A State-of-the-Art Review on Theory and Engineering Applications of Eigenvalue and Eigenvector Derivatives

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Abstract

Eigenvalue and eigenvector derivatives with respect to system design variables and their applications have been and continue to be one of the core issues in the design, control and identification of practical engineering systems. Many different numerical methods have been developed to compute accurately and efficiently these required derivatives from which, a wide range of successful applications have been established. This paper reviews and examines these methods of computing eigenderivatives for undamped, viscously damped, nonvsicously damped, fractional and nonlinear vibration systems, as well as defective systems, for both distinct and repeated eigenvalues. The underlying mathematical relationships among these methods are discussed, together with new theoretical developments. Major important applications of eigenderivatives to finite element model updating, structural design and modification prediction, performance optimization of structures and systems, optimal control system design, damage detection and fault diagnosis, as well as turbine bladed disk vibrations are examined. Existing difficulties are identified and measures are proposed to rectify them. Various examples are given to demonstrate the key theoretical concepts and major practical applications of concern. Potential further research challenges are identified with the purpose of concentrating future research effort in the most fruitful directions.

1. Introduction

Eigenvalue problems are often introduced in the context of linear algebra or matrix theory, even though they arose first in the study of quadratic forms and differential equations. Over centuries of developments, many prominent mathematicians have made significant contributions to the subject, such as Euler [1], Lagrange [2], Cauchy [2], Fourier [3], Liouville [3] and Hilbert [3]. In particular, the eigenvalue perturbation problem, which is the earlier form of eigenderivative analysis, was first popularized by Lord Rayleigh [4], in his investigation of harmonic vibrations of a string perturbed by small inhomogeneities, and was later further developed by Schrödinger in his fundamental series quantum theory [5]. The first numerical algorithm for computing eigenvalues and eigenvectors, the well-known power method, was proposed by von Mises [6] in 1929 and the powerful QR method was developed by Francis [7] in 1961. Since then the eigenvalue problem and its related eigenderivative analysis have undergone thorough development as evidenced by the more recent excellent treatises by Wilkinson [8] on the theoretical frameworks and by Golub and van Loan [9] on numerical implementation.

It should be noted that the present paper is mainly concerned with eigenvalue and eigenvector derivatives, sometimes referred to as local sensitivities, as opposed to global sensitivities whose detailed discussions can be found in [10-12]. Whereas an output determined using local sensitivity is confined to the region of linear behavior close to the current value of a system parameter, global sensitivity is a measure of the contribution that a change of parameter makes across a range of values to the variance of an output. Global sensitivities are used in uncertainty quantification of dynamic systems in engineering (as well as in other disciplines) and are considered to be outside the scope of the present review.

The most general form of eigenvalue problem of a linear vibration system may be written as,

$$\mathbf{A}(p,\lambda)\,\mathbf{X} = \lambda\,\mathbf{B}(p,\lambda)\,\mathbf{X} \tag{1}$$

where $\mathbf{A}(p,\lambda)$ and $\mathbf{B}(p,\lambda)$ are the general system complex matrices which are functions of design variable p as well as eigenvalue λ in a most general case though for many practical systems, the system matrices are independent of λ. The required eigenvalue derivative $\partial\lambda/\partial p$ and eigenvector derivative $\partial\mathbf{X}/\partial p$ of interest can then either be analytically derived or numerically computed from (1).

Nowadays, eigenvalue and eigenvector derivatives with respect to system design variables have become an important and versatile routine tool in the optimal design, control and identification of most practical engineering systems. Major numerical methods have been developed for the accurate and efficient computations of eigenderivatives of dynamic systems which are undamped, damped, fractional or nonlinear in vibration response characteristics. A wide range of important practical applications of eigenderivatives to finite element model updating, structural design and modification prediction, performance optimization of structures and systems, optimal control system design, damage detection and fault diagnosis, and turbine bladed disk vibrations have been firmly established. The growing importance and practical significance of the subject area has been witnessed by the large number of publications and their high citations over the recent decades, yet there remains a lack of comprehensive review articles on the subject to date. The present work therefore seeks to address such a need by critically reviewing existing numerical methods for computing eigenderivatives for both distinct and repeated eigenvalues in order to fully establish their advantages, application requirements and potential pitfalls. The underlying mathematical relationships among those methods are discussed and new theoretical developments required are proposed. The current success and limitation of the various important practical applications of eigenderivatives are examined and existing difficulties are identified with proposed measures to rectify them to ensure best possible outcome. Numerical examples are given to demonstrate the important theoretical concepts and major practical applications. Further research works are suggested with the purpose of concentrating research effort in the most fruitful directions.

2. Theoretical Development of Eigenvalue and Eigenvector Derivatives

Eigenvalue and eigenvector derivatives are important for many practical applications and their accurate and efficient computation has since become an interesting and important area of research in dynamics analysis of structures and systems. Numerically, once effective eigensolution methods are developed, the required derivatives can be accurately obtained based on the rudimentary finite difference scheme developed primarily on Taylor's Theorem [3]. Suppose that the system is perturbed by a small change in design variable Δp, then the perturbed eigenvalue problem of (1) becomes,

$$\mathbf{A}(p+\Delta p, \tilde{\lambda})\ \tilde{\mathbf{X}} = \tilde{\lambda}\ \mathbf{B}(p+\Delta p, \tilde{\lambda})\ \tilde{\mathbf{X}} \tag{2}$$

The required eigenvalue and eigenvector derivatives can then be obtained numerically as,

$$\frac{\partial \lambda}{\partial p} \equiv \lambda' = \frac{\tilde{\lambda} - \lambda}{\Delta p}; \qquad \frac{\partial \mathbf{X}}{\partial p} \equiv \mathbf{X}' = \frac{\tilde{\mathbf{X}} - \mathbf{X}}{\Delta p} \tag{3}$$

Provided the magnitude of Δp is chosen to be sufficiently small, the derivatives obtained using such a finite difference scheme can become very accurate. However, such a simple method is seldom used in practice due to its computational inefficiency. As a result, many other methods have been developed over the years and these can be roughly categorized as (i) eigenderivatives with distinct eigenvalues, (ii) eigenderivatives with repeated eigenvalues, (iii) eigenderivatives of general viscously damped systems, (iv) eigenderivatives of general nonviscously damped systems, (v) eigenderivatives of non-self-adjoint and defective systems and, (vi) eigenderivatives of fractional vibration systems. For each case, basic theory will be discussed and relevant existing works will be critically reviewed, together with new developments and proposals for possible further improvement.

2.1. Eigenderivatives with Distinct Eigenvalues

Eigenderivatives with distinct eigenvalues are one of the most thoroughly studied areas in dynamical analysis of structural systems. In the general case of structural damping, the complex eigenvalue problem may be written as,

$$\mathbf{K}\,\mathbf{X}_k - \lambda_k\,\mathbf{M}\,\mathbf{X}_k = \mathbf{0} \tag{4}$$

$$\mathbf{X}_k^T\,\mathbf{M}\,\mathbf{X}_k = 1 \tag{5}$$

where **M** is the system mass matrix, generally assumed to be positive definite, and **K** the complex stiffness matrix including structural damping. Both **M** and **K** are assumed to be symmetric with dimensions of $N \times N$. For such a general eigenvalue problem, there exist mainly 3 major approaches for the computation of eigenderivatives with distinct eigenvalues which are (i) the modal method [13], (ii) the improved modal method [14] and, (iii) Nelson's method [15], though many variations and modifications have been discussed in literature which will be discussed shortly. For convenience of subsequent discussions, prior to proper review of existing methods, the theoretical aspects of these 3 major approaches are briefly discussed first since they form the basis for more advanced techniques for the cases of degenerate eigenvalue problems and defective systems.

The modal method seeks to derive the eigenvector derivatives in terms of known modal data. Upon differentiating (4) with respect to design variable p, it is found that,

$$(\mathbf{K} - \lambda_k\,\mathbf{M})\,\mathbf{X}_k' = -(\mathbf{K}' - \lambda_k\,\mathbf{M}')\mathbf{X}_k + \lambda_k'\,\mathbf{M}\,\mathbf{X}_k \tag{6}$$

Pre-multiplying both sides of (6) by $\mathbf{X}_k^T$ and using (4) and (5), as well as the symmetric nature of the system matrices, we have eigenvalue derivative as,

$$\lambda_k' = \mathbf{X}_k^T\,(\mathbf{K}' - \lambda_k\,\mathbf{M}')\,\mathbf{X}_k \tag{7}$$

Since **M** is symmetric and positive definite, it can be proven that the complete set of eigenvectors of the system forms mathematically a complete linearly independent vector base [16]. So, without loss of generality, the eigenvector derivative may be written as,

$$\mathbf{X}_k' = \sum_{j=1}^{N} \beta_{kj}\,\mathbf{X}_j \tag{8}$$

To compute β_{kj}, substitute the assumed solution (8) into (6), which leads to,

$$(\mathbf{K} - \lambda_k \mathbf{M}) \sum_{j=1}^{N} \beta_{kj} \mathbf{X}_j = -(\mathbf{K}' - \lambda_k \mathbf{M}') \mathbf{X}_k + \lambda_k' \mathbf{M} \mathbf{X}_k \tag{9}$$

Pre-multiply both sides of (9) by $\mathbf{X}_i^T$ for $i \neq k$, thereby obtaining β_{ki} in the form,

$$\beta_{ki} = \frac{\mathbf{X}_i^T (\mathbf{K}' - \lambda_k \mathbf{M}') \mathbf{X}_k}{\lambda_k - \lambda_i} = \frac{\mathbf{X}_i^T \mathbf{F}_k}{\lambda_k - \lambda_i} \qquad (\forall i = 1, 2, \ldots, N;\ i \neq k) \tag{10}$$

where $\mathbf{F}_k$ is defined as $\mathbf{F}_k \equiv (\mathbf{K}' - \lambda_k \mathbf{M}') \mathbf{X}_k$. For $i = k$, β_{kk} can be obtained from the mass normalization of (5). Differentiating (5) with respect to *p* leads to,

$$\mathbf{X}_k^T \mathbf{M}' \mathbf{X}_k + 2 \mathbf{X}_k^T \mathbf{M} \mathbf{X}_k' = 0 \tag{11}$$

Upon substitution of the assumed solution $\mathbf{X}_k'$ of (8), β_{kk} may be solved such that,

$$\beta_{kk} = -\frac{1}{2} \mathbf{X}_k^T \mathbf{M}' \mathbf{X}_k \tag{12}$$

Having computed all the coefficients β_{ki}, the eigenvector derivative $\mathbf{X}_k'$ may be found based on (8). This completes the modal method for the derivation of eigenderivatives with distinct eigenvalues.

It is apparent that the modal method requires all the eigenvalues and eigenvectors of the system to be known a priori. In practical vibration analysis of large structural systems, usually only partial eigensolutions are available. The improved modal method seeks to approximate the eigenvector derivatives by using only the available lower *m* ($m \ll N$) modes. The eigenvector derivatives may now be written as,

$$\mathbf{X}_k' = \sum_{j=1}^{N} \beta_{kj} \mathbf{X}_j = -\frac{1}{2} \mathbf{X}_k^T \mathbf{M}' \mathbf{X}_k \mathbf{X}_k + \sum_{j=1;\ j \neq k}^{N} \frac{\mathbf{X}_j \mathbf{X}_j^T}{\lambda_k - \lambda_j} \mathbf{F}_k \tag{13}$$

The second summation term can be expressed as the sum of contributions from lower available modes and higher modes separately, and upon considering that when $j > m$, $\lambda_j >> \lambda_k$ such that $\lambda_k - \lambda_j \approx -\lambda_j$,

$$\begin{aligned}
\sum_{j=1;\, j\neq k}^{N} \frac{\mathbf{X}_j \mathbf{X}_j^T}{\lambda_k - \lambda_j} \mathbf{F}_k &= \sum_{j=1;\, j\neq k}^{m} \frac{\mathbf{X}_j \mathbf{X}_j^T}{\lambda_k - \lambda_j} \mathbf{F}_k + \sum_{j=m+1}^{N} \frac{\mathbf{X}_j \mathbf{X}_j^T}{\lambda_k - \lambda_j} \mathbf{F}_k \\
&\approx \sum_{j=1;\, j\neq k}^{m} \frac{\mathbf{X}_j \mathbf{X}_j^T}{\lambda_k - \lambda_j} \mathbf{F}_k + \sum_{j=m+1}^{N} \frac{\mathbf{X}_j \mathbf{X}_j^T}{-\lambda_j} \mathbf{F}_k \\
&= \sum_{j=1;\, j\neq k}^{m} \frac{\mathbf{X}_j \mathbf{X}_j^T}{\lambda_k - \lambda_j} \mathbf{F}_k + \sum_{j=1}^{N} \frac{\mathbf{X}_j \mathbf{X}_j^T}{-\lambda_j} \mathbf{F}_k - \sum_{j=1}^{m} \frac{\mathbf{X}_j \mathbf{X}_j^T}{-\lambda_j} \mathbf{F}_k \\
&= \sum_{j=1;\, j\neq k}^{m} \frac{\mathbf{X}_j \mathbf{X}_j^T}{\lambda_k - \lambda_j} \mathbf{F}_k - \mathbf{K}^{-1} \mathbf{F}_k - \sum_{j=1}^{m} \frac{\mathbf{X}_j \mathbf{X}_j^T}{-\lambda_j} \mathbf{F}_k
\end{aligned} \tag{14}$$

where the inverse of the system complex stiffness matrix, from spectral decomposition, becomes,

$$\mathbf{K}^{-1} = \sum_{j=1}^{N} \frac{\mathbf{X}_j \mathbf{X}_j^T}{\lambda_j} \tag{15}$$

Equation (14) forms the essence of the improved modal method [14].

The Nelson's method [15] seeks to compute the eigenvalue and eigenvector derivatives by using just the modal data of the mode itself. To solve for $\mathbf{X}_k'$, rewrite (6) as a set of linear algebraic equations as,

$$(\mathbf{K} - \lambda_k \mathbf{M})\, \mathbf{X}_k' \equiv \mathbf{G}\mathbf{X}_k' = \mathbf{F} \equiv -(\mathbf{K}' - \lambda_k \mathbf{M}')\, \mathbf{X}_k + \lambda_k' \mathbf{M}\, \mathbf{X}_k \tag{16}$$

Since $\mathbf{G}$ has rank N-1 and a null space $\mathbf{X}_k$, then for any particular solution $\boldsymbol{\sigma}$ that satisfies $\mathbf{G}\boldsymbol{\sigma} = \mathbf{F}$, $\boldsymbol{\sigma} + \gamma\, \mathbf{X}_k$ is also a solution where γ is a complex constant. The approach is first to find some particular solution $\boldsymbol{\sigma}$, then to find γ so that $\boldsymbol{\sigma} + \gamma\, \mathbf{X}_k = \mathbf{X}_k'$. Nelson suggested the following procedure to determine the required particular solution $\boldsymbol{\sigma}$:

1) Find the index r such that $|(\mathbf{X}_k)_r| = \|\mathbf{X}_k\|_\infty = \max_i |(\mathbf{X}_k)_i|$ where $(\mathbf{X}_k)_i$ is the i-th element of eigenvector $\mathbf{X}_k$.

2) Zero the r-th row and column and assign 1 to the r-th diagonal element of $\mathbf{G} \rightarrow \tilde{\mathbf{G}}$ and zero the k-th element of $\mathbf{F} \rightarrow \tilde{\mathbf{F}}$.

3) Solve the modified resultant linear algebraic equations $\tilde{\mathbf{G}}\,\boldsymbol{\sigma} = \tilde{\mathbf{F}}$.

It remains to find the complex constant γ, which is done by substituting $\mathbf{X}'_k = \boldsymbol{\sigma} + \gamma\,\mathbf{X}_k$ into (11) to obtain,

$$\mathbf{X}_k^T\,\mathbf{M}'\,\mathbf{X}_k + 2\,\mathbf{X}_k^T\,\mathbf{M}\left(\boldsymbol{\sigma} + \gamma\,\mathbf{X}_k\right) = 0 \tag{17}$$

$$\gamma = -\,\mathbf{X}_k^T\,\mathbf{M}\,\boldsymbol{\sigma} - \frac{1}{2}\,\mathbf{X}_k^T\,\mathbf{M}'\,\mathbf{X}_k \tag{18}$$

This completes Nelson's method [15] for computing eigenvector derivatives. Although a set of linear algebraic equations of system dimension need to be solved, Nelson's method only requires the modal data of the particular mode in question.

