I_1 of $U(2, 1)$ fix p_U and hence each have a natural projective action on $L_{[p_U]}$. We are going to identify those two actions in the chart $\psi_{p'_U, p''_U}$. We begin with U :

Remark 10.9 The action of U on $\mathbb{C}\mathbb{P}^2$ fixes $[p_U]$, $[p'_U]$ and $[p''_U]$. Hence it acts on $L_{[p_U]}$ by fixing $\psi(p'_U)$ and $\psi(p''_U)$. In the chart $\psi_{p'_U, p''_U}$, the action is hence given by either a rotation or a homothety with center 0. We will give some details later on the corresponding angle or ratio.

Consider now the action of the involution I_1 .

Lemma 10.10 *The action of I_1 on $L_{[p_U]}$ is given, in the chart $\psi_{p'_U, p''_U}$, by $z \mapsto 1/z$.*

Proof The involution I_1 satisfies $I_1 p'_U = p''_U$, $I_1 p''_U = p'_U$ and $I_1 p_B = p_B$. Since $\psi(p'_U)$ and $\psi(p''_U)$ equal 0 and ∞ , and $\psi(p_B) = 1$, the involution I_1 acts on the chart $\psi_{p'_U, p''_U}$ by a projective map fixing 1 and exchanging 0 and ∞ . Hence, it is $z \mapsto 1/z$. □

10.4 The loxodromic side

In this subsection, we study the case where $[U]$ is loxodromic. We are going to study the relative position of the disks D_k^\pm in $L_{[p_U]}$, which will be sufficient to prove the global combinatorics of the intersections of the bisectors \mathcal{J}_k^\pm . We are going to show that the disks are arranged as in [Figure 12](#).

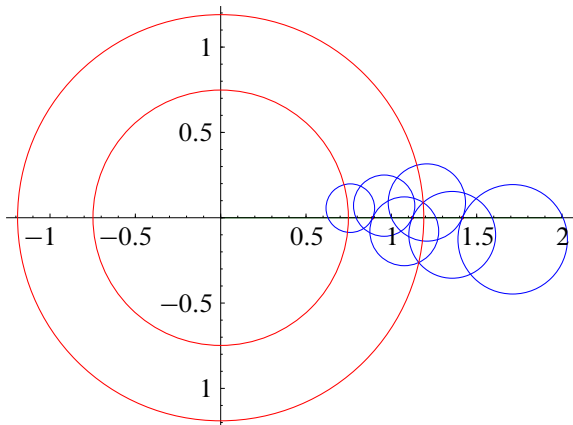


Figure 12: The disks D_k^\pm (in blue) for $\alpha_2 = 0.91$.

We consider here the parameters $\alpha_2 \in]\frac{\pi}{6}, \alpha_2^{\text{lim}}[$. By [Remark 6.6](#), we know that for such α_2 , the element U is loxodromic and has eigenvalues 1, e^l and e^{-l} . The

length $l \in]0, \operatorname{acosh}(\frac{5}{2})[$ can also be used as a parameter for the deformation. It is related to α_2 by the equation

$$2 \cosh(l) + 1 = \operatorname{tr}(U) = 8 \cos^2(\alpha_2).$$

In this case, recall also that, by setting $\delta = 2 \sinh(l)$, we have

$$[p'_U] = \begin{bmatrix} 2(2e^{2i\alpha_2} + 1) \\ -\sqrt{2}e^{i\alpha_2}(2e^{2i\alpha_2} + 1 + \delta) \\ -(8 \cos^2(\alpha_2) + 1) - \delta \end{bmatrix} \quad \text{and} \quad [p''_U] = \begin{bmatrix} 2(2e^{2i\alpha_2} + 1) \\ -\sqrt{2}e^{i\alpha_2}(2e^{2i\alpha_2} + 1 - \delta) \\ -(8 \cos^2(\alpha_2) + 1) + \delta \end{bmatrix}.$$

Remark 10.11 In the loxodromic case, $[p'_U], [p''_U] \in \partial_\infty \mathbb{H}^2_{\mathbb{C}}$, and $[p_U], [p'_U]$ and $[p''_U]$ form an autopolar triangle. We compute the following Hermitian products:

$$\begin{aligned} \langle p'_U, p'_U \rangle &= \langle p''_U, p''_U \rangle = 0, \\ \langle p_U, p_U \rangle &= \cosh(l) - 1, \\ \langle p_U, p'_U \rangle &= \langle p_U, p''_U \rangle = 0, \\ \langle p'_U, p''_U \rangle &= -16 \sinh^2(l). \end{aligned}$$

