Classification of Singularities of Functions and Mappings Via Non Standard Equivalence Relations

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Contents

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Abstract

Classification of Singularities of Functions and Mappings

Via Non Standard Equivalence Relations

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The thesis is devoted to the classification of simple germs of functions and mappings with respect to several new non-standard equivalence relations. They are more rough than the standard classifications of functions via the group of the diffeomorphisms preserving a certain variety or respectively the group of the diffeomorphisms preserving given projection. The goal is to show some useful applications and various interesting properties of them.

We obtain the list of all simple "in the sense of Arnold" classes of singularities of function germs with respect to non standard equivalence relations and discuss their relation with the singularities of Lagrangian projections with borders. Also, we describe the bifurcation diagrams and caustics of simple quasi boundary and quasi corner singularities and algebraic invariants of simple quasi border classes. In addition, we classify all simple classes with respect to quasi projection of graphs of germ mappings from the plane to the plane and graphs of parametrized curves in the plane.

Introduction

The famous results of singularity theory and its applications due to R. Thom, V.I. Arnold and others are based on the classification of smooth functions and mappings with respect to the action of the group of diffeomorphisms of the source space (right equivalence) or the group of diffeomorphisms of the source and the target (left-right equivalence). This is an example of infinite dimensional Lie group acting on an infinite dimensional space. The germs of diffeomorphisms act on germs of functions or on germs of mappings.

The very fruitful idea of V.I. Arnold was to consider so called simple singularities (the orbits which have a neighborhood intersecting with finitely many other orbits only). Simple classes have nice algebraic and topological properties. For example, in the case of functions, they are related to A_k, D_k and E_k Weyl group. The complement to the collection of simple classes has codimension 7 in the space of germs of functions. Starting with 7, some orbits form families depending on modules (continuous invariants) and are non-simple.

On the other hand, there is a classification of functions and mappings with respect to Thom-Boardman classes $\sum^{i_1,\dots,i_k}$. These classes are not the orbits of any group action and they are discrete. No continuous invariants are involved. Thom-Boardman classes represent germs with given values of some invariants like ranks of differentials and dimensions of quotient spaces of some ideals generated by derivatives of the mappings. One can say that Thom-Boardman classification is rougher than the standard right or left right classification since germs from the same standard orbit belong to the same Thom-Boardman class.

The idea of the present work is to investigate other possible useful examples of classifications of functions and mappings which are rougher than the standard ones and which are defined by some conditions on jets of functions or mappings.

We consider two settings. In chapters 1-7 we consider equivalence relations which play an intermediate role between the right action of all diffeomorphisms and the action of diffeomorphisms which preserve a given hypersurface in the source space (called border).

Then in chapters 8-11 we consider projections of submanifolds embedded into $\mathbb{R}^n$ to the base $\mathbb{R}^k$ and introduce special equivalence for them.

So, in **chapters 1-7** we consider the space $\mathbb{R}^n$ with some fixed hypersurface Γ (which can be regular, singular and reducible). The hypersurface will be called *a border.* We consider germs of smooth functions on this space and introduce the following basic definition.

Two function germs are pseudo border equivalent, if there is a diffeomorphism acting as a change of variables, taking one germ to the other and satisfying the following condition: if one of these functions has a critical point at the border then its image (or, respectively, the inverse image) also belongs to the border. After a natural modification, this equivalence relation behaves well when functions depend on parameters. The modified definition is called quasi border equivalence.

We use four different examples of borders: smooth border(called boundary), corner, cusp and cone. The union of two transversal intersecting smooth hypersurfaces is called a corner. In these cases, we classify discrete (simple) equivalence classes and describe corresponding bifurcation diagrams and caustics.

In spite of rather artificial nature of the definitions, quasi border singularities have very natural applications. Their discriminants show the behavior of critical points of a function (for example, its global extremum) inside and on certain domain with a border.

Moreover the above mentioned notion of non-standard equivalence relations have **direct application** in symplectic geometry. They are used in classifying singularities of Lagrangian projections with a border [9, 36].

Arnold's classical boundary singularities of functions depending on n parameters are related to projections of a pair of Lagrangian submanifolds (of dimension n) which have $(n-1)$-dimensional regular intersection and are transversal in complementary directions [3]. Our quasi-equivalence relation (introduced for generating families of functions) keeps information only about one Lagrangian submanifold of the pair and about its intersection with the second component. So, it is more adequate model for applications when we need the precise notion of Lagrangian submanifold with boundary.

Thus, a pair (L, Γ) of a Lagrangian submanifold $L^n \subset M = T^*\mathbb{R}^n$ and an $(n-1)$-dimensional isotropic variety $\Gamma \subset L$ is called a *Lagrangian submanifold with a border.* It arises in various singularity theory applications to differential equations and variational problems [8]. Isotropic submanifolds play the role of the initial data set with some inequality constraints.

An important **example** of a Lagrangian submanifold with a regular boundary or a corner is presented by a set of Hamilton vector field trajectories issued from an initial set being an isotropic submanifold subset determined by some inequalities. This construction is needed for various setting in geometry and physics. For example, given an initial hypersurface H with a boundary H_1 in Euclidean space, the envelope of the family of normals to H forms the ordinary caustic and the union of normals to H at the points of H_1 forms the second component of the caustic of the projection of the respective Lagrange submanifold with a boundary. Other motivations to study singularities of Lagrange projections with boundaries are mentioned in [20].

More complicated borders appear in various applications in physics. For example, the Lagrangian manifold with a corner is the solution of Hamilton-Jacobi equation with the initial data embedded into the cotangent bundle of the configuration space as a manifold with boundary or corner [3].

We show that the singularities of the projection to the base space of Lagrangian submanifolds with border are closely related to quasi border singularities of functions. In particular, the list of simple stable classes of these projections is exactly the list of simple quasi border classes.

So, as an **application** of our theory we get the classification of simple classes of Lagrangian projections with boundary or corner. We describe the bifurcation diagrams and caustics of simple quasi boundary and corner singularities.

The quasi-border bifurcation diagrams of function germ deformation consist of two strata. The first one is the ordinary discriminant which corresponds to all critical points of the deformation. The second stratum is the subset of the first one which corresponds to the critical points on the border (it satisfies extra equations which define the border). So the strata have different dimensions. The caustics of quasi border deformation functions also consist of two strata (or more). The first one is the ordinary caustic while the other stratum is the projection (to the base of the reduced deformation) of the subset of the bifurcation diagram which corresponds to the other stratum(of lower dimension). However, their dimensions are equal.

There are series of papers (e.g. [37, 20, 33]) on the classification of caustics and Lagrangian projections of different types. Some of our results coincide with the known ones but our methods are new and universal and worth to be compared with the other approaches.

Standard classification of singularities of functions up to diffeomorphisms which preserve a distinguished hypersurface (boundary singularities) is closely related to singularities of functions invariant under reflection. Similarly, a classification of functions which are invariant under reflections in two transversal hypersurfaces give rise to the classification of function germs with respect to diffeomorphisms which preserve the corner (union of two transversal hyperplanes). The list of simple and unimodal boundary and corner singularities was obtained by Dirk Siersma [29] in 70ths. Later the unimodal and bimodal corner singularities were listed in [21].

Comparing with standard corner singularities, obtained by Dirk Siersma (his list starts with unimodal singularity), we see that all Siersma's singularities become simple with respect to quasi corner equivalence relation.

The lists of simple quasi boundary and corner classes are clearly organized. Dropping the boundary, or the corner, any simple class belongs to some A_k- right equivalence class. We get the nice algebraic description of all simple classes. In fact, each simple class corresponds to a pair, consisting of a local algebra of A_k type and an ideal in it.

In chapters 8-11 we turn to the study of projections of submanifolds.

Recall that the starting point of singularity theory in the middle of the 20th century was the classical Whitney theorem stating that generic singularities of a generic projection of a two-dimensional surface in $\mathbb{R}^3$ onto a plane are fold or a pleat [1].

However, choosing the direction of the projection in a special way, one can obtain non-standard projections of a generic surface. In a more general context, the singularities of projections were studied later by D. Shaffer, V.Arnold [7] and V.Goryunov [18] as orbits in the space of germs of complete intersections embedded into a given bundle space of the action of the diffeomorphisms which preserve the bundle structure [6].

The classification of singularities of projections of a two-surface embedded into RP^3 to a plane was a nice generalization of Whitney theorem. The surface is assumed to be generic, and centre of projection can vary in RP^3. The famous hierarchy of germs of projections of a surface according to calculations of O.A. Platonova [27], V. Arnold [7], O.P.Shcherbak [30] is as follows [6]:

$$
\begin{array}{lll}
P_1 : f = x, & P_2 : f = x^2, & P_3 : f = x^3 + xy, \\
P_4 : f = x^3 \pm xy^2, & P_5 : f = x^3 + xy^3, & P_6 : f = x^4 + xy, \\
P_7 : f = x^4 \pm x^3y + xy, & P_8 : f = x^5 \pm x^3y + xy, & P_9 : f = x^3 \pm xy^4, \\
P_{10} : f = x^4 + x^2y + xy^3, & P_{11} : f = x^5 + xy, &
\end{array}
$$

where,

$$
\begin{array}{ccccccccc}
P_1 & \leftarrow & P_2 & \leftarrow & P_3 & \leftarrow & P_6 & \leftarrow & P_8 \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & P_4 & \leftarrow & P_7 & & P_{11} \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & P_5 & \leftarrow & P_{10} & & \\
 & & & & \uparrow & & & & \\
 & & & & P_9 & & & &
\end{array}
$$

Some of these classes are non weighted homogeneous however all are simple. The equivalence here is the diffeomorphism of the domain of the ambient space containing the germ of the surface and not containing the center of projection. The diffeomorphism preserves the fibration over the two dimensional plane base of the projection.

This classification was used later by many authors in various applications in differential and algebraic geometry [2].

In the 80 th, the singularities of projections of surfaces with boundaries were studied and classified by J.Bruce, P.Giblin [13] and V.Goryunov [19].

Giblin and Bruce [13] considered the classifications of singularities when a generic smooth surface in three space with a boundary is projected along a parallel beam of rays to a plane. The low codimensional normal forms written as projections $(x, y, z) \mapsto (y, z)$ of a graph $z = f(x, y)$ with the boundary $z = f(x, y), g(x, y) = 0$ are as follows:

$$f = x, g = x; \quad f = x^2 + xy, g = x; \quad f = x^3 + xy, \ g = x;$$
$$f = \pm x^6 + x^4 + xy, \ g = x; \quad f = \pm xy^2 + x^2, g = x; \quad f = x^2 + y^3 x, \ g = x;$$
$$f = xy^2 + x^2 y + ax^3 + x^4, \ g = x; \quad f = x^2, \ g = y + x^3; \quad f = x^2, \ g = y + x^5;$$
$$f = xy + ax^3 \pm x^5, \ g = y \pm x^2.$$

In the paper [38], another idea of a non-standard equivalence relation related to projections of submanifolds was introduced. Namely two surfaces are called pseudo-equivalent if there is a diffeomorphism of the domain of the ambient space mapping one sumanifold onto the other and satisfying the following property: if the projection ray is tangent to one of the sumanifold at a point then at the image (or at the inverse image, respectively) of the point the other surface is also tangent to the ray passing through it.

After a modification of this equivalence to get better properties with respect to parameter dependence in J. Damon's sense [16], the following list of generic quasi-singularities of projections $Q_i, i = 1, \ldots, 9$, of surfaces embedded in $\mathbb{R}^3$ is obtained in [38]:

$$\begin{array}{ccccccccc}
Q_1 & \leftarrow & Q_2 & \leftarrow & Q_3 & \leftarrow & Q_6 & \leftarrow & Q_8 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & Q_4 & \leftarrow & Q_7 & & \\
 & & & & \uparrow & & & & \\
 & & & & Q_5 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & Q_9 & & & &
\end{array}$$

where

$$\begin{array}{lll}
Q_1 : f = x, & Q_2 : f = x^2, & Q_3 : f = x^3 + xy, \\
Q_4 : f = x^3 \pm xy^2, & Q_5 : f = x^3 + xy^3, & Q_6 : f = x^4 + xy, \\
Q_7 : f = x^4 + x^2 y, & Q_8 : f = x^5 + xy, & Q_9 : f = x^3 \pm xy^4.
\end{array}$$

Comparing these relations, P_8 and P_{11} merge into the single class Q_8, while P_7 and P_{10} merge into Q_7. All remaining Q classes coincide with respective P classes with equal subscripts.

In chapter 9, we give the details of the previous construction and state all results from [38]. We classify simple classes of quasi projections of surfaces to the plane embedded in three space. The results in that paper are outlines, so we aslo give the complete proofs as we will use the new idea and results to develop similar constructions.

In chapter 10 we introduce similar definition which holds for surfaces with boundaries (that is curves embedded into surfaces): additionally we require that the diffeomorphisms of the ambient space send boundary to the boundary. Again, we modify this equivalence to get a better equivalence which behaves regularly when the function defining the surface depends on extra parameters. The improved equivalence relation is called quasi projection equivalence. We distinguish two different notions of quasi projection equivalence of surfaces with boundaries: the strong and weak equivalences.

Finally, in chapter 11, by similar ideas, we classify simple singularities with respect to quasi projections of parametrized curves $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ and quasi projections of mappings $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

Our main results are:

1. Theorems [3.1.6],[4.1.6],[5.1.5],[6.1.4] and [6.1.8] on the classification of simple quasi corner, quasi cusp, quasi cone, quasi complete flag and quasi non-complete flag singularities, respectively.

2. Propositions [3.4.2] and [6.2.1] on description of bifurcation diagrams and caustics of quasi corner and quasi complete flags singularities, respectively.

3. Propositions [7.2.1] and [7.2.2] on algebraic invariants of simple quasi border classes.

4. Theorem [10.2.4] on classification of simple quasi projections of surfaces embedded into three-space with boundaries.

5. Theorem [11.1.2] on classification of quasi projections of graphs of parameterized curve germs $F : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$.

6. Theorems [11.2.2] and [11.2.3] on classification of simple quasi projections of graphs of germ mappings $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

The brief description of the main results is given in the conclusion of the thesis.

We hope that exploring similar non-standard equivalence relations in other settings will help to better understand the geometry beyond standard simple classes in various singularity theory problems and applications.

The main technique which is used in the classification is the standard Moser's homotopy method. Also, we use an adopted version of Arnold's spectral sequence method [1]. We prove Lemma 1.3.5 which is valid for smooth $\mathbf{C}^\infty$ and is based on Malgrange preparation theorem. This completes Arnold's description which was given for power series only. We apply special criterions and introduce prenormal forms of function germs to simplify the classification.

The initial approach to the study of such non-standard equivalence relations was given in papers [36, 38] by Vladimir Zakalyukin.

The results of the thesis were published in [9, 10, 11] and presented at the international conference on differential equations and dynamical systems in Suzdal

(Russia), July 2008, and at the first workshop on singularities in generic geometry and applications in Valencia (Spain), April 2009.

Chapter 1

Preliminaries

1.1 Basic concepts in singularity theory

In this section we review the standard notations in singularity theory, which we will use later.

1.1.1 Germs, jets and ideal of finite codimension

Throughout the thesis we consider C^∞ (or smooth) maps, that is maps which has derivatives of all orders.

Definition 1.1.1 Two maps $f, g : \mathbb{R}^n \to \mathbb{R}^p$ are said to be germ equivalent at $a \in \mathbb{R}^n$ if a is in the domain of both and there is a neighbourhood U of a such that the restricitions to U coincide, $f_U = g_U$: that is $\forall x \in U, f(x) = g(x)$.

This relation is an equivalence relation. A map-germ or a function-germ at a point a is an equivalence class of germ equivalent maps. If χ is such an equivalence class then any $f \in \chi$ is called a representative of χ. It will be denoted as $[f]_a$ and written as

$$[f]_a : (\mathbb{R}^n, a) \to \mathbb{R}^p, \quad \text{or} \quad [f]_a : (\mathbb{R}^n, a) \to (\mathbb{R}^p, b),$$

where $b = f(a)$. For simplicty we write this as

$$f : (\mathbb{R}^n, a) \to \mathbb{R}^p, \quad \text{or} \quad f : (\mathbb{R}^n, a) \to (\mathbb{R}^p, b),$$

Denote by $\mathbf{C}_p^n$ the space of smooth map-germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}^p$. If $p = 1$ then we denote by $\mathbf{C}_x = \mathbf{C}_1^n$ the space of all smooth functions-germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, with local coordinates x. Also, denote by $\mathcal{M}_x$ (or $\mathcal{M}_n$) the maximal ideal in the space $\mathbf{C}_x$.

Definition 1.1.2 The k-jet space $J^k(n,p)$ is the vector space of all polynomial maps of degree k from $\mathbb{R}^n$ to $\mathbb{R}^p$.

Definition 1.1.3 Let $f \in \mathbf{C}_p^n$. The k-jet, $j_p^k f$ of f at a point $a \in \mathbb{R}^n$ is the Taylor expansion of f about the point a truncated at degree k.

Definition 1.1.4 An ideal $I \subset \mathbf{C}_x$ is of finite codimension if $\mathbf{C}_x/I$ is a finite dimensional space over $\mathbb{R}$.

This means that there is a finite dimensional real vector subspace V of $\mathbf{C}_w$ such that $\mathbf{C}_x = V + I$, so that any germ $f \in \mathbf{C}_x$ has the form $f = g + h$ where $g \in V$ and $h \in I$.

Proposition 1.1.1 [24] *An ideal I is of finite codimension if and only if there is $r \in \mathbb{N}$ such that $\mathcal{M}_x^r \subset I$.*

1.1.2 The standard Mather groups and tangent spaces

The standard Mather groups which are denoted by $\mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C}$ and $\mathcal{K}$ are defined as follows.

The group $\mathcal{R}$ is defined to be the group of germs of diffeomorphisms $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $\mathcal{L}$ is the group of germs of diffeomorphisms $(\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$, and $\mathcal{A}$ is the direct product $\mathcal{A} = \mathcal{R} \times \mathcal{L}$. The group $\mathcal{C}$ is defined to be the group of germs of diffeomorphisms $(\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ which project to the identity on $\mathbb{R}^n$ and leave locally fixed the subspace $\mathbb{R}^n \times \{0\}$. Thus, if $H \in \mathcal{C}$ then H takes the form:

$$H(x,y) = (x, \widetilde{H}(x,y)), \qquad (*)$$

where $\widetilde{H} : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ and $\widetilde{H}(x, 0) = 0$ for $x \in \mathbb{R}^n$ near zero. The group $\mathcal{K}$ is obtained by replacing (*) by

$$H(x, y) = (g(x), \widetilde{H}(x, y)).$$

Mather groups acts on $\mathbf{C}_p^n$ naturally. All actions will be given below. We start with the following.

Definition 1.1.5 Two function germs $f, g : (\mathbb{R}^n, 0) \to \mathbb{R}$ are right equivalent or $\mathcal{R}-$equivalent if there is a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \phi$.

Definition 1.1.6 Two function germs $f, g : (\mathbb{R}^n, 0) \to \mathbb{R}$ are $\mathcal{R}^+$ equivalent if there is a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $b \in R^n$ such that $f = g \circ \phi + b$.

Let $T(\mathbb{R}^n, 0)$ be the tangent bundle of the germ $(\mathbb{R}^n, 0)$ and $\pi_n : T(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the natural projection.

Definition 1.1.7 A vector field along a map-germ f is a map $v : \mathbb{R}^n \to T(\mathbb{R}^p, 0)$ such that to each $x \in \mathbb{R}^n$ it assigns a vector based at $f(x)$ (so $v(x) \in T_{f(x)}\mathbb{R}^p$).

Such vector fields arise from perturbations of the map f. Let f_t be a one parameter family of maps $f_t : \mathbb{R}^n \to \mathbb{R}^p$ such that $f_0 = f$. Then for each $x \in \mathbb{R}^n$, let

$$v(x) = \left. \frac{df_t(x)}{dt} \right|_{t=0}.$$

Then, $v(x)$ is a vector field along f. Here we identify the tangent space $T_0\mathbb{R}^p$ with the space $\mathbb{R}^p$ itself.

Denote by $\theta(f)$ the set of vector fields along a map-germ f.

Let θ_p denote the module of smooth vector fields on $(\mathbb{R}^p, 0)$ and θ_n denote the module of smooth vector fields on $(\mathbb{R}^n, 0)$.

Example: Let $wf : \theta_p \to \theta(f)$ and $tf : \theta_n \to \theta(f)$ be given as follows: $wf(\eta) = \eta \circ f$ and $tf(\xi) = df \circ \xi$. Then, wf and tf are vector fields along f.

The tangent spaces to the orbits of group actions at a map-germ f are very important notions in singularity theory. For example, they are used to classify map-germs. Aslo, they are used to determine the versality of a map-germ. The notion of versality will be explained below.

Denote by $\mathcal{G}$ one of the groups $\mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{K}$ and $\mathcal{C}$.

Let $\phi_s \in \mathcal{G}$ be a smooth curve in the group acting on $\mathbf{C}_p^n$. Let ϕ_0 be the identity of the group. Then, the set of all tangent vectors

$$\left.\frac{d(f \cdot \phi_s)}{ds}\right|_{s=0}$$

at f to such curves define **the tangent space** to $\mathbf{C}_p^n$ at f. Here $f \cdot \phi_s$ is the orbit under the action of ϕ_s of a given $f \in \mathbf{C}_p^n$.

Now, for the $\mathcal{R}$-equivalence, let $\gamma(s) = f \circ \phi_s$. Then a direct calculation shows that the tangent space to the orbit of the $\mathcal{R}$-equivalence at f takes the form

$$T\mathcal{R}.f = \mathcal{M}.Jf = \left\{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i(x)\right\} = tf(\mathcal{M}_n\theta_n),$$

where $v_i \in \mathbf{C}_w$.

Here, we require that $\phi(0) = 0$ hence $v_i \in \mathcal{M}_x$. However, if we do not require that $\phi(0) = 0$, then we obtain the ideal Jf generated by $\frac{\partial f}{\partial i}$ over $\mathbf{C}_x$.

Definition 1.1.8 The ideal Jf is the extended right tangent space an is denoted by

$$T\mathcal{R}_e.f = Jf = tf(\theta_n).$$

Definition 1.1.9 Two map-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are said to be left equivalent (or $\mathcal{L}$-equivalent) if there is a diffeomorphism germ $\psi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that $g = \psi \circ f$.

The left tangent space is defined to be

$$T\mathcal{L}.f = wf(\mathcal{M}_p\theta_p) = \{\mathbf{w}\,(f) : \mathbf{w} \in \mathcal{M}_p\theta_p\} \subset \mathcal{M}_n\theta(f).$$

Remark: The extended left tangent space is $T\mathcal{L}_e.f = wf(\theta_p)$.

Definition 1.1.10 Two map-germs $F_1, F_2 : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are called right-left or $\mathcal{A}$-equivalent if there exist diffeomorphisms $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\varphi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$, such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^n, 0) & \xrightarrow{F_1} & (\mathbb{R}^p, 0) \\ \downarrow \psi & & \downarrow \varphi \\ (\mathbb{R}^n, 0) & \xrightarrow{F_2} & (\mathbb{R}^p, 0). \end{array}$$

Then, the tangent space to $\mathcal{A}$-orbit of f at f is given by the formula:

$$T_{\mathcal{A}} = wf(\theta_p) + tf(\theta_n).$$

Definition 1.1.11 Two map-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are $\mathcal{K}-$equivalent or contact equivalent if there exists a diffeomorphism $\Psi : (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^p, (0,0))$ of the form $(x, y) \mapsto (\phi(x), \psi(x, y))$ such that

1. $\Psi(\Gamma_0) = \Gamma_0$ and
2. $\Psi(\Gamma_f) = \Gamma_g$,

where Γ_f denotes the graph of f, so Γ_0 is the graph of the zero map, $\Gamma_0 = \mathbb{R}^n \times \{0\}$. More explicitly, these two conditions are equivalent to

1. $\psi(x, 0) = 0$,
2. $g \circ \phi(x) = \psi(x, f(x))$.

Remark: If in the previous definition $\phi(x) = x$, then we say that f and g are $\mathcal{C}$-equivalent.

There is an alternative definition for contact equivalence.

Definition 1.1.12 Two map-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are $\mathcal{K}-$equivalent or contact equivalent if there exist a diffeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a matrix $M \in GL_p(\mathbf{C}_x)$ such that $f \circ \varphi(x) = M(x)g(x)$, where $f(x)$ and $g(x)$ are written as column vectors and $M(x)g(x)$ is the usual product of a matrix times vector.

In the definition above, if φ is the identity, then we get the definition of $\mathcal{C}-$equivalence.

If f and g are $\mathcal{K}-$equivalent (or $\mathcal{C}-$equivalent) then $g(x) = 0 \Leftrightarrow f(\varphi(x)) = 0$, so that $\varphi(g^{-1}(0)) = f^{-1}(0)$. This implies

Proposition 1.1.2 [35] *If f and g are $\mathcal{K}-$equivalent then their zero sets are diffeomorphic.*

The $\mathcal{K}-$ tangent space of a map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ takes the form

$$T\mathcal{K}.f = tf(\mathcal{M}_n B) + I_f\theta(f),$$

where $I_f = \{f_1, \ldots, f_p\}$ is the ideal generated by the components of f, while the $\mathcal{C}-$ tangent space takes the form:

$$T\mathcal{C}.f = I_f\theta(f).$$

The extended $\mathcal{K}-$ tangent space is defined to be

$$T\mathcal{K}_e.f = tf(B) + I_f\theta(f).$$

1.1.3 Versal deformations

Definition 1.1.13 An s-parameter deformation of a map-germ $f_0 : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a map germ

$$F : (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}^p, 0)$$
$$(x, \lambda) \mapsto F(x, \lambda)$$

such that $f_0(x) = F(x, 0)$.

Sometimes we write $F(x, \lambda) = F_\lambda(x)$.

The notion of versal deformation, introduced by G.Tyurina [34, 5], is a very useful concept in many applications of singularity theory.

For shortness, we consider right equivalence case to discuss the versality concept and treat by similar method the versality of $\mathcal{A}$ and $\mathcal{K}$ equivalences.

Let $F : (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}, 0)$ be a deformation of a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. A deformation F' is **right equivalent** to F if

$$F'(x, \lambda) = F(g(x, \lambda), \lambda)),$$

where $g : (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}^n, 0)$ is a smooth germ with $g(x, 0) = x$.

The deformation F' is **induced** from F if

$$F'(x, \lambda') = F(x, \varphi(\lambda')),$$

where $\varphi : (\mathbb{R}^{s'}, 0) \to (\mathbb{R}^s, 0)$ is a smooth germ.

The deformation F of f is said to be **$\mathcal{R}$-versal** if any deformation F' of this germ has a representation in the form

$$F'(x, \lambda') = F(g(x, \lambda'), \varphi(\lambda')), \qquad g(x, 0) = x, \qquad \varphi(0) = 0. \tag{1.1}$$

In the case of $\mathcal{A}$-equivalence relation, we need to replace the relation (1.1) by

$$F'(x,\lambda') = k(F(g(x,\lambda'),\varphi(\lambda')),\lambda'), \quad g(x,0) = x, \quad k(y,0) = y \quad \varphi(0) = 0.$$

Here k is a parameter λ depending family of diffeomorphisms of the target space $\mathbb{R}^p$.

The $\mathcal{K}$-equivalence of two deformations F and F' (of one and the same distinguished germ f) is defined by the condition

$$F'(x,\lambda) = M(x,\lambda)F(g(x,\lambda),\lambda).$$

Here $M(x,\lambda)$ is the germ of invertible matrix $M : \mathbb{R}^m \times \mathbb{R}^s \to GL(\mathbb{R}^n)$.

Therefore a deformation F of a germ f is said to be $\mathcal{K}$-versal if any deformation of this germ can be written in the form in

$$F'(x,\lambda') = M(x,\lambda')F(g(x,\lambda'),\varphi(\lambda')),$$

where M is a parameter depending family of smooth mappings of the source space to the space of non-degenerate $(m \times m)$-matrices whose entries depend on x.

Remark: A versal deformation with least possible number of parameters is known as **a miniversal deformation**.

Definition 1.1.14 Let F be a deformation of f. The **initial velocities** of F are the germs

$$\dot{F} = \left.\frac{\partial F(x,\lambda_1,\ldots,\lambda_l)}{\partial \lambda_i}\right|_{\lambda=0}, \quad i = 1,\ldots,l.$$

Definition 1.1.15 A deformation F of the germ f is said to be *infinitesimally* versal if its initial velocities together with tangent space to the orbit of f generate the whole space of variations of f, that is the space of all germs of mappings.

For each type of equivalence this can be specified as follows.

Theorem 1.1.3 *The conditions for the infinitesimal versality of a deformation F of $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ for right, right-left and contact equivalence consist in the existence for each map germ α (variation of f) of the representation*

$$\alpha(x) = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i}\right) h_i(x) + \sum_{i=1}^{l} c_i \dot{F}_i(x) \qquad (\mathcal{R}\text{-versality;})$$

$$\alpha(x) = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i}\right) h_i(x) + k(f(x)) + \sum_{i=1}^{l} c_i \dot{F}_i(x) \qquad (\mathcal{A} - \text{versality});$$

$$\alpha(x) = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i}\right) h_i(x) + \sum_{j=1}^{n} f_j(x) k_j(x) + \sum_{i=1}^{l} c_i \dot{F}_i(x) \qquad (\mathcal{K}\text{-versality}).$$

Remarks:

1. For all three cases ($\mathcal{R}$- , $\mathcal{A}$- or $\mathcal{K}$-equivalence) a versal deformation is infinitesimally versal.

2. A deformation $F(x, \lambda)$ of a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ $(\lambda = (\lambda_1, \ldots, \lambda_s) \in (\mathbb{R}^s, 0))$ is infinitesimally versal with respect to $\mathcal{R}$-equivalence if the germs $\frac{\partial F(x,\lambda)}{\partial \lambda_i}|_{\lambda=0}$ $(i = 1, \ldots, s)$ generate the local algebra $\mathbf{C}_w / T\mathcal{R}_e.f$ of the germ f as a vector space.

For each of the three cases ($\mathcal{R}$- , $\mathcal{A}$- or $\mathcal{K}$-equivalence) we have the following:

Theorem 1.1.4 [1][Versality Theorem,p151] *An infinitesimally versal deformation is versal.*

1.1.4 Finite determinacy

Finite determinacy is a useful notion in singularity theory when classifying maps. For example, it allows one to study a smooth function germ by replacing it with a polynomial which is right equivalent to it.

Definition 1.1.16 We say that $f \in \mathbf{C}_p^n$ is $k - \mathcal{G}$-determined if any map germ with $j^k g = j^k f$ is $\mathcal{G}$-equivalent to f.

The main theorems on finite determinacy with respect to the equivalences $\mathcal{R}, \mathcal{L}$ and $\mathcal{A}$ are:

Theorem 1.1.5 (Finite determinacy for right equivalence) [35] *Let $f \in \mathbf{C}_w$. If $\mathcal{M}_n^{k+1} \subset \mathcal{M}_w^2 Jf$ then f is $k-\mathcal{R}$-determined.*

Theorem 1.1.6 (Finite determinacy for left equivalence) [35] *If $f \in \mathbf{C}_p^n$ satisfies $\mathcal{M}_n^{k+1}\theta(f) \subset T\mathcal{L}.f$ then it is $(2k+1)-\mathcal{L}$-determined.*

Theorem 1.1.7 (Finite determinacy for right-left equivalence) [35] *If $f \in \mathbf{C}_p^n$ satisfies $\mathcal{M}_n^{k+1}\theta(f) \subset T\mathcal{A}.f$ then it is $(2k+1)-\mathcal{A}$-determined.*

1.1.5 Arnold's simple classes

We recall well known result on classification of simple germs of functions with respect to right equivalence. This result will be used later for classification of germs with respect to the non-standard equivalence relations.

A map $f : \mathbb{R}^n \to \mathbb{R}^p$ has a singularity at a point a if the rank of its differential at a is not maximal.

Definition 1.1.17 *A smooth function $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, is called quasi homogeneous function of degree d with exponents $d_1, \ldots, d_n$ related to the coordinates $x_1, \ldots, x_n$ if $f(\lambda^{d_1}x_1, \ldots, \lambda^{d_n}x_n) = \lambda^d f(x_1, \ldots, x_n)$ for all $\lambda > 0$. The exponents d_i are also called the weights of the variables x_i.*

Assume that $\mathbf{v} = (d_1, \ldots, d_n)$ and $\mathbf{k} = (k_1, \ldots, k_n)$. Then, in terms of the Taylor series $\sum f_{\mathbf{k}} x^{\mathbf{k}}$ of f the quasihomogeneisty condition means that the exponents of the non-zero terms of the series lie in the hyperplane

$$L = \{\mathbf{k} : d_1k_1 + \cdots + d_nk_n = d\}.$$

Definition 1.1.18 *The monomial* $\mathbf{x}^{\mathbf{k}}$ *is said to have degree* d *if* $<\mathbf{v},\mathbf{k}>= d_1k_1 + \cdots + d_nk_n = d$.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ. Then,

Definition 1.1.19 The local gradient algebra corresponding to f is the algebra

$$\mathcal{Q}_f = \mathbf{C}_x/Jf \quad \text{or} \quad \mathcal{Q}_f = \mathbb{R}[[x]]/Jf,$$

where $\mathbb{R}[[x]]$ is the ring of formal power series in the variables x over $\mathbb{R}$. If the algebra $\mathcal{Q}_f$ is of finite dimension, these two algebras coincide. The local multiplicity (or the Milnor number) μ_f of the germ f is the dimension of the local gradient algebra $\mathcal{Q}_f$ (as a real vector space).

Definition 1.1.20 A germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ has modality $\leq s$ with respect to right equivalence if all germs close to f are $\mathcal{R}$-equivalent to germs from a finite number of families each of which depends on less or equal than s parameters.

Remark: Singularities of modality 0 are called simple.

Definition 1.1.21 Two function germs said to be stably equivalent if they become $\mathcal{R}-$equivalent after the addition of quadratic forms in an appropriate number of extra variables.

Theorem 1.1.8 [1] *Simple function-germs* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *are stably* $\mathcal{R}$*-equivalent to the following ones:*

1. $A_k : \pm x^{k+1} \pm y^2, \quad k \geq 1;$
2. $D_k : x^2y \pm y^{k-1}, \quad k \geq 4;$
3. $E_6 : x^3 \pm y^4;$
4. $E_7 : x^3 + xy^3;$
5. $E_8 : x^3 + y^5.$

Here, $(x, y) \in \mathbb{R}^2$ *and* $n \geq 2$.

Remarks:

1. In the previous theorem if $n = 1$, then a simple germ is stably $\mathcal{R}-$ equivalent to $A_k : \pm x^{k+1}, \quad k \geq 1$.

2. Singularities of different types are not $\mathcal{R}$-equivalent to each other.

3. The subscript in the notations is equal to the Milnor number of the germ.

Let X and Y be singularities, with respect to right equivalence. Then,

Definition 1.1.22 We say that X is adjacent to Y (we write $X \longrightarrow Y$) if there exist a family of germs $f_\lambda : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$, such that f_0 has type X and f_λ has type Y for $\lambda \neq 0$ small enough.

Remarks:

1. $X \longrightarrow Y \Rightarrow \mu_X \geq \mu_Y$.

2. Simple $\mathcal{R}$-singularities posses the following adjacencies $X_\mu \longrightarrow Y_{\mu-1}$:

$$A_k \longrightarrow A_{k-1} \qquad D_k \longrightarrow D_{k-1} \qquad D_k \longrightarrow A_{k-1}$$
$$E_6 \longrightarrow D_5 \qquad E_6 \longrightarrow A_5 \qquad E_7 \longrightarrow D_6$$
$$E_7 \longrightarrow A_6 \qquad E_8 \longrightarrow D_7 \qquad \text{and} \quad E_8 \longrightarrow A_7.$$

1.2 Pseudo and quai border equivalence relation: The non-standard equivalence relation

Consider the space $\mathbb{R}^n = \{w = (x, y)\}$, where $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ and $y = (y_1, y_2, \dots, y_{n-m}) \in \mathbb{R}^{n-m}$. Let a hypersurface Γ be given by the equation $h(x) = 0$, where h is a smooth function. We call the hypersurface Γ *a border.*

In our examples Γ can be regular or singular (reducible or irreducible). In fact, we consider the following shapes of Γ.

1) The hypersurface is smooth, in which case we set $\Gamma = \Gamma_b = \{x_1 = 0\}$.

2) The hypersurface is a union of two transversal hypersurfaces, called a corner,in which case we set $\Gamma = \Gamma_c = \{x_1 x_2 = 0\}$.

3) The hypersurface is a cusp $\Gamma = \Gamma_{csp} = \{x_2^2 - x_1^s = 0 : \text{for some } s \geq 3\}$.

4) The hypersurface is a cone $\Gamma = \Gamma_{cn} = \{x_1 x_2 - x_3^2 = 0\}$.

Definition 1.2.1 Two functions $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}$ are called *pseudo-border* equivalent if there exists a diffeomorphisim $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_1 \circ \theta = f_0$, and if a critical point c of the function f_0 belongs to the border Γ then $\theta(c)$ also belongs to Γ and vice versa, if c is a critical point of f_1 and belongs to Γ then $\theta^{-1}(c)$ also belongs to Γ.

Similar definitions can be stated for germs of functions.

We consider germs at the origin of C^∞ -smooth functions $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, with local coordinates w as above. Denote by $\mathbf{C}_w$ the ring of all these germs at the origin with a unit.

Remarks:

1. In the definition above, the diffeomorphism θ will be called *admissible.*

2. The general statements below are valid for reasonably good hypersurfaces. For rigorousness, we assume that the hypersurface Γ is a stratified set, and the stratification satisfies Whitney condition 1. Also, we assume in the definition that if the critical point c belongs to some stratum I then $\theta(c)$ belongs to the same stratum.

3. Pseudo-border equivalence will be also called pseudo-boundary or pseudo-corner, etc ... for respective type of Γ.

4. Clearly, pseudo border is an equivalence relation: If $f_1 \sim f_2$ and $f_2 \sim f_3$ then $f_1 \sim f_3$. However, this relation is not given by a group action as the set of admissible diffeomorphisms depends on a function.

5. The ideal I generated by $g_1, g_2, \ldots, g_k$ will be denoted as $I = \{g_1, g_2, \ldots, g_k\}$.

Definition 1.2.2 [24] Let J be an ideal in $\mathbf{C}_w$, then we define the *radical* $Rad(J)$ of the ideal J as the set of all germs in $\mathbf{C}_w$ vanishing on common all zero points for germs in J:

$$Rad(J) = I(V(J)),$$

where

$$V(J) = \{w = (x, y) : h(w) = 0 \text{ for any } h \in J\},$$

and

$$I(V(J)) = \{\varphi \in \mathbf{C}_w : \varphi(w) = 0 \text{ for all } w \in V(J)\}.$$

Example 1: Consider the ideal $J = \{(x-y)^2\} = (x-y)^2 A(x,y)$ with $(x,y) \in \mathbb{R}^2$ and $A \in \mathbf{C}_{x,y}$. Then, $Rad(J) = \{(x-y)\}$. Note that always $J \subset Rad(J)$.

Sometimes the radical of an ideal behaves badly when the ideal depends on a parameter.

Example 2: Consider the family of ideals depending on ε with a constant $a \in \mathbb{R}$ and a variable $x \in \mathbb{R}$

$$J_\varepsilon = \{(x-a)(x-(1+\varepsilon)a)\}.$$

Then,

$$Rad(J_\varepsilon) = \begin{cases} J_\varepsilon & \text{if} \quad \varepsilon \neq 0, \\ \{(x-a)\} & \text{if} \quad \varepsilon = 0. \end{cases}$$

Hence, the dimension of the quotient space $\mathbf{C}_x / Rad(J_\varepsilon)$ varies with ε:

$$dim[\mathbf{C}_x / Rad(J_\varepsilon)] = \begin{cases} 2 & \text{if} \quad \varepsilon \neq 0, \\ 1 & \text{if} \quad \varepsilon = 0. \end{cases}$$

Remarks on properties of the radical of an ideal[24]:

In fact, there is another definition of a radical of an ideal I in the space $\widetilde{\mathbf{C}}_w$ of algebraic or analytic functions:

$$Rad(I) = \{h \in \mathbf{C}_w : h^n \in I \text{ for some positive integer } n\}.$$

In general, this definition is not equivalent to the previous one for an arbitrary ideal in the space of smooth functions. However, the definitions coincide foran ideal J with finitely generated quotient space $\mathbf{C}_w/J$.

Recall that the vector field v preserves the hypersurface $\Gamma = \{h(x) = 0\}$ if the Lie derivative $L_v h$ belongs to the principal ideal $\{h(x)\}$. Vector fields v are tangent to Γ. The module $\mathbb{S}_\Gamma$ of all vector fields preserving the hypersurface Γ is the Lie algebra of the group of diffeomorphims preserving Γ. The module $\mathbb{S}_\Gamma$ is called the *stationary algebra* of Γ.

Let $\Gamma_x = \{h(x) = 0\} \subset \mathbb{R}^n$ be a hypersurface with an isolated singular point at zero, defined by a quasi-homogeneous function $h(x)$ with weights of variables d_i and degree d. Then the following holds:

Lemma 1.2.1 (O.V. Lyasko) [22] *The module of tangent vector field $\mathbb{S}_\Gamma$ is generated by the Euler field $v_0 = d_1 w_1 \frac{\partial}{\partial x_1} + \cdots + d_n w_n \frac{\partial}{\partial x_n}$ and the Hamiltonian field $v_{ij} = H_i \frac{\partial}{\partial x_j} - H_j \frac{\partial}{\partial x_i}$, where $H_i = \frac{\partial h}{\partial x_i}$.*

Using the previous Lemma, we deduce the following.

- If $\Gamma_b = \{x_1 = 0\}$ then

$$\mathbb{S}_{\Gamma_b} = \left\{ x_1 h_1 \frac{\partial}{\partial x_1} + \sum_{i=1}^{n-1} k_i \frac{\partial}{\partial y_i} \right\},$$

for arbitrary function germs $h_1, k_i \in \mathbf{C}_w$. Here $x_1 \in \mathbb{R}, y \in \mathbb{R}^{n-1}$.

- If $\Gamma_c = \{x_1x_2 = 0\}$ then

$$\mathbb{S}_{\Gamma_c} = \left\{ x_1h_1\frac{\partial}{\partial x_1} + x_2h_2\frac{\partial}{\partial x_2} + \sum_{i=1}^{n-2} k_i\frac{\partial}{\partial y_i} \right\},$$

  for arbitrary function germs $h_1, h_2, k_i \in \mathbf{C}_w$. Here $x = (x_1, x_2) \in \mathbb{R}^2, y \in \mathbb{R}^{n-2}$.

- If the border is a cusp $\Gamma_{csp} = \{x_2^2 - x_1^s = 0 : \text{for some } s \geq 3\}$ then

$$\mathbb{S}_{\Gamma_{csp}} = \left\{ (\frac{x_1}{s}h_1 + 2x_2h_2)\frac{\partial}{\partial x_1} + (\frac{x_2}{2}h_1 + sx_1^{s-1}h_2)\frac{\partial}{\partial x_2} + \sum_{i=1}^{n-2} k_i\frac{\partial}{\partial y_i} \right\},$$

  for arbitrary function germs $h_1, h_2, k_i \in \mathbf{C}_w$. Here $x = (x_1, x_2) \in \mathbb{R}^2, y \in \mathbb{R}^{n-2}$.

- If the border is a cone $\Gamma_{cn} = \{x_1x_2 - x_3^2 = 0\}$ then

$$\begin{aligned} \mathbb{S}_{\Gamma_{cn}} &= \Big\{ (x_1h_1 - x_1h_2 + 2x_3h_3)\frac{\partial}{\partial x_1} + (x_2h_1 + x_2h_2 + 2x_3h_4)\frac{\partial}{\partial x_2} \\ &+ (x_3h_1 + x_2h_3 + x_1h_4)\frac{\partial}{\partial x_3} + \sum_{i=1}^{n-3} k_i\frac{\partial}{\partial y_i} \Big\}, \end{aligned}$$

  for arbitrary function germs $h_1, h_2, h_3, h_4, k_i \in \mathbf{C}_w$. Here $x = (x_1, x_2, x_3) \in \mathbb{R}^2, y \in \mathbb{R}^{n-3}$.

Suppose that all function germs in a smooth family f_t are pseudo-border equivalent to the function germ f_0,i.e. $f_t \circ \theta_t = f_0,\ t \in [0,1]$, with respect to a smooth family $\theta_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ of germs of diffeomorphisms such that $\theta_0 = id$ and $t \in [0,1]$. Then we obtain the derivative equation:

$$-\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial x}\dot{X} + \frac{\partial f_t}{\partial y}\dot{Y},$$

where the vector field $v = \dot{X}\frac{\partial}{\partial x} + \dot{Y}\frac{\partial}{\partial y}$ generates the phase flow θ_t.

Denote by $\mathcal{V}_{Rad(f_t)}$ the set of vector field germs, each component of which belongs to the radical of the gradient ideal I of the function f_t. We have the following

Proposition 1.2.2 *The vector field v generates the family of admissible diffeomorphisms θ_t as above if and only if*

$$v \in \mathbb{S}_\Gamma + \mathcal{V}_{Rad(f_t)}.$$

Proof. If m is not a critical point of f_t then $Rad\{\frac{\partial f_t}{\partial w_i}\}$ coincides with the space of all function germs at m so v could be any vector field and therefore there are no conditions on v. In fact, at least one of derivative is $\frac{\partial f_t}{\partial w_i} \neq 0$, hence $J = \left\{\frac{\partial f_t}{\partial w}\right\} = \mathbf{C}_w$.

We have assumed that Γ is a stratified manifold satisfying Whitney condition 1 (this covers all the cases we need and in all our settings this condition holds). Also, we assumed that the factor algebra $\mathbf{C}_w/Rad\{\frac{\partial f_t}{\partial w}\}$ has finite multiplicity. This implies that there are finitely many isolated critical points of f_t.

Denote by $\sum_t = \{m_1, \dots, m_s\}$ the set of critical points of f_t. Some of m_i belong to Γ and others do not.

For simplicity we consider the boundary case only. The other cases can be treated similarly using the stratification properties. So, we suppose that $\Gamma = \{w_1 = 0\}$. In this case $\widetilde{v} \in \mathbb{S}_\Gamma$ takes the form

$$\widetilde{v} = w_1 \Phi_1(w) \frac{\partial}{\partial w_1} + \sum_{i=2}^{n} W_i(w) \frac{\partial}{\partial w_i}, \quad \text{where } w = (w_1, w_2, \dots, w_n),$$

for some smooth functions $W_i(w)$.

Let $m_0 \in \Gamma \cap \sum_0$ then by definition of pseudo equivalence the trajectory $\theta_t(m_0) = m_t$ lies in Γ, so the vector field has the form:

$$v = W_1(w) \frac{\partial}{\partial w_1} + \sum_{i=2}^{n} W_i(w) \frac{\partial}{\partial w_i},$$

with $W_1(m_0) = 0$ (and also $W_1(m_t) = 0$ for all t). Thus, we have a component function W_1 which vanishes at $m_t \in \Gamma \cap \sum_t$.

For other examples of borders considered in the thesis the proof is similar. In the proof we use the following fact: any finite number of vectors tangent to strata of the border can be extended to a smooth vector field from the stationary algebra which is tangent to the border everywhere and coincide with given vectors where they are attached. Obviously this property holds for corner, cusps and cone shape borders. In particular, this property implies Whitney 1 condition: the tangent space to a smaller stratum must be in the limit of the tangent spaces of the larger strata. However this is not sufficient.

To prove "the only case" it suffices to prove that for given function $W_1(w)$ with $W_1(m_0) = 0$ there exists a smooth function $\Phi_1(w)$ and $\chi \in Rad\{\frac{\partial f_t}{\partial w}\}$ so that

$$W_1(w) = w_1\Phi_1(w) + \chi.$$

Since the set $\sum_t$ of all critical points of f_t is finite then there is a polynomial of the form

$$P_j(w) = w_1\Lambda_1(w)\ldots\Lambda_{j-1}(w)\Lambda_{j+1}(w)\ldots\Lambda_s(w),$$

where $\Lambda_k(w)$ is an affine function $\Lambda_k(w)$ in w which vanishes at m_k and does not vanish at other points m_i $(i \neq k)$. That is $P_j(m_k) = 0$ and $P_j(m_j) \neq 0$ for $j = 1, \ldots, s$ except for $m_j = m_0$.

Consider a linear combination of these polynomials

$$\mathbb{P} = \sum_{j=1}^{s} A_j P_j(w)$$

with coefficients A_j, so that $\mathbb{P}(m_i) = W_1(m_i)$. On the other hand

$$\mathbb{P} = w_1\left(\sum_{j=1}^{s} A_j \prod \Lambda_l(w)\right) = w_1\Phi(w).$$

Now we have that $W_1(w) - \mathbb{P}$ vanishes at each critical point of f_t. Hence $W_1(w) - \mathbb{P} = \chi \in Rad\{\frac{\partial f_t}{\partial w}\}$. So we get the first component of the vector field as $v_1 = w_1\Phi(w) + \chi$, where $w_1\Phi(w) \in \mathbb{S}_\Gamma$.

The following proof of the "if" claim is valid for any border. We know that $f_t \circ \theta_t = f_0$, then $-\frac{\partial f_t}{\partial t} = \sum \frac{\partial f_t}{\partial w_i}\dot{W}_i$ such that $v = \sum \dot{W}_i \frac{\partial}{\partial w_i} \in \mathbb{S}_\Gamma + \mathcal{V}_{Rad(f_t)}$. Let

$v_1 \in \mathbb{S}_\Gamma$ and $v_2 \in \mathcal{V}_{Rad(f_t)}$. Consider the trajectory $m_t = \theta_t(m_0)$. At each t, we have $v(m_t) = v_1(m_t)$ as m_t remains critical for f_t. Thus, the trajectory $\theta_t(m_0)$ is an integral line for both vector fields (v_1 and v_2). So consider only $v_1 \in \mathbb{S}_\Gamma$. Then, if $m_0 \in \Gamma$ then a trajectory of m_0 lies on Γ.

∎

We modify pseudo equivalence relation to have better properties with respect to parameter dependence, replacing the radical $Rad\{\frac{\partial f_t}{\partial w}\}$ by the ideal $\{\frac{\partial f_t}{\partial w}\}$ itself in the definition of pseudo border equivalence.

Denote by V_I the ideal of the algebra of germs of vector fields, each component of which belongs to the gradient ideal I of the function f_t.

Definition 1.2.3 Two functions $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}$ are called *quasi border equivalent*, if they are pseudo border equivalent and there is a family of function germs f_t which continuously depend on parameter $t \in [0,1]$ and a continuous piece-wise smooth family of diffeomorphisms $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ depending on parameter $t \in [0,1]$ such that:$f_t \circ \theta_t = f_0$, $\theta_0 = id$ and the vector field v generated by θ_t on each segment of smoothness satisfies the inclusion

$$v \in \mathbb{S}_\Gamma + V_I.$$

The diffeomorphisms θ_t generated by the vector field v as well as the vector field itself will be called admissible for the family f_t.

The previous definition implies that the formulas of the quasi border tangent spaces $TQ\Gamma_{f_t}$ to the quasi border orbits at an admissible deformations f_t are given as follows:

- If $\Gamma_b = \{x_1 = 0\}$ then

$$TQB_{f_t} = \left\{ \frac{\partial f_t}{\partial x_1} \left(x_1 h_1 + \frac{\partial f_t}{\partial x_1} A_1 \right) + \sum_{i=1}^{n-1} \frac{\partial f_t}{\partial y_i} k_i \right\},$$

for arbitrary function germs $h_1, A_1, k_i \in \mathbf{C}_w$. Here $x_1 \in \mathbb{R}, y \in \mathbb{R}^{n-1}$.

- If $\Gamma_c = \{x_1 x_2 = 0\}$ then

$$\begin{aligned} TQC_{f_t} &= \left\{ \frac{\partial f_t}{\partial x_1}\left(x_1 h_1 + \frac{\partial f_t}{\partial x_1} A_1 + \frac{\partial f_t}{\partial x_2} A_2\right) + \frac{\partial f_t}{\partial x_2}\left(x_2 h_2 + \frac{\partial f_t}{\partial x_1} B_1 + \frac{\partial f_t}{\partial x_2} B_2\right) \right. \\ &+ \left. \sum_{i=1}^{n-2} \frac{\partial f_t}{\partial y_i} k_i \right\}, \end{aligned}$$

for arbitrary function germs $h_i, A_i, B_i, k_i \in \mathbf{C}_w$. Here $x = (x_1, x_2) \in \mathbb{R}^2$, $y \in \mathbb{R}^{n-2}$.

- If the border is a cusp $\Gamma_{csp} = \{x_2^2 - x_1^s = 0 : \text{for some } s \geq 3\}$ then

$$\begin{aligned} TQCU_{f_t} &= \left\{ \frac{\partial f_t}{\partial x_1}\left(\frac{x_1}{s} h + 2x_2 k + \frac{\partial f_t}{\partial x_1} A_1 + \frac{\partial f_t}{\partial x_2} A_2\right) \right. \\ &+ \left. \frac{\partial f_t}{\partial x_2}\left(\frac{x_2}{2} h + s x_1^{s-1} k + \frac{\partial f_t}{\partial x_1} B_1 + \frac{\partial f_t}{\partial x_2} B_2\right) + \sum_{i=1}^{n-2} \frac{\partial f_t}{\partial y_i} C_i. \right\} \end{aligned}$$

for arbitrary function germs $h, k, A_i, B_i, C_i \in \mathbf{C}_w$. Here $x = (x_1, x_2) \in \mathbb{R}^2$, $y \in \mathbb{R}^{n-2}$.

- If the border is a cone $\Gamma_{cn} = \{x_1 x_2 - x_3^2 = 0\}$ then

$$\begin{aligned} TQCO_{f_t} &= \left\{ \frac{\partial f_t}{\partial x_1}\left(x_1 h_1 - x_1 h_2 + 2x_3 h_3 + \frac{\partial f_t}{\partial x_1} B_1 + \frac{\partial f_t}{\partial x_2} B_2 + \frac{\partial f_t}{\partial x_3} B_3\right) \right. \\ &+ \frac{\partial f}{\partial x_2}\left(x_2 h_1 + x_2 h_2 + 2x_3 h_4 + \frac{\partial f_t}{\partial x_1} C_1 + \frac{\partial f_t}{\partial x_2} C_2 + \frac{\partial f_t}{\partial x_3} C_3\right) \\ &+ \left. \frac{\partial f_t}{\partial x_3}\left(x_3 h_1 + x_2 h_3 + x_1 h_4 + \frac{\partial f_t}{\partial x_1} D_1 + \frac{\partial f_t}{\partial x_2} D_2 + \frac{\partial f_t}{\partial x_3} D_3\right) + \sum_{i=1}^{n-3} \frac{\partial f_t}{\partial y_i} E_i \right\}, \end{aligned}$$

for arbitrary function germs $h_i, A_i, B_i, C_i, E_i \in \mathbf{C}_w$. Here $x = (x_1, x_2, x_3) \in \mathbb{R}^2, y \in \mathbb{R}^{n-3}$.

1.3 Basic techniques and prenormal forms

1.3.1 Moser's homotopy method and spectral sequence

Moser's homotopy method is the main technique we use to prove quasi border equivalence between two function germs f_0 and f_1. The idea is as follows: introduce a family of function germs f_t with $t \in [0,1]$, joining f_0 and f_1. We are trying now to find a family of admissible diffeomorphisms θ_t with $t \in [0,1]$, $\theta_0 = id$ and $f_t \circ \theta_t = f_0$.

We differentiate this relation with respect to t to get

$$-\frac{\partial f_t}{\partial t} = \sum \frac{\partial f_t}{\partial w_i} \dot{W}_i,$$

where all $\frac{\partial f_t}{\partial w_i}$ and the components of the vector field $v = \sum \dot{W}_i \frac{\partial}{\partial w_i}$ are evaluated at $\theta_t(w)$. Therefore θ_t is the phase flow of the vector field v. If for a given function $\frac{\partial f_t}{\partial t}$ we can find a decomposition in the above form, then the vector field v with the components $\dot{W}_i$ in the (w,t)-space can be integrated to obtain the family of diffeomorphisms θ_t. Of course, we need to be sure that the germs of diffeomorphisms are defined on some neighborhood of the base point. This is usually achieved if at the base point the vector field vanishes.

Recall now some basic results in singularity theory which will be used intensively in the following sections.

The Malgrange Preparation Theorem and *Nakayama Lemma* are important tools to prove some results on prenormal forms. Here we state them.

Theorem 1.3.1 (Malgrange Preparation Theorem) [24] *Let $\mathbf{C}_x$ be the algebra of germs at the origin of smooth functions in $x \in \mathbb{R}^m$. Let M be a finitely generated $\mathbf{C}_x$- module and $f : (x,0) \to (y(x),0)$ be a germ of a C^∞ mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$. If I_f is the ideal in $\mathbf{C}_x$ generated by the components of f and the quotient algebra $M/I_f M$ is isomorphic to some finitely generated real vector space, with generators $[g_1(x)], \dots [g_k(x)]$, then M regarded as an $\mathbf{C}_{y(x)}$-module is generated by $g_1, \dots, g_k$. Here $\mathbf{C}_{y(x)}$ is the algebra of smooth function germs at the origin composed with the components of the map f, i.e. $\mathbf{C}_{y(x)} = \{h(y_1(x), \dots, y_n(x))\}$, where h is a smooth function germ at the origin n in variables $y_1, \dots, y_n$.*

Let A be a commutative ring with a unit. Let $I \subset A$ be an ideal with the following property: for every $\alpha \in I$, $1 + \alpha$ is invertible in A. For example, $A = \mathbf{C}_x$ and I is any proper ideal of A.

Lemma 1.3.2 (Nakayama Lemma) [13] *Let M be a finitely generated A-module and let N be a submodule of M. Then, the condition*

$$N + I.M = M$$

implies that $N = M$.

Lemma 1.3.3 (Hadamard's lemma) [13] *A smooth function $f(x, y)$ with local coordinates $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$ which vanishes on the coordinate subspace $x = 0$ can be written in the form $f = \sum_{i=1}^{n} x_i h_i(x, y)$ for certain smooth functions h_i.*

The following property of quasi border multiplicity is needed to prove further results.

Proposition 1.3.4 *If a function germ f has a finite (right) multiplicity for some border, then f has finite quasi border multiplicity. In other words, the quasi border tangent space has finite codimension over $\mathbb{R}$ in the space $\mathbf{C}_w$ of all germs.*

Proof. Let $I = \{\frac{\partial f}{\partial w_i}\}$ be the gradient ideal of f. As f has finite right multiplicity, its local algebra $\mathcal{Q} = \mathbf{C}_w/I = \mathbb{R}\{\rho_0 = 1, \rho_1(w), \ldots, \rho_k(w)\}$ for some smooth functions $\rho_i(w)$ is finitely generated over $\mathbb{R}$.

This means that for any function germ $\varphi(w)$, there is a decomposition of the form

$$\varphi(w) = \sum_{i=1}^{n} \frac{\partial f}{\partial w_i} A_i(w) + \sum_{j=0}^{k} c_j \rho_j(w),$$

where c_j are constants and $A_i(w)$ are smooth function germs

Now, for any A_i, we can write

$$A_i = \sum_{s=1}^{n} \frac{\partial f}{\partial w_s} Y_{si} + \sum_{m=0}^{k} \beta_{mi} \rho_m.$$

This yields

$$\varphi(w) = \sum_{i,s=1}^{n} \frac{\partial f}{\partial w_i} \frac{\partial f}{\partial w_s} Y_{si} + \sum_{i=1}^{n} \sum_{m=0}^{k} \frac{\partial f}{\partial w_i} \rho_m \beta_{m_i} + \sum_{j=0}^{k} c_j \rho_j.$$

Notice that the square of the gradient ideal $I^2 = \sum_{i,s=1}^{n} \frac{\partial f}{\partial w_i} \frac{\partial f}{\partial w_s} Y_{si}$ belongs to the quasi border tangent space TQ_f. Therefore, we see that

$\mathbf{C}/TQ_f \subset \mathbf{C}_w/I^2 = \mathbb{R}\{1, \rho_1, \dots, \rho_k, \dots, \frac{\partial f}{\partial w_1}\rho_s, \dots, \frac{\partial f}{\partial w_n}\rho_s, \dots\}$, where $s = 0, \dots, k$. This completes the proof.