Whitesell [17] developed a procedure for the computation of eigenvector derivatives based on the coordinate transformation,

$$\mathbf{X} = \mathbf{M}^{-1/2}\,\mathbf{U} \tag{19}$$

where $\mathbf{M}^{-1/2}$ is the positive definite square root of $\mathbf{M}^{-1}$. Then, the eigenvalue problem may be written in the standard form,

$$\mathbf{A}\,\mathbf{U} = \lambda\,\mathbf{U} \tag{20}$$

where symmetric matrix $\mathbf{A}$ is given by,

$$\mathbf{A} = \mathbf{M}^{-1/2}\,\mathbf{K}\,\mathbf{M}^{-1/2} \tag{21}$$

Similar to (10), but now in terms of eigenvectors $\mathbf{U}$ rather than $\mathbf{X}$, Whitesell [17] showed that,

$$\mathbf{U}'_k = \mathbf{D}_k^{+}\,\mathbf{A}'\,\mathbf{U} \tag{22}$$

where the pseudo inverse $\mathbf{D}_k^{+}$ of $\left(\mathbf{A} - \lambda_k\,\mathbf{I}\right)$ is given by,

$$\mathbf{D}_k^+ = -\sum_{j=1;\, j\neq k}^{N} \frac{1}{\lambda_j - \lambda_k} \mathbf{U}_j \mathbf{U}_j^T \tag{23}$$

Equations (22) and (23) are however not used explicitly. Instead the approximation $\bar{\lambda}_k = \lambda_k + \varepsilon$ is used to find the eigenvector $\mathbf{U}_k$ using inverse iteration, and the pseudo inverse is estimated by LU decomposition of $\left(\mathbf{A} - \bar{\lambda}_k \mathbf{I}\right)$. The full complement of rank-1 eigenvector products gives $\sum_{j=1}^{N} \mathbf{U}_j \mathbf{U}_j^T = \mathbf{I}$, so that $\sum_{j=1;j\neq k}^{N} \mathbf{U}_j \mathbf{U}_j^T$ may be replaced by $\left(\mathbf{I} - \mathbf{U}_k \mathbf{U}_k^T\right)$, thereby confining the solution to k^{th} eigenpair. Whitesell [17] describes an efficient numerical algorithm as follows:

1) Form an LU decomposition of $\left(\mathbf{A} - \bar{\lambda}_k \mathbf{I}\right)$.
2) Use the LU decomposition in an inverse iteration scheme to find the eigenvector $\mathbf{U}_k$.
3) Solve for $\mathbf{Z}_k$ in $\mathbf{A}_L \mathbf{Z}_k = \left(\mathbf{I} - \mathbf{U}_k \mathbf{U}_k^T\right)\mathbf{A}' \mathbf{U}_k$.
4) Solve for $\mathbf{Y}_k$ in $\mathbf{A}_U \mathbf{Y}_k = \mathbf{Z}_k$.
5) $\mathbf{U}_k' = \left(\mathbf{I} - \mathbf{U}_k \mathbf{U}_k^T\right)\mathbf{Y}_k = \mathbf{Y}_k - \left(\mathbf{U}_k^T \mathbf{Y}_k\right)\mathbf{U}_k + c_k \mathbf{U}_k$.

where $c_k \mathbf{U}_k \in \mathrm{null}\left(\mathbf{A} - \bar{\lambda}_k \mathbf{I}\right)$.

$\mathbf{A}_U$ and $\mathbf{A}_L$ denote the upper and lower triangular matrices from the LU decomposition.

Over the last half a century, great efforts have been made to design and develop accurate and efficient numerical methods for computing eigenderivatives. Eigenvalue derivatives were first derived and utilized to analyze steady-state oscillations and to estimate flutter velocities [18-19]. The method was further improved subsequently, requiring one left and one right eigenvector to calculate eigenderivatives [20-21]. Wittrick [22] derived expressions for first derivatives of eigenvalues for buckling and vibration problems. Rogers [23] extended the modal method first proposed by Fox and Kapoor [13] to compute eigenvalue and eigenvector derivatives of dynamic systems with general asymmetric system matrices. Eigenvalue sensitivity of a general eigenvalue problem was discussed [24] and an effective numerical algorithm was implemented. The sensitivity of matrix eigenvalues with respect to numerical conditioning of a given matrix was examined and quantifications were made through condition numbers [25]. For distributed parameter systems on the other hand, derivatives of eigenvalues as well as those of eigenfunctions were investigated [26-28]. Matrix perturbation theory [29] was often employed in the analysis of eigenderivatives from which several different methods were developed. Eigenvalue sensitivity under elementary matrix perturbations of various forms was examined [30]. The quantitative relationship between eigenvector derivatives and matrix perturbations was analyzed [31]. An improved method was presented for eigenvector sensitivities using matrix perturbation analysis [32]. Further, a mathematical formula for eigenvalue sensitivities under a special form of rank-1 matrix perturbations was developed and its potential applications to structural dynamics were discussed [33-34]. A systematic study on sensitivity and accuracy of eigenvalues under various forms and magnitudes of given perturbations was conducted [35]. Subsequently, a general framework for the sensitivity and backward perturbation analysis of linear as well as nonlinear multiparameter eigenvalue problems was established [36]. In order to improve computational efficiency while maintaining sufficient accuracy for practical applications, approximate numerical methods for computing eigenderivatives were also sought and investigated [37-38].

Alongside these studies, iterative methods were proven to be practically very useful. An effective iteration method for the simultaneous estimations of eigenvalue and eigenvector derivatives was developed and was found to be numerically accurate and computationally efficient [39-41]. A scheme of iteration [42-43] was introduced into the existing improved modal method proposed in [14] to further enhance numerical accuracy, however the method was only efficient for the case of limited numbers of design variables since separate iterations were required for each design variable. Also, an iterative least squares scheme was proposed [44] which employed the band structure of the stiffness matrix and an efficient use of the Householder transformation to improve computational efficiency. A simultaneous iteration method was presented for computing mixed partial derivatives of several eigenvalues and the corresponding eigenvectors of a general matrix [45]. The computation of eigenvector derivatives for middle eigenvalues in real symmetric eigensystems was discussed by introducing a numerical shift of eigenvalues [46] in order to minimize errors.

Considering the practical design case where only partial eigensolutions are often performed, methods have been developed to compensate for the contributions of higher unavailable modes to the required eigenderivatives. The contribution of the truncated modes to the eigenvector derivative was expressed as a convergent series that could be evaluated by a simple iterative procedure [47-48]. The method substantially improved the numerical accuracy and provided error estimates for the computed results. Also, practical algorithms for the efficient computation of eigenvector sensitivities in the case of close modes and localized design changes were proposed [49-50] by considering the residual terms associated with higher unavailable modes. Further, a dynamic flexibility method with variable frequency shifting was developed to accurately estimate the contributions of higher modes to eigenderivatives [51-53]. For free-free structures with rigid body modes and hence the non-existence of system flexibility matrix, alternative methods were devised to derive the required eigenderivatives [54-55]. The subspace iteration method, which was first developed for partial eigensolutions, was seamlessly extended to the computations of eigenderivatives with compensations for higher unavailable vibration modes [56-58].

Higher order eigenderivatives have also been examined in order to improve accuracy of structural dynamic analysis. Second-order eigensensitivities of matrices with distinct eigenvalues were derived using matrix calculus and the algebra of Kronecker products and were expressed by concise matrix equations [59]. Exact analytical nth-order derivatives of eigenvalues and eigenvectors of distinct eigenvalues and corresponding eigenvectors were established for general linear and nonlinear eigenvalue problems [60]. The iteration based method discussed in [39] was further developed and extended to the computation of second-order eigenderivatives [61]. Nelson's method, originally developed for the calculation of the first-order eigenvector derivatives, was generalized so that it becomes applicable to second- and higher-order eigenvector derivatives [62]. To overcome the rank deficiency of the coefficient matrix, normalization constraints were developed and incorporated into the set of linear algebraic equations which became rank full and hence could be solved for the eigenvector derivatives required [63-64]. The QR algorithm was further developed [65], a Lanczos-vectors based numerical procedure was devised [66] and Davidson's method [67] was formulated for eigenvector derivatives. Comparison studies were carried out to critically assess the numerical accuracies and computational efficiencies of some of the commonly used methods [68-69]. For matrix functions and functions of eigenvalues, eigensensitivity analyses were carried out and numerical procedures for computing these sensitivities were accordingly developed [70-72].

Furthermore, for second order dynamic systems, approximate formulations were developed using a modal expansion which required less computing time than the corresponding generalized formulations [73], as well as using structure-preserving equivalence [74]. Two problems on sensitivity of eigenvalues and eigen-decompositions of matrices due to Wilkinson and Demmel were solved and solutions were provided [75]. Eigenvalue-eigenvector sensitivity analyses of linear time-invariant multivariable singular systems were conducted [76]. An additional proof to previously published results concerning direct computation of derivatives of eigenvalues and eigenvectors of parameter-dependent matrices was discussed [77]. Sensitivity analysis of a general eigenvalue problem with matrices depending on parameters was considered and linear and quadratic parts in series of eigenvalues were obtained [78]. The derivative of an orthogonal matrix of eigenvectors of a real symmetric matrix was derived and a real symmetric random matrix was then used to get the asymptotic distribution of the orthogonal eigen-matrix [79]. In addition, a preconditioned conjugate projected gradient-based technique was developed by leveraging the existing factorizations in eigensolution methods such as Lanczos or subspace iteration to improve computational efficiency [80]. For the eigenvalue problem of a Reissner-Mindlin system arising in the study of the free vibration modes of an elastic clamped plate, quantitative estimates for eigenvalue and mode shape derivatives were obtained [81]. Rotating clamped Euler-Bernoulli beams were studied using asymptotic numerical method with derived equations transformed into a linear gyroscopic eigenvalue problem from which eigensensitivity analyses were carried out [82].

2.2 Eigenderivatives with Repeated Eigenvalues

Many structural systems in practice possess some forms of spatial symmetries which then result in repeated eigenvalues. In the case of repeated eigenvalues, the associated eigenvector space becomes mathematically degenerate, in which any linear combinations of the eigenvectors associated with the repeated eigenvalue is also a valid eigenvector, and some special numerical procedures are required in order to compute the required eigenvector derivatives correctly. For this reason, methods for computing eigenderivatives with repeated eigenvalues are often discussed and developed separately from those for distinct eigenvalues. Suppose λ is a repeated eigenvalue of multiplicity *m*, then the repeated eigenvalue and its associated eigenvectors satisfy,

$$\mathbf{K}\,\mathbf{X} = \mathbf{M}\,\mathbf{X}\,\mathbf{\Lambda} \tag{24}$$

$$\mathbf{X}^T\,\mathbf{M}\,\mathbf{X} = \mathbf{I} \tag{25}$$

where $\mathbf{X}$ is the eigenvector matrix of dimensions $N\times m$ associated with the repeated eigenvalue λ, $\mathbf{I}$ is a unit matrix of dimensions $m\times m$ and $\mathbf{\Lambda}$ is a diagonal eigenvalue matrix $\mathbf{\Lambda}=\lambda\mathbf{I}$. Since in the case of repeated eigenvalues, the eigenvector space is degenerate, resulting in any linear combinations of the columns of $\mathbf{X}$ also a valid eigenvector. However, the eigenvector space generally splits into as many as *m* distinct eigenvectors as the design variable *p* varies. For eigenvector derivatives to exist in the case of repeated eigenvalues, we must restrict our choice of these eigenvectors so that they lie geometrically adjacent to those *m* distinct eigenvectors obtainable as *p* varies. Such choices of eigenvectors are necessary to ensure that there is no discontinuity in eigenvectors as *p* varies gradually away from the case of repeated eigenvalues.

Such adjacent eigenvectors needed can be obtained through an orthonormal transformation: $\mathbf{Z}=\mathbf{X}\mathbf{\Gamma}$ where columns of $\mathbf{Z}$ are the adjacent eigenvectors for which derivatives can be defined and $\mathbf{\Gamma}$ is an orthonormal matrix of dimensions $m\times m$. The orthognormality of $\mathbf{\Gamma}$ is apparent since $\mathbf{Z}$, as eigenvectors, must satisfy (25),

$$\mathbf{Z}^T\,\mathbf{M}\,\mathbf{Z} = \mathbf{I} \quad \Rightarrow \quad \mathbf{\Gamma}^T\mathbf{X}^T\,\mathbf{M}\,\mathbf{X}\,\mathbf{\Gamma} = \mathbf{I} \quad \Rightarrow \quad \mathbf{\Gamma}^T\,\mathbf{\Gamma} = \mathbf{I} \tag{26}$$

The problem is to first find $\mathbf{\Gamma}$ to obtain $\mathbf{Z}$, then to proceed to derive eigenvalue and eigenvector derivatives $\mathbf{\Lambda}'$ and $\mathbf{Z}'$. Replacing $\mathbf{X}$ by $\mathbf{Z}$ in (24) and taking the derivative with respect to *p*,

$$(\mathbf{K}-\lambda\mathbf{M})\,\mathbf{Z}' = -(\mathbf{K}'-\lambda\mathbf{M}')\,\mathbf{Z} + \mathbf{M}\,\mathbf{Z}\,\mathbf{\Lambda}' \tag{27}$$

where $\mathbf{\Lambda}' \equiv \mathrm{diag}(\lambda_1', \lambda_2', \cdots, \lambda_m')$ is the eigenvalue derivative matrix associated with the repeated eigenvalue. Thus, by pre-multiplying (27) by $\mathbf{X}^T$ and letting $\mathbf{Z}=\mathbf{X}\mathbf{\Gamma}$, it is found that,

$$\mathbf{X}^T(\mathbf{K}'-\lambda\mathbf{M}')\,\mathbf{X}\,\mathbf{\Gamma} = \mathbf{\Gamma}\,\mathbf{\Lambda}' \tag{28}$$

This is a standard eigenvalue problem of a known symmetric matrix $\mathbf{D}\equiv\mathbf{X}^T(\mathbf{K}'-\lambda\mathbf{M}')\mathbf{X}$ of dimension $m\times m$ whose eigenvalues form the diagonal matrix $\mathbf{\Lambda}'$ and eigenvectors $\mathbf{\Gamma}$. So, the required $\mathbf{\Lambda}'$ and $\mathbf{\Gamma}$ can be obtained directly from solving the known eigenvalue problem of $\mathbf{D}$. It remains to find $\mathbf{Z}'$ which can be solved from (27), rewritten in the form,

$$(\mathbf{K}-\lambda\mathbf{M})\,\mathbf{Z}' \equiv \mathbf{G}\mathbf{Z}' = \mathbf{E} \equiv -(\mathbf{K}'-\lambda\mathbf{M}')\,\mathbf{Z} + \mathbf{M}\,\mathbf{Z}\,\mathbf{\Lambda}' \tag{29}$$

Since $\mathbf{\Lambda}'$ is known from the eigensolution of **D**, the coefficient matrix **G** and the right hand side matrix **E** of (29) are known and the eigenvector derivatives $\mathbf{Z}'$ can be solved. However, since λ is a repeated eigenvalue of multiplicity *m*, matrix **G** becomes rank-*m* deficient and as a result, infinitely many solutions of $\mathbf{Z}'$ exist, all satisfying (29). Suppose now **W** is one of the particular solutions of (29) so that **W** satisfies $\mathbf{G}\mathbf{W}=\mathbf{E}$, then it can be proven that for any matrix **C** of arbitrary complex numbers, $\mathbf{W}+\mathbf{Z}\mathbf{C}$ is also a solution of (29), since the added part satisfies the eigenvalue equation $(\mathbf{K}-\lambda\mathbf{M})\,\mathbf{Z}\mathbf{C}=\mathbf{0}$,

$$(\mathbf{K}-\lambda\mathbf{M})\,(\mathbf{W}+\mathbf{Z}\mathbf{C}) = (\mathbf{K}-\lambda\mathbf{M})\,\mathbf{W}+(\mathbf{K}-\lambda\mathbf{M})\,\mathbf{Z}\mathbf{C} = \mathbf{G}\mathbf{W} = \mathbf{E} \tag{30}$$

The particular solution **W** required can be solved from (29) by reconfiguring the coefficient matrix **G** to eliminate the singularity based on the extended Nelson's procedure similar to the case of distinct eigenvalues [15]. Once **W** is established, the matrix **C** required can be directly determined by taking first differentiation of the orthonormal constraints of (25) and the second differentiation of the eigenvalue equation (24) as discussed in detail in the works of Dailey [83] and Mills-Curran [84],

$$c_{ii} = -\mathbf{W}_i^T\,\mathbf{M}\,\mathbf{Z}_i - \mathbf{Z}_i^T\,\mathbf{M}\,\mathbf{W}_i - \mathbf{Z}_i^T\,\mathbf{M}'\,\mathbf{Z}_i \qquad (\forall i=1,2,...,m) \tag{31}$$

$$c_{ij} = \frac{\mathbf{Z}_i^T(\mathbf{K}''-\lambda\mathbf{M}'')\mathbf{Z}_j+2\mathbf{Z}_i^T(\mathbf{K}'-\lambda\mathbf{M}')\mathbf{W}_j-2\lambda_j'\mathbf{Z}_i^T\mathbf{M}'\mathbf{Z}_j-2\lambda_j'\mathbf{Z}_i^T\mathbf{M}\,\mathbf{W}_j}{2\,(\lambda_j'-\lambda_i')} \tag{32}$$

$$(\forall i,\ j=1,2,..,m,\ i\neq j)$$

However, such a method to compute **W** is computationally very inefficient since a set of linear algebraic equations of system dimension has to be solved for each repeated eigenvalue with respect to each design variable. In addition, the proposed procedure in [83] can fail sometimes as rightly pointed out by Mills-Curran [85], leading to repeated reconfigurations and re-factorizations of the coefficient matrix **G**. Fortunately, such required particular solution **W** can be more efficiently obtained as proposed by two of the present authors in their recent work [86].