Remark 10.12 In the case where U is loxodromic, we have

$$\psi(p_A) = \frac{\operatorname{tr}(U) + 1 - \delta}{\operatorname{tr}(U) + 1 + \delta} = \frac{1 + e^{-l}}{1 + e^l} = e^{-l}.$$

Remark 10.13 The action of U on $L_{[p_U]}$ fixes $l_{[p_U],[p'_U]}$ and $l_{[p_U],[p''_U]}$. In the chart $\psi_{p'_U, p''_U}$, since $Up'_U = e^l p'_U$ and $Up''_U = e^l p''_U$, the action of U is a homothety centered at 0 and with ratio e^{2l} ; it is given by $z \mapsto e^{2l} z$.

Remark 10.14 Since $\psi(p_B) = 1$, $\psi(p_A) = e^{-l}$ and U acts by a homothety of ratio e^{2l} , we deduce that $\psi(U^k p_B) = e^{2kl}$ and $\psi(U^k p_A) = e^{(2k-1)l}$ for all $k \in \mathbb{Z}$.

We deduce from the actions of U and I_1 a result on the intersections:

Proposition 10.15 *The disks D_0^+ and D_1^- are tangent at the point $\pi_{[p_U]}(l_{[p_U],[Up_A]})$. The disks D_0^+ and D_{-2}^- are tangent at the point $\pi_{[p_U]}(l_{[p_U],[U^{-1}p_B]})$.*

Proof Recall that D_0^+ is a disk whose boundary contains $\psi(U^{-1}p_B)$ and $\psi(Up_A)$, and that D_1^- is a disk whose boundary contains $\psi(U^2p_B)$ and $\psi(Up_A)$. Let θ be the angle at $\psi(Up_A)$ between ∂D_0^+ and the real axis. Since the map $z \mapsto 1/z$ is conformal, and it exchanges D_0^+ and D_0^- , the angle between ∂D_0^- and the real axis at

$\psi(p_A) = 1/\psi(U_{p_A})$ also equals θ . Since D_1^- is obtained from D_0^- by a homothety of ratio e^{2l} , the angle between the real axis and ∂D_1^- at $\psi(U_{p_A})$ equals θ . Since ∂D_1^- and ∂D_0^+ intersect the real axis $\psi(U_{p_A})$ with the same angle, they are tangent.

The proof for the other tangency is analogous. □

Corollary 10.16 *The bisectors \mathcal{J}_0^+ and \mathcal{J}_1^- are tangent at $[U_{p_A}]$. The bisectors \mathcal{J}_0^+ and \mathcal{J}_{-2}^- are tangent at $[U^{-1} p_B]$.*

Proof By the proposition above, we know that the intersection of \mathcal{J}_0^+ and \mathcal{J}_1^- is contained in the line $l_{[p_U],[U_{p_A}]}$. By Lemma 8.14, this line is tangent to $\partial_\infty \mathcal{J}_0^+$ at $[U_{p_A}]$, and hence the intersection contains exactly one point. □

The following proposition is a key result for determining the relative position of the disks D_k^\pm . We are going to show that each disk is contained in an annulus centered at 0 with explicit radii; we will see later that the annuli are obtained by homotheties with center 0. One of the annuli that we consider is pictured in Figure 12. The effective bounds that we give in the following proposition will be enough to prove the global combinatorics of the intersections of the disks D_k^\pm and the bisectors \mathcal{J}_k^\pm .

Proposition 10.17 *The disk D_0^+ is contained in the annulus of center 0 and radii $e^{-\frac{5}{2}l}$ and $e^{\frac{3}{2}l}$.*

The proof of Proposition 10.17 is rather technical and needs many computations. We prove first three technical lemmas that we will use in the proof.

For $r \geq 0$, $\theta \in \mathbb{R}$ and $\mu \in \mathbb{C}$, let $q_{r,\theta,\mu} = p''_U + r e^{i\theta} p'_U + \mu p_U$. The set $\{q_{r,\theta,\mu} : \mu \in \mathbb{C}\}$ is the preimage of $r e^{i\theta}$ by the map $\psi_{p'_U, p''_U}$.