∎

Recall that a smooth function $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, is called quasi homogeneous function of degree d with exponents $d_1, \dots, d_n$ related to the coordinates $x_1, \dots, x_n$ if $f(\lambda^{d_1} x_1, \dots, \lambda^{d_n} x_n) = \lambda^d f(x_1, \dots, x_n)$ for all $\lambda > 0$. The exponents d_i are also called the weights of the variables x_i.

Let $\mathbf{v} = (d_1, \dots, d_n)$ and $\mathbf{k} = (k_1, \dots, k_n)$. Then, in terms of the Taylor series $\sum f_{\mathbf{k}} x^{\mathbf{k}}$ of f the quasihomogeneisty condition means that the exponents of the non-zero terms of the series lie in the hyperplane

$$L = \{\mathbf{k} : d_1 k_1 + \dots + d_n k_n = d\}.$$

The monomial $\mathbf{x}^{\mathbf{k}}$ is said to have degree d if $< \mathbf{v}, \mathbf{k} >= d_1 k_1 + \dots + d_n k_n = d$.

The Newton polyhedron Γ_j of a power series may be defined as the convex hull of the union of the positive quadrants $\mathbb{R}^n_+$ with vertices at the indices of the monomials belonging to the series with non zero coefficient. The Newton diagram Γ is the union of the compact faces of this polyhedron.

Let Γ be a Newton diagram. Then, each face specifies quasihomogeneity type $\mathbf{v}_j$, in which the degree of the monomials with exponents lying on Γ_j is equal one: $<\mathbf{k}, \mathbf{v}_j>= 1$ [1].

Let us fix a Newton diagram Γ. A monomial $\mathbf{x}^{\mathbf{k}}$ is said to have Newton degree d if $d = min_j < \mathbf{k}, \mathbf{v}_j >$. In other words, the Newton degree of a monomial is the smallest of its degrees in any of the quasi homogeneous filtrations defined by the faces of the diagram Γ.

The Newton order γ_i of a function is the smallest of the Newton degrees of the monomials that appear in it.

The functions of order at least γ_i form an ideal S_{γ_i} in the ring $\mathbf{C}_w$. The ideals S_{γ_i} yield the Newton filtration in the ring of $\mathbf{C}_w$. The sum f_0 of the terms of Newton degree γ_i of a function f of order γ_i will be referred to as the principal part of f.

Assume that $S_0 \supset S_{\gamma_1} \supset \cdots \supset S_{\gamma_p} \supset \ldots$ is a semi quasi homogeneous filtration on $\mathbf{C}_w$ defined by the Newton diagram ([1]). Let $f = f_0 + f_*$ be a decomposition of a function germ f into its principal part f_0 of the fixed lowest degree terms defined by the diagram and higher terms $f_* \in S_{\gamma_1}$ and f_0 has a finite multiplicity.

Lemma 1.3.5 *Suppose that $e_1(w), e_2(w), \ldots, e_s(w)$ are quasi homogeneous polynomials of various degrees $N + p_i$, where $p_i > 0$, $\mathbb{R}$-generating the quotient space $\mathbf{C}_w/TQf_0$, where TQf_0 is the tangent space to the orbit of f_0 with respect to quasi border equivalence.*

Suppose that for any term $\varphi \in S_{\gamma_p} \setminus S_{>\gamma_p}$:

1. *There is a quasi admissible vector field $\dot{W} = \sum \dot{w}_i \frac{\partial}{\partial w_i}$, where*

$$\dot{w} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_m, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_{n-m}),$$

$$\dot{x}_1 = \widetilde{x}_1 + \sum_{i=1}^{n} A_i^{(1)} \frac{\partial f_0}{\partial w_i}, \quad \ldots, \dot{x}_m = \widetilde{x}_m + \sum_{i=1}^{n} A_i^{(m)} \frac{\partial f_0}{\partial w_i},$$

and

$$\dot{y}_{m+1}, \ldots, \dot{y}_n \in \mathbf{C}_w$$

with $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_m) \in \mathbb{S}_\Gamma$ *and* $A_i^{(j)}(w) \in \mathbf{C}_w$*, such that*

$$\varphi = \sum_{i=1}^{n} \frac{\partial f_0}{\partial w_i} \dot{w}_i + \widehat{\varphi} + \sum_{i=1}^{s} c_i e_i(w),$$

where $\widehat{\varphi} \in S_{>\gamma_p}$ *and* $c_i \in \mathbb{R}$.

2. *Moreover, for any* S_{γ_i} *and* $\psi \in S_{\gamma_i}$ *the expression*

$$E(\psi, \varphi) = \sum_{i=1}^{m} \frac{\partial \psi}{\partial x_i} \left[\dot{x}_i + \sum_{j=1}^{n} A_j^{(i)} \frac{\partial \psi}{\partial w_j} \right] + 2 \sum_{i=1}^{m} \frac{\partial f_0}{\partial x_i} \left[\sum_{j=1}^{n} A_j^{(i)} \frac{\partial \psi}{\partial w_j} \right] + \sum_{i=m+1}^{n} \frac{\partial \psi}{\partial y_i} \dot{y}_i$$

belongs to $S_{>\gamma_i}$.

Then any germ f *is quasi border equivalent to the germ* $f_0 + \sum_{i=1}^{s} c_i e_i$*, where* $c_i \in \mathbb{R}$.

Proof. Consider the family $F_0 = f_0 + \sum_{i=1}^{s} \lambda_i e_i(w)$ of functions with parameters $\lambda \in \mathbb{R}^s$. Consider a homotopy $F_t = f_0 + t f_* + \sum_{i=1}^{s} \lambda_i e_i(w)$ with $t \in [0, 1]$. We will prove that the homotopy is quasi border admissible. The homological equation takes the form

$$-\frac{\partial F_t}{\partial t} = \sum \frac{\partial F_t}{\partial w_i} \dot{w}_i + \sum_{i=1}^{s} \dot{\Lambda}_i e_i(w), \tag{1.2}$$

where $\dot{w} = \dot{w}_i(x, \lambda) \frac{\partial}{\partial w_i}$ is quasi border admissible and $\dot{\Lambda}_i = \dot{\Lambda}_i(\lambda, t)$.

Due to Malgrange preparation theorem, it is enough to solve (1.2) for $\lambda = 0$ (we show the details below). For $\lambda = 0$, the homological equation becomes

$$-f_* = \sum \left(\frac{\partial f_0}{\partial w_i} + t \frac{\partial f_*}{\partial w_i} \right) \dot{w}_i + \dot{\Lambda}_i(0, t) e_i(w).$$

Since the quasi degree $d(f_*) > d(f_0)$, then according to the conditions of the Lemma, there exists a quasi border admissible vector field $w_{(1)} = \sum \dot{w}_{i(1)} \frac{\partial}{\partial w_i}$ and a function $\varphi_1(w) \in S_{\gamma_1}$ with $d(\varphi_1) > d(f_*)$ such that

$$-f_* = \sum \frac{\partial f_0}{\partial w_i} \dot{w}_{i(1)} + \sum b_i^{(1)} e_i(w) + \varphi_1(w).$$

This is equivalent to

$$-f_* = \sum\left(\frac{\partial f_0}{\partial w_i} + t\frac{\partial f_*}{\partial w_i}\right)\dot{w}_{i(1)} + \sum b_i^{(1)} e_i(w) + \varphi_1(w) - \sum t\frac{\partial f_*}{\partial w_i}\dot{w}_{i(1)}. \qquad (1.3)$$

Since $\dot{w}_{i(1)}$ is admissible for f_0 we see that

$$\dot{w}_{i(1)} = \dot{x}_{i(1)} = \widetilde{x}_{i(1)} + \sum_{j=1}^{n} A_{j(1)}^{(i)} \frac{\partial f_0}{\partial w_j} \quad \text{for } i \in \{1, \ldots, m\} \quad \text{and } \widetilde{x}_{i(1)} \in \mathbb{S}_\Gamma$$

and

$$\dot{w}_{i(1)} = \dot{y}_{i(1)} \quad \text{for } i \in \{m+1, \ldots, n\}.$$

Denote by

$$\dot{w}_{i(1)}^* = \widetilde{x}_{i(1)} + \sum_{j=1}^{n} A_{j(1)}^{(i)} \frac{\partial (f_0 + tf_*)}{\partial w_j}, \quad \text{for } i \in \{1, \ldots, m\}$$

and

$$\dot{w}_{i(1)}^* = \dot{w}_i = \dot{y}_{i(1)} \quad \text{for } i \in \{m+1, \ldots, n\}.$$

So we have that the vector field $\dot{w}_{i(1)}^*$ is admissible for $f_0 + tf_*$ and we get using (1.3):

$$-f_* = \sum\left(\frac{\partial f_0}{\partial w_i} + t\frac{\partial f_*}{\partial w_i}\right)\dot{w}_{i(1)}^* + \sum b_i^{(1)} e_i(w) + \varphi_1(w) - tE; \qquad (1.4)$$

where

$$E = \sum_{i=1}^{m} \frac{\partial f_*}{\partial x_i}\left[\dot{x}_{i(1)} + \sum_{j=1}^{n} A_{j(1)}^{(i)} \frac{\partial f_*}{\partial w_j}\right] + 2\sum_{i=1}^{m} \frac{\partial f_0}{\partial x_i}\left[\sum_{j=1}^{n} A_{j(1)}^{(i)} \frac{\partial f_*}{\partial w_j}\right] + \sum_{i=m+1}^{n} \frac{\partial f_*}{\partial y_i}\dot{y}_{i(1)}.$$

The expression E is exactly the expression $E = E(f_*, tf_*)$. So by second condition of the Lemma we see that $E(f_*, tf_*) \in S_{>\gamma_1}$. Thus, the equation (1.4) implies that

$$-f_* = \sum \frac{\partial F_t}{\partial w_i}\dot{w}_{i(1)}^* + \sum b_i^{(1)} e_i + \widetilde{\varphi}_1,$$

where $\widetilde{\varphi}_1 = \varphi_1 - tE(f_*, tf_*) \in S_{>\gamma_1}$.

Now again consider a similar decomposition for $\widetilde{\varphi}_1$

$$\widetilde{\varphi}_1 = \sum \frac{\partial F_t}{\partial w_i} \dot{w}^*_{i(2)} + \sum b_i^{(2)} e_i + \widetilde{\varphi}_2,$$

where $\widetilde{\varphi}_2 \in S_{\gamma_2}$ and $\gamma_2 > deg(\widetilde{\varphi}_1)$. Thus, by induction after several steps, we will have

$$f_* = \sum \frac{\partial F_t}{\partial w_i} [\dot{w}^*_{i(1)} + \dot{w}^*_{i(2)} + \dots] + \sum [b_i^{(1)} + b_i^{(2)} + \dots] e_i + \Phi,$$

with the degree of Φ being sufficiently large. Since f_0 has a finite multiplicity then $f_0 + tf_*$ also has a finite multiplicity. Hence, for some large power N of the maximal ideal $\mathcal{M}_w^N$, we have that $\mathcal{M}^N \subset \{\frac{\partial F_t}{\partial w_i}\}$. Moreover, we have $\mathcal{M}^{2N} \subset \{\frac{\partial F_t}{\partial w_i}\}^2 = I^2$. Thus, $\mathcal{M}^{2N}$ belongs to the quasi fixed tangent space. So $\Phi = \sum \frac{\partial F_t}{\partial w_i} \dot{\widehat{w}}_i$ (We shall give the details below). Therefore, we see that

$$f_* = \sum \frac{\partial F_t}{\partial w_i} [\dot{w}^*_{i(1)} + \dot{w}^*_{i(2)} + \dots + \dot{\widehat{w}}_i] + \sum [b_i^{(1)} + b_i^{(2)} + \dots] e_i.$$

Thus, we have shown that the homological equation (1.2) is solvable.

Now consider the family of quasi border equivalent germs $F_t \circ \Theta_t = F_0$ or equivalently $F_t = F_0 \circ \Theta_t^{-1}$, where

$$\Theta_t : (w, \lambda, t) \mapsto (t, W_t(w, \lambda), \Lambda_t(\lambda)).$$

Thus, in particular $F_1 = F_0 \circ \Theta_1^{-1}$. Consider the restriction of Θ_1^{-1} to the subspace $\lambda = 0$. We obtain

$$f_0(w) + f_*(w) = f_0(\widetilde{W}(w, 0)) + \sum \widetilde{\Lambda}_1(0) e_i(\widetilde{W}(w, 0)).$$

The Lemma is proven.

∎

Lemma 1.3.6 [24] *Assume we have smooth functions $\Phi_i(w,\lambda)$ and $e_j(w)$. If for any $\varphi(w,0)$ there is a decomposition $\varphi(w,0) = \sum \Phi_i(w,0)\dot{w}_i(w) + \sum c_j e_j(w)$, then for any smooth function $\varphi(w,\lambda)$ there is a decomposition*

$$\varphi(w,\lambda) = \sum \Phi_i(w,\lambda)\dot{\tilde{w}}_i(w,\lambda) + \sum c_j(\lambda)e_j(w),$$

with $\dot{\tilde{w}}_i(w,0) = \dot{w}_i(w)$ and $c_j(0) = c_j$ and for λ close to zero.

Proof. Consider the mapping

$$M : (w,\lambda) \mapsto (\Phi_i, \lambda_1, \ldots, \lambda_s).$$

By H'Adamard Lemma we have

$$\begin{aligned}
\varphi(w,\lambda) &= \varphi(w,0) + \sum \lambda_i H_i(w,\lambda) \\
&= \sum \Phi_i(w,\lambda)\dot{w}_i(w) + \sum c_j e_j(w) + \sum \lambda_i H_i(w,\lambda) - \sum \lambda_i \widetilde{\Phi}_i(w,\lambda)\dot{w}_i(w) \\
&= \sum \Phi_i(w,\lambda)\dot{w}_i(w) + \sum c_j e_j(w) + \sum \lambda_i \left(H_i(w,\lambda) - \widetilde{\Phi}_i(w,\lambda\right) \dot{w}_i(w).
\end{aligned}$$

This is exactly the form we need to use Malgrange theorem as e_j form a basis for the local algebra $\mathcal{Q} = \mathbf{C}_{w,\lambda}/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by Φ_i and λ_i (the components of the mapping M above). Applying Magrange preparation theorem for the mapping M, we obtain that for any smooth function $\varphi(w,\lambda)$

$$\varphi(w,\lambda) = \sum e_j(w)C_j\left(\Phi_i,\lambda\right).$$

Therefore, using H'Adamard Lemma we see that

$$\varphi(w,\lambda) = \sum e_j(w)C_j(0,\lambda) + \sum \Phi_i \left(\sum e_j(w)\widetilde{C}_{i,j}(w,\lambda)\right).$$

as required.

∎

Remark: In fact, in lemma 1.3.5, we have used Tougeron's theorem which states that:

Theorem 1.3.7 [1] *If a function f has finite multiplicity μ, that is the local algebra $\mathcal{A} = \mathbf{C}_w/\{I_f\} = \mathbb{R}\{\delta_1, \ldots, \delta_\mu\}$ is of dimension μ, then $\mathcal{M}^N \subset I_f$ for any $N > \mu$.*

Proof. Suppose that the local algebra $\mathcal{A} = \mathbf{C}_w/I_f = \mathbb{R}\{\delta_1, \ldots, \delta_\mu\}$ is of dimension μ.

Consider the ideal $\mathcal{M}^N = \{w_1^{k_1} w_2^{k_2} \ldots w_n^{k_n}\}$ with $\sum k_i = N$.

Take some monomial $h_{(k_1,\ldots k_n)} = h_N = w_1^{k_1} w_2^{k_2} \ldots w_n^{k_n}$, where $\sum k_i = N$. Take a sequence $h_j = h_{j-1}.b_j$ where $b_j \in \mathcal{M}$.

Denote by $[a(w)] = [C_w/I_f]$ the class in the factor algebra $\mathcal{A}$ and consider the sequence

$$[h_1], [h_2], \ldots [h_N].$$

So if $N > \mu$, then the classes $[h_i]$ are linearly dependent . This means that

$$\alpha_p[h_p] + \alpha_{p+1}[h_{p+1}] + \cdots + \alpha_N[h_N] = [0]$$

for some $[h_p]$, where $\alpha_p \neq 0$. Since $h_{p+1} = h_p.b_{p+1}$ with $b_{p+1} \in \mathcal{M}$, we get

$$\alpha_p[h_p]\left\{1 + \frac{\alpha_{p+1}}{\alpha_p}[b_{p+1}] + \ldots\right\} = [0].$$

This yields that $[h_p] \equiv [0]$. So $h_p \in I_f$. Hence, $h_N = h_p.B \in I_f$.

∎

Remark: Dropping the condition that f_0 has finite multiplicity, the proof of Lemma 1.3.5 implies the existence of similar prenormal form

$$f_0 + \sum \lambda_i c_i e_i + \widetilde{\psi},$$

for any function $f_0 + f_*$ up to the addition of an error term $\widetilde{\psi}$ which belongs to a sufficiently large power of the maximal ideal.

1.3.2 Quasi fixed equivalence

In this section we introduce another non-standard equivalence relation. It is "weaker" than quasi border equivalence. However, it is universal and does not depend on the border germ.

Consider the space $\mathbb{R}^n$ with local coordinates $x = (x_1, x_2, \ldots, x_n)$. Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a function germ defined on the space $\mathbb{R}^n$.

Definition 1.3.1 Two functions $f_0, f_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ are called *pseudo fixed* equivalent if there exists a diffeomorphisim $\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f_1 \circ \theta = f_0$, and if c is a critical point of the function f_0, then θ fixes the point c (maps it to itself) and vice versa if c is a critical point of the function f_1 then θ^{-1} also fixes c.

Now assume we have a family f_t of function germs which are pseudo fixed equivalent: $f_t \circ \theta_t = f_0$, $t \in [0, 1]$ with respect to a smooth family $\theta_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of diffeomorphisms, then we have the derivative equation:

$$-\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial x}\dot{X},$$

where the vector field $v = \dot{X}_1 \frac{\partial}{\partial x_1} + \cdots + \dot{X}_n \frac{\partial}{\partial x_n}$ on $\mathbb{R}^n$ generates the phase flow θ_t and its components satisfy the conditions:

$$\dot{X}_i \in \mathbf{C}_{x,t}\left\{Rad\{\frac{\partial f_t}{\partial x}\}\right\}.$$

As before we replace $Rad\{\frac{\partial f_t}{\partial x}\}$ by the ideal $\{\frac{\partial f_t}{\partial x}\}$ itself and get the following definition.

Definition 1.3.2 Two functions f_0 and f_1 are called *quasi fixed equivalent*, if they are pseudo fixed equivalent and there is a family of function germs f_t which continuously depends on parameter $t \in [0, 1]$ and a continuous piece-wise smooth family of diffeomorphisms $\theta_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ depending on parameter $t \in [0, 1]$ such that: $f_t \circ \theta_t = f_0$ and $\theta_0 = id$ and the components of the vector field v satisfy:

$$\dot{X}_i \in \mathbf{C}_{x,t}\left\{\frac{\partial f_t}{\partial x}\right\}.$$

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ of the form $f = f_2(x) + f^*(x)$, where f_2 is a quadratic form in x and $f^* \in \mathcal{M}_x^3$. We start with redducing the quadratic form f_2.

Lemma 1.3.8 *If $f_2(x)$ is a non-degenerate quadratic form with a critical point at the origin then $f_2(x)$ is quasi fixed equivalent to $\sum_{i=1}^{n} \pm x_i^2$.*

Proof. The proof is straightforward. However, we give full details as simple example of application of Moser homotopy method.

Take a homotopy F_t joining f_2 and standard quadratic form $\sum_{i=1}^{n} \pm x_i^2$ such that F_t is a non-degenerate form in x for any $t \in [0,1]$ and F_t is of the form $F_t = \sum_{i,j=1}^{n} a_{i,j}(t) x_i x_j$, where $F_1 = Q_2$ and $F_2 = f_2$.

We now use Moser's homotopy method and consider the homological equation

$$-\frac{\partial F_t}{\partial t} = \frac{\partial F_t}{\partial x_1}\left\{\sum_{i=1}^{n} \frac{\partial F_t}{\partial x_i} A_i^{(1)}\right\} + \cdots\cdots + \frac{\partial F_t}{\partial x_n}\left\{\sum_{i=1}^{n} \frac{\partial F_t}{\partial x_i} A_i^{(n)}\right\}.$$

Note that all $\frac{\partial F_t}{\partial x_i}$ are independent linear forms. Therefore, let $\widetilde{x}_i = \frac{\partial F_t}{\partial x_i}$ and up to linear transformation we get the equivalent homological equation

$$-\frac{\partial F_t}{\partial t} = \widetilde{x}_1\left\{\sum_{i=1}^{n} \widetilde{x}_i \widetilde{A}_i^{(1)}\right\} + \cdots\cdots + \widetilde{x}_n\left\{\sum_{i=1}^{n} \widetilde{x}_i \widetilde{A}_i^{(n)}\right\}.$$

On the other hand, we have

$$-\frac{\partial F_t}{\partial t} = \sum_{i,j=1}^{n} b_{i,j}(t) \widetilde{x}_i \widetilde{x}_j.$$

Thus, given $b_{i,j}(t)$ for any $i, j \in \{1, \ldots, n\}$ one can find easily solutions for the homological equation. Hence we conclude that the previous homological equation is solvable. Thus all F_t are quasi fixed equivalent. In particular $f_2(x)$ is quasi fixed equivalent to $\sum_{i=1}^{n} \pm x_i^2$.

∎

The previous Lemma implies the following

Theorem 1.3.9 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ at the origin with a non-degenerate quadratic form $f_2(x)$ then f is quasi fixed equivalent to $\sum_{i=1}^{n} \pm x_i^2$.*

Proof. Lemma 1.3.8 shows that f is quasi fixed equivalent to a germ

$$G(x) = \sum_{i=1}^{n} \pm x_i^2 + Q(x), \quad \text{where} \quad Q(x) \in \mathcal{M}_x^3.$$

Consider the quasi homogeneous part $Q_2(x, y) = \sum_{i=1}^{n} \pm x_i^2$ being the principal part with weights $w_{x_i} = \frac{1}{2}$ for all i. The tangent space at Q_2 with respect to quasi fixed equivalence takes the form

$$TQF_{Q_2} = \pm 2x_1 \left\{ \sum_{i=1}^{n} \pm 2x_i A_i^{(1)} \right\} \pm \cdots \pm 2x_n \left\{ \sum_{i=1}^{n} \pm 2x_n A_i^{(n)} \right\}.$$

Then for any monomial of degree greater that 1 for example $g^* = x_i^2 x_1^{i_1} \dots x_n^{i_n}$, divisible by x_i^2, there is $A_i^{(i)}$ such that $g^* = \pm 4x_i^2 A_i^{(i)}$ with $A_i^{(i)} = \frac{\pm 1}{4} x_1^{i_1} \dots x_n^{i_n} \in \mathcal{M}_x$. Similar argument holds for other monomials $g^* = x_i x_j A$. Now using Lemma (1.3.5) we see that that the germ

$$\Phi = A_i^{(i)} \frac{\partial g^*}{\partial x_i} \frac{\partial g^*}{\partial x_i} + A_i^{(i)} \frac{\partial Q_2}{\partial x_i} \frac{\partial g^*}{\partial x_i},$$

has quasi degree $d(\Phi)$ greater than the quasi degree $d(g^*)$. Hence we conclude that G is quasi fixed equivalent to the germ Q_2 as required. ∎

1.3.3 Quasi partially x-fixed equivalence

Consider a version of quasi fixed equivalence. This yet another special equivalence relation happens to be useful in many proofs. It allows us to get a prenormal form of a function germ with respect to quasi border equivalence. So, we introduce the following:

Consider the space $\mathbb{R}^n = \{w = (x, y)\}$, where $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $y = (y_1, y_2, \ldots, y_{n-m}) \in \mathbb{R}^{n-m}$. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ defined on the space $\mathbb{R}^n$.

Definition 1.3.3 Two functions $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}$ are called *pseudo partially x-fixed* equivalent if there exists a diffeomorphisim $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_1 \circ \theta = f_0$, and if $c = (x_0, y_0)$ is a critical point of the function f_0 , then θ sends c to the point $\widetilde{c} = (x_0, Y(x_0, y_0))$ and vice versa if $c = (x_0, y_0)$ is a critical point of the function f_1 then θ^{-1} sends c to $\widetilde{c} = (x_0, Y(x_0, y_0))$. In other words, it is pseudo border equivalence when the border is the $(0, y)$ coordinate subspace.

Now assume we have a family f_t of function germs which are pseudo partially x-fixed equivalent: $f_t \circ \theta_t = f_0,\ t \in [0, 1]$ with respect to a smooth family $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ of diffeomorphisms, then we have the derivative equation:

$$-\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial x}\dot{X} + \frac{\partial f_t}{\partial y}\dot{Y},$$

where the vector field $v = \dot{X}\frac{\partial}{\partial x} + \dot{Y}\frac{\partial}{\partial y}$ on $\mathbb{R}^n$ generates the phase flow θ_t and its components satisfy the conditions:

$$\dot{Y} \in \mathbf{C}_{x,y,t}, \quad \dot{X} \in \mathbf{C}_{x,y,t}\left\{Rad\{\frac{\partial f_t}{\partial w}\}\right\}.$$

As before we replace $Rad\{\frac{\partial f_t}{\partial w}\}$ by the ideal $\{\frac{\partial f_t}{\partial w}\}$ itself and get the following definition.

Definition 1.3.4 Two function f_0 and f_1 are called *quasi partially x-fixed equivalent* if they are pseudo partially x-fixed equivalent and there is a family of function germs f_t which continuously depends on parameter $t \in [0, 1]$ and a continuous piece-wise smooth family of diffeomorphisms $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ depending on parameter $t \in [0,1]$ such that:$f_t \circ \theta_t = f_0$ and $\theta_0 = id$ and the components of the vector field v satisfy:

$$\dot{Y} \in \mathbf{C}_{x,y,t}, \quad \dot{X} \in \mathbf{C}_{x,y,t}\left\{\frac{\partial f_t}{\partial w}\right\}.$$

Remark: Apparently, quasi partially x-fixed equivalence implies quasi border equivalence, as the quasi border tangent space contains the quasi partially x-fixed tangent space, provided that the border contains the $(0, y)$- coordinate subspace. Hence, all quasi x-fixed partially equivalence properties are valid for quasi border equivalence. The simple classes for quasi partially x-fixed equivalence (e.g Morse functions) remain simple for quasi border equivalence.

Lemma 1.3.10 *Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function germ with an isolated critical point at the origin. Then, f is quasi fixed (or quasi partially x-fixed equivalent) for each $t \in [0,1]$ to the function germ $g_t(w) = f(w) + th(w)$ with $h(w) \in \{\frac{\partial f}{\partial w_i}\}^2$, provided that the rank of the second differential $d_0^2 g_t$ of g_t at the origin is constant.*

Proof. At first we claim that if the rank of $d_0^2 g_t$ is constant then for different t the gradient ideals $I_t = \left\{\frac{\partial g_t}{\partial w}\right\}$ coincide. In fact, this claim does not depend on the choice of local coordinates so we may assume that the second differential at the origin of f has diagonal form $d_0^2 f = \sum_{i=1}^{r} \varepsilon_i w_i^2$, where $\varepsilon_i = \pm 1$ for $i = 1, \dots, r$ and $\varepsilon_i = 0$ for $i > r$.

The second jet of g_t at the origin takes the form

$$d_0^2 g_t = \sum_{i,j=1}^{r} \left(\varepsilon_i \delta_{ij} w_i^2 + 4th_{i,j}(0)\varepsilon_i \varepsilon_j w_i w_j\right) = \sum_{i,j=1}^{r} D_{i,j}^{(t)} w_i w_j$$

where $h_{i,j},\ i,j = 1, \dots, n$, are coefficients of the decomposition

$$h(w) = \sum_{i,j=1}^{n} h_{i,j}(w) \frac{\partial f}{\partial w_i} \frac{\partial f}{\partial w_j}$$

of the function h, $\delta_{i,j}$ is the Kronecker symbol, and $D_{i,j}^{(t)} = \varepsilon_i \delta_{i,j} + 4t\varepsilon_i \varepsilon_j h_{i,j}(0)$. We can assume here that $h_{i,j} = h_{j,i}$.

The $r \times r$ matrix with entries $D^{(t)}_{i,j}$ at any t is invertible since the rank of $d^2_0 g_t$ is r. Reversing signs of some rows of this matrix $D^{(t)}_{i,j}$ we see that the $n \times n$ matrix with the entries $\hat{D}^{(t)}_{i,j} = \delta_{i,j} + 4t\varepsilon_j h_{i,j}(0)$ for $i, j = 1, \dots, r$ and $\hat{D}^{(t)}_{i,j} = \delta_{i,j}$ otherwise, is also invertible.

The derivative

$$\frac{\partial g_t}{\partial w_i} = \frac{\partial f}{\partial w_i} + 2t \sum_{k,m=1}^{n} \left(2h_{k,m} \frac{\partial^2 f}{\partial w_k \partial w_i} + \frac{\partial h_{k,m}}{\partial w_i} \frac{\partial f}{\partial w_k} \right) \frac{\partial f}{\partial w_m}$$

implies that $\{\frac{\partial g_t}{\partial w}\} \subset \{\frac{\partial f}{\partial w}\}$. This derivative can be written as

$$\frac{\partial g_t}{\partial w_i} = \sum_{j=1}^{n} (\delta_{i,j} + 4t\varepsilon_j h_{i,j}(0) + R_{i,j}) \frac{\partial f}{\partial w_j} = \sum_{j=1}^{n} \left(\hat{D}_{i,j} + R_{i,j} \right) \frac{\partial f}{\partial w_j},$$

where the functions $R_{i,j}$ vanish at the origin. So in some (smaller) neighborhood of the origin $\hat{D}_{i,j} + R_{i,j}$ is invertible. This implies that $\{\frac{\partial f}{\partial w}\} \subset \{\frac{\partial g_t}{\partial w}\}$

Therefore the derivatives $\frac{\partial g_t}{\partial w_i}$ also form a basis for the gradient ideal $I_0 = \{\frac{\partial f}{\partial w}\}$ as claimed.

Now the homological equation $-\frac{\partial g_t}{\partial t} = \sum_{i=1}^{n} \frac{\partial g_t}{\partial w_i} V_i$ can be solved for the unknown functions V_i belonging for any t to the gradient ideal I_t, since the left hand side belongs to the square of this ideal. The phase flow of the vector field $\sum V_i \frac{\partial}{\partial w_i}$ leaves all critical points of g_t fixed.

∎

In fact, we have proven also the following useful Lemma.

Lemma 1.3.11 *If $G_t(w)$ is quasi fixed (or quasi partially x-fixed) admissible family of germs then for any function $H(t, w) \in \{\frac{\partial G_t}{\partial w}\}^2$ the family $\widetilde{G}_t(w) = G_t(w) + H(t, w)$ is also admissible and $\widetilde{G}_t$ is quasi partially fixed equivalent to G_t for each value of t provided that the rank of the second differential of $G_t + \tau H$ is constant for any $t, \tau \in [0, 1]$.*

Lemma 1.3.12 *If $\Phi_t(x,y)$ is an admissible deformation with respect to quasi partially x-fixed equivalence(or quasi fixed equivalence) and $w = (x,y) \mapsto (X_t(x), Y_t(x,y))$ is a family of diffeomorphisms of $\mathbb{R}^n$, which preserve the fibration $((x,y),0) \mapsto (x,0)$, then $G_t(x,y) = \Phi_t(X_t(x), Y_t(x,y))$ is also an admissible family.*

Proof. The claim that the deformation $\Phi_t(X,Y)$ is admissible means that

$$-\frac{\partial\Phi_t(X,Y)}{\partial t} = \sum \frac{\partial\Phi_t(X,Y)}{\partial X_i}\left\{\sum A_i(X,Y,t)\frac{\partial\Phi_t(X,Y)}{\partial X_i}\right\} + \sum \frac{\partial\Phi_t(X,Y)}{\partial Y_i}\dot{Y}_i,$$

with some smooth functions $A_i, \dot{Y}_i$.

The matrices $\frac{\partial X}{\partial x}$ and $\frac{\partial Y}{\partial y}$ are invertible and $\frac{\partial\Phi_t}{\partial x} = \frac{\partial\Phi_t}{\partial X}\frac{\partial X}{\partial x}$, $\frac{\partial\Phi_t}{\partial y} = \frac{\partial\Phi_t}{\partial Y}\frac{\partial Y}{\partial y}$. Hence, the decomposition can be written in the form

$$\begin{aligned} -\frac{\partial\Phi_t(X(x),Y(x,y))}{\partial t} &= \sum \frac{\partial\Phi_t(X(x),Y(x,y))}{\partial x_i}\Big\{\sum \widetilde{A}_i(X(x),Y(x,y),t) \\ \frac{\partial\Phi_t(X(x),Y(x,y))}{\partial x_i}\Big\} &+ \sum \frac{\partial\Phi_t(X(x),Y(x,y))}{\partial y_i}\widetilde{Y}_i, \end{aligned}$$

with some smooth functions $\widetilde{Y}_i, \widetilde{A}_i$. This means that the family G_t is admissible. ∎

Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function germ with a critical point at the origin. Denote by $f^*(y) = f|_{x=0}$, the restriction of function f to the y coordinates subspace. Denote by r^* the rank of the second differential $d_0^2 f^*$ at the origin and set $c = n - m - r^*$.

Lemma 1.3.13 *(Stabilization) The function germ $f(x,y)$ is quasi partially x-fixed equivalent to $\sum_{i=1}^{r^*} \pm y_i^2 + g(x,\widetilde{y})$, where $\widetilde{y} \in \mathbb{R}^c$ and $g^* \in \mathcal{M}_{\widetilde{y}}^3$. For quasi partially x-fixed equivalent f germs, the respective reduced germs g are quasi partially x-fixed equivalent.*

Proof. Since the rank of the second differential $(d_0^2 f^*)$ is equal r^*, then after an admissible linear transformation the function germ f can be written as:

$$f_1 = \sum_{i=1}^{r^*} \pm y_i^2 + \sum_{i=1}^{n-m} \sum_{j=1}^{m} a_{i,j} y_i x_j + Q_2(x) + f_3(x,y),$$

with $f_3 \in \mathcal{M}_{x,y}^3$ and Q_2 is a quadratic form in x only.

Let $\hat{y} = (y_1, y_2, \dots, y_{r^*})$ and $\widetilde{y} = (y_{r^*+1}, \dots, y_{n-m}) \in \mathbb{R}^{n-m-r^*}$. Then, the previous form can be written as:

$$f_1 = \sum_{i=1}^{r^*} \pm y_i^2 + \varphi(x, \hat{y}, \widetilde{y}) + \widetilde{f}_1(x, \widetilde{y}),$$

where

$$\varphi = \sum_{i=1}^{r^*} \sum_{j=1}^{n} a_{i,j} y_i x_j + \sum_{l=1}^{r^*} y_l \widetilde{\varphi}(x,y) \quad \text{with} \quad \widetilde{\varphi} \in \mathcal{M}_{x,y}^2,$$

and

$$\widetilde{f}_1 = Q_2(x) + \sum_{i=r^*+1}^{n-m} \sum_{j=1}^{n} b_{i,j} y_i x_j + \widetilde{f}_3(x, \widetilde{y}) \quad \text{with} \quad \widetilde{f} \in \mathcal{M}_{x,\widetilde{y}}^3.$$

Now we try to find the family of diffeomorphisms which preserve the fibration

$$\theta_t : (x,y) \mapsto \left(x, \hat{Y}(t, x, \hat{y}), \widetilde{Y}(x, \widetilde{y})\right).$$

Take the family

$$f_t = \sum_{i-1}^{r^*} \pm y_i^2 + t\varphi(x, \hat{y}, \widetilde{y}) + \widetilde{f}_t(x, \widetilde{y}),$$

which joins f_1 and $f_0 = \sum_{i=1}^{r^*} \pm y_i^2 + \widetilde{f}_0(x, \widetilde{y})$ with $t \in [0,1]$ and $\widetilde{f} = \widetilde{f}_1$. Here, $\widetilde{f}_t$ and $\widetilde{f}_0$ are unknown. So, we want to solve the homological equation for $\dot{y}$ and also for $\widetilde{f}_t$ simultaneously.

The homological equation takes the form

$$-\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial x}\dot{x} + \sum_{i=1}^{r^*} \frac{\partial f_t}{\partial y_i}\dot{y}_i + \sum_{j=r^*+1}^{n-m} \frac{\partial f_t}{\partial y_j}\dot{y}_j.$$

Note here that $\dot{x} = \dot{y}_j = 0$ for $j = r^* + 1, \ldots, n - m$ as they do not depend on t.

Thus the previous equation can be written as:

$$-(\varphi + \frac{\partial \widetilde{f}_t}{\partial t}) = \sum_{i=1}^{r^*} (\pm 2y_i + t\frac{\partial \varphi}{\partial y_i})\dot{y}_i.$$

Set $w_i = \pm 2y_i + t\frac{\partial \varphi}{\partial y_i}$, $i = 1, \ldots, r^*$, which are the known functions. Note that the matrix $(\frac{\partial w}{\partial \hat{y}})$ has the maximal rank at the origin for any value of t . Hence we can take the new coordinates w_i instead of $\hat{y}$. Let $w = (w_1, w_2, \ldots, w_{r^*})$. Thus the previous equation takes the form:

$$-(\varphi + \frac{\partial \widetilde{f}_t}{\partial t}) = \sum_{i=1}^{r^*} w_i \dot{y}_i$$

Using H'Adamard Lemma, we can write this as:

$$\sum_{i=1}^{r^*} w_i \psi_i(x, w, \widetilde{y}, t) + \phi(x, \widetilde{y}, t) + \frac{\partial \widetilde{f}_t}{\partial t} = \sum_{i=1}^{r^*} -w_i \dot{y}_i$$

By taking $\psi_i = -\dot{y}_i$ and $\frac{\partial \widetilde{f}_t}{\partial t} = -\phi$, we have shown that the homological equation is solvable. Note that the vector field $\dot{v} = \sum_{i=1}^{n} y_i$ is defined in some neighborhood of the origin as $w_i(0) = 0$. Hence all f_t are quasi equivalent. In particular, the function germ f_1 is quasi partially x-fixed equivalent to f_0. The last step is to find $\widetilde{f}_0$. This can be done using the following relation:

$$-\int_0^1 \phi dt = \int_0^1 \frac{\partial \widetilde{f}_t}{\partial t} dt = \widetilde{f}_1 - \widetilde{f}_0.$$

The second claim can be deduced directly using Lemma 1.3.12 as $\theta_t : (x, y) \mapsto \left(x, \hat{Y}(t, x, \hat{y}), \widetilde{Y}(x, \widetilde{y})\right)$ preserves the projection $(x, \hat{y}, \widetilde{y}) \mapsto (x, \widetilde{y})$.

∎

In fact, Lemma 1.3.11 implies the following improved stabilization splitting

Lemma 1.3.14 *There is a non-negative integer $s \leq r - r^*$ such that the function* *germ $f(x,y)$ is quasi partially fixed equivalent to $\sum_{i=1}^{r^*+s} \pm y_i^2 + \widetilde{f}(x,\widetilde{y})$, where $\widetilde{y} \in \mathbb{R}^{c-s}$ and $\widetilde{f}$ is a sum of a function germ from $\mathcal{M}^3_{x,\widetilde{y}}$ and a quadratic form in x only. For quasi partially fixed equivalent f germs, the respective reduced germs $\widetilde{f}$ are quasi partially fixed equivalent.*

Proof. After the stabilization procedure via Lemma (3.1.1) the quadratic form $f_2 = \sum_{i=1}^{r^*} \pm y_i^2 + x_1 \sum_{i=r^*+1}^{n-m} \alpha_i^{(1)} y_i + \cdots + x_m \sum_{i=r^*+1}^{n-m} \alpha_i^{(m)} y_i + g_2(x)$ with some coefficients $\alpha_i^{(j)}$ and quadratic form g_2 in x only. Let some of these coefficients, for example $\alpha_{r^*+1}^{(1)} \neq 0$, then summing up the function f with $\delta \left(\frac{\partial f}{\partial x_1}\right)^2$ for sufficiently small δ, Lemma 1.3.11 yields a new function $\widehat{f}$ which is quasi partially fixed equivalent to f and has non-zero quadratic term with $y_{r^*+1}^2$. Therefore the rank of the restriction of $\widehat{f}$ to $x = 0$ subspace becomes larger that r^*. Repeating the procedure several times, if needed, we get the function germ with some larger value of r^* and without any $\alpha_i^{(j)}$ coefficients. This is exactly the required form.

∎

Chapter 2

Quasi boundary singularites

2.1 The classification of simple classes

Following Arnold [1], we discuss the description of simple classes. A function germ is called simple if its neighborhood in the space of function germs contains only a finite number of quasi equivalence classes.

Apparently the quasi border classification of critical points outside the border Γ coincides with the standard right equivalence. Hence the standard classes A_k, D_k, E_6, E_7 and E_8 form the list of simple classes in this case. Also by definition non-critical points are all equivalent wherever they are. So we classify only critical points.

For a function germ $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$, which has a critical point at the origin, denote by f_2 its quadratic form. So we assume that f has the form $f = f_2 + \widetilde{f}_3$ and $\widetilde{f}_3 \in \mathcal{M}^3$.

In this chapter the coordinates are as follows: $\mathbb{R}^n = \{(x, y)\}$, where $x = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. We consider germs of C^∞ -smooth functions $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, with the boundary Γ_b given by the equation $y = 0$.

The quasi boundary tangent space to an admissible deformation f_t at the origin takes the form

$$TQB_{f_t} = \left\{ \sum_{i=1}^{n-1} \frac{\partial f_t}{\partial x_i} A_i + \frac{\partial f_t}{\partial y} \left(yB_1 + \frac{\partial f_t}{\partial y} B_2 + \sum_{i=1}^{n-1} \frac{\partial f_t}{\partial x_i} \widetilde{A}_i \right) \right\}$$

for arbitrary smooth functions A_i, B_1, B_2 and $\widetilde{A}_i$.

For convenience, we rewrite the auxiliary Lemmas of the section 1.3 in chapter 1 and specialise properties of quasi border equivalence for the case of quasi boundary singularities in the new coordinates.

Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function germ with a critical point at the origin. Denote by $f^*(x) = f|_{y=0}$, the restriction of the function f to the x coordinates subspace.

Denote by r^* the rank of the second differential $d_0^2 f^*$ at the origin and set $c = n - 1 - r^*$. Let r be the rank of the second differential $d_0^2 f$ and $k = n - r$ the respective corank, then

Lemma 2.1.1 *(Stabilization) The function germ $f(x, y)$ is quasi boundary equivalent to $\sum_{i=1}^{r^*} \pm x_i^2 + g(\hat{x}, y)$, where $\hat{x} \in \mathbb{R}^c$ and the restriction $g^*(\hat{x}) = g(x, 0) \in \mathcal{M}_{\hat{x}}^3$. For quasi boundary equivalent f germs, the respective reduced germs g are quasi boundary equivalent.*

Lemma 2.1.2 *There is a non-negative integer $s \leq r - r^*$ such that the function germ $f(x, y)$ is quasi boundary equivalent to $\sum_{i=1}^{r^*+s} \pm x_i^2 + \widetilde{f}(\widetilde{x}, y)$, where $\widetilde{x} \in \mathbb{R}^{c-s}$ and $\widetilde{f} = ay^2 + g(\widetilde{x}, y)$, here $g \in \mathcal{M}_{\widetilde{x},y}^3$ and $a \in \mathbb{R}$. For quasi boundary equivalent f germs, the respective reduced germs $\widetilde{f}$ are quasi boundary equivalent.*

The main prenormal forms of quasi boundary singularities are given in the following:

Lemma 2.1.3 *1. If $k = 0$, then f is quasi boundary equivalent to Morse function $\mathcal{B}_2 : \pm y^2 + \sum_{i=1}^{n-1} \pm x_i^2$.*

2. If $k = 1$, then f is quasi boundary equivalent to either $\sum_{i=1}^{n-1} \pm x_i^2 + \widetilde{f}(y)$ with $\widetilde{f} \in \mathcal{M}_y^3$ or to $\sum_{i=2}^{n-1} \pm x_i^2 + ay^2 + \widetilde{f}(x_1, y)$ with some $a \neq 0, a \in \mathbb{R}$ and $\widetilde{f} \in \mathcal{M}_{x_1,y}^3$.

3. If $k \geq 2$, then f is a non-simple germ.

Proof. For $k = 0$, we have $n = r$. By Lemma 2.1.2, there is a non-negative number s such that $s \leq r - r^* = n - r^*$. We shall consider all possible choices for s. If $s = n - r^*$, then $c - s = n - 1 - r^* - n + r^* = -1$. So this choice is not possible. Next, if $s = n - r^* - 1$, then $c = s$. Hence f is quasi boundary equivalent to the germ $\widetilde{F} = \sum_{i=1}^{n-1} \pm x_i^2 + ay^2 + \widetilde{f}(y)$ with $a \neq 0, a \in \mathbb{R}$ and $\widetilde{f} \in \mathcal{M}_y^3$. By standard boundary equivalence (also, the germ is quasi simple fixed singularity, hence simple with respect to quasi boundary equivalence), we see that $\widetilde{F}$ is quasi boundary equivalent to $\sum_{i=1}^{n-1} \pm x_i^2 \pm y^2$. If $s = n - r^* - 2$, then $c - s = 1$. Hence f is quasi boundary equivalent to the germ $\widetilde{G} = \sum_{i=2}^{n-1} \pm x_i^2 + ay^2 + \widetilde{f}(x_1, y)$ with $a \in \mathbb{R}$ and $\widetilde{f} \in \mathcal{M}_{y,x_1}^3$. However, the germ $\widetilde{G}$ has total rank equal to $n - 1$. The total rank is quasi boundary invariant for equivalent germs. Hence, the choice of $s \leq n - 2 - r^*$ is not possible in this case.

For $k = 1$, we have $r = n - 1$. If $s = n - 1 - r^*$, then $c = s$. Hence f is quasi boundary equivalent to the germ $\widetilde{F} = \sum_{i=1}^{n-1} \pm x_i^2 + g(y)$ with $g \in \mathcal{M}_y^3$. If $s = n-2-r^*$, then $c - s = 1$. Thus f is quasi boundary equivalent to $\widetilde{G} = \sum_{i=2}^{n-1} \pm x_i^2 + ay^2 + \widetilde{g}(x_1, y)$ with $a \neq 0, a \in \mathbb{R}$ and $\widetilde{g} \in \mathcal{M}_{x_1,y}^3$. Finally, if $s = n - 3 - r^*$, then $c - s = 2$. Thus f is quasi equivalent to $\widetilde{H} = \sum_{i=3}^{n-1} \pm x_i^2 + ay^2 + \widetilde{g}(x_1, x_2, y)$ with $a \neq 0, a \in \mathbb{R}$. However, $\widetilde{H}$ has total rank r equal to $n - 2$. Hence, the choice of $s \leq n - 3 - r^*$ is not possible in this case.

For $k \geq 2$, consider first the case $k = 2$. If $s = n - 2 - r^*$, then $c - s = 1$. Hence f is quasi boundary equivalent to the germ $\sum_{i=2}^{n-1} \pm x_i^2 + \widetilde{f}(x_1, y)$ with $\widetilde{f} \in \mathcal{M}_{x_1,y}^3$. If $s = n - 3 - r^*$, then $c - s = 2$. Hence, f is quasi boundary equivalent to the germ $\sum_{i=3}^{n-1} \pm x_i^2 + ay^2 + \widetilde{f}(x_1, x_2, y)$, with $a \neq 0, a \in \mathbb{R}$ and $\widetilde{f} \in \mathcal{M}_{x_1,x_2,y}^3$. The choice of $s \leq n - 4 - r^*$ is not compatible with $k = 2$. Following the procedures similar to the case $k = 2$ we see that for any $k \geq 2$ the function germ f is quasi boundary equivalent either to the germ $G = \sum_{i=k+1}^{n-1} \pm x_i^2 + ay^2 + \widetilde{f}(x_1, x_2, \ldots, x_k, y)$ or to the germ $F = \sum_{i=k}^{n-1} \pm x_i^2 + \widetilde{f}(x_1, x_2, \ldots, x_{k-1}, y)$ where $\widetilde{f} \in \mathcal{M}_{x_1,x_2,\ldots,x_k,y}^3$ and $a \neq 0, a \in \mathbb{R}$. The germs G and F are non-simple by the following Lemma.

∎

Lemma 2.1.4 *The function germs of the form:*

1. $F(x,y) = \sum_{i=k}^{n-1} \pm x_i^2 + \widetilde{f}(x_1, x_2, \ldots, x_{k-1}, y)$,

2. $G(x,y) = \sum_{i=k+1}^{n-1} \pm x_i^2 + ay^2 + \widetilde{g}(x_1, x_2, \ldots, x_k, y)$,

where $\widetilde{f} \in \mathcal{M}^3_{x_1,x_2,\ldots,x_{k-1},y}$, $\widetilde{g} \in \mathcal{M}^3_{x_1,x_2,\ldots,x_k,y}$ *and* $a \neq 0, a \in \mathbb{R}$ *are non simple with respect to quasi boundary equivalence, for* $k \geq 2$.

Proof. Consider the function germ F. Then, the tangent space to the orbit of quasi equivalence at the germ $\widetilde{f}$ at the origin takes the form:

$$TQB_{\widetilde{f}} = \sum_{i=1}^{k-1} \frac{\partial \widetilde{f}}{\partial x_i} A_i + \frac{\partial \widetilde{f}}{\partial y} \left\{ yB_1 + \frac{\partial \widetilde{f}}{\partial y} B_2 + \sum_{i=1}^{k-1} \frac{\partial \widetilde{f}}{\partial x_i} C_i \right\}.$$

Consider the projection of $TQB_{\widetilde{f}}$ to the 3-jet space at the origin. The cubic terms in $TQB_{\widetilde{f}}$ depend only on $\sum_{i=1}^{k-1} \frac{\partial \widetilde{f}}{\partial x_i} A_i$, where $A_i = a_0 y + \sum_{i=1}^{k-1} a_i x_i$, and $\frac{\partial \widetilde{f}}{\partial y} y B_1$, where $a_0, a_i, B_1 \in \mathbb{R}$. So the projection coincides with the projection of 3-jet of the tangent space of standard orbit of Arnold boundary equivalence. They form a subspace of dimension $k(k-1)+1$. This dimension is less than the $M = \frac{(k+2)(k+1)k}{6}$. Here, M stands for the dimension of all homogeneous cubic terms of the variables $x_1, x_2, \ldots, x_{k-1}$ and y. Hence cubic terms can not belong to finitely many orbits.

Consider the function germ $G(x,y) = \sum_{i=k+1}^{n-1} \pm x_i^2 + ay^2 + \widetilde{g}(x_1, x_2, \ldots, x_k, y)$. Let $\widetilde{G}_\delta = \sum_{i=k+1}^{n-1} \pm x_i^2 + a(y + \delta x_k)^2 + \widetilde{g}(x_1, x_2, \ldots, x_k, y)$ for sufficiently small δ. Note that all $\widetilde{G}_\delta$ are in the same orbit and G is in the closure of $\widetilde{G}_\delta$. By stabilization lemma, $\widetilde{G}$ is quasi boundary equivalent to the germ $\widetilde{F} = \sum_{i=k}^{n-1} \pm x_i^2 + \hat{g}(x_1, x_2, \ldots, x_{k-1}, y)$. The germ $\widetilde{F}$ is non-simple. ∎

Lemma 2.1.5 *The function germ* $F(x_1, y) = ay^2 + \widetilde{f}(x_1, y)$ *with some* $a \neq 0, a \in \mathbb{R}$ *and* $\widetilde{f} \in \mathcal{M}^3_{x_1,y}$ *is quasi boundary equivalent to the germ* $\pm y^2 + y\phi_1(x_1) + \phi_2(x_1)$, *where* $\phi_1 \in \mathcal{M}^2_{x_1}$ *and* $\phi_2 \in \mathcal{M}^3_{x_1}$.

Proof. By scaling a to ± 1, the germ F can be written in the form $\widetilde{F}(x_1, y) = \pm y^2 + \widetilde{f}(x_1, y)$. Consider the deformations $G_0 = \pm y^2 + y\lambda_1 + \lambda_2$ and $G_1 = \pm y^2 + g(x_1, y) + y\lambda_1 + \lambda_2$, where $g \in \mathcal{M}^3_{x_1,y}$. We deal with y as a variable and x_1, λ_1

and λ_2 as parameters. Take the homotopy $G_t = G_0 + tg(x_1, y)$, joining G_0 and G_1, where $t \in [0,1]$. We want to prove that all G_t are quasi boundary equivalent as deformations. So we need to find a family of diffeomorphisms of the form

$$\Phi_t : (x_1, y, \lambda_1, \lambda_2) \mapsto (X(x_1, \lambda_1, \lambda_2, t), Y(x_1, y, \lambda_1, \lambda_2, t), \Lambda_1(x_1, \lambda_1, \lambda_2, t), \Lambda_2(x_1, \lambda_1, \lambda_2, t)),$$

such that: $G_t \circ \Phi_t = G_0$. The respective homological equation takes the form:

$$-\frac{\partial G_t}{\partial t} = -g(x_1, y) = \frac{\partial G_t}{\partial x}A + \frac{\partial G_t}{\partial y}\left\{yB_1 + \frac{\partial G_t}{\partial x}B_2 + \frac{\partial G_t}{\partial y}B_3\right\} + y\dot{\lambda}_1 + \dot{\lambda}_2,$$

where $\dot{\lambda}_i = \frac{\partial \Lambda_i}{\partial t}$ for $i = 1, 2$.

Set $B_2 = B_3 = A = 0$ and solve the homological equation for $B_1, \dot{\lambda}_1$ and $\dot{\lambda}_2$. Let $\mathbb{P} = \mathbf{C}_{x_1, y, \lambda_1, \lambda_2, t}$. Consider the mapping:

$$H : (x_1, y, \lambda_1, \lambda_2, t) \mapsto (h_1 = y\frac{\partial G_t}{\partial y}, h_2 = x_1, h_3 = \lambda_1, h_4 = \lambda_2, h_5 = t).$$

Then, $\mathbb{P}/I\mathbb{P} = \{\alpha_1 + \alpha_2 y\}$ where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $I\mathbb{P}$ is the ideal generated by the components of the mapping H. Thus, by Malgrange preparation theorem, we get, for any $P \in \mathbb{P}$, the following decomposition:

$$\begin{aligned} P &= 1.\widetilde{K}_1(h_1, x_1, \lambda_1, \lambda_2, t) + y.\widetilde{K}_2(h_1, x_1, \lambda_1, \lambda_2, t) \\ &= h_1 K_1(h_1, x_1, \lambda_1, \lambda_2, t) + yK_2(x_1, \lambda_1, \lambda_2, t) + K_3(x_1, \lambda_1, \lambda_2, t). \end{aligned}$$

Thus the homological equations is solvable by setting $B_1 = K_1, \dot{\lambda}_1 = K_2$ and $\dot{\lambda}_2 = K_3$. The restriction of Φ_1 to the subspace $\{\lambda_1 = \lambda_2 = 0\}$ provides a quasi boundary equivalence of G_1 with the family $\pm y^2 + y\Lambda_1(x_1) + \Lambda_2(x_1)$.

∎

The full classifications of simple quasi boundary singularities is given in the following:

Theorem 2.1.6 [36] *A simple quasi boundary singularity class for the boundary ($y = 0$) is a class of stabilizations of one of the following germs:*

1. B_k : $\pm x_1^2 \pm y^k$, $k \geq 2$ k,
2. $F_{k,m}$: $\pm(y \pm x_1^k)^2 \pm x_1^m$, $2 \leq k < m$ $k + m - 1$.

The orbit codimension in the space of germs is shown in the right column.

Remarks:

1. The classes $D_{k,l}$ listed in theorem 2.6 in [36] are in fact uni-modal.
2. The germs with corank greater or equal to 2 of f_2 are non-simple. The germs of corank 1 which are non-simple belongs to a subset of infinite codimension.
3. The classes B_k can be written in the form $\pm(y \pm x_1)^2 \pm x_1^k$ and can be included in the series $F_{k,m}$ as $F_{1,k}$.
4. Notice that classes $F_{3,4}$ and $F_{2,4}$ are pseudo equivalent but not quasi-equivalent. In fact, the transformation $(x_1, y) \mapsto (x_1, y + x_1^2)$ is pseudo equivalence between $y^2 + x_1^4$ of $F_{3,4}$ type and $(y + x_1^2)^2 + x_1^4$ which is of $F_{2,4}$ type. However their codimensions of quasi-tangent spaces are different, and the classes are not quasi-equivalent.

Proof of Theorem 2.1.6.

Lemmas 2.1.3 and 2.1.5 show that we need to consider the function germs of the form $F = \sum_{i=1}^{n-1} x_i^2 + \widetilde{f}(y)$ and $G_1(x_1, y) = \pm y^2 + \phi_1(x_1) + b_m x_1^m + \phi_2(x_1)$ to discuss simple quasi boundary germs. Start with the germ F, then properties of standard boundary equivalence imply that the germ F is quasi boundary equivalent to some class of the simple series of classes B_k : $\pm x_1^2 \pm y^k$, $k \geq 2$.

Now consider the germ G_1 and let

$$G_1(x_1, y) = \pm y^2 + a_k y x_1^k + y\varphi_1(x_1) + b_m x_1^m + \varphi_2(x_1),$$

$$\text{where} \quad a_k \neq 0, b_m \neq 0 \quad \text{and} \quad \varphi_1 \in \mathcal{M}_{x_1}^{k+1}, \varphi_2 \in \mathcal{M}_{x_1}^{m+1}.$$

We distinguish the following cases:

1) If $k \geq m-1$, then G_1 is quasi boundary equivalent to the germ $\widetilde{G}_0 = \pm y^2 \pm x_1^m$. To prove this, consider the germ $G_0 = \pm y^2 + b_m x_1^m$. Assign weights $w_y = \frac{1}{2}$ and $w_{x_1} = \frac{1}{m}$. The tangent space to the quasi boundary orbit at G_0 takes the form:

$$TQB_{G_0} = \pm 2y \{yB_0 + yB_1\} + mb_m x_1^{m-1} C_1.$$

There exist solutions for any term $g_1^* = \alpha_n y x_1^n$, where $n \geq k$ with quasi degree $d(g_1^*) = \frac{1}{2} + \frac{n}{m}$ by setting

$$B_0 = B_1 = 0 \quad \text{and} \quad C_1 = \frac{\alpha_n y x_1^n}{mb_m x_1^{m-1}} = \frac{\alpha_n}{mb_m} y x_1^{n-m+1}.$$

Then, the germ $\frac{\partial g_1^*}{\partial x_1} C_1 = \frac{n\alpha_n^2}{mb_m} y^2 x^{2n-m}$ has quasi degree equal to $1 + \frac{2(n+1)}{m}$ which is greater than $d(g_1^*)$.

Similarly, for any term $g_2^* = \beta_l x_1^l$ where $l \geq m$ with quasi degree $d(g_2^*) = \frac{l}{m}$, one can find solutions for g_2^* by setting:

$$B_0 = B_1 = 0 \quad \text{and} \quad C_1 = \frac{\beta_l x_1^l}{mb_m x_1^{m-1}} = \frac{\beta_l}{mb_m} x_1^{l-m+1}.$$

The germ $\frac{\partial g_1^*}{\partial x_1} C_1 = \frac{l\beta^2}{mb_m} x_1^{2l-m}$ has quasi degree equal to $\frac{2l-m}{m}$ which is greater than $d(g_2^*)$ when $l > m$.

Thus, by Lemma 1.3.5, we conclude that G_1 is quasi boundary equivalent to the germ G_0. Note that there is a solution for the term $g_2^* = b_m x_1^m$. This means that the orbit is simple with respect to quasi boundary equivalence. Rescaling b_m to ± 1, we get the classes: $\widetilde{G}_0 = \pm y^2 \pm x_1^m$ with $m \geq 3$.

On the other hand, $\widetilde{G}_0$ can be written in the form $\pm y^2 \pm 2yx_1^{m-1} \pm x_1^{2(m-1)} \pm x_1^m = \pm(y \pm x_1^{m-1})^2 \pm x_1^m$.