Reduced eigenvalue and eigenvector matrices may be written as,

$$\overline{\mathbf{\Lambda}} \equiv diag\left[\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_k \quad \lambda_{k+m+1} \quad \lambda_{k+m+2} \quad \cdots \quad \lambda_N\right] \tag{33a}$$

$$\overline{\mathbf{X}} \equiv \left[\mathbf{X}_1 \quad \mathbf{X}_2 \quad \cdots \quad \mathbf{X}_k \quad \mathbf{X}_{k+m+1} \quad \mathbf{X}_{k+m+2} \quad \cdots \quad \mathbf{X}_N\right] \tag{33b}$$

in which the repeated eigenvalue and its associated eigenvectors of multiplicity *m*, $\lambda_{k+1},\cdots,\lambda_{k+m}$ and $\mathbf{X}_{k+1},\cdots,\mathbf{X}_{k+m}$ are not included. Then by constructing $\mathbf{W} = \overline{\mathbf{X}}[\overline{\mathbf{\Lambda}} - \lambda\mathbf{I}]^{-1}\overline{\mathbf{X}}^T\mathbf{E}$ one may proceed to prove that **W** readily becomes a particular solution which satisfies (29). It may be seen that $\overline{\mathbf{X}}^T$ is of rank *N-m* since it includes all the eigenvectors except those associated with the repeated eigenvalue. The coefficient matrix **G** of (29) so that pre-multiplication of **G** by $\overline{\mathbf{X}}^T$ represents a full rank transformation [9] which does not change the nature of the solution of (29). Thus, after the pre-multiplication by $\overline{\mathbf{X}}^T$, (29) may be written as,

$$\overline{\mathbf{X}}^T\mathbf{G}\,\mathbf{W} = \overline{\mathbf{X}}^T\mathbf{E} \tag{34}$$

Substituting the proposed particular solution **W** into (34), the LHS of (34) then becomes,

$$\begin{aligned}\overline{\mathbf{X}}^T\mathbf{G}\,\overline{\mathbf{X}}[\overline{\mathbf{\Lambda}} - \lambda\mathbf{I}]^{-1}\overline{\mathbf{X}}^T\mathbf{E} &= \overline{\mathbf{X}}^T[\mathbf{K} - \lambda\mathbf{M}]\,\overline{\mathbf{X}}[\overline{\mathbf{\Lambda}} - \lambda_k\mathbf{I}]^{-1}\overline{\mathbf{X}}^T\mathbf{E} \\ &= [\overline{\mathbf{\Lambda}} - \lambda\mathbf{I}]\,[\overline{\mathbf{\Lambda}} - \lambda\mathbf{I}]^{-1}\overline{\mathbf{X}}^T\mathbf{E} \\ &= \overline{\mathbf{X}}^T\mathbf{E}\end{aligned} \tag{35}$$

where the relation $\overline{\mathbf{X}}^T[\mathbf{K} - \lambda\mathbf{M}]\,\overline{\mathbf{X}} = [\overline{\mathbf{\Lambda}} - \lambda\mathbf{I}]$ is used. Indeed, the thus constructed **W** is a particular solution of (29). Then, the complete solution for eigenvector derivatives become,

$$\mathbf{Z}' = \mathbf{W} + \mathbf{Z}\mathbf{C} = \overline{\mathbf{X}}[\overline{\mathbf{\Lambda}} - \lambda\mathbf{I}]^{-1}\overline{\mathbf{X}}^T\mathbf{E} + \mathbf{Z}\mathbf{C} \tag{36}$$

which is the true mode superposition for of eigenvector derivatives in the case of repeated eigenvalues [86].

In the practical case where only *M* lower modes are available from partial eigensolution, (36) may be expanded to become,

$$\mathbf{Z}' = \left(\sum_{i=1}^{k} \frac{\mathbf{X}_i\mathbf{X}_i^T}{\lambda_i - \lambda} + \sum_{i=k+m+1}^{M} \frac{\mathbf{X}_i\mathbf{X}_i^T}{\lambda_i - \lambda} + \sum_{i=M+1}^{N} \frac{\mathbf{X}_i\mathbf{X}_i^T}{\lambda_i - \lambda} \right)\mathbf{E} + \mathbf{Z}\mathbf{C} \tag{37}$$

The last summation term, which represents the contributions from unavailable higher modes, can be approximated as, since $\lambda_i >> \lambda$ when $i > M > k$,

$$\sum_{i=M+1}^{N} \frac{\mathbf{X}_i \mathbf{X}_i^T}{\lambda_i - \lambda} \doteq \sum_{i=M+1}^{N} \frac{\mathbf{X}_i \mathbf{X}_i^T}{\lambda_i} = \sum_{i=1}^{N} \frac{\mathbf{X}_i \mathbf{X}_i^T}{\lambda_i} - \sum_{i=1}^{M} \frac{\mathbf{X}_i \mathbf{X}_i^T}{\lambda_i}$$
$$= [\mathbf{K} + j\mathbf{H}]^{-1} - \sum_{i=1}^{M} \frac{\mathbf{X}_i \mathbf{X}_i^T}{\lambda_i} \tag{38}$$

The proposed method is numerically accurate even when very few lower modes are available as shown in Table 1 in which some of the *v* coordinates of the eigenvector derivatives of the first 2 repeated modes with respect to the design variable as the moment of area I_z of the 10 elements between elements 11 and 20 as shown in Fig. 1. The dimensions of the beam are 20mm×20mm×3000mm modeled as 300 elements with each node having 4 DOFs of *v* (displacement in *y* direction), θ_z (rotation about *z* axis), *w* (displacement in z direction) and θ_y (rotation about *y* axis), as shown in Figure 1.

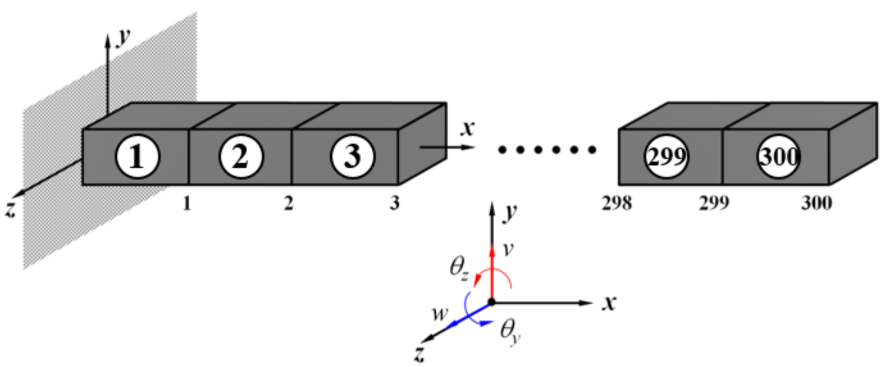

Figure 1 – Finite element modeling of a square beam

Table 1 – Effect of mode truncation on eigenvector derivatives with repeated eigenvalue

v_i	Z' (exact $\times 10^5$)	Z' (4 modes $\times 10^5$)	% error	Z' (10 modes $\times 10^5$)	% error ($\times 10^{-4}$)
v_{30}	1.7515384	1.7496412	-0.11088253	1.7515304	-4.5674134
v_{60}	4.4752669	4.4705114	-0.10626182	4.4752592	-2.34909172
v_{90}	5.7535179	5.7480437	-0.09514527	5.7535305	2.1899645
v_{120}	5.8371540	5.8338164	-0.05717855	5.8371549	0.15418473
v_{150}	4.9740957	4.9744309	0.00673891	4.9740838	-2.3923947
v_{180}	3.4029824	3.4064025	0.10050302	3.4029854	0.88157964
v_{210}	1.3445202	1.3487667	0.31583758	1.3445301	7.3632215
v_{240}	-1.0088935	-1.0065555	-0.23173903	-1.0088987	5.1541615
v_{270}	-3.5047305	-3.5063179	0.04529307	-3.5047373	1.9402348
v_{300}	-6.0426141	-6.0489100	0.01041917	-6.0425997	-2.3830746

Eigenvalue and eigenvector derivatives with repeated eigenvalues have attracted intensive research interest over the years. Systematic sensitivity analysis of multiple eigenvalues was conducted for a symmetric eigenvalue problem depending on several parameters [87-89]. An explicit formula was developed using singular value decomposition to compute required bases for eigenspaces, as well as to keep track of the dimensions of state variables and the conditioning of the state equations [90]. The existing method proposed in [83-84] was further extended to compute the derivatives for the case where both repeated eigenvalues and repeated eigenvalue derivatives are present [91]. Extensions were made to the modal expansion procedure originally formulated by Fox and Kapoor [13] to allow the computation of sensitivity information for mode shapes associated with repeated roots and repeated root derivatives [92]. Further, a modal expansion method was discussed and applied to compute eigenderivatives of repeated eigenvalues [93]. Also, eigenderivatives with repeated eigenvalues of the generalized nondefective eigenproblem was examined in what was essentially an extension of Dailey's work [94]. A generalized inverse was employed for determining the particular solution of eigenvector derivatives with and without repeated eigenvalues [95]. The simultaneous iteration method first proposed for distinct eigenvalues was further developed for derivatives of eigenvectors in which the dominant eigenvalue was repeated [96]. For the special case where design variable change affects the stiffness in only one direction but not the mass matrix, a much simplified method was developed for computing eigenderivatives of doubly repeated eigenvalues [97]. A novel extension to Nelson's method was discussed and used to calculate the first order derivatives of eigenvectors when the derivatives of the associated eigenvalues are also equal, together with continuity of eigenvectors [98]. A set of nonmodal vectors were obtained from the modes associated with the repeated eigenvalue which were orthogonal to the eigenvectors and were used to compute eigenvector derivatives [99].

Following a different approach, an iterative procedure with guaranteed convergence was proposed in which a shift in each eigenvalue was introduced to avoid the singularity [100]. The modal mass is important in characterizing the dynamical behavior of a base driven structure and the calculation of the effective mass sensitivities was generalized to the case of repeated eigenvalues [101]. Conditions on the parameterization were derived and formulated as theorems, which ensured the existence of derivatives of eigenvectors with respect to these parameters [102]. A combined method based on the constrained generalized inverse of the frequency-shifted stiffness matrix was formulated which became applicable to all nondefective systems [103]. Further, a new algorithms for computing derivatives, to an arbitrary order, of eigenvalues and eigenvectors of a general eigenvalue problem was introduced and was demonstrated to be valid under more general conditions than existing algorithms [104]. Without explicit use of the eigenvectors, a novel algorithm was designed to calculate derivatives of eigenvalues with respect to system model parameters for both distinct and repeated eigenvalues [105]. To further improve accuracy, a higher-precision dynamic flexibility expression was proposed based on a geometrical series expansion which possesses excellent convergence [106]. Also, by employing symmetry properties of cyclic structures in order to reduce computational effort, a method was presented for eigenvector derivatives which was suitable for parallel implementation [107]. The phenomena of eigenvalue curve veering and mode localization were addressed with relation to eigenvalue and eigenvector derivatives as two vibration modes approach each other [108]. Derivatives of a repeated eigenvalue of viscously damped vibrating systems with respect to a parameter were successfully implemented on the subspace spanned by the eigenvectors [109].

To avoid the use of second order differentiations, an adjoint method was proposed to compute adjoint variables from the simultaneous linear system equation [110]. Nelson’s method was extended to the case of repeated eigenvalues for symmetric real eigensystems with improved condition number of the coefficient matrix [111]. The governing equations for the particular solutions of eigenvector derivatives were augmented by requiring the solution to be mass orthogonal with respect to the repeated modes and adjusting the corresponding coefficients so that the coefficient matrix of the augmented system became non-singular and had reduced smaller condition numbers [112]. A method of eigenvector-sensitivity was proposed for repeated eigenvalues and eigenvalue derivatives in which an extended system with a nonsingular coefficient matrix was constructed to derive the needed particular solutions [113]. Similarly, for asymmetric quadratic eigenvalue problems with repeated eigenvalues, a numerical algorithm was developed for eigenvector derivatives by introducing additional normalization conditions to calculate the required particular solution [114]. Also, a preconditioned conjugate gradient method was proposed for repeated eigenvalues by leveraging on the iterative eigensolutions such as Lanczos or subspace iteration methods [115]. A general framework for computing eigenvector sensitivity was discussed whenever tracking specific mode shapes selected beforehand in the case of multiple eigenvalues [116]. In an effort to construct a non-singular coefficient matrix, an effective strategy was proposed which had the advantage of only requiring eigenvalues and eigenvectors of the mode of interest [117].