Lemma 10.18 *If $q_{r,\theta,\mu} \in \mathfrak{B}(p_U, p_V)$, then*

$$|\langle p''_U + r e^{i\theta} p'_U \rangle|^2 \leq 16r(2 \cosh(l) + 1)^2 \cosh^2(\frac{1}{2}l) \cos(\theta).$$

Proof Suppose that $q_{r,\theta,\mu} \in \mathfrak{B}(p_U, p_V)$. We have two conditions for $q_{r,\theta,\mu}$ to belong to the bisector:

- (8) $\langle q_{r,\theta,\mu}, q_{r,\theta,\mu} \rangle \leq 0,$
- (9) $|\langle q_{r,\theta,\mu}, p_U \rangle| = |\langle q_{r,\theta,\mu}, p_V \rangle|.$

They can be written as

$$(10) \quad 2 \operatorname{Re}(re^{i\theta} \langle p''_U, p'_U \rangle) + |\mu|^2 \langle p_U, p_U \rangle \leq 0,$$

$$(11) \quad |\mu| |\langle p_U, p_U \rangle| = |\langle p''_U + re^{i\theta} p'_U \rangle + \bar{\mu} \langle p_U, p_U \rangle|.$$

By the computations above, the conditions become

$$(12) \quad |\mu|^2 \leq 2r \frac{16 \sinh^2(l)}{\cosh(l) - 1} \cos(\theta) = 32r(\cosh(l) + 1) \cos(\theta),$$

$$(13) \quad (\cosh(l) - 1)|\mu| = |\langle p''_U + re^{i\theta} p'_U \rangle - \frac{3}{2}\bar{\mu}|.$$

We make some extra computations in order to have a more explicit second condition. It implies, by the triangle inequality, that

$$(14) \quad (\cosh(l) - 1)|\mu| \geq |\langle p''_U + re^{i\theta} p'_U \rangle| - \frac{3}{2}|\mu|.$$

We deduce that

$$(15) \quad (\cosh(l) + \frac{1}{2})|\mu| \geq |\langle p''_U + re^{i\theta} p'_U \rangle|.$$

By squaring (15) and using (12), we deduce that

$$(16) \quad |\langle p''_U + re^{i\theta} p'_U \rangle|^2 \leq 32r(\cosh(l) + \frac{1}{2})^2(\cosh(l) + 1) \cos(\theta).$$

This last equation can be rewritten as

$$(17) \quad |\langle p''_U + re^{i\theta} p'_U \rangle|^2 \leq 16r(2 \cosh(l) + 1)^2 \cosh^2(\frac{1}{2}l) \cos(\theta). \quad \square$$

Lemma 10.19 *Let $h_1 = \langle p'_U, p_V \rangle$ and $h_2 = \langle p''_U, p_V \rangle$. Then*

$$|h_1|^2 = 12e^{\frac{1}{2}l}(2 \cosh(l) + 1) \cosh(\frac{1}{2}l), \quad |h_2|^2 = 12e^{-\frac{1}{2}l}(2 \cosh(l) + 1) \cosh(\frac{1}{2}l),$$

$$h_1 \bar{h}_2 = 2(2 \cosh(l) + 1)((5 - 2 \cosh(l))(\cosh(l) + 1) - 4i \sin(2\alpha_2) \sinh(l)).$$

Proof We have

$$h_1 = \langle p'_U, p_V \rangle = (2 \sinh(l) + 3 + 4i \sin(2\alpha_2))(e^{2i\alpha_2} + 1)$$

$$= 2 \cos(\alpha_2) e^{i\alpha_2} (2 \sinh(l) + 3 + 4i \sin(2\alpha_2)),$$

$$h_2 = \langle p''_U, p_V \rangle = (-2 \sinh(l) + 3 + 4i \sin(2\alpha_2))(e^{2i\alpha_2} + 1)$$

$$= 2 \cos(\alpha_2) e^{i\alpha_2} (-2 \sinh(l) + 3 + 4i \sin(2\alpha_2)).$$

We easily deduce, replacing $\sin(2\alpha_2)^2$ by $4 \cos^2(\alpha_2)(1 - \cos^2(\alpha_2))$ and $\cos^2(\alpha_2)$ by $\frac{1}{8}(2 \cosh(l) + 1)$, that

$$(18) \quad \begin{aligned} |h_1|^2 &= 12e^{\frac{1}{2}l}(2 \cosh(l) + 1) \cosh\left(\frac{1}{2}l\right), \\ |h_2|^2 &= 12e^{-\frac{1}{2}l}(2 \cosh(l) + 1) \cosh\left(\frac{1}{2}l\right). \end{aligned}$$