2) If $m > k+1$ and $\pm a_k^2 + 4b_m \neq 0$ when $m = 2k$ then G_1 is quasi boundary equivalent to the germ $\widetilde{G}_0 = \pm y^2 \pm yx_1^k \pm x_1^m$. To prove this, consider the tangent space to the quasi orbit at $G_0 = \pm y^2 + a_k y x_1^k + b_m x_1^m$.

$$\begin{aligned} TQB_{G_0} &= (\pm 2y + a_k x_1^k)\left\{yB_0 + (\pm 2y + a_k x_1^k)B_1 + (ka_k y x_1^{k-1} + mb_m x_1^{m-1})B_2\right\} \\ &+ (ka_k y x_1^{k-1} + mb_m x_1^{m-1})C_1. \end{aligned}$$

This space is equivalent to

$$TQB_{G_0} = (\pm 2y + a_k x_1^k)\left\{y\widetilde{B}_0 + x_1^k\widetilde{B}_1\right\} + (ka_k y x_1^{k-1} + mb_m x_1^{m-1})\widetilde{C}_1.$$

We have $mod\ TQB_{G_0}$:

$$ka_k y x_1^{k-1} + mb_m x_1^{m-1} \equiv 0, \tag{2.1}$$

$$\pm y^2 + a_k y x_1^k \equiv 0, \tag{2.2}$$

and

$$\pm 2y x_1^k + a_k x_1^{2k} \equiv 0. \tag{2.3}$$

If we multiply the equation (2.1) by x_1, we get:

$$ka_k y x_1^k + mb_m x_1^m \equiv 0. \tag{2.4}$$

If we substitute yx_1^k from the equation (2.3) in the equation (2.4), we get:

$$\mp\frac{ka_k^2}{2}x_1^{2k} + mb_m x_1^m \equiv 0. \tag{2.5}$$

The relation (2.5) yields that $x_1^m \equiv 0$ and $x_1^{2k} \equiv 0$. Hence $yx_1^k \equiv 0$ and $y^2 \equiv 0$. Thus, there exist solutions for any term of the form $g_1^* = \alpha_n y x_1^n$ with $n \geq k$ or of the form $g_2^* = \beta_l x_1^l$ with $l \geq m$. In particular, assign weights $w_y = \frac{1}{2}$, $w_{x_1} = \frac{1}{2k}$. Then, G_0 is semi quasi homogeneous and the germs g_1^* and g_2^* have quasi degree $d(g_1^*) = \frac{1}{2} + \frac{n}{2k}$ and $d(g_1^*) = \frac{l}{2k}$, respectively.

Assume that $2k \geq m$ (similar argument holds when $2k < m$). Then, for any monomial of the form g_1^*, we can set $\widetilde{B}_0 = 0$ and take $\widetilde{B}_1$ and $\widetilde{C}_1$ such that

$$\alpha_n y x_1^n = (\pm 2y + a_k x_1^k)x_1^k\widetilde{B}_1 + (ka_k y x_1^{k-1} + mb_m x_1^{m-1})\widetilde{C}_1.$$

This yields that

$$\alpha_n y x_1^n = \pm 2 y x_1^k \widetilde{B}_1 + k a_k y x_1^{k-1} \widetilde{C}_1 \qquad \text{and} \qquad a_k x_1^{2k} \widetilde{B}_1 + m b_m x_1^{m-1} \widetilde{C}_1 = 0.$$

Hence,

$$\widetilde{C}_1 = \frac{-a_k}{m b_m} x_1^{2k-m+1} \widetilde{B}_1 \quad \text{and} \quad \widetilde{B}_1 = \frac{\alpha_n y x_1^{n-k}}{\pm 2 - \frac{k a_k^2}{m b_m} x_1^{2k-m}}.$$

The germ $\Phi_1 = \alpha_n x_1^n . x_1^k \widetilde{B}_1 + (\pm 2y + a_k x_1^k)\alpha_n x_1^n . \widetilde{B}_1 + (n\alpha_n y x_1^{n-1})\widetilde{C}_1$ has quasi degree greater than $d(g_1^*)$ when $n > k$.

Similarly, for any monomial of the form $g_2^* = \beta_l x_1^l$ for $l \geq m$, we can set $\widetilde{B}_0 = 0$ and take $\widetilde{B}_1$ and $\widetilde{C}_1$ such that

$$\beta_l x_1^l = (\pm 2y + a_k x_1^k) x_1^k \widetilde{B}_1 + (k a_k y x_1^{k-1} + m b_m x_1^{m-1})\widetilde{C}_1.$$

This gives

$$\widetilde{C}_1 = \mp \frac{2}{k a_k} x_1 \widetilde{B}_1 \quad \text{and} \quad \widetilde{B}_1 = \frac{\beta_l x_1^{l-m}}{a_k x_1^{2k-m} \mp \frac{2 m b_m}{k a_k}}.$$

The germ $\Phi_2 = l\beta_l x_1^{l-1}\widetilde{C}_1$ has quasi degree greater than $d(g_2^*)$ when $l > m$.

Thus, Lemma 1.3.5 shows that G_1 is quasi boundary equivalent to the germ G_0. Rescaling a_k and b_m to ± 1, we get the classes: $\widetilde{G}_0 : \pm y^2 \pm y x_1^k \pm x_1^m$. The classes can be written in the form $\widetilde{G} : \pm y^2 \pm 2 y x_1^k \pm x_1^m \pm x_1^{2k} = \pm(y \pm x_1^k) \pm x_1^m$.

Note that if $m = 2k$ then G_1 is quasi boundary equivalent to the germ $\widetilde{G}_0 = \pm y^2 \pm y x_1^k$.

3) If $m > k+1$, $\pm a_k^2 + 4b_m = 0$ and $m = 2k$. Then, the germ G_1 takes the form $G_1 = \pm(y \pm \frac{a_k}{2} x_1^k)^2 + y\widetilde{\varphi}_1(x_1) + \widetilde{\varphi}_2(x_1)$ with $\widetilde{\varphi}_1 \in \mathcal{M}_{x_1}^{k+1}$ and $\widetilde{\varphi}_2 \in \mathcal{M}_{x_1}^{2k+1}$. Let $c_k = \pm\frac{a_k}{2}$. The function germ G_1 can be reduced to the form $\widetilde{G}_2 = \pm(y + c_k x_1^k)^2 + \phi(x_1)$ where $\phi \in \mathcal{M}^{2k+1}$. To prove this claim, consider the principal part $f_0 = \pm(y + c_k x_1^k)^2$ and take the tangent space to the quasi boundary orbit at f_0.

$$TQB_{f_0} = \pm 2(y + c_k x_1^k)\left\{yB_0 \pm (y + c_k x_1^k)B_1\right\} + [\pm 2kc_k x_1^{k-1}(y + c_k x_1^k)]C.$$

Then we have *mod* TQB_{f_0}: $y^2 + c_k y x_1^k \equiv 0$ and $yx_1^{k-1} + c_k x_1^{2k-1} \equiv 0$. This yields that $\mathbf{C}_w/TB_{f_0} = \mathbb{R}\{1, x_1, x_1^2, \dots\}$. Assign weights $w_y = \frac{1}{2}$ and $w_{x_1} = \frac{1}{2k}$. Let $g^* = d_s y x^s$, where $s \geq k+1$ with quasi degree $d(g^*) = \frac{1}{2} + \frac{s}{2k}$. Then, we see that g^* belongs to TBQ_{f_0} up to higher quasi degree terms. In particular, $g^* = y(y + c_k x_1^k)B_0 - \frac{d_s}{c_k} y^2 x_1^{s-k}$ where $B_0 = \frac{d_s}{c_k} x_1^{s-k}$. Now the germ

$$\Phi = d_s y x_1^s A_0 = \frac{d_s^2}{c_k} y x_1^{2s-k},$$

has quasi degree $d(\Phi) = \frac{1}{2} + \frac{2s-k}{2k}$ which is clearly greater than $d(g^*)$.

Now normalize c_k to ± 1 and let $\phi(x_1) = e_s x_1^s + \widetilde{\phi}(x_1)$ where $e_s \neq 0, s \geq 2k+1$ and $\widetilde{\phi} \in \mathcal{M}_{x_1}^{2k+2}$. Consider the tangent space at $\widetilde{f}_0 = \pm(y \pm x_1^k) + e_s x_1^s$.

$$TQB_{\widetilde{f}_0} = \pm 2(y \pm x_1^k)\left\{yB_0 + \pm 2(y \pm x_1^k)B_1\right\} + [\pm 2kx_1^{k-1}(y \pm x_1^k) + se_s x_1^{s-1}]C.$$

We have *mod* $TQ_{\widetilde{f}_0}$:

$$y^2 \pm y x_1^k \equiv 0, \tag{2.6}$$

$$y x_1^k \pm x_1^{2k} \equiv 0, \tag{2.7}$$

and

$$\pm 2kx_1^{k-1}(y \pm x_1^k) + se_s x_1^{s-1} \equiv 0. \tag{2.8}$$

Multiply the equation (2.8) by x_1 to get

$$\pm 2kyx_1^k + 2kx_1^{2k} + se_s x_1^s \equiv 0. \tag{2.9}$$

Substitute yx_1^k from the equation (2.7) into the equation (2.9) to get $x_1^s \equiv 0$. This means that there exist solutions for any term $g_1^* = \beta_l x_1^l$ with $l \geq s$.

Assign weights $w_y = \frac{1}{2}$ and $w_{x_1} = \frac{1}{2k}$. Then, g_1^* has quasi degree $d(g_1^*) = \frac{l}{2k}$. One can find solutions for any monomial g_1^* by taking B_0, B_1 and C such that

$$\pm B_0 + 2B_1 = 0, \quad \pm x_1 B_0 \pm 8x_1 B_1 \pm 2kC = 0,$$

$$4B_1 + s e_s x_1^{s-1-2k} C = \beta_l x_1^{l-2k}.$$

Now comparing the quasi degree of the germ

$$\phi = l\beta_l x_1^{l-1} C,$$

with $d(g_1^*)$, we conclude that $\pm(y \pm x_1^k)^2 + e_s x_1^s + \widetilde{\phi}(x_1)$ can be reduced to the form $\pm(y \pm x_1^k)^2 \pm x_1^s$ with $s \geq 2k+1$.

These classes are the only simple classes. Other germs are either adjacent to non simple classes or have codimension infinity. This completes the proof of the classification theorem.

2.2 Adjacency of lower codimension classes

The construction of the table of adjacencies is based on Lemmas 2.1.3, 2.1.5 and the proof of the theorem 2.1.6.

We describe first the adjacency of lower codimension classes for quasi boundary when the critical points lie on the boundary.

Let $f = f_2(x_1, y) + \varphi(x_1, y)$, where f_2 is a quadratic form in y and x_1 and $\varphi \in \mathcal{M}^3_{x_1,y}$. If f_2 is non-degenerate then f is contained in the class $B_2 : \pm x_1^2 \pm y^2$.

However, if f is degenerate of corank 1 then f can be written in the form $f = \pm(a_1 x_1 + a_2 y)^2 + \varphi(x_1, y)$

If $a_1 \neq 0$ then f is quasi boundary equivalent to the germ $\widetilde{f} = \pm x_1^2 + \widetilde{\varphi}(y)$ with $\widetilde{\varphi} \in \mathcal{M}^3_y$. Thus, we get the series of classes $B_k : \pm x_1^2 \pm y^k$ with $k \geq 3$. Hence, we obtain the following adjacent classes:

$$B_2 \leftarrow B_3 \leftarrow B_4 \leftarrow B_5 \leftarrow \dots$$

If $a_1 = 0$ and $a_2 \neq 0$ then f is quasi boundary equivalent to a germ of the form $F = \pm y^2 + y\phi_1(x_1) + \phi_2(x_1)$. Let

$$F = \pm y^2 + y(c_2x_1^2 + c_3x_1^3 + \dots) + c_3'x_1^3 + c_4'x_1^4 + \dots.$$

If $c_3' \neq 0$ then we get the class $F_{2,3} : \pm y^2 \pm x_1^3 \sim \pm(y \pm x_1^2)^2 \pm x_1^3$.

Note that the class B_3 can be written as $g(x_1, y) = \pm(ax_1 + by)^2 \pm y^3 \pm x_1^3$. Thus, when $a = 0$, we get the class $F_{2,3}$. This means that the class $F_{2,3}$ is adjacent to the class B_3.

If $c_3' = 0$, $c_2 \neq 0$ and $c_4' \neq 0$, then we distinguish the following:

If $c_4' \neq \pm\frac{1}{4}c_2^2$, then F is quasi equivalent to the class $F_{2,4} : \pm y^2 \pm yx_1^2 \sim \pm(y \pm x_1^2)^2 \pm x_1^4$. Thus, the class $F_{2,4}$ is adjacent to the class $F_{2,3}$. On the other hand, the class $F_{2,4}$ is adjacent to B_4 as the class B_4 can be written in the form $g(x_1, y) = \pm(ax_1 + by)^2 \pm y^4 \pm x_1^4$. Hence, when $a = 0$ we get the class $F_{2,4}$. So, we get, up to this stage , the following table of adjacent classes:

$$\begin{array}{ccccccccc} B_2 & \leftarrow & B_3 & \leftarrow & B_4 & \leftarrow & B_5 & \leftarrow & \dots \\ & & \uparrow & & \uparrow & & & & \\ & & F_{2,3} & \leftarrow & F_{2,4} & & & & \end{array}$$

If $c_4' = \pm\frac{1}{4}c_2^2$, then F is quasi equivalent to a germ of the form $\widetilde{F} = \pm(y \pm x_1^2)^2 + \phi(x_1)$ with $\phi \in \mathcal{M}_{x_1}^5$ which is adjacent to the class $F_{2,4}$. Thus, we get the series of classes $F_{2,m} : \pm(y \pm x_1^2) \pm x_1^m$ with $m \geq 5$. Hence, we obtain the following adjacent classes:

$$F_{2,4} \leftarrow F_{2,5} \leftarrow F_{2,6} \leftarrow F_{2,7} \leftarrow \dots$$

Note that the classes $B_m : \pm x_1^2 \pm y^m$ can be written in the form $g(x_1, y) = \pm(ax_1 + by)^2 \pm y^m \pm x_1^m$. Hence, when $a = 0$ then, we obtain the classes $F_{2,m} : \pm(y \pm x_1^2)^2 \pm x_1^m$ which adjacent to B_m.

If $c_3' = c_2 = 0$, $c_3 \neq 0$ and $c_4' \neq 0$, then we obtain the class $F_{3,4} : \pm y^2 \pm yx_1^3 \pm y^4 \sim \pm(y^2 \pm x_1^3)^2 \pm x_1^4$ which is adjacent to the class $F_{2,4}$.

If $c_3' = c_2 = c_4' = 0$, $c_3 \neq 0$ and $c_5' \neq 0$, then we get the class $F_{3,5}$ which is adjacent to the class $F_{3,4}$. On the the other hand, the class $F_{2,5}$ has the form $\pm(y + ax_1^2 + bx_1^3) \pm x_1^5$. Thus, when $a = 0$, the we obtain the class $F_{3,5}$. This means that the class $F_{3,5}$ is adjacent to the class $F_{2,5}$

If $c_3' = c_2 = 0 = c_4' = c_5' = 0$ and $c_3 \neq 0$ but $c_6' \neq 0$, then follow the procedure of the previous case when $c_3' = 0$, $c_2 \neq 0$ and $c_4' \neq 0$.

Assume now that the second jet of f is zero then f is contained in the non-simple class $\widetilde{S}_5 : y^3 + x_1^3 + ay^2x_1$, where $a \in \mathbb{R}$. Clearly,

$$F_{2,3} \leftarrow \widetilde{S}_5.$$

Therefore, the table of adjacencies of low dimension is given as follows:

$$\begin{array}{ccccccccccc}
B_2 & \leftarrow & B_3 & \leftarrow & B_4 & \leftarrow & B_5 & \leftarrow & B_6 & \leftarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & F_{2,3} & \leftarrow & F_{2,4} & \leftarrow & F_{2,5} & \leftarrow & F_{2,6} & \leftarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \widetilde{S}_5 & & F_{3,4} & \leftarrow & F_{3,5} & \leftarrow & F_{3,6} & \leftarrow & \dots \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & \dots & & \dots & & \dots
\end{array}$$

Now if the critical points lie outside the boundary, then the classes B_k are adjacent to the classes A_{k-1}. In fact, the classes $B_k : \pm x_1^2 \pm y^k$ can be written in the form $G = \pm(x_1 + y)^2 \pm y^k$. The germ G is adjacent to the germ $G_\varepsilon = \pm(x_1 + \varepsilon + y)^2 \pm y^k$, for sufficiently small ε. Note that G_ε has critical points outside the boundary and quasi boundary equivalent to a germ contained in one of the series of classes A_{k-1}. Hence, the full table of the adjacency of lower codimension classes for quasi boundary singularities is shown in the table.

$$
\begin{array}{ccccccccccccc}
A_0 & \leftarrow & A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & A_4 & \leftarrow & A_5 & \leftarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & B_2 & \leftarrow & B_3 & \leftarrow & B_4 & \leftarrow & B_5 & \leftarrow & B_6 & \leftarrow & \dots \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & F_{2,3} & \leftarrow & F_{2,4} & \leftarrow & F_{2,5} & \leftarrow & F_{2,6} & \leftarrow & \dots \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \widetilde{S}_5 & \leftarrow & F_{3,4} & \leftarrow & F_{3,5} & \leftarrow & F_{3,6} & \leftarrow & \dots \\
 & & & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & & & \dots & & \dots & \dots &
\end{array}
$$

Remarks:

1. The top row of the table of adjacencies consists of standard right singularities outside the boundary.

2. Any germ f with corank of f_2 greater or equal 2 is non-simple. In particular, the uni-modal ($a \in \mathbb{R}$) classes $\widetilde{S}_5 : y^3 + x_1^3 + ay^2x_1$ and $\widetilde{E}_{2,6} : (y + x_1^2 + ax_1x_2)^2 + x_1^4 + x_2^3y$ are adjacent to $F_{2,3}$ and $F_{2,6}$ respectively.

2.3 Comparison of quasi boundary and standard boundary singularities

From the definition of pseudo boundary equivalence, standard Arnolds boundary equivalence (right action of diffeomorphisims preserving the boundary) implies quasi boundary equivalence. So simple Arnolds boundary classes $B_k : \pm x^2 \pm y^k$, where $k \geq 2$, $C_k : xy \pm x^k$, where $k \geq 2$ and $F_4 : \pm y^2 + x^3$ remain simple for quasi classification, but some classes can merge together.

The quasi boundary class $B_2 : x^2 - y^2$ has another equivalent form $C_2 : xy$ (which represents a single quasi boundary class containing all ordinary C_k boundary classes). So all B_k classes remain non-equivalent but all C_k classes become equivalent to C_2. The classes $F_{k,k}$ have equivalent forms $\pm y^2 \pm x^{k+1}$. In particular, $F_{2,2} : \pm y^2 + x^3$ coincides with F_4 the ordinary boundary singularity class F_4. Other $F_{i,3}$ classes contain non-simple ordinary boundary singularities.

2.4 The caustics and bifurcation diagrams of simple quasi boundary singularities

The quasi border bifurcation diagram of a function deformation germ $F(w, \lambda)$ depending on parameters λ is the set of points in the base $\{\lambda\}$ consisting of several strata. The first stratum W_0 is the projection to the base $\{\lambda\}$ of the subset X_0 in the total space (w, λ) given by the equations: $F = 0$ and $\frac{\partial F}{\partial w} = 0$. The other strata W_i are projections of subsets in X_0 satisfying the extra equations which define the border $h(x) = 0$. In other words, the first stratum W_0 is the set of parameters which correspond to the critical points of the functions $F(\cdot, \lambda)$ with zero critical value while W_i are the subsets of W_0 corresponding to critical points on the border.

The quasi border caustic of a function germ deformation $F(w, \widetilde{\lambda}) + \lambda_0$, which has an additive constant λ_0 as one of the parameters and satisfies $F(0, \widetilde{\lambda}) = 0$, is the subset of points in the reduced deformation base $\{\widetilde{\lambda}\}$ consisting of several strata. The first stratum Σ_0 is the image of the singular points of the first stratum of the bifurcation diagram W_0 under the projection π_0 to the reduced base which forgets λ_0. The other strata of the caustics Σ_i are the images $\pi_0(W_i)$.

In contrast to pseudo border equivalence we claim (and this is easy to prove using techniques of section 1.3) that the versality theorem holds for the quasi border equivalence, the versal deformation of a function germ f with respect to the quasi border equivalence can be taken as the deformation $F(x, \lambda) = f(x) + \sum_{i=0}^{m-1} \lambda_i \varphi_i(x)$ where $\lambda = (\lambda_0, \ldots, \lambda_{m-1}) \in (\mathbb{R}^m, 0)$ and the germs φ_i at zero form a linear basis of the local algebra $Q = \mathbf{C}_w/TQ_f$, where TQ_f is the quasi border tanget space at the germ f. The proof of this versality theorem is exactly the same as the standard proof of the versality for right equivalence. The tangent space to the quasi orbit is a finitely generated module over the algebra of functions in main variables and parameters. The complete details based on the application of Malgrange preparation theorem, can be reproduced following the proof given in the paper [39].

The dimension μ of the local algebra Q will be called the quasi border multiplicity. It is convenient to choose $\varphi_0 \equiv 1$ and φ_i vanishing at the base point for $i = 1, \ldots, (\mu - 1)$. The space $\mathbb{R}^m$ is the base of the versal deformation, whereas the space $\mathbb{R}^{\mu-1} = (\lambda_1, \ldots, \lambda_{\mu-1})$ is the base of the reduced versal deformation.

Here are some properties of quasi bifurcation diagrams for simple quasi boundary classes. First note that all the quasi boundary singularities are reduced to A_k singularity with respect to the standard right equivalence, provided that the border is forgotten. Hence the first component of the bifurcation diagram of a function germ deformation F is a product a generalized swallow tail and $\mathbb{R}^{n-k}$, where n stands for dimension of the base of the versal deformation.

Recall that the versal deformation of the $A_k : \pm x^{k+1} + \sum y_i^2$ singularity takes the form

$$F(x,y) = \pm x^{k+1} + \sum y_i^2 + \sum^{k-1} \lambda_i x^i,$$

and the set

$$\Lambda = \{(\lambda_0, \dots, \lambda_{k-1}) : \frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y_i} = 0\}$$

is called generalized swallow tail.

Proposition 2.4.1 *The quasi boundary mini-versal deformations of the simple quasi boundary classes are as follows:*

1. $B_2 : \quad \pm x_1^2 \pm y^2 + \lambda_0 + \lambda_1 y,$
2. $B_k : \quad \pm x_1^2 \pm y^k + \sum_{i=0}^{k-1} \lambda_i y^i, \quad k \geq 3;$
3. $F_{k,m} : \pm(y \pm x_1^k)^2 \pm x_1^m + \sum_{i=0}^{m-2} \lambda_i x_1^i + \sum_{j=0}^{k-1} \mu_j y x_1^j \qquad 2 \leq k < m.$

Proof. For B_k classes, consider the tangent space to quasi boundary orbit at $f(x_1, y) = \pm x_1^2 \pm y^k$, where $k \geq 2$.

$$TQB_f = x_1 A \pm k y^{k-1} \{yB + x_1 C\}.$$

Then, we have *mod* $TB_f : x \equiv 0$ and $y^k \equiv 0$. Hence, the monomials $1, y, y^2, \dots, y^{k-1}$ form a basis of the local algebra $\mathcal{Q} = \mathbf{C}_{x_1,y}/TQB_f$.

For the $F_{k,m}$ classes, let $f(x_1, y) = \pm(y \pm x_1^k)^2 \pm x_1^m$. Then we distinguish the following cases:

• If $k \geq m-1$, then f is quasi boundary equivalent to the germ $g = \pm y^2 \pm x_1^m$. Note that g belongs to the class $F_{m-1,m}$. Consider the tangent space to the quasi boundary orbit at g

$$TQB_g = \pm 2y\left\{yB_0 + x_1^{m-1}B_1\right\} + x_1^{m-1}C.$$

Thus, *mod* TQB_g: $y^2 \equiv 0$, and $x_1^{m-1} \equiv 0$. Therefore, the monomials

$$1, x_1, x_2^2, \ldots, x_1^{m-2}, y, yx_1, \ldots, yx_1^{m-2}$$

form a basis of the local algebra $\mathcal{Q} = \mathbf{C}_{x_1,y}/TQB_g$

• If $2k \geq m > k+1$, then f can take the form $\widetilde{f} = \pm y^2 \pm yx_1^k \pm x_1^m$. Take the tangent space to the quasi boundary orbit at $\widetilde{f}$

$$TQB_{\widetilde{f}} = [\pm kyx_1^{k-1} \pm mx_1^{m-1}]A + (\pm 2y \pm x_1^k)\left\{yB + x_1^kC\right\}.$$

Thus, we get the following relations *mod* $TQB_{\widetilde{f}}$:

$$\pm 2y^2 \pm yx_1^k \equiv 0, \tag{2.10}$$

$$\pm yx_1^k \equiv \mp\frac{1}{2}x_1^{2k}, \tag{2.11}$$

and

$$\pm kyx_1^{k-1} \pm mx_1^{m-1} \equiv 0. \tag{2.12}$$

If we multiply the last equation by x_1, we obtain

$$\pm kyx_1^k \pm mx_1^m \equiv 0. \tag{2.13}$$

Substitute yx_1^k from the equation (2.11) in the equation (2.13) to get:

$$\mp\frac{k}{2}x_1^{2k} \pm mx_1^m \equiv 0. \tag{2.14}$$

The equation (2.14) yields that: $x_1^m \equiv 0$ and $x_1^{2k} \equiv 0$. Hence , $yx_1^k \equiv 0$ and $y^2 \equiv 0$. Now, if we substiute $\pm 2kyx_1^{k-1} \equiv \mp mx^{m-1}$ in the local algebra $\mathcal{Q} = \mathbf{C}_{x_1,y}/TB_{\widetilde{f}}$, then the monomials:

$$1, x_1, x_1^2, \ldots, x_1^{m-2}, y, yx_1, yx_1^2, \ldots, yx_1^{k-1}$$

form a basis for $\mathcal{Q}$.

• If $m \geq 2k+1$, then consider the tangent space to the orbit at $f = \pm(y \pm x_1^k)^2 \pm x_1^m$

$$TQB_f = [\pm 2kx_1^{k-1}(y \pm x_1^k) \pm mx^{m-1}]A \pm 2(y \pm x_1^k)\left\{yB + x_1^k C\right\}.$$

Then, we obtain the following relations *mod* TQB_f :

$$\pm 2kyx_1^{k-1} + 2kx_1^{2k-1} \pm mx_1^{m-1} \equiv 0, \tag{2.15}$$

$$y^2 \pm yx_1^k \equiv 0, \tag{2.16}$$

and

$$yx_1^k \equiv \mp x_1^{2k}. \tag{2.17}$$

If we multiply the equation (2.15) by x_1, we get:

$$\pm 2kyx_1^k + 2kx_1^{2k} \pm mx^m \equiv 0. \tag{2.18}$$

Substitute yx_1^k from the equation (2.17) in the equation (2.18), we see that $x^m \equiv 0$.

If we substitute $y^2 \equiv \mp yx_1^k$ and $\pm 2kyx_1^{k-1} \equiv \mp 2kx_1^{2k-1} \mp mx^{m-1}$ in the local algebra $\mathcal{Q} = \mathbf{C}_{x_1,y}/TQB_f$, then again the monomials:

$$1, x_1, x_1^2, \ldots, x_1^{m-2}, y, yx_1, yx_1^2, \ldots, yx_1^{k-1}$$

form a basis for $\mathcal{Q}$.

Thus the deformation $H(x_1, y) = \pm(y \pm x_1^k)^2 \pm x_1^m + \sum_{i=0}^{m-2} \lambda_i x_1^i + \sum_{j=0}^{k-1} \mu_j yx_1^j$ is a mini versal deformation for the classes $F_{k,l}$.

Remark: Notice that, the deformation H is quasi boundary equivalent to the deformation

$$\begin{aligned}\widetilde{H} &= \pm(y \pm x_1^k)^2 \pm 2y \sum_{j=0}^{k-1} \mu_j x_1^j \pm 2 \sum_{j=0}^{k-1} \mu_j x_1^{j+k} \pm x_1^m + \sum_{i=0}^{m-2} \lambda_i x_1^i \\ &= \pm(y \pm x_1^k)^2 \pm 2(y \pm x_1^k) \sum_{j=0}^{k-1} \mu_j x_1^j \pm x_1^m + \sum_{i=0}^{m-2} \lambda_i x_1^i.\end{aligned}$$

On the other hand, adding the terms $\Lambda = \left(\sum_{j=0}^{k-1} \mu_j x_1^j\right)^2$ to $\widetilde{H}$ does not affect the versality of $\widetilde{H}$ as $\frac{\partial \Lambda}{\partial \mu_i}|_{\mu=0}$. Hence, we get the following alternative form of the versal deformation of the classes $F_{k,m}$:

$$G(x_1, y) = \pm(y \pm x_1^k + \sum_{j=0}^{k-1} \mu_j x_1^j)^2 \pm x_1^m + \sum_{i=0}^{m-2} \lambda_i x_1^i.$$

∎

The formulas of versal deformations listed in proposition 2.4.1 provide the explicit description of simple bifurcation diagrams and caustics.

Before we give the precise description, we introduce the following.

Definition 2.4.1 The image of the mapping

$$\begin{pmatrix} x \in \mathbb{R} \\ (\lambda, \mu) \in \mathbb{R}^m; m = k_1 + k_2 - 3 \end{pmatrix} \mapsto \begin{pmatrix} x^{k_1} + \sum_{i=1}^{k_1-1} \lambda_i x^i \\ x^{k_2} + \sum_{j=1}^{k_2-2} \mu_j x^j \\ \lambda \\ \mu \end{pmatrix}$$

is called Morin stable mapping or generalized Whitney umbrella mapping.

Example: The standard Whitney umbrella is the image of the mapping.

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ x^2 + \lambda x \\ \lambda \end{pmatrix}$$

Proposition 2.4.2 *1. The hypersurface component of the bifurcation diagram for B_k series is a product of generalized swallow tail and a line. The second component is the maximal smooth submanifold passing through the vertex of the generalized swallow tail times a line. In particular, the bifurcation diagram of B_2 in (λ_0, λ_1)-plane is a smooth curve and a distinguished point on it. The bifurcation diagram of $B_3 \subset \mathbb{R}^3$ is a cuspidal cylinder and a line in it which is tangent to the ridge.*

2. *The caustic of singularity B_k is a union of cylinder over generalized swallow tail (with one-dimensional generator) and a smooth hypersurface having smooth $(k-3)$-dimensional intersection with the first component. In particular, the B_3 caustic is the union of two tangent lines, for B_4 this is a seimicubic cylinder and a plane (the configuration is isomorphic to the discriminant of the standard C_3 boundary singularity). See figures 2.1 and 2.2.*

3. *The caustic of $F_{k,l}$ singularity is a union of a cylinder over a generalized swallow tail of type A_l and an image of Morin stable mapping (generalized Whitney umbrella) being the set of common zeros of two polynomials of degree l and k. In particular, the caustic of $F_{2,2}$ is the union of Whitney umbrella which is the second component, and a smooth tangent surface which is the caustic of the A_2 singularity. See figure 2.3.*

Proof. Start with B_k singularity. Let $F(x_1, y, \lambda) = \pm x_1^2 \pm y^k + \sum_{i=0}^{k-1} \lambda_i y^i$ be its versal deformation. Clearly, the versal deformation with respect to quasi equivalence coincides with the versal deformation with respect to the standard right equivalence with an extra parameter λ_{k-1}. Thus, the first component of the bifurcation diagram is a product of generalized swallow tail and a line.

Explicitly, solve simultaneously the equations $\frac{\partial F}{\partial x} = \pm 2x = 0$, $\frac{\partial F}{\partial y} = 0$ and $F = 0$. Thus, one of the stratum is parametrized by the mapping:

$$\Phi : \begin{pmatrix} y \\ \lambda_2 \\ \vdots \\ \lambda_{k-1} \end{pmatrix} \mapsto \begin{pmatrix} \lambda_0 = \pm(k-1)y^k + \lambda_2 y^2 + 2\lambda_3 y^3 + \cdots + (k-2)\lambda_{k-1}y^{k-1} \\ \lambda_1 = \mp k y^{k-1} - 2\lambda_2 y - 3\lambda_3 y^2 - \cdots - (k-1)\lambda_{k-1}y^{k-2} \\ \lambda_2 \\ \vdots \\ \lambda_{k-2} \end{pmatrix}.$$

On the other hand, the special case which we preserve via quasi equivalences occurs when the critical point lies on the boundary. Hence, we can again restrict ourselves to the zero level set of the function. Thus, to get this stratum, we have to consider an extra equation $y = 0$. The union of these two strata (mind that they have different dimensions and the second stratum is a subset of the first one) form the required bifurcation diagram. Thus, the second stratum of the bifurcation diagram is obtained by restricting the mapping Φ to $y = 0$. This gives the space $\Lambda = \{(0, 0, \lambda_2, \lambda_3, \ldots, \lambda_{k-1})\}$.

The critical points of the projection of the first stratum is given by the equation $Hessian(F) = 0$ or equivalently by the equation $\frac{\partial^2 F}{\partial y^2} = 0$. Thus, in our case, the critical points is given by the equation

$$\pm k(k-1)y^{k-2} + 2\lambda_2 + 6\lambda_3 y + \cdots + (k-1)(k-2)\lambda_{k-1}y^{k-3} = 0$$

or equivalently, $\lambda_2 = \mp\frac{1}{2}k(k-1)y^{k-2} - 3\lambda_3 y - \cdots - \frac{1}{2}(k-1)(k-2)\lambda_{k-1}y^{k-3}$.

Hence, the critical points of the projection are parametrized by the mapping:

$$\theta : \begin{pmatrix} y \\ \lambda_3 \\ \vdots \\ \lambda_{k-1} \end{pmatrix} \mapsto \begin{pmatrix} \lambda_0 = \pm(k-1)(1-\frac{1}{2}k)y^k - \lambda_3 y^3 + \cdots + \frac{1}{2}(k-2)(3-k)\lambda_{k-1}y^{k-1} \\ \lambda_1 = \pm k(k-2)y^{k-1} + 3\lambda_3 y^2 + \cdots + (k-1)(k-3)\lambda_{k-1}y^{k-2} \\ \lambda_2 = \mp\frac{1}{2}k(k-1)y^{k-2} - 3\lambda_3 y - \cdots - \frac{1}{2}(k-1)(k-2)\lambda_{k-1}y^{k-3} \\ \lambda_3 \\ \vdots \\ \lambda_{k-1} \end{pmatrix}.$$

This gives the ridge of the cylinder over the swallowtail which is clearly tangent to the space Λ.

The caustic of B_k consist of two strata. The projection of the ridge along the λ_0 axis gives a generalized swallowtail times a line which is parametrized as follows:

$$\theta : \begin{pmatrix} y \\ \lambda_3 \\ \vdots \\ \lambda_{k-1} \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 = \pm k(k-2)y^{k-1} + 3\lambda_3 y^2 + \cdots + (k-1)(k-3)\lambda_{k-1}y^{k-2} \\ \lambda_2 = \mp\frac{1}{2}k(k-1)y^{k-2} - 2\lambda_3 y - \cdots - \frac{1}{2}(k-1)(k-2)\lambda_{k-1}y^{k-3} \\ \lambda_3 \\ \vdots \\ \lambda_{k-1} \end{pmatrix}.$$

This is the first stratum of the caustic. The projection of the second stratum of the bifurcation diagram along λ_0-axis gives the space $\widetilde{\Lambda} = \{(0, \lambda_2, \ldots, \lambda_{k-1})\}$- the second stratum of the caustic.

In particular, consider B_2 singularity. Let $F(x_1, y, \lambda) = \pm x_1^2 \pm y^2 + \lambda_0 + \lambda_1 y$ be its versal deformation. Thus, one of the stratum of the bifurcation diagram is the parabola $\lambda_0 = \frac{\pm 1}{4}\lambda_1^2$. For the other strata, we have to consider an extra equation $y = 0$. This gives the origin point in the parameter plane. The union of these two strata forms the required bifurcation diagram. Note that they have different dimensions 1 and 0 and the second component is a subset of the first one.

For the class $B_3 : \pm x_1^2 \pm y^3$, consider the versal deformation $F(x_1, y, \lambda) = \pm x_1^2 \pm y^3 + \lambda_0 + \lambda_1 y + \lambda_2 y^2$. Again, the bifurcation diagram consists of two strata. First, we need to solve simultaneously the equations $\frac{\partial F}{\partial x} = \pm 2x = 0$, $\frac{\partial F}{\partial y} = \pm 3y^2 + \lambda_1 + 2\lambda_2 y = 0$ and $F = 0$. Hence, one of the stratum is the cuspidal cylinder which is parametrized by the mapping:

$$\Phi : (y, \lambda_2) \mapsto (\lambda_0 = \pm 2y^3 + \lambda_2 y^2, \lambda_1 = \mp 3y^2 - 2\lambda_2 y, \lambda_2).$$

The second stratum is obtained by substituting $y = 0$ in the previous mapping. This gives the λ_2 axis in the parameter space which is tangent to the ridge.

The critical points of the projection (the ridge of the cylinder) are parametrized by the mapping:

$$\theta : (y, \lambda_2) \mapsto (\lambda_0 = \mp y^3, \lambda_1 = \pm 3y^2, \lambda_2 = \mp 3y).$$

This ridge is tangent to λ_2 axis. Recall that the total bifurcation diagram is cuspidal cylinder and a distinguished line in it. If we project this along λ_0-axis then the axis λ_2 remains a straight line but the ridge of the cylinder gives a curved line (the parabola: $\lambda_1 = \pm 3y^2, \lambda_2 = \mp 3y$). Hence, the total caustic of B_3 class is a union of two tangent lines. The general 3-dimensional sketch of the caustic is shown in the Figure 2.1.

We pass now to the class $B_4 : \pm x_1^2 \pm y^4$. Let $F(x_1, y, \lambda) = \pm x_1^2 \pm y^4 + \lambda_0 + \lambda_1 y + \lambda_2 y^2 + \lambda_3 y^3$ be its miniversal deformation. Thus, the standard stratum is a cylinder over the standard swallowtail which is parametrized by the mapping:

$$\Phi : (y, \lambda_2, \lambda_3) \mapsto (\lambda_0 = \pm 3y^4 + \lambda_2 y^2 + 2\lambda_3 y^3, \lambda_1 = \mp 4y^3 - 2\lambda_2 y - 3\lambda_3 y^2, \lambda_2, \lambda_3).$$

The second stratum is obtained by substituting $y = 0$ in the previous mapping. This gives the $\lambda_2 - \lambda_3$–plane in the parameter space which is tangent to the ridge.

The ridge of the cylinder is cuspidal cylinder which is parametrized by the mapping:

$$\theta : (y, \lambda_3) \mapsto (\lambda_0 = \mp 3y^4 - \lambda_3, \lambda_1 = \pm 8y^3 + 3\lambda_3 y^2, \lambda_2 = \mp 6y^2 - 3\lambda_3 y).$$

If we project the ridge along λ_0-axis then we get the cuspidal cylinder:

$$\theta : (y, \lambda_3) \mapsto (\lambda_1 = \pm 8y^3 + 3\lambda_3 y^2, \lambda_2 = \mp 6y^2 - 3\lambda_3 y).$$

The projection of the second stratum gives the plane $\lambda_1 = 0$. The general 3-dimensional sketch of the caustic is shown in Figure 2.2.

Consider $F_{k,m}$ singularity. It is clear that $F_{k,m}$ is equivalent to the standard A_{m-1} singularity with respect to the standard right equivalence . As its versal deformation takes the form: $\pm(y \pm x_1^k)^2 \pm x_1^m + \sum_{i=0}^{m-2} \lambda_i x_1^i + \sum_{j=0}^{k-1} \mu_j y x_1^j$, the first stratum of the bifurcation diagram of $F_{k,m}$ classes (and hence the first stratum of the caustic) is a product of a generalized swallowtail with $\mathbb{R}^k$ space.

Consider the alternative versal deformation of the form:

$$F(x_1, y, \lambda, \mu) = \pm(y \pm x_1^k + \sum_{j=0}^{k-1} \mu_j x_1^j)^2 \pm x_1^m + \sum_{i=0}^{m-2} \lambda_i x_1^i.$$

Let $A = \pm x_1^k + \sum_{j=0}^{k-1} \mu_j x_1^j$ and $B = \pm x_1^m + \sum_{i=0}^{m-2} \lambda_i x_1^i$. Then, $F(x_1, y, \lambda, \mu) = \pm(y + A)^2 + B$.

Thus, the second stratum of the caustics is given by the equations:

$$\frac{\partial F}{\partial x} = \pm 2(y + A)\frac{\partial A}{\partial x} + \frac{\partial B}{\partial x} = 0 \quad , \frac{\partial F}{\partial y} = \pm 2(y + A) = 0 \quad \text{and} \quad y = 0.$$

These equations are equivalent to :

$$\frac{\partial B}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = A = 0.$$

Thus, the second stratum can be given as the image of the following mapping

$$\Gamma = \begin{pmatrix} x_1 \in \mathbb{R} \\ \mu^* = (\mu_1, \mu_2, \dots, \mu_{k-2}) \\ \lambda^* = (\lambda_2, \lambda_3, \dots, \lambda_{l-2}) \end{pmatrix} \mapsto \begin{pmatrix} \mu_0 = \mp x_1^k - \sum_{i=1}^{k-1} \mu_j x_1^j \\ \lambda_1 = \mp m x_1^{m-1} - \sum_{i=1}^{m-3} (i+1)\lambda_{i+1} x_1^i \\ \mu^* \\ \lambda^* \end{pmatrix}.$$

In fact, Γ is the Morin stable mapping (generalized Whitney umbrella).

In particular, consider the particular $F_{2,3}$ class. Consider its miniversal deformation:

$$F(x_1, y, \lambda) = x^3 + y^2 + \lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy.$$

The solution of the equations $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = F = 0$ with respect to the four parameters gives the first stratum of the bifurcation diagram:

$$\left\{(\lambda) : \lambda_0 = 2x_1^3 + y^2 + \lambda_3 x_1 y, \lambda_1 = -3x^2 - \lambda_3 y, \lambda_2 = -2y - \lambda_3 x\right\}. \tag{2.19}$$

Now, we have to calculate the critical values of the mapping:

$$(x, y, \lambda_3) \mapsto (2x_1^3 + y^2 + \lambda_3 x_1 y, -3x^2 - \lambda_3 y, -2y - \lambda_3 x, \lambda_3).$$

The Jacobi matrix is

$$J = \begin{bmatrix} 6x^2 + \lambda_3 y & 2y + \lambda_3 x & xy \\ -6x & -\lambda_3 & -y \\ -\lambda_3 & -2 & -x \\ 0 & 0 & 1 \end{bmatrix}.$$

The critical points are determined by the condition $rank(J) \leq 2$. This is the same as having all its three order 2 minors equal to zero. Thus, the critical points are the set $\lambda_3^2 - 12x = 0$. We can use λ_3 and y to parametrize the caustic. By setting $x = \frac{\lambda_3^2}{12}$, the critical values are the set:

$$\left\{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) : \lambda_0 = \frac{\lambda_3^6}{854} + y^2 + \frac{\lambda_3^3}{12}y, \lambda_1 = \frac{-3\lambda_2^4}{144} - \lambda_3 y, \lambda_2 = -2y - \frac{\lambda_3^3}{12}\right\}.$$

Project this along λ_0. Then, the first component of the caustic is a smooth surface:

$$\left\{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1 = \frac{-3\lambda_2^4}{144} - \lambda_3 y, \lambda_2 = -2y - \frac{\lambda_3^3}{12}\right\}.$$

To get the second stratum of the bifurcation diagram, set $y = 0$ in the equation (2.19). This gives:

$$\left\{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) : \lambda_0 = 2x^3, \lambda_1 = -3x^2, \lambda_2 = -\lambda_3 x\right\}.$$

Thus, the second stratum of the caustic is Whitney Umbrella parametrized as follows:

$$\left\{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1 = -3x^2, \lambda_2 = -\lambda_3 x\right\}.$$

The smooth surface (the first components) is tangent to Whitney Umbrella along a smooth curved line parametrized by $\lambda_1 = \frac{\lambda_3^4}{48}$ and $\lambda_2 = \frac{-\lambda_3^3}{12}$, where $\lambda_3 \in \mathbb{R}$.

The general sketch of $F_{2,3}$ caustics is shown in Figure2.3. ∎

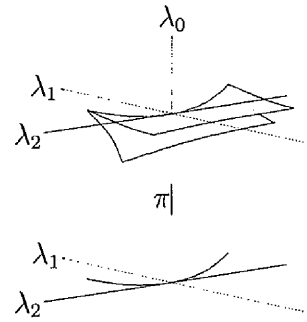

Figure 2.1: The bifurcation diagram and caustics of B_3.

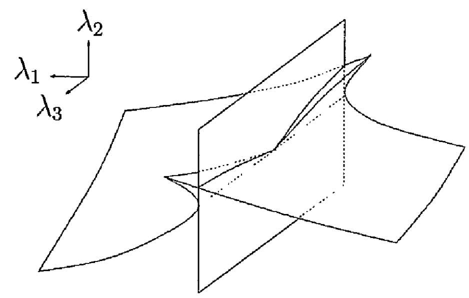

Figure 2.2: The caustics of B_4.

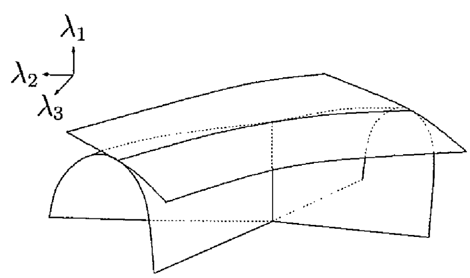

Figure 2.3: The caustics of $F_{2,3}$.

Chapter 3

Quasi corner singularites

3.1 The classification of simple classes

In this chapter the coordinates are as follows: $\mathbb{R}^n = \{w = (x, y, z) : x, y \in \mathbb{R}, z = (z_1, \ldots, z_{n-2}) \in \mathbb{R}^{n-2}\}$. We consider germs of C^∞ -smooth functions $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ of the form $f = f_2 + f^*$ where f_2 is a quadratic form in w and $f^* \in \mathcal{M}_w^3$, equipped with the corner $\Gamma_c = \{xy = 0\}$.

Recall that the quasi corner tangent space to an admissible deformation f_t at the origin takes the form

$$\begin{aligned} TQC_{f_t} &= \left\{ \frac{\partial f_t}{\partial x} \left(xh_1 + \frac{\partial f_t}{\partial x} A_1 + \frac{\partial f_t}{\partial y} A_2 \right) + \frac{\partial f_t}{\partial y} \left(yh_2 + \frac{\partial f_t}{\partial x} B_1 + \frac{\partial f_t}{\partial y} B_2 \right) \right. \\ &+ \left. \sum_{i=1}^{n-2} \frac{\partial f_t}{\partial z_i} k_i \right\}, \end{aligned}$$

for arbitrary function germs $h_i, A_i, B_i, k_i \in \mathbf{C}_w$.

If the function germs base point is at a regular point of the cross Γ_c outside the intersection of the components, then the quasi corner equivalence coincides with quasi boundary equivalence . Hence, the simple quasi corner classes in this case are the simple quasi boundary classes: $F_{k,m} : \pm(y \pm x_1^k)^2 \pm x_1^m, \quad 1 \leq k < m$. Mind that the classes B_m are included in $F_{k,m}$ as $F_{1,k}$.

The remaining case of the function germ having a critical base point at the intersection of the components is the main subject of the present chapter.

We now specialise properties of quasi border equivalence for the case of quasi corner singularities in the new coordinates.

Denote by $f^*(z) = f|_{x=y=0}$, the restriction of function f to the z coordinates subspace. Denote by r^* the rank of the second differential $d_0^2 f^*$ at the origin and set $c = n - 2 - r^*$.

Lemma 3.1.1 *(Stabilization) The function germ $f(x, y, z)$ is quasi corner equivalent to $\sum_{i=1}^{r^*} \pm z_i^2 + g(x, y, \hat{z})$, where $\hat{z} \in \mathbb{R}^c$ and $g^* \in \mathcal{M}_{\hat{z}}^3$. For quasi corner equivalent germs f, the respective reduced germs g are quasi corner equivalent.*

Lemma 3.1.2 *There is a non-negative integer $s \leq r - r^*$ such that the function germ $f(x, y, z)$ is quasi corner equivalent to $\sum_{i=1}^{r^*+s} \pm z_i^2 + \widetilde{f}(x, y, \widetilde{z})$, where $\widetilde{z} \in \mathbb{R}^{c-s}$ and $\widetilde{f}$ is a sum of a function germ from $\mathcal{M}_{x,y,\widetilde{z}}^3$ and a quadratic form in x and y only. For quasi corner equivalent f germs, the respective reduced germs $\widetilde{f}$ are quasi corner equivalent.*

Lemmas 3.1.1 and 3.1.2 imply the following preliminary classification result.

Lemma 3.1.3 *Let $k = n - r$ be the corank of the second differential $d_0^2 f$ at the origin.*

1. *If $k = 0$, then f is quasi corner equivalent to $\pm x^2 \pm y^2 + \sum_{i=1}^{n-2} \pm z_i^2$.*

2. *If $k = 1$, then f is quasi corner equivalent to either $\sum_{i=1}^{n-2} (\pm z_i^2) + \widetilde{f}(x, y)$ with* rank $d_0^2 \widetilde{f}(x, y)) = 1$ *or to $\sum_{i=2}^{n-2} (\pm z_i^2) + \widetilde{f}(x, y, z_1) \pm x^2 \pm y^2$ with $\widetilde{f}(x, y, z_1) \in \mathcal{M}_{x,y,z_1}^3$.*

3. *If $k \geq 2$, then f is non-simple.*

Proof. If $k = 0$, then $n = r$. Lemma 3.1.2 yields that there is a non-negative number s such that $s \leq r - r^* = n - r^*$. Consider all possible choices for s. If $s = n - r^*$ or $s = n - r^*$ then $c - s = -2$ or $c - s = -1$, respectively. So these choices are not acceptable. Next, if $s = n - r^* - 2$, then $c - s = 0$. Hence f is quasi corner equivalent to the germ $\widetilde{F} = \sum_{i=1}^{n-2} \pm z_i^2 + f_2(x, y) + \widetilde{f}(x, y)$ where f_2 is a non-degenerate quadratic form and $\widetilde{f} \in \mathcal{M}^3_{x,y}$. By quasi fixed equivalence, the germ $\widetilde{F}$ is quasi corner equivalent to the simple germ: $\pm x^2 \pm y^2 + \sum\limits_{i=1}^{n-2} \pm z_i^2$.

Let $k = n - r = 1$. Then, the total rank $r = n - 1$. Thus, there is s such that $s \leq r - r^* = n - 1 - r^*$. Take $s = n - 1 - r^*$. Then $c - s = n - 2 - r^* - n + 1 + r^* = -1$. So this choice is not possible. Take $s = n - 2 - r^*$. Then $c - s = n - 2 - r^* - n + 2 + r^* = 0$. So f is quasi corner equivalent to $\sum\limits_{i=1}^{n-2}(\pm z_i^2) + \widetilde{f}(x, y)$. Note that the total rank $r = n - 1$ and the rank of $\sum\limits_{i=1}^{n-2}(\pm z_i^2)$ is $n - 2$. Hence rank $d_0^2 \widetilde{f}(x, y) = 1$. Take $s = n - 3 - r^*$. Then, $c - s = 1$. Thus, f is quasi corner equivalent to $\sum\limits_{i=1}^{n-3}(\pm z_i^2) + f_1(x, y, z_{n-2}) + f_2(x, y)$ where $f_1 \in \mathcal{M}^3_{x,y,z_{n-2}}$ and f_2 is a quadratic form in x, y. Note that the rank of $\sum\limits_{i=1}^{n-3}(\pm z_i^2)$ is $n - 3$. This means that rank $d_0^2 f_2(x, y)$ is 2. Hence, f_2 can be reduced to $\pm x^2 \pm y^2$.

Consider $s = n - 4 - r^*$. Then $c - s = 2$ and f is quasi corner equivalent to $\sum\limits_{i=1}^{n-4}(\pm z_i^2) + f_1(x, y, z_{n-3}, z_{n-2}) + f_2(x, y)$ where $f_1 \in \mathcal{M}^3_{x,y,z_{n-2}}$. Note again that the rank of $\sum\limits_{i=1}^{n-4}(\pm z_i^2)$ is $n - 4$. Hence rank $d_0^2 f_2(x, y)$ is 3. However, this is not true as rank $d_0^2 f_2(x, y) \leq 2$. This is not possible either.

Finally, let $k \geq 2$. Then, Lemma 3.1.2 and similar argument as above shows that the function germ f is quasi corner equivalent to one of the following germs:

1. $\sum_{i=k+1}^{n-2} \pm z_i^2 + f_2(x, y) + \widetilde{f}(z_1, z_2, \ldots, z_k, x, y)$, where f_2 is a non-degenerate quadratic form and $\widetilde{f} \in \mathcal{M}_{z_1, z_2, \ldots, z_k, x, y}$ or

2. $\sum_{i=k}^{n-2} \pm z_i^2 + f_2(x, y) + \widetilde{f}(z_1, z_2, \ldots, z_{k-1}, x, y)$, where f_2 is a degenerate quadratic

form of rank one and $\widetilde{f} \in \mathcal{M}_{z_1,z_2,\dots,z_{k-1},x,y}$ or.

3. $\sum_{i=k-1}^{n-2} \pm z_i^2 + \widetilde{f}(z_1, z_2, \dots, z_{k-2}, x, y)$, where $\widetilde{f} \in \mathcal{M}_{z_1,z_2,\dots,z_{k-2},x,y}$

These reduced germs are non-simple by the following Lemma. ∎

Lemma 3.1.4 *The function germs of the form:*

1. $F_1(x, y, z) = \sum_{i=k-1}^{n-2} \pm z_i^2 + \widetilde{f}(z_1, z_2, \dots, z_{k-2}, x, y)$ *, where* $\widetilde{f} \in \mathcal{M}_{z_1,z_2,\dots,z_{k-2},x,y}$
2. $F_2(x, y, z) = \sum_{i=k}^{n-2} \pm z_i^2 + f_2(x, y) + \widetilde{f}(z_1, z_2, \dots, z_{k-1}, x, y)$ *, where* f_2 *is a degenerate quadratic form of rank one and* $\widetilde{f} \in \mathcal{M}_{z_1,z_2,\dots,z_{k-1},x,y}$
3. $F_3(x, y, z) = \sum_{i=k+1}^{n-2} \pm z_i^2 + f_2(x, y) + \widetilde{f}(z_1, z_2, \dots, z_k, x, y)$ *, where* f_2 *is a non-degenerate quadratic form and* $\widetilde{f} \in \mathcal{M}_{z_1,z_2,\dots,z_k,x,y}$

are non simple, if $k \geq 2$.

Proof. Consider the germ F_1. Then, the tangent space to the orbit at $\widetilde{f}$ takes the form:

$$\begin{aligned} TQC_{\widetilde{f}} &= \sum_{i=1}^{k-2} \frac{\partial \widetilde{f}}{\partial z_i} A_i + \frac{\partial \widetilde{f}}{\partial x} \left\{ x B_1 + \frac{\partial \widetilde{f}}{\partial x} B_2 + \frac{\partial \widetilde{f}}{\partial y} B_3 \right\} \\ &+ \frac{\partial \widetilde{f}}{\partial y} \{ y C_1 + \frac{\partial \widetilde{f}}{\partial x} C_2 + \frac{\partial \widetilde{f}}{\partial y} C_3 \}. \end{aligned}$$

The cubic terms in $TQC_{\widetilde{f}}$ are obtained from $\sum_{i=1}^{k-2} \frac{\partial \widetilde{f}}{\partial z_i} A_i$ ($A_i = a_0 x + b_0 y + \sum_{i=1}^{k-2} a_i z_i$) , $\frac{\partial \widetilde{f}}{\partial x} x B_1$ and $\frac{\partial \widetilde{f}}{\partial y} y C_1$ ($a_0, b_0, a_i, B_1, C_1 \in \mathbb{R}$) which form a subspace of dimension $k(k-2)+2$. This dimension is less than the $M = \frac{(k+2)(k+1)k}{6}$-the dimension of all homogeneous quadraic term terms in z_i, x and y.

The function germ F_2 can be written in the form $F_2(x, y, z) = \sum_{i=k}^{n-2} \pm z_i^2 \pm (ax + by)^2 + \widetilde{f}(z_1, z_2, \dots, z_{k-1}, x, y)$. Thus, the germ F_2 is adjacent to the germ

$F_2(x, \delta z_{k-1}+y, z) = \sum_{i=k}^{n-2} \pm z_i^2 + (ax + b(\delta z_{k-1} + y))^2 + \widetilde{f}(z_1, z_2, \dots, z_{k-1}, x, y)$ for sufficiently small δ. Stabilization Lemma 3.1.1 yields that $F_2(x, \delta z_{k-1}+y, z)$ is quasi corner equivalent to the germ $F_1(x, y, z) = \sum_{i=k-1}^{n-2} \pm z_i^2 + \widetilde{f}(z_1, z_2, \dots, z_{k-2}, x, y)$ where $\widetilde{f} \in \mathcal{M}^3_{z_1,z_2,\dots,z_{k-2},x,y}$. The germ F_1 is non-simple.

Similarly, the function germ F_3 can be written in the form $F_3 = \sum_{i=k+1}^{n-2} \pm z_i^2 \pm x^2 \pm y^2 + \widetilde{f}(z_1, z_2, \dots, z_k, x, y)$. The germ F_3 is adjacent to the germ $F_3(\delta z_k + x, y, z) = \sum_{i=k+1}^{n-2} \pm z_i^2 \pm (\delta z_k + x)^2 \pm y^2 + \widetilde{f}(z_1, z_2, \dots, z_k, x, y)$ for sufficiently small δ. Stabilization Lemma shows that $F_3(\delta z_k + x, y, z)$ is quasi corner equivalent to the germ $F_2(x, y, z) = \sum_{i=k}^{n-2} \pm z_i^2 + f_2(x, y) + \widetilde{f}(z_1, z_2, \dots, z_{k-1}, x, y)$, where f_2 is a degenerate quadratic form of rank 1 and $\widetilde{f} \in \mathcal{M}^3_{z_1,z_2,\dots,z_{k-1},x,y}$. The germ F_2 is non-simple.

∎

Lemma 3.1.5 *1. Let the function germ $f(x, y)$ with a critical point at the origin has the quadratic form f_2 of rank 1. Then f is quasi corner equivalent to either $\pm(x \pm y)^2 + \varphi(y)$ or up to permutation of x and y to $\pm x^2 + xg_1(y) + g_2(y)$ where $\varphi, g_2 \in \mathcal{M}^3_y$ and $g_1 \in \mathcal{M}^2_y$.*

2. The germ $\pm x^2 \pm y^2 + \widetilde{f}(x, y, z_1)$ which is described in Lemma (3.1.3) is quasi corner equivalent to the germ $\pm x^2 \pm y^2 + xh_1(z_1) + yh_2(z_1) + h_3(z_1)$ with $h_1, h_2 \in \mathcal{M}^2_{z_1}$, $h_3 \in \mathcal{M}^3_{z_1}$ and $\widetilde{f} \in \mathcal{M}^3_{x,y,z_1}$.

Proof. **1)** By an appropriate scaling of the coordinates we can reduce the quadratic part to either $\pm(x \pm y)^2$ or $\pm x^2$ (permuting if needed x and y). We treat the two cases separately.

i. Consider a deformation

$$H(x, y, \lambda) = \pm(x \pm y)^2 + \varphi(x, y) + \lambda$$

of functions in x with parameters y and λ. We shall prove that these deformations for any φ are quasi corner equivalent. Take a homotopy $H_t(x, y, \lambda) = \pm(x \pm y)^2 + t\varphi(x, y) + \lambda$ between $H(x, y, \lambda)$ and $H_0(x, y, \lambda) = (x \pm y)^2 + \lambda$, we prove that H_t are all quasi corner equivalent as deformations.

For $H_t(x,y,\lambda) = \pm(x\pm y)^2 + t\varphi(x,y) + \lambda$ we seek θ_t such that $H_t \circ \theta_t = H_0$ with a family of admissible diffeomorphisms $\theta_t : (x,y,\lambda) \longmapsto (\widetilde{x}(x,y,\lambda,t), \widetilde{y}(y,\lambda,t), \widetilde{\lambda}(y,\lambda,t))$.

Th homological equation takes the form

$$-\varphi(x,y) = -\frac{\partial H_t}{\partial t} = \frac{\partial H_t}{\partial x}\left(xa + \frac{\partial H_t}{\partial x}b + \frac{\partial H_t}{\partial y}c\right) + \frac{\partial H_t}{\partial y}\left(ya^* + \frac{\partial H_t}{\partial x}b^* + \frac{\partial H_t}{\partial y}c^*\right) + \dot{\lambda},$$

for smooth functions a, b, c, a^*, b^*, c^* and $\dot{\lambda}$.