2.3 Eigenderivatives of General Viscously Damped Systems

The eigenvalue problem of vibration systems with viscous damping can in general be transformed into standard general complex eigenvalue problem with symmetric system matrices using augmented state space formulation [16], though non-symmetric formulation [118] has also be attempted which will be discussed later. The equations of motion in this case become,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{39}$$

where **K** in this case is the real symmetric system stiffness matrix without structural damping since damping is assumed to be viscous and is represented by viscous damping matrix **C**. This can be transformed into an augmented state space form of $\mathbf{A}\dot{\mathbf{y}} + \mathbf{B}\mathbf{y} = \mathbf{0}$ in which the system matrices **A** and **B** and vector **y** are defined as,

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{B} \equiv \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \qquad \mathbf{y} \equiv \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{Bmatrix}.$$

The corresponding eigenvalue problem in state space representation therefore becomes,

$$(s_k \mathbf{A} + \mathbf{B})\, \boldsymbol{\varphi}_k = \mathbf{0} \qquad (\forall k = 1, 2, ..., 2N) \tag{40}$$

The eigenvalues s_k and eigenvectors $\boldsymbol{\varphi}_k$ can be solved from (40) using standard eigensolution methods and the eigenvectors are normally normalized to **A** matrix so that $\boldsymbol{\varphi}_k^T \mathbf{A} \boldsymbol{\varphi}_k = 1 \ (\forall k = 1, 2, ..., 2N)$. The eigenvalues are generally complex with their imaginary parts associated with the frequencies of vibrations and real parts the damping decay rates. By differentiating (40), it is found that,

$$s_k' \mathbf{A} \boldsymbol{\varphi}_k + s_k \mathbf{A}' \boldsymbol{\varphi}_k + \mathbf{B}' \boldsymbol{\varphi}_k + (s_k \mathbf{A} + \mathbf{B})\, \boldsymbol{\varphi}_k' = \mathbf{0} \tag{41}$$

Then, pre-multiplication of (41) by $\boldsymbol{\varphi}_k^T$ and using (40) and the symmetric properties of **A** and **B**, leads to,

$$s_k' = -\boldsymbol{\varphi}_k^T (s_k \mathbf{A}' + \mathbf{B}') \boldsymbol{\varphi}_k \tag{42}$$

Equation (41) may be re-written in the following familiar linear equation form,

$$(s_k \mathbf{A} + \mathbf{B}) \boldsymbol{\varphi}_k' \equiv \hat{\mathbf{G}}_k \boldsymbol{\varphi}_k' = -(s_k' \mathbf{A} \boldsymbol{\varphi}_k + s_k \mathbf{A}' \boldsymbol{\varphi}_k + \mathbf{B}' \boldsymbol{\varphi}_k) \equiv \hat{\mathbf{F}}_k \tag{43}$$

The eigenvector derivative $\boldsymbol{\varphi}_k'$ can be solved using either the complex version Nelson's method [119], or the complex modal method [120]. In the case of the complex modal method, the eigenvector derivative becomes,

$$\boldsymbol{\varphi}_k' = \sum_{j=1,\, j\neq k}^{2N} \left[\frac{\boldsymbol{\varphi}_j \boldsymbol{\varphi}_j^T}{s_k - s_j} + \frac{\boldsymbol{\varphi}_j^* \boldsymbol{\varphi}_j^*}{s_k - s_j^*} \right] \hat{\mathbf{F}}_k + \frac{\boldsymbol{\varphi}_k^* \boldsymbol{\varphi}_k^*}{s_k - s_k^*} \hat{\mathbf{F}}_k - \frac{1}{2} \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^T \mathbf{A}' \boldsymbol{\varphi}_k \tag{44}$$

which is a form of mode superposition in state space representation. A computationally more efficient method has been developed to deal with mode truncation considering in practice when only the partial eigensolution is usually carried out [121].

Alongside these studies, several other methods have been discussed for viscously damped vibration systems. A sensitivity analysis of the algebraic eigenvalue problem for non-Hermitian matrices was presented by introducing better normalizing condition, together with discussions on numerical accuracy and computational efficiency [122]. Based on the Kronecker algebra and matrix calculus, the derivatives of eigenvalues with respect to the model parameters for linear damped systems was proposed [123]. Eigensensitivity analysis was carried out to predict eigenvectors and eigenvalues of the non-proportionally damped structure due to the change in structural mass, damping and stiffness through iterations [124]. First-order derivatives of complex eigenvectors of general non-defective eigensystems were examined by introducing new normalization conditions [125]. The sensitivity and stability of the linearization of transforming a nonlinear quadratic eigenvalue problem to a general linear complex eigenvalue problem were examined [126]. The derivatives of semi-simple eigenvalues and the corresponding eigenvectors of the quadratic matrix polynomial were discussed, together with proposed measures to improve the condition of the coefficient matrix [127]. Also, the rounding errors associated with derivatives of a simple eigenvalue of quadratic eigenvalue problem were established and ways to minimize these errors were suggested [128]. Further, an iterative method was proposed to compute partial derivatives of eigenvectors of quadratic eigenvalue problems with respect to system parameters, together with a criterion of convergence [129].

For eigenderivatives with repeated eigenvalues of viscously damped systems, different methods have also been developed. A simplified method for the computation of first-, second- and higher-order derivatives of eigenvalues and eigenvectors associated with repeated eigenvalues was presented and was found to be numerically stable and efficient [130]. A procedure was formulated to compute the derivatives of repeated eigenvalues and the corresponding eigenvectors of damped systems by avoiding the rather undesirable state space representation [131]. For quadratic eigenvalue problems, new algorithms, which were valid much more generally, were proposed, analyzed, and tested [132]. The computation of eigensolution sensitivity of viscously damped eigensystems with repeated eigenvalues was examined and an efficient algorithm was proposed which worked within the physical N-space without resorting to state-space equations [133]. Further, by employing the constrained generalized inverse, an efficient algorithm to compute the particular solutions of the governing equations of the derivatives of eigenvectors was developed [134].

2.3 Eigenderivatives of General Non-viscously Damped Systems

Though it is mathematically convenient and very often, it leads to simplified vibration analysis, viscous damping model does not seem to describe the actual damping behavior well in some systems such as those with viscoelastic damping frequently encountered in engineering practice [135]. As a result, nonviscous damping models have been developed and introduced to improve vibration analysis in order to obtain better and more accurate damping estimates which are very important to dynamic response prediction and stability assessment. Nonviscous damping models possess certain memory effects so as to depend on the past history of motion through a convolution integral over suitable kernel functions. Mathematically, vibration systems with nonviscous damping can be generally described as,

$$\mathbf{M}\,\ddot{\mathbf{x}}(t) + \int_0^t \tilde{\boldsymbol{\mathcal{G}}}(t-\tau)\,\dot{\mathbf{x}}(\tau)\,d\tau + \mathbf{K}\,\mathbf{x}(t) = \boldsymbol{f}(t) \tag{45}$$

where $\boldsymbol{f}(t)$, $\mathbf{x}(t)$ are the input force and output vibration response vectors, $\mathbf{M}$, $\mathbf{K}$, $\tilde{\boldsymbol{\mathcal{G}}}(t)$ are the system mass, stiffness and damping kernel function matrices. The damping kernel function matrix is normally defined in frequency domain. Upon taking Laplace transform of (45) and assuming zero initial conditions of displacements and velocities, we have,

$$\mathbf{D}(s)\,\mathbf{X}(s) = \mathbf{F}(s) \tag{46}$$

where the dynamic stiffness matrix $\mathbf{D}(s)$ is defined as,

$$\mathbf{D}(s) \equiv [\, s^2 \mathbf{M} + s\tilde{\mathbf{G}}(s) + \mathbf{K} \,] \tag{47}$$

where $\tilde{\mathbf{G}}(s)$ is the Laplace transform of $\tilde{\mathcal{G}}(t)$. Mathematically, any general form of $\tilde{\mathbf{G}}(s)$ is admissible as long as it represents the causal dissipative nature of damping. As a result, by specifying appropriate forms of $\tilde{\mathbf{G}}(s)$, a wide variety of damping models can be treated as special cases of this general nonviscous damping model. For example, by setting $\tilde{\mathbf{G}}(s) = \mathbf{C}$ to be a constant matrix, (45) then reduces to that of the classical *viscous damping model* case with a viscous damping matrix **C**. For the convenience of discussion, we consider the case of symmetric system matrices **M**, **K** and $\tilde{\mathbf{G}}(s)$ so that the left and right eigenvectors become identical, though more general treatment can be made.

The eigenvalue problem associate with (45) becomes,

$$[\, s_j^2 \mathbf{M} + s_j \tilde{\mathbf{G}}(s_j) + \mathbf{K} \,]\, \mathbf{u}_j \equiv \mathbf{0} \tag{48}$$

where s_j and $\mathbf{u}_j$ are the jth eigenvalue and eigenvector, respectively. Such a general frequency dependent eigenvalue problem can be solved using iterative method [136]. In addition, it is worth mentioning that due to the fractional form of the general nonviscous damping model [137], nonviscous modes can exist which are the modes with negative real eigenvalues, leading to a unique behavior of a nonviscous system having more modes than that of the system dimension [138] since $\tilde{\mathbf{G}}(s)$ is a function of s with fractional terms appearing in the denominators, resulting in a determinant with powers of s greater than $2N$. These nonviscous modes behave similarly to rigid body vibration modes since their imaginary parts which are the frequencies are zero, as shown in Fig. 2, [138].

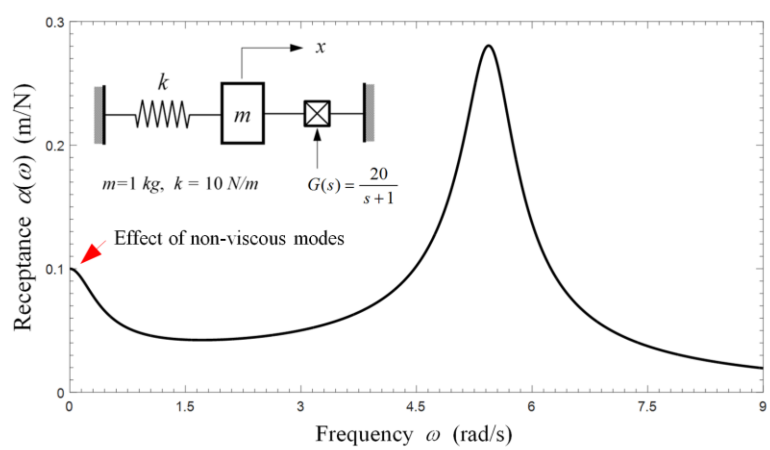

Figure 2 – Nonviscous modes as "rigid body" contributions to FRF

The eigenvector $\mathbf{u}_j$ is assumed to be normalized to the derivative of the dynamic stiffness matrix with respect to s evaluated at $s = s_j$. Thus,

$$\mathbf{u}_j^T \left[\left. \frac{\partial \mathbf{D}(s)}{\partial s} \right|_{s = s_j} \right] \mathbf{u}_j = \mathbf{u}_j^T \left[2 s_j \mathbf{M} + \tilde{\mathbf{G}}(s_j) + \left. \frac{\partial \tilde{\mathbf{G}}(s)}{\partial s} \right|_{s = s_j} \right] \mathbf{u}_j = 1 \tag{49}$$

Then, by differentiating (48) with respect to design variable p, it is found that,

$$[\, s_j^2 \mathbf{M} + s_j \tilde{\mathbf{G}}(s_j) + \mathbf{K} \,] \mathbf{u}'_j = [\, 2 s_j s'_j \mathbf{M} + s_j^2 \mathbf{M}' + s'_j \tilde{\mathbf{G}}(s_j) + s_j \tilde{\mathbf{G}}'(s_j) + \mathbf{K}' \,] \mathbf{u}_j \tag{50}$$

where the term $\tilde{\mathbf{G}}'(s_j)$ appearing in (50) can be expressed as,

$$\tilde{\mathbf{G}}'(s_j) \equiv \frac{\partial \tilde{\mathbf{G}}(s_j)}{\partial p} = \frac{\partial s_j}{\partial p} \left. \frac{\partial \tilde{\mathbf{G}}(s)}{\partial s} \right|_{s = s_j} + \left. \frac{\partial \tilde{\mathbf{G}}(s)}{\partial p} \right|_{s = s_j} = s'_j \left. \frac{\partial \tilde{\mathbf{G}}(s)}{\partial s} \right|_{s = s_j} + \left. \tilde{\mathbf{G}}'(s) \right|_{s = s_j} \tag{51}$$

Pre-multiplying (50) by $\mathbf{u}_j^T$ and using the relations of (48) and (49), the eigenvalue derivative becomes,

$$s'_j = \mathbf{u}_j^T \, [\, s_j^2 \mathbf{M}' + s_j \left. \tilde{\mathbf{G}}'(s) \right|_{s = s_j} + \mathbf{K}' \,] \mathbf{u}_j \tag{52}$$

Once s'_j is computed, the RHS of (50) is known and the eigenvector derivative $\mathbf{u}'_j$ can be solved either using the complex version of Nelson's method [139], or the complex modal method [140]. In the case of modal method, $\mathbf{u}'_j$ may be derived as,

$$\mathbf{u}'_j = \sum_{k=1,\, k \neq j}^{N+m} \frac{\mathbf{u}_k^T \, [\, s_j^2 \mathbf{M}' + s_j \left. \tilde{\mathbf{G}}'(s) \right|_{s = s_j} + \mathbf{K}' \,] \mathbf{u}_j \, \mathbf{u}_k}{s_k - s_j} - \frac{1}{2} \mathbf{u}_j^T \left. \frac{\partial \mathbf{D}'(s)}{\partial s} \right|_{s = s_j} \mathbf{u}_j \, \mathbf{u}_j \tag{53}$$

where $\partial \mathbf{D}'(s)/\partial s$ is defined as,

$$\frac{\partial \mathbf{D}'(s)}{\partial s} \equiv 2s\,\mathbf{M}' + \tilde{\mathbf{G}}'(s) + s\frac{\partial \tilde{\mathbf{G}}'(s)}{\partial s} \tag{54}$$

The case of repeated eigenvalues of nonviscously damped systems was examined and the mode truncation effect was also discussed [138].

Furthermore, complex eigenvalues and eigenvectors of viscoelastically damped systems were updated within the desired tolerance by an adaptive step-size control scheme using the first- and higher order eigenderivatives [141]. The algebraic method was further extended to both symmetric and asymmetric systems with nonviscous damping [142]. To circumvent the problem of numerical ill-conditioning and hence to improve numerical accuracy, an algorithm for calculating the eigensensitivity of asymmetric nonviscous damped systems without using the left eigenvector was proposed [143]. To eliminate the influence of the higher modes on the eigensensitivity of nonviscously damped systems, a method based on system flexibility was discussed [144]. Furthermore, an efficient algorithm was derived for the computation of eigenvalue and eigenvector derivatives of symmetric nonviscously damped systems with repeated eigenvalues [145]. By employing adjoint variable method, a design sensitivity analysis for the transient vibration response of a nonviscously damped system was considered [146].

2.4 Eigenderivatives of Non-self-adjoint and Defective Systems

For a general eigenvalue problem $\mathbf{Ax} = \lambda \mathbf{Bx}$ of a system, if any of the system matrices $\mathbf{A}$ or $\mathbf{B}$ does not satisfy $\mathbf{A}^H = \mathbf{A}$ or $\mathbf{B}^H = \mathbf{B}$, then the system is said to be non-self-adjoint where H is a complex conjugate transpose, or Hermitian, operator. In the case of real system matrices, being non-self-adjoint implies that at least one of these matrices is asymmetrical. A non-self-adjoint system often possesses two distinct sets of eigenvectors called left and right eigenvectors. The right eigenvectors physically represent the particular states of the motion of the system. The left eigenvectors however are only mathematical constructs that do not have direct physical interpretations. They may be interpreted as some aggregate behavior of all possible motion states of the system. Generally, for non-self-adjoint systems, some special normalization requirements need to be introduced in order to uniquely define eigenvalue and eigenvector derivatives due to the arbitrary scaling of both right and left eigenvectors [147].

By extending the method proposed by Fox and Kapoor [13], similar expressions for the rates of change of eigenvalues and eigenvectors with respect to parameters in the system in the case non-self-adjoint systems were obtained [148]. A procedure was developed for determining the sensitivities of a discrete linear-vibration system with non-symmetric structural matrices [149]. A complex transcendental eigenvalue problem, which arises in the dynamic stability analysis of non-conservative physical systems, flutter analysis of aeroelastic systems, was discussed and a method for computing eigenvalue sensitivities was proposed [150]. Based on perturbation approach, an algorithm was developed for evaluating eigenvalue and eigenvector derivatives of nonsymmetric systems [150]. For non-proportionally damped vibration systems, expressions for the first- and second-order derivatives of the eigenvalues and eigenvectors were derived when systems matrices were asymmetric [152]. An algebraic method was presented for the simultaneous computation of the derivatives of eigenvalues and those of their associated right and left eigenvectors, for asymmetric damped systems [153]. Similarly, Nelson's approach was extended to the computation of second-order derivatives of the eigenvalues and eigenvectors of asymmetric damped systems [154]. Further, an efficient algebraic method for the computation of eigensolution derivatives of asymmetric damped systems was presented by introducing an additional and new normalization condition to remove the singularity of the coefficient matrix [155]. Eigensolution derivatives of asymmetric damped systems were obtained by allowing the left and right eigenvector derivatives to be computed separately and independently so that parallel computational implementation became possible [156]. Following a different approach, several relative eigenvalue condition numbers were derived that exploited the triangular factorizations instead of the matrix entries and shed light on when eigenvalues were less sensitive to perturbations of factored forms than to perturbations of the matrix entries [157]. In addition, an improved modal truncation method with arbitrarily high order accuracy was developed for calculating the second- and third-order eigenvalue derivatives and the first- and second-order eigenvector derivatives of an asymmetric dynamic system [158].