Furthermore, a direct computation gives

$$(19) \quad h_1 \bar{h}_2 = 2(2 \cosh(l) + 1)((5 - 2 \cosh(l))(\cosh(l) + 1) - 4i \sin(2\alpha_2) \sinh(l)). \quad \square$$

Lemma 10.20 *Let $l > 0$. Then $18 \cosh^2(2l) > 8 \cosh(3l) + 8 \cosh(2l) - 8 \cosh(l) + 10$.*

Proof On the one hand, since the function \cosh is convex, we have

$$8 \cosh(4l) + 8 \cosh(l) \geq 8 \cosh(3l) + 8 \cosh(2l).$$

On the other hand, $\cosh(4l) > 1$. Thus, by adding the two inequalities, we obtain

$$\begin{aligned} 9 \cosh(4l) + 8 \cosh(l) &> 8 \cosh(3l) + 8 \cosh(2l) + 1, \\ 9 \cosh(4l) + 9 &> 8 \cosh(3l) + 8 \cosh(2l) - 8 \cosh(l) + 10, \\ 18 \cosh^2(2l) &> 8 \cosh(3l) + 8 \cosh(2l) - 8 \cosh(l) + 10. \quad \square \end{aligned}$$

Using the last three technical lemmas, we are able to prove [Proposition 10.17](#).

Proof of Proposition 10.17 We are going to show that if a point of the form $q_{r,\theta,\mu} = p'_U + r e^{i\theta} p'_U + \mu p_U$, where $r \geq 0$, $\theta \in \mathbb{R}$ and $\mu \in \mathbb{C}$, belongs to the bisector $\mathfrak{B}(p_U, p_V)$, then $e^{-\frac{5}{2}l} < r < e^{\frac{3}{2}l}$. Suppose that $q_{r,\theta,\mu} \in \mathfrak{B}(p_U, p_V)$. Rewriting [Lemma 10.18](#) with the notation of [Lemma 10.19](#) gives

$$(20) \quad |h_2 + r e^{i\theta} h_1|^2 \leq 16r(2 \cosh(l) + 1)^2 \cosh^2\left(\frac{1}{2}l\right) \cos(\theta).$$

The left side of the inequality can be written in the form

$$(21) \quad |h_2|^2 + 2r \operatorname{Re}(e^{i\theta} h_1 \bar{h}_2) + r^2 |h_1|^2.$$

Using [Lemma 10.19](#), we compute

$$\begin{aligned} &2 \operatorname{Re}(e^{i\theta} h_1 \bar{h}_2) \\ &= 4(2 \cosh(l) + 1) \times ((5 - 2 \cosh(l))(\cosh(l) + 1) \cos(\theta) + 4 \sin(2\alpha_2) \sinh(l) \sin(\theta)) \\ &= 8(2 \cosh(l) + 1) \cosh\left(\frac{1}{2}l\right) \\ &\quad \times ((5 - 2 \cosh(l)) \cosh\left(\frac{1}{2}l\right) \cos(\theta) + 4 \sin(2\alpha_2) \sinh\left(\frac{1}{2}l\right) \sin(\theta)). \end{aligned}$$

Replacing the terms of (20) with the values computed below and in Lemma 10.19, and by dividing by $4(2 \cosh(l) + 1) \cosh(\frac{1}{2}l)r$, we obtain

$$(22) \quad 3r^{-1}e^{-\frac{1}{2}l} + 3re^{\frac{1}{2}l} + 2((5 - 2 \cosh(l)) \cosh(\frac{1}{2}l) \cos(\theta) + 4 \sin(2\alpha_2) \sinh(\frac{1}{2}l) \sin(\theta)) \leq 4(2 \cosh(l) + 1) \cosh(\frac{1}{2}l) \cos(\theta).$$