We want solve the previous equation for given $\varphi(x,y,)$. Note that $\widetilde{y}$ depends on parametres y and λ only. Thus, we should set $b^* = c^* = 0$ and mind that $a^* \in \mathbf{C}_{y,\lambda}$. Now and try to find a, b, a^*, c and $\dot{\lambda}$ where $\dot{\lambda} = \frac{\partial \widetilde{\lambda}}{\partial t}$. Therefore, it is sufficient to solve:

$$\begin{aligned} -\varphi(x,y) &= [\pm 2(x\pm y) + t\frac{\partial\varphi}{\partial x}]\left\{xa + [\pm 2(x\pm y) + t\frac{\partial\varphi}{\partial x}]b + [\pm 2(x\pm y) + t\frac{\partial\varphi}{\partial y}]c\right\} \\ &+ [\pm 2(x\pm y) + t\frac{\partial\varphi}{\partial y}]\{ya^*\} + \dot{\lambda}. \qquad (*) \end{aligned}$$

We shall use Malgrange preparation theorem. Let $\mathbb{P} = \mathbf{C}_{x,y,\lambda,t}$ and consider the mapping

$$\begin{aligned} G : (x,y,\lambda,t) &\mapsto (g_1, g_2, g_3, g_4, g_5) \\ &= ([\pm 2(x\pm y) + t\frac{\partial\varphi}{\partial x}]x, [\pm 2(x\pm y) + t\frac{\partial\varphi}{\partial x}].[\pm 2(x\pm y) + t\frac{\partial\varphi}{\partial x}], \lambda, t, y) \end{aligned}$$

Let I be the ideal generated by components of the mapping G. That is $I\mathbb{P} = g_1h_1 + g_2h_2 + \lambda h_3 + th_4 + yh_5$ for some $h_1, h_2, h_3, h_4, h_5 \in \mathbb{P}$. It follows that $\mathbb{P}/I\mathbb{P} = \{\alpha_1 + x\alpha_2\}$ or equivalently $\mathbb{P}/I\mathbb{P} = \{\alpha_1 + (2(x\pm y) + t\frac{\partial\varphi}{\partial y})\alpha_2\}$ with α_1 ,$\alpha_2 \in \mathbb{R}$.
Thus by Malgrange preparation theorem we get $\mathbb{P} = \mathbf{C}_G\{1,x\}$. So for any $P \in \mathbb{P}$

we have

$$\begin{aligned}
P &= 1.\beta_1(g_1, g_2, \lambda, t, y) + [\pm 2(x \pm y) + t\frac{\partial\varphi}{\partial y}].\beta_2(g_1, g_2, \lambda, t, y) \\
&= g_1 A(g_1, g_2, \lambda, t, y) + g_2 B(g_1, g_2, \lambda, t, y) + C(\lambda, t, y) + g_1 \widehat{A}(g_1, g_2, \lambda, t, y)x \\
&+ g_2 \widehat{B}(g_1, g_2, \lambda, t, y)x + y[\pm 2(x \pm y) + t\frac{\partial\varphi}{\partial y}]\widehat{C}(\lambda, t, y) + [\pm 2(x \pm y) + t\frac{\partial\varphi}{\partial y}]D(t, \lambda) \\
&= g_1 \widetilde{A} + g_2 \widetilde{B} + C(\lambda, t, y) + y[\pm 2(x \pm y) + t\frac{\partial\varphi}{\partial y}]\widetilde{C}(\lambda, t, y) + [\pm 2(x \pm y) + t\frac{\partial\varphi}{\partial y}]D(t, \lambda)
\end{aligned}$$

If we replace φ in (*) by P, then the homological equation becomes solvable by taking $a = \widetilde{A}$, $b = \widetilde{B}$, $\dot{\lambda} = C$, $a^* = \widetilde{C}$ and $c = 0$. Note here that $\varphi \in \mathcal{M}^3_{x,y}$ does not depend on t and λ. So we can assume that $D(t, \lambda) \equiv 0$.

Thus we have shown that H_t is quasi corner equivalent to H_0.

The restriction of θ_1 to the subspace $\lambda = 0$ provides a quasi corner equivalence of H_1 with the family $H_0 + \Lambda(y)$ for some function Λ.

ii. The proof of the second claim is similar to the previous argument. Consider the deformations $F_0 = \pm x^2 + \lambda_1 x + \lambda_2$ and $F = \pm x^2 + \varphi(x, y) + \lambda_1 x + \lambda_2$ which depend on three parameters λ_1, λ_2 and y, but x is considered as a variable.

We take the family of deformations $F_t = F_0 + t\varphi(x, y)$ with $t \in [0, 1]$ and show that all F_t are quasi corner equivalent as deformations.

Thus, we consider $F_t \circ \theta_t = F_0$ with

$$\theta_t : (x, y, \lambda_1, \lambda_2, t) \longmapsto (\widetilde{X}(x, y, \lambda_1, \lambda_2, t), \widetilde{Y}(y, \lambda_1, \lambda_2, t), \widetilde{\Lambda}_1(y, \lambda_1, \lambda_2, t), \widetilde{\Lambda}_2(y, \lambda_1, \lambda_2, t)).$$

We solve the homological equation

$$-\frac{\partial F_t}{\partial t} = -\varphi(x, y) = \frac{\partial F_t}{\partial x}\{xa + \frac{\partial F_t}{\partial x}b + \frac{\partial F_t}{\partial y}c\} + \frac{\partial F_t}{\partial y}\{ya^* + \frac{\partial F_t}{\partial x}b^* + \frac{\partial F_t}{\partial y}c^*\} + x\dot{\lambda}_1 + \dot{\lambda}_2,$$

where $\dot{\lambda}_i = \frac{\partial \widetilde{\Lambda}_i}{\partial t}, i = 1, 2$ and a, b, c, a^*, b^* and c^* are smooth functions. We want to solve the previous equation for given $\varphi(x, y)$. Thus, we set $b = c = b^* = a^* = c^* = 0$ and try to solve $-\frac{\partial F_t}{\partial t} = (\pm 2x + t\frac{\partial\varphi}{\partial x} + \lambda_1)\{xa\} + x\dot{\lambda}_1 + \dot{\lambda}_2$.

Let $\mathbb{P} = \mathbf{C}_{x,y,\lambda_1,\lambda_2,t}$. Consider the mapping

$$G : (x, y, \lambda_1, \lambda_2, t) \longmapsto (\pm 2x + t\frac{\partial\varphi}{\partial x} + \lambda_1)x, \lambda_1, \lambda_2, y, t).$$

Then $\mathbb{P}/I\mathbb{P} = \mathbf{C}_x/\{x^2\} = \{\alpha_1 + \alpha_2 x\}$ with α_1, $\alpha_2 \in \mathbb{R}$. Thus by Malgrange preparation theorem we get for any $P \in \mathbb{P}$

$$\begin{aligned} P &= 1.h_1(g_1, \lambda_1, \lambda_2, y, t) + xh_2(g_1, \lambda_1, \lambda_2, y, t) \\ &= g_1\widehat{h}_1(g_1, \lambda_1, \lambda_2, y, t) + \widehat{h}_2(\lambda_1, \lambda_2, y, t) + x\widehat{h}_3(\lambda_1, \lambda_2, y, t), \end{aligned}$$

where $g_1 = (\pm 2x + t\frac{\partial\varphi}{\partial x} + \lambda_1)x$.

Thus, the previous homological equation is solvable by replacing φ by P and taking $a = \widehat{h}_1$, $\dot{\lambda}_2 = \widehat{h}_2$ and $\dot{\lambda}_1 = \widehat{h}_3$.

The restriction of θ_1 to the subspace $\lambda_1 = \lambda_2 = 0$ provides a quasi corner equivalence of H_1 with the family $\pm x^2 + x\Lambda_1(y) + \Lambda_2(y)$ for some functions Λ_1 and Λ_2.

2) Take the deformations $F_1 = \pm x^2 \pm y^2 + Q_3(x, y, z_1) + x\lambda_1 + y\lambda_2 + \lambda_3$ and $F_0 = \pm x^2 \pm y^2 + x\lambda_1 + y\lambda_2 + \lambda_3$, where x and y are considered as variables but $z_1, \lambda_1, \lambda_2$ and λ_3 as parameters.

Construct the homotopy $F_t = \pm x^2 \pm y^2 + tQ_3(x, y, z_1) + x\lambda_1 + y\lambda_2 + \lambda_3$ joining F_0 and F_1, with $t \in [0, 1]$. We shall prove that all F_t are quasi corner equivalent. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$.

Let us consider $F_t \circ \theta_t = F_0$ where θ_t are admissible and takes the form

$$\begin{aligned} \theta_t : (x, y, z_1, \lambda, t) \;&\mapsto\; \widetilde{X}(x, y, z_1, \lambda, t), \widetilde{Y}(x, y, z_1, \lambda, t), \widetilde{Z}(z_1, \lambda, t), \\ &\quad \widetilde{\Lambda}_1(z_1, \lambda, t), \widetilde{\Lambda}_2(z_1, \lambda, t), \widetilde{\Lambda}_3(z_1, \lambda, t). \end{aligned}$$

We need now to solve the homological equation

$$-Q(x, y, z_1) = \frac{\partial F_t}{\partial x}\{x\alpha + \frac{\partial F_t}{\partial x}\beta + \frac{\partial F_t}{\partial y}\gamma\} + \frac{\partial F_t}{\partial y}\{y\alpha' + \frac{\partial F_t}{\partial x}\beta' + \frac{\partial F_t}{\partial y}\gamma'\} + \frac{\partial F}{\partial z_1}A + x\dot{\lambda}_1 + \dot{\lambda}_2 y + \dot{\lambda}_3,$$

where $\dot{\lambda}_i = \frac{\partial\widetilde{\lambda}_i}{\partial t}, i = 1, 2, 3.$

Let $\mathbb{P} = \mathbf{C}_{x,y,z_1,\lambda,t}$ and take the mapping:

$$G : (x, y, z_1, \lambda, t) \mapsto ((\pm 2x + t\frac{\partial Q}{\partial x} + \lambda_1)x, (\pm 2y + t\frac{\partial Q}{\partial y} + \lambda_2)y, z_1, \lambda, t).$$

$$\begin{aligned} G : (x, y, z_1, \lambda, t) \quad \mapsto \quad & \Big((\pm 2x + t\frac{\partial Q}{\partial x} + \lambda_1)x, (\pm 2y + t\frac{\partial Q}{\partial y} + \lambda_2)y, z_1, \lambda, t \\ & (\pm 2x + t\frac{\partial Q}{\partial x} + \lambda_1)(\pm 2y + t\frac{\partial Q}{\partial y} + \lambda_2)\Big). \end{aligned}$$

Thus $\mathbb{P}/I\mathbb{P} = \mathbf{C}_{x,y}/\{x^2, y^2\} = \mathbb{R}\{1, x, y\}$. Here I is the ideal generated by the components of the mapping G.

Hence, according to Malgrange preparation theorem we get $\mathbb{P} = \mathbf{C}_G\{1, x, y\}$. Thus for $P \in \mathbb{P}$

$$\begin{aligned} P &= H_1(g_1, g_2, z_1, \lambda, t) + xH_2(g_1, g_2, z_1, \lambda, t)) + yH_3(g_1, g_2, z_1, \lambda, t) \\ &= g_1\widetilde{H_1}(g_1, g_2, z_1, \lambda, t) + g_2\widetilde{H_2}(g_1, g_2, z_1, \lambda, t) + x\widetilde{H_3}(z_1, \lambda, t) + \\ &+ y\widetilde{H_4}(z_1, \lambda, t) + g_3\widetilde{H_6}(g_1, g_2, z_1, \lambda, t) + \widetilde{H_5}(z_1, \lambda), \end{aligned}$$

where $g_1 = (\pm 2x + t\frac{\partial Q}{\partial x} + \lambda_1)x$, $g_2 = (\pm 2y + t\frac{\partial Q}{\partial y} + \lambda_2)y$ and $g_3 = (\pm 2x + t\frac{\partial Q}{\partial x} + \lambda_1)(\pm 2y + t\frac{\partial Q}{\partial y} + \lambda_2)$.

Therefore, the homological equation is solvable by setting $\alpha = \widetilde{H_1}, \alpha' = \widetilde{H_2}, \dot{\lambda}_1 = \widetilde{H_3}, \dot{\lambda}_2 = \widetilde{H_4}, \dot{\lambda}_3 = \widetilde{H_5}$, $\beta = \gamma = \gamma' = A = 0$ and $\beta' = \widetilde{H_6}$.

The restriction of θ_1 to the subspace $\lambda_1 = \lambda_2 = \lambda_3 = 0$ provides a quasi corner equivalence of F_1 with the family $\pm x^2 \pm y^2 + x\Lambda_1(z_1) + y\Lambda_2(z_1) + \Lambda_3(z_1)$ for some functions Λ_1, Λ_2 and Λ_3.

∎

Theorem 3.1.6 *Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a simple germ with respect to the quasi corner equivalence. Then the following is true:*

• *If f_2 is a non-degenerate form, then f is quasi corner equivalent to the Morse function* $\mathcal{B}_2 : \pm x^2 \pm y^2 + \sum_{i=1}^{n-2} \pm z_i^2$.

• *If f_2 is a degenerate form of corank 1 then f is stably quasi corner equivalent to one of the following simple classes:*

1. $\mathcal{B}_m : \pm(x \pm y)^2 \pm y^m, \quad m \geq 3, \qquad m+1;$

2. $\mathcal{F}_{k,m} : \pm(x \pm y^k)^2 \pm y^m, \quad m > k \geq 2, \qquad k+m;$

3. $\mathcal{H}_{m,n,k} : \pm(x \pm z_1^m)^2 \pm (y \pm z_1^n)^2 \pm z_1^k$, *where* $k > n \geq m, \geq 2 \quad m+n+k-1.$

The orbit codimension in the space of germs is shown in the right column.

Remark: Any germ f with corank of f_2 being greater or equal 2 is non-simple. Any germ of corank 1 either is simple (and hence is quasi corner equivalent to one of the germs stated in the theorem) or belongs to a subset of infinite codimenesion in the space of all germs.

Proof of Theorem 3.1.6.

Lemma 3.1.4 implies that we need to consider germs of the forms stated in Lemma 3.1.5 to classify possible simple classes. So start with function germ of the form: $H(x,y) = \pm(x \pm y)^2 + \varphi(y)$ with $\varphi(y) \in \mathcal{M}_y^3$. Assume that $\varphi(y) = a_k y^k + \widetilde{\varphi}(y)$, where $a_k \neq 0$ and $\widetilde{\varphi}(y) \in \mathcal{M}_y^{k+1}$. Let $H_0(x,y) = \pm(x \pm y)^2 + a_k y^k$. Consider the tangent space of the quasi corner orbit at H_0.

$$TQC_{H_0} = \pm 2(x \pm y)\left\{xA_0 + yA_1\right\} + \left[\pm 2(x \pm y) + k a_k y^{k-1}\right]\left\{yB_0 + xB_1\right\}.$$

We have *mod* TQC_{H_0}

$$\pm(x \pm y)y \equiv 0. \tag{3.1}$$

Also,

$$\pm 2(x \pm y)y + k a_k y^k \equiv 0. \tag{3.2}$$

Multiply the equation (3.2) by y to obtain

$$\pm 2(x \pm y)y + ka_k y^k \equiv 0. \tag{3.3}$$

If we substitute (3.1) in (3.3) then we get $y^k \equiv 0$. This yields the existence of solutions of the equations for any term $g^* = b_s y^s$ with $s \geq k$. In particular, one can find solution for g^* by setting $A_0 = B_1 = 0$ and $A_1 = -B_0 = -\frac{b_s}{ka_k} y^{s-k}$. Assign weights $w_x = w_y = \frac{1}{2}$. Then, g^* has quasi degree $d(g^*) = \frac{s}{2}$. Note if $s > k$ then $A_1, B_0 \in \mathcal{M}_y$. The germ $\Phi = sb_s y^s B_0$ has quasi degree greater than $d(g^*)$ when $s > k$. Hence, by Lemma (1.3.5), we conclude that H is quasi corner equivalent to the germ H_0. Note that there are solutions for the term $g^* = a_k y^k$. Hence the class is simple. Rescaling a_k to ± 1 we get the classes $\mathcal{B}_m : \pm(x \pm y)^2 \pm y^m$ with $m \geq 3$.

Now consider the germ of the form

$$F_1(x, y) = \pm x^2 + x\varphi_1(y) + \varphi_2(y), \tag{3.4}$$

where $\varphi_1 \in \mathcal{M}_y^2$ and $\varphi_2 \in \mathcal{M}_y^3$. Let $\varphi_1(y) = a_k y^k + \widetilde{\varphi}_1(y)$ with $a_k \neq 0, k \geq 2, \widetilde{\varphi}_1 \in \mathcal{M}_y^{k+1}$. Let $\varphi_2(y) = b_m y^m + \widetilde{\varphi}_2(y)$ with $b_m \neq 0, m \geq 3, \widetilde{\varphi}_2 \in \mathcal{M}_y^{m+1}$.

We distinguish the following cases:

1. If $k \geq m - 1$ then the germ F_1 is quasi corner equivalent to the germ $\widetilde{G}_0 = \pm x^2 \pm y^m$. To prove this claim, consider the tangent space to the quasi corner orbit at $F_0 = \pm x^2 + b_m y^m$ which takes the form:

$$TQC_{F_0} = \pm 2x \left\{ xA^* + y^{m-1}C \right\} + mb_m y^{m-1} \left\{ y\widetilde{A}^* + x\widetilde{B} \right\}.$$

For any term $g_1^* = e_s x y^s$ with $s \geq k$, set $A^* = \widetilde{A}^* = \widetilde{B} = 0$ and take $C = \frac{\mp e_s}{2} y^{s-m+1}$.

For any term $g_2^* = d_s y^s$ with $s \geq m$, set $A^* = C = \widetilde{B} = 0$ and take $\widetilde{A}^* = \frac{\mp d_s}{mb_m} y^{s-m}$. Note that $\widetilde{A}^* \in \mathcal{M}_y$ when $s > m$.

Assign weights $w_x = \frac{1}{2}$ and $w_y = \frac{1}{m}$. Then, g_1^* has quasi degree $d(g_1^*) = \frac{1}{2} + \frac{s}{m}$. Moreover, the germ

$$\Phi_1 = e_s y^s (y^{m-1}C + se_s x y^{s-1} C) = (e_s y^{s+m-1} + se_s^2 x y^{2s-1})C,$$

where C is as above, has quasi degree $d(\Phi_1)$ greater than $d(g_1^*)$.

Similarly, g_2^* has quasi degree $d(g_2^*) = \frac{s}{m}$ and the germ

$$\Phi_2 = sd_s y^{s-1}(y\widetilde{A}^*) = sd_s y^s \widetilde{A}^*,$$

where $\widetilde{A}^*$ is as above, has quasi degree $d(\Phi_2)$ greater than $d(g_2^*)$.

Thus, Lemma 1.3.5 shows that F_1 is quasi corner equivalent to the germ $F_0 = \pm x^2 + b_m y^m$. Moreover, F_0 is simple as there are solutions for the term $b_m y^m$. Normalize b_m to ± 1 to get the equivalent germ $\widetilde{G}_0 = \pm x^2 \pm y^m, \quad m \geq 3$. Note that the germ $\widetilde{G}_0$ is quasi corner equivalent to the germ $\pm x^2 \pm 2xy^{m-1} \pm y^{2(m-1)} \pm y^m = \pm(x \pm y^{m-1})^2 \pm y^m$.

2. If $m > k+1$ and $\mp a_k^2 + 4b_m \neq 0$ when $m = 2k$, then F_1 is quasi corner equivalent to the germ $\widetilde{G}_0 = \pm x^2 \pm xy^k \pm y^m$. To prove this claim consider the tangent space to the quasi corner orbit at $F_0 = \pm x^2 + a_k xy^k + b_m y^m$.

$$TQC_{F_0} = (\pm 2x + a_k y^k)\left\{xA^* + y^k B\right\} + (ka_k xy^{k-1} + mb_m y^{m-1})\left\{y\widetilde{A}^* + x\widetilde{B}\right\}.$$

We have *mod* TQC_{F_0}:

$$\pm 2x^2 + a_k xy^k \equiv 0, \tag{3.5}$$

$$xy^k \equiv \mp\frac{a_k}{2}y^{2k}, \tag{3.6}$$

and

$$ka_k xy^k + mb_m y^m \equiv 0. \tag{3.7}$$

If we substitute xy^k from the equation (3.6) in the equation (3.7), we get:

$$\mp\frac{ka_k^2}{2}y^{2k} + mb_m y^m \equiv 0. \tag{3.8}$$

The last relation yields that $y^{2k} \equiv 0$ and $y^m \equiv 0$. Hence : $xy^k \equiv 0$ and $x^2 \equiv 0$. This implies the existence for solutions for any term of the form $g_1^* = e_s xy^s$ with $s \geq k$ and $g_2^* = d_l y^l$ with $l \geq m$.

In particular, one can set $A^* = \widetilde{B} = 0$ and take $\widetilde{A}^*$ and B such that:

$$g_1^* + g_2^* = (\pm 2xy^k + a_k y^{2k})B + (ka_k xy^k + mb_m y^m)\widetilde{A}^*$$

or, equivalently

$$g_1^* + g_2^* = x[\pm 2y^k B + ka_k y^k \widetilde{A}^*] + [a_k y^{2k} B + mb_m y^m \widetilde{A}^*].$$

Assume that $2k \geq m$ (similar argument holds when $2k < m$).

Thus, the solution for any term g_1^* can be found by setting $A^* = \widetilde{B} = 0$ and taking $\widetilde{A}^*$ and B such that

$$\widetilde{A}^* = \frac{-a_k}{mb_m} y^{2k-m} B,$$

and

$$B = \frac{e_s y^{s-k}}{\pm 2 - \frac{ka_k^2}{mb_m} y^{2k-m}}.$$

Similarly, one can find solutions for any term g_2^* by setting $A^* = \widetilde{B} = 0$ and taking $\widetilde{A}^*$ and B such that

$$\widetilde{A}^* = \frac{\mp 2}{ka_k} B,$$

and

$$B = \frac{d_l y^{l-m}}{\frac{\mp 2mb_m}{ka_k} + a_k y^{2k-m}}.$$

Now assign weights $w_x = \frac{1}{2}$ and $w_y = \frac{1}{2k}$. Then the germ g_1^* has quasi degree $d(g_1^*) = \frac{1}{2} + \frac{s}{2k}$. The germ

$$\Phi_1 = e_s y^s [y^k B + e_s y^s] + se_s xy^{s-1}[y\widetilde{A}^*]$$

has quasi degree greater than $d(g_1^*)$ when $s > k$.

On the other hand, g_2^* has $d(g^*) = \frac{l}{2k}$. The quasi degree of germ

$$\Phi_2 = ld_l y^{l-1}[y\widetilde{A}^*],$$

is greater than $d(g_2^*)$ for $l > m$.

Lemma 1.3.5 shows that F_1 is quasi corner equivalent to F_0. If we normalize a_k and b_m to ± 1 then we get the equivalent germ $\widetilde{G}_0 = \pm x^2 \pm xy^k \pm y^m$. Note that the germ G_0 is quasi corner equivalent to the germ $\pm x^2 \pm 2xy^k \pm y^{2k} \pm y^m = \pm(x \pm y^k)^2 \pm y^m$.

Note that if $m = 2k$ then similar calculations show that F_1 is quasi corner equivalent to the germ $\widetilde{G}_0 = \pm x^2 \pm xy^k$.

3. If $m > k+1$, $m = 2k$ and $\mp a_k^2 + 4b_m = 0$, then the function germ F_1 takes the form:

$$F_1(x,y) = \pm(x \pm \frac{1}{2}a_k y^k)^2 + x\varphi_1(y) + \widetilde{\varphi}_2(y),$$

where $\varphi_1 \in \mathcal{M}_y^{k+1}$ and $\widetilde{\varphi}_2 \in \mathcal{M}^{2k+1}$. Similar argument to proof of the first statement of Lemma (3.1.5) (or using Lemma 1.3.5) shows that F_1 is quasi corner equivalent to the germ $\widetilde{F} = \pm(x \pm y^k)^2 + \widetilde{\phi}(y)$, where $\widetilde{\phi} \in \mathcal{M}_y^{2k+1}$.

Let $\widetilde{\phi} = \alpha_s y^s + h(y)$ where $\alpha_s \neq 0, s \geq 2k+1$ and $h \in \mathcal{M}_y^{2k+2}$. Then, again similar argument to proof of the first statement in the present theorem proves that $\widetilde{F}$ is quasi corner equivalent to the germ $\pm(x \pm y^k)^2 \pm y^s$.

Finally, consider the function germ $F_1 = \pm x^2 \pm y^2 + xh_1(z_1) + yh_2(z_1) + h_3(z_1)$. Let

$$F_1 = \pm x^2 \pm y^2 + a_m x z_1^m + x\varphi_1(z_1) + b_n y z_1^n + y\varphi_2(z_1) + c_k z_1^k + \varphi_3(z_1),$$

with $a_m \neq 0, m \geq 2, b_n \neq 0, n \geq 2, c_k \neq 0$ and $k \geq 3$. Suppose that $n \geq m$ (Up to the permutation of x and y, if needed) . Then, we distinguish the following cases:

1) If $m, n \geq k-1$, $n, m \geq 2$ and $k \geq 3$, then, F_1 is quasi corner equivalent to the germ $G_0 = \pm x^2 \pm y^2 \pm z_1^k$. To see this, consider the tangent space to the quasi corner orbit $F_0 = \pm x^2 \pm y^2 + c_k z_1^k$.

$$TQC_{F_0} = \pm x\left\{xA + yB\right\} \pm 2y\left\{y\widetilde{A} + x\widetilde{B}\right\} + kc_k z_1^{k-1}C.$$

Then, one can find solutions for any term $g_1^* = \alpha_l x z_1^l$ with $l \geq m$, by setting:

$$A = B = \widetilde{A} = \widetilde{B} = 0 \quad \text{and} \quad C = \frac{\alpha_l}{kc_k} x z_1^{l-k+1}.$$

Similarly, one can find solutions for any term $g_2^* = \beta_s y z_1^s$ with $s \geq n$, by setting:

$$A = B = \widetilde{A} = \widetilde{B} = 0 \text{ and } C = \frac{\beta_s}{kc_k} y z_1^{s-k+1}.$$

Also, for any term $g_3^* = \gamma_p z_1^p$ with $p \geq k$, one can set:

$$A = B = \widetilde{A} = \widetilde{B} = 0 \text{ and } C = \frac{\gamma_l}{kc_k} z_1^{p-k+1}.$$

Assign weights $w_x = w_y = \frac{1}{2}$ adn $w_{z_1} = \frac{1}{k}$. Then, $d(g_1^*) = \frac{1}{2} + \frac{l}{k}$ and the germ $\Phi_1 = l\alpha_l x z_1^{l-1} C$ has quasi degree greater than $d(g_1^*)$. Similarly, $d(g_2^*) = \frac{1}{2} + \frac{s}{k}$ and the germ $\phi_2 = s\beta_s y z_1^{s-1} C$ has quasi degree greater than $d(g_2^*)$. For g_3, we have $d(g_3^*) = \frac{p}{k}$ and $C \in \mathcal{M}_{z_1}^2$. The germ $\phi_3 = p\gamma_p z_1^{p-1} C$ has quasi degree greater than $d(g_3^*)$.

Thus by lemma 1.3.5, we see that F_1 is quasi corner equivalent to the germ F_0. Rescaling c_k to ± 1, we get the classes $\pm x^2 \pm y^2 \pm z_1^k$ which has an alternative form $\pm(x \pm z_1^{k-1})^2 \pm (y \pm z_1^{k-1})^2 \pm z_1^k$.

2) If $k > m+1$, $n \geq k-1$ and $\mp a_m^2 + 4c_k \neq 0$ when $2m = k$, then the function germ F_1 is quasi corner equivalent to the germ $G_0 = \pm x^2 \pm y^2 \pm x z_1^m \pm z_1^k$. To see this, consider the tangent space at $F_0 = \pm x^2 \pm y^2 + a_m x z_1^m + c_k z_1^k$.

$$\begin{aligned} TQC_{F_0} &= (\pm 2x + a_m z_1^m)\left\{xA_0 + (\pm 2x + a_m z_1^m)A_1 + yA_2\right\} \\ &\quad \pm 2y\left\{yB_0 + (\pm 2x + a_m z_1^m)B_1 + yB_2\right\} + (m a_m x z_1^{m-1} + kc_k z_1^{k-1})C_1. \end{aligned}$$

Assign weights $w_x = w_y = \frac{1}{2}$ and $w_{z_1} = \frac{1}{k}$. For any term of the form $g_1^* = d_l x z_1^l$ with $l \geq m$, there are solutions for the homological equation with respect to F_0 by setting $A_2 = B_0 = B_1 = B_2 = 0$ and taking A_0, A_1 and C_1 such that

$$\pm A_0 + 2A_1 = 0, \quad kc_k z_1^{k-1} C_1 + a_m^2 z^{2m} A_1 = 0, \quad \widetilde{A} = d_l z_1^{l-m+1},$$

where

$$\widetilde{A} = a_m z_1 A_0 \pm 4a_m z_1 A_1 + m a_m C_1.$$

Note that $A_0, A_1, C_1 \in \mathcal{M}_{x,y,z_1}$. Now we have the germ

$$\phi_1 = d_l z_1^l [xA_0 + (\pm 2x + a_m z_1^m)A_1 + d_l z_1^l A_1] + (\pm 2x + a_m z_1^m)[d_l z_1^l A_1] + l d_l x z_1^{l-1} C_1.$$

For any term of the form $g_2^* = e_s y z_1^s$ with $s \geq k-1$, there are solutions for the homological equation with respect to F_0 by setting $A_2 = B_0 = B_1 = B_2 = 0$ and taking A_0, A_1 and C_1 such that

$$a_m z_1 A_0 \pm 4a_m z_1 A_1 + m a_m C_1 = 0, \quad \pm A_0 + 2A_1 = 0, \quad C_1 = \frac{e_s}{kc_k} y z_1^{s-k+1}.$$

Note that g_2^* belongs to TQC_{F_0} up to higher quasi degree term. That is $g_2^* = a_m^2 z_1^{2m} A_1 + kc_k z_1^{k-1} C_1$. Also, note that and $A_0, A_1, C_1 \in \mathcal{M}_{y,z_1}$. Now we have the germ $\Phi_2 = s e_s z_1^{s-1} C_1$.

Finally for any term $g_3^* = e_i z_1^i$ for $i \geq k$ then set $A_2 = B_0 = B_1 = B_2 = 0$ and take A_0, A_1 and C_1 such that such that

$$m a_m C_1 \pm 4A_1 + a_m z_1 A_0 = 0, \quad \pm 2A_0 + A_1 = 0, \quad C_1 = \frac{e_i}{kc_k} z_1^{i-k+1}.$$

Note here also that $A_0, A_1, C_1 \mathcal{M}_{y,z_1}$. Now we have the germ $\Phi_3 = i e_i z_1^{i-1} C_1$.

Comparing the quasi degree of ϕ_1, Φ_2 and ϕ_3 with the quasi degree of g_1^*, g_2^* and g_3^*, respectively we conclude that F is quasi corner equivalent to the germ F_0 which can be written in the form $\pm(x \pm z_1^m)^2 \pm (y \pm z_1^{k-1})^2 \pm z_1^k$ after rescaling a_m and c_k to ± 1.

3) If $k = 2m$, $n \geq k-1$ (so $n \geq 2m-1$) and $\mp a_m^2 + 4c_k = 0$, then F_1 takes the form

$$F_1 = \pm y^2 \pm (x + \widetilde{a} z_1^m)^2 + x\varphi_1(z_1) + b_n y z_1^n + y\varphi_2(z_1) + \varphi_3(z_1),$$

where $\widetilde{a} = \pm\frac{1}{4} a_m$, $\varphi_1 \in \mathcal{M}_{z_1}^{m+1}, \varphi_2 \in \mathcal{M}_{z_1}^{n+1}$ and $\varphi_3 \in \mathcal{M}_{z_1}^{2m+1}$. Let $\varphi_3(z_1) = e_s z_1^s +$

$\widetilde{\varphi}_3(z_1)$ where $s \geq 2m+1$ and $\widetilde{\varphi}_3 \in \mathcal{M}^{s+1}$. Consider the germ $F_0 = \pm y^2 \pm (x + \widetilde{a}z_1^m)^2 + b_n y z_1^n + e_s z_1^s$. Then the quasi corner tangent space to the orbit at F_0 takes the form:

$$\begin{aligned} TQC_{F_0} &= \pm 2(x+\widetilde{a}z_1^m)\{xA_0 + [\pm 2(x+\widetilde{a}z_1^m)]A_1 + (\pm 2y + b_n z_1^n)A_2\} \\ &+ (\pm 2y + b_n z_1^n)\{yB_0 + [\pm 2(x+\widetilde{a}z_1^m)]B_1 + (\pm 2y + b_n z_1^n)B_2\} \\ &+ [\pm 2\widetilde{a}_m z_1^{m-1}(x+\widetilde{a}z_1^m) + nb_n y z_1^{n-1} + se_s z_1^{s-1}]C_1. \end{aligned}$$

Set $A_2 = B_1 = 0$ and consider the subspace

$$\begin{aligned} \Phi &= x^2(\pm 2A_0 + 4A_1) + xz_1^{m-1}[8\widetilde{a}z_1A_1 \pm 2z_1A_0 \pm 2\widetilde{a}mC_1] \\ &+ y^2(\pm 2B_0 + 4B_2) + yz_1^{n-1}[nb_nC_1 \pm 4b_n z_1 B_2 \pm z_1 B_0] \\ &+ [4\widetilde{a}^2 z_1^{2m}A_1 + b_n^2 z_1^{2n}B_2 + se_s z_1^{s-1}C_1]. \end{aligned}$$

Then any term of the form $g_1^* = d_l x z_1^l$ with $l \geq m$ can be obtained from the subspace Φ by setting:

$$A_0 = \mp 2A_1 \quad B_0 = \mp 2B_2, \quad C_1 = \frac{\mp 2}{n} z_1 B_2,$$

$$A_1 = \frac{1}{\widetilde{a}}[b_n^2 z_1^{2n-2m} \mp \frac{2se_s}{n} z_1^{s-2m}], \quad B_2 = \frac{d_l z_1^{l-m}}{\frac{4}{\widetilde{a}}\Lambda \mp \frac{4\widetilde{a}m}{n}},$$

where $\Lambda = b_n^2 z_1^{2n-2m} \mp \frac{2se_s}{n} z_1^{s-2m}$. Note that if $l > m$, then $A_0, A_1, B_0, B_2 \in \mathcal{M}_{z_1}$ and $C_1 \in \mathcal{M}_{z_1}^2$. Now the respective germ takes the form:

$$\Phi_1 = d_l z_1^l [xA_0 + z_1^l A_1] + (x + \widetilde{a}z_1^m) z_1^l A_1 + l d_l x z_1^{l-1} C_1.$$

Similar argument can be carried out for any term of the form $g_2^* = \alpha_k y z_1^k$ where $k \geq n$ or of the form $g_3^* = \beta_k z_1^k$ where $k \geq s$. Comparing the quasi degrees of the germs g_i^* and the respective germs Φ_i, $i = 1, 2, 3$ with respect to weights $w_x = w_y = \frac{1}{2}$ and $w_{z_1} = \frac{1}{2m}$, we conclude that the germ F_1 is quasi corner equivalent to the germ $\widetilde{F} = \pm y^2 \pm (x \pm z_1^m)^2 \pm y z_1^n \pm z_1^s$. Note that $\widetilde{F}$ can be written in the form

$\pm(x \pm z_1^m)^2 \pm (y \pm z_1^n)^2 \pm z_1^s$.

4) If $k > n+1 \geq m+1$ and $\mp\frac{ma_m^2}{2} \mp \frac{nb_n^2}{2} + kc_k \neq 0$ when $2m = 2n = k$, then, the germ F_1 is quasi corner equivalent to the germ $F_0 = \pm x^2 \pm y^2 \pm xz_1^m \pm yz_1^n \pm z_1^k$. To prove this claim, consider the tangent space to the quasi corner orbit at $F_0 = \pm x^2 \pm y^2 + a_m xz_1^m + b_n yz_1^n + c_k z_1^k$.

$$\begin{aligned} TQC_{F_0} &= (\pm 2x + a_m z_1^m)\{xA_0 + (\pm 2x + a_m z_1^m)A_1 + (\pm 2y + b_n z_1^n)A_2\} \\ &+ (\pm 2y + b_n z_1^n)\{yB_0 + (\pm 2x + a_m z_1^m)B_1 + (\pm 2y + b_n z_1^n)B_2\} \\ &+ (ma_m xz_1^{m-1} + nb_n xz_1^{n-1} + kc_k z_1^{k-1})C_1. \end{aligned}$$

Set $A_2 = B_1 = 0$ and consider the subspace

$$\Omega = x^2[\pm 2A_0 + 4A_1] + xz_1^{m-1}[a_m z_1 A_0 \pm 4a_m z_1 A_1 + ma_m C_1]$$

$$+y^2[\pm 2B_0 + 4B_2] + yz_1^{n-1}[b_n z_1 B_0 \pm 4b_n z_1 B_2 + nb_n C_1]$$

$$+[kc_k z_1^{k-1} C_1 + a_m^2 z_1^{2m} A_1 + b_n^2 z_1^{2n} B_2]$$

Thus,the term $g_1^* = e_s xz_1^s$ is obtained from Ω by choosing A_0, A_1, B_0, B_2 and C_1 which satisfy the following relations:

$$A_0 = \mp 2A_1, \quad B_0 = \mp 2B_2, \quad C_1 = \frac{\mp 2}{n} z_1 B_2,$$

$$\mp \frac{2kc_k}{n} z_1^k B_2 + a_m^2 z_1^{2m} A_1 + b_n^2 z_1^{2n} B_2 = 0,$$

and

$$\pm 2a_m A_1 \mp \frac{ma_m}{n} B_2 = e_s z_1^{s-m}.$$

Note that if $s > m$ then $A_0, A_1, B_0, B_2 \in \mathcal{M}_{z_1}$ and $C_1 \in \mathcal{M}_{z_1}^2$.

Assign weights $w_x = w_y = \frac{1}{2}$ and $w_{z_1} = \frac{1}{k}$. Then the respective germ takes the form

$$\Phi_1 = se_s xz_1^{s-1} C_1 + e_s z_1^s [xA_0 + (\pm 2x + a_m z_1^m)A_1] + (\pm 2x + a_m z_1^m) e_s z_1^s A_1.$$

By similar argument we can show that any term of the form $g_2^* = d_l y z_1^l$ is obtained from Ω.

Finally, let $g_3^* = \tilde{e}_s z_1^s$ where $s \geq k$. Choose A_0, A_1, B_0, B_2 and C_1 such that

$$A_0 = \mp 2A_1, \quad B_0 = \mp 2B_2 \quad , C_1 = \frac{\mp 2}{n} z_1 B_2, \quad A_1 = \frac{m}{n} B_2$$

and

$$\tilde{e}_s z_1^s = [\mp \frac{2kc_k}{n} z_1^k + \frac{ma_m^2}{n} z_1^{2m} + b_n^2 z_1^{2n}] B_2$$

Note that if $s > m$ then $A_0, A_1, B_0, B_2 \in \mathcal{M}_{z_1}$ and $C_1 \in \mathcal{M}_{z_1}^2$. The respective germ takes the form

$$\Phi_3 = s\tilde{e}_s z_1^{s-1} C_1.$$

Comparing the quasi degrees $d(\Phi_i)$ with $d(g_i^*)$, $i = 1, 2, 3$, we conclude that F_1 is quasi corner equivalent to the germ $\widetilde{F}_0 = \pm x^2 \pm y^2 \pm x z_1^m \pm y z_1^n \pm z_1^k$ which can be written in an alternative form as $\pm(x \pm z_1^m)^2 \pm (y \pm z_1^n)^2 \pm z_1^k$.

5) If $2m = 2n = k$ and $\mp \frac{ma_m^2}{2} \mp \frac{nb_n^2}{2} + kc_k = 0$, then, the function germ F_1 can be written as

$$F_1 = \pm(x \pm \frac{a_k}{2} z_1^m)^2 \pm (x \pm \frac{b_m}{2} z_1^m)^2 + x\varphi_1(z_1) + y\varphi_2(z_1) + \varphi_3(z_1),$$

where $\varphi_1, \varphi_2 \in \mathcal{M}_{z_1}^{m+1}$ and $\varphi_3 \in \mathcal{M}_{z_1}^{2m+1}$. Let $\widetilde{a}_m = \pm\frac{a_m}{2}$, $\widetilde{b}_m = \pm\frac{b_m}{2}$ and consider $F_0 = \pm(x + \widetilde{a}_m z_1^m)^2 \pm (x\widetilde{b}_m z_1^m)^2$. Then the tangent space at F_0 takes the form

$$\begin{aligned} TQC_{F_0} &= \pm 2(x + \widetilde{a}_m z_1^m) \left\{ xA_0 \pm 2(x + \widetilde{a}_m z_1^m) A_1 \pm 2(y + \widetilde{b}_m z_1^m) A_2 \right\} \\ &\pm 2(y + \widetilde{b}_m z_1^m) \left\{ yB_0 \pm 2(x + \widetilde{a}_m z_1^m) B_1 \pm 2(y + \widetilde{b}_m z_1^m) B_2 \right\} \\ &+ [\pm 2m\widetilde{a}_m z_1^{m-1}(x + \widetilde{a}_m z_1^m) \pm 2m\widetilde{b}_m z_1^{m-1}(y + \widetilde{b}_m z_1^m)] C. \end{aligned}$$

There are 6 relations *mod* TQC_{F_0}:

$$x^2 + \widetilde{a}_m x z_1^m \equiv 0, \quad x z_1^m + \widetilde{a}_m z_1^{2m} \equiv 0, \quad y^2 + \widetilde{b}_m y z_1^m \equiv 0, \quad y z_1^m + \widetilde{b}_m z_1^{2m} \equiv 0,$$

$(x+\widetilde{a}_m z_1^m)(y+\widetilde{b}_m z_1^m) \equiv 0$ and $[\pm 2m\widetilde{a}_m z_1^{m-1}(x+\widetilde{a}_m z_1^m) \pm 2m\widetilde{b}_m z_1^{m-1}(y+\widetilde{b}_m z_1^m)] \equiv 0.$

These relations yield that $\mathbf{C}_{x,y,z_1}/TQC_{F_0} \equiv \mathbb{R}\{1, z_1, z_2^2, \dots\}$. This means that there are solutions for any term of the form $g_1^* = e_s x z_1^s$ or of the form $g_2^* = d_l y z^l$ where $s, l \geq m$ (some terms belong to TQC_{F_0} up to terms of higher quasi degree).

In particular, consider g_1^* (similar argument holds for the any term g_2^*) and take A_0, A_1 such that

$$4\widetilde{a}_m z_1^{2m} A_1 + g_1^* = x^2(\pm 2A_0 + 4A_1) + 2\widetilde{a}_m y z_1^m(\pm A_0 + 4A_1) + 4\widetilde{a}_m z_1^{2m} A_1 - 4\widetilde{a}_m z_1^{2m} A_1.$$

Thus we need to set $A_0 = \frac{\mp e_s}{2\widetilde{a}_m} z_1^{s-m}$ and $A_1 = \mp\frac{1}{2}A_0$. Note that $g_1^* \in TQC_{F_0}$ up to the term $4\widetilde{a}_m z_1^{2m} A_1$ of higher quasi degree. Also note that $A_0, A_1 \in \mathcal{M}_{z_1}$ when $s > m$.

Assign weights $w_x = w_y = \frac{1}{2}$ and $w_{z_1} = \frac{1}{2m}$. Then g_1^* has quasi degree $d(g_1^*) = \frac{1}{2} + \frac{s}{2m}$. Consider the germ

$$\Phi_1 = e_s z_1^s [x A_0 + e_s z_1^s A_1] \pm 2(x + \widetilde{a} z_1^m)[e_s z_1^s A_1].$$

Comparing the quasi degree $d(\Phi)$ of the respective germs with $d(g_1^*)$, we conclude that F_1 is quasi corner equivalent to a germ of the form $\widetilde{F}_1 = \pm(x \pm z_1^m)^2 \pm (y \pm z_1^m)^2 + \widetilde{\varphi}(z_1)$ where $\widetilde{\varphi} \in \mathcal{M}^{2m+1}$.

Let $\widetilde{\varphi} = \widetilde{c}_s z_1^s + \phi(z_1)$ where $s \geq 2m+1$ and $\phi \in \mathcal{M}_{z_1}^{s+1}$. Consider the germ $\widetilde{F}_0 = \pm(x \pm z_1^m)^2 \pm (y \pm z_1^m)^2 + \widetilde{c}_s z_1^s$. Then

$$\begin{aligned} TQC_{\widetilde{F}_0} &= \pm 2(x \pm z_1^m)\{x A_0 \pm 2(x \pm z_1^m) A_1 \pm 2(y \pm z_1^m) A_2\} \\ &\pm\ 2(y \pm z_1^m)\{y B_0 \pm 2(x \pm z_1^m) B_1 \pm 2(y \pm z_1^m) B_2\} \\ &+\ [\pm 2m z_1^{m-1}(x \pm z_1^m) \pm 2m z_1^{m-1}(y \pm z_1^m) + s\widetilde{c}_s z_1^{s-1}]C. \end{aligned}$$

Let $g_3^* = e_l z_1^l$ where $l \geq s$. Set $A_2 = B_1 = 0$ and choose A_0, A_1, B_0, B_2 and C such that

$$A_0 = \mp 2A_1, \quad B_0 = \mp 2B_2, \quad A_1 = B_2, \quad C = \frac{\mp 2}{m} z_1 B_2,$$

and

$$B_2 = \frac{e_l z_1^{l-2m}}{8 \pm \frac{2s\tilde{c}_s}{m} z_1^{s-2m}}.$$

Note that $A_0, A_1, B_0, B_2, C \in \mathcal{M}_{z_1}$. The respective germ is $\Phi_3 = se_l z_1^{l-1}C$.

Comparing the quasi degree $d(\Phi_3)$ with $d(g_3^*)$ with respect to weights $w_x = w_y = \frac{1}{2}$ and $w_{z_1} = \frac{1}{2m}$, we see that $\widetilde{F}_1$ is quasi corner equivalent to the germ $\pm(x \pm z_1^m)^2 \pm (y \pm z_1^m)^2 \pm z_1^s$.

These are the only simple classes. Other germs are either adjacent to non-simple classes or have infinite codimension. This complete the proof of the theorem.

Proposition 3.1.7 *The singularities following special cases $\mathcal{F}_{k,m}$ can be written in the alternative way as follows:*

1. *The class $\mathcal{F}_{m-1,m}$ is quasi corner equivalent to* $\pm x^2 \pm y^m$, *for* $m \geq 3$.
2. *The class $\mathcal{F}_{k,2k}$ is quasi corner equivalent to* $\pm x^2 \pm xy^k$, *for* $k \geq 2$.
3. *The class $\mathcal{F}_{k,m}$ is quasi corner equivalent to* $\pm x^2 \pm xy^k \pm y^m$, *for* $3 \leq k+1 < m \leq 2k-1$.

Proof.

This result follows immediately from the proof of the theorem 3.1.6.

∎

Remarks:

1. Stabilization Lemma 3.1.1 yields that for specific small values of c these simple classes can be written alternatively, for example:

For $c = 1$, the simple function germs $\pm z_1(x \pm y) \pm y^k, k \geq 2,$ $\pm z_1(x \pm y^m) \pm y^k, m > k \geq 2$, are stably quasi corner equivalent to $\mathcal{B}_k$ and $\mathcal{F}_{m,k}$ respectively.

For $c = 2$, the simple function germ

$$\pm z_1 x \pm y^2 \pm y z_2^m \pm z_1 z_2^n \pm z_2^k$$

is stably quasi corner equivalent to $\mathcal{H}_{m,n,k}$.

For $c = 3$, a simple function germ

$$\pm z_1 x \pm z_2 y \pm z_1 z_3^m \pm z_2 z_3^n \pm z_3^k \quad \text{or} \quad \pm z_1 x \pm z_2 y + z_3^2$$

are stably quasi corner equivalent to $\mathcal{H}_{m,n,k}$ and $\mathcal{B}_2$ respectively.

Any germ with $c \geq 4$ is non-simple with respect to quasi corner equivalence.

2. The rank r of the second differential $d^2 f(x, y, z_1)$ is quasi corner equivalence invariant while the ranks of the second differentials $d^2 f(0, 0, z_1)$ and $d^2 f(x, y, 0)$ are not. For example, the function germs $f(x, y, z_1) = zx$ and $g(x, y, z_1) = z_1^2 - x^2$ are quasi corner equivalent. However, rank $d_0^2 f(0, 0, z_1) = 0$ but rank $d_0^2 g(0, 0, z_1) = 1$.

3. Notice, that the formulas for quasi corner classes $\mathcal{F}, \mathcal{B}$ coincide with quasi boundary classes F, B. However they have larger codimensions since the quasi corner equivalences preserve the origin and is finer than the quasi boundary one.

3.2 Adjacency of lower codimension classes

The construction of the table of adjacencies is based on the proof of the theorem 3.1.6.

Let $f(x, y) = f_2(x, y) + \varphi(x, y)$ where $\varphi \in \mathcal{M}_{x,y}^3$ and f_2 is a quadratic form. If f_2 is non-degenerate quadratic form, then f is contained in the class $\mathcal{B}_2$. If f_2 is a degenerate quadratic form of rank one, then f can be written in the form $\widetilde{f} = \pm(ax + by)^2 + \widetilde{\varphi}(x, y)$ where $\widetilde{\varphi} \in \mathcal{M}_{x,y}^3$ which is adjacent to the class $\mathcal{B}_2$. Thus, if $a \neq 0$ and $b \neq 0$, then , $\widetilde{f}$ is contained in the adjacent classes $\mathcal{B}_k$ with $k \geq 3$.

$$\mathcal{B}_2 \leftarrow \mathcal{B}_3 \leftarrow \mathcal{B}_4 \leftarrow \mathcal{B}_5 \leftarrow \dots$$

If $a = 0$ and $b \neq 0$, then f is quasi corner equivalent to a germ of the form $F = \pm y^2 + y\varphi_1(x) + \varphi_2(x)$ where $\varphi_1 \in \mathcal{M}_x^2$ and $\varphi_2 \in \mathcal{M}_x^3$. Let $F(x, y) = \pm y^2 + y(c_2 x^2 + c_3 x^3 + \dots + c_i x^i + \dots) + d_3 x^3 + d_4 x^4 + \dots + d_i x^i + \dots$.

If $d_3 \neq 0$, then we get the class $\mathcal{F}_{2,3} : \pm y^2 \pm x^3 \sim \pm(y \pm x^2)^2 \pm x^3$. Note that the class $\mathcal{B}_3$ has an alternative form $h(x, y) = \pm(ax + by \pm x^2)^2 \pm y^3 \pm x^3$. Thus, when $a = 0$, we obtain the class $\mathcal{F}_{2,3}$. This means that $\mathcal{F}_{2,3}$ is adjacent to the class $\mathcal{B}_3$.

Next, if $d_3 = 0$, $c_2 \neq 0$ and $d_4 \neq 0$, then we distinguish the following:

If $d_4 \neq \pm\frac{1}{4}c_2^2$, then F is contained in the class $\mathcal{F}_{2,4} : \pm y^2 \pm yx^2 \sim \pm(y \pm x^2)^2 \pm x^4$. Hence, the class $\mathcal{F}_{2,4}$ is adjacent to the class $\mathcal{F}_{2,3}$. On the other hand, the class $\mathcal{F}_{2,4}$ is adjacent to $\mathcal{B}_4$ as the class $\mathcal{B}_4$ has the an alternative form $h(x,y) = \pm(ax + by \pm x^2)^2 \pm y^4 \pm x^4$. Thus, if $a = 0$ we get the class $\mathcal{F}_{2,4}$. Hence, we get, up to now, the following table of adjacent classes:

$$\begin{array}{ccccccccc}
\mathcal{B}_2 & \leftarrow & \mathcal{B}_3 & \leftarrow & \mathcal{B}_4 & \leftarrow & \mathcal{B}_5 & \leftarrow & \ldots \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{F}_{2,3} & \leftarrow & \mathcal{F}_{2,4} & & & &
\end{array}$$

If $d_4 = \pm\frac{1}{4}c_2^2$, then F is quasi equivalent to a germ of the form $\widetilde{F} = \pm(y \pm x^2)^2 + \phi(x)$ where $\phi \in \mathcal{M}_x^5$ which is adjacent to the class $\mathcal{F}_{2,4}$. Therefore, we obtain the series of classes $\mathcal{F}_{2,m} : \pm(y \pm x^2)^2 \pm x^m$ with $m \geq 5$. Hence, we get the following adjacent classes:

$$\mathcal{F}_{2,4} \leftarrow \mathcal{F}_{2,5} \leftarrow \mathcal{F}_{2,6} \leftarrow \mathcal{F}_{2,7} \leftarrow \ldots$$

Note that the classes $\mathcal{B}_m : \pm x^2 \pm y^m$ can be written in an alternative form as $h(x,y) = \pm(ax + by \pm x^2)^2 \pm y^m \pm x^m$. Thus, if $a = 0$ then we obtain the classes $\mathcal{F}_{2,m} : \pm(y \pm x^2)^2 \pm x^m$. This means that the classes $\mathcal{F}_{2,m}$ are adjacent to the classes $\mathcal{B}_m$ with $m \geq 5$.

Suppose now that $d_3 = c_2 = 0$ and $c_3 \neq 0$ but $d_4 \neq 0$, then F is contained in the class $\mathcal{F}_{3,4} : \pm y^2 \pm yx^3 \pm y^4 \sim \pm(y^2 \pm x^3)^2 \pm x^4$ which is adjacent to the class $\mathcal{F}_{2,4}$.

If $c_2 = d_3 = d_4 = 0 =$ and $c_3 \neq 0$ but $d_5 \neq 0$, then we get the class $\mathcal{F}_{3,5}$ which is adjacent to the class $\mathcal{F}_{3,4}$. On the the other hand, the class $\mathcal{F}_{2,5}$ has the form $\pm(y + ax^2 + bx^3) \pm x^5$. Hence, when $a = 0$ then we get the class $\mathcal{F}_{3,5}$.

If $c_2 = d_3 = d_4 = d_5 = 0$ and $c_3 \neq 0$ but $d_6 \neq 0$, then follow the previous discussion of the case $d_3 = 0$, $c_2 \neq 0$ and $d_4 \neq 0$.

Therefore, the table of adjacencies of low dimension is as follows:

$$\begin{array}{ccccccccccc}
\mathcal{B}_2 & \leftarrow & \mathcal{B}_3 & \leftarrow & \mathcal{B}_4 & \leftarrow & \mathcal{B}_5 & \leftarrow & \mathcal{B}_6 & \leftarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}_{2,3} & \leftarrow & \mathcal{F}_{2,4} & \leftarrow & \mathcal{F}_{2,5} & \leftarrow & \mathcal{F}_{2,6} & \leftarrow & \dots \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{F}_{3,4} & \leftarrow & \mathcal{F}_{3,5} & \leftarrow & \mathcal{F}_{3,6} & \leftarrow & \dots \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & \dots & & \dots & & \dots
\end{array}$$

Remarks:

1. In the previous discussion, note that any adjacency between simple classes is a consequence of the following ones:

$$\mathcal{F}_{k+1,m} \longrightarrow \mathcal{F}_{k,m} \longleftarrow \mathcal{F}_{k,m+1}.$$

2. Similar argument shows that any adjacency of singularities $\mathcal{H}_{m,n,k}$ is generated by the following basic ones:

$$\begin{array}{ccccc}
H_{m,n+1,k} & \longrightarrow & H_{m,n,k} & \longleftarrow & H_{m+1,n,k} \\
 & & \uparrow & & \\
 & & H_{m,n,k+1} & &
\end{array}$$

3.3 Comparison of quasi corner and standard corner singularities

The standard classification of singularities of the standard action on functions of diffeomorphisms preserving the corner $\Gamma_c = \{xy = 0\}$ was obtained by D.Siersma and others in [29, 21]for functions on the corner. There are no simple classes and the classification starts with unimodal singularities. The comparison between these singularities and quasi corner singularities is given in the following table.

The unimodal and bimodal corner singularities	The normal form with respect to quasi corner equivalence
$\pm x^2 + axy \pm y^2, \quad a^2 \neq 4$	$\mathcal{B}_2$
$x^n + y^m + axy, \quad a \neq 0$	$\mathcal{B}_2$
$\pm(x+y)^2 + ay^3, \quad a \neq 0$	$\mathcal{B}_3$
$\pm(x+y)^2 + ay^4, \quad a \neq 0$	$\mathcal{B}_4$
$\pm x^2 + axy^2 \pm y^3, \quad a \neq 0$	$\mathcal{F}_{2,3}$
$\pm z(x \pm y) + z^3 + ay^2, \quad a \neq 0$	$\mathcal{B}_2$
$\pm z(x \pm y) + z^3 + ay^3, \quad a \neq 0$	$\mathcal{B}_3$
$\pm z(x \pm y) + az^4 \pm y^2, \quad a \neq 0$	$\mathcal{B}_2$
$\pm zx + az^2y \pm y^2 + z^3, \quad a \neq 0$	$\mathcal{B}_2$
$z^3 + zx \pm y^2 + axy, \quad a \neq 0$	$\mathcal{B}_2$
$z^m + zx + zy + axy^n, \quad a \neq 0,\ m \geq 2$	$\mathcal{B}_{n+1}$
$\pm y^4 + \pm x^2 + axy^2 + bxy^3, \quad a^2 \neq 4, b \neq 0$	$\mathcal{F}_{2,4}$
$\pm x^n + yx^2 + ay^2 + byx^3, \quad a \neq 0, n > 4$	$\mathcal{F}_{2,4}$
$\pm(y^2 + x)^2 + (a + by)xy^{n-2}, \quad a \neq 0, n > 4$	$\mathcal{F}_{2,n}$
$y^5 \pm x^2 + (a + by)xy^3, \quad a, b \neq 0$	$\mathcal{F}_{3,5}$
$\pm z^4 + zx + \pm y^2 + ayz^2 + byx, \quad a^2 \neq 4, b \neq 0$	$\mathcal{B}_2$
$\pm z^n + zx + yz^2 + ay^2 + byx, \quad a \neq 0, n > 4$	$\mathcal{B}_2$
$\pm(z^2 + y)^2 + zx + azy^n + byx, \quad a \neq 0, n > 2$	$\mathcal{B}_2$
$z^5 \pm y^2 + zx + ayz^3 + byx, \quad a, b \neq 0$	$\mathcal{B}_2$
$z^3 \pm x^2 \pm y^2 + axy + bxyz, \quad a^2 \neq 4, b \neq 0$	$\mathcal{H}_{2,2,3}$

3.4 The caustics and bifurcation diagrams of simple quasi corner singularities

We start with the following.

Proposition 3.4.1 *The quasi corner mini-versal deformations with parameters* λ, β, γ *(of the respective dimensions) of the simple quasi corner classes can be chosen in the following form:*

1. $\mathcal{B}_2$: $\quad \pm x^2 \pm y^2 + \lambda_0 + \lambda_1 x + \lambda_2 y;$

2. $\mathcal{B}_m$: $\quad \pm(x \pm y)^2 \pm y^m + \lambda_1 x + \sum_{i=0}^{m-1} \beta_i y^i, \quad m \geq 3;$

3. $\mathcal{F}_{k,m}$: $\quad \pm(x \pm y^k + \sum_{j=0}^{k-1} \lambda_j y^j)^2 \pm y^m + \sum_{i=0}^{m-1} \beta_i y^i \quad m > k \geq 3;$

4. $\mathcal{H}_{m,n,k}$: $\quad \pm(x \pm z_1^m + \sum_{i=0}^{m-1} \lambda_i z_1^i)^2 \pm (y \pm z_1^n + \sum_{j=0}^{n-1} \beta_j z_1^j)^2 \pm z^k + \sum_{l=0}^{k-2} \gamma_l z_1^l,$
$k > n \geq m \geq 2.$

Proof.

1. For $\mathcal{B}_2$ class, the quasi corner tangent space to the orbit at $f(x,y) = \pm x^2 \pm y^2$ is

$$TQC_f = \pm x\{xA + yB\} \pm y\{y\widetilde{A} + x\widetilde{B}\}.$$

Then, $x^2 \equiv 0, xy \equiv 0$ and $y^2 \equiv 0$. Thus, clearly $1, x$ and y form a basis of the local algebra $Q = \mathbf{C}_{x,y}/TQC_f$.