Alongside these studies, methods have also been developed for eigenderivatives of defective eigensystems. A defective matrix is a square matrix that does not have a complete basis of eigenvectors, and is therefore not diagonalizable. In particular, an $N \times N$ matrix is defective if and only if it does not have N linearly independent eigenvectors. A complete basis however can be formed by augmenting the eigenvectors space with generalized eigenvectors, which are necessary for solving defective systems of ordinary differential equations and other engineering problems. Defective matrices occur in areas such as automatic control and system theory. In dynamic problems of structure and fluid interactions such as flutter of airplane and missile wings or long blades of turbines, the corresponding system matrices can become defective. Other examples of defective systems include the eigenvalue problem of the zeros, or anti-resonances of frequency response functions. Their sensitivities [159] were found to be given by a linear combination of the conventional eigenvalue and eigenvector sensitivities. Characteristics of anti-resonances are often easier to measure in practice than mode shapes and can amplify information contributed by the modes beyond the measurement frequency range. Friswell *et al*. [160] showed that viscously damped dynamic systems modified by a rank-1 viscous damping matrix will become defective if repeated damped vibration modes occur due to such rank-1 damping modification.

Based on the exact modal expansion method, an arbitrary high-order approximate method was developed for calculating the second-order eigenvalue derivatives and the first-order eigenvector derivatives of a defective matrix [161-162]. Also, a direct method was developed for calculating the first- to third-order eigenvalue derivatives and first- to second-order eigenvector derivatives of a defective matrix with a zero first-order eigenvalue derivative associated with Jordan blocks of order higher than the lowest [163]. A fractional perturbation power series approach was used to estimate eigenderivatives of defective systems for the cases of distinct and repeated eigenvalues [164-165]. Based on Puiseux expansions of perturbation parameter for the solution of a perturbed problem, a modal expansion method for the eigensensitivity analysis of a defective matrix was developed, in which any of the eigenvector derivatives was expressed as a linear combination of all the eigenvectors and the principal vectors of the matrix itself [166]. Further, a novel approach was introduced to address the problem of the existence of differentiable eigenvectors for a given defective matrix that have repeated eigenvalues [167]. Finitely bounded sensitivity of a defective eigenvalue with respect to perturbations that preserve the geometric multiplicity and the smallest Jordan block size was derived and examined [168]. Following a similar approach, sensitivity of defective multiple eigenvalues of reducible matrix pencil of a quadratic eigenvalue problem dependent on several parameters was investigated [169]. In addition, a generalized eigenproblem was formulated together with normalization constraints for a Hessian matrix, which is a square matrix of second-order partial derivatives of a scalar field, and the associated eigenderivatives were derived [170].

2.5 Eigenderivatives of Fractional Vibration Systems

Fractional derivatives arise in engineering practice due to the various structural and material properties which result in a complex force-displacement-velocity relationship which not only involves integer ordered displacement and velocity terms, but also fractional ordered displacement and velocity. Mathematically, such a general modeling approach which includes integer order as a special case can naturally be considered as a generalization and a further improvement on the classical vibration analysis and has subsequently been found to lead to more accurate predictions of vibration characteristics. Such a class of vibration problems can in general be described by the following differential equation,

$$\mathbf{M}\,\ddot{\mathbf{x}}(t) + \mathbf{C}\,\dot{\mathbf{x}}(t) + \mathbf{K}\,\mathbf{x}(t) + \sum_{s=1}^{m_1} \mathbf{E}_s\, \mathrm{D}^{\alpha_s}\mathbf{x}(t) = \boldsymbol{f}(t) \tag{55}$$

where **M**, **K**, **C** are the mass, stiffness and viscous damping matrices of the vibration system and $\boldsymbol{f}(t)$ is the input force. The additional new terms $\mathrm{D}^{\alpha_s}\,\mathbf{x}(t)$ are the fractional derivatives applied to the displacement vector $\mathbf{x}(t)$ of order α_s $(0 \le \alpha_s \le 1)$ and $\mathbf{E}_s$ are the coefficient matrices associated with these fractional derivatives. Depending on the physical nature of the problem, generally a different number of fractional derivative terms m_1 are required to accurately describe the vibration characteristics of the system under consideration.

The matrix representation of the eigenvalue problem of a general fractional vibration system of (55) may be written as,

$$[\, -\lambda^2\, \mathbf{M} \,+\, \mathrm{i}\,\lambda\, \mathbf{C} \,+\, \mathbf{K} \,+\, \sum_{s=1}^{m_1} \lambda^{\alpha_s} e^{\mathrm{i}\,\alpha_s\,\pi/2}\, \mathbf{E}_s \,]\, \mathbf{X} \,=\, \mathbf{0} \tag{56}$$

Such a general eigenvalue problem cannot possibly be solved directly or using classical state space formulations but has to be solved based on numerical iterations as discussed in detail in [170]. Further, the eigenvectors are no longer orthonormal to system mass matrix **M**, nor to system matrix **B** in the case of state space formulations and as a result, new normalization and orthogonality equations, which become very important in the subsequent derivations of the eigenvector derivatives, need to be first established. For modes *j* and *k*, the eigenvalues and their corresponding eigenvectors satisfy the following,

$$[\, -\lambda_j^2\, \mathbf{M} \,+\, \mathrm{i}\,\lambda_j\, \mathbf{C} \,+\, \mathbf{K} \,+\, \sum_{s=1}^{m_1} \lambda_j^{\alpha_s} e^{\mathrm{i}\,\alpha_s\,\pi/2}\, \mathbf{E}_s \,]\, \mathbf{X}_j \,=\, \mathbf{0} \tag{57}$$

$$[\, -\lambda_k^2\, \mathbf{M} \,+\, \mathrm{i}\,\lambda_k\, \mathbf{C} \,+\, \mathbf{K} \,+\, \sum_{s=1}^{m_1} \lambda_k^{\alpha_s} e^{\mathrm{i}\,\alpha_s\,\pi/2}\, \mathbf{E}_s \,]\, \mathbf{X}_k \,=\, \mathbf{0} \tag{58}$$

Pre-multiplying (57) by $\mathbf{X}_k^T$ and (58) by $\mathbf{X}_j^T$, and then taking their difference, we have,

$$\mathbf{X}_k^T\, [\, (\lambda_k^2 - \lambda_j^2)\, \mathbf{M} \,+\, \mathrm{i}\,(\lambda_j - \lambda_k)\, \mathbf{C} \,+\, \sum_{s=1}^{m_1} (\lambda_j^{\alpha_s} - \lambda_k^{\alpha_s})\, e^{\mathrm{i}\,\alpha_s\,\pi/2}\, \mathbf{E}_s \,]\, \mathbf{X}_j \,=\, 0 \tag{59}$$

This is the general orthogonality equation for modes of fractional vibration systems. By dividing each term by a common factor, $\lambda_j - \lambda_k$, gives,

$$\mathbf{X}_k^T\, [\, -(\lambda_j + \lambda_k)\, \mathbf{M} \,+\, \mathrm{i}\, \mathbf{C} \,+\, \sum_{s=1}^{m_1} \frac{\lambda_j^{\alpha_s} - \lambda_k^{\alpha_s}}{\lambda_j - \lambda_k}\, e^{\mathrm{i}\,\alpha_s\,\pi/2}\, \mathbf{E}_s \,]\, \mathbf{X}_j \,=\, 0 \tag{60}$$

In the case of repeated eigenvalue, as λ_k approaches to λ_j to their common value λ, while $\mathbf{X}_k$ and $\mathbf{X}_j$ remain distinct eigenvectors associated with the repeated eigenvalue λ, the matrix inside the square brackets becomes,

$$\lim_{\substack{\lambda_k \to \lambda \\ \lambda_j \to \lambda}} [\,-(\lambda_j + \lambda_k)\,\mathbf{M} + \mathrm{i}\,\mathbf{C} + \sum_{s=1}^{m_1} \frac{\lambda_j^{\alpha_s} - \lambda_k^{\alpha_s}}{\lambda_j - \lambda_k}\, e^{\mathrm{i}\,\alpha_s \pi/2}\, \mathbf{E}_s\,] = -2\lambda\,\mathbf{M} + \mathrm{i}\,\mathbf{C} + \sum_{s=1}^{m_1} \alpha_s\, \lambda^{\alpha_s - 1} e^{\mathrm{i}\,\alpha_s \pi/2}\, \mathbf{E}_s \tag{61}$$

Therefore, it becomes logical and appropriate that the eigenvectors should be normalized in the case of distinct eigenvalue λ_j such that,

$$\mathbf{X}_j^T\, [\,-2\lambda_j\, \mathbf{M} + \mathrm{i}\,\mathbf{C} + \sum_{s=1}^{m_1} \alpha_s\, \lambda_j^{\alpha_s - 1}\, e^{\mathrm{i}\,\alpha_s \pi/2}\, \mathbf{E}_s\,]\, \mathbf{X}_j = 1 \tag{62}$$

and in the case of repeated eigenvalue λ of multiplicity m,

$$\mathbf{X}^T\, [\,-2\lambda\, \mathbf{M} + \mathrm{i}\,\mathbf{C} + \sum_{s=1}^{m_1} \alpha_s\, \lambda^{\alpha_s - 1} e^{\mathrm{i}\,\alpha_s \pi/2}\, \mathbf{E}_s\,]\, \mathbf{X} = \mathbf{I} \tag{63}$$

where $\mathbf{X}$ is the eigenvector matrix associated with the repeated eigenvalue λ and $\mathbf{I}$ is a unit matrix of dimension $m \times m$. In fact, the eigenvectors associated with the repeated eigenvalue λ only become orthogonal to the matrix defined in the square bracket of (63) which is the only orthonormal constraints one can obtain for a fractional vibration system in the case of repeated eigenvalue. Such orthonormal constraints need to be imposed before meaningful eigenvalue and eigenvector derivatives can be established.

Differentiating (56), then leads to,

$$
\begin{aligned}
&[-\lambda_j^2 \mathbf{M}' + \mathrm{i}\,\lambda_j \mathbf{C}' + \mathbf{K}' + \sum_{s=1}^{m_1} \lambda_j^{\alpha_s} e^{\mathrm{i}\,\alpha_s \pi/2} \mathbf{E}'_s]\, \mathbf{X}_j \\
&\quad + [-2\lambda_j \lambda'_j \mathbf{M} + \mathrm{i}\,\lambda'_j \mathbf{C} + \sum_{s=1}^{m_1} \alpha_s \lambda_j^{\alpha_s - 1} \lambda'_j e^{\mathrm{i}\,\alpha_s \pi/2} \mathbf{E}_s]\, \mathbf{X}_j \\
&\quad\quad + [-\lambda_j^2 \mathbf{M} + \mathrm{i}\,\lambda_j \mathbf{C} + \mathbf{K} + \sum_{s=1}^{m_1} \lambda_j^{\alpha_s} e^{\mathrm{i}\,\alpha_s \pi/2} \mathbf{E}_s]\, \mathbf{X}'_j = \mathbf{0}
\end{aligned}
\tag{64}
$$

Pre-multiplying both sides of (64) by $\mathbf{X}_j^T$ and using (56) and (62), the eigenvalue derivative λ'_j may be obtained as,

$$
\lambda'_j = -\mathbf{X}_j^T \, [-\lambda_j^2 \mathbf{M}' + \mathrm{i}\,\lambda_j \mathbf{C}' + \mathbf{K}' + \sum_{s=1}^{m_1} \lambda_j^{\alpha_s} e^{\mathrm{i}\,\alpha_s \pi/2} \mathbf{E}'_s]\, \mathbf{X}_j \tag{65}
$$

Since (64) is a set of linear algebraic equations, the required eigenvector derivative $\mathbf{X}'_j$ can be solved using the extended modal method, Nelson's method, as well as eigenvector derivatives in the case of repeated eigenvalues as discussed in detail in [172].

The fractional eigenvalue problem and its derivatives only recently became an active area of research owing to the ever increasing applications and improved numerical modeling tools to effectively deal with fractional derivatives. Eigensensitivities were derived and used to identify parameters of time invariant linear dynamical systems with fractional derivative damping models in an asymmetric generalized eigenvalue problem [173]. A singular eigenvalue problem for a higher order fractional differential equation involving Riemann–Liouville fractional derivatives was investigated [174]. Based on the perturbation of the coefficients of a part of the differential operator by piecewise constant functions, a new algorithm for eigenvalue problems for linear differential operators with fractional derivatives was developed [175]. Further, linear and nonlinear fractional eigenvalue problems involving the Atangana-Baleanu fractional derivative of the order $1 < \alpha < 2$ were examined by estimating the fractional derivatives of a function at its extreme points and applying them to obtain a maximum principle for the linear fractional boundary value problem [176].

3. Engineering Applications of Eigenvalue and Eigenvector Derivatives

Inverse sensitivity analysis of eigenvalues and eigenvectors has emerged as a fruitful area of engineering research. The increased interest witnessed is due to the recognition of the variety of applications in which eigenderivatives have been employed. Sensitivity analysis, at its early stage of development, found its predominant use in assessing the effect of varying system parameters in mathematical models of vibration and control systems. Early interests in optimal control and automated structural optimization led to the use of gradient-based mathematical programming methods in which derivatives were used to find search directions toward optimum solutions. More recently, there has been strong interest in promoting systematic structural optimization as a useful design tool for the practicing structural design engineers for large structural problems. To date, a wide range of engineering applications have been established in which eigensensitivities play a vital role in ensuring their successes. Some of these major successful applications including (i) finite element model updating, (ii) structural modification and redesign including damping design, (iii) performance optimization of structural systems, (iv) control system design, (v) structural damage detection and fault diagnosis and, (v) vibration of bladed disk structures are discussed in this paper, together with comprehensive reviews on the latest progress and potential future research directions.

3.1 Finite Element Model Updating

Model updating [177] is a process whereby an analytical finite element model is improved/updated using measured vibration test data. Many model updating methods have since been developed, but the inverse eigensensitivity based methods remain as the most effective methods [178] to date. As discussed in Section 2, given analytical finite element mass and stiffness matrices, an eigenvalue problem can be defined from which eigensensitivities can be computed. The corrections to the model parameters required $\Delta p_i \ (i = 1, 2, ..., m)$ and the difference between the measured and analytical modal data rth mode can be related, to a first order approximation, as,

$$\begin{bmatrix} \partial\{\boldsymbol{\varphi}\}_\mathrm{r}/\partial p_1 & \partial\{\boldsymbol{\varphi}\}_\mathrm{r}/\partial p_2 & \cdots & \partial\{\boldsymbol{\varphi}\}_\mathrm{r}/\partial p_m \\ \partial\lambda_\mathrm{r}/\partial p_1 & \partial\lambda_\mathrm{r}/\partial p_2 & \cdots & \partial\lambda_\mathrm{r}/\partial p_m \end{bmatrix} \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \vdots \\ \Delta p_m \end{Bmatrix} = \begin{Bmatrix} \{\boldsymbol{\varphi}_\mathrm{x}\}_\mathrm{r} - \{\boldsymbol{\varphi}_\mathrm{a}\}_\mathrm{r} \\ (\lambda_\mathrm{x})_\mathrm{r} - (\lambda_\mathrm{a})_\mathrm{r} \end{Bmatrix} \quad (66)$$

where subscripts “a” and “x” denote data from the analytical and the experimental models respectively.