Taking the terms in θ to the right-hand side, we have

$$(23) \quad 3r^{-1}e^{-\frac{1}{2}l} + 3re^{\frac{1}{2}l} \leq 6(2 \cosh(l) - 1) \cosh(\frac{1}{2}l) \cos(\theta) - 8 \sin(2\alpha_2) \sinh(\frac{1}{2}l) \sin(\theta).$$

Now, the right-hand side is of the form $a \cos(\theta) + b \sin(\theta)$. Its maximum value is $\sqrt{a^2 + b^2}$. In particular, we have

$$(24) \quad (3r^{-1}e^{-\frac{1}{2}l} + 3re^{\frac{1}{2}l})^2 \leq 36(2 \cosh(l) - 1)^2 \cosh^2(\frac{1}{2}l) + 64 \sin^2(2\alpha_2) \sinh^2(\frac{1}{2}l).$$

By replacing $\sin(2\alpha_2)^2$ by its expression in terms of $\cosh(l)$, we see that the right-hand side, after linearization, equals $16 \cosh(3l) + 16 \cosh(2l) - 16 \cosh(l) + 20$. By Lemma 10.20, and since all terms are positive, we obtain

$$(25) \quad (3r^{-1}e^{-\frac{1}{2}l} + 3re^{\frac{1}{2}l})^2 < 36 \cosh(2l)^2,$$

$$(26) \quad r^{-1}e^{-\frac{1}{2}l} + re^{\frac{1}{2}l} < 2 \cosh(2l).$$

Therefore, we get the following inequality of degree 2 in r :

$$(27) \quad e^{-\frac{1}{2}l} - 2r \cosh(2l) + r^2 e^{\frac{1}{2}l} < 0.$$

Notice that the equality holds for $r = e^{-\frac{5}{2}l}$ and $r = e^{\frac{3}{2}l}$, and that all coefficients are positive. Thus, (27) implies $e^{-\frac{5}{2}l} < r < e^{\frac{3}{2}l}$. □

Since D_1^- is the image of D_0^+ by $z \mapsto 1/z$, we have the following corollary:

Corollary 10.21 *The disk D_0^- is contained in the open annulus of center 0 and radii $e^{-\frac{3}{2}l}$ and $e^{\frac{5}{2}l}$*

Proposition 10.22 *The disks D_k^\pm intersect in the following way:*

- (1) D_0^+ intersects D_k^+ if and only if $k \in \{-1, 0, 1\}$.
- (2) D_0^- intersects D_k^- if and only if $k \in \{-1, 0, 1\}$.
- (3) D_0^+ intersects D_k^- if and only if $k \in \{-2, -1, 0, 1\}$. Furthermore, D_0^+ is tangent to D_1^- and D_{-2}^- .

Proof By an immediate induction, we know that the disk D_k^+ is contained in the open annulus of radii $e^{2kl+\frac{3}{2}l}$ and $e^{2kl-\frac{5}{2}l}$ and that the disk D_k^- is contained in the open annulus of radii $e^{2kl+\frac{5}{2}l}$ and $e^{2kl-\frac{3}{2}l}$. If $|k| > 1$, the disks D_k^+ and D_0^+ are in two disjoint annuli. The same happens for the disks D_k^- and D_0^- , as well as for D_k^- and D_0^+ . Hence it only remains to prove that the expected intersections exist. By [Proposition 10.1](#), we know that

- \mathcal{J}_0^+ intersects \mathcal{J}_1^+ and \mathcal{J}_{-1}^+ ,
- \mathcal{J}_0^- intersects \mathcal{J}_1^- and \mathcal{J}_{-1}^- ,
- \mathcal{J}_0^+ intersects \mathcal{J}_0^- and \mathcal{J}_{-1}^- ,

which proves the incidences. Finally, by [Proposition 10.15](#), we know that D_0^+ is tangent to D_1^- and D_{-2}^- . \square

With the projection described above, we obtain [Proposition 10.1](#).

10.5 The elliptic side

As stated before, a complete proof of [Proposition 10.2](#) can be found in the article of Parker, Wang and Xie [17]. The corresponding notation is $\mathcal{J}_k^+ = \mathcal{B}_{-2k}$ and $\mathcal{J}_k^- = \mathcal{B}_{-2k-1}$.