2. For $\mathcal{B}_m$ classes, let $f(x,y) = \pm(x \pm y)^2 \pm y^m$. Then, the tangent space of the quasi corner orbit at f takes the form

$$TQC_f = \pm 2(x \pm y)\{xA_0 + yA_1\} + [\pm 2(x \pm y) \pm my^{m-1}]\{yB_0 + xB_1\}.$$

Thus , we get the following relations *mod* TQC_f

$$x(x \pm y) \equiv 0, \tag{3.9}$$

$$y(x \pm y) \equiv 0, \tag{3.10}$$

and

$$\pm 2y(x \pm y) \pm my^m \equiv 0, \Rightarrow y^m \equiv 0. \tag{3.11}$$

Thus, if we use the two relations (3.9) and (3.10) in the local algebra $\mathcal{Q} = \mathbf{C}_{x,y}/TQC_f$ we see that monomials $1, y, y^2, \ldots, y^{m-1}$ form a basis for $\mathcal{Q}$.

3. For $\mathcal{F}_{k,m}$ classes. Let $f(x,y) = \pm(x \pm y^k)^2 \pm y^m$. Then, the quasi corner tangent space to the orbit at f has the form

$$TQC_f = \pm 2(x \pm y^k)\{xA_0 + y^k A_1\} + [\pm 2ky^{k-1}(x \pm y^k) \pm my^{m-1}]\{yB_0 + xB_1\}$$

Thus, we obtain the following relations *mod* TQC_f:

$$xy^k \pm y^{2k} \equiv 0 \Rightarrow xy^k \equiv \mp y^{2k}, \tag{3.12}$$

$$x^2 \pm xy^k \equiv 0 \Rightarrow x^2 \equiv \mp xy^k, \tag{3.13}$$

$$\pm 2kxy^k + 2ky^{2k} \pm my^m \equiv 0, \tag{3.14}$$

and

$$\pm 2kx^2y^{k-1} + 2kxy^{2k-1} \pm mxy^{m-1} \equiv 0. \tag{3.15}$$

If we substitute $xy^k \equiv \mp y^{2k}$ in the equation (3.14), we obtain $y^m \equiv 0$. Also, substituting $x^2 \equiv \mp xy^k$ in the equation (3.15), we get $xy^{m-1} \equiv 0$.

Now we distinguish the following cases:

i) If $m \leq 2k$, then we see that $y^{2k} \equiv 0$ and hence $xy^k \equiv 0$ and $x^2 \equiv 0$. Thus, the monomials: $1, y, y^2, \ldots, y^{m-1}, x, xy, xy^2, \ldots, xy^{k-1}$ form a basis for the local algebra $\mathcal{Q} = \mathbf{C}_{x,y}/TQC_f$.

ii) If $m > 2k$, then use the two relations $x^2 \equiv \mp xy^k$ and $xy^k \equiv \mp y^{2k}$ in the local algebra $\mathcal{Q}$. Thus, again the monomials: $1, y, y^2, \ldots, y^{m-1}, x, xy, xy^2, \ldots, xy^{k-1}$ form a basis of $\mathcal{Q}$.

Thus, the deformation $F(x,y) = \pm(x \pm y^k)^2 + x\sum_{j=0}^{k-1} \lambda_j y^j \pm y^m + \sum_{i=0}^{m-1} \beta_i y^i$ is a mini versal deformation with respect to quasi corner equivalence for the classes $\mathcal{F}_{k,m}$.

On the other hand, the deformation F is quasi corner equivalent to the deformation

$$\begin{aligned}\widetilde{F} &= \pm(x \pm y^k)^2 \pm 2x\sum_{j=0}^{k-1}\lambda_j y^j + 2\sum_{j=0}^{k-1}\lambda_j y^{j+k} \pm y^m + \sum_{i=0}^{m-1}\beta_i y^i \\ &= \pm(x \pm y^k)^2 \pm 2(x \pm y^k)\sum_{j=0}^{k-1}\lambda_j y^j \pm y^m + \sum_{i=0}^{m-1}\beta_i y^i.\end{aligned}$$

Notice that adding the terms $\Lambda = \pm\left(\sum\limits_{j=0}^{k-1}\lambda_j y^j\right)^2$ to $\widetilde{F}$ does not affect the versality of $\widetilde{F}$ as $\frac{\partial\Lambda}{\partial\lambda_i}|_{\lambda=0}$. Hence, we get the following an alternative form of the versal deformation of the classes $\mathcal{F}_{k,m}$:

$$G(x,y) = \pm(x \pm y^k + \sum_{j=0}^{k-1}\lambda_j y^j)^2 \pm y^m + \sum_{i=0}^{m-1}\beta_i y^i.$$

For $\mathcal{H}_{m,n,k}$ classes, consider the tangent space to the quasi corner orbit at $g(x,y,z_1) = \pm(x \pm z_1^m)^2 \pm (y \pm z_1^n)^2 \pm z_1^k$.

$$\begin{aligned}TQC_g &= \pm 2(x \pm z_1^m)\{xA_0 + z_1^m A_1 + (y \pm z_1^n)A_2\} \\ &\quad \pm 2(y \pm z_1^n)\{yB_0 + (x \pm z_1^m)B_1 + z_1^n B_2\} \\ &\quad [\pm 2mz_1^{m-1}(x \pm z_1^m) \pm 2nz_1^{n-1}(y \pm z_1^n) \pm kz_1^{k-1}]C.\end{aligned}$$

Thus, we obtain the following relations $mod\ TQC_f$

$$x^2 \pm xz_1^m \equiv 0 \Rightarrow x^2 \equiv \mp xz_1^m, \tag{3.16}$$

$$xz_1^m \pm z_1^{2m} \equiv 0 \Rightarrow xz_1^m \equiv \mp z_1^{2m}, \tag{3.17}$$

$$y^2 \pm yz_1^n \equiv 0 \Rightarrow y^2 \equiv \mp yz_1^n, \tag{3.18}$$

$$yz_1^n \pm z_1^{2n} \equiv 0 \Rightarrow yz_1^n \equiv \mp z_1^{2n}, \tag{3.19}$$

$$xy \pm yz_1^m \pm xz_1^n \pm z_1^{m+n} \equiv 0, \tag{3.20}$$

and

$$\pm 2mxz_1^{m-1} + 2mz_1^{2m-1} \pm 2nyz_1^{n-1} + 2nz_1^{2n-1} \pm kz_1^{k-1} \equiv 0. \tag{3.21}$$

If we substitute xz_1^m and yz_1^n using the relations (3.17) and (3.19) respectively in the equation (3.21), we get $z_1^{k-1} \equiv 0$.

Now we distinguish the following cases:

i) If $2n \geq 2m \geq k$, then we see that $xz_1^m \equiv 0$ and $yz_1^n \equiv 0$. Therefore, we get $x^2 \equiv 0$ and $y^2 \equiv 0$. If we use the relations (3.20) and (3.21) in the local algerba $\mathcal{Q} = \mathbf{C}_{x,y,z_1}/TQC_f$ we see that the monomials

$$1, x, xz_1, xz_1^2, \ldots, xz_1^{m-1}, y, yz_1, yz_1^2, \ldots, yz_1^{n-1}, z_1, z_1^2, \ldots, z_1^{k-2}$$

form a basis of $\mathcal{Q}$.

ii) If $2n \geq k$ and $2m < k$, then we see that $yz_1^n \equiv 0$ and $y^2 \equiv 0$. Thus, if we use the relations (3.16), (3.17), (3.20) and (3.21) in the local algebra $\mathcal{Q}$ we see that the monomials

$$1, x, xz_1, xz_1^2, \ldots, xz_1^{m-1}, y, yz_1, yz_1^2, \ldots, yz_1^{n-1}, z_1, z_1^2, \ldots, z_1^{k-2}$$

form a basis of $\mathcal{Q}$.

iii) If $k \geq 2m+1, 2n+1$, then using the relations (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21) in the local algerba $\mathcal{Q}$ we see that again the monomials

$$1, x, xz_1, xz_1^2, \ldots, xz_1^{m-1}, y, yz_1, yz_1^2, \ldots, yz_1^{n-1}, z_1, z_1^2, \ldots, z_1^{k-2}$$

form a basis of $\mathcal{Q}$.

Similar arguments as in the previous case, we can take the deformation:

$$G(x,y,z_1) = \pm(x \pm z^m + \sum_{i=0}^{m-1} \lambda_i z^i)^2 \pm (y \pm z^n + \sum_{j=0}^{n-1} \beta_j z^j)^2 \pm z^k + \sum_{l=0}^{k-2} \gamma_l z^l,$$

as a mini-versal deformation for the classes $\mathcal{H}_{m,n,k}$.

∎

The geometrical description of the bifurcation diagrams and caustics of some simple quasi corner singularities is given in the following:

Proposition 3.4.2 *1. The first stratum of the bifurcation diagram (caustic) of any simple quasi corner singularity is a cylinder over standard bifurcation diagram (caustic) of the standard right A_k singularity of function. In particular, the bifurcation diagram of $\mathcal{B}_2$ is a smooth surface with two transversal lines in it. See Figure 3.1. The first stratum of the bifurcation diagram of $\mathcal{B}_3$ is a product of a cusp and a plane in $\mathbb{R}^4$. Two other strata are smooth surfaces inside the first one. They are tangent to the cuspidal ridge.*

2. *The caustics of $\mathcal{B}_k$ is a union of a cylinder over a generalized swallow tail and two smooth hypersurfaces tangent to the first stratum. In particular, the caustics of $\mathcal{B}_3$ consists of three strata which are smooth pairwise tangent surfaces in 3-space. See Figure 3.2.*

3. *The caustics of $\mathcal{F}_{k,m}$ is a union of a cylinder over a generalized swallow tail, a smooth hypersurface and a generalized Whitney umbrella multiplied by a line. In particular, the caustics of $\mathcal{F}_{2,3}$ is a union of two smooth hypersurfaces in $\mathbb{R}^4$ and a Whitney umbrella multiplied by a line.*

4. *The caustics of $\mathcal{H}_{k,m,n}$ is a union of a cylinder over a generalized swallow tail and two generalized Whitney umbrellas of respective dimensions.*

Proof.

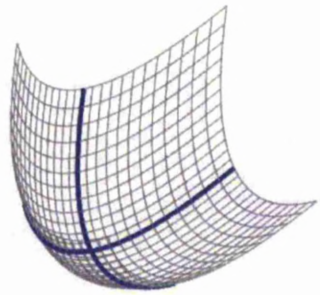

Figure 3.1: The bifurcation diagram of $\mathcal{B}_2$.

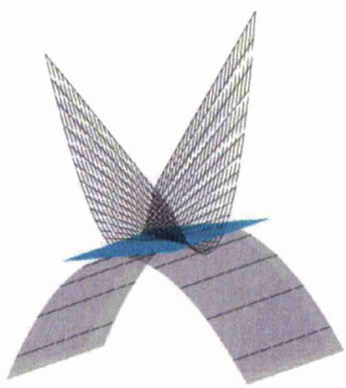

Figure 3.2: The caustic of $\mathcal{B}_3$.

1. In fact, any simple quasi corner singularity has corank 1 of the second differential and is reduced to A_k singularity. So W_0 is a product a generalized swallow tail and $\mathbb{R}^{m-k}$, where m is the quasi corner multiplicity.

For $\mathcal{B}_2$ class, consider its versal deformation $F(x,y) = \pm x^2 \pm y^2 + \lambda_0 + \lambda_1 x + \lambda_2 y$. Then, the first stratum is given by $F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$. Hence W_0 is the smooth surface $\{(\lambda_0, \lambda_1, \lambda_2) : \lambda_0 = \pm x^2 \pm y^2, \lambda_1 = \mp 2x, \lambda_2 = \mp 2y\}$. The two other strata W_1, W_2 are the subsets of W_0 which satisfy the extra equations $x = 0$ and $y = 0$, respectively. Thus, $W_1 = \{(\lambda_0, \lambda_1, \lambda_2) : \lambda_0 = \pm y^2, \lambda_1 = 0, \lambda_2 = \mp 2y\}$ and $W_1 = \{(\lambda_0, \lambda_1, \lambda_2) : \lambda_0 = \pm x^2, \lambda_1 = \mp 2x, \lambda_2 = 0\}$.

For $\mathcal{B}_3$ class, let $F(x, y) = (x+y)^2 + y^3 + \lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 y^2$ be its mini-versal deformation. Then, clearly that $\mathcal{B}_3$ can be reduced to A_2 singularity with respect to standard right equivalence. Therefore, the first stratum W_0 is a product of a cusp and a plane in $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$-space. Explicitly,

$$\begin{aligned} W_0 &= \{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) : \lambda_0 = (x+y)^2 + 2y^3 + 2\lambda_3 y^2, \lambda_1 = -2(x+y), \\ &\quad \lambda_2 = -2(x+y) - 3y^2 - 2\lambda_3 y\}. \end{aligned}$$

The ridge of W_0 satisfies the following

$$\begin{vmatrix} \frac{\partial^2 F}{\partial x^2}, & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x}, & \frac{\partial^2 F}{\partial y^2} \end{vmatrix} = 0.$$

This gives $\lambda_3 = -3y$. Hence the ridge is the smooth set:

$$\mathcal{R} = \{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) : \lambda_0 = (x+y)^2 - 4y^3, \lambda_1 = -2(x+y), \lambda_2 = -2(x+y) + 3y^2, \lambda_3 = -3y\}.$$

The strata corresponding to $x = 0$ and $y = 0$ are smooth and are given respectively as follows:

$$W_1 = \{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) : \lambda_0 = y^2 + 2y^3 + 2\lambda_3 y^2, \lambda_1 = -2y, \lambda_2 = -2y - 3y^2 - 2\lambda_3 y\},$$

$$W_2 = \{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) : \lambda_0 = x^2, \lambda_1 = -2x, \lambda_2 = -2x\}.$$

The ridge $\mathcal{R}$ intersects the second stratum W_1 when $x = 0$ and $\lambda_3 = -3y$. Hence their intersection is the smooth line $L_1 = \{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) : \lambda_0 = y^2 - 4y^3, \lambda_1 = -2y, \lambda_2 = -2y + 3y^2, \lambda_3 = -3y\}$. This means that the ridge is tangent to W_1 along the curve L_1.

Similarly, the ridge $\mathcal{R}$ is tangent to W_2 along the line $L_2 = \{(\lambda_0, \lambda_1, \lambda_2) : \lambda_0 = x^2, \lambda_1 = -2x, \lambda_2 = -2x, \lambda_3 = 0\}$.

Now if we project the ridge and the two strata W_1 and W_2 to the space $(\lambda_1, \lambda_2, \lambda_3)$, we get the configuration of the caustic of $\mathcal{B}_3$ singularity.

2. Consider the mini-versal deformation $F(x, y, \lambda, \mu) = (x + y)^2 + y^m + \lambda x + \sum_{i=0}^{m-1} \mu_i y^i$ of the singularity $\mathcal{B}_m$. Clearly, $\mathcal{B}_m$ is reduced to standard A_{m-1} singularity. Hence the first stratum of the caustic is a cylinder over a generalized swallowtail.

For the second stratum, we need to consider the following conditions $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x} = x = 0$. This yields that this is a smooth hypersurfaces, given as follows:

$$\{(\lambda, \mu_1, \mu_2, \dots, \mu_{m-1}) : \lambda = -2y, \mu_1 = -2y - my^{m-1} - 2\mu_2 y - \dots - (m-1)\mu_{m-1} y^{m-2}\}.$$

Similarly, we see that the third strttum is a smooth hypersurfaes given by :

$$\{(\lambda, \mu_1, \mu_2, \dots, \mu_{m-1}) : \lambda = -2x, \mu_1 = -2x\}.$$

3. Consider the mini-vesaral deformation of $\mathcal{F}_{k,l}$ classes:

$$F(x, y, \lambda, \mu) = (x + y^k + \sum_{i=0}^{k-1} \lambda_i y^i)^2 + y^m + \sum_{j=0}^{m-1} \mu_j y^j.$$

Then clearly that $\mathcal{F}_{k,m}$ can be reduced to the standard A_{m-1} singularity. Hence the first stratum of the caustic is a cylinder over a generalized swallowtail.

Let $Q_k(y, \lambda) = y^k + \sum_{i=0}^{k-1} \lambda_i y^i$ and $P_m(y, \mu) = y^m + \sum_{j=0}^{m-1} \mu_j y^j$. Then,

$$F(x, y, \lambda, \mu) = (x + Q_k)^2 + P_m.$$

The second stratum satisfies $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = x = 0$. That is $\frac{\partial F}{\partial x} = 2(x + Q_k) = Q_k = 0$ and $\frac{\partial F}{\partial y} = 2\frac{\partial Q_k}{\partial y}(x + Q_k) + \frac{\partial P_m}{\partial y} = \frac{\partial P_m}{\partial y} = 0$.

Thus the second stratum is given as follows:

$$\begin{aligned}\wedge_1 &= \{(\lambda_0, \lambda_1, \ldots, \lambda_{k-1}, \mu_1, \mu_2, \ldots, \mu_{m-1}) : \lambda_0 = -y^k - \lambda_{,1} y - \lambda_2 y^2 - \cdots - \lambda_{k-1} y^{k-1} \\ &\mu_1 = -my^{m-1} - 2y\mu_2 - \cdots - (m-1)\mu_{m-1} y^{m-2}\}.\end{aligned}$$

The set $\wedge_1$ is the image of the Morin mapping with one extra parameter μ_{m-1}.

The third stratum is the set, satisfying $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = y = 0$. Hence, it is the smooth hypersurface, given as follows:

$$\{(\lambda_0, \lambda_1, \ldots, \lambda_{k-1}, \mu_1, \mu_2, \ldots, \mu_{m-1}) : \lambda_0 = -x\}.$$

4. For $\mathcal{H}_{m,n,k}$ classes, consider its mini-versal deformation

$$F(x, y, z_1, \lambda, \mu, \gamma) = (x + z_1^m + \sum_{i=0}^{m-1} \lambda_i z^i)^2 + (y + z_1^n + \sum_{j=0}^{n-1} \mu_j z_1^j)^2 \pm z_1^k + \sum_{l=0}^{k-2} \gamma_l z^l.$$

Then $\mathcal{H}_{m,n,k}$ is reduced to the standard singularity A_{k-1}. Hence, the first stratum of the caustic is a cylinder over a generalized swallowtail.

On the boundary $\{x = 0\}$, we can use the following transformation:

$$\widetilde{Y} = y + z^n + \sum_{j=0}^{n-1} \mu_j z^j, \quad \widetilde{X} = x, \quad \widetilde{\lambda} = \lambda, \quad \widetilde{\mu} = \mu, \quad \widetilde{\gamma} = \gamma,$$

to get the equivalent versal deformation:

$$\widetilde{F}(x, y, z_1, \lambda, \mu, \gamma) = (x + z_1^m + \sum_{i=0}^{m-1} \lambda_i z_1^i)^2 + y^2 \pm z_1^k + \sum_{l=0}^{k-2} \gamma_l z^l.$$

Hence, by similar argument to $\mathcal{F}_{k,l}$ case, we see that the second and the third stratum of the caustic are images of the cylinder over the Morin mappings. ∎

Chapter 4

Quasi cusp singularites

4.1 The classification of simple classes

In this chapter the coordinates are denoted as follows $\mathbb{R}^n = \{w = (x, y, z)\}$, where $x, y \in \mathbb{R}$ and $z = (z_1, \ldots, z_{n-2}) \in \mathbb{R}^{n-2}$. We consider germs of C^∞ -smooth functions $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, with a distinguished cusp $\Gamma_{csp} = \{x^s - y^2 = 0 : \text{for some } s \geq 3\}$. Notice that if $s = 2$ then hypersurface $\{x^2 - y^2 = 0\}$ is diffeomorphic to the corner $xy = 0$.

Recall that the quasi cusp tangent space to an admissible deformation f_t takes the form

$$\begin{aligned} TQCU_{f_t} &= \left\{ \frac{\partial f_t}{\partial x} \left(\frac{x}{s} h + 2yk + \frac{\partial f_t}{\partial x} A_1 + \frac{\partial f_t}{\partial y} A_2 \right) \right. \\ &+ \left. \frac{\partial f_t}{\partial y} \left(\frac{y}{2} h + s x^{s-1} k + \frac{\partial f_t}{\partial x} B_1 + \frac{\partial f_t}{\partial y} B_2 \right) + \sum_{i=1}^{n-2} \frac{\partial f_t}{\partial z_i} C_i . \right\} \end{aligned}$$

for arbitrary function germs $h, k, A_i, B_i, C_i \in \mathbf{C}_w$.

Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function germ with a critical point at the origin.

If the base point of the function germ is at the regular point of the border Γ_{cp} outside the cusp point (the origin), then the quasi cusp equivalence coincides with quasi boundary equivalence. Hence, the list of simple quasi cusp classes in this case is the same as the list of quasi boundary classes. The remaining case of the function germ having a critical base point at the cusp component is main object of the current chapter.

Denote by $f^*(z) = f|_{x=y=0}$, the restriction of function f to the z coordinates subspace. Denote by r^* the rank of the second differential $d_0^2 f^*$ at the origin and set $c = n - 2 - r^*$.

We restate the prenormal forms in the new coordinates.

Lemma 4.1.1 *(Stabilization) The function germ $f(x, y, z)$ is quasi cusp equivalent to $\sum_{i=1}^{r^*} \pm z_i^2 + g(x, y, \hat{z})$, where $\hat{z} \in \mathbb{R}^c$ and $g^* \in \mathcal{M}_{\hat{z}}^3$. For quasi cusp equivalent f germs, the respective reduced germs g are quasi cusp equivalent.*

Lemma 4.1.2 *There is a non-negative integer $s \leq r - r^*$ such that the function germ $f(x, y, z)$ is quasi cusp equivalent to $\sum_{i=1}^{r^*+s} \pm z_i^2 + \widetilde{f}(x, y, \widetilde{z})$, where $\widetilde{z} \in \mathbb{R}^{c-s}$ and $\widetilde{f}$ is a sum of a function germ from $\mathcal{M}_{x,y,\widetilde{z}}^3$ and a quadratic form in x and y only. For quasi cusp equivalent f germs, the respective reduced germs $\widetilde{f}$ are quasi cusp equivalent.*

These Lemmas imply the following preliminary classification results.

Lemma 4.1.3 *Let $m = n - r$ be the corank of the second differential $d_0^2 f$ at the origin.*

1. *If $m = 0$, then f is quasi cusp equivalent to $\sum_{i=1}^{n-2} \pm z_i^2 + f_2(x, y) + \widetilde{f}(x, y)$, where f_2 is non-degenerate quadratic form and $\widetilde{f} \in \mathcal{M}_{x,y}^3$.*

2. *If $m = 1$, then f is quasi cusp equivalent to either $\sum_{i=1}^{n-2} (\pm z_i^2) + \widetilde{f}(x, y)$ with $\mathrm{rank} d_0^2 \widetilde{f}(x, y)) = 1$ or to $\sum_{i=2}^{n-2} (\pm z_i^2) + \widetilde{f}(x, y, z_1) \pm x^2 \pm y^2$ with $\widetilde{f}(x, y, z_1)) \in \mathcal{M}_{x,y,z_1}^3$.*

3. *If $m \geq 2$, then f is non-simple.*

Proof. Lemmas 4.1.2 and 4.1.1 imply that any germ can be reduced to one of the following germs form:

1. $F_1 = \sum_{i=m-1}^{n-2}(\pm z_i^2) + \widetilde{f}(x, y, z_1, z_2, \dots, z_{m-2})$ where $\widetilde{f} \in \mathcal{M}^3_{x,y,z_1,z_2,\dots,z_{m-2}}$, or

2. $F_2 = \sum_{i=m}^{n-2}(\pm z_i^2) + f_2(x, y) + \widetilde{f}(x, y, z_1, z_2, \dots, z_{m-1})$ where $\widetilde{f} \in \mathcal{M}^3_{x,y,z_1,z_2,\dots,z_{m-1}}$ and f_2 is a degenerate quadratic form of rank one.

3. $F_3 = \sum_{i=m+1}^{n-2}(\pm z_i^2) + f_2(x, y) + \widetilde{f}(x, y, z_1, z_2, \dots, z_m)$ where $\widetilde{f} \in \mathcal{M}^3_{x,y,z_1,z_2,\dots,z_m}$ and f_2 is a non-degenerate quadratic form.

Thus, the results follow for the first two statements.

Suppose that $m \geq 2$. Consider the germ F_1. Then, the tangent space to the quasi cusp orbit at the germ $\widetilde{f}$ takes the form:

$$\begin{aligned} TQCU_{\widetilde{f}} &= \frac{\partial \widetilde{f}}{\partial x}\left\{\frac{x}{s}h + 2yk + \frac{\partial \widetilde{f}}{\partial x}A_1 + \frac{\partial \widetilde{f}}{\partial y}A_2\right\} \\ &+ \frac{\partial \widetilde{f}}{\partial y}\left\{\frac{y}{2}h + sx^{s-1}k + \frac{\partial \widetilde{f}}{\partial x}B_1 + \frac{\partial \widetilde{f}}{\partial y}B_2\right\} + \sum_{i=1}^{m-2}\frac{\partial \widetilde{f}}{\partial z_i}C_i. \end{aligned}$$

The cubic terms which belongs to $TQCU_{\widetilde{f}}$ are obtained from $\sum_{i=1}^{m-2}\frac{\partial \widetilde{f}}{\partial z_i}C_i$, $(\frac{\partial \widetilde{f}}{\partial x}\frac{x}{s} + \frac{\partial \widetilde{f}}{\partial y}\frac{y}{2})h$ and $(\frac{\partial \widetilde{f}}{\partial x}2y + \frac{\partial \widetilde{f}}{\partial y}sx^{s-1})k$, where C_i are linear forms and $h, k \in \mathbb{R}$. These terms form a subspace of dimension $m(m-2)+2$ which is less than $M = \frac{m(m+1)(m+2)}{6}$- the dimension of all cubic terms in the variables $x, y, z_1, z_2 \dots, z_{m-2}$. Hence all cubic terms can not belong to finitely many orbits. This means that the germ F_1 is non-simple.

The germ F_2 can take the form $F_2 = \pm(ax+by)^2 + f_3(x, y, z_1, z_2, \dots, z_{m-1})$. Note that F_2 is adjacent to the germ $\widetilde{F_2} = \pm(ax+by+\delta z_{m-1})^2 + f_3(x, y, z_1, z_2, \dots, z_{m-1})$, for sufficiently small δ. Lemma 4.1.1 shows that $\widetilde{F_2}$ is quasi cusp equivalent to the germ $G = \pm z_{m-1}^2 + \hat{f}(x, y, z_1, \dots, z_{m-2})$ where $\hat{f} \in \mathcal{M}^3_{x,y,z_1,\dots,z_{m-1}}$. By the previous argument, the germ G is non-simple. Similar argument shows that F_3 is adjacent to the germ F_2 and the result follows.

∎

Lemma 4.1.4 *Let $f : (\mathbb{R}^3, 0) \mapsto (\mathbb{R})$ be a function germ with critical point at the origin. If the quadratic form f_2 of f has rank 1 then f is quasi cusp equivalent to either $\pm x^2 + \varphi(y)$ where $\varphi \in \mathcal{M}_y^3$ or $\pm y^2 + \varphi(x, y)$ where $\varphi \in \mathcal{M}_{x,y}^3$. Moreover, if $s = 3$ then $\pm y^2 + \varphi(x, y)$ is quasi cusp equivalent to $\pm y^2 + \phi(x)$ where $\phi \in \mathcal{M}_x^3$.*

Proof. The function germ f takes the form $f = \pm(ax + by)^2 + \widetilde{f}(x, y)$ where $\widetilde{f} \in \mathcal{M}_{x,y}^3$. Consider the quadratic terms $Q = \pm(ax+by)^2$. Suppose that $a \neq 0$. Take the homotopy $Q_t = \pm(ax + tby)^2$ where $t \in [0, 1]$. Then the respective homological equation takes the form:

$$\begin{aligned} \mp 2by(ax + tby) &= \pm 2a(ax + tby)\{\frac{x}{s}h + 2yk + (ax + tby)A\} \pm 2tb(ax + by)\left\{\frac{y}{2}h\right. \\ &+ \left. sx^{s-1}k + (ax + tby)B\right\} \end{aligned}$$

This is equivalent to:

$$\mp 2by = \pm 2a\{\frac{x}{s}h + 2yk + (ax + tby)A\} \pm 2tb\{\frac{y}{2}h + sx^{s-1}k + (ax + tby)B\}.$$

The homological equation is solvable by setting $h = B = 0$ and taking A, k such that:

$$\pm 2a^2 A \pm 2tbsx^{s-2}k = 0 \quad \text{and} \quad \pm 4ak \pm 2atbA = \mp 2b.$$

Thus, all Q_t are quasi cusp equivalent. In particular, $Q = \pm(ax + by)^2$ is quasi cup equivalent to $Q_0 = \pm x^2$.

Now, consider the germ $F = \pm x^2 + f_3(x, y)$ where $f_3 \in \mathcal{M}_{x,y}^3$. Let $F_0 = \pm x^2$. Then, the quasi cusp tangent space at F_0 takes the from

$$TQCU_{F_0} = \pm 2x\left\{\frac{x}{s}h + 2yk + xA_1\right\}.$$

Thus, we get *mod* $TQCU_{F_0} : x^2 \equiv 0$ and $xy \equiv 0$. Hence, $\mathbf{C}_{x,y}/TQCU_{F_0} \equiv \phi(y)$. Lemma 1.3.5 shows that F is quasi cusp equivalent to the germ $G = \pm x^2 + \varphi(y)$ with $\varphi \in \mathcal{M}_y^3$.

Now, if $a = 0$ and $b \neq 0$ then f takes the form $f = \pm y^2 + \widetilde{f}(x, y)$.

Suppose that $s = 3$ and consider the germ $f_0 = \pm y^2$. The quasi cusp tangent space to the orbit at f_0 takes the form

$$TQCU_{f_0} = \pm 2y \left\{ \frac{y}{2} h + 3x^2 k + yB_0 \right\}.$$

Then, we obtain $mod\, TQCU_{f_0}$: $y^2 \equiv 0$ and $yx^2 \equiv 0$. Hence, $\mathbf{C}_{x,y}/TQCU_{F_0} \equiv \varphi(x)$. Again Lemma 1.3.5 shows that f is quasi cusp equivalent to $G = \pm y^2 + \phi(x)$, where $\phi \in \mathcal{M}_x^3$.

∎

Lemma 4.1.5 *The function germ* $\widetilde{f}(x, y, z_1) \pm x^2 \pm y^2$ *where* $\widetilde{f}(x, y, z_1) \in \mathcal{M}^3_{x,y,z_1}$ *is quasi cusp equivalent to the germ* $\pm x^2 \pm y^2 + xh_1(z_1) + yh_2(z_1) + h_3(z_1)$ *where* $h_1, h_2 \in \mathcal{M}^2_{z_1}$ *and* $h_3 \in \mathcal{M}^3_{z_1}$.

Proof. Let $f = \pm x^2 \pm y^2 + \widetilde{f}(x, y, z_1)$. Consider the germ $f_0 = \pm x^2 \pm y^2$. Then, the quasi cusp tangent space to the orbit at f_0 takes the form

$$TQCU_{f_0} = \pm 2x \left\{ \frac{x}{s} h + 2yk + xA_1 + yA_2 \right\} \pm 2y \left\{ \frac{y}{2} h + sx^{s-1} k + xB_1 + yB_2 \right\}.$$

Thus, we get $mod\, TQCU_{f_0}$: $x^2 \equiv 0$, $y^2 \equiv 0$ and $xy \equiv 0$. Hence, $\mathbf{C}_{x,y,z_1}/TQCU_{f_0} \equiv x\varphi_1(z_1) + y\varphi_2(z_1) + \varphi(z_1)$. Lemma (1.3.5) yields that f is quasi cusp equivalent to $G = \pm x^2 \pm y^2 + xh_1(z_1) + yh_2(z_1) + h_3(z_1)$ where $h_1, h_2 \in \mathcal{M}^2_{z_1}$ and $h_3 \in \mathcal{M}^3_{z_1}$. ∎

Theorem 4.1.6 *Let* $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ *be simple with respect to the quasi cusp equivalence. Then, either* f_2 *is a non-degenerate form and hence* f *is quasi cusp equivalent to* $\mathcal{L}_2 : \pm x^2 \pm y^2 + \sum_{i=1}^{n-2} \pm z_i^2$ *or* f_2 *is a degenerate form of corank 1 and hence* f *is stably quasi cusp equivalent to one of the following simple classes:*

1. $\mathcal{L}_k : \pm x^2 \pm y^k$, $\quad k \geq 3$; $\quad k+1$.
2. $\mathcal{M}_k : \pm y^2 \pm x^k$, $\quad k \geq 3$, *when* $s = 3$; $\quad k+2$.

3. $\mathcal{M}_3 : \pm y^2 + x^3$, *when* $s \geq 4$; 5.

4. $\mathcal{N}_{2,2,3} : \pm(x \pm z_1^2)^2 \pm (y \pm z_1^2)^2 \pm z_1^3$, *when* $s \geq 3$; 7.

5. $\mathcal{N}_{m,2,k} : \pm(x \pm z_1^m)^2 \pm (y \pm z_1^2)^2 \pm z_1^k$, $k \geq m > 2$, *when* $s = 3$; $m + k + 3$.

The orbit codimension in the space of germs is shown in the right column.

Remark: The non-simple classes f either have corank of f_2 greater or equal 2 or belong to a subspace of infinite codimension in $\mathbf{C}_w$.

Proof Theorem 4.1.6

We shall use Lemma 1.3.5 and prove the theorem only for necessary the condition. That is the homological equation is solvable for $t = 0$.

Lemmas 4.1.3, 4.1.4 and 4.1.5 shows that all possible simple germs can be obtained from one of the following germs $G_1 = \pm x^2 + \varphi(y)$ or $G_2 = \pm y^2 + \varphi(x, y)$ or $G_3 = \pm x^2 \pm y^2 + xh_1(z_1) + yh_2(z_1) + h_3(z_1)$.

Consider the germ G_1. Let $\varphi(y) = a_k y^k + \widetilde{\varphi}(y)$ where $k \geq 3$ and $\widetilde{\varphi} \in \mathcal{M}_y^{k+1}$. Consider the germ $g_0 = \pm x^2 + a_k y^k$. Then, the quasi cusp tangent space to the orbit at g_0

$$TQCU_{g_0} = \pm 2x \left\{ \frac{x}{s}h + 2yk + xA_1 + y^{k-1}A_2 \right\} + ka_k y^{k-1} \left\{ \frac{y}{2}h + sx^{s-1}k + xB_1 + y^{k-1}B_2 \right\}.$$

Thus, we get *mod* $TCCU_{f_0}$: $x^2 \equiv 0$, $y^k \equiv 0$, $xy \equiv 0$. This means that the simplified homological equation is solvable for a given germ $a_k y^k + \widetilde{\varphi}(y)$. Thus, G_1 is quasi cusp equivalent to the germ $\pm x^2 \pm y^k$ with $k \geq 3$. Note that the monomials $1, y, y^2, \ldots, y^{k-1}$ form a basis for the local algebra $\mathcal{Q} = \mathbf{C}_{x,y}/TQCU_{G_1}$.

Suppose that $s = 3$, then G_2 is quasi cusp equivalent to $\widetilde{G} = \pm y^2 + \phi(x)$ by Lemma 4.1.4. Similar argument to the previous one shows that $\widetilde{G}$ is quasi cusp equivalent to the germ $\pm y^2 \pm x^k$ where $k \geq 3$.

Suppose that $s \geq 4$. Then, consider the germ $H(x, y) = \pm y^2 + \varphi(x, y)$. Let $\varphi(x, y) = ax^3 + bxy^2 + cx^2y + dy^3 + \widetilde{\varphi}(x, y)$ where $\widetilde{\varphi} \in \mathcal{M}_{x,y}^4$. If $a \neq 0$, then H is quasi cusp equivalent to the germ $\pm y^2 + x^3$. Again, the proof of this claim is similar to previous cases. If $a = 0$, then consider the quasi homogeneous terms

$H_0 = \pm y^2 + cyx^2 + ex^4$ with respect to the weights $w_x = \frac{1}{4}$ and $w_y = \frac{1}{2}$. The quasi tangent space to H_0 takes the form

$$\begin{aligned} TQCU_{H_0} &= (2cyx + 4ex^3)\left\{\frac{x}{s}h + 2yk + (2cyx + 4ex^3)A_1 + (\pm 2y + cx^2)A_2\right\} \\ &+ (\pm 2y + cx^2)\left\{\frac{y}{2}h + sx^{s-1}k + (2cyx + 4ex^3)B_1 + (\pm 2y + cx^2)B_2\right\}. \end{aligned}$$

Then, the terms which belong to H_0 and obtained from $TQCU_{H_0}$ are $\chi_1 = [(2cyx + 4ex^3)\frac{x}{s} + (\pm 2y + cx^2)\frac{y}{2}]h$ and $\chi_2 = (\pm 2y + cx^2)^2 B_2$ where $h, B_2 \in \mathbb{R}$. Hence, χ_1 and χ_2 form a subspace of dimension 2. This means that H is non-simple germ in this case.

Finally, consider the reduced germ $G_3 = \pm x^2 \pm y^2 + xh_1(z_1) + yh_2(z_1) + h_3(z_1)$. For simplicity, consider the the equivalent form

$$G = \frac{1}{2}(x + H_1(z))^2 + \frac{1}{2}(y + H_1(z))^2 + H_3(z).$$

Let $a = x + H_1(z)$ and $b = y + H_2(z)$. Consider the deformation within functions in z of the form:

$$F_t = \frac{1}{2}[x + H_1(z_1, t)]^2 + \frac{1}{2}[y + H_2(z_1, t)]^2 + H_3(z_1, t).$$

Then, the respective homological equation takes the form

$$-\frac{\partial F_t}{\partial t} = -[2a\dot{H}_1(z_1, t) + 2b\dot{H}_2(z_1, t) + \dot{H}_3(z_1, t)] = TQCU_{F_t},$$

where

$$\begin{aligned} TQCU_{F_t} &= a\left\{[a - H_1]\frac{h}{s} + 2(b - H_2)k + aA_1 + bB_1\right\} \\ &+ b\left\{[b - H_2]\frac{h}{2} + s(a - H_1)^{s-1}k + aA_2 + bB_2\right\} \\ &+ \left(\frac{\partial H_3}{\partial z_1} + a\frac{H_1}{\partial z_1} + b\frac{\partial H_2}{\partial z_1}\right)V. \end{aligned}$$

To solve the homological equation , we need to find functions in $TQCU_{F_t}$ of the form

$$\psi = a\alpha(z) + b\beta(z) + \gamma(z) \qquad (*).$$

Thus, let $h = h_0(z_1)+ah_1(a,b,z_1)+bh_2(a,b,z_1)$, $k = k_0(z_1)ak_1(a,b,z_1)+bk_2(a,b,z_1)$, and $V = V_0(z_1) + aV_1(a,b,z_1) + bV_2(a,b,z_1)$. Hence, the quasi cusp tangent space takes the form:

$$\begin{aligned} TQCU_{F_t} &= a^2\left[\frac{h_0}{s} - \frac{1}{s}H_1h_1 - 2H_2k_1 + \cdots + A_1\right] + b^2\left[\frac{h_0}{2} - \frac{1}{2}H_2h_2 + \cdots + B_2\right] \\ &+ ab\left[2k + \left(s(a-H_2)^{s-1} - s(-H_2)^{s-1}\right)k + \ldots B_1 + A_2\right] \\ &+ a\left[-H_1\frac{h_0}{s} - 2H_2k_0 + \frac{\partial H_1}{\partial z_1}V_0\right] + b\left[-H_2\frac{h_0}{2} + s(-H_1)^{s-1}k_0 + \frac{\partial H_2}{\partial z_1}V_0\right] \\ &+ \frac{\partial H_3}{\partial z_1}V_0(z_1). \end{aligned}$$

with some smooth functions $h_0, h_1, h_2, k_0, k_1, k_2, V_0, V_1, V_2$. To get terms in the required form (*), we always can set

$$A_1 = -\left[\frac{h_0}{s} - \frac{1}{s}H_1h_1 - 2H_2k_1 + \ldots\right],$$

$$A_2 = -\left[2k + \left(s(a-H_2)^{s-1} - s(-H_2)^{s-1}\right)k + \ldots\right],$$

$$B_2 = -\left[\frac{h_0}{2} - \frac{1}{2}H_2h_2 + \ldots\right], \quad \text{and} \quad B_1 = 0.$$

Let $H_1 = c_m z_1^m + t\widetilde{H}_1(z_1)$, $H_2 = d_n z_1^n + t\widetilde{H}_2(z_1)$ and $H_3(z_1) = e_k z_1^k + t\widetilde{H}_2(z_1)$, where $c_m \neq 0, b_n \neq 0$, $e_k \neq 0$ and $\widetilde{H}_1 \in \mathcal{M}_{z_1}^{m+1}$, $\widetilde{H}_2 \in \mathcal{M}_{z_1}^{n+1}$ $\widetilde{H}_3 \in \mathcal{M}_{z_1}^{k+1}$.

Consider the germ $F_0 = \pm(x + c_2z_1^2)^2 \pm (y + d_2z_1^2)^2 + e_kz^k$ with $k \geq 3$.

Thus, we get *mod* $TQCU_{F_0}$:

$$[-\frac{1}{s}ac_2z_1^2 - \frac{1}{2}bd_2z_1^2]h_0(z_1) \equiv 0, \tag{4.1}$$

$$[-2ad_2z_1^2 \pm sc_2^{s-1}bz_1^{2(s-1)}]k_0(z_1) \equiv 0, \tag{4.2}$$

and

$$[2c_2az_1 + 2d_2bz_1 + ke_kz_1^{k-1}]V_0(z_1) \equiv 0. \tag{4.3}$$

Clearly, the equations (4.1) and (4.2) yields that $az_1^2 \equiv 0$ and $bz_1^2 \equiv 0$. Hence, we get that $z_1^k \equiv 0$. Therefore, we get the series of classes

$$\mathcal{N}_{2,2,k}: \quad \pm(x \pm z_1^2)^2 \pm (y \pm z_1^2)^2 \pm z_1^k, \text{ with } k \geq 3, \text{ for any } s \geq 3.$$

However, the classes $\mathcal{N}_{2,2,4}$ for $s \geq 4$ are adjacent to the classes $\pm y^2 + \alpha yx^2 + x^4$ which are non-simple. Hence, the only simple class we get in this case is $\mathcal{N}_{2,2,3}$ for $s \geq 3$ (mind that this class is adjacent to the simple class $\mathcal{M}_3$.)

Assume that $s = 3$. Consider the germ $F_{m,n,k} = (x + c_mz_1^m)^2 + (y + d_nz_1^n)^2 + e_kz^k$. Then, we obtain $mod\ TQCU_{F_0}$:

$$-\frac{1}{3}c_maz_1^m - \frac{1}{2}bd_nz_1^n \equiv 0, \tag{4.4}$$

$$-2ad_nz_1^n + 3bc_m^2z_1^{2m} \equiv 0, \tag{4.5}$$

and

$$amc_mz_1^{m-1} + bnd_nz_1^{n-1} + ke_kz_1^{k-1} \equiv 0, \tag{4.6}$$

The equation (4.4) implies that $az_1^m \equiv -\frac{3d_n}{2c_m}bz_1^n$. Thus, we obtain

$$3\frac{d_n^2}{c_m}bz_1^{2n-m} + 3c_m^2bz_1^{2m} \equiv 0. \tag{4.7}$$

If we consider the germ $F_{2,3,3}$ and suppose that $s\frac{d_3^2}{c_2} + 3c_2^2 = 0$, then the left hand side from the equation (4.7) vanishes. This yields that the germ $F_{2,3,3}$ is non-simple. Other germs $F_{m,3,k}$ with different values of m and k are discrete set of orbits in the stratum. However, all these germs $F_{m,3,k}$ are adjacent to the non-simple germ $F_{2,3,3}$. Thus, germs $F_{m,2,k}$ with $m > 2$ and $k \geq m$ are the only simple ones in this case. Hence, we get the class: $\mathcal{N}_{m,2,k}$ with $m > 2$ and $k \geq m$. This completes the proof of the theorem.

The proof of the theorem yields the following

Proposition 4.1.7 *The formulas of quasi cusp versal deformations of the simple quasi cusp classes are listed as follows:*

1. $\mathcal{L}_k : \pm x^2 \pm y^k + \mu x + \sum_{i=0}^{k-1} \lambda_i y^i, \quad k \geq 3.$

2. $\mathcal{M}_k : \pm y^2 \pm x^k + \mu y + \gamma xy + \sum_{i=0}^{k-1} \lambda_i x^i, \quad k \geq 3.$

3. $\mathcal{N}_{2,2,3} : \pm(x \pm z_1^2)^2 \pm (y \pm z_1^2)^2 \pm z_1^3 + \lambda_0 + \lambda_1 x + \lambda_2 x z_1 + \mu_1 y + \mu_2 y z_1 + \gamma_1 z_1 + \gamma_2 z_1^2.$

4. $\mathcal{N}_{m,2,k} : \pm(x \pm z_1^m)^2 \pm (y \pm z_1^2)^2 \pm z_1^k + \sum_{i=0}^{m-1} \lambda_i x z_1^i + \mu_0 + \mu_1 y z_1 \sum_{j=0}^{k-1} \gamma_i z_1^i, \; k \geq m > 2\ .$

Chapter 5

Quasi cone singularites

5.1 The classification of simple classes

Assume that $\Gamma_{cn} = \{xy - z^2 = 0\}$ and the local coordinates are $\mathbb{R}^n = \{(x, y, z, w)\}$ where $(x, y, z) \in \mathbb{R}^3$ and $w = (w_1, \ldots, w_{n-3}) \in \mathbb{R}^{n-3}$.

If the function germ base point is at the regular point of the border Γ_{cn} outside the singular component, then quasi cone equivalence coincides with quasi boundary equivalence. Hence, the list of simple quasi cone classes in this case is the same as the list of quasi boundary classes. In what followed we consider the remaining case of the function germ having a critical base point at the cone component is given in the following theorem.

Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function germ with a critical point at the origin.

Recall that the quasi cone tangent space to an admissible deformation f_t takes the form

$$\begin{aligned} TQCO_{f_t} &= \left\{\frac{\partial f_t}{\partial x}\left(xh_1 - xh_2 + 2zh_3 + \frac{\partial f_t}{\partial x}B_1 + \frac{\partial f_t}{\partial y}B_2 + \frac{\partial f_t}{\partial x}B_3\right)\right. \\ &+ \frac{\partial f}{\partial y}\left(yh_1 + yh_2 + 2zh_4 + \frac{\partial f_t}{\partial x}C_1 + \frac{\partial f_t}{\partial y}C_2 + \frac{\partial f_t}{\partial x}C_3\right) \\ &+ \left.\frac{\partial f_t}{\partial z}\left(zh_1 + yh_3 + xh_4 + \frac{\partial f_t}{\partial x}D_1 + \frac{\partial f_t}{\partial y}D_2 + \frac{\partial f_t}{\partial x}D_3\right) + \sum_{i=1}^{n-3}\frac{\partial f_t}{\partial w_i}A_i,\right\} \end{aligned}$$

with some arbitrary smooth functions $h_1, h_2, h_3, B_i, C_i, D_i$ and A_i.

Denote by $f^*(w) = f|_{x=y=z=0}$, the restriction of function f to the $(w_1, w_2, \dots, w_{n-3})$ coordinates subspace. Denote by r^* the rank of the second differential $d_0^2 f^*$ at the origin and set $c = n - 3 - r^*$.

We restate the prenormal forms in the new coordinates.

Lemma 5.1.1 *(Stabilization) The function germ $f(x, y, z, w)$ is quasi cone equivalent to $\sum_{i=1}^{r^*} \pm w_i^2 + g(x, y, z, \hat{w})$, where $\hat{w} \in \mathbb{R}^c$ and $g^* \in \mathcal{M}_{\hat{w}}^3$. For quasi cone equivalent f germs, the respective reduced germs g are quasi cone equivalent.*

Lemma 5.1.2 *There is a non-negative integer $s \le r - r^*$ such that the function germ $f(x, y, z, w)$ is quasi cone equivalent to $\sum_{i=1}^{r^*+s} \pm w_i^2 + \widetilde{f}(x, y, z, \widetilde{w})$, where $\widetilde{w} \in \mathbb{R}^{c-s}$ and $\widetilde{f}$ is a sum of a function germ from $\mathcal{M}_{x,y,z,\widetilde{w}}^3$ and a quadratic form in x, y and z only. For quasi cone equivalent f germs, the respective reduced germs $\widetilde{f}$ are quasi cone equivalent.*

The main preliminary classification results are the following.

Lemma 5.1.3 *Let $k = n - r$ be the corank of the second differential $d_0^2 f$ at the origin.*

1. *If $k = 0$, then f is quasi cone equivalent to $\pm x^2 \pm y^2 \pm z^2 + \sum_{i=1}^{n-3} \pm w_i^2$.*
2. *If $k = 1$, then f is quasi cone equivalent to either $\sum_{i=1}^{n-3} (\pm w_i^2) + \widetilde{f}(x, y, z)$ with* rank *$d_0^2 \widetilde{f}(x, y, z) = 2$ or to $\sum_{i=2}^{n-3} (\pm w_i^2) + \widetilde{f}(x, y, z, w_1) \pm x^2 \pm y^2 \pm z^2$ with $\widetilde{f}(x, y, z, w_1) \in \mathcal{M}_{x,y,z,w_1}^3$.*
3. *If $k \ge 2$, then f is non-simple.*

Proof. Using properties of fixed and partially x-fixed equivalences, the dimension of the intersection of the kernel subspace of f_2 with the coordinate subspace $x = y = z = 0$ is the only invariant for second order jet. So, Lemmas 5.1.1 and 5.1.2 yield that f is quasi cone equivalent to one of the following germs:

1. $F_1 = \sum_{i=k-2}^{n-3}(\pm w_i^2) + \widetilde{f}(x,y,z,w_1,w_2,\ldots,w_{k-3})$ where $\widetilde{f} \in \mathcal{M}^3_{x,y,z,w_1,w_2,\ldots,w_{k-3}}$, or

2. $F_2 = \sum_{i=k-1}^{n-3}(\pm w_i^2) + f_2(x,y,z) + \widetilde{f}(x,y,z,w_1,w_2,\ldots,w_{k-2})$ where f_2 is a quadratic form of rank one and $\widetilde{f} \in \mathcal{M}^3_{x,y,z,w_1,w_2,\ldots,w_{k-2}}$, or

3. $F_3 = \sum_{i=k}^{n-3}(\pm w_i^2) + f_2(x,y,z) + \widetilde{f}(x,y,z,w_1,w_2,\ldots,w_{k-1})$ where f_2 is a quadratic form of rank two and $\widetilde{f} \in \mathcal{M}^3_{x,y,z,w_1,w_2,\ldots,w_{k-1}}$, or

4. $F_4 = \sum_{i=k+1}^{n-3}(\pm w_i^2) + f_2(x,y,z) + \widetilde{f}(x,y,z,w_1,w_2,\ldots,w_k)$ where f_2 is a non-degenerate quadratic form and $\widetilde{f} \in \mathcal{M}^3_{x,y,z,w_1,w_2,\ldots,w_k}$.

Hence, the first two statements follow.

Now, suppose that $k \geq 3$. Consider the germ F_1. The quasi cone tangent space to the orbit at $\widetilde{f}$ takes the form:

$$\begin{aligned}
TQCO_{\widetilde{f}} &= \sum_{i=1}^{k-3}\frac{\partial\widetilde{f}}{\partial w_i}A_i + \frac{\partial\widetilde{f}}{\partial x}\{xh_1 - xh_2 + 2zh_3 + \frac{\partial\widetilde{f}}{\partial x}B_1 + \frac{\partial\widetilde{f}}{\partial y}B_2 + \frac{\partial\widetilde{f}}{\partial x}B_3\} \\
&+ \frac{\partial\widetilde{f}}{\partial y}\{yh_1 + yh_2 + 2zh_4 + \frac{\partial\widetilde{f}}{\partial x}C_1 + \frac{\partial\widetilde{f}}{\partial y}C_2 + \frac{\partial\widetilde{f}}{\partial x}C_3\} \\
&+ \frac{\partial\widetilde{f}}{\partial z}\{zh_1 + yh_3 + xh_4 + \frac{\partial\widetilde{f}}{\partial x}D_1 + \frac{\partial\widetilde{f}}{\partial y}D_2 + \frac{\partial\widetilde{f}}{\partial x}D_3\}.
\end{aligned}$$

The cubic terms in the tangent space are obtained from:$\sum_{i=1}^{k-3}\frac{\partial\widetilde{f}}{\partial w_i}A_i$, $(x\frac{\partial\widetilde{f}}{\partial x} + y\frac{\partial\widetilde{f}}{\partial y} + z\frac{\partial\widetilde{f}}{\partial z})h_1$,$(-x\frac{\partial\widetilde{f}}{\partial x} + y\frac{\partial\widetilde{f}}{\partial y})h_2$, $(2z\frac{\partial\widetilde{f}}{\partial x} + y\frac{\partial\widetilde{f}}{\partial z})h_3$, and $(2z\frac{\partial\widetilde{f}}{\partial y} + x\frac{\partial\widetilde{f}}{\partial z})h_4$. Note that $A_i's$ are linear forms and $h_1, h_2, h_3, h_4 \in \mathbb{R}$. Thus, These terms form a subspace of dimension $k(k-3)+4$ which is less than $M = \frac{k(k+1)(k+2)}{6}$-the dimension of all cubic terms in the variables $x, y, z, w_1, w_2, \ldots, w_{k-3}$. Hence, F_1 is non-simple.

Standard argument as before shows the following adjacencies

$$F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow F_4.$$

This yields that F_2, F_3 and F_4 are non-simple.

If $k = 2$, then f is quasi cone equivalent to one of the following germs:

1. $H_1 = \sum_{i=1}^{n-3}(\pm w_i^2) + f_2(x,y,z) + \widetilde{f}(x,y,z)$ where f_2 is a quadratic form of rank one and $\widetilde{f}(x,y,z) \in \mathcal{M}^3_{x,y,z}$, or

2. $H_2 = \sum_{i=2}^{n-3}(\pm w_i^2) + f_2(x,y,z) + \widetilde{f}(x,y,z,w_1)$ where f_2 is a quadratic form of rank two and $\widetilde{f}(x,y,z,w_1) \in \mathcal{M}^3$, or

3. $H_3 = \sum_{i=3}^{n-3}(\pm w_i^2) + f_2(x,y,z) + \widetilde{f}(x,y,z,w_1,w_2)$ where f_2 is a non-degenerate quadratic form and $\widetilde{f} \in \mathcal{M}^3_{x,y,z,w_1,w_2}$.

Again comparing the dimensions of the orbits with their quasi cone tangent space and constructing adjacencies between them yield that H_1, H_2 and H_3 are non-simple. For example, consider the germ H_1 and let $H_0 = ax^2 + bz^3 + cy^3 + dyz^2 + ey^2z$ be the lowest quasi homogeneous part with respect to the weights $w_x = \frac{1}{2}, w_y = w_z = \frac{1}{3}$. Then, the dimension of the subspace which contribute to the quasi homogeneous part is 3.

∎

Lemma 5.1.4 *1. Let $f : (\mathbb{R}^3, 0) \mapsto (\mathbb{R})$ be a function germ with critical point at the origin. If the quadratic form f_2 of f has rank 2 then f is quasi cone equivalent to either $\pm x^2 \pm y^2 + \varphi_1(z)$ or $\pm(x-y)^2 \pm z^2 + \varphi_2(y)$ or $\pm x^2 \pm z^2 + x\phi_1(y) + \phi_2(y)$ with $\varphi_1 \in \mathcal{M}^3_z$, $\phi_1 \in \mathcal{M}^2_y$, $\varphi_2, \phi_2 \in \mathcal{M}^3_y$.*

2. The germ $\widetilde{f}(x,y,z,w_1) \pm x^2 \pm y^2 \pm z^2$ with $\widetilde{f}(x,y,z,w_1) \in \mathcal{M}^3_{x,y,z,w_1}$ is quasi cone equivalent to the germ: $\pm x^2 \pm y^2 \pm z^2 + xH_1(w_1) + yH_2(w_1) + zH_3(w_1) + H_4(w_1)$, with $H_1, H_2, H_3 \in \mathcal{M}^2_{w_1}$ and $H_4 \in \mathcal{M}^3_{w_1}$.

Proof. Let $f : (\mathbb{R}^3, 0) \mapsto (\mathbb{R})$ be a function germ with critical point at the origin. Suppose that the quadratic form f_2 of f has rank 2. Then, f_2 can be written as

$$f_2(x,y,z) = \pm L_1^2 \pm L_2^2,$$

where $L_1 = a_1x + b_1y + c_1z$ and $L_2 = a_2x + b_2y + c_2z$ are linearly independent linear forms. Then, the line $\{L_1 = 0, L_2 = 0\}$ is the kernel subspace of f_2. By quasi-fixed equivalence we can replace L_1 and L_2 by their linear combinations $\widetilde{L}_1$ and $\widetilde{L}_2$ with the same kernel subspace. Then, the kernel subspace by cone preserving transformation can be reduced to a fixed normal position. More precisely, up to permutation of x and y, we distinguish the following cases:

1) if $a_1 \neq 0$ and $b_2 \neq 0$, then f_2 can be reduced to the form $\widetilde{f_2} = \pm x^2 \pm y^2$. To proof this claim, take the family $f_t = \pm(a_1x + t(b_1y + c_1z))^2 \pm (b_2y + t(a_2x + c_2z))^2$ and that the matrix $\begin{pmatrix} a_1 & tb_1 \\ ta_2 & b_2 \end{pmatrix}$ is non-degenerate and belongs to a connected components of identity matrix in the set of non-degenerate matrices. Then, it can be proven that the homological equation $-\frac{\partial f_t}{\partial t} = TQCO_{f_t}$ within the quadratic form is solvable for any t.

Now consider the germ $F = \pm x^2 \pm y^2 + f_3(x, y, z)$ where $f_3 \in \mathcal{M}^3_{x,y,z}$. Take the quasi cone tangent space to the orbit at $f_0 = \pm x^2 \pm y^2$.

$$\begin{aligned} TQCO_{f_0} &= \pm 2x\{xh_1 - xh_2 + 2zh_3 + xA_1 + yA_2\} \\ &\quad \pm 2y\{yh_1 + yh_2 + 2zh_4 + xB_1 + yB_2\}. \end{aligned}$$

Thus we obtain *mod* $TQCO_{f_0}$: $x^2 \equiv 0, y^2 \equiv 0, xy \equiv 0, xz \equiv 0$ and $yz \equiv 0$. Hence, $\mathbf{C}_{x,y,z}/TQCO_{f_0} \equiv G(z)$. Therefore, the germ $H = \varphi_1(x, y) + z\varphi_2(x, y)$ belongs to $TQCO_{f_0}$ where $\varphi_1 \in \mathcal{M}^3_{x,y}$ and $\varphi_2 \in \mathcal{M}^2_{x,y}$.

In particular, let $\varphi_1(x, y) = x^2K_1(x, y) + y^2K_2(x, y)$ where $K_1, K_2 \in \mathcal{M}_{x,y}$. Let $\varphi_2(x, y) = xK_3(x, y) + y^2K_4(x, y)$ where $K_3 \in \mathcal{M}_{x,y}$ and $K_4 \in \mathbf{C}_{x,y}$. Then, the terms which are divisible by x^2 (divisible by y^2, respectively) in the germ φ_1 can be solved by setting $h_1 = h_2 = h_3 = h_4 = A_2 = B_1 = B_2 = 0$ and taking
$K_1 = \pm 2A_1$ ($h_1 = h_2 = h_3 = h_4 = A_1 = A_2 = B_1 = 0$, $K_2 = \pm 2B_2$, respectively).

Also for the terms which are divisible by x in the germ $z\varphi_2(x, y)$ (divisible by y^2, respectively), one can set $h_1 = h_2 = h_4 = A_1 = A_2 = B_1 = B_2 = 0$ and take $h_3 = \pm\frac{1}{4}K_3$ ($h_1 = h_2 = h_3 = A_1 = A_2 = B_1 = B_2 = 0$, $h_4 = \mp\frac{1}{4}yK_4$, respectively).