Equation (66) is formulated using modal data of rth mode only and when in practice, L modes have been assumed to have been measured, (66) can be expanded to become,

$$
\begin{bmatrix}
\partial\{\boldsymbol{\varphi}\}_1/\partial p_1 & \partial\{\boldsymbol{\varphi}\}_1/\partial p_2 & \cdots & \partial\{\boldsymbol{\varphi}\}_1/\partial p_m \\
\partial\lambda_1/\partial p_1 & \partial\lambda_1/\partial p_2 & \cdots & \partial\lambda_1/\partial p_m \\
\vdots & \vdots & \vdots & \vdots \\
\partial\{\boldsymbol{\varphi}\}_{\mathrm{L}}/\partial p_1 & \partial\{\boldsymbol{\varphi}\}_{\mathrm{L}}/\partial p_2 & \cdots & \partial\{\boldsymbol{\varphi}\}_{\mathrm{L}}/\partial p_m \\
\partial\lambda_{\mathrm{L}}/\partial p_1 & \partial\lambda_{\mathrm{L}}/\partial p_2 & \cdots & \partial\lambda_{\mathrm{L}}/\partial p_m
\end{bmatrix}
\begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta p_3 \\ \vdots \\ \Delta p_m \end{Bmatrix}
=
\begin{Bmatrix} \{\boldsymbol{\varphi}_{\mathrm{x}}\}_1 - \{\boldsymbol{\varphi}_{\mathrm{a}}\}_1 \\ (\lambda_{\mathrm{x}})_1 - (\lambda_{\mathrm{a}})_1 \\ \vdots \\ \{\boldsymbol{\varphi}_{\mathrm{x}}\}_{\mathrm{L}} - \{\boldsymbol{\varphi}_{\mathrm{a}}\}_{\mathrm{L}} \\ (\lambda_{\mathrm{x}})_{\mathrm{L}} - (\lambda_{\mathrm{a}})_{\mathrm{L}} \end{Bmatrix}
\tag{67}
$$

which can be solved using least squares procedure [16]. Since the (67) is formulated in theory based first-order eigensensitivity analysis, iterations are often required in order to get more accurate estimates of the corrections to the parameters required [16]. A typical case study of model updating using inverse eigensensivity method based on the GARTEUR AG11 is shown in Fig. 3 in which the updated model reproduces the FRFs of the simulated "experimental" FRFs, [16].

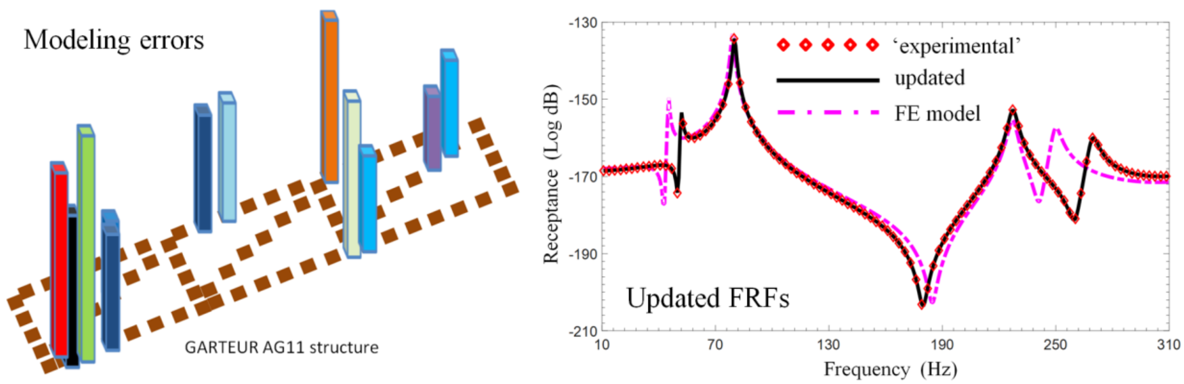

Fig. 3 – Numerical case study of model updating using inverse eigensensitivity method [16]

A matrix perturbation based sensitivity analysis method was proposed to compute the Jacobian matrix involved in parameter estimation to update an analytical model [179]. Further, sensitivity parameters were used to remove the difficulties encountered in matching vibration modes and convergence when the structure to be analyzed has quasi-multiple eigenvalues [180]. Inverse eigensensitivity method was further developed to allow large magnitudes of modeling errors and to improve the speed of convergence [181]. Also, an effective method was discussed and was applied to identify structural damping models by correlating analytical mass and stiffness matrices which were assumed to contain modeling errors, with measured complex eigenvalues and eigenvectors of a test structure [182].

To update models with repeated eigenvalues, a procedure was proposed for extending the inverse sensitivity method to systems described by analytical conservative mechanical models [183]. The mathematical relationship between the two more powerful methods for updating analytical models, the inverse eigensensitivity method and the FRF sensitivity method, was examined with an objective to demonstrate their advantages and disadvantages, as well as their specific practical application requirements [184]. A strategy for the enrichment of experimental data was examined in connection with the problem of finite element model updating based on the inverse eigensensitivity formulation [185]. Also, the application of inverse sensitivity method to self-adjoint systems with repeated eigenvalues was discussed [186]. Different model-updating experiments were carried out using different sets of updating parameters based on a three-storey aluminium space frame with unrepresentative rigid joints [187]. A practical method for finite element model updating technique for real structures using ambient vibration test results was discussed [188]. The performance of the inverse eigensensitivity method was compared with the response function method for analytical model updating using simulated experimental test data [189]. An eigensensitivity-based finite element model updating procedure was applied to the model updating of an existing 310m tall tower based on ambient vibration measurements [190]. A model updating based structural assessment of highway bridges using measured vibration acceleration and strain data was conducted during structural upgrading including strengthening [191]. Based on the computation of the error measures on the constitutive relations using classical eigensensitivity analysis, a parametric model updating method using modal test results, which allows the corrections to be made in both the stiffness and the mass matrices, was developed [192]. Also, a substructure-based finite element model updating technique was proposed using a few eigenmodes of the independent substructures and their associated derivative matrices to be assembled into a reduced eigenequation to recover the eigensensitivities of the global structure [193]. To ensure physically meaningful updated models and to improve the numerical conditions associated with the updating processes, effective parametrization and regularization techniques were proposed [194]. A response surface-based finite element model updating procedure for civil engineering structures was investigated in which quadratic polynomial response surface was assumed and constructed from vibration test data to be used for physical parameter estimation [195].

Alongside these studies, zeros of frequency response functions, or anti-resonance, sensitivities were applied in model updating [196-197]. Further, modern developments in optical measurement have made full-field vibration test data [198] available (i.e. not limited to a finite number of acceleration sensors). Issues of redundancy arising from huge volumes of data were addressed by projecting the measured image onto an orthonormal basis and removing those small coefficients deemed to be insignificant. The measurement was then available in the shape descriptor domain and modal analysis and model updating (using eigendata sensitivities) might be carried out [199]. For model updating of nonlinear structures, location and identification of mathematical model of structural nonlinearities were examined [200-201]. Amplitude-dependent natural frequencies and damping values were used to update nonlinear Iwan-joint models [202] and using time-frequency data, instantaneous frequencies and amplitudes were used to update nonlinear FE models [203].

Another approach was to use FRF sensitivities obtained from constant-input stepped-sine tests to obtained nonlinear updated models [204-205]. Furthermore, for stochastic model updating, eigenvalue sensitivities were applied using perturbation and interval methods [206-208] and in covariance updating by minimizing the Frobenius norm of the matrix of covariance residuals, dependent upon eigenvalue and eigenvector derivatives [209].

3.2 Structural Modification and Redesign

The ability to predict accurately the effect of given known modifications to a structure becomes very important to structural operations and designs. These modifications may be imposed by external factors such as design alterations for operational reasons and in this case, it will generally be necessary to determine what changes in vibration properties will ensue from introducing the modifications as these might be detrimental, for example by moving closer to a resonance condition. Another possibility is that it may be required to change the vibration properties themselves, perhaps to avoid a resonance or to add more damping, and then it is important to know how best to go about modifying the structure so as to bring about the desired changes in vibration behavior without at the same time introducing some new unwanted effects. Eigensensitivities can enable us to accomplish these important tasks since these provide the direct relationships between vibration properties and changes in system parameters. Structural design is an iterative process in which initial designs are continuously modified and redesigned before desired optimal designs are obtained. With the help of eigensensitivities, modern structural design can become a systematic process and can even be made automatic. To date, numerous works have been carried out on the applications of eigensensitivities to optimal structural dynamics design.

Derivatives of eigenvectors of a locally modified structure with respect to a design variable was presented and used for structural design [210]. Equations were derived based upon the continuum approach for eigenvalue sensitivity analysis of planar frame structures with variable joint and support locations and design sensitivity analyses were carried out [211]. The stability behavior of a cantilever beam, carrying a concentrated mass at its free end and subjected to a uniformly distributed follower force, was studied using eigenvalue sensitivity with respect to the follower force [212]. Following a sensitivity analysis approach, based on a sparse formulation, a useful method was presented for the design of power systems [213]. Eigenvalue sensitivities with respect to location of discrete in-span occurrence for structure members were derived using a normal mode method for design calculations of concentrated mass and inertia, rigid supports, elastic supports and appended spring-mass systems [214]. On the other hand, sensitivities with respect to specific design variables were found without forming an auxiliary eigenequation and were employed for structural design in the case of repeated eigenvalues [215]. A unified continuum-based sizing design sensitivity analysis of eigenvectors was presented by adding load-dependent Ritz vector to the basis to improve the accuracy of design sensitivity of eigenvectors [216]. Further, closed-form approximate solutions for uniaxial and biaxial compressive buckling of rectangular symmetric angle-ply laminates were developed based on eigensensitivity analysis [217]. Eigenvalue sensitivities with respect to the changes in the magnitude of the damping constant, tip mass ratio and location of the damper attachment point of a beam structure were examined [218]. The fundamental frequency of a rectangular isotropic plate with a linear thickness variation was predicted using eigensensitivity analysis [219].

Eigensensitivities have been widely employed in structural design optimizations. Optimal distribution of a viscoelastic damping layer was sought for the suppression of the transient vibration of a flexible beam using eigensensitivity analysis [220]. A closed-form solution was presented to compute the moisture-related buckling loads of symmetric angle-ply laminates as well as plate under combined bending and compression [221-222]. Optimal damping element locations in trusses were discussed based on calculating the effectiveness indices obtained from the derivatives of the eigenfrequencies of the dynamic eigenproblem with regard to structural parameters [223]. Also, by combining the adjoint variable method with complex variables, shape and size sensitivities were obtained for structural optimizations [224]. Normalized frequencies were computed for a rectangular, isotropic plate resting on elastic supports using eigensensitivity analysis [225]. Adjoint-based first-order sensitivity theory was applied to estimate the sensitivity of eigenvalues to system geometric dimensions [226]. Further, a tool was developed to efficiently estimate the dynamic properties of variable geometry trusses throughout their movement based on modal superposition of eigensensitivities [227]. The eigenvalue sensitivities of a combined system consisting of an arbitrarily supported linear structure carrying various lumped attachments and viscoelastic solids were analyzed [228-229]. Free vibration analysis and eigenvalue sensitivity analysis of composite laminates with interfacial imperfections were investigated based on the radial point interpolation method (RPIM) in a Hamiltonian system [230]. The theory and implementation details of a generalized perturbation free approach were described, together with error analyses [231]. An optimization technique for lightweight design of composite laminated beams with a big ratio of span to thickness was presented using eigenvalue sensitivity with respect to fiber volume fractions [232].

3.3 Performance Optimization of Structural Systems

Stable economic growth and social development of most countries are intimately dependent upon the reliable and durable performance of their structures and infrastructures. Natural hazards, ageing, and operational fluctuations can inflict detrimental effects on the performance of structural systems during their life-cycles. The accurate modeling of structures and the loading conditions to which they are expected to be exposed during their life-cycle as well as their possible deterioration mechanisms are major issues of structural and engineering mechanics. Through accurate modeling and detailed design, structural performance can be optimized to prolong smooth operations and service life span.

Eigensensitivity based structural performance optimization has gained much momentum over the last 4 decades and has delivered very promising results. Eigenvalue sensitivity in a nuclear reactor system to variations of the system parameters was a critical factor measuring the danger of system excursion and a state variable feedback system was introduced to reduce eigenvalue sensitivity to simultaneous deviations of the system parameters to enhance system performance [233]. Repeated eigenvalues were shown to be only directionally differentiable with respect to design and such characteristics were demonstrated to bear substantial consequences in classes of optimal design problems in which the fundamental eigenvalue is known to be repeated at an optimum design [234]. Based on Rayleigh's quotient, a method for second-order sensitivity analysis of multimodal eigenvalues was proposed and applied to formulate a sequential quadratic programming of multimodal optimal design of structures [235]. Also, based on eigenvalue sensitivity, the straight line stability of an automobile with steering wheel fixed was examined [236]. A simple and easily implemented method to calculate the eigenvalue derivatives of a geared shaft system with respect to design parameters were proposed [237]. Errors associated with semi-analytical method of finite element based design sensitivity analysis were analyzed when linearly elastic bending of beam, plate and shell structures was considered [238]. It was demonstrated that the eigenvalue sensitivity could be used to identify an association between the eigenvalues and the parameters of the system and that it could provide useful guidance in selecting the system parameters [239].

To design against uncertainties, stochastic sensitivity analysis was examined and adapted to existing finite element programs for structural performance optimization [240]. Nonlocal nonlinear perturbation bounds were derived for an eigensystem with distinct eigenvalues based on asymptotic expansions [241]. An effective method was developed to derive structural design sensitivities from limited vibration test data which were found to be more accurate than their analytical counterparts [242]. Based on generalized variational principle, sensitivity of eigenvalues to the location of bracing and boundary support was formulated [243]. A Kelvin formulation was used to study the sensitivity of anisotropic materials with woven composites [244]. Based on the modal truncation method of eigenderivatives, an optimization algorithm was developed to become an effective tool for engineering performance designs [245]. The sensitivity of the critical eigenvalue to hydrogen density changes in a homogeneous sphere containing low-enriched uranium was computed and showed that changes in uranium (U-238) shielded cross sections caused by perturbations in hydrogen concentrations were important components in the overall eigenvalue sensitivity coefficient [246]. Methodologies to calculate adjoint-based first-order sensitivity coefficients with multigroup Monte Carlo methods were developed, implemented, and tested [247]. For serpentine belt drive systems, closed-form expressions for the eigensensitivities were obtained using perturbation methods [248]. For vibration analysis of microelectromechanical systems (MEMS), eigensensitivities were derived and employed for improved dynamic performance [249]. Further, analytically derived eigenderivatives were utilized in mode-tracing in elastic guided waves including a study of dispersion curve representation [250]. In the context of Euclidean Jordan algebras, derivative of eigenvalues of the elements of a Jordan frame associated with the spectral decomposition were examined [251]. The allowance distribution state of thin-walled workpiece was determined and optimized in semi-finishing machining, based on the eigenvalue sensitivity analysis method [252].

Performance optimization of electrical power systems has been an important area of research in which eigensensitivities have been extensively employed. Eigenvalue sensitivities of interconnected power systems were presented for a wide variety of control equipment and varying degrees of modeling complexity [253]. Second order eigenvalue sensitivities with respect to actual system parameters were formulated for lightly loaded hydro generator equipped with a stabilizer and connected to a local non-linear load [254]. Power system stabilizer (PSS) optimization through extensive application and development of techniques of modal and eigenvalue-sensitivity analyses were carried out [255]. A technique for the design of optimal modulation controllers for multi-area ac/dc systems was developed based on optimal pole assignment using eigenvalue sensitivities [256]. When an unrestricted rank constant state feedback was used for pole assignment in a multi-input linear system, an additional design freedom was produced which was then utilized to minimize the sensitivity of the closed loop poles to system variations [257]. A combined procedure for coordinated application of power system stabilizers (PSSs) in order to improve the overall dynamic stability of power systems was described based on eigenvalue sensitivity analysis [258]. A method to determine the distance to voltage collapse was proposed by using second-order eigenvalue sensitivity technique [259].

Furthermore, the effects of general system parameters were accurately assessed by the first and second order eigenvalue sensitivities with respect to any arbitrary system operating parameter and power network parameter [260]. The concepts of asymptotic stability robustness for model and parameter uncertainty of power systems were presented using eigenvalue sensitivity matrices [261]. Also, the conventional eigenvalue sensitivity analysis was extended to probabilistic environment of PSS parameter uncertainties [262]. A Newton-Raphson method was employed to improve the harmonic voltage performance of a system in which eigenvalue sensitivity coefficients were used to determine the elements of the required Jacobian matrix [263]. Similarly, a method for PSS parameter tuning based system identification using a Kalman filter was discussed and the results were compared with those obtained using eigenvalue sensitivity [264]. A detailed mathematical model for small-signal stability analysis of a dc-distribution electrical power system (EPS) was developed and a comprehensive modal analysis was performed to enhance system performance [265]. Some practical tools to evaluate the impact of the delays on system modes were developed by analytically computing the eigenvalue sensitivity [266]. Further, an algorithm was proposed for determining the minimum generation redispatch capable of moving specific oscillation modes to the closest small-signal security boundary that guarantees them a desired damping factor [267].