Nevertheless, the projection method used for the loxodromic side can be adapted for proving [Proposition 10.2](#) for $n \geq 9$. We give here a brief summary of the key steps and the main differences with the loxodromic case.

The computations are almost the same, but we have to replace δ by $2i \sin(\beta)$ instead of $2 \sinh(l)$, and hence deal with usual trigonometry instead of hyperbolic trigonometry. The projection of the bisectors will still be the disks D_k^\pm , and the tangencies remain the same. However, the action of $[U]$ on the chart ψ of $L_{[pU]}$ is no longer by homotheties but by rotations of angle 2β . Pictures of the projections are given in [Figure 13](#).

It is then possible to prove that if $n \geq 9$, each disk is contained in an angular sector of angle 4β . They play the role of the annuli of [Section 10.4](#); and this proves that the intersections are exactly the expected ones, at least for bisectors which are close in the orbit. For a full proof, a refinement is still necessary, since the orbit of the disks D_k^\pm turns twice around the origin. The problem can be solved by considering the real visual sphere of $[pU]$, and proving that the bisectors at the opposite sides of the orbit, which project to same disk, cannot intersect in $\mathbb{H}_{\mathbb{C}}^2$.

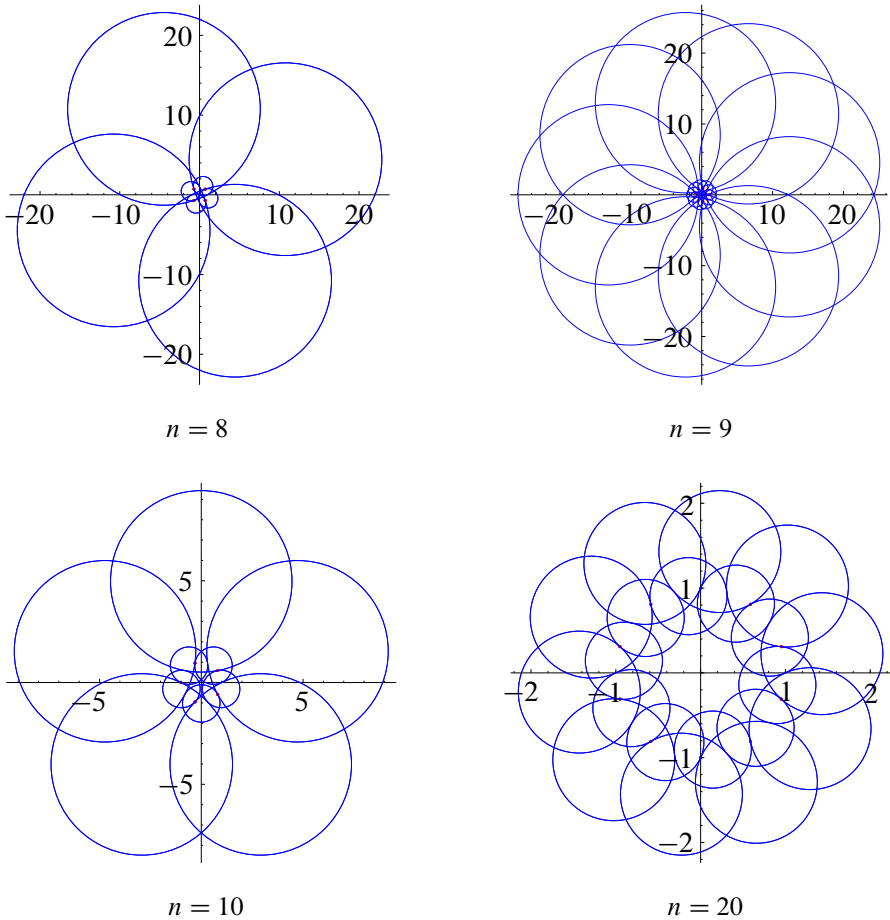


Figure 13: Projection on the visual sphere for $n = 8, 9, 10$ and 20 .

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*Unité de recherche en Mathématiques, Université du Luxembourg, Campus Belval
Esch-sur-Alzette, Luxembourg*

miguel.acosta@uni.lu

Proposed: Walter Neumann

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