Now assign weights $w_x = w_y = w_z = \frac{1}{2}$. Then any term monomial in the germ

$\varphi_1(x, y)$ of the form $g_1^* = a_{2+i,j}x^{2+i}y^j$ has the quasi degree $d(g_1^*) = \frac{2+i+j}{2}$. The germ

$$\Phi_1 = (2+i)a_{2+i,j}x^{2+i-1}y^j[xA_1 + (2+i)a_{2+i,j}x^{2+i-1}y^jA_1] + (2+i)a_{2+i,j}x^{2+i}y^jA_1,$$

clearly has quasi degree Φ_1 greater than $d(g_1^*)$.

Similar argument can be carried ou for all terms which belongs to $TQCO_{F_0}$.

Hence, Lemma 1.3.5 shows that that F is quasi cone equivalent to a germ of the form $\widetilde{F} = \pm x^2 \pm y^2 + \widetilde{\varphi}_1(z)$ with $\widetilde{\varphi}_1 \in \mathcal{M}_z^3$. In this case, the kernel line $\{L_1 = 0, L_2\}$ is outside the cone.

2) if $a_1 \neq 0, c_2 \neq 0$ and $b_1 = a_2 = b_2 = 0$(or $b_1 = b_2 = 0$ but $a_1 \neq 0$, $c_2 \neq 0$), then f_2 can be reduced to the form $\widetilde{f}_2 = \pm x^2 \pm z^2$. In this case, the kernel line is in the cone.

Now consider the germ $F = \pm x^2 \pm z^2 + f_3(x, y, z)$ with $f_3 \in \mathcal{M}_{x,y,z}^3$. Take the tangent space to the orbit at $f_0 = \pm x^2 \pm z^2$.

$$\begin{aligned} TQCO_{f_0} &= \pm 2x\{xh_1 - xh_2 + 2zh_3 + xA_1 + zA_3\} \\ &\quad \pm 2z\{zh_1 + yh_3 + xh_4 + xC_1 + zC_3\}. \end{aligned}$$

Thus we obtain *mod* $TQCO_{f_0}$: $x^2 \equiv 0, xz \equiv 0, z^2 \equiv 0$ and $yz \equiv 0$. Hence, $\mathbf{C}_{x,y,z}/TQCO_{f_0} \equiv xH_1(y) + H_2(y)$.

Now assign weights $w_x = w_y = w_z = \frac{1}{2}$. Then,comparison of the quasi degrees $d(g_i^*)$ of $g_i^* \in TQCO_{f_0}$ with the quasi degrees of the respective germs $d(\Phi_i)$ yield that F is quasi cone equivalent to a germ of the form $\widetilde{F} = \pm x^2 \pm z^2 + x\phi_1(y) + \phi_2(y)$ with $\phi_1 \in \mathcal{M}_y^2$ and $\phi_2 \in \mathcal{M}_y^3$.

3) if $a_1 \neq 0, b_1 \neq 0, c_2 \neq 0$ and $a_2 = b_2 = 0$, then f_2 can be reduced to the form $\widetilde{f}_2 = \pm(x \pm y)^2 \pm z^2$. Hence, considering the principal part $\widetilde{f}_2$ yields that the germ $F = \pm(x \pm y)^2 \pm z^2 + f_3(x, y, z)$ with $f_3 \in \mathcal{M}_{x,y,z}^3$ is quasi cone equivalent to the germ $\widetilde{F} = \pm(x \pm y)^2 \pm z^2 + \varphi_1(y)$. The argument to prove this claim is similar to the previous cases **1)** and **2)**. Notice that the germ $\widetilde{F}^- = \pm(x - y)^2 \pm z^2 + \varphi_1(y)$ corresponds to the case when the kernel subspace of the quadratic form of $\widetilde{F}^-$ lies inside the cone. It can be reduced to normal fixed position by cone preserving transformation.

For the second claim, consider the tangent space to the orbit at $f_0 = \pm x^2 \pm y^2 \pm z^2$.

$$\begin{aligned} TQCO_{f_0} &= \pm 2x\{xh_1 - xh_2 + 2zh_3 + xA_1 + yA_2 + zA_3\} \\ &\quad \pm 2y\{yh_1 + yh_2 + 2zh_4 + xB_1 + yB_2 + zB_3\} \\ &\quad \pm 2z\{zh_1 + yh_3 + xh_4 + xC_1 + yC_2 + zC_3\}. \end{aligned}$$

Thus, we obtain *mod* $TQCO_{f_0}$: $x^2 \equiv 0, y^2 \equiv 0, z^2 \equiv 0, xy \equiv 0, xz \equiv 0$ and $yz \equiv 0$. Hence, $\mathbf{C}_{x,y,z,z_1}/TQCO_{f_0} \equiv xG_1(w_1) + yG_2(w_1) + zG_3(w_1) + G_4(w_1)$.

Similar argument as in Lemma 4.1.5 (by quasi partially x-fixed equivalence) shows that $F = \pm x^2 \pm y^2 \pm z^2 + \widetilde{f}(x, y, z, w_1)$ is quasi cone equivalent to a germ of the form $\widetilde{F} = \pm x^2 \pm y^2 \pm z^2 + xH_1(w_1) + yH_2(w_1) + zH_3(w_1) + H_4(w_1)$ with $H_1, H_2, H_3 \in \mathcal{M}^2_{w_1}$ and $H_4 \in \mathcal{M}^3_{w_1}$.

∎

Theorem 5.1.5 *Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be simple with respect to the quasi cone equivalence. Then, either f_2 is a non-degenerate form and hence f is quasi cone equivalent to $\mathcal{P}_2 : \pm x^2 \pm y^2 \pm z^2 + \sum_{i=1}^{n-3} \pm w_i^2$ or f_2 is a degenerate form of corank 1 and hence f is stably quasi cone equivalent to one of the following simple classes (up to a possible permutation of x, y coordinates and up to the addition with a quadratic form in some extra variables):*

1. $\mathcal{P}_k : \pm x^2 \pm y^2 \pm z^k$, $k \geq 3$; k.
2. $\mathcal{O}_m : \pm z^2 \pm (x-y)^2 \pm y^m$, $m \geq 3$; $m+2$.
3. $\mathcal{S}_{k,m} : \pm z^2 \pm (x \pm y^k)^2 \pm y^m$, $m > k \geq 2$; $k+m+1$.
4. $\mathcal{Y}_l : \pm x^2 \pm y^2 \pm z^2 \pm w^l$, $l \geq 3$; $4l-4$.
5. $\mathcal{W}_{k,l} : \pm x^2 \pm y^2 \pm (z + w_1^k)^2 \pm z^l$, $l > k$; $3k+l-1$.
6. $\mathcal{Q}_{m,l} : \pm(x \pm w_1^m)^2 \pm (y \mp w_1^m)^2 \pm z^2 \pm w_1^l$, $l > m \geq 2$; $3m+l-1$.

7. $\mathcal{V}_{m,n,l}: \quad \pm(x+w_1^m)^2 \pm y^2 \pm (z+w_1^n)^2 \pm w_1^l, \quad l > n > m; \qquad 2m+n+l-1.$

The orbit codimension in the space of germs is shown in the right column.

Remark: The non-simple classes f either have corank of f_2 greater or equal 2 or belongs to a subspace of infinite codimension in $\mathbf{C}_w$.

Proof of Theorem 5.1.5

We prove the theorem for the necessary condition (the first condition in Lemma 1.3.5). The second condition is straightforward.

Consider the reduced germs in Lemma 5.1.4.

1) Consider the germ $F = \pm x^2 \pm y^2 + \varphi_1(z)$. Suppose that $\varphi_1(z) = a_k z^k + \widetilde{\varphi}_1(z)$ where $a_k \neq 0, \varphi_1 \in \mathcal{M}_z^{k+1}$ and $k \geq 3$. Let $F = f_0 + \widetilde{\varphi}_1(z)$ where $f_0 = \pm x^2 \pm y^2 + a_k z^k$. The quasi cone tangent space to the orbit at f_0 has the form

$$\begin{aligned} TQCO_{f_0} &= \pm 2x\left\{xh_1 - xh_2 + 2zh_3 + xA_1 + yA_2 + z^{k-1}A_3\right\} \\ &\quad \pm 2y\left\{yh_1 + yh_2 + 2zh_4 + xB_1 + yB_2 + z^{k-1}B_3\right\} \\ &+ \ ka_k z^{k-1}\left\{zh_1 + yh_3 + xh_4 + xC_1 + yC_2 + z^{k-1}C_3\right\}. \end{aligned}$$

Then, we get *mod* $TQCO_{f_0}$: $x^2 \equiv 0, yx \equiv 0, yz \equiv 0, y^2 \equiv 0, xz \equiv 0$ and $z^k \equiv 0$. This implies existence of solutions of the homological equation of the given φ_1. Note that the monomial $1, z, z^2, \ldots, z^{k-1}$ form a basis for the local algebra $\mathcal{Q} = \mathbf{C}_{x,y,z}/TQCN_{f_0}$.

The simplified homological equation is solvable for the term $a_k z^k$. Hence f_0 is simple and after rescaling a_k to ± 1, we get the classes $\pm x^2 \pm y^2 \pm z^k$ with $k \geq 3$.

2) Consider the germ $F = \pm z^2 \pm x^2 + x\phi_1(y) + \phi_2(y)$. Assume that $\phi_1(y) = a_k y^k + \widetilde{\phi}_1(y)$ and $\phi_2(y) = b_m y^m + \widetilde{\phi}_2(y)$ where $a_k \neq 0, b_m \neq, k \geq 2, m \geq 3$ and $\widetilde{\phi}_1 \in \mathcal{M}_y^{k+1}, \widetilde{\phi}_2 \in \mathcal{M}_y^{m+1}$. Then, we distinguish the following cases:

i. If $k \geq m-1$, then F is quasi cone equivalent to the germ $\pm z^2 \pm x^2 \pm y^m$. To prove this claim, consider the tangent space to the quasi cone orbit at $f_0 = \pm z^2 \pm x^2 + b_m y^m$.

$$\begin{aligned} TQCO_{f_0} &= \pm 2x\left\{xh_1 - xh_2 + 2zh_3 + xA_1 + y^{m-1}A_2 + zA_3\right\} \\ &\quad + mb_m y^{m-1}\left\{yh_1 + yh_2 + 2zh_4 + xB_1 + y^{m-1}B_2 + zB_3\right\} \\ &\quad \pm 2z\left\{zh_1 + yh_3 + xh_4 + xC_1 + y^{m-1}C_2 + zC_3\right\}. \end{aligned}$$

Then we get $mod\ TQCN_{f_0} : x^2 \equiv 0, xy^{m-1} \equiv 0, xz \equiv 0, y^m \equiv 0, z^2 \equiv 0$ and $yz \equiv 0$.

Thus, the germ $x\phi_1(y) + \phi_2(y)$ belongs to $TQCO_{f_0}$. Note that, the monomials $1, z, y, y^2, \ldots, y^{m-1}, x, xy, xy^2, \ldots, xy^{m-2}$ form a basis for the local algebra $\mathcal{Q} = \mathbf{C}_{x,y,z}/TQCN_{f_0}$. Thus, we get the classes $\pm z^2 \pm x^2 \pm y^m$ which is equivalent to the form $\pm z^2 \pm (x \pm y^{m-1})^2 \pm y^m$ with $m \geq 3$.

ii. If $m > k+1$ and $\mp a_k^2 + 4b_m \neq 0$ when $m = 2k$, then F is quasi cone equivalent to the germ $\pm z^2 \pm x^2 \pm xy^k \pm y^m$. To prove this claim, consider the tangent space to the quasi cone orbit at $f_0 = \pm z^2 \pm x^2 + a_k xy^k + b_m y^m$.

$$\begin{aligned} TQCO_{f_0} &= (\pm 2x + a_k y^k)\left\{xh_1 - xh_2 + 2zh_3 + (\pm 2x + a_k y^k)A_1\right. \\ &+ \left.(ka_k xy^{k-1} + mb_m y^{m-1})A_2 + zA_3\right\} \\ &+ (ka_k xy^{k-1} + mb_m y^{m-1})\left\{yh_1 + yh_2 + 2zh_4 + (\pm 2x + a_k y^k)B_1\right. \\ &+ \left.(ka_k xy^{k-1} + mb_m y^{m-1})B_2 + zB_3\right\} \\ &\quad \pm 2z\left\{zh_1 + yh_3 + xh_4 + (\pm 2x + a_k y^k)C_1 + (ka_k xy^{k-1} + mb_m y^{m-1})C_2 + zC_3\right\}. \end{aligned}$$

Thus, we obtain $mod\ TQCO_{f_0}$: $z^2 \equiv 0$. Also,

$$\pm 2x^2 + a_k xy^k + ka_k xy^k + mb_m y^m \equiv 0 \tag{5.1}$$

and

$$\mp 2x^2 - a_k xy^k + ka_k xy^k + mb_m y^m \equiv 0. \tag{5.2}$$

The equations (5.1) and (5.2) yield that

$$ka_k xy^k + mb_m y^m \equiv 0, \tag{5.3}$$

and

$$\pm 2x^2 + a_k xy^k \equiv 0 \Rightarrow x^2 \equiv \mp \frac{1}{2} a_k xy^k. \tag{5.4}$$

Also, we have

$$(\pm 2x + a_k y^k)^2 = 4x^2 \pm 4a_k xy^k + a_k^2 y^{2k} \equiv 0. \tag{5.5}$$

Substitute x^2 from the equation (5.4) in the equation (5.5) to obtain

$$\pm 2a_k xy^k + a_k^2 y^{2k} \equiv 0 \Rightarrow xy^k \equiv \mp \frac{1}{2} a_k y^{2k}. \tag{5.6}$$

Substitute xy^k from the equation (5.6) in the equation (5.3) to get $y^{2k} \equiv 0$ and $y^m \equiv 0$. This yields that $xy^k \equiv 0$ and $x^2 \equiv 0$. Thus, there are solutions for the homological in the simplified case.

Notice that the germ $\pm z^2 \pm x^2 \pm xy^k \pm y^m$ can be written in the form $\pm z^2 \pm (x \pm y^k)^2 \pm y^m$.

Also, note that $xz \equiv 0$ and $yz \equiv 0$. Hence, the local algebra $\mathcal{Q} = C_{x,y,z}/TQCO_F$ is generated by the monomials $1, z, y, y^2, \ldots, y^{m-1}$ and $x, xy, xy^2, \ldots, xy^{k-1}$.

iii. If $m = 2k$ and $\mp a_k^2 + 4b_m = 0$, then F takes the form $F = \pm z^2 \pm (x \pm \frac{1}{2} a_k y^k)^2 + x\widetilde{\phi}_1(y) + \widetilde{\phi}_2(y)$ where $\widetilde{\phi}_1(y) \in \mathcal{M}_y^{k+1}$ and $\widetilde{\phi}_2(y) \in \mathcal{M}^{2k+1}$. Suppose that $\widetilde{\phi}_2(y) = d_s y^s + \varphi(y)$ where $s \geq 2k+1$ and $\varphi \in \mathcal{M}_y^{s+1}$. Then, F is quasi cone equivalent to the germ $\pm z^2 \pm (x \pm y^k)^2 \pm y^s$. To prove this claim, consider the tangent space to the quasi cone orbit at $f_0 = \pm z^2 \pm (x + \widetilde{a}_k y^k)^2 + d_s y^s$ where $\widetilde{a} = \frac{1}{2} a_k$.

$$\begin{aligned} TQCO_{f_0} &= \pm 2(x + \widetilde{a}_k y^k) \left\{ xh_1 - xh_2 + 2zh_3 + (x + \widetilde{a}_k y^k) A_1 + y^{s-1} A_2 + z A_3 \right\} \\ &+ [\pm 2\widetilde{a}_k k y^{k-1} (x + \widetilde{a}_k y^k) + s d_s y^{s-1}] \left\{ yh_1 + yh_2 + 2zh_4 + (x + \widetilde{a}_k y^k) B_1 \right. \\ &+ \left. y^{s-1} B_2 + z B_3 \right\} \pm 2z \left\{ zh_1 + yh_3 + xh_4 + (x + \widetilde{a}_k y^k) C_1 + y^{s-1} C_2 + z C_3 \right\}. \end{aligned}$$

We have *mod* $TQCO_{f_0}$: $z^2 \equiv 0$, $zy \equiv 0$ and $zx \equiv 0$. Also,

$$\pm 2x^2 \pm \widetilde{a}_k xy^k \pm 2\widetilde{a}_k k xy^k \pm 2\widetilde{a}_k k y^{2k} + s d_s y^s \equiv 0, \tag{5.7}$$

and

$$\mp 2x^2 \mp \widetilde{a}_k xy^k \pm 2\widetilde{a}_k kxy^k \pm 2\widetilde{a}_k ky^{2k} + sd_s y^s \equiv 0. \tag{5.8}$$

The equations (5.7) and (5.8) yields that

$$x^2 + \widetilde{a}_k xy^k \equiv 0 \Rightarrow x^2 \equiv -\widetilde{a}_k xy^k, \tag{5.9}$$

and

$$\pm 2\widetilde{a}_k kxy^k \pm 2\widetilde{a}_k^2 ky^{2k} + sd_s y^s \equiv 0. \tag{5.10}$$

Also, we have

$$(x + \widetilde{a}_k y^k)^2 = x^2 + 2\widetilde{a}_k xy^k + \widetilde{a}^2 y^{2k} \equiv 0. \tag{5.11}$$

Substitute x^2 from the equation (5.9) in the equation (5.11) to obtain

$$xy^k + \widetilde{a}_k y^{2k} \equiv 0 \Rightarrow xy^k \equiv -\widetilde{a}_k y^{2k}. \tag{5.12}$$

Now substitute xy^k from the equation (5.12) in the equation (5.10) to get $y^s \equiv 0$.

If we use the two relations (5.9) and (5.12) in the local algebra $\mathcal{A} = \mathbf{C}_{x,y,z}/TQCO_{f_0}$, we see that the monomials

$$1, z, xy, xy^2, \dots, xy^{k-1}, y, y^2, \dots, y^{s-1}$$

form a basis for $\mathcal{A}$.

3) Consider the germ $F = \pm z^2 \pm (x \pm y)^2 + \varphi_2(y)$. Suppose that $\varphi_2(y) = e_s y^s + \widetilde{\varphi}_2(y)$ where $s \geq 3$, $e_s \neq 0$ and $\widetilde{\varphi}_2 \in \mathcal{M}_y^{s+1}$. Take the tangent space to the quasi cone orbit at $\widetilde{f}_0 = \pm z^2 \pm (x \pm y)^2 + e_s y^s$. Then, the proof is a special case from **2)-iv** when $k = 1$. Hence, we get the classes $\pm z^2 \pm (x \pm y)^2 \pm y^s$ where $s \geq 3$.

4) Finally, consider the germ $F = \pm x^2 \pm y^2 \pm z^2 + x\widetilde{H}_1(w_1) + y\widetilde{H}_2(w_1) + z\widetilde{H}_3(w_1) + \widetilde{H}_4(w_1)$, where $\widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_3 \in \mathcal{M}_{w_1}^2$ and $\widetilde{H}_4 \in \mathcal{M}_{w_1}^3$.

Following similar methods to the last case in the proof of quasi cusp classification theorem, consider the equivalent deformations form

$$G_t = \pm(x + H_1(w,t))^2 \pm (y + H_2(w,t))^2 \pm (z + H_3(w,t))^2 + H_4(w).$$

Let $A = x + H_1(w,t)$, $B = y + H_2(w,t)$ and $C = z + H_3(w,t)$. Then, the homological equation takes the form:

$$-\frac{\partial G_t}{\partial t} = -2[A\dot{H}_1 + B\dot{H}_2 + C\dot{H}_3] + \dot{H}_4 = TQCO_{G_t}. \quad (*)$$

Consider the subspace $T^*QCO_{G_t} \subset TQCO_{G_t}$

$$\begin{aligned} T^*QCO_{G_t} &= A\left[\frac{h_1 - h_2}{2}H_1 + 2H_3h_3 + 2\frac{\partial H_1}{\partial w_1}V_0 + \frac{\partial H_4}{\partial w_1}V_1\right] \\ &+ B\left[\frac{h_1 + h_2}{2}H_2 + 2H_3h_4 + 2\frac{\partial H_2}{\partial w_1}V_0 + \frac{\partial H_4}{\partial w_1}V_2\right] \\ &+ C\left[h_1H_3 - h_3H_2 - h_4H_1 + 2\frac{\partial H_3}{\partial w_1}V_0 + \frac{\partial H_4}{\partial w_1}V_3\right] \\ &+ \frac{\partial H_4}{\partial w_1}V_0, \end{aligned}$$

for some smooth functions $h_1(w_1), h_2(w_1), h_3(w_1), h_4(w_1), V_i(w_1, A, B, C)$ and $V_0(w)$.

Let $H_1 = \alpha_m w_1^m + \ldots$, $H_2 = \beta_n w_1^n + \ldots$, $H_3 = \gamma_k w_1^k + \ldots$ and $H_4 = \delta_l w_1^l + \ldots$. Assume that at least some of the coefficient $\alpha_m, \beta_n, \gamma_k$ and δ_l are non-zero.

If $l - 1 \leq m, n, k$, then working for example with the initial coordinates we get the simple classes $\pm x^2 \pm y^2 \pm z^2 \pm w^l$ where $l \geq 3$.

For the sake of simplicity, we describe here only the lowest case like $m = 2$.

Consider the germ $F_{2,2,2,3} = \pm(x + \alpha_2 w_1^2)^2 \pm (y + \beta_2 w_1^2)^2 \pm (z + \gamma_2 w_1^2)^2 + \delta_3 w_1^3$. Then, we obtain *mod* $T^*QCO_{F_{2,2,2,3}}$

$$\delta_3 w_1^2 AV_1(w) \equiv 0, \quad \delta_3 w_1^2 BV_2(w) \equiv 0, \quad \delta_3 w_1^2 CV_3(w) \equiv 0,$$

$$3\delta_3 w_1^2 + 4\alpha_2 w_1 A + 4\beta_2 B w_1 + 4\gamma_2 w_1 \equiv 0.$$

Multiplying the last relation by w_1 yields that the deformation $A\dot{H}_1 + B\dot{H}_2 + C\dot{H}_3 + \dot{H}_4$ is contained in the space $T^*QCO_{F_{2,2,2,3}}$. Hence, we obtain the simple class $\mathcal{T}_{2,2,2,3} : \widetilde{F} = \pm(x \pm w_1^2)^2 \pm (y \pm w_1^2)^2 \pm (z \pm w_1^2)^2 \pm w_1^3$. Notice here that if we consider the function $F = \pm x^2 \pm y^2 \pm z^2 + \delta w_1^3$, then we get the same conclusion. That is the deformation $A\dot{H}_1 + B\dot{H}_2 + C\dot{H}_3 + \dot{H}_4$ is contained in the space $T^*QCO_{F_{2,2,2,3}}$. Therefore, the simple class $\mathcal{T}_{2,2,2,3}$ can be reduced further to the form $\widetilde{F} = \pm x \pm y^2 \pm z^2 \pm w_1^3$.

Now assume that $l > m, n, k$ and consider the germ $F_{2,2,2,4} = \pm(x + \alpha_2 w_1^2)^2 \pm (y + \beta_2 w_1^2)^2 \pm (z + \gamma_2 w_1^2)^2 + \delta_4 w_1^4$. Then, we obtain *mod* $T^*QCO_{F_{2,2,2,4}}$:

$$\frac{A}{2}\alpha_2 w^2 + \frac{B}{2}\beta_2 w^2 + C\gamma w^2 \equiv 0, \tag{5.13}$$

$$\frac{-A}{2}\alpha_2 w^2 + \frac{B}{2}\beta_2 w^2 \equiv 0, \tag{5.14}$$

$$2A\gamma_2 w^2 - C\beta_2 w^2 \equiv 0, \tag{5.15}$$

$$2B\gamma_2 w^2 - C\alpha_2 w^2 \equiv 0. \tag{5.16}$$

These relations corresponds to h_1, h_2, h_3 and h_4. Multiplying the equation (5.14) by -1 then adding the equations (5.13) we obtain

$$A\alpha_2 w_1^2 + C\gamma_2 w_1^2 \equiv 0. \tag{5.17}$$

Adding the equations (5.13) and (5.14) we get

$$0 + B\beta_2 w_1^2 + C\gamma_2 w_1^2 \equiv 0. \tag{5.18}$$

We can put the coefficients of the relations (5.15),(5.16),(5.17) and (5.18) in the following matrix

$$M = \begin{pmatrix} \alpha_2 & 0 & 2\gamma_2 & 0 \\ 0 & \beta_2 & 0 & 2\gamma_2 \\ \gamma_2 & \gamma_2 & -\beta_2 & -\alpha_2 \end{pmatrix}.$$

Notice that the matrix M of second order terms has 3 rank when $\alpha_2\beta_2 + 2\gamma_2^2 \neq 0$. Call the set of points in the space of coefficients $\alpha_2, \beta_2, \gamma_2$ given by the relations above and defined by the equation $\{\alpha_2\beta_2 + 2\gamma_2^2 = 0\}$ - the *dual cone* . Consider now the generic case. That is the points $(\alpha_2, \beta_2, \gamma_2)$ with $\alpha_2\beta_2 + 2\gamma_2^2 \neq 0$.

Lemma 5.1.6 *The germ* $G = \pm(x+\alpha_2 w_1^2+\dots)^2 \pm (y+\beta_2 w_1^2+\dots)^2 \pm (z+\gamma_2 w_1^2+\dots)^2+\delta_2 w_1^4+\dots$ *is simple and is equivalent to the either* $\widetilde{F} = \pm x^2 \pm y^2 \pm (z+w_1^2)^2 \pm w_1^4$ *or* $\widetilde{F} = \pm(x\pm w_1^2)^2 \pm (y \mp w_2^2)^2 \pm z^2 \pm w_1^4$*. Here* $\dots$ *denotes terms of higher degrees.*

Proof.

As the rank of the matrix M is 3, the four relations (5.15),(5.16),(5.17) and (5.18) yield that the space $T^*QCO_{G_t}$ contains the second order terms Aw_1^2, Bw_1^2 and Cw_1^2. Moreover, the relations

$$A\delta_4 w_1^3 V_1 \equiv 0,\ \ B\delta_4 w_1^3 V_2 \equiv 0,\ \ C\delta_4 w_1^3 V_3 \equiv 0,$$

and

$$2\alpha_2 w_1 A + 2\beta_2 w_1 B + 2\gamma_2 C + 4\delta_4 w_1^3 \equiv 0$$

give the complete solutions for the respective homological equations. Now if the point $P = (\alpha_2, \gamma_2, \beta_2)$ lies outside the dual cone, then P can be reduced to $(0,0,1)$ which gives the first form. If the point $P = (\alpha_2, \gamma_2, \beta_2)$ lies inside the dual cone, then we can reduce P to $(0,0,1)$ which gives the other forms. The lemma is proven.

Assume now that $(\alpha_2, \beta_2, \gamma_2)$ belongs to to the dual cone $\alpha_2\beta_2 + 2\gamma_2^2 = 0$.

Lemma 5.1.7 *The linear span of the columns of the matrix* M *(the images of* MD *where* $D \in \mathbb{R}^4$ *is a column vector) coincides with the tangent plane to the dual cone at the point* $(\alpha_2, \beta_2, \gamma_2)$.

Proof. Take a curve $t \mapsto (\alpha_2 = \alpha_2(t), \beta_2 = \beta_2(t), \gamma_2 = \gamma_2(t))$ lying on the dual cone $\alpha_2\beta_2 + 2\gamma_2^2 = 0$. The derivative equation takes the form

$$\dot\alpha_2\beta_2 + \alpha_2\dot\beta_2 + 2\gamma_2\dot\gamma_2 = 0. \qquad (*)$$

Take for example the column vector $D = (1,0,0,0)^T$. Then $MD = (\alpha_2, 0, \gamma_2)^T = (\dot\alpha_2, \dot\beta_2, \dot\gamma_2)$. The vector MD satisfies the relation $(*)$. This means that the columns of the matrix M span the tangent plane to the dual cone at $(\alpha_2, \beta_2, \gamma_2)$.

∎

Lemma 5.1.7 implies that the germ $G = \pm(x + \alpha_2 w_1^2 + \dots)^2 \pm (y + \beta_2 w_1^2 + \dots)^2 \pm (z + \gamma_2 w_1^2 + \dots)^2 + \delta_2 w_1^4 + \dots$ is simple and is equivalent to the germ $\widetilde{F} = \pm(x + w_1^2)^2 \pm y^2 \pm (z + w^3)^2 \pm w_1^4$.

We sum up our observation geometrically. The classification of the germ $G = \pm(x + \alpha_2 w_1^2 + \dots)^2 \pm (y + \beta_2 w_1^2 + \dots)^2 \pm (z + \gamma_2 w_1^2 + \dots)^2 + \delta_2 w_1^4 + \dots$ with $\widetilde{F}_{2,2,2,4}$ being the principal part splits into the following classes

$$\pm x^2 \pm y^2 \pm (z + w_1^2)^2 \pm z^4,$$

$$\pm(x \pm w_1^2)^2 \pm (y \mp w_1^2)^2 \pm z^2 \pm w_1^4,$$

$$\pm(x + w_1^2)^2 \pm y^2 \pm (z + w_1^3)^2 \pm w_1^4.$$

Following similar arguments we arrive increasing orders to the following classes

1. $\mathcal{Y}_l: \ \pm x^2 \pm y^2 \pm z^2 \pm w^l$ where $l \geq 3$;
2. $\mathcal{W}_{k,l}: \ \pm x^2 \pm y^2 \pm (z + w_1^k)^2 \pm z^l, \ l > k$;
3. $\mathcal{Q}_{m,l}: \ \pm (x \pm w_1^m)^2 \pm (y \mp w_1^m)^2 \pm z^2 \pm w_1^l, \ l > m \geq 2$;
4. $\mathcal{V}_{m,n,l}: \ \pm (x + w_1^m)^2 \pm y^2 \pm (z + w_1^n)^2 \pm w_1^l, \ l > n > m$,

which depends on natural values of m, n, k and k.

The theorem is proven.

The proof of the theorem yields the following

Proposition 5.1.8 *The formulas of quasi cone versal deformations of the simple quasi cone classes are listed as follows:*

1. $\mathcal{P}_2: \quad \pm x^2 \pm y^2 \pm z^2 + \lambda_0 + \lambda_1 z.$

2. $\mathcal{P}_k: \quad \pm x^2 \pm y^2 \pm z^k + \sum_{i=0}^{k-1} \lambda_i z^i, \quad k \geq 3.$

3. $\mathcal{O}_k$: $\pm(x+y)^2 \pm z^2 \pm y^k + \mu x + \gamma z + \sum_{i=0}^{k-1} \lambda_i y^i, \quad k \geq 3.$

4. $\mathcal{S}_{k,m}$: $\pm(x+y^k)^2 \pm z^2 \pm y^m + \gamma z + \sum_{i=0}^{k-1} \mu_i x y^i + \sum_{j=0}^{m-1} \lambda_j y^j, \quad m > k \geq 2.$

5. $\mathcal{Y}_l$: $\pm x^2 \pm y^2 \pm z^2 \pm w_1^l + \sum_{i=0}^{l-2} x w_1^i + \sum_{j=0}^{l-2} y w_1^j + \sum_{p=0}^{l-2} z w_1^p + \sum_{q=0}^{l-2} w_1^q,$ *where* $l \geq 3.$

6. $\mathcal{W}_{k,l}$: $\pm x^2 \pm y^2 \pm (z+w_1^k)^2 \pm w_1^l + \sum_{i=0}^{k-1} x w_1^i + \sum_{j=0}^{k-1} y w_1^j + \sum_{p=0}^{k-1} z w_1^p + \sum_{q=0}^{l-2} w_1^q,$ *where* $l > k.$

7. $\mathcal{Q}_{m,l}$: $\pm (x \pm w_1^m)^2 \pm (y \mp w_1^m)^2 \pm z^2 \pm w_1^l + \sum_{i=0}^{m-1} x w_1^i + \sum_{j=0}^{m-1} y w_1^j + \sum_{p=0}^{m-1} z w_1^p + \sum_{q=0}^{l-2} w_1^q, \; l > m \geq 2.$

8. $\mathcal{V}_{m,n,l}$: $\pm (x + w_1^m)^2 \pm y^2 \pm (z + w_1^n)^2 \pm w_1^l + \sum_{i=0}^{m-1} x w_1^i + \sum_{j=0}^{m-1} y w_1^j + \sum_{p=0}^{n-1} z w_1^p + \sum_{q=0}^{l-2} w_1^q, \; l > n > m.$

Chapter 6

Quasi flag singularites

In this short chapter we outline another useful type of similar non standard classifications. Instead of hypersurfaces we can consider sets of nested smooth submanifolds called flags. We consider two easiest cases.

Consider the space $\mathbb{R}^n = \{w = (x, y, z)\}$, where $x = (x_1, x_2, \dots, x_{n-2}) \in \mathbb{R}^{n-2}$ and $(y, z) \in \mathbb{R}^2$. Also, consider the flag $\mathbb{R}^n \supset P_1 = \{y = 0\} \supset P_2 = \{y = z = 0\}$ and call it *a complete flag.* Let $\mathbb{R}^n \supset P_2 = \{y = z = 0\}$ be a flag defined by a single stratum. It is called *a non-complete flag.*

Definition 6.0.1 Two function-germs $f_0, f_1 : (\mathbb{R}^n, 0) \mapsto \mathbb{R}$ are called pseudo complete flag (pseudo non-complete flag, respectively) equivalent if there exists a diffeomorphism $\Theta : (\mathbb{R}^n, 0) \mapsto (\mathbb{R}^n, 0)$ such that $f_1 \circ \Theta = f_0$ and if m is a critical point of f_0 and it belongs to the flag $P_1 \supset P_2$ (P_2, respectively) then $\Theta(m)$ also belongs to the flag $P_1 \supset P_2$ (P_2, respectively) and vice versa.

Remark: Notice that the diffemorphism Θ does not need to preserve the flag but need to shift the critical points lying on components of the flag along the components.

Suppose we have a family f_t of function germs which are pseudo-equivalent such that $f_t \circ \Theta_t = f_0, t \in [0,1]$, where $\Theta_t : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a family of smooth diffeomorphisms. Then the derivative equation takes the form:

$$-\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial x}\dot{X} + \frac{\partial f_t}{\partial y}\dot{Y} + \frac{\partial f_t}{\partial z}\dot{Z},$$

and the components of the vector field $v = \dot{X}\frac{\partial}{\partial x} + \dot{Y}\frac{\partial}{\partial y} + \dot{Z}\frac{\partial}{\partial z}$ satisfy the following condition: for pseudo complete flag equivalence:

$$\dot{X} \in \mathbf{C}_{x,y,z,t} \quad , \quad \dot{Y} \in \{y\mathbf{C}_{x,y,z,t} + Rad\{I\}\},$$
$$\dot{Z} \in \{y\mathbf{C}_{x,y,z,t} + z\mathbf{C}_{x,y,z,t} + Rad\{I\}\},$$

and for pseudo non-complete flag equivalence:

$$\dot{X} \in \mathbf{C}_{x,y,z,t} \quad , \quad \dot{Y} \in \{y\mathbf{C}_{x,y,z,t} + z\mathbf{C}_{x,y,z,t} + Rad\{I\}\},$$
$$\dot{Z} \in \{y\mathbf{C}_{x,y,z,t} + z\mathbf{C}_{x,y,z,t} + Rad\{I\}\},$$

where $I = \left\{\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}\right\}$.

As usual we need to replace $Rad\{I\}$ by the ideal I. Hence we get the improved definition:

Definition 6.0.2 Two function germs $f_0, f_1 : (\mathbb{R}^n, 0) \mapsto \mathbb{R}$ are called quasi complete flag (respectively, quasi non-complete flag) equivalent, if they are pseudo complete flag equivalent (respectively, pseudo non-complete flag) and there is a family f_t of function germs which continuously depends on parameter $t \in [0, 1]$ and a continuous piece-wise smooth family of diffeomorphisms $\Theta_t : \mathbb{R}^n \mapsto \mathbb{R}^n$ depending on parameter $t \in [0, 1]$ such that $f_t \circ \Theta_t = f_0, \Theta_0 = id$ and:

1. For the complete flag case:

$$\dot{X} \in \mathbf{C}_{x,y,z,t}, \; \dot{Y} \in \{y\mathbf{C}_{x,y,z,t} + I\}, \; \dot{Z} \in \{y\mathbf{C}_{x,y,z,t} + z\mathbf{C}_{x,y,z,t} + I\}.$$

2. For non-complete flag case:

$$\dot{X} \in \mathbf{C}_{x,y,z,t}, \; \dot{Y} \in \{y\mathbf{C}_{x,y,z,t} + z\mathbf{C}_{x,y,z,t} + I\}, \; \dot{Z} \in \{y\mathbf{C}_{x,y,z,t} + z\mathbf{C}_{x,y,z,t} + I\}.$$

The family Θ_t will be called admissible for the family f_t.

6.1 The classification of simple quasi flag singularities

The quasi complete flag singularities outside the flag $P_1 \supset P_2$ coincide with standard right ones. Hence, the classes A_k, D_k and E_k form the simple quasi complete flag classes in this case. If the function germ has a critical point on $P_1 - P_2$, then the quasi complete flag equivalence coincides with the quasi boundary equivalence. Hence, the simple quasi boundary classes B_k and $F_{k,l}$ form the simple quasi complete flag singularities.

Similarily, the classification of the critical points outside P_2 with respect to quasi non-complete flag equivalence coincide with the classes A_k, D_k and E_k.

Thus, in what follows we shall consider the case when the critical point lies in the intersection of the spaces P_1 and P_2 and the case when the critical points lying on P_2 to discuss the remaining quasi complete flag singularities and quasi non-complete flag singularities, respectively.

Let $f : (\mathbb{R}^n, 0) \mapsto \mathbb{R}$ be a function germ with a critical point at the origin. Recall that the quasi complete flag tangent space takes the form:

$$TQCF_f = \sum_{i=1}^{n-2} \frac{\partial f}{\partial x_i} A_i + \frac{\partial f}{\partial y} \left\{ yB_1 + \frac{\partial f}{\partial y} D_1 + \frac{\partial f}{\partial z} D_2 \right\}$$

$$+ \frac{\partial f}{\partial z} \left\{ zB_2 + yB_3 + \frac{\partial f}{\partial y} D_3 + \frac{\partial f}{\partial z} D_4 \right\},$$

for arbitrary smooth functions A_i, B_j and D_j.

The quasi non-complete flag tangent space is given by the formula

$$TQNF_f = \sum_{i=1}^{n-2} \frac{\partial f}{\partial x_i} A_i + \frac{\partial f}{\partial y} \left\{ yB_1 + zB_2 + \frac{\partial f}{\partial y} D_1 + \frac{\partial f}{\partial z} D_2 \right\}$$

$$+ \frac{\partial f}{\partial z} \left\{ yB_3 + zB_4 + \frac{\partial f}{\partial y} D_3 + \frac{\partial f}{\partial z} D_4 \right\},$$

for arbitrary smooth functions A_i, B_j and D_j.

Notice that the quasi corner tangent space with respect to the corner $\{yz = 0\}$ is contained in the quasi complete (non-complete) flag tangent space.

Denote by $f^*(x) = f_{|y=z=0}$, the restriction of the function f to the x coordinates subspace. Denote by r^* the rank of the second differential $d_0^2 f^*$ at the origin and set $c = n - 2 - r^*$. Let r be the rank of the second differential $d_0^2 f$ and $k = n - r$ the respective corank.

The quasi partially fixed tangent space is contained in the quasi complete (non-complete) flag tangent space. Hence, we use Lemmas on partially fixed equivalence to obtain prenomal forms up to quasi complete (non-complete) equivalence.

The following Lemma describes the main prenormal forms of quasi complete and non-complete flag singularities.

Lemma 6.1.1 *1. If $k = 0$, then f is quasi complete(non-complete) flag equivalent to $\sum_{i=1}^{n-2} \pm x_i^2 \pm y^2 \pm z^2$.*

2. If $k = 1$, then f is quasi complete(non-complete) flag equivalent to $\sum_{i=1}^{n-2} x_i^2 + g(y, z)$, with rank $d_0^2 g$ is 1 or to $\sum_{i=2}^{n-2} \pm x_i^2 \pm y^2 \pm z^2 + g(x_1, y, z)$ where $g \in \mathcal{M}^3_{x_1,y,z}$.

3. If $k = 2$, the f is quasi complete(non-complete) equivalent to the germ $G_1 = \sum_{i=3}^{n-2} \pm x_i^2 + g_2(y, z) + \widetilde{g}(x_1, x_2, y, z)$ where g_2 is non-degenerate quadratic form and $\widetilde{g} \in \mathcal{M}^3_{x_1,x_2,y,z}$, or $G_2 = \sum_{i=2}^{n-2} \pm x_i^2 + g_2(y, z) + \widetilde{g}(x_1, y, z)$ where g_2 is degenerate quadratic form of corank one and $\widetilde{g} \in \mathcal{M}^3_{x_1,y,z}$, or $G_3 = \sum_{i=1}^{n-2} \pm x_i^2 + \widetilde{g}(y, z)$ where $\widetilde{g} \in \mathcal{M}^3_{y,z}$.

Proof. The Lemma is proven using Lemmas 1.3.13 and 1.3.14. ∎

We start with classifying quasi complete flag singularities.

Lemma 6.1.2 *If $k \geq 2$, then f is non-simple with respect to quasi complete flag equivalence.*

Proof. As $k \geq 2$, Lemmas 1.3.13 and 1.3.14 yields that f is quasi complete flag equivalent to one of the following function germs:

1. $F_1 = \sum_{i=k+1}^{n-2} \pm x_i^2 + f_2(y,z) + \widetilde{f}(x_1, x_2, \ldots, x_k, y, z)$ where h is non-degenerate form, or
2. $F_2 = \sum_{i=k}^{n-2} \pm x_i^2 + f_2(y,z) + \widetilde{f}(x_1, x_2, \ldots, x_{k-1}, y, z)$ where h is degenerate form of corank 1, or
3. $F_3 = \sum_{i=k-1}^{n-2} \pm x_i^2 + \widetilde{f}(x_1, x_2, \ldots, x_{k-2}, y, z)$ where $\widetilde{f} \in \mathcal{M}^3_{x_1, x_2, \ldots, x_{k-2}, y, z}$.

Consider the germ F_3. The tangent space to the orbit at $\widetilde{f}$ is given as follows:

$$TQCF_{\widetilde{f}} = \sum_{i=1}^{k-2} \frac{\partial \widetilde{f}}{\partial x_i} A_i + \frac{\partial \widetilde{f}}{\partial y} \left\{ yB_1 + \frac{\partial \widetilde{f}}{\partial y} D_1 + \frac{\partial \widetilde{f}}{\partial z} D_2 \right\} + \frac{\partial \widetilde{f}}{\partial z} \left\{ zB_2 + yB_3 + \frac{\partial \widetilde{f}}{\partial y} D_3 + \frac{\partial \widetilde{f}}{\partial z} D_4 \right\}.$$

The cubic terms in the tangent space are obtained from $\sum_{i=1}^{k-2} \frac{\partial \widetilde{f}}{\partial x_i} A_i$, $y \frac{\partial \widetilde{f}}{\partial y} B_1$, $z \frac{\partial \widetilde{f}}{\partial z} B_2$ and $y \frac{\partial \widetilde{f}}{\partial z} B_3$. These subspaces form together a $k(k-2)+3$-dimensional subspace which is less than the dimension $M = \frac{(k+2)(k+1)k}{6}$ of all homogeneous cubic terms of the variables: $x_1, x_2, \ldots, x_{k-2}, y$ and z. Hence, the germ F_3 is non-simple with respect to quasi complete flag equivalence.

The germ $f_2(y,z) + \widetilde{f}$ in F_2 can be written as $f_2(y,z) + \widetilde{f} = \pm(ay + bz)^2 + \widetilde{f}(x_1, x_2, \ldots, x_{k-1}, y, z)$. The germ F_2 is adjacent to the germ $G = \pm(ay + bz + \delta x_{k-1})^2 + \widetilde{f}(x_1, x_2, \ldots, x_{k-1}, y, z)$ for sufficiently small δ. By stabilization lemma, G is quasi complete flag equivalent to the $\widetilde{G} = \pm x_{k-1}^2 + \hat{f}(x_1, x_2, \ldots, x_{k-2}, y, z)$. The germ $\widetilde{G}$ is non simple with respect to quasi complete flag equivalence.

The germ F_1 can take the form $F_1 = ay^2 + bz^2 + \widetilde{f}(x_1, x_2, \ldots, x_k, y, z)$. The germ F_1 is adjacent to the germ $G = a(y + \delta_1 x_k)^2 + bz^2 + \widetilde{f}(x_1, x_2, \ldots, x_k, y, z)$ for sufficiently small δ_1. Stabilization lemma yields that G is quasi complete flag equivalent to the germ $\widetilde{G} = \pm x_k^2 + bz^2 + \hat{f}(x_1, x_2, \ldots, x_{k-1}, y, z)$. If we repeat the previous argument as before, we see that the germ $\widetilde{G}$ is adjacent to the germ of the form $H = \pm x_k^2 \pm x_{k-1}^2 + \hat{f}(x_1, x_2, \ldots, x_{k-2}, y, z)$. The germ H is non simple.

∎

Lemma 6.1.3 *1. The function germ $g(y,z)$ of corank 1 is quasi complete flag equivalent to either the germ $\pm z^2 + \vartheta_0(y)$, where $\vartheta \in \mathcal{M}_y^3$ or to the germ $\pm y^2 + y\vartheta_1(z) + \vartheta_2(z)$, where $\vartheta_1 \in \mathcal{M}_z^2$ and $\vartheta_2 \in \mathcal{M}_z^3$.*

2. The function germ $\pm y^2 \pm z^2 + g(x_1, y, z) \in \mathcal{M}_{x_1,y,z}^3$ is quasi complete flag equivalent to the germ: $\pm y^2 \pm z^2 + yh_1(x_1) + zh_2(x_1) + h_3(x_1)$ where $h_1, h_2 \in \mathcal{M}_{x_1}^2$ and $h_3 \in \mathcal{M}_{x_1}^3$.

Proof. In fact, the reduction can be done by restricting the quasi flag complete tangent space to the subspace which coincides with the quasi corner tangent space with respect to the corner $\{yz = 0\}$. Hence, up to permutation between y and z, the function germ $g(y,z)$ is quasi complete flag equivalent to either $G(y,z) = \pm(z \pm y)^2 + \vartheta(y)$, where $\vartheta \in \mathcal{M}_y^3$ or to the germ $\widetilde{G} = \pm y^2 + y\vartheta_1(z) + \vartheta_2(z)$ where $\vartheta_1 \in \mathcal{M}_z^2$ and $\vartheta_2 \in \mathcal{M}_z^3$.

Now consider the quadratic form $G_2 = \pm(z \pm y)^2$ of the function germ $G(y,z)$. Take the family $G_t = \pm(z \pm ty)^2$, joining G_2 and $G_0 = \pm z^2$. The homological equation takes the form:

$$\pm 2y(z \pm ty) = \pm 2t(z \pm ty)\{yB_1 + (z \pm ty)D_1\} \pm 2(z \pm ty)\{zB_2 + yB_3 + (z \pm ty)D_2\}.$$

Equivalently, this can be written as:

$$\pm 2y = \pm 2t\{yB_1 + (z \pm ty)D_1\} \pm 2\{zB_2 + yB_3 + (z \pm ty)D_2\}.$$

Thus, the homological equation can be solved by setting $B_1 = B_2 = D_1 = D_2 = 0$ and $B_3 = \pm 1$. Hence, all G_t are quasi complete flag equivalent. Thus the germ G can be reduced further to the form $\widetilde{G} = \pm z^2 + \varphi_0(y)$.

∎

Theorem 6.1.4 *A simple (with respect to quasi complete flag equivalence) function germ $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ with a critical point at the origin is quasi complete flag equivalent (up to addition with a quadratic form in some extra variables x) to the germ of one of the classes :*

1. $\mathbb{B}_k : \pm z^2 \pm y^k,\quad k \geq 2\quad k+1,$
2. $\mathbb{F}_{k,m} : \pm(y \pm z^k)^2 \pm y^m\quad m > k, m \geq 3, k \geq 2\quad m+k,$
3. $\mathbb{H}_{m,n,k} : \pm(y \pm x_1^m)^2 \pm (z \pm x_1^n)^2 \pm x_1^k,\quad k > n \geq m\quad m+n+k-1,$

and therefore has corank 1 of the second differential. The orbit codimension in the space of germs is shown in the right column.

Remark: All germs with corank greater or equal 2 of the second differential are non-simple.

Proof of Theorem 6.1.4

The proof of the theorem is based on restricting the quasi flag complete tangent space to the quasi corner tangent subspace and Lemma 6.1.3. Thus, we get the classes:

1. $F_1(y, z) = \pm z^2 \pm y^k,\quad k \geq 2,$
2. $F_2(y, z) = \pm(y \pm z^k)^2 \pm y^m\quad m > k, m \geq 3, k \geq 2,$
3. $F_3(y, z, x_1) = \pm(y \pm x_1^m)^2 \pm (z \pm x_1^n)^2 \pm x_1^k,\quad k \geq n \geq m.$

The quasi complete flag tangent space to orbits at F_2 is

$$\begin{aligned} TQCF_{F_2} &= [\pm 2(y \pm z^k) \pm my^{m-1}] \left\{yB_1 + [\pm 2(y \pm z^k) \pm my^{m-1}]B_2\right. \\ &+ \left.[\pm 2kz^{k-1}(y \pm z^k)]B_3\right\} + [\pm 2kz^{k-1}(y \pm z^k)] \left\{zD_1 + yD_2\right. \\ &+ \left.[\pm 2(y \pm z^k) \pm my^{m-1}]D_3 + [\pm 2kz^{k-1}(y \pm z^k)]D_4\right\}. \end{aligned}$$

or equivalently

$$TQCF_{F_2} = [\pm 2(y \pm z^k) \pm my^{m-1}]\{y\widetilde{B}_1 + z^k\widetilde{B}_2\} + [\pm 2kz^{k-1}(y \pm z^k)]\{y\widetilde{B}_3 + \widetilde{B}_4\}.$$

Notice that the space $TQCF_{F_2}$ coincides with the quasi corner tangent subspace at F_2. Similarly, the space $TQCF_{F_3}$ coincides with the quasi corner tangent subspace at the same germ F_3. Thus, the germs F_1, F_2 and F_3 form the list of simple quasi complete flag classes.

Proposition 6.1.5 *The quasi complete flag miniversal deformations of the simple classes from the theorem can be chosen in the following form:*

1. $\mathbb{B}_k : \pm z^2 \pm y^k + \lambda_1 z + \sum_{i=1}^{k-1} \beta_i y^i$,
2. $\mathbb{F}_{k,m} : \pm(y \pm z^k + \sum_{i=0}^{k-1} \lambda_i z_i)^2 \pm z^m + \sum_{j=0}^{m-1} \mu_j z^j$,
3. $\mathbb{H}_{m,n,k} : \pm(y \pm x_1^m + \sum_{i=0}^{m-1} \lambda_i x_1^i)^2 + \pm(z \pm x_1^n + \sum_{j=0}^{n-1} \mu_j x_1^j)^2 \pm x_1^k + \sum_{l=0}^{k-2} \beta_l x_1^l$.

Proof. As the quasi complete flag tangent space to the classes $\mathbb{F}_{k,m}$ and $\mathbb{H}_{m,n,k}$ coincides with the quasi corner tangent subspace at the same germs, the result follows.

For the classes $\mathbb{B}_k$, let $F_1(y,z) = \pm z^2 \pm y^k$. Then, the the quasi complete flag tangent space to the orbit at F_1 takes the form:

$$TQCF_{F_1} = \pm k y^{k-1}\{yB_1 + zB_2\} \pm 2z\{zB_4 + yB_4\}.$$

Thus, clearly the monomials $1, z, y, y^2, \ldots, y^{k-1}$ form a basis for the local algebra $\mathbf{C}_w/TQCF_{F_1}$.

∎

We turn now to classifying quasi non-complete flag singularities.

Lemma 6.1.6 *1. If $k = 2$ and f is quasi non-complete flag equivalent to one of the following germs,*

(a) $F_1 = \sum_{i=2}^{n-2} \pm x_i^2 + g_2(y,z) + \widetilde{g}(x_1, y, z)$, *where g_2 is a degenerate quadratic form of corank 1 and $\widetilde{g} \in \mathcal{M}^3_{x_1,y,z}$, or*

(b) $F_2 = \sum_{i=3}^{n-2} \pm x_i^2 + g_2(y,z) + \widetilde{g}(x_1, x_2, y, z)$ *where g_2 is non-degenerate quadratic form and $\widetilde{g} \in \mathcal{M}_{x_1,x_2,y,z}$,*

then f is non-simple.

2. *If $k \geq 3$, then f is non simple.*

Proof. Consider the germ $F_1 = \sum_{i=2}^{n-2} \pm x_i^2 + g_2(y,z) + \widetilde{g}(x_1,y,z)$ with g_2 is a degenerate quadratic form in y and z and $\widetilde{g} \in \mathcal{M}^3_{x_1,y,z}$. Let $G = g_2(y,z) + \widetilde{g}(x_1,y,z)$. Then, clearly the germ g_2 can be reduced to $\pm y^2$, up to permutation of y and z. Thus, The quasi non-complete flag tangent space to the orbit at $\widetilde{G} = \pm y^2 + g(x_1,y,z)$ takes the form:

$$\begin{aligned} TQNF_{\widetilde{G}} &= \frac{\partial g}{\partial x_1}A_1 + (\pm 2y + \frac{\partial g}{\partial y})\left\{yB_1 + zB_2 + (\pm 2y + \frac{\partial g}{\partial y})B_3 + \frac{\partial g}{\partial z})B_4\right\} \\ &+ \frac{\partial g}{\partial z}\left\{yC_1 + zC_2 + (\pm 2y + \frac{\partial g}{\partial y})D_3 + \frac{\partial g}{\partial z})D_4\right\} \end{aligned}$$

Comparing the dimension of cubic terms in x_1 and z in $TQNF_G$ with the dimension of all homogeneous cubic terms, we see that germ is non-simple.

Let $F_2 = \sum_{i=3}^{n-2} \pm x_i^2 + g_2(y,z) + \widetilde{g}(x_1,x_2,y,z)$ with g_2 is non-degenerate quadratic form in y and z and $\widetilde{g} \in \mathcal{M}^3_{x_1,x_2,y,z}$. Set $G = g_2(y,z) + \widetilde{g}(x_1,x_2,y,z)$. Then, G can be reduced to the form $\widetilde{G} = \pm y^2 \pm z^2 + g(x_1,x_2,y,z)$. The germ $\widetilde{G}$ is adjacent to the germ $H(x_1,y,z) = \pm y^2 \pm (z + \delta x_2)^2 + g_3(x_1,x_2,y,z)$, for sufficiently small δ. Stabilization Lemma yields that the germ H can be reduced to the form: $\widetilde{H} = \pm y^2 \pm x_2^2 + h(x_1,y,z)$. Similar argument as the first part of the proof we see that $\widetilde{H}$ is non-simple.

Now, suppose that $k \geq 3$. Then, using Lemmas on partially fixed equivalence, the germ f is quasi non-complete flag equivalent to one of the following function germs:

1. $F_1 = \sum_{i=k+1}^{n-2} \pm x_i^2 + g_2(y,z) + \widetilde{g}(x_1,x_2,\ldots,x_k,y,z)$ where g_2 is non-degenerate quadratic form and $\widetilde{g} \in \mathcal{M}^3_{x_1,x_2,\ldots,x_k,y,z}$, or
2. $F_2 = \sum_{i=k}^{n-2} \pm x_i^2 + g_2(y,z) + \widetilde{g}(x_1,x_2,\ldots,x_{k-1},y,z)$ where g_2 is a degenerate quadratic form of corank 1 and $\widetilde{g} \in \mathcal{M}^3_{x_1,x_2,\ldots,x_{k-1},y,z}$, or
3. $F_3 = \sum_{i=k-1}^{n-2} \pm x_i^2 + \widetilde{g}(x_1,x_2,\ldots,x_{k-2},y,z)$ where $\widetilde{g} \in \mathcal{M}^3_{x_1,x_2,\ldots,x_{k-2},y,z}$.

Consider the germ F_3. The tangent space to the orbit at $\widetilde{g}$ is given as follows:

$$TQNF_{\widetilde{g}} = \sum_{i=1}^{k-2} \frac{\partial \widetilde{g}}{\partial x_i} A_i + \frac{\partial \widetilde{g}}{\partial y} \left\{ yB_1 + zB_2 + \frac{\partial \widetilde{g}}{\partial y} B_3 + \frac{\partial \widetilde{g}}{\partial z} B_4 \right\}$$
$$+ \frac{\partial \widetilde{g}}{\partial z} \left\{ zB_5 + yB_4 + \frac{\partial \widetilde{g}}{\partial y} B_6 + \frac{\partial \widetilde{g}}{\partial z} B_7 \right\}.$$

If we compare the dimension of cubic terms in $x_1, x_2, \dots x_{k-2}, y$ and z which belong to $TQNF_{\widetilde{g}}$ with the dimension of all homogeneous cubic terms, we conclude that germ is non-simple.

The cubic terms in $x_1, x_2, \dots x_{k-2}, y$ and z which belong to $TQNF_{\widetilde{g}}$ are obtained from $\sum_{i=1}^{k-2} \frac{\partial \widetilde{g}}{\partial x_i} A_i$, $y \frac{\partial \widetilde{g}}{\partial y} B_1$,$y \frac{\partial \widetilde{g}}{\partial y} B_2$, $z \frac{\partial \widetilde{g}}{\partial z} B_4$ and $y \frac{\partial \widetilde{g}}{\partial z} B_5$. These subspaces form together a $k(k-2)+4$-dimensional subspace which is less than the dimension $M = \frac{(k+2)(k+1)k}{6}$ of all homogeneous cubic terms of the variables: $x_1, x_2, \dots, x_{k-2}, y$ and z. Hence cubic terms can not belong to finitely many orbits.

The germ F_2 can be written in the form $\widetilde{F}_2 = \pm(ay+bz)^2 + \widetilde{f}(x_1, x_2, \dots, x_{k-1}, y, z)$. The germ $\widetilde{F}_2$ is adjacent to the germ $H = \pm(ay+bz+\delta x_{k-1})^2 + \widetilde{f}(x_1, x_2, \dots, x_{k-1}, y, z)$, for sufficiently small δ. By stabilization lemma, the germ H is quasi non-complete flag equivalent to the germ $\pm x_{k-1}^2 + \hat{f}(x_1, x_2, \dots, x_{k-2}, y, z)$ which is non simple by the previous argument.

Finally, the germ F_1 can be reduced to $\widetilde{F}_3 = \pm y^2 \pm z^2 + \widetilde{f}(x_1, x_2, \dots, x_k, y, z)$. The germ $\widetilde{F}_3$ is adjacent to the germ $H = \pm(y+\delta x_k)^2 \pm z^2 + \widetilde{f}(x_1, x_2, \dots, x_k, y, z)$, for sufficiently small δ. The germ H is quasi non-complete flag equivalent to the germ $\widetilde{H} = \pm x_k^2 \pm z^2 + \hat{f}(x_1, x_2, \dots, x_{k-1}, y, z)$. If we repeat the previous argument as before, we see that the germ $\widetilde{F}_3$ is adjacent to the germ $\pm x_k^2 \pm x_{k-1}^2 + \hat{f}(x_1, x_2, \dots, x_{k-2}, y, z)$ which is non simple.

∎

Lemma 6.1.7 *The germ $G = \pm y^2 \pm z^2 + g(x_1, y, z)$, where $g \in \mathcal{M}^3_{x_1,y,z}$ is quasi non-complete flag equivalent to the germ $\widetilde{G} = \pm y^2 \pm z^2 + yh_1(x_1) + zh_2(x_1) + h_3(x_1)$ where $h_1, h_2 \in \mathcal{M}^2_{x_1}$ and $h_3 \in \mathcal{M}^3_{x_1}$*

Proof. The Lemma is proven by restricting the quasi non-complete flag tangent space to the quasi corner subspace. ∎

Theorem 6.1.8 *A simple (with respect to quasi* <u>*non-complete*</u> *flag equivalence) function* $f : (\mathbb{R}^n, 0) \mapsto \mathbb{R}$ *with a critical point at the origin is quasi non-complete flag equivalent to the germ of one of the classes (up to permutation of* y *and* z*, and stabilization in* x*),*

1. $g(y,z)$; $\quad g \in A_k : y^2 + z^{k+1}; k \geq 1, \quad D_k : z^2y + y^{k-1}; k \geq 4,$ $E_6 : z^3 + y^4, \quad E_7 : z^3 + zy^3, \quad E_8 : z^3 + y^5,$
2. $\mathbf{H}_{m,n,k} : \pm(y \pm x_1^m)^2 \pm (z \pm x_1^n)^2 \pm x_1^k, k \geq m \geq n \qquad k + m + n - 1.$

Remarks:

1. The codimension of the classes $g(y,z)$ is equal to the codimension of the standard Arnold's singularities: A_k, D_k and E_k plus one.