For optimal performance design of plate structures, closed-form expressions were developed to predict the vibration of rectangular angle-ply antisymmetric laminated composite plates and sensitivity derivatives were used to derive series representation of the laminate's frequencies [268-269]. Eigenvalue sensitivity of a stiffened plate with respect to stiffener location was analyzed based on the generalized Rayleigh quotient of the combined plate beam system [270]. Design sensitivity analysis and topology optimization were carried out for piezoelectric resonators [271]. Similarly, the desired change of each natural frequency by modifying an arbitrary parameter of a system with a known amount was achieved for laminated composite plates with piezoelectric patches [272]. Also, a sensitivity-based approach to characterize viscoelastic material property variation with frequency was outlined [273]. Further, an iterative complex eigensensitivity-based characterization method was presented based on real eigensolution performed using finite-element analysis [274].

3.4 Design of Control Systems

The design of control systems is one of the most important in engineering designs. The goal of control engineering design is to obtain the configuration, specifications, and identification of the key parameters of a proposed system to meet actual performance requirements. To optimize a control system, eigensensitivities are often needed to tune system parameters as well as to tailor system overall input/output characteristics.

Eigenstructure assignment is one of the effective state space based control design method for linear multi-input multi-output systems. Using derivatives of the eigenvalues with respect to the gains, eigenstructure assignment was considered with gain suppression in which selected entries in the output feedback gain matrix were eliminated [275]. The problem of eigenvalue assignment in active vibration control was addressed using system receptance in a single-input state feedback and demonstrated that not only could assignment be applied to the poles of the system but also to the sensitivities of the poles [276]. Based on the notions of spectrum sensitivities, a novel optimization approach to deal with robustness in the closed-loop eigenvalues for partial quadratic eigenvalue assignment problem arising in active vibration control was proposed [277]. For friction contact problems, a robust stabilization method that assigns both desirable eigenvalues and their sensitivities and thus renders assigned eigenvalues stable and insensitive to perturbations in uncertain contact parameters was implemented [278]. In stability analysis of control system, eigenvalue sensitivity matrices were used to design state feedback controllers that gave the closed loop system a prescribed set of pre-assigned eigenvalues, thereby minimizing a quadratic cost functional [279]. Also, an optimal control strategy for improved dynamic performance of integrated ac/dc systems, based on power modulation techniques, was discussed [280]. The finite-element method and the technique of eigenvalue sensitivity were both applied to the stability analysis of a cantilever column subjected to a follower force at its free end [281-282]. A robust stability analysis tool for a two-dimensional discrete system by using eigenvalue sensitivity was developed and used to predict system stability margins [283]. A sensitivity analysis of the critical eigenvalues was performed to explore stability characteristics in the neighborhood of the critical point in the parameter space [284]. Further, the modeling and oscillatory stability analysis of a wind turbine with doubly fed induction generator were carried out based on the loci of the system Jacobian's eigenvalues [285]. Experimental and theoretical investigation on the stability of high-

Reynolds-number low-density reacting wakes near a hydrodynamic Hopf bifurcation was conducted [286].

In determining the sensitivity of the response of a linear time-invariant dynamic system, eigenvalue and eigenvector sensitivity were employed which were derived from a closed-form expression [287]. A new measure of sensitivity specifically applicable to the realization of a linear discrete system was proposed and the sensitivity of the eigenvalues to parameter inaccuracies in the realization was found to depend strongly on the choice of the state variables [288]. Sensitivity theory was used to differentiate between alternative plants from the point of view of the sensitivity of their eigenvalues to loop-gain variations [289]. On the other hand, the sensitivity of eigenproblem as viewed from numerical analysis, perturbation theory and linear systems theory was examined to avoid obscurities which might have resulted from the existence of these different but logically equivalent solutions of the same problem [290]. Further, a simple method was presented for minimizing the deviation in the eigenvalues of the computer realization when all elements of the system matrix were subject to simultaneous variations [291]. The application of an eigenvalue sensitivity method to a linear, time-invariant model of the high-temperature gas-cooled reactor was discussed [292]. Similarly, the application of second-order eigenvalue sensitivities to multivariable control systems was examined and its practical importance was demonstrated by performing dynamic stability calculations for a fourteenth-order power system [293]. A design of a suboptimal controller for linear time invariant multivariable systems, which assigns the closed loop eigenvalues at desired locations and minimizes their sensitivity with respect to plant parameters, was outlined [294].

Following an iterative strategy, a method for designing an optimal constant gain feedback controller for a linear system to achieve minimum eigenvalue sensitivity to parameter variations was presented [295]. A digital computer based method for generating the sensitivities of the eigenvalues of a linear, time invariant, lumped parameter dynamic system to perturbations in its design parameters was described [296]. Also, a new method for designing modalized observers that achieve both small eigenvalue sensitivity and attenuation of the estimation error caused by an initial condition mismatch was developed [297]. The sensitivity of the closed-loop eigenvalues and eigenvectors of actively controlled flexible structures with distinct eigenvalues was discussed [298]. Further, methods to study the application of controllable series capacitors for damping power system electromechanical oscillations were developed based on small signal models of the power system and the corresponding eigenvalue sensitivities [299]. Eigen-sensitivity of an augmented matrix was examined for optimal tuning of control parameters and determining locations of compensating devices for stability enhancement [300]. By minimizing the least-squared error between the achievable and the desired eigenspace to obtain mode decoupling, a design method was introduced to enable the closed-loop system insensitive to perturbations or parameter variations [301]. A method of eigensensitivity analysis was developed and applied to a functionally graded material plate actively controlled by piezoelectric sensor/actuators [302]. An eigensensitivity analysis was employed to analyze a transcendental eigenvalue problem arising in the dynamics of a bridge deck subjected to aerodynamic forces [303]. A fully discrete formalism was introduced to perform stability analysis of a turbulent compressible flow whose dynamics were modeled with the Reynolds-Averaged Navier–Stokes equations [304]. Based on the theory of complex perturbations, an explicit consideration of the sampling rate and its relevant expressions were discussed for closed loop systems [305].

3.5 Damage Detection and Fault Diagnosis

Structural health monitoring refers to the process of implementing a damage detection and characterization strategy for engineering structures. Damages often occur due to changes in material and geometric properties of a structural system, including changes to the boundary conditions and system connectivity, which adversely affect the system's performance. Measurements are made over time using periodically sampled responses from an array of sensors to extract damage-sensitive features and statistical analysis to determine the current state of system health. Quantitative methods such as eigensensitivity based damage detection and fault diagnosis have long been used to evaluate structures for their capacity to serve their intended purpose [306].

A concept of selective sensitivity was applied to the adaptive diagnosis of the stiffness abnormality based on hierarchical halving, whereby a sequence of forces were applied in such a way as to efficiently home in on the failure location [307]. Modal characteristics extracted from vibration tests were used together with finite element model to detect damages based on eigensensitivity and multiple-constraint matrix adjustment [308]. A genetic algorithm was applied to the problem of damage detection using vibration data in which eigensensitivity method was used to estimate the extent of the imminent damage [309]. Eigenparameters were used to study the sensitivity of a system to perturbations, due perhaps to damage incurred by one or more discrete elements [310]. A structural modeling methodology was proposed based on the concept of Damage-Detection-Orientated-Modeling, using modal sensitivity of the structural model to physical changes at the sub-element level [311]. Also, inverse eigensensitivity method was applied to the damage identification of a full-scale, 2-storey steel-concrete frame structure characterized by partial-strength beam-to-column joints, designed to localize plastic phenomena in end-plate connections and in column shear panels [312]. Further, a two-stage eigensensitivity-based procedure was developed for structural damage detection for the IASC-ASCE structural health monitoring benchmark steel structure on the basis of ambient vibration measurements [313]. In order to diagnose the location and the extent of damage in steel braced space frame structures, a two-stage damage diagnosis approach was proposed using eigensensitivity analysis [314]. Modal parameters were obtained by employing the natural excitation technique in conjunction with the eigensystem realization algorithm from acceleration responses and were used for damage assessment [315]. Closed form of the sensitivity of modal flexibility was derived based on the algebraic eigensensitivity method and was employed to effectively identify single and multiple damage events occurring in structural systems [316].

3.6 Mistuned Bladed Disk Vibration Problems

Accurate and effective vibration modeling and characterization of bladed disk assemblies have been and remain as one of the central issues of gas turbine engine design and development. Small variations of vibration characteristics of individual blades arise in practice due to manufacturing tolerances, in-service wear, as well as lack of material homogeneity, leading to inevitable mistuning of practical bladed disk assemblies. It is well known that the dynamic response of a mistuned bladed disk can become significantly different from that of its tuned counterpart so that each individual blade may experience large stress variations within the same assembly [317]. As a result, our ability to predict vibration responses of blades given known mistuning patterns within a disk assembly becomes paramount in assessing accurately and reliably the integrity and fatigue life of engines as well as possibly controlling the mistuning effect to enhance engine performance. To this end, many methods have been developed and we only examine how eigensensitivity analysis can be employed to predict the effects of mistuning. Due to the cyclic symmetry, bladed disk structures possess many doubly repeated eigenvalues and before eigenderivatives can be applied to the prediction of mistuning effects, the concept of global design variables needs to be established.

As discussed in Section 2, for repeated eigenvalues, the eigenvector space becomes degenerate, leading to generally different differentiable eigenvectors with respect to different design variables. To illustrate this important point, a simple discrete model of a bladed disk shown in Fig. 4 with 36 blades is considered. The differentiable eigenvectors for the first pair of doubly repeated non-rigid body modes (modes 2 and 3) with respect to the stiffness of blade 1 and that of blade 5 are shown in Fig. 5, indicating that they are very different [318].

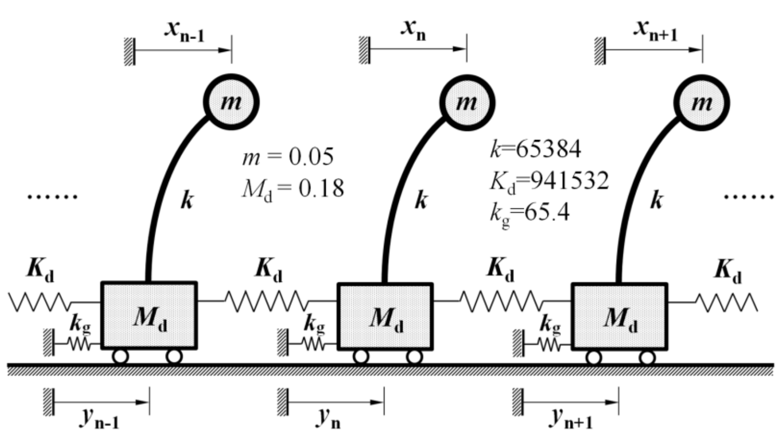

Fig. 4 – Discrete model of a bladed disk with trolleys connected in a ring

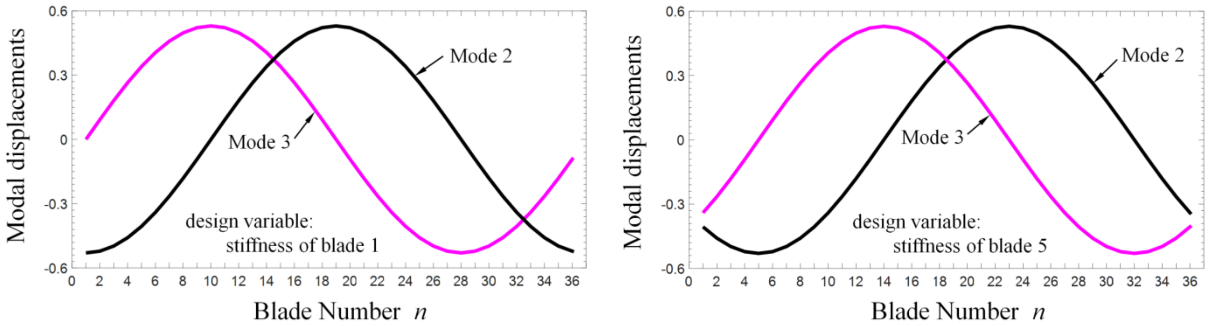

Figure 5 – Different differentiable eigenvectors with respect to different design variables

Such non-uniqueness of differentiable eigenvectors causes difficulties in the direct application of eigensensitivity method and requires the introduction of a global design variable which groups all the original design variables as,

$$\mathbf{p} = \mathbf{p}_0 + \sigma \Delta \mathbf{p} \tag{68}$$

A value of $\sigma = 0$ corresponds to the current design and $\sigma = 1$ represents the modified design whose vibration properties need to be predicted. The required differentiations of system matrices become,

$$\mathbf{K}' \equiv \frac{d\mathbf{K}}{d\sigma} = \frac{\partial \mathbf{K}}{\partial \mathbf{p}} \frac{d\mathbf{p}}{d\sigma} = \frac{\partial \mathbf{K}}{\partial \mathbf{p}} \Delta \mathbf{p} = \sum_{k=1}^{n} \frac{\partial \mathbf{K}}{\partial p_k} \Delta p_k \tag{69a}$$

$$\mathbf{M}' \equiv \frac{d\mathbf{M}}{d\sigma} = \sum_{k=1}^{n} \frac{\partial \mathbf{M}}{\partial p_k} \Delta p_k \, . \tag{69b}$$

$$\mathbf{K}'' \equiv \frac{d^2\mathbf{K}}{d\sigma^2} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \mathbf{K}}{\partial p_j \partial p_k} \Delta p_j \Delta p_k \tag{69c}$$

$$\mathbf{M}'' \equiv \frac{d^2\mathbf{M}}{d\sigma^2} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \mathbf{M}}{\partial p_j \partial p_k} \Delta p_j \Delta p_k \tag{69d}$$

The eigenvalue and eigenvector derivatives with respect to σ for repeated modes can now be computed. The predicted eigenvalues and eigenvectors of the modified structure at $\sigma = 1$ ($\Delta\sigma = 1$) become,

$$
\begin{cases}
\lambda_i(\mathbf{p}) \;=\; \lambda_i(\mathbf{p}_0) \;+ \lambda_i' \,\Delta\sigma \;=\; \lambda_i(\mathbf{p}_0) \;+\; \lambda' \\
\mathbf{Z}_i(\mathbf{p}) \;=\; \mathbf{Z}_i(\mathbf{p}_0) \;+\; \mathbf{Z}_i'\Delta\sigma \;=\; \mathbf{Z}_i(\mathbf{p}_0) \;+\; \mathbf{Z}_i'
\end{cases}
\tag{70}
$$

For a given pattern of blade mistuning (variations in blade cantilever frequencies), the predicted eigenvectors of the mistuned assembly become very accurate, as shown in Fig. 6.

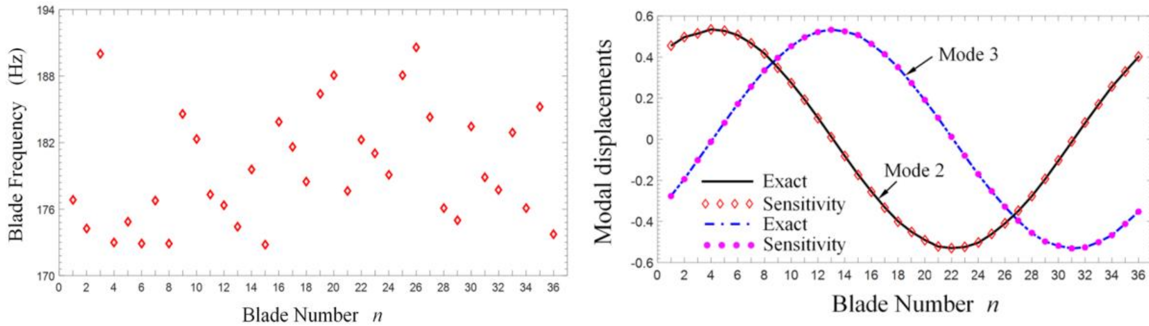

Fig. 6 – Blade mistuning and prediction of mistuning effects [317]

Alongside these studies, an approach was presented here that was both accurate and efficient for calculating modal sensitivities by formulating a modal expansion method for blisk models [319]. Sensitivity analysis of vibration response amplitude to parameters of nonlinear friction dampers [320], as well to the maximum forced response levels for mistuned bladed disks [321] were investigated.