2. The codimension of the classes $\mathbf{H}_{m,n,k}$ is equal $m + k + n - 1$.

Proof of Theorem 6.1.8

Lemmas 6.1.6,6.1.7, 6.1.1 and 6.1.7 yield that we need to consider the germs of the form $G_1 = g(y,z)$, with rank $d_0^2 g$ is 0 or 1 and $G_2 = \pm y^2 \pm z^2 + yh_1(x_1) + zh_2(x_1) + h_3(x_1)$ where $h_1, h_2 \in \mathcal{M}_{x_1}^2$ and $h_3 \in \mathcal{M}_{x_1}^3$ to discuss simple quasi non-complete flag singularities.

Let $\widetilde{f}_t : (\mathbb{R}^2, 0) \mapsto (\mathbb{R}, 0)$ be a function germ in the variables y and z. The tangent space to the orbit at the family $\widetilde{f}_t$ is:

$$TQNF_{\widetilde{f}_t} = \frac{\partial \widetilde{f}_t}{\partial y}\left\{yA_1 + zA_2 + \frac{\partial \widetilde{f}_t}{\partial y}A_3 + \frac{\partial \widetilde{f}_t}{\partial y}A_4\right\} + \frac{\partial \widetilde{f}_t}{\partial z}\left\{yB_1 + zB_2 + B_3\frac{\partial \widetilde{f}_t}{\partial y}B_3 + \frac{\partial \widetilde{f}_t}{\partial y}B_4\right\}.$$

Notice that $TQNF_{\widetilde{f}_t}$ coincides with the module $\left\{\frac{\partial \widetilde{f}_t}{\partial y}, \frac{\partial \widetilde{f}_t}{\partial z}\right\}$ over $\mathcal{M}_{y,z}$. This module is the standard tangent space with respect to right equivalence without constants terms. These terms does not affect on the classification of standard simple classes but makes difference in calculating the codimensions. Hence we get the simple classes A_k, D_k, E_6, E_7, and E_8.

Now consider the reduced germ which is obtained in Lemma 6.1.7. Again, if we restrict the quasi non-complete flag tangent space to the quasi corner subspace, then the classes: $\mathbf{H}_{m,n,k} : \pm(y \pm x_1^m)^2 \pm (z \pm x_1^n)^2 \pm x_1^k$, with $k > m \geq n$ form the simple quasi non-complete classes in this case. In fact, the quasi non-complete flag tangent space of the classes $\mathbf{H}_{m,n,k}$ coincides with the quasi corner tangent subspace. The theorem is proven.

6.2 The caustics and bifurcation diagrams of simple quasi complete flag singularities

The bifurcation diagrams of quasi complete flag singularities consists of three strata. The first stratum is the ordinary one given by the equations: $\frac{\partial F}{\partial w_i} = 0$ and $F = 0$. The second stratum is a subset of the first one which satisfies an extra equation: $y = 0$. The third stratum is a subset of the second one and satisfies an extra equation: $z = 0$.

Similarly, the caustics of simple quasi flag complete singularities consist of three strata. The first one is the ordinary one. The second and third ones are the projections to the reduced base of the deformation of the second and third strata of the bifurcation diagrams described above.

Proposition 6.2.1 • *The first stratum of the bifurcation diagram (caustic) of any simple quasi flag complete singularity is a cylinder over the generalized swallowtail.*

• *In particular, the first stratum of the bifurcation diagram of the class $\pm z^2 \pm y^3$ is the product of a cusp and a plane in $\mathbb{R}^4$. The second stratum is a smooth surface inside the first one. The third stratum is a line inside the second stratum. The second and third strata are tangent to the cuspidal edge.*

• *The first and second strata of the caustic of the class $\pm z^2 \pm y^3$ are smooth tangent surfaces in $\mathbb{R}^3$ and their intersection is exactly the third component.*

• *The caustic of $\pm z^2 \pm y^k, k \geq 3$ is a union of a cylinder over a generalized swallowtail and smooth hypersurafaces and $(k-2)$-dimensional space. The second and third strata are tangent to the first one.*

• *The caustic of the class* $\pm(y\pm z^k)^2\pm z^m$ *is a union of a cylinder over a generalized swallowtail and a generalized Whitney umbrella times a line and a* $(k+m-3)$-*dimensional space.*

• *The caustic of the class* $\pm(y\pm x_1^m)^2\pm(z\pm x_1^n)^2\pm x_1^k$ *is a union of a cylinder over a generalized swallowtail and a generalized Whitney umbrella and intersection of two generalized Whitney umbrellas.*

Proof. The proof is based on the proof of describing the bifurcation diagrams and caustics of simple quasi corner singularities. Mind that, we need to add an extra equation to one of the strata of the bifurcation diagrams (or caustics) of simple quasi complete flag singularities. Thus, the proof for the first five statements are straightforward.

For the classes $\mathbb{H}_{m,n,k}$, consider the miniversal deformation

$$F=\pm(y\pm x_1^m+\sum_{i=0}^{m-1}\lambda_i x_1^i)^2+\pm(z\pm x_1^n+\sum_{j=0}^{n-1}\mu_j x_1^j)^2\pm x_1^k+\sum_{l=0}^{k-2}\beta_l x_1^l.$$

We will construct the third stratum of the caustics of F.

Let $P_1(y,z,x_1,\lambda)=y\pm x_1^m+\sum_{i=0}^{m-1}\lambda_i x_1^i$, $P_2(y,z,x_1,\mu)=z\pm x_1^n+\sum_{j=0}^{n-1}\mu_j x_1^j$ and $Q=\pm x_1^k+\sum_{l=0}^{k-2}\beta_l x_1^l$. Thus, we get the dervatives $\frac{\partial F}{\partial y}=\pm 2P_1(y,z,x_1,\lambda)=0$ and $\frac{\partial F}{\partial z}=\pm 2P_2(y,z,x_1,\mu)=0$. Hence, we obtain $\frac{\partial Q}{\partial x_1}=Q(x_1,\beta)=0$. Therefore, the equations $P_1(0,0,x_1,\lambda)=0$, $P_2(0,0,x_1,\mu)=0$ and $\widetilde{Q}(x_1,\beta)=0$ define the third component of the caustic which is an intersection of a cylinder over the generalized Whitney umbrella $W_1=\{P_1(0,0,x_1,\lambda)=0,\widetilde{Q}(x_1,\beta)=0\}$, and the generalized Whitney umbrella $W_2=\{P_2(0,0,x_1,\mu)=0,\widetilde{Q}(x_1,\beta)=0\}$.

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Chapter 7

Applications and invariants

7.1 Lagrangian projections with a border

We recall some standard notions on Lagrangian singularities [1]. All the standard materials in this chapter are from [2]. We srart with basic definitions.

Let M be a smooth manifold. At any point $p \in M$, a k-form α is defined to be an alternating multilinear map

$$\alpha_p : \underbrace{T_pM \times \cdots \times T_pM}_{k} \to R,$$

where T_pM is the tanget space to M at p.

The wedge (or exterior) product of $k-$form α and l- form β is a $(k+l)-$form denoted by $\alpha \wedge \beta$. If $k = l = 1$, then $\alpha \wedge \beta$ is the 2-form whose value at a point p is the alternating bilinear form defined by

$$(\alpha \wedge \beta)_p(v, w) = \alpha_p(v)\beta_p(w) - \alpha_p(w)\beta_p(v),$$

for $v, w \in T_pM$.

Assume that $x_1, x_2, \ldots x_n$ are local coordiantes on M, then the $k-$ form α takes the expression $\alpha = \sum_I f_I dx_i$. Here I stands for a multi-index $(i_1, i_2, \ldots, i_k)$ and $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$.

If f is a 0-form, that is a smooth function, then we define df to be the 1−form

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

Suppose that $\alpha = \sum_I f_I dx_i$ is a k-form and each component f_I is a smooth function, then we defin $d\alpha$ to be $(k+1)-$ form

$$d\alpha = \sum_I df_I \wedge dx_I.$$

The operation d is called exterior differentaion.

Definition 7.1.1 A symplectic form ω on an even dimensional manifold M^{2m} is a closed differential 2-form which is non-degenerate (as a skew-symmetric bilinear form on the tangent space at each point). A manifold M equipped with a symplectic form is called *symplectic.*

The non-degeneracy condition means that for all $p \in M$ there does not exist non-zero $v \in T_pM$ such that $\omega(v, w) = 0$ for all $w \in T_pM$.

The skew-symmetric means that for all $p \in M$ we have $\omega(v, w) = -\omega(w, v)$ for all $v, w \in T_pM$. Recall that in odd dimensions skew-symmetric matrices are not invertible. Since ω is a differential 2-form the skew-symmetric condition implies that M has even dimension. The closed condition means that the exterior derivative of ω is identically zero.

Remarks:

1. In the above definition ω^n is a volume form . This means that $\omega^n = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n}$ is an n-form.

2. If the form ω is exact, $\omega = d\lambda$, the manifold M is called *exact symplectic.*

Example 1: The basic model of a symplectic space is the vector space

$$K = \mathbb{R}^{2n} = \{q_1, \ldots, q_n, p_1, \ldots, p_n\}$$

with the form

$$\lambda = pdq = \sum_{i=1}^{n} p_i dq_i\,, \qquad \omega = d\lambda = dp \wedge dq\,.$$

In these coordinates the form ω is constant. The corresponding bilinear form on the tangent space at a point is given in coordinates (q,p) by the matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Example 2: The cotangent bundle T^*Q with coordinates q in the base Q and p is its dual coordinates on the fibers of thw projection $\pi : T^*Q \to Q$ is a symplectic manifold. Its symplectic structure is $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$.

Definition 7.1.2 A diffeomorphism $\varphi : M_1 \to M_2$ which sends the symplectic structure ω_2 on M_2 to the symplectic structure ω_1 on M_1,

$$\varphi^*\omega_2 = \omega_1 \,,$$

is called a *symplectomorphism* between (M_1, ω_1) and (M_2, ω_2). When the manifolds (M_i, ω_i) coincide, a symplectomorphism preserves the symplectic structure. In particular, it preserves the volume form ω^n.

Definition 7.1.3 A submanifold L of a symplectic manifold M is called *isotropic* if the symplectic form induces the null form on it. That is $\omega_L = 0$.

Example 3: In the basic model example, the plane $q = q_0 = const$ is isotropic, as $\omega = d\lambda = \sum_{i=1}^{n} dp_i \wedge dq_{0_i} = 0$.

Definition 7.1.4 The isotropic submanifolds of the maximal possible dimension (equal to n, the half of the dimension of the symplectic manifold) are said to be Lagrangian.

Example 4: In example 1: The isotropic submanifold is Lagrangian.

Example 5:. A family of functions $F(x, \lambda)$ depending on parameters $(\lambda_1, \ldots, \lambda_n)$ defines a Lagrangian submanifold L in the cotangent bundle T^*Q (see example 2:) by standard Hormander formulas for generating families [1] :

$$L = \left\{ (\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^n : \exists x \in \mathbb{R}^m, \frac{\partial F}{\partial x_i} = 0, \mu = \frac{\partial F}{\partial \lambda} \right\},$$

provided that Morse non-degeneracy condition (the matrix $[\frac{\partial^2 F}{\partial x_i \partial x_j} \quad \frac{\partial^2 F}{\partial x_i \partial \lambda_j}]$ is non-degenerate) holds. The condition guarantees L being a smooth manifold.

Definition 7.1.5 Two family-germs $F_i(x, q)$, $x \in \mathbb{R}^k$, $q \in \mathbb{R}^n$, $i = 1, 2$, at the origin are called $\mathcal{R}$-equivalent if there exists a diffeomorphism $\mathcal{T} : (x, q) \mapsto (X(x, q), q)$ (i.e. preserving the fibration $\mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$) such that $F_2 = F_1 \circ \mathcal{T}$.

Definition 7.1.6 Two family-germs $F_i(x, q)$, $x \in \mathbb{R}^k$, $q \in \mathbb{R}^n$, $i = 1, 2$, at the origin are called $\mathcal{R}_+$-equivalent if there exists a diffeomorphism $\Phi : (x, q) \mapsto (X(x, q), Q(q))$ and a smooth function of parameters $\Theta(q)$ such that $F_2(x, q) = F_1(X(x, q), Q(q)) + \Theta(q)$.

Definition 7.1.7 The family $\Phi(x, y, q) = F(x, q) \pm y_1^2 \pm \ldots, \pm y_m^2$ is called a stabilization of F.

Definition 7.1.8 Two family-germs are called stably $\mathcal{R}$-equivalent if they are $\mathcal{R}$-equivalent to appropriate stabilizations of the same family (in a lower number of variables).

Lemma 7.1.1 [1] *Up to addition of a constant, any two generating families of the same germ L of a Lagrangian submanifold are stably $\mathcal{R}$-equivalent.*

Definition 7.1.9 A fibre bundle $\pi : E^{2n} \to B^n$ is said to be Lagrangian if the space E is equipped with a symplectic form and the fibres are Lagrangian submanifolds.

Examples6: The cotangent bundle $\pi : T^*N \to N, \quad (p,q) \mapsto q$ is Lagrangian.

Definition 7.1.10 Let $\psi : L \to E^{2n}$ be a Lagrangian embedding and $\rho : E^{2n} \to B$ the fibration. The product $\rho \circ \psi : L \to B$ is called a *Lagrangian mapping* (or Lagrangian projection).

Definition 7.1.11 The equivalence of Lagrangian mappings is defined up to fibre-preserving symplectomorphisms of the ambient symplectic space. So a Lagrange equivalence is a commutative diagram

$$\begin{array}{ccccc} L_1 & \xrightarrow{i_1} & E^{2n} & \xrightarrow{\pi} & B \\ \downarrow{\scriptstyle \Theta_1} & & \downarrow{\scriptstyle \Theta_2} & & \downarrow{\scriptstyle \Theta_3} \\ L_2 & \xrightarrow{i_2} & E^{2n} & \xrightarrow{\pi} & B \end{array}$$

where i_1 and i_2 are embeddings , Θ_2 is a symplectomorphism and Θ_1, Θ_3 are diffeomorphisims.

Definition 7.1.12 The set of critical values

$$\Sigma_L = \{q \in B | \exists p : (p,q) \in L, \text{ rank } d(\rho \circ \psi)|_{(p,q)} < n\}$$

form the *caustic* of the Lagrangian mapping $\rho \circ \psi : L \to N$.

Lemma 7.1.2 [1] *Two germs of Lagrangian maps are Lagrangian equivalent if and only if the germs of their generating families are stably $\mathcal{R}_+$ equivalent.*

Definition 7.1.13 A Lagrangian mapping is said to be Lagrangian stable if every nearby Lagrangian is Lagrange equivalent to it.

Singularities of Lagrangian projections (mappings) are essentially the singularities of their generating families treated as families of functions depending on parameters and considered up to right equivalence depending on parameters and addition with functions in parameters. In particular, the caustic $\Sigma(L)$ of Lagrangian submanifold L projection coincides with the stratum of the bifurcation diagram of the generating family $F(x, q)$ which is the collection of parameter q values such that the restriction $F(\cdot, q)$ has a non-Morse critical point.

Stability of Lagrangian projections with respect to symplectomorphisms preserving the fibration structure corresponds to the versality of the generating family with respect to the $\mathcal{R}_+$- equivalence group.

A pair (L, Γ) of a Lagrangian submanifold $L^n \subset M$ and an $(n-1)$-dimensional isotropic variety $\Gamma \subset L$ is called a *Lagrangian submanifold with a border* Γ.

Projection π restricted to L defines the Lagrange mapping ρ of the pair $(L, \Gamma.)$ The caustic of a Lagrange projection with border is the union of the ordinary caustic of L (being the set of critical values of ρ) and the ρ image of the border Γ.

The Lagrange projections of two Lagrange submanifolds with borders (L_i, Γ_i), $i = 1, 2$ are *equivalent* if some symplectomorphism of the ambient space M preserves the π-bundle structure and sends one pair (L_i, Γ_i) to the other.

The notions of stability and simplicity of Lagrangian submanifolds with border with respect to Lagrange equivalence are straightforward.

Locally any Lagrangian submanifold of the cotangent bundle M can be determined by the generating family of functions $F(w, q)$ in variables $w \in \mathbb{R}^n$ and parameters q (satisfying Morse non-degeneracy conditions) according to standard formulas which is given in example 5.

Up to a Lagrange equivalence we may assume that in a vicinity of a base point the tangent space to L has regular projection to the fiber of π and the coordinates p can be taken as coordinates w on the fibers of the source space of the generating family.

Generating family is defined up to $\mathcal{R}_+$ equivalence. So having two Lagrange equivalent pairs (L_i, Γ_i) we can choose generating family for one of those in coordinates

(p,q) and the generating family for the second pair in transformed coordinates $\widetilde{P}(p)$ so that the projection of Γ_1 to p-coordinate subspace coincide with the projection of Γ_2 to the $\widetilde{P}(p)-$ coordinate subspace.

Assume that Γ_i are borders. Rename the coordinates p by w and q by λ. Let $g(w)=0$ be the equation of the border.

Now we get generating families $F_i(w,\lambda)$ for both submanifolds such that the critical points of F_i with respect to variables w at the set $\{g(w)=0\}$ correspond to the Lagrangian border Γ_i.

Hence, Lagrangian equivalence of pairs (L_i,Γ_i), $i=1,2$ gives rise to a $\mathcal{R}_+$ equivalence of the generating families F_i which is a pseudo border equivalence and addition with a function in parameters.

Moreover the following holds

Proposition 7.1.3 *Let (L_t,Γ_t), $t\in[0,1]$ be a family of equivalent pairs of Lagrangian submanifolds with a border, then the respective generating families are* **quasi border** *equivalent up to addition of functions depending on the parameters.*

Proof. We shall prove the statement when Γ is the union of two regular components I_1, I_2 which are mutually transversal in L.

Consider the family of Lagrange equivalences of L_t, joining L_1 and L_2. Construct a family of respective generating families $f_t(w,\lambda)$ of L_t which all are $\mathcal{R}_+$ equivalent:

$$f_t\left(\widetilde{w}_t(w,\lambda),\Lambda_t(\lambda)\right)=f_0(w,\lambda)+\Psi_t(\lambda)$$

and the critical points subsets $\left\{\frac{\partial f_t}{\partial w}=0\right\}$ correspond to the Lagrangian submanifolds L_t.

Notice that the generating families are pseudo equivalent since the critical points on the corner remain on the corner. Differentiating the previous equation by t provides

$$-\frac{\partial f_t}{\partial t}=\frac{\partial f_t}{\partial w}\dot{w}_t+\frac{\partial f_t}{\partial\Lambda}\dot{\Lambda}_t+\dot{\Psi}_t(\lambda),$$

where $\dot{w}=(\dot{x},\dot{y},\dot{z}_1,\dots\dot{z}_{n-2})$.

The components $(\dot{x}, \dot{y})$ of the vector field vanish at the critical points lying on the corner $xy = 0$. Since the subsets $\{x = 0, \frac{\partial f_t}{\partial w}\}$ and $\{y = 0, \frac{\partial f_t}{\partial w}\}$ are regular (due to Morse non-degeneracy conditions), then by Hadamard lemma:

$$\dot{x} = xA(w, \lambda) + \sum \frac{\partial f_t}{\partial w_i} B_i(w, \lambda),$$

$$\dot{y} = y\widetilde{A}(w, \lambda) + \sum \frac{\partial f_t}{\partial w_i} \widetilde{B}_i(w, \lambda)$$

for some smooth functions $A, \widetilde{A}, B_i, \widetilde{B}_i$. This yields that all f_t all are quasi corner + equivalent as required.

∎

This result and the classification of simple quasi border singularities imply the following theorem

Theorem 7.1.4 *1. A germ (L, Γ) is stable if and only if its arbitrary generating family is versal with respect to quasi border equivalence and addition with functions in parameters.*

2. Any stable and simple projection of a Lagrangian submanifold with a border is symplectically equivalent to the projection determined by the generating families which are quasi border reduced-versal deformations of the simple quasi border classes. In particular, any stable and simple projection of a Lagrangian submanifold with a boundary is symplectically equivalent to the projection determined by one of the following generating families which are quasi- R+-versal deformations of the classes from the theorem 2.1.6.

- B_2 : $\pm x_1^2 \pm y^2 + \lambda_1 y$.
- B_k : $x_1^2 \pm y^k + \sum_{i=1}^{k-1} \lambda_i y^i, \quad k \geq 3$.
- $F_{k,l}$: $\pm(y \pm x_1^k)^2 \pm x_1^l + \sum_{i=1}^{l-2} \lambda_i x_1^i + \sum_{j=0}^{k-1} \mu_j y x_1^j, \qquad 2 \leq k < l$.

Proof. Suppose that the germ (L_0, Γ_0) is stable. Then for any germ $(\widetilde{L}, \widetilde{\Gamma})$ close (L_0, Γ_0), there is a Lagrange equivalence.

Assume we have a family (L_t, Γ_t) of deformations of (L_0, Γ_0), with $t \in [0, 1]$. Also assume that there is a family of diffeomorphism $\theta_t : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ which preserve Lagrange fibration and the sympletic form and maps L_t, Γ_t to L_0, Γ_0.

Consider families depending on t of respective generating families $G_t(w, \lambda)$ of L_t, Γ_t with $t \in [0,1]$ and G_0 is a generating family of the pair L_0, Γ_0. Thus,

$$G_t\left(\widetilde{w}_t(w,\lambda), \Lambda_t(\lambda)\right) = G_0(w,\lambda) + \Psi_t(\lambda)$$

and the critical points subsets $\left\{\frac{\partial G_t}{\partial w} = 0\right\}$ correspond to the Lagrangian submanifolds L_t. By proposition 7.1.3, all G_t are quasi border + equivalent up to addition of a function in parameters. This implies , in particular, that G_0 is versal with respect to quasi border equivalence.

By reversing the previous argument we prove the reciprocal claim.

The second part is a consequence of classifying the function germs with respect to quasi border equivalence. ∎

7.2 Algebraic invariants of simple quasi border classes

We start with a general construction which is useful in various settings of singularity theory. It is very basic, however we could not find it explicitly in the literature.

Given the germ at the origin of a smooth mapping $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $g : x \mapsto Y(x)$, consider the local algebra $Q_{g,0} = \mathbf{C}_x/\mathbf{C}_x\{Y_1(x), \ldots, Y_n(x)\}$ being the factor space of the space of germs at the origin on the source space. It is (up to isomorphism) invariant under the right-left (and even contact) transformations of the mapping. It was used by J.Mather [25] in his classical papers to classify nice dimensions and stable map germs.

The subgroup of right-left diffeomorphisms of the target and source spaces preserving some distinguished subset $B \subset \mathbb{R}^n$ (border) in the source space defines isomorphisms not only between the local algebras $Q_{g,0}$ but also between the ideals I_B in the algebras formed by the classes in $Q_{g,0}$ of the germs of functions $h(X)$ which vanish on B. The pair $(Q_{g,0}, I_B)$ will be called *local pair* for a mapping g and a border B.

Apply this construction to the Lagrangian mapping associated to a family of quasi border singularities.

Given a quasi border orbit of the germ $f(w)$ with the border B, consider its versal deformation $F(w,\lambda)$ and the Lagrangian submanifold L given by the generating family F. Consider also the submanifold

$$\widetilde{L} = \{(w,\lambda) : \frac{\partial F(w,\lambda)}{\partial w} = 0\}.$$

The Lagrangian projection of L is equivalent to the projection of $\widetilde{L}$ along w-coordinate fibers. Let

$$\widetilde{B} = \{(w,\lambda) \in \widetilde{L} : \frac{\partial F(w,\lambda)}{\partial w} = 0, \quad w \in B\} \subset \widetilde{L}$$

be its subset corresponding to the boundary.

Denote by $Q_{f,B}$ the local pair associated to $(\widetilde{L}, \widetilde{B})$.

Remark: In fact, the definitions imply that $Q_{f,B}$ remains the same not only for different choices of versal deformation F but also for any deformations of f satisfying Morse non-degeneracy conditions and $\mathcal{R}_+$ equivalent to each other. The validity of this claim is implied by the following. In fact what follows , the other definition of the pair $Q_{f,B}$ shows explicitely the claim.

The local pair can be defined equivalently as follows. Take the set of all functions on $\widetilde{L}$ given by

$$\mathcal{A}_{\widetilde{L}} = \mathbf{C}_{w,\lambda}/\{\frac{\partial F(w,\lambda)}{\partial w}\}.$$

Restricting to the set $\{\lambda = \lambda_0\}$, we get the local algebra

$$Q_{f,0} = \mathcal{A}_{\widetilde{L}\cap\{\lambda=\lambda_0\}} = \mathbf{C}_{w,\lambda}/\{\frac{\partial F(w,\lambda)}{\partial w}, \lambda - \lambda_0\} = \mathbf{C}_w/\{\frac{\partial F(w,\lambda_0)}{\partial w}\}.$$

The set of functions $\widehat{g}$ on $\widetilde{L}$ which vanish on the border gives rise to the ideal I_B of the local algebra $\mathcal{A}_{\widetilde{L}}$.

The proposition 7.1.3 implies that the local pair is the invariant of the border orbit.

Proposition 7.2.1 *If f_1 and f_2 are quasi border equivalent, then their local pairs are isomorphic.*

Proof. Suppose that f_1 and f_2 are quasi border equivalent. Take $F_1(w,\lambda)$ and $F_2(w,\lambda)$ as versal deformations (with respect to quasi border equivalence) of f_1 and f_2, respectively. These deformations are quasi border equivalent. This implies that the respective Lagrange projections of the two Lagrange submanifolds with borders (L_1,Γ_1) and (L_2,Γ_2) (which are determined by the generating families of functions $F_1(w,\lambda)$ and $F_2(w,\lambda)$, respectively) are equivalent. The previous discussion and construction yield that the respective local pairs are isomorphic.

∎

The classification of simple quasi border classes has a nice description in terms of associated local pairs.

Consider the local algebra $Q_{A_k} = \mathbf{C}_t/\{t^k\} = \mathbb{R}\{1, t, \dots, t^{k-1}\}$ of the standard A_k singularity which is isomorphic to the algebra of truncated polynomials in t of degree $k-1$. All ideals of Q_{A_k} are principle and form a discrete family $I_s = t^s Q_{A_k}$. Note that:

$$Q_{A_k} \supset I_1 \supset I_2 \supset \cdots \supset I_{k-1}.$$

The normal forms of simple classes yield the following

Proposition 7.2.2 *The associated local pair of the simple quasi boundary singularity B_k is (Q_{A_k}, I_1). The associated local pair of the simple quasi boundary singularity $F_{k,m}$ is $(Q_{A_m}, I_k.)$ The associated local triple of the simple quasi corner singularity $\mathcal{H}_{m,n,k}$ consisting of the local algebra and two ideals corresponding to two sides $x = 0,\ y = 0$ of the corner is (Q_{A_k}, I_n, I_m). For $n = 1$ we get the triple of $\mathcal{F}_{m,k}$, and for $n = 1, m = 1$ we get the triple of $\mathcal{B}_k$.*

Proof. Consider the quasi boundary singularity $B_k : x_1^2 + y^{k+1}$. Its versal deformation is $F(x_1,y) = x_1^2 + y^{k+1} + \sum_{i=1}^{k} \lambda_i y^i$ for $k \geq 1$. Firstly, we calculate the local algebra $Q_{F,0}$:

$$Q_{F,0} = \mathbf{C}_w, \lambda / \left\{ \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial y}, \lambda - \lambda_0 \right\} \cong \{1, y, y^2, \dots, y^{k-1}\},$$

where λ_0 are constants.

Clearly $Q_{F,0}$ corresponds to the local algebra Q_{A_k}. The set of elements of the local algebra $Q_{F,0}$ which vanish on the boundary $y = 0$ is the ideal I_1 generated by y. Thus, the associated local pair of the simple quasi boundary singularity B_k is Q_{A_k}, I_1.

For the quasi boundary singularity $F_{k,m}$: $(y + x_1^k)^2 + x_1^{m+1}$ with $m \geq k \geq 2$. Consider its versal $F = (y + x_1^k)^2 + x_1^{m+1} + \sum_{i=1}^{m-1} \lambda_i x_1^i + \sum_{j=0}^{k-1} \mu_j y x_1^j$. Let $\mathcal{I} = \{\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial y}, \lambda - \lambda_0\}$. Then, similar argument as above shows that

$$Q_{F,0} \cong \{1, x_1, x_1^2, \ldots, x_1^{m-1}\}$$

which corresponds to Q_{A_m}. Notice that in this case we have

$$y + x_1^k \cong 0 \; mod \; \mathcal{I}.$$

It follows that the set of elements which vanish on the boundary satisfies

$$x^k \cong 0 \; mod \; \mathcal{I}.$$

This gives the ideal I_k generated by x_1^k. Hence, the associated local pair of the simple quasi boundary singularity $F_{k,m}$ is Q_{A_m}, I_k.

For quasi corner singularities, start with $\mathcal{H}_{m,n,k}$: $(x + z_1^m)^2 + (y + z_1^n)^2 + z_1^{k+1}$ with $k \geq m \geq n \geq 2$. Let $\mathcal{I} = \{\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z_1}, \lambda - \lambda_0\}$. Then,

$$Q_{F,0} = \mathbf{C}_{x,y,z_1,\lambda}/\mathcal{I} \cong \mathbf{C}_{x,y,z_1}/\{x + z_1^m, y + z_1^n, z_1^k\} \cong \{1, z_1, z_1^2, \ldots, z_1^{k-1}\}.$$

The corner $xy = 0$ is a union of two transversal boundaries $x = 0$ and $y = 0$. The elements of $Q_{F,0}$ wihch vanish on the boundaries $x = 0$ and $y = 0$ satisfy

$$z_1^m \cong 0 \; mod \; \mathcal{I} \quad \text{and} \quad z_1^n \cong 0 \; mod \; \mathcal{I},$$

respectively. They correspond to the ideals $I_m = \{z_1^m, z_1^{m+1}, \ldots, z_1^{k-1}\}$ and $I_n = \{z_1^n, z_1^{n+1}, \ldots, z_1^{k-1}\}$. Thus, the triple $\mathcal{Q}_{A_k}, I_m, I_n$ is associated to the simple quasi corner singularity $\mathcal{H}_{m,n,k}$ consisting of the local algebra and two ideals corresponding to two sides $x = 0$, $y = 0$ of the corner.

For the singularity $\mathcal{F}_{m,k}$: $(x + y^m)^2 + y^{k+1}$ with $k \geq m \geq 2$, notice that $\mathcal{F}_{m,k}$ can be written in an alternative form $(x + y^m)^2 + y^k + (z_1 + y)^2$. Consider the ideal $\mathcal{I} = \{x + y^m, y^k, z_1 + y\}$. It follows that the associated local triple of the simple quasi corner singularity $\mathcal{F}_{m,k}$ is Q_{A_k}, I_m, I_1.

The calculation of the local triple of the $\mathcal{B}_k$ is straightforward.

∎

Remarks:

1. The proposition implies that all these classes are distinct.

2. Notice that all other local gradient algebras of isolated function singularities have continuous systems of principle ideals. For example D_4- type local algebra $Q_{D_4} = C_{x,y}/\{x^2, y^2\}$ contains a projective line of ideals of functions being multiples of a fixed linear term $\alpha x + \beta y$, where $\alpha, \beta \in \mathbb{R}$.

3. Recall that the ideal structure of local algebras of simple function singularities A,D,E can be represented by the graph of the shape similar to the standard Dynkin diagram of the singularity. So the classification of local pairs with simple Lagrange projections for all Lagrangian boundary pairs is (even for non-simple ones) straightforward.

4. A homotopy of Lagrange equivalences of stable Lagrange mapping given by miniversal deformation of an isolated function singularity with itself induces the identity isomorphism of the local algebra [37]. This is the consequence of the uniqueness of the analytic function representation by the class in the local gradient algebra. Therefore, simple quasi border singularities can occur only for A_k-type local algebras.

5. For simple classes the codimension of quasi corner or quasi boundary singularity is equal to the sum of the dimension of local algebra with the codimensions of the ideals $I_{x=0}$ and $I_{y=0}$ in the space of all principle ideals of the local algebra. We conjecture that the formula remains true for arbitrarily quasi boundary or quasi corner class.

7.3 Quasi-contact border equivalence

The contact equivalence is an important tool to classify the singularities of so called Legendrain mappings. For more details, see for example [1].

Definition 7.3.1 Let $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}$ be smooth functions, then the two hypersurfaces $f_0 = 0$ and $f_1 = 0$ in $\mathbb{R}^n$ are called pseudo border equivalent if they are diffeomorphic via a diffeomorphism which maps critical points of the first hypersurface belonging to a distinguished border Γ to those of the second hypersurface also in the border Γ and vice versa.

The definition implies the existence of a diffeomorphisim $\theta : \mathbb{R}^n \to \mathbb{R}^n$ and a smooth function $H \in \mathbf{C}_w$ such that $f_1 \circ \theta = H f_0$ and $H(0) \neq 0$, and if a critical point c of the function f_0 belongs to the border Γ then $\theta(c)$ also belongs to Γ and vice versa, if c is a critical point of f_1 and belongs to Γ then $\theta^{-1}(c)$ also belongs to Γ. The functions f_0 and f_1 are said to pseudo-contact border equivalent.

Recall that $\mathcal{V}_{Rad(f_t)}$ denotes the set of vector field germs, each component of which belong to the radical of the gradient ideal I of the function f_t and V_I denotes the ideal of the algebra of germs of vector fields, each component of which belongs to the gradient ideal I of the function f_t.

Proposition 7.3.1 *The tangent space to the pseudo-cntact border orbit $f : \mathbb{R}^n \to R$ takes the form*

$$TCP_f = fA + \sum_{i=1}^{n} \frac{\partial f}{\partial w_i} \dot{v}_i,$$

where $A \in \mathbf{C}_w$ and $v = \sum_{i=1}^{n} \frac{\partial f}{\partial w_i} \dot{v}_i \in \mathbb{S}_\Gamma + \mathcal{V}_{Rad(f_t)}$.

Proof. The proof is the same as proof of proposition 1.2.2. ∎

Similar to pseudo border equivalence, we modify pseudo-contact equivalence relation to have better property with respect to parameter dependence replacing the radical $Rad\{\frac{\partial f_t}{\partial w}\}$ by the ideal $\{\frac{\partial f_t}{\partial w}\}$ itself in the definition of pseudo-contact border equivalence.

Definition 7.3.2 Two functions $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}$ are called *quasi-contact border equivalent*, if they are pseudo-contact border equivalent and there is a family of function germs f_t which continuously depends on parameter $t \in [0,1]$, a continuous piece-wise smooth family of diffeomorphisms $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ depending on parameter $t \in [0,1]$ and a non-vanishing family $H_t \in \mathbf{C}_w$ such that: $H_t(f_t \circ \theta_t) = f_0$, $\theta_0 = id$, $H_0 = 1$ and the vector field v generated by θ_t on each segment of smoothness satisfies the inclusion

$$v \in \mathbb{S}_\Gamma + V_I.$$

The previous definition implies that in particular the formula of quasi-contact boundary tangent space $TCQB_{f_t}$ (the boundary is $\Gamma_b = \{x_1 = 0\}$) to the quasi-contact boundary orbits at an admissible deformations f_t takes the form:

$$TCQB_{f_t} = \left\{ fA_0 + \frac{\partial f_t}{\partial x_1}\left(x_1 h_1 + \frac{\partial f_t}{\partial x_1}A_1\right) + \sum_{i=1}^{n-1} \frac{\partial f_t}{\partial y_i} k_i \right\},$$

for arbitrary function germs $h_1, A_1, k_i, A_0 \in \mathbf{C}_w$. Here $x_1 \in \mathbb{R}, y \in \mathbb{R}^{n-1}$.

Similar formulas can be obtained for other borders.

In all our simple quasi border classes from theorems stated before the singularities are weighted homogeneous (however, the homogeneous coordinates are not the original coordinates). This fact implies the following

Proposition 7.3.2 *The list of simple classes with respect to quasi-***contact** *border equivalence coincides with the simple quasi border classes.*

Proof. We will prove the proposition for the quasi-contact boundary equivalence only as the arguments for other cases are similar. We prove that quasi-contact boundary tangent space of the simple and non-simple classes which are obtained in Theorem 2.1.6 space coincides with the quasi boundary tangent space of the same classes.

Recall that the quasi-contact boundary tangent space at the germ $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ takes the form

$$TQCB_f = fH + \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i}\dot{X}_i + \frac{\partial f}{\partial y}\dot{Y},$$

where $\dot{Y} = yB_0 + \frac{\partial f}{\partial y}B_1 + \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i}\widetilde{B}_i$ and $H, \dot{X}_i, B_0, B_1, \widetilde{B}_i \in \mathbf{C}_w$.

The classes B_k and the non-simple classes are quasi homogeneous. Hence, the tangent spaces of these classes with respect to quasi boundary and quasi-contact boundary coincides.

For the classes $F_{k,m} : G = \pm(y \pm x_1^k)^2 \pm x_1^m$, let $w = y \pm x_1^k$. Then, $G = \pm w^2 \pm x_1^m$. Thus, the derivatives takes the form

$$\frac{\partial G}{\partial y} = \pm 2w \quad \text{and} \quad \frac{\partial G}{\partial x} = \pm 2kwx_1^{k-1} \pm mx_1^{m-1}.$$

As G is quasi homogeneous in the new coordinates w and x, we see that

$$G = \frac{w}{2}\frac{\partial G}{\partial y} + \frac{x}{m}(\pm mx_1^{m-1}) = \frac{w}{2}\frac{\partial G}{\partial y} + \frac{x}{m}(\frac{\partial G}{\partial x_1} \mp 2kwx_1^{k-1}),$$

or, equivalently

$$G = \frac{\partial G}{\partial y}(\frac{w}{2} \pm \frac{kx^k}{m}) + \frac{x}{m}\frac{\partial G}{\partial x_1}.$$

Notice that $\dot{Y} = yB_0 + wB_1 + B_3(\pm mx_1^{m-1} \pm 2kwx_1^{k-1}) = w\widetilde{B}_0 + x_1^k\widetilde{B}_1$. Clearly, the term $(\frac{w}{2} \mp \frac{2kx^k}{m})$ from G is contained in $\dot{Y}$ and hence the result follows.

∎

Chapter 8

Basics of projections

8.1 Introduction

We start by revising the main definitions.

Let $V \subset \mathbb{R}^n$ be the germ of zeros of smooth maping $f : (\mathbb{R}^k, 0) \to (\mathbb{R}^m)$, $n \geq m$.

Definition 8.1.1 [18]

V is called a complete intersection if codim $V = m$.

Definition 8.1.2 [18]The projection of a submanifold V from a bundle space E onto the base B is a triple $V \to E \to B$, where the first arrow is the embedding and second arrow is the projection.

Definition 8.1.3 [18]Two projections $V_i \to E_i \to B_i$, $i = 1, 2$ are equivalent if the following 3×2 diagram:

$$\begin{array}{ccccc} V_1 & \xrightarrow{i_1} & E_1 & \xrightarrow{\pi} & B_1 \\ \downarrow\phi & & \downarrow h & & \downarrow k \\ V_2 & \xrightarrow{i_2} & E_2 & \xrightarrow{\pi} & B_2 \end{array}$$

commutes. Here ϕ, h and k are diffeomorphisms, $i_{1,2}$ are embeddings.

Locally a bundle $E \to B$ is isomorphic to the trivial bundle

$$\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p : (x, y) \mapsto y.$$

Instead of a submanifold V we can consider more general setting. Let V be a complete intersection. Recall that this means that V is diffeomorphic to an analytic variety. However, the variety V need not to be analytic [18]. The codimension of the germ V in E is equal to m and V is given by a system of m analytic equations:

$$V = \{(x, y) : f_1 = f_2 = \cdots = f_m = 0\}.$$

This system is determined up to multiplication by a germ on E of a non degenerate $m \times m$ matrix $M(x, y)$.

In this case the equivalence of given systems $f = 0$ and $g = 0$ of germs of projections from $\mathbb{R}^n \times \mathbb{R}^p$ onto $\mathbb{R}^p$, $(x, y) \mapsto y$ means that there exists a local diffeomorphism of the form

$$h(x, y) = (X(x, y), Y(y)),$$

for which $h^*g = Mf$.

Let $\mathbf{C}_{x,y}$ be the space of germs at zero of $\mathbf{C}^\infty$- function germs in variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. Let $\mathbf{C}^m_{x,y}$ be the space of germs of $\mathbf{C}^\infty$- mappings from $\mathbb{R}^{n+p}$ to $\mathbb{R}^m$, defining embedding of complete intersections. Denote by O_f the equivalence class of the variety $f = 0$.

Assume that $f_t = 0$ is 1-parameter family of equivalent germs of projections where $f_t : \mathbb{R}^{n+p} \longrightarrow \mathbb{R}^m$ and $f_0 = f$. This implies the existence of a 1-parameter family $M_t(x, y)$ of $m \times m$ matrices and 1-paramter family of diffeomorphisms ϕ_t : $\mathbb{R}^{n+p} \longrightarrow \mathbb{R}^{n+p}$ of the form $\phi_t(x, y) = (X_t(x, y), Y_t(y))$ with $t \in [0, 1]$ such that

$$f_t = M_t f \circ \phi_t, \quad (*)$$

where $M_0 = id_m$, $\phi_0 = (x, y)$, $X_t(x, y) = (X_1(t, x, y), \ldots, X_n(t, x, y))$ and $Y_t(y) = (Y_1(t, y), \ldots, Y_p(t, y))$. If we differentiate (*) with respect to t, we obtain

$$\frac{d}{dt} f_t = \dot{M}_t f_t \circ \phi_t + M_t \left(\sum_{i=1}^{n} \frac{\partial f_t}{\partial x_i} \circ \phi_t \frac{dX_i}{dt} + \sum_{j=1}^{p} \frac{\partial f_t}{\partial y_j} \circ \phi_t \frac{dY_j}{dt} \right) \quad (**)$$

where $\dot{M}_t = \frac{d}{dt} M_t$.

Let $v_i(t, X_t, Y_t) = \frac{dX_i}{dt}$ and $\widetilde{v}_j(t, Y_t) = \frac{dY_j}{dt}$. Then , (**) can be written as

$$\frac{d}{dt} f_t = \dot{M}_t f_t \circ \phi_t + M_t \left(\sum_{i=1}^{n} \frac{\partial f_t}{\partial x_i} \circ \phi_t v_j(t, X_t, Y_t) + \sum_{j=1}^{p} \frac{\partial f_t}{\partial y_i} \circ \phi_t \widetilde{v}_j(t, Y_t) \right).$$

Suptituting $t = 0$ gives

$$\left. \frac{d}{dt} f_t \right|_{t=0} = \dot{M}_0 f + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i(x, y) + \sum_{j=1}^{p} \frac{\partial f}{\partial y_j} \widetilde{v}_j(y).$$

Notice that $v(0) = \widetilde{v}(0) = 0$. Denote by $\mathbf{C}_{x,y}\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\}$ the ideal generated by $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ over $\mathbf{C}_{x,y}$ and denote by $\mathbf{C}_y\{\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_p}\}$ the ideal generated by $\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_p}$ over $\mathbf{C}_y$. Then, the extended tangent space (or just the tangent space) TO_f of the orbit O_f of the projection of $f \in \mathbf{C}^m_{x,y}$ is given by the formula:

$$TO_f = Mf + \mathbf{C}_{x,y}\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\} + \mathbf{C}_y\{\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_p}\},$$

where M is $m \times m$-matrix and its entries belongs to $\mathbf{C}_{x,y}$.

Let the map germ $F : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ be given in the local coordinates as follows:

$$u = (u_1, u_2, \ldots, u_n) \mapsto z = (z_1 = f_1(u), z_2 = f_2(u), \ldots, z_p = f_p(u)).$$

Let $\Gamma = \{(u, z) : z_1 = f_1(u), \ldots, z_p = f_p(u)\} \subset \mathbb{R}^n \times \mathbb{R}^p$ be the graph of the mapping F.

The classification of map germs with respect to right-left equivalence is equivalent to the classification of the projection germs $\Gamma \hookrightarrow \mathbb{R}^n \times \mathbb{R}^p \xrightarrow{\pi} \mathbb{R}^p$. In fact,

Proposition 8.1.1 [26] *The tangent space of the orbit of the projection of* Γ*, coincides with tangent space of the orbit of right-left equivalence of the respective mapping* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0);\ u \mapsto z = f(u)$.

Proof. Let M_t be a family of $p \times p$ matrices and its entries belong to $\mathbf{C}_{u,z}$. Let P_t be an one parameter family of systems defining the graphs Γ_t:

$$P_t = \begin{pmatrix} z_1 - f_1(u,t) \\ \vdots \\ z_p - f_p(u,t) \end{pmatrix}.$$

Let $\Phi_t : (\mathbb{R}^n \times \mathbb{R}^p, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}^p, 0)$ be a one parameter family of diffeomorphisms such that $\Phi_0 = id$ and has the form $(u, z, t) \mapsto (U_t(u, z), Z_t(z))$, where $t \in [0, 1]$. Consider, the family of equivalent projections:

$$M_t.(P_t \circ \Phi_t) = P_0.$$

If we differentiate this relation with respect to t and then substitute $t = 0$, we get:

$$\begin{aligned} -\frac{\partial P_t}{\partial t}|_{t=0} &= \widetilde{M} \begin{pmatrix} z_1 - f_1(u,0) \\ \vdots \\ z_p - f_p(u,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1(u,0)}{\partial u_1} & \dots & \frac{\partial f_1(u,0)}{\partial u_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n(u,0)}{\partial u_1} & \dots & \frac{\partial f_n(u,0)}{\partial u_n} \end{pmatrix} \begin{pmatrix} \dot{U}_1(u,z) \\ \vdots \\ \dot{U}_n(u,z) \end{pmatrix} \\ &+ \begin{pmatrix} \dot{Z}_1(z) \\ \vdots \\ \dot{Z}_p(z) \end{pmatrix}. \end{aligned}$$

Here, $\dot{U} = \frac{\partial U}{\partial t}|_{t=0}$ and $\widetilde{M}$ is $p \times p-$ matrix

Substitute $z = f$ in the previous formula ,we get the tangent space of the orbit at f with respect to right-left equivalence:

$$-\frac{\partial P_t}{\partial t}|_{t=0} = \begin{pmatrix} \frac{\partial f_1(u,0)}{\partial u_1} & \dots & \frac{\partial f_1(u,0)}{\partial u_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n(u,0)}{\partial u_1} & \dots & \frac{\partial f_n(u,0)}{\partial u_n} \end{pmatrix} \begin{pmatrix} \dot{U}_1(u) \\ \vdots \\ \dot{U}_n(u) \end{pmatrix} + \begin{pmatrix} \dot{Z}_1(f) \\ \vdots \\ \dot{Z}_p(f) \end{pmatrix}.$$

The last formula is the tangent space to the right left orbit at the mapping $u \mapsto z = f(u,0)$ as required.

∎

Let $\Phi_t : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0); u = (u_1, \dots, u_n) \mapsto Z_t = (Z_t^{(1)}(u), \dots, Z_t^{(n)}(u))$ be an one parameter family of diffeomorphisms and $\Phi_0 = id$. Let $\Phi_t^{-1} : Z_t^{(1)}(u), \dots, Z_t^{(n)}(u) \mapsto (z_t^{-(1)}(u), \dots, z_t^{-(n)}(u))$ be the inverse image of Φ_t. Let $V_t = \dot{Z}_t = (\dot{Z}_t^{(1)}, \dots, \dot{Z}_t^{(n)})$, where $\dot{Z}_t^{(i)} = \frac{\partial Z_t^{(i)}}{\partial t}$, $i = 1, \dots n$. Let $(a_{i,j}) = (\frac{\partial Z_t^{(i)}}{\partial u_j})$ be the matrix of the differential of Φ_t. Let $\xi = (\xi_1, \dots, \xi_n)$ be an arbitrary vector field on $\mathbb{R}^n$. Then,

Proposition 8.1.2 $\frac{\partial}{\partial t}(\Phi_t^*(\xi))_{|_{t=0}} = -[V_0, \xi]$, *where* $[,]$ *is the Lie bracket of vector fields.*

Proof. We want first to calculate $\frac{\partial}{\partial t}\Phi_t^*(\xi)$. Note that:

$$\Phi_t^*(\xi) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \xi_j(\Phi_t^{-1}(Z_t)) a_{i,j} \right) \frac{\partial}{\partial u_i}.$$

Differentiate the last equation with respect to t to get:

$$\frac{\partial}{\partial t}\Phi_t^*(\xi) = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \xi_j(\Phi_t^{-1}(Z_t))\dot{a}_{i,j} - \left(\sum_{m=1}^{n} \frac{\partial \xi_j(\Phi_t^{-1}(Z_t))}{\partial u_m} . z_t(u)^{-(m)} \circ [\dot{Z}_t^{(m)} \circ \Phi_t^{-1}] \right) a_{i,j} \right] \frac{\partial}{\partial u_i},$$

where $(\dot{a}_{i,j}) = \frac{\partial}{\partial t}(a_{i,j})$.

If we substitute $t = 0$, in the last equation, we get:

$$\frac{\partial}{\partial t}\left[\Phi_t^*(\xi)\right]_{|_{t=0}} = \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(\xi_j \dot{\tilde{a}}_{i,j} - \frac{\partial \xi_j}{\partial u_i}\dot{Z}_0^{(i)}\right)\right]\frac{\partial}{\partial u_i} = -[V,\xi],$$

where $\dot{\tilde{a}}_{i,j} = \frac{\partial}{\partial t}(a_{i,j})_{|_{t=0}}$.

Note here that : $\Phi_t \circ \Phi_t^{-1} = id$, if we differentiate this relation with respect to t, we get:

$$(\frac{\partial}{\partial t}\Phi_t) \circ \Phi_t^{-1} + \Phi_t \circ (\frac{\partial}{\partial t}\Phi_t^{-1}) = 0.$$

Hence,

$$\frac{\partial \Phi_t^{-1}}{\partial t} = -\Phi_t^{-1} \circ [(\frac{\partial \Phi_t}{\partial t} \circ \Phi_t^{-1})].$$

Also note that $\dot{a}_{i,j} = \frac{\partial}{\partial t}(a_{i,j}) = \frac{\partial}{\partial t}\frac{\partial Z_t^{(i)}}{\partial u_j} = \frac{\partial}{\partial u_j}\frac{\partial Z_t^{(i)}}{\partial t} = \frac{\partial \dot{Z}_t^{(i)}}{\partial u_j}$. ∎

Remark: We need the detail of the previous two statements as it is necessary in our further considerations.

8.2 The classifications of singularities of projections of surfaces

The classification of singularities of projections of a two-surface embedded into RP^3 to a plane obtained by V.I.Arnold [7], O.Platonova [27], V.Goryunov [18] and O.Scherback [30] at the beginning of 80-th was a nice generalization of Whitney theorem [1]. The surface is assumed to be generic, and centre of projection can vary in RP^3.

Theorem 8.2.1 [2] *For a generic surface, any projection from any point (outside the surface) is locally equivalent to a projection of one of the 14 surfaces ($z = f(x,y)$) in the following list at zero by a pencil of lines parallel to the x-axis:*

$$
\begin{array}{cc}
P_1 : f = x & P_2 : f = x^2 \\
P_3 : f = x^3 + xy & P_4 : f = x^3 \pm xy^2 \\
P_5 : f = x^3 + xy^3 & P_6 : f = x^4 + xy \\
P_7 : f = x^4 \pm x^3y + xy, & P_8 : f = x^5 \pm x^3y + xy \\
P_9 : f = x^3 \pm xy^4, & P_{10} : f = x^4 + x^2y + xy^3, \\
 & P_{11} : f = x^5 + xy.
\end{array}
$$

The hierarchy of germs of projection of a surface according to calculations of O.A. Platonova, V. Arnold and O.P.Shcherbak are as follows:

$$
\begin{array}{ccccccccc}
P_1 & \leftarrow & P_2 & \leftarrow & P_3 & \leftarrow & P_6 & \leftarrow & P_8 \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & P_4 & \leftarrow & P_7 & & P_{11} \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & P_5 & \leftarrow & P_{10} & & \\
 & & & & \uparrow & & & & \\
 & & & & P_9 & & & &
\end{array}
$$

Later on, meeting the needs of several application in geometry and differential equations authors considered also projections of submanifolds with boundaries. In particular, singularities of projections of surfaces with boundaries were studied and classified by J.Bruce P.Giblin [15], V.Goryunovin [19]in 80-th and F.Tari [32] in 90-th. They considered the classifications of singularities when a generic surface in three space with a boundary is projected to a plane along a parallel beam of rays.

Let $\mathbf{C}_{x,y,z}$ be the space of germs at zero of $\mathbf{C}^\infty$- function germs in variables $(x, y, z) \in \mathbb{R}^3$. Let $\mathbf{C}^2_{x,y,z}$ be the space of germs of $\mathbf{C}^\infty$- mappings from $\mathbb{R}^3$ to $\mathbb{R}^2$. Consider the projection $\pi : (x, y, z) \mapsto (y, z)$. Suppose that the surface Γ is embedded in $\mathbb{R}^3$ and is given by the equation $\Gamma = \{g_1(x, y, z) = 0\}$. Also, assume that its boundary is given by the equation $B = \{g_1(x, y, z) = g_2(x, y, z) = 0\}$. Denote by G the pair $G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ the germ at zero of the surface Γ with its boundary B.

The classification of the pairs G was considered for example in [19] up to diffeomorphisms of $\mathbb{R}^3$ of the form

$$h : (x, y, z) \mapsto (h_1(x, y, z), h_2(y, z), h_3(y, z))$$

fibered over $\mathbb{R}^2$ and the transformations $(g_1, g_2) \mapsto (ag_1, bg_1 + cg_2)$, where $a, b, c \in \mathbf{C}_{x,y,z}$. More precisely,

Definition 8.2.1 [19] The projection of two pairs $G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ and $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ are equivalent if there exists a diffeomorphism of the form

$$h : (x, y, z) \mapsto (h_1(x, y, z), h_2(y, z), h_3(y, z))$$

and a matrix

$$M = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

where where $a, b, c \in \mathbf{C}_{x,y}$ with $a(0)c(0) \neq 0$ such that $G = MF \circ h$.

The tangent space of the orbit of the projection of the pair G is given by the formula:

$$T_G = \mathbf{C}_{x,y,z} \left\{ \begin{array}{c} g_1, \frac{\partial g_1}{\partial x} \\ g_1, g_2, \frac{\partial g_2}{\partial x} \end{array} \right\} + \mathbf{C}_{y,z} \left\{ \begin{array}{c} \frac{\partial g_1}{\partial y}, \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial y}, \frac{\partial g_2}{\partial z} \end{array} \right\},$$

Remark: We will use similar definition later.

The normal forms of the projection of the pair $G = (g_1, g_2)$ where g_1 is a smooth surface and given as a graph $z = f(x, y)$ (so $g_1 = z - f(x, y)$) is given as follows [15, 19]:

$$f = x, g_2 = x; \quad f = x^2 + xy, g_2 = x; \quad f = x^3 + xy, \ g_2 = x;$$

$$f = \pm x^6 + x^4 + xy, \ g_2 = x; \quad f = \pm xy^2 + x^2, g_2 = x; \quad f = x^2 + y^3 x, \ g_2 = x;$$

$$f = xy^2 + x^2 y + ax^3 + x^4, \ g_2 = x; \quad f = x^2, \ g_2 = y + x^3; \quad f = x^2, \ g_2 = y + x^5;$$

$$f = xy + ax^3 \pm x^5, \ g_2 = y \pm x^2.$$

Chapter 9

Quasi projections of hypersurfaces

9.1 Introduction

In this chapter, we classify simple singularities of projections of hypersurfaces up to a special equivalence relation [38] which is more rough than the standard one which was discussed in the previous chapter.

We give here the complete proofs of the theorems stated in that paper as some proofs are outlined there . On the other hand, our methods and results in the next chapters are based on the constructions and results of the paper.

Consider the trivial bundle $\mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$; $(x, y) \mapsto y$.

Definition 9.1.1 Given a variety $V \subset \mathbb{R}^n \times \mathbb{R}^p$, a point $b \in V$ is called *critical* if the fiber through b is not transversal to V at b. In particular, b can be a singular point of V.

Definition 9.1.2 Two varieties V_1 and V_2 embedded in $\mathbb{R}^n \times \mathbb{R}^p$ are called *pseudo-equivalent* if there is a diffeomorphism $\theta : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}^p$ such that $V_1 = \theta(V_2)$, the set of critical points of V_2 is mapped onto the set of critical points of V_1, and the differential of θ at any critical point maps the direction of the projection to that at the image of the point.

Obviously, this is an equivalence relation. We will denote by P_V the equivalence class of a germ V and call it the *pseudo-orbit* of V.

For simplicity, we consider only the case of analytic hypersurface $V = \{(x, y) : f(x, y) = 0\}$ given by a single equation $f = 0$. Also, we assume that the fibers are one dimensional $x \in \mathbb{R}$, $n = 1$.

Denote by TP_f the tangent space at $V = \{(x, y) : f(x, y) = 0\}$ (or just at f) to the orbit P_V. Denote by $\mathrm{Rad}(J_f)$ the ideal consisting of function germs $h(x, y)$ whose certain power h^m belongs to the ideal J_f generated by $\frac{\partial f}{\partial x}$ and f. Denote by $\mathrm{IRad}(J_f)$ the module over the algebra $\mathbf{C}_{x,y}$ of function germs g such that the derivative $\frac{\partial g}{\partial x} \in \mathrm{Rad}(J_f)$. Denote by IJ_f the integral of the ideal J_f consisting of all function germs h such that $\frac{\partial h}{\partial x} \in J_f$. Clearly, the functions in $\mathbf{C}_y$ which do not depend on x are in IJ_f for any germ f. Denote by f_x the partial derivative of f with respect to x (i.e $f_x = \frac{\partial f}{\partial x}$).

In fact below, we replace the algebraic notion of the radical by the geometric one (similar to the idea which was introduced in chapter 1), assuming that f and g are diffeomorphic to analytic maps.

Differentiating upon parameter all deformations within the pseudo-orbit of a given germ f we obtain the tangent space TP_f at f to the orbit P_V.

Proposition 9.1.1 *The tangent space TP_f of the pseudo orbit P_V at f is given by the formula*

$$TP_f = Af + f_x\dot{X} + \sum_{i=1}^{n-1} f_{y_i}\dot{Y}_i,$$

where $A, \dot{X} \in \mathbf{C}_{x,y}$ and $\dot{Y}_i \in IRad(J_f)$.

Proof. Lemma 8.1.2 yields that the vector field $v = v_0\frac{\partial}{\partial x} + \sum_{i=1}^{n-1} v_i\frac{\partial}{\partial y_i}$, the flow of which preserves the chosen direction $e = \frac{\partial}{\partial x}$, along a trajectory γ of a point satisfies the relation $[v, e] = he$ with some factor h. The decomposition of this relation is

$$[v, e] = -\left(\frac{\partial v_0}{\partial x}\right)\frac{\partial}{\partial x} - \left(\sum_{i=1}^{n-1}\left(\frac{\partial v_i}{\partial x}\right)\frac{\partial}{\partial y_i}\right) = h\frac{\partial}{\partial x}.$$

This means that the derivatives with respect to x of the components v_i of the vector field corresponding to the coordinates $y_i, i = 1, \ldots, n-1$, vanish at the points of γ. In the analytic case this means that these derivatives belong to the radical of the ideal defining the critical locus. This proves the proposition.

∎

Unfortunately, this relation does not satisfy the properties of a geometrical subgroup of equivalences in J. Damon sense [16](see **Example 1:** in chapter 1). In particular, the versality theorem can fail. To avoid this difficulty we use a subspace of the tangent space, which behaves regularly when the function f depends on extra parameters. Namely take the following sub-module

$$TQ_f = Af + f_x \dot{X} + \sum_{i=1}^{n-1} f_{y_i} \dot{Y}_i \subset TP_f, \qquad (*)$$

where $A, \dot{X} \in \mathbf{C}_{x,y}$ and $\dot{Y}_i \in IJ_f$, as the set of admissible infinitesimal deformations of a function. Hence, we introduce the respective notion Q_f of the class of a quasi equivalence relation.