4. Eigenderivatives and Model Reductions

In practical structural designs, full scale FE models often need to be reduced for computational reasons. There are many ways a model can be reduced and eigensensitivity analysis can be performed based on the reduced model. There are however existing issues associated with model reduction and one of the most difficult is the spread of local stiffness changes in the reduced model. Take the Guyan reduction [322] for example, if we introduce localized stiffness error in a single element between nodes 35 and 36, as shown in Fig. 7, by overestimating the cross-sectional area of the element by 100%, the resulting stiffness modeling error will be confined within the 6 coordinates associated with the 2 nodes of the element in the full FE model. Upon Guyan reduction however, the errors spread almost all over the reduced model, as shown in Fig. 8, [323]. Such characteristics of model reductions have serious ramifications as far as model updating and structural modification prediction based on reduced model are concerned. It becomes much more difficult, if not impossible, to obtain accurate updated models or accurate predictions of structural modifications when reduced models are employed.

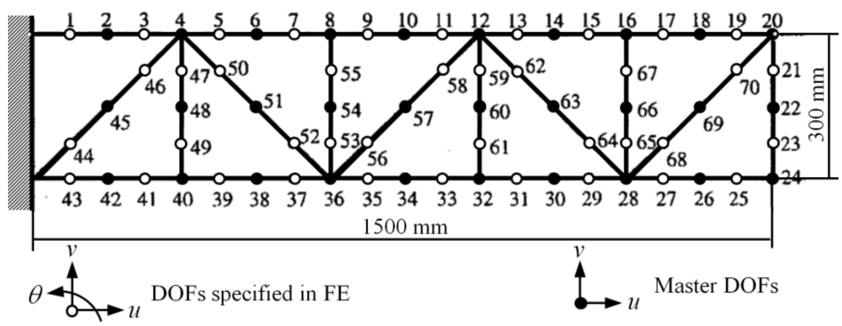

Fig. 7 – Finite element model of a truss structure

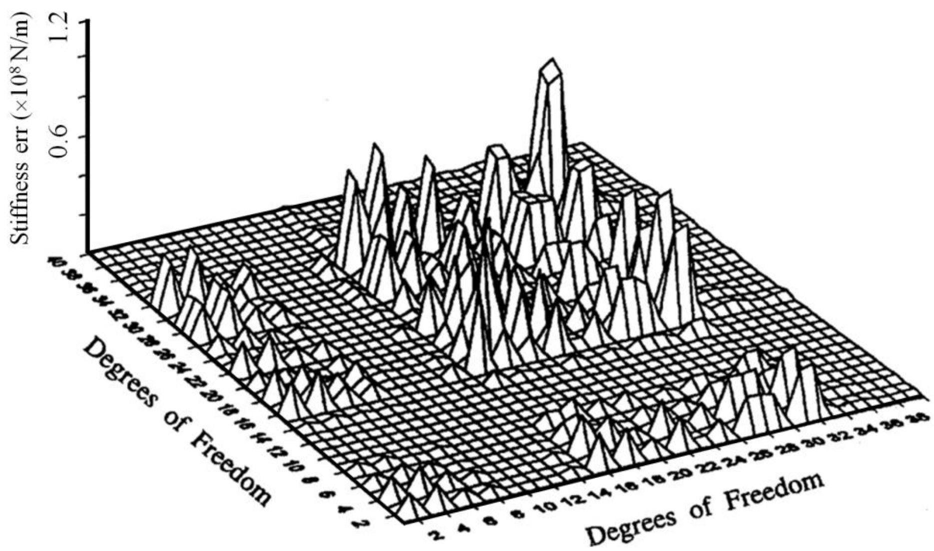

Fig. 8 – Spreading of stiffness errors in model reduction [300]

However, there are other niche applications where eigensensitivities of reduced models are desired. Based on sub-structural coupling, Kron's eigenvalue procedure was found to be more attractive in design applications in which eigenvalue and vector sensitivity formulae were developed by using the properties of the "intersection" matrix only [324-325]. Expressions for eigenvalue and eigenvector sensitivities were derived with respect to the singular perturbation parameter whose variation changes the order of the system [326]. Sensitivity of the closed-loop response to structural changes was calculated for a multi-span beam with direct-rate feedback using co-located velocity sensors and point actuators based on reduced models [327]. A method was presented for generating mode shapes for model order reduction in a way that led to accurate calculation of eigenvalue derivatives and eigenvalues for a class of control augmented structures [328]. It was shown that for some control laws, only right eigenvectors were required if derivatives were obtained from the second order form of the equations of motion by examining reduced basis approximations for the eigenvalues and their derivatives [329]. By making use of a set of theorems and definitions, an efficient procedure was proposed for eigensensitivity based on physical model reduction [330]. Further, Kron's substructuring method to compute the first-order derivatives of the eigenvalues and eigenvectors with respect to structural parameters was examined [331]. An iterative substructuring method was proposed to accurately obtain the eigensolutions and eigensensitivities of structures which converged to their exact values of the original structure [332]. Similarly, based on substructuring, a method for computing the first, second and higher order eigensensitivities was developed in which eigensensitivity of global structure with respect to a design parameter was calculated from the eigensensitivity of a particular substructure that contained the design parameter [333]. Also, based on dynamic condensation technique, eigensensitivities of a structure were computed by iteratively updating the derivatives of the condensed system matrices, together with a transformation matrix [334].

5. Eigenderivatives for Nonlinear Vibration Systems

Dynamic behavior of general nonlinear vibration systems can become extremely complex in theory depending the nature of nonlinearities and system inputs, encompassing the whole spectrum from periodic behavior to period doubling and bifurcation, to internal resonances and unpredictable chaos [335]. In practice however, most engineering structural systems, albeit nonlinear, are operating at relatively low vibration amplitudes and as a result, their nonlinear dynamic behavior can be closely modeled as a perturbation to their linear counterparts [336]. It is under this understanding that it becomes meaningful to define the eigenvalue problem and its associated eigenderivative analysis for nonlinear vibration systems. One of the general observations of practical nonlinear structural systems is that the system stiffness and/or damping matrices vary with the actual amplitudes of vibrations. Such a dependency on vibration amplitude is not difficult to comprehend since the nonlinear stiffness and/or damping elements involved will show different effective stiffness and/or damping values under different vibration levels. As a result, nonlinear vibration systems possess in general eigenvalues and eigenvectors that depend upon vibration amplitudes. For a given amplitude, the system is said to be linearized at the said vibration amplitude and the system matrices corresponding to that particular vibration amplitude can be established from which an eigenvalue problem can be defined. For different vibration amplitudes however, these system matrices are different, leading to different eigenvalues and eigenvectors. Mathematically, such an approach seeks to approximate a nonlinear system by a series of linearized systems corresponding to different vibration amplitudes. Given the physical characteristics of structural nonlinearities involved, the analytical forms of amplitude dependency of stiffness and/or damping matrices can be established using describing functions [337] by considering the fundamental frequency component present in the response spectrum. The general eigenvalue problem of a nonlinear system can then be defined as,

$$\mathbf{A}(\lambda, \hat{\mathbf{X}})\,\mathbf{X} = \lambda\,\mathbf{B}(\lambda, \hat{\mathbf{X}})\,\mathbf{X} \tag{71}$$

in which $\hat{\mathbf{X}}$ is the vibration amplitudes which is related to the eigenvector $\mathbf{X}$ of the mode to be analyzed by some pre-defined relationship. To solve such eigenvalue problem of a nonlinear system, iterations are required until the eigenvalues and eigenvectors of interest converge. Despite its practical significance, eigenvalue problems and their derivatives of nonlinear dynamical systems have not been fully developed to date due perhaps to the existing difficulties in analytically analyzing and experimentally characterizing practical nonlinear systems with desired accuracy. A beam with immovable ends under large amplitude oscillations subjected to geometrical nonlinearity was considered and a computational scheme using the finite element method was developed to calculate the design sensitivities of the eigenvalues and eigenvectors [338]. On the other hand, for nonlinear flutter and divergence problems, a procedure was presented for the determination of eigenvalue and eigenvector derivatives [339]. Further, a continuum formulation for design sensitivity analysis of critical loads was developed for nonlinear structural systems that were subjected to conservative loading with both geometric and material nonlinearities [340].

Furthermore, in truly nonlinear systems the concept of nonlinear normal modes (NNMs) has been studied intensively in recent years - a helpful introductory text was provided by Kerschen et al [341]. Such modes may or may not develop straightforwardly from linear normal modes and may arise as a result of internal resonance phenomena. Like linear normal modes, NNMs are characterised by synchronous motion so that all the degrees of freedom reach maximum values at the same time and pass through zero together. If an initial condition is on the manifold of a NNM, then the motion will remain on the same manifold and other NNMs will not be excited. NNMs differ from linear normal modes in that they depend not only on frequency, but also on the total energy of the system in motion. This means the linear principle of othogonality no longer applies. It is usual to illustrate NNM behaviour in a frequency energy plot (FEP) that may be generated numerically from an analytical model by using continuation methods. Kurt *et al.* [342] considered the model updating of a simple bi-linear beam with a softening stiffness characteristic using NNMs. Thus, there were two stiffness parameters and a switching parameter to be estimated by either local or global optimisation based upon the experimental backbone curve from the FEP and an initial model – both approaches resulted in almost identical results even under high energy conditions. The initial model of the first (low-energy) stiffness term was obtained using the conventional eigensensitivity approach. The generation of analytical backbone-curve models for generation of the updating residual was achieved by using continuation methods. However, despite its practical significance of NNMs, detailed review of the subject is outside the scope of the present review.

Finally, it is also perhaps worth mentioning that the eigenvalue problem of a nonlinear system is a totally different concept to the nonlinear (or polynomial) eigenvalue problem. A nonlinear eigenvalue problem is generally characterized by the frequency λ dependency of system matrices. The damped vibration systems and fractional vibration systems discussed earlier are examples of linear vibration systems with nonlinear eigenvalue problems. For analysis of eigenderivatives of nonlinear eigenvalue problems, numerous research works have been accomplished [343]. General *n*-th derivatives of eigenvalues and eigenvectors were derived [344]. Frequency dependent nonlinear eigenvalue problems were considered and numerical computations of partial derivatives of eigenvalues and eigenvectors were presented [345]. An explicit formula for the first-order derivatives of distinct eigenvalues was derived for nonlinear eigenproblems and computational methods for the derivatives of eigenvectors were proposed [346]. For nonlinear eigenvalue problem of fractional vibration systems, a procedure was proposed to obtain closed-form solutions of eigenvalues and eigenvectors and their associated derivatives [347]. A general nonlinear eigenproblem was discussed in which the widely used undamped, viscously or nonviscously damped eigenproblems were considered as special cases of the more general nonlinear eigenproblem [348].

6. Future Research Challenges

While much progress has been made on the theory and engineering applications of eigenderivatives to date as discussed herein, many challenges still remain. These include (i) development of effective methods for eigenvalue and eigensensitivity analysis of nonlinear structural system, (ii) sensitivity analysis of nonlinear normal modes (NNM), (iii) unified approach for damping modeling and identification using eigensensitivity analysis, (iv) inclusion of higher-order eigensensitivityies to further improve numerical accuracy, (v) modeling error location and model updating using macro element based inverse eigensensitivities, (vi) model updating and system identification of nonlinear structural systems and, (vii) inverse eigensensitivity for system identification based on reduced analytical models.

Eigensensitivities of a nonlinear structural system with respect to operating deflections will provide valuable quantification on how natural frequencies and modeshapes change with operating conditions. Such characteristics are believed to be vitally important in the modeling and identification, as well as damage detection and fault diagnosis of practical nonlinear structural systems whose nonlinearities are found to be strong and are often related to the state of structural health. Further, eigensensitivities with respect to nonlinear structural elements can be utilized to improve structural stability analysis such as flutter and limit cycle behavior of nonlinear aeroelastic structural systems in which nonlinearities often lead to steady limit cycle oscillations, thereby avoiding flutter failures. Commonly encountered structural nonlinearities such as the smooth polynomial stiffness and damping nonlinearities, as well as the non-smooth nonlinearities (friction, piece-wise linear stiffness, freeplay and hysteresis) need to be carefully examined and the eigenderivatives with respect to nonlinear system parameters need to be established which will become important not only to the designs of such nonlinear systems, but also to their accurate modeling and identifications.

Further research on nonlinear normal modes (NNMs) is required. Present methods are restricted to systems with a small number of nonlinearities spread over just a few degrees of freedom. The work of Kurt *et al.* [342] indicated that local optimization methods were effective even under high energy conditions, although some caution should probably be exercised. Sombroek *et al.* [349] addressed the problem of high computational expense by developing reduced order models that included model derivatives. Further research in this area is clearly needed if NNMs are to become a practical tool in modern industry.

Various damping models such as viscous, non-viscous and fractional damping are used in practical vibration analysis. Though methods have been developed to deal with specific damping models, in practice however, damping can appear be any combinations of these and as a result, a more general and unified approach becomes necessary. Structural materials with better damping capacities are increasingly being deployed and the accurate derivation of eigensensitivities becomes ever so important in stability analysis, vibration response prediction, damping design and identification of damping using vibration tests.

Most current eigensensitivity analyses are restricted to only first order approximations. Though they are adequate for many practical applications, they nevertheless lead to numerical problems such divergence in FE model updating as the magnitudes of modeling errors become substantial, and inaccuracy in predicting vibrations when large modifications are made. To further improve these applications, higher order eigenderivatives are required to be included in the formulation so that it can better handle large modeling errors as well as large structural modifications.

Considering the practical limitations of measured modal data, model updating problems formulated based on inverse eigensensitivity are generally underdetermined. As a result, the location of localized modeling errors in the overall analytical model needs to be pinpointed first in order to reduce the number of unknowns involved to possibly turn the problem to become deterministic. To this end, eigensensitivities with respect to macro elements (a group of elements) need to be developed for the purpose of error locations. This approach needs to be applied to different levels of aggregate elements until successful error locations are achieved. Final model updating can then be made using inverse eigensensitivity based on the location information modeling errors that need to be corrected.

Model updating of nonlinear structural systems has been gaining momentum over recent years. As one of the major model updating methods, inverse eigensensitivity method needs to be further developed so that it can be applied to model updating of nonlinear structural systems. Model updating in this case has to address two main issues: to estimate the modeling errors and to identify changes in stiffness (or damping) properties due to nonlinearities. The key to success will be the ability to progressively update the eigensensitvity coefficients at each stage of updating and for each measured data at a given vibration amplitude. Since both modeling errors and stiffness/damping changes due to nonlinearities are present, the starting analytical model could be quite far from the target updated models and to ensure convergence, it perhaps becomes necessary to include higher order eigensenstivities in the improved formulation of inverse eigensensitivity method.

Finally, when in practice reduced models are sought and need to be updated, eigensensitivity analysis based on reduced models must be formulated. Though several methods have been developed to compute eigensensivities of reduced models, applications of these sensitivities to model updating and structural modification may not be straightforward due to the simple fact that localized modeling errors/modifications tend to spread upon model reduction. Additional research is required on how to use these eigensensitivities from the reduced model to reformulate the methods which can then be applied to model updating and modification prediction of reduced model.

7. Conclusions

In this paper, we have sought to perform a comprehensive review on the state-of-the-art research of eigenvalue and eigenvector derivatives and their important engineering applications. Major existing methods for different eigenvalue problems have been introduced and theoretically discussed, together with their inherent advantages and disadvantages. The underlying mathematical relationships between these methods have been examined and new theoretical developments have been made to alleviate existing difficulties and to enhance performance. Important practical applications of eigensensitivity to analytical model updating, structural design, performance optimization of structures and systems, optimal control system design, damage detection and fault diagnosis, as well as bladed disk vibrations have been discussed and major existing methods to each of these applications critically reviewed. Key application requirements have been identified and discussed, and new theoretical development needed has been introduced. Furthermore, to illustrate key theoretical concepts and major practical applications, numerical case studies have been included. Potential future research challenges have been identified with a view to pooling further research efforts in the area in a more fruitful direction.

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