Definition 9.1.3 Two hypersurfaces $V_1 = \{f_1 = 0\}$, $V_2 = \{f_2 = 0\}$ are called quasi-equivalent if there is a family of diffeomorphisms $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ continuously and piece-wise smoothly depending on parameter $t \in [1, 2]$ and a family h_t of continuous piece-wise smooth non-vanishing functions such that $f_t = h_t(f_2 \circ \theta_t)$, θ_2 is the identity mapping, $h_2 = 1$, and for any $t \in [1, 2]$ the components of the vector field $V = \frac{d\theta_t}{dt} \circ \theta_t^{-1}$ takes the form:

$$\dot{X} \in \mathbf{C}_{x,y} \quad \text{and} \quad \dot{Y}_i \in \left\{ c_i(y) + \int_0^x (a_i(x,y) f_t + \frac{\partial f_t}{\partial x} b_i(x,y)) dx \right\},$$

where $a_i, b_i \in \mathbf{C}_{x,y}$. The family of functions f_t being the homotopy between f_1 and f_2 is called admissible.

Remarks:

1. It is obvious that the quasi-equivalence of two functions implies their pseudo equivalence.

2. The hypersurfaces which are O-equivalent (and belong to one connected component of the orbit) are quasi equivalent, since the functions from C_y belong to IJ_f for any f.

3. Similarly to the remarks on page 54, it is easy to see that the versality theorem holds for the quasi equivalence. As usual, a miniversal deformation of the germ of a hypersurface $V = \{f(x,y) = 0\}$ is a family of hypersurfaces determined by a family of functions that is a sum of an organizing center $f(x,y)$ and a linear combination of functions whose classes form a basis over $\mathbb{R}$ of the quotient space $\mathbf{C}_{x,y}/TQ_f$. .

4. The definitions imply that quasi-equivalent hypersurfaces have diffeomorphic sets Σ of critical points. Moreover, the Thom-Boardman-type stratification of the critical locus Σ is preserved. Let $\Sigma_1 \subset \Sigma$ be the subset of points s at which the critical set is tangent to the direction $\frac{\partial}{\partial x}$. In other words, the direction belongs to the tangent cone to Σ at s. Define by induction subsets $\Sigma_i \subset \Sigma_{i-1}$ consisting of points at which $\frac{\partial}{\partial x}$ is tangent to Σ_{i-1}. All of them are preserved by the quasi-equivalence.

5. Assuming that the critical distinguished point remains at the origin for any value of the parameter deformation, we can apply only admissible vector fields which vanish at the origin. In a number of cases, this allows us to show the jets of quasi-orbits of some order coincide with the jets of standard O-orbits. For example, if all components of the singularity germ belong to the cube of the maximal ideal, then the terms which are in the quasi-orbit but not in the ordinary orbit belong to the forth power of the maximal ideal. So in this case the 3-jet of the standard orbit coincide with the 3-jet of quasi-orbit.

9.2 Basic techniques: Spectral sequence method

In what follows, we use mainly Moser homotopy method which was explained in chapter 1, standard spectral sequence method [1] and sometimes our modification, given in Lemma 1.3.5 in chapter 1.

We describe the standard technique here briefly.

Assume we have an $\mathbb{A}$-equivalence relation. Here $\mathbb{A}$ stands either for right-left equivalence or some quasi-equivalence described above. Let the space $T\mathbb{A}_g$ be the tangent space to the orbit at a germ g. Here we consider formal power series. Let a function $g = g_0 + g_1 + \ldots$ be the decomposition of g into its quasi-homogeneous of degrees $N, N+1, \ldots$. Then, any power series with the principal part g_0 can be reduced to the form $g_0 + \sum c_i e_i$ with respect to $\mathbb{A}$-equivalence relation, where the e_i form the part of a monomial basis of $\mathbf{C}_{x,y}/T\mathbb{A}_{g_0}$, of degrees greater than the degree of g_0.

9.3 Prenormal forms of quasi projection classes

In many cases we can find an appropriate prenormal form of a germ.

Consider the trivial bundle $\mathbb{R}^1 \times \mathbb{R}^p \longrightarrow \mathbb{R}^p : (x, y) \mapsto y$. Consider the tangent space to the quasi projection orbit at f, given by the formula

$$TQ_f = Af + f_x\dot{X} + \sum_{i=1}^{n-1} f_{y_i}\dot{Y}_i, \qquad (*)$$

where $A, \dot{X} \in \mathbf{C}_{x,y}$, and $\dot{Y}_i \in IJ_f$.

Proposition 9.3.1 *The module TQ_f given by the previous formula can be equivalently written as*

$$TQ_f = fC_{x,y} + f_x\mathbf{C}_{x,y} + \sum_{i=1}^{n-1} f_{y_i}\{I_0J_f\} \subset TP_f \qquad (**)$$

or

$$TQ_f = fC_{x,y} + f_x\mathbf{C}_{x,y} + \sum_{i=1}^{n-1} f_{y_i}\{I_1J_f\} \subset TP_f, \qquad (***)$$

where I_0J_f and I_1J_f are submodules of functions $h(x, y)$ such that $h_x(x, y)$ belongs to the principal ideal generated only by f or f_x, respectively.

Proof. Applying integration by parts, we see that

$$\int fadx = f\int adx - \int f_x\left(\int adx\right)dx.$$

Thus,

$$\int (fa + f_x b)dx = f\int adx + \int f_x \left(b - \int adx\right) dx.$$

Similarly,

$$\int f_x b dx = fb - \int f b_x dx.$$

Hence,

$$\int (fa + f_x b)dx = fb + \int f(a - b_x)dx.$$

These formulas provide the required identities. Here a and b are smooth functions.

∎

Lemma 9.3.2 *If $\Phi_t(x,y)$ is an admissible deformation of functions for quasi projection and $(x,y) \mapsto (X_t(x,y), Y_t(y))$ is a family of diffeomorphisms of $\mathbb{R}^n$ that preserve the fibration $(x,y),0 \mapsto y,0$, then $G_t(x,y) = \Phi_t(X_t(x,y), Y_t(y))$ is also admissible deformation.*

Proof. The fact that the deformation $\Phi_t(X,Y)$ is admissible means that

$$\begin{aligned}\frac{\partial \Phi(X,Y)}{\partial t} &= \Phi_t(X,Y)D(X,Y,t) + \frac{\partial \Phi_t(X,Y)}{\partial X}\dot{X} + \sum_{i=1}^{n-1} \frac{\partial \Phi_t(X,Y)}{\partial Y_i}\Bigg\{ A_i(Y,t) \\ &+ \int \left(B_i(X,Y,t)\frac{\partial \Phi_t(X,Y)}{\partial X} + \Phi_t(X,Y)C_i(X,Y,t)\right) dX \Bigg\},\end{aligned}$$

with some smooth functions A_i, B_i, C_i, and $\dot{X}$.

The matrix $\frac{\partial Y}{\partial y}$ is invertible, $\frac{\partial X}{\partial x} \neq 0$ and $\frac{\partial \Phi_t}{\partial x} = \frac{\partial \Phi_t}{\partial X}\frac{\partial X}{\partial x}, \frac{\partial \Phi_t}{\partial y} = \frac{\partial \Phi_t}{\partial Y}\frac{\partial Y}{\partial y}$. The functions Y_i depend only on y_i and t. Hence the decomposition can be written in the form:

$$
\begin{aligned}
\frac{\partial\Phi(X(x,y),Y(y))}{\partial t} &= \Phi_t(X(x,y),Y(y))\widetilde{D}(x,y,t)+\frac{\partial\Phi_t(X(x,y),Y(y))}{\partial x}\widetilde{X} \\
&+ \sum_{i=1}^{n-1}\frac{\partial\Phi_t(X(x,y),Y(y))}{\partial y_i}\left\{\widetilde{A}_i(y,t)+\int(B_i(x,y,t)\frac{\partial\Phi_t(X(x,y),Y(y))}{\partial x}\right. \\
&+ \left.\Phi_t(X(x,y),Y(y))C_i(x,y,t)dx)\right\},
\end{aligned}
$$

with some smooth functions $\widetilde{X}, \widetilde{A}_i, \widetilde{B}_i, \widetilde{C}_i$ and $\widetilde{D}_i$. This means that the family G_t is admissible.

∎

Lemma 9.3.3 *If $G_t(x,y)$ is an admissible family, then for an arbitrary function $H(t,x,y)$ the family $\widetilde{G}_t(x,y) = G_t(x,y) + H\left(\frac{\partial G_t}{\partial x}\right)^2$ is also admissible and $\widetilde{G}_t$ is quasi equivalent to G_t for each value of t.*

Proof. The fact that $\widetilde{G}_t(x,y)$ is admissible means that:

$$
TQ_{\widetilde{G}_t} = \widetilde{G}_t\widetilde{A}(x,y)+\frac{\partial\widetilde{G}_t}{\partial x}\dot{\widetilde{X}}(x,y)+\sum_{i=1}^{n-1}\frac{\partial\widetilde{G}_t}{\partial y_i}\left\{\widetilde{B}_i(y)+\int_0^x\frac{\partial\widetilde{G}_t}{\partial x}\widetilde{C}_i(x,y)dx\right\},
$$

with some smooth functions $\widetilde{A}, \dot{\widetilde{X}}, \widetilde{B}_i(y)$ and $\widetilde{C}_i$.

Note that:

$$
\frac{\partial\widetilde{G}_t}{\partial x} = \frac{\partial G_t}{\partial x}+\frac{\partial H}{\partial x}\left(\frac{\partial G_t}{\partial x}\right)^2+2H\frac{\partial^2 G_t}{\partial x^2}\frac{\partial G_t}{\partial x} = \frac{\partial G_t}{\partial x}K_1(x,y).
$$

Also, note that:

$$
\frac{\partial\widetilde{G}_t}{\partial y} = \frac{\partial G_t}{\partial y}+\frac{\partial H}{\partial y}\left(\frac{\partial G_t}{\partial x}\right)^2+2H\frac{\partial^2 G_t}{\partial x\partial y}\frac{\partial G_t}{\partial x} = \frac{\partial G_t}{\partial y}+\frac{\partial G_t}{\partial x}K_2(x,y).
$$

Hence, $TQ_{\tilde{G}_t}$ takes the form:

$$TQ_{\tilde{G}_t} = G_t A(x,y) + \frac{\partial G_t}{\partial x}\dot{X}(x,y) + \sum_{i=1}^{n-1} \frac{\partial G_t}{\partial y_i}\left\{B_i(y) + \int_0^x \frac{\partial G_t}{\partial x} C_i(x,y)dx\right\}.$$

This space coincides with the tangent space to the orbit at G_t. Hence, $\widetilde{G}_t$ is also admissible.

∎

Lemma 9.3.4 (Stabilization) *If the second derivative $f_{xx} \neq 0$, then the germ $f(x,y)$ is quasi projection equivalent to $x^2 + \widetilde{f}(y)$. For quasi projection equivalent germs f, the respective reduced germs $\widetilde{f}$ are quasi equivalent.*

Proof. We apply the standard O-equivalence. As $f_{xx} \neq 0$, then the germ f, in fact, is a deformation of A_1 singularity in x and with parameters y. Thus,

$$f(x,y) = x^2 + \varphi(x,y) \quad \text{where} \quad f(x,0) = f_0(x) = x^2.$$

The germ f_0 has a miniversal deformation of the form $F(x,\lambda) = x^2 + \lambda$. Hence, any deformation of f_0, can be induced from F and has the form : $\widetilde{f} = x^2 + \lambda(y)$.

Now, suppose that $f_t(x,y)$ is admissible deformation of functions with $\frac{\partial^2 f_t}{\partial x^2} \neq 0$, then f_t can be reduced to the form $\widetilde{f} = x^2 + \lambda_t(y)$. By lemma (9.3.2), the family $\lambda_t(y)$ is admissible. Hence all λ_t are quasi equivalent.

∎

Let V be a germ of regular hypersurface at a critical point of the projection. Up to a permutation of indices of y coordinates and up to a multiplication by a non vanishing factor the equation of V takes the form

$$f(x,y) = g(x,\widetilde{y}) + z$$

where $\widetilde{y} = (y_1, \ldots, y_{n-2})$, $z = y_{n-1}$ and $g \in \mathcal{M}^2_{x,\widetilde{y}}$.

The following lemma relates the tangent space TQ_f with that of the derivative $f_x = g_x$.

Lemma 9.3.5 *Let $\{TQ_f\}_x$ be the set of the derivatives with respect to x of germs from TQ_f then*

$$TQ_{f_x} \subset \{TQ_f\}_x \subset TQ_{f_x} + fC_{x,y}.$$

Remark. Restrict the germs from the modules, mentioned in the statement of this lemma, to the hypersurface V (that is make the substitution $y_{n-1} = -g(x, \widetilde{y})$). We get the nature inclusions of these tangent spaces into the modules over the algebra of functions in x and $\widetilde{y}$ related to the projections of hypersurfaces in $\mathbb{R}^{n-1}$:

$$TQ_{g_x} \subset \{TQ_g\}_x \subset TQ_{g_x} + gC_{x,\widetilde{y}}.$$

Proof. A function germ $h(x, y) \in TQ_f$ if it admits a decomposition

$$h = fa + f_x b + \sum_{i=1}^{n-1} f_{y_i} \left(c_i + \int_0^x f_x e_i dx \right), \qquad (*)$$

with some smooth functions $a, b \in \mathbf{C}_{x,y}$ and $e_i \in \mathbf{C}_y$, $i = 1, \ldots, n-1$. The differentiation of the equation (*) with respect to x yields

$$h_x = fa_x + f_x a + f_x b_x + f_{xx} b + \sum_{i=1}^{n-2} f_{xy_i} \left(c_i + \int_0^x f_x e_i dx \right) + \sum_{i=1}^{n-2} f_{y_i} f_x e_i + f_x e_{n-1},$$

or equivalently

$$h_x = (y_{n-1}+g)a_x+\widetilde{g}(e_{n-1}+a+b_x+\sum_{i=1}^{n-2} f_{y_i} e_i)+\widetilde{g}_x b+\sum_{i=1}^{n-2} \widetilde{g}_{y_i} \left(c_i(\widetilde{y}, y_{n-1}) + \int_0^x \widetilde{g} e_i dx \right),$$

where $\widetilde{g} = g_x$. The factors $y_{n-1} + g, \widetilde{g}$, and $\widetilde{g}_x$ are independent functions; hence the right hand side of the last decomposition represents an arbitrary function from

$TQ_{\tilde{g}} + f\mathbf{C}_{x,y}$. Setting $y_{n-1} = -g(x, \tilde{y})$, we get the required inclusions in the space of germs in $x, \tilde{y}$ and consequently in the space of germs in x, y. Note that, Hadamard Lemma yields that $c_i(\tilde{y}, g) = c_i(\tilde{y}, 0) + g\tilde{c}_i(\tilde{y}, g)$. Let $V = g\tilde{c}_i(\tilde{y}, g)$, then $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial g} \cdot \frac{\partial g}{\partial x}$. ∎

Corollary 9.3.6 *Assume that the germ $\tilde{g}$ is quasi projection simple in the space of projections of hypersurfaces in $\mathbb{R}^{n-1}$. In other words, a neighborhood of $\tilde{g}$ consists of finitely many quasi-orbits. Then f is quasi-simple.*

Proof. Indeed, a neighborhood of a regular germ f is the space of primitives of functions close to $\tilde{g}$. Due to the left inclusion of the lemma, an admissible deformation of the derivatives produces an admissible deformation of the primitives. ∎

Assign some positive weights $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ to the variables $x, y_1, \ldots, y_{n-1}$.

Definition 9.3.1 A system of weights α_i is called *adopted* if the derivative of any weighted homogeneous germ f of degree d along any vector field that preserve the direction $\frac{\partial}{\partial x}$ and vanish at a distinguished point has degree $\geq d$

Lemma 9.3.7 *Let $f_t, 0$ be an admissible deformation of function germs. Let the basic points be at the origin for any t. Let α be an adopted system of weights. Then the non zero terms of f_t of the lowest α-degree are equivalent with respect to the standard projection.*

Proof. Take the lowest order terms of f_t. The order does not depend on t. Take the lowest order terms in the decomposition

$$h = af + f_x b + \sum_{i=1}^{n-1} f_{y_i} \left(c_i + \int_0^x f_x e_i dx \right), \qquad (*).$$

Notice that the integral terms that do not belong to the tangent space of the quasi equivalence can not enter because of higher degree. Hence the weighted homogeneous lowest order terms on the right-hand side of (*) belong to the O-tangent space, which proves the lemma.

∎

Corollary 9.3.8 *Under the conditions of Lemma 9.3.7, assume that α-lowest part f_0 of f is "parabolic," that is the space of weighted homogeneous deformations of f_0 contains a continuous family of O-orbits. Then, f is not quasi simple.*

9.4 Classification of simple classes

The classification of simple classes in low dimensions is given in the following theorems.

Theorem 9.4.1 *If $n = 2$ the list of simple classes is the same as for standard O-group of foliation preserving diffeomorphisms of the plane acting on the germs of curves.*

$$A_k : f = x^{k+1} + y; \quad k = 0, 1, \dots$$

$$B_k : f = x^2 \pm y^k, \quad C_k : f = xy + x^k, \quad k = 2, 3, \dots,$$

$$F_4 : f = x^3 + y^2.$$

Proof of Theorem 9.4.1

We start with O-classifications. We will use the spectral sequence method. Let $f(x, y) = a_1 x + a_2 y + b_1 x^2 + b_2 xy + b_3 y^2 + \dots$. Then, we distinguish the following cases:

• If $\frac{\partial f}{\partial x}(0) \neq 0$, then consider the principal part $f_0 = x$. Clearly, f is equivalent to $f_0 = x$ as the tangent space coincides with the space of all germs. Note that the germ f can be written as equivalent to $A_0 : g = x + y$.

• If $\frac{\partial f}{\partial x}(0) = 0$ but $\frac{\partial f}{\partial y}(0) \neq 0$, then consider the principal part $f_0 = y$. The tangent space contains all germs which divisible by y. Hence, the germ f is equivalent to

$\widetilde{f} = y + \varphi(x)$, with $\varphi \in \mathcal{M}_x^2$. Suppose that, $\varphi(x) = \alpha_k x^k + \alpha_{k+1}x^{k+1} + \ldots$, with $\alpha_k \neq 0$ and $k \geq 2$. Consider the germ $\widetilde{f}_0 = y + \alpha_k x^k$. Then, the tangent space to the orbit at $\widetilde{f}_0$ is

$$TQ_{\widetilde{f}_0} = (y + \alpha_k x^k)a(x,y) + x^{k-1}b(x,y) + c(y).$$

Hence, $\widetilde{f}$ after normalization α_k is equivalent to $A_{k-1}: \quad g(x,y) = y + x^k, k \geq 2$.

• If $\frac{\partial f}{\partial y}(0) = \frac{\partial f}{\partial x} = 0$ and $\frac{\partial^2 f}{\partial x^2}(0) \neq 0$, then Lemma (9.3.4) yields that f is equivalent to $\widetilde{f} = x^2 + \varphi(y)$ with $\varphi(y) \in \mathcal{M}_y^2$. Let $\varphi(y) = \beta_k y^k + \beta_{k+1}y^{k+1} + \ldots$ with $\beta_k \neq 0$ and $k \geq 2$. Consider the principal part $\widetilde{f}_0 = x^2 + \beta_k y^k$. Then, the tangent space to the orbit at $\widetilde{f}_0$ is

$$TQ_{\widetilde{f}_0} = (x^2 + \beta_k y^k)a(x,y) + xb(x,y) + y^{k-1}c(y).$$

Hence, $\widetilde{f}$ is equivalent to , after normalization β_k, $B_k: \quad g = x^2 \pm y^k, k \geq 2$.

• If $\frac{\partial f}{\partial y}(0) = \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}(0) = 0$ but $\frac{\partial^2 f}{\partial x \partial y}(0) \neq 0$, then consider the principal part $f_0 = xy$. The tangent space to the orbit at f_0 takes the form:

$$TQ_{f_0} = xyA(x,y) + yB(x,y) + xD(y).$$

Hence, f is equivalent to $\widetilde{f} = xy + \varphi(x)$ with $\varphi(x) \in \mathcal{M}_x^3$. Let $\varphi(x) = \alpha_k x^k + \alpha_{k+1}x^{k+1} + \ldots$ with $\alpha_k \neq 0$ and $k \geq 3$. Consider the main part $\widetilde{f}_0 = xy + \alpha_k x^k$. The tangent space to the orbit at $\widetilde{f}$ has the form

$$TQ_{\widetilde{f}_0} = (xy + \alpha_k x^k)a(x,y) + (y + k\alpha_k x^{k-1})b(x,y) + xc(y).$$

If we substitute *mod* $TQ_{\widetilde{f}_0}$: $y \equiv -k\alpha_k x^{k-1}$ in $xy + \alpha_k x^k \equiv 0$, we get: $x^k \equiv 0$. Hence, $xy \equiv 0$ and $y \equiv 0$. Thus, after normalization β_k the germ $\widetilde{f}$ is equivalent to $C_k: \quad g = xy \pm x^k, k \geq 3$.

• If $\frac{\partial f}{\partial y}(0) = \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}(0) = \frac{\partial^2 f}{\partial x \partial y}(0) = 0$, but $\frac{\partial^2 f}{\partial y^2}(0) \neq 0$, then consider the principal part $f_0 = y^2$. The quotient space $\mathbf{C}[x,y]/TQ_{f_0}$ is generated by $y\widetilde{h}_1(x)$ and $\widetilde{h}_2(x)$. Hence, f is equivalent to $\widetilde{f} = y^2 + yh_1(x) + h_2(x)$ with $h_1 \in \mathcal{M}_x^2$ and $h_2 \in \mathcal{M}_x^3$. Let $h_1 = a_2x^2 + a_3x^3 + \ldots$ and $h_2 = b_2x^3 + b_4x^4 + \ldots$.

Suppose that $b_2 \neq 0$ and consider the main part $\widetilde{f}_0 = y^2 + b_3x^3$. Then, the tangent space to the orbit at $\widetilde{f}$ takes the form:

$$TQ_{\tilde{f}_0} = (y^2 + a_3x^3)A(x, y) + x^2B(x, y) + yC(y).$$

We have $mod\, TQ_{\tilde{f}_0}$: $x^2B \equiv 0$. Hence, $y^2 \equiv 0$. Thus, The quotient space $\mathbf{C}[x, y]/TQ_{\tilde{f}_0}$ is generated by $1, x, y$ and xy. Hence, the germ $\widetilde{f}$ is equivalent, after normalization a_3, to $F_4 : g = y^2 + x^3$.

Other germs are adjacent either to the class G with zero 2-jet or to the class $F_5 : \quad f(x, y) = y^2 + y(a_2x^2 + a_3x^3 + \dots) + b_4x^4 + b_5x^5 + \dots$.

In the first case consider the 3-jet which is the lowest quasi homogeneous part $f_3 = \hat{a}_1x^3 + \hat{a}_2x^2y + \hat{a}_3xy^2 + \hat{a}_4y^3$. Then, the tangent space with respect to the standard O-projection at f_3 takes the form:

$$TQ_{f_3} = f_3A(x, y) + (3\hat{a}_1x^2 + 2\hat{a}_2xy + a_3y^2)B(x, y) + (\hat{a}_2x^2 + 2\hat{a}_3xy + 3\hat{a}_4y^2)C(y).$$

or equivalently,

$$TQ_{f_3} = (3\hat{a}_1x^2 + 2\hat{a}_2xy + a_3y^2)\widetilde{B}(x, y) + (\hat{a}_2x^2 + 2\hat{a}_3xy + 3\hat{a}_4y^2)\widetilde{C}(y),$$

as f_3 is quasi homogeneous.

The cubic terms are obtained from the tangent space if $\widetilde{B} = \tilde{a}_0x + \tilde{b}_0y$ and $\widetilde{C} = \tilde{c}_0y$ where $\tilde{a}_0, \tilde{b}_0, \tilde{c}_0 \in \mathbb{R}$. Hence, the the dimension of the subgroup of linear transformations with an eigenvector along the x-axis which is generated by $\widetilde{B}$ and $\widetilde{C}$ is 3 which is less than the dimension 4 of the 3-jets of the functions from the class G. Hence, the class G is non-simple with respect to the standard O-equivalence equivalence. By lemma, 9.3.7, the class G remains non-simple with respect to quasi projection equivalence as the lowest quasi homogeneous part is non-simple with respect to the standard O-equivalence.

In the second case, consider the lowest quasi homogeneous part: $f_4 = y^2 + a_2x^2y + b_4x^4$, with quasi degree 1 with respect of weight of x being $\frac{1}{4}$ and weight of y being

$\frac{1}{2}$. Then, the tangent space at f_4 takes the form:

$$TQ_{f_4} = (2a_2xy + 4b_4x^3)B(x,y) + (2y + a_2x^2)C(y).$$

The quasi homogeneous part is obtained from the tangent space if $B = \alpha_0 x$ and $C = \gamma_0 y$ where $\alpha_0, \beta_0 \in \mathbb{R}$. Hence, the the dimension of the subgroup of the linear transformations with an eigenvectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ which is generated by B and C is 2 which is less than the dimension 3 of the quasi homogeneous part f_4 of the functions from the class F_5. Hence, the class F_5 is non-simple with respect to the standard O-projection equivalence. So, Lemma 9.3.7 yields that the class G remains non-simple with respect to quasi projection equivalence as the lowest quasi homogeneous part is non-simple with respect to the standard O-equivalence.

Thus, the classes A_{k-1}, B_k, C_k and F_4 are the only simple quasi projection classes. The theorem is proven.

Theorem 9.4.2 *For $n = 3$ the list of simple quasi projection of regular hypersurfaces singularities consists of*

$$\widetilde{A}_k : f = x^{k+1} + y_1x + y_2, \quad k \geq 0,$$

$$\widetilde{B}_k : f = x^3 + y_1^k x + y_2, \quad k \geq 2,$$

$$\widetilde{C}_k : f = x^{k+1} + x^2y_1 + y_2, \quad k \geq 2,$$

$$\widetilde{F}_4 : f = x^4 + y_1^2 x + y_2.$$

The list of simple quasi projections of singular hypersurfaces is as follows:

$$A_k^*,\ D_k^*,\ E_s^*,\ s = 6,7,8 : f = x^2 + g(y_1, y_2)$$

where g is one of the standard simple A, D, E function germ in y;

$$A_2^{**} : f = x^3 + y_1x + y_2^2;$$

$$A_2^{(k)} : f = x^3 + y_1^k x + y_1^2 + y_2^2,\ k \geq 2.$$

Proof of Theorem 9.4.2

• In the case of regular hypersurfaces, the result follows from Lemmas 9.3.2, 9.3.5 and 9.3.7. Thus, the integration of simple classes of Theorem 9.4.1 gives simple classes for $n = 3$.

$$\widetilde{A}_k : f = x^{k+1} + y_1 x + y_2, \quad k \geq 0,$$

$$\widetilde{B}_k : f = x^3 + y_1^k x + y_2, \quad k \geq 2,$$

$$\widetilde{C}_k : f = x^{k+1} + x^2 y_1 + y_2, \quad k \geq 2,$$

$$\widetilde{F}_4 : f = x^4 + y_1^2 x + y_2.$$

These classes are nonequivalent due to multiplicity reasons. The remaining classes after differentiation contain the germs of functions in the plane that either have zero 2-jets or have degree ≥ 1 with respect to adopted weights $\frac{1}{4}$ and $\frac{1}{2}$ for x and y, respectively. Lemmas 9.3.5, 9.3.7 and the proof of the previous theorem yields that these classes are non-simple in this case.

For singular hypersurfaces, we start with the O-classifications, using Lemma 9.3.2, we distinguish the following cases:

• If a germ f is a deformation of A_1 singularity in x with parameters y_1 and y_2, then Lemma 9.3.4 yields that f is O-equivalent to the germ: $\widetilde{f} = x^2 + g(y_1, y_2)$. Hence, we need to classify the germs $g(y_1, y_2)$. The tangent space to the orbit at g coincides with the tangent space with respect to standard right equivalence. Thus, the germ g belongs to one of the classes A_k, D_k or E_s, $s = 6, 7, 8$. Thus, the following classes

$$A_k^*, \; D_k^*, \; E_s^*, \; s = 6, 7, 8 : f = x^2 + g(y_1, y_2)$$

where g is one of the standard simple A_k, D_k, E_s function germ in (y_1, y_2); remain simple with respect to quasi equivalence.

All other germs are adjacent to the nonsimple class

$$J_{10} : \widetilde{g} = y_1^3 + y_2^6 + a y_1^2 y_2^2, \quad \text{with} \quad 4a^3 + 27 \neq 0.$$

The class is quasi homogeneous with respect to the adopted weights 2 for y_1 and 1 for y_2. By lemma 9.3.7, the class J_{10} remains nonsimple with respect to quasi equivalence.

• If a germ f is a deformation of A_2 singularity in x with parameters y_1 and y_2. That is $f(x, y_1, y_2) = x^3 + \varphi(x, y_1, y_2)$ and $f(x, 0, 0) = f_0(x) = x^3$. Then, the germ $f_0(x)$ has a miniversal deformation $\Phi(x, \lambda_0, \lambda_1) = x^3 + \lambda_1 x + \lambda_0$. Hence, any deformation is induced from Φ and has the form:

$$\widetilde{f}(x, y_1, y_2) = x^3 + h_1(y_1, y_2)x + h_2(y_1, y_2) \quad \text{with} \quad h_1 \in \mathcal{M}_{y_1,y_2} \quad \text{and} \quad h_2 \in \mathcal{M}^2_{y_1,y_2}.$$

Let $h_1(y_1, y_2) = a_1y_1 + a_2y_2 + b_1y_1^2 + b_2y_1y_2 + b_3y_2^2 + \dots$.

Up to permutations between y_1 and y_2, suppose that $a_1 \neq 0$, then f can be reduced to the form $\widetilde{f} = x^3 + \widetilde{a}_1y_1x + \widetilde{h}(y_1, y_2)$ where $\widetilde{h} \in \mathcal{M}^2_{y_1,y_2}$. Assume that $\widetilde{h}(y_1, y_2) = c_1y_1^2 + c_2y_1y_2 + c_3y_2^2 + \dots$ and $c_3 \neq 0$. Consider the main quasi homogeneous part: $\widetilde{f_0} = x^3 + \widetilde{a}_1y_1x + c_3y_2^2$. Then, the tangent space at $\widetilde{f_0}$ takes the form:

$$TQ_{\widetilde{f_0}} = (x^3 + \widetilde{a}_1y_1x + c_3y_2^2)A(x, y_1, y_2) + (3x^2 + \widetilde{a}_1y_1)B(x, y_1, y_2) + xC(y_1, y_2) + y_2D(y_1, y_2).$$

Then, we obtain *mod* $TQ_{\widetilde{f_0}}$:

$$y_1 \equiv \frac{-3}{a_1}x^2, \tag{9.1}$$

This relation yields that

$$\mathbf{C}_{x,y_1,y_2}/TQ_{\widetilde{f_0}} \cong \mathbf{C}_{x,y_2}/T^*Q_{f_0} \cong \mathbb{R}\{1, x^2\},$$

where $T^*Q_{f_0} = (-2x^3 + c_3y_2^2)A(x, y_2) + xC(x^2, y_2) + y_2D(x^2, y_2)$.

Thus, after normalization of a_1 and c_3, $\widetilde{f}$ becomes equivalent to the germ $F_1 = x^3 + y_1x + y_2^2$.

Next, suppose that $a_1 = a_2 = 0$ and h_2 is non-degenerate function, then f can be reduced to the form: $\widetilde{f} = x^3 + xH(y_1, y_2) + y_1^2 + y_2^2$ where $H \in \mathcal{M}^2_{y_1,y_2}$. Consider the main quasi homogeneous part: $\widetilde{f_0} = x^3 + c_kxy_1^k + y_1^2 + y_2^2$ where $k \geq 2$. The tangent space at $\widetilde{f_0}$ contains all deformation of the form $xH_t(y_1, y_2)$. To prove this claim, consider the tangent space at $\widetilde{f}_0$

$$\begin{aligned} TQ_{\widetilde{f}_0} &= (x^3 + c_k x y_1^k + y_1^2 + y_2^2)A(x, y_1, y_2) + (3x^2 + c_k y_1^k)B(x, y_1, y_2) \\ &+ (2y_1 + k c_k x y_1^{k-1})C(y_1, y_2) + y_2 D(y_1, y_2). \end{aligned}$$

Let $A = c_k y_1^k$ and $B = y_1^2 + y_2^2$. Then the the function $\widetilde{f}_0 = x^3 + Ax + B$ is quasi homogeneous with respect to weights $w_x = \frac{1}{3}$, $A = \frac{2}{3}$ and $B = 1$. Thus, we can write $\widetilde{f}_0 = \frac{1}{3} x \frac{\partial \widetilde{f}_0}{\partial x} + \frac{2}{3} Ax + B$. Thus, we obtain *mod* $TQ_{\widetilde{f}_0}$

$$\widetilde{f}_0 - \frac{1}{3} x \frac{\partial \widetilde{f}_0}{\partial x} = \frac{2}{3} Ax + B \equiv 0, \tag{9.2}$$

Multiplying the equation (9.2) by x we get

$$\frac{2}{3} Ax^2 + Bx \equiv 0. \tag{9.3}$$

Substituting $x^2 \equiv \frac{-c_k}{3} y_1^k$ in the equation (9.3) we obtain

$$\frac{-2c_k}{9} y_1^k A + Bx \equiv 0. \tag{9.4}$$

Also, we have

$$x[k c_k y^{k-1}] + 2y_1 \equiv 0, \tag{9.5}$$

and

$$y_2 \equiv 0. \tag{9.6}$$

Now all terms in $x H_t(y_1, y_2)$ belong to $TQ_{\widetilde{f}_0}$, using the relations (9.2), (9.4), (9.5) and (9.6). Hence, we conclude that $\widetilde{f}$ is equivalent $\hat{f} = x^3 + x y_1^k + y^2 + y_2^2$.

Other germs are adjacent to a germ of the form:

$$\widetilde{f} = ax^3 + by_1 x + c y_1 y_2 + d y_2^3 + \varphi(y_1, y_2)$$

of quasi degree ≥ 1 with respect to the weights of x and y_2 being $\frac{1}{3}$ and y_1 being $\frac{2}{3}$. Let $f_0 = ax^3 + by_1x + cy_1y_2 + dy_2^3$. Then, the tangent space at f_0 takes the form:

$$TQ_{f_0} = (3ax^2 + by_1)B(x, y_1, y_2) + (bx + cy_2)C(y_1, y_2) + (cy_1 + 3dy_2^2)D(y_1, y_2).$$

The terms of f_0 are obtained from if $B = \beta x$, $C = \alpha y_1$ and $D = \gamma y_2$ where $\beta, \alpha, \gamma \in \mathbb{R}$. Hence the dimension of the subgroup of the linear transformation generated by B, C and D is 3 which is less than the dimension of the quasi homogeneous part f_0. Thus, the germ $\widetilde{f}$ is non-simple with respect to the standard O-equivalence. Lemma 9.3.7 yields that the germ $\widetilde{f}$ is also non-simple with respect to quasi equivalence.

- The deformation of A_3 (or of A_k, $k > 3$) are adjacent to

$$f = x^4 + y_1x^2 + y_2x + \varphi(y_1, y_2) \quad \text{with} \quad \varphi \in \mathcal{M}^2_{y_1,y_2}.$$

Consider the lowest quasi homogeneous part $g = ax^4 + by_1x^2 + cy_2x + dy_1^2$ with respect to the weights $w_x = \frac{1}{4}, w_{y_1} = \frac{1}{2}$ and $w_{y_2} = \frac{3}{4}$. Take the tangent space at g:

$$TQ_g = (4ax^3 + 2by_1x + cy_2)B(x, y_1, y_2) + (bx^2 + 2dy_1)C(y_1, y_2) + xD(y_1, y_2).$$

The dimension of the subgroup of the linear transformation generated by $B = \beta x$, $C = \alpha y_1$ and $D = \gamma y_2$ where $\beta, \alpha, \gamma \in \mathbb{R}$ is 3 which is less than the dimension of g. Thus, the germ f is non-simple with respect to the standard O-equivalence. By Lemma (9.3.7), we conclude that the germ f is also non-simple with respect to quasi equivalence. This finishes the proof of the theorem.

9.5 Quasi vf projection

In [36], another example of non-stanard equivalence in the projection theory was introduced. It is called pseudo and quasi vf equivalences.

Let v be a non-singular field in the space where a function or a complete intersection is defined.

Definition 9.5.1 Two functions $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2$ are called *pseudo vf-equivalent* if there is a diffeomorphism $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_2 = f_1 \circ \Theta$ and if m is a critical point of f_2 then the linear part of Θ at m maps the direction of the vector field v to the direction of v at the image $\Theta(m)$.

Denote by Pv_f the equivalence class of a germ f and call it the *pseudo-vf-orbit* of f.

This equivalence takes an intermediate place between the standard right-equivalence and the right action of fibration preserving diffeomorphisms $\theta : \mathbb{R}^n \to \mathbb{R}^n, \theta_* v = hv$, for some non-zero factor $h : \mathbb{R}^n \to \mathbb{R}$.

One of the possible applications of vf singularities is the classification of vertical vector fields on Lagrangian submanifolds. The setting is as follows: Critical points of a function depending on parameters define Lagrangian submanifold. Vector field defines the flow of right-equivalences, which defines a family of Lagrange equivalences (without changing parameters). Therefore vectors of the vector fields evaluated at the critical points of the function define a vector field on Lagrange manifold which is along the fibers of Lagrange projection to the base (parameter space). The singularities of these vector fields are of interest in variational problems with constraints.

We shall consider the simplest case of non-singular vector field $v = \frac{\partial}{\partial x}$ where $x \in \mathbb{R}, \mathbb{R}^n = \{(x, y)\}$ and $y \in \mathbb{R}^{n-1}$. Notice that the diffeomorphisims θ which preserve the fibration $\pi : (x, y) \mapsto y$ takes the form $\theta : (x, y) \mapsto (X(x, y), Y(y))$.

Denote by J_∇ the ideal generated by the derivatives of the function f.

Differentiating upon parameter all deformations within the pseudo-vf-orbit of a given germ f, we obtain the tangent space TPv_f at f to the orbit Pv_f.

Proposition 9.5.1 *The tangent space TPv_f is given by the formula*

$$TPv_f = f_x \dot{X} + \sum_{i=1}^{n-1} f_{y_i} \dot{Y}_i,$$

where $\dot{X} \in \mathbf{C}_{x,y}$ and $\dot{Y}_i \in$ $\mathrm{IRad}(J_\nabla)$.

Proof. The proof is similar to the proof of proposition 9.1.1.

∎

As in the previous construction of quasi projection to get better properties with respect to parameter dependence, take the following sub-module

$$TQv_f = f_x\dot{X} + \sum_{i=1}^{n-1} f_{y_i}\dot{Y}_i \subset TPv_f, \qquad (*)$$

where $\dot{X} \in \mathbf{C}_{x,y}$ and $\dot{Y}_i \in IJ_\Delta$, as the set of admissible infinitesimal deformations of a function, and introduce the respective notion Qv_f of the class of quasi equivalence relation which is finer than the pseudo-equivalence class.

Definition 9.5.2 Two functions $f_1, f_2 : (\mathbb{R}^n, 0) \to \mathbb{R}$ are called *quasi-vf-equivalent* if there is a family of function germs f_t which continuously depends on parameters $t \in [1,2]$ and a continuous piece-wise smooth family of diffeomorphisms $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ depends on parameters $t \in [1,2]$, such that: $f_t \circ \theta_t = f_1$, $\theta_1 = id$ and the components of the vector field generated by θ are of the form:

$$\dot{X} \in \mathbf{C}_{x,y} \quad \forall i, \dot{Y}_i \in \left\{ c_i(y) + \int_0^x \left(\frac{\partial f_t}{\partial x} a_i(x,y) + \sum_{j=1}^{n-1} \frac{\partial f_t}{\partial y_j} b_{i,j}(x,y) \right) dx \right\},$$

where $a_i, b_{i,j} \in \mathbf{C}_{x,y}$ and $c_i \in \mathbf{C}_y$.

The classification of simple quasi vf projection is given as follows.

Theorem 9.5.2 *The simple quasi vf projection classes of function germs are given by the following list:*

1. $A_0 : f = x$,
2. $\widetilde{A}, \widetilde{D}, \widetilde{E} : f = x^2 + f(y)$, *where* $\widetilde{f}(y)$- *is a standard simple singularity class* A_k, D_k, E_k,
3. $C_2 : f = xy_1 \pm y_2^2 \cdots \pm y_{n-1}^2$,

4. $F_{n+1} : f = x^3 \pm y_1^2 \cdots \pm y_{n-1}^2$.

Remarks:

1. The fencing non simple classes are $\widetilde{F} : x^4 + ax^2y_2^2 \pm y_1^2 \cdots \pm y_{n-1}^2$ and C : $xy_1 + ay_2^3 \pm y_3^2 \cdots \pm y_{n-1}^2$.

2. The adjacency of the classes of low codimension in the plane $(x, y) \in \mathbb{R}^2$ is shown in the table:

$$\begin{array}{ccccccc} A_0 & \leftarrow & \widetilde{A}_2 & \leftarrow & \widetilde{A}_2 & \leftarrow & \ldots \\ & & \uparrow & & & & \\ & & C_2 & \leftarrow & F_{n+1} & \leftarrow & \widetilde{F} \end{array}$$

3. The contact quasi vf projection classification coincides with the right one. This is because all classes given in the theorem are weighted homogeneous.

Proof of Theorem 9.5.2

If the function germ f is non-singular then we obtain $x \pm y_1^2 \cdots \pm y_{n-1}^2$. Suppose that the germ f is singular and consider the two jet of f, restricted to the subspace $y = 0$. If $J^2(f)$ contains αx^2 then the function is quasi vf-equivalent to the germ $F = \pm x^2 + g(y)$. Two functions of these type are quasi vf- equivalent if the respective germs $g(y)$ are right equivalent. If $J^2(f)$ contains $xg(y)$ term (where $g(y)$ is linear in y) then the function germ is quasi vf-equivalent to $G = xy_1 + g(y)$ where $g \in \mathcal{M}^2_{y_2,\ldots,y^{n-1}}$. The germ G is quasi vf-equivalent to the simple germ $\widetilde{F} = xy_1 \pm y_2^2 \cdots \pm y_{n-1}^2$. Finally, the function germ f with zero two jet and has the form $f = \alpha x^3 + g(y)$ is quasi vf-equivalent to the simple germ $x^3 \pm y_1^2 \cdots \pm y_{n-1}^2$. Other germs are adjacent to non-simple classes with respect to quasi vf-equivalence relation.

Chapter 10

Quasi projection with boundaries

10.1 Introduction

In this chapter, we classify simple singularities of projections to a plane of surfaces embedded into three-space and equipped with a boundary. We will use two special equivalence relations which are more rough than the standard one and similar to quasi projection of hypersurfaces. They are called quasi strong and quasi weak equivalence relations. We shall classify the simple quasi strong singularities only.

Consider the space $\mathbb{R}^n = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}^{n-1}\}$ and the trivial bundle structure defined by the projection $(x,y) \mapsto y$. So the fibres here are one-dimensional.

Again for simplicity, we will consider the analytic case. So, let $V = \{(x,y) : f(x,y) = 0\}$ be an analytic hypersurface given by a single equation. Let $B = \{(x,y) : f(x,y) = g(x,y) = 0\}$ be its distinguished boundary which is subvariety in V of codimension 1. Then, the pair (V, B) is called a hypersurface with boundary. Denote by $G = (f, g)$ the pair of the equations which define the hypersurface V with the boundary B.

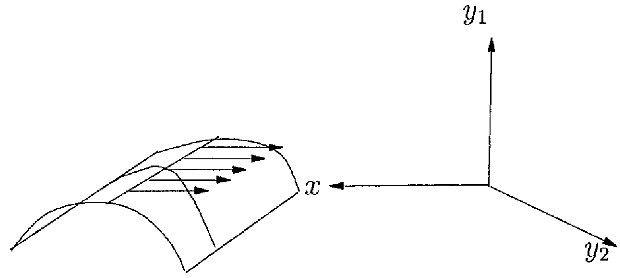

Figure 10.1: Strong quasi projection with boundary .

10.2 The strong equivalence relation

Definition 10.2.1 Two hypersurfaces with boundaries $G_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$ and $G_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$ are called pseudo-strong equivalent if there is a diffeomorphism $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that:

1. $G_1 = MG_2 \circ \theta$, where $M = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, where $a, b, c \in \mathbf{C}_{x,y}$ with $a(0)c(0) \neq 0$.
2. The set of critical points of the projection of V_2 is mapped by θ onto the set of critical points of the projection of V_1.
3. The differential of θ at any critical point maps the direction of the projection to that of the image of the point.

Remark: We call this equivalence **strong** because we preserve the direction at all critical points of the projection.

Differentiating all deformations within the pseudo-strong-orbit of a given pair $G = \begin{pmatrix} f \\ g \end{pmatrix}$ with respect to a parameter, we get the tangent space $TPS_G = \begin{pmatrix} TPS_f \\ TPS_g \end{pmatrix}$ to the pseudo-strong-orbit PS_G at the pair G.

Proposition 10.2.1 *The tangent space TPS_G is given by the formula*

$$TPS_G = \begin{pmatrix} TPS_f \\ TPS_g \end{pmatrix} = \begin{pmatrix} fA + f_x\dot{X} + \sum_{i=1}^{n-1} f_{y_i}\dot{Y}_i. \\ fB + gC + g_x\dot{X} + \sum_{i=1}^{n-1} g_{y_i}\dot{Y}_i. \end{pmatrix}$$

where $A, B, C, \dot{X} \in \mathbf{C}_{x,y}$, $\dot{Y}_i \in IRad(J_f)$.

Proof. The proof is similar to the proof of proposition 9.1.1.

∎

Again, this equivalence relation does not satisfy the properties of a geometrical subgroup of equivalences in the Damon sense [16]. So, as before we use a subspace of TPS_G that behaves regularly when the pair G depends on extra parameters. Namely, take the submodule

$$TQS_G = \begin{pmatrix} TQS_f \\ TQS_g \end{pmatrix} = \begin{pmatrix} fA + f_x\dot{X} + \sum_{i=1}^{n-1} f_{y_i}\dot{Y}_i. \\ fB + gC + g_x\dot{X} + \sum_{i=1}^{n-1} g_{y_i}\dot{Y}_i. \end{pmatrix} \subset TPS_G,$$

where $A, B, C, \dot{X} \in \mathbf{C}_{x,y}$, $\dot{Y}_i \in IJ_f$.

Definition 10.2.2 Two hypersurfaces with boundaries $G_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$ and $G_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$, are called QS-equivalent if there exists a family of diffeomorphisms $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ continuously and piecewise smoothly depending on parameter $t \in [1, 2]$ and a family of matrices

$$M_t = \begin{pmatrix} a_t & 0 \\ b_t & c_t \end{pmatrix},$$

where $a_t, b_t, c_t \in \mathbf{C}_{x,y,t}$, $a_t(0)c_t(0) \neq 0$,$M_2 = I_2$ and $\theta_2 = id$ and a family of pairs $G_t = \begin{pmatrix} f_t \\ g_t \end{pmatrix}$, with $t \in [1, 2]$, such that: For any $t \in [1, 2]$ we have $G_t = M_t G_2 \circ \theta_t$

and the vector field $V = (\dot{X}, \dot{Y}_i)$ generated by θ_t is of the following form: $\dot{X} \in \mathbf{C}_{x,y}$ and $\dot{Y}_i \in IJ_{f_t}$.

We start with the case of the regular hypersurface $V = \{(x, y, z) \in \mathbb{R}^3 : \widetilde{f}(x, y, z) = f(x, y) + z = 0\}$. Let the boundary be $B = \{(x, y, z) \in \mathbb{R}^3 : \widetilde{f}(x, y, z) = g(x, y, z) = 0\}$ and the natural projection be $(x, y, z) \mapsto (y, z)$. The classification of QS- simple classes is carried out in the following order. At first, we classify the surfaces with respect to quasi projection equivalence. Hence, we get the simple classes listed in the theorem 9.4.2. Secondly, we classify the boundary B for each class obtained in the first step. This means that we need to calculate the stationary algebra of the admissible vector fields

$$W = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z},$$

the flow of which provides a quasi equivalence of the surface V with itself. Then, we classify the orbits of its action on the equations $g(x, y) = 0$ of the boundary modulo the equation of the surface.

Definition 10.2.3 The vector field W is called *stationary* with respect to quasi projection of the surface V, if the diffeomorphism generated by W preserves the surface V and the direction of projection at the critical pints of the projection.

The stationary vector field W which is tangent to V satisfies

$$H(x, y, z)(f(x, y) + z) + f_x\dot{x} + f_y\dot{y} + \dot{z} = 0, \tag{10.1}$$

for some function H. Its components $\dot{x}$, $\dot{y}$ and $\dot{z}$ satisfy: $\dot{x} \in \mathbf{C}_{x,y}$ and $\dot{y}, \dot{z} \in IJ_f$. It follows that the components $\dot{y}$ and $\dot{z}$ have the forms

$$\dot{y} = Y_0(y, z) + \int_0^x ((f(x, y) + z)h_1(x, y, z) + f_x(x, y)h_2(x, y, z))\, dx, \tag{10.2}$$

$$\dot{z} = Z_0(y, z) + \int_0^x ((f(x, y) + z)h_3(x, y, z) + f_x(x, y)h_4(x, y, z))\, dx, \tag{10.3}$$

for some smooth functions $Y_0, Z_0, h_i, i = 1, \dots, 4$.

Denote by $\widetilde{W} = \dot{x}(x, y, -f(x, y))\frac{\partial}{\partial x} + \dot{y}(x, y, -f(x, y))\frac{\partial}{\partial y}$ the projection of the vector field W to the x, y coordinate plane. Denote its components by $\dot{X}, \dot{Y}$.

Differentiation of (10.1) with respect to x provides the proof of the following:

Lemma 10.2.2 *Vector field W is stationary if and only if $\widetilde{W}$ is tangent to the critical point locus, that is,*

$$A(x, y)f_x + f_{xx}\dot{X} + f_{xy}\dot{Y} = 0,$$

for some function A, and the component $\dot{Y}$ has the form

$$\dot{Y} = Y_1(y) + \int_0^x f_x(x, y)h(x, y)dx, \tag{10.4}$$

with some smooth functions Y_1, h.

Proof. Differentiate (10.1) with respect to x and get

$$(H_x + h_3 + f_y h_1)(f(x, y) + z) + f_x(\dot{x}_x + H + h_4 + f_y h_2) + f_{xx}\dot{x} + f_{xy}\dot{y} = 0.$$

The relation (10.2) and integration by parts imply

$$\dot{y} = Y_0(y, z) + \widetilde{h}_1(x, y, z)(f(x, y) + z) + \int_0^x f_x(h_2 - \widetilde{h}_1\,(x, y, z))\,dx,$$

with some smooth function $\widetilde{h}_1$. Restricting the last formulas to the surface V we get the required relations. Note that for the second relation, we get:

$$\dot{y} = Y_0(y, -f(x, y)) + \int_0^x f_x(h_2 - \widetilde{h}_1\,(x, y, -f(x, y))\,dx.$$

H'Adamard Lemma yields that $Y_0(y, -f(x, y)$ can be written as follows: $Y_0(y, -f(x, y)) = Y_0(y, 0) + fY_1(y, f)$. The second summand belongs to $\int_0^x f_x N(x, y)dx$ as $\frac{\partial}{\partial x}(fY_1(y, f)) = \frac{\partial}{\partial f}(fY_1(y, f).f_x$, due to the chain rule.

Obviously the converse is also true.

∎

Lemma 10.2.2 implies the following:

Lemma 10.2.3 *The stationary algebra is given by the following formulas:*

1. *For* $\widetilde{A}_k$: $\widetilde{f} = \frac{1}{k+1}x^{k+1} + yx + z, \quad k \geq 0,$

$$\begin{aligned} \dot{X} &= \tfrac{1}{k}\left[xS(y) + E(x,y)x^2 - M(x,y)(y + x^k)\right]; \\ \dot{Y} &= yS(y) + \tfrac{1}{k}x[x^k + (k+1)y]E(x,y) + \tfrac{1}{k}x^2(y + x^k)E_x(x,y), \end{aligned} \tag{10.5}$$

where $S(y), E(x,y), M(x,y)$ *are arbitrary function germs and* $\frac{\partial E}{\partial x} = E_x$.

2. *For* $\widetilde{B}_k$: $\widetilde{f} = \frac{1}{3}x^3 + xy^k + z, \quad k \geq 2,$

$$\dot{X} = \frac{k}{2}xS(y) + \frac{k}{2}E(x,y)y^{k-1}x^2 - N(x,y)(x^2 + y^k);$$

$$\dot{Y} = yS(y) + \frac{1}{2}x(x^2 + 3y^k)E(x,y) + \frac{1}{2}x^2(x^2 + y^k)E_x(x,y),$$

where $S(y), E(x,y), M(x,y)$ *are arbitrary functions and* $\frac{\partial E}{\partial x} = E_x$.

3. *For* $\widetilde{C}_k$: $\widetilde{f} = \frac{1}{k+1}x^{k+1} + x^2y + z, \quad k \geq 2,$

$$\dot{X} = \frac{1}{k-1}xS(y) + N(x^k + xy) + x^3Q(x,y) + x^4Q_x(x,y);$$

$$\dot{Y} = yS(y) + x[2x^k + (k+1)xy]Q(x,y) + x^2[4x^k + (k+3)xy]Q_x(x,y) + x^3(x^k + xy)Q_{xx}(x,y),$$

where $S(y), Q(x,y), M(x,y)$ *are arbitrary functions and* $\frac{\partial Q}{\partial x} = Q_x$, $\frac{\partial^2 Q}{\partial x^2} = Q_{xx}$.

4. *For* $\widetilde{F}_4$: $\widetilde{f} = x^4 + y^2x + z$,

$$\dot{X} = -2N(x,y)(y^2 + x^3) + \frac{2}{3}xS(y) + \frac{2}{3}x^2yE(x,y);$$

$$\dot{Y} = yS(y) + \frac{1}{3}x(x^3 + 4y^2)E(x,y) + \frac{1}{3}x^2(x^3 + y^2)E_x(x,y),$$

where $S(y), E(x,y), M(x,y)$ *are arbitrary functions.*

Proof.

1. In the case of $\widetilde{A}_k : f = \frac{1}{k+1}x^{k+1} + xy, k \geq 2$, the derivative equation takes the form:

$$A(x,y)(y + x^k) + kx^{k-1}\dot{x} + \dot{y} = 0,$$

where $\dot{y} = S_0(y) + \int\limits_0^x (y + x^k)a(x,y)dx$.

By integration by parts, the last formula can be written as

$$\dot{y} = S_0(y) + (x^k + y)xb - x^{k+1}E, \tag{10.6}$$

with some functions $E(x,y), S_0(y)$ and $\int_0^x adx = xb$, $\int\limits_0^x kx^k bdx = x^{k+1}E$.

Setting $x = y = 0$ we get $S_0(0) = 0$. Hence $S_0(y) = yS(y)$ for a smooth function $S(y)$. Hence, the derivative equation can be written in the form

$$\widetilde{A}(x^k + y) + kx^{k-1}\dot{x} + yS(y) - x^{k+1}E = 0,$$

where $\widetilde{A} = A + xb$. This formula can be rewritten as:

$$x^{k-1}[k\dot{x} + x\widetilde{A} - x^2E] + y[S(y) + \widetilde{A}] = 0.$$

This yields to the existence of a smooth function $M(x,y)$ such that

$$k\dot{x} + x\widetilde{A} - x^2E = -My, \tag{10.7}$$

and

$$S(y) + \widetilde{A} = Mx^{k-1}. \tag{10.8}$$

If we substitute $\widetilde{A}$, from the equation (10.8), into the equation (10.7), we get:

$$\dot{x} = \frac{-1}{k}M(y + x^k) + \frac{1}{k}xS(y) + \frac{1}{k}x^2E.$$

Now differentiating the relation $\int\limits_0^x kx^k bdx = x^{k+1}E$ with respect to x gives $b = \frac{k+1}{k}E + \frac{1}{k}xE_x$. Hence, if we substitute b in the equation (10.6), we obtain

$$\dot{y} = yS(y) + \frac{1}{k}[x^k + (k+1)y]E + \frac{1}{k}x^2(y + x^k)E_x,$$

as required.

2. In the case of $\widetilde{B}_k : f = \frac{1}{3}x^3 + y^k x, k \geq 2$, the derivative equation takes the form:

$$A(x^2 + y^k) + 2x\dot{x} + ky^{k-1}\dot{y} = 0,$$

where $\dot{y} = S_0(y) + \int\limits_0^x (x^2 + y^k)a(x,y)dx$. By integration by parts, the $\dot{y}$ component of the stationary algebra can be written as follows:

$$\dot{y} = S_0(y) + (x^2 + y^k)xb(x,y) - x^3 E(x,y), \tag{10.9}$$

with some functions $b(x,y)$ and $E(x,y)$ and $\int_0^x a dx = xb$, $\int\limits_0^x 2x^2 b(x,y)dx = x^3 E$. Thus the stationary algebra can be written as:

$$A(x^2 + y^k) + 2x\dot{x} + ky^{k-1}\left(S_0(y) + (x^2 + y^k)xb(x,y) - x^3 E(x,y)\right) = 0,$$

or equivalently

$$\widetilde{A}(x^2 + y^k) + 2x\dot{x} + ky^{k-1}\left(S_0(y) - x^3 E(x,y)\right) = 0,$$

where $\widetilde{A} = A + xb(x,y)$. This formula can be rewritten as:

$$2x[\dot{x} + \frac{1}{2}\widetilde{A}x - \frac{k}{2}x^2 y^{k-1}E] + y^{k-1}[y\widetilde{A} + kS_0(y)] = 0.$$

This tells us that there exists a smooth function $M(x,y)$ such that:

$$\dot{x} + \frac{1}{2}\widetilde{A}x - \frac{k}{2}x^2 y^{k-1}E = My^{k-1} \tag{10.10}$$

and

$$y\widetilde{A} + kS_0(y) = -2xM. \tag{10.11}$$

If we set $x = y = 0$ in the equation (10.11), we see that $S_0(0) = 0$. This means that $S_0(y) = yS(y)$. Thus we can write the the equation (10.11) as:

$$y\left[\widetilde{A} + kS(y)\right] + 2xM = 0,$$

which is equivalent to the existence of a smooth function $N(x, y)$ such that:

$$\widetilde{A} + kS(y) = 2xN,$$

and

$$M = -yN.$$

Substituting $\widetilde{A}$ and M in the equation (10.10) gives the $\dot{x}$ component of the stationary algebra as follows:

$$\dot{x} = -N(x^2 + y^k) + \frac{k}{2}xS(y) + \frac{k}{2}y^{k-1}x^2E.$$

If we differentiate $\int\limits_0^x 2x^2b(x, y)dx = x^3E$ with respect x, we get: $b = \frac{3}{2}E + \frac{1}{2}E_x$, where $\frac{\partial E}{\partial x} = E_x$. Thus:

$$\dot{y} = yS_0(y) + \frac{1}{2}x(x^2 + 3y^k)E + \frac{1}{2}x^2(x^2 + y^k)E_x.$$

3. For the singularity $\widetilde{C} : f = \frac{1}{k+1}x^{k+1} + \frac{1}{2}x^2y, k \geq 2$, the derivative equation has the form:

$$A(x^k + xy) + (kx^{k-1} + y)\dot{x} + x\dot{y} = 0,$$

where $\dot{y} = S_0(y) + \int\limits_0^x (x^k + xy)a(x, y)dx$. By integration by parts, the $\dot{y}$ component can be written as

$$\dot{y} = S_0(y) + (x^k + xy)xb(x, y) - \int\limits_0^x (kx^{k-1} + y)xb(x, y)dx,$$

where $xb = \int_0^x a dx$. Integration by parts again gives

$$\dot{y} = S_0(y) + (x^k + xy)xb - (kx^{k-1} + y)x^2E + x^{k+1}Q,$$

with smooth functions $E(x, y)$ and $Q(x, y)$ and the relations $\int\limits_0^x xbdx = x^2E$ and $\int\limits_0^x k(k-1)x^k E dx = x^{k+1}Q$.

Thus the stationary algebra takes the form:

$$A(x^k + xy) + (kx^{k-1} + y)\dot{x} + x[S_0(y) + (x^k + xy)xb - (kx^{k-1} + y)x^2E + x^{k+1}Q] = 0.$$

Equivalently, this formula can be written as:

$$\widetilde{A}(x^k + xy) + \dot{x}(kx^{k-1} + y) + x\left[S_0(y) - (kx^{k-1} + y)x^2E + x^{k+1}Q\right] = 0,$$

with $\widetilde{A} = A + x^2b$. We get the following equivalent formula

$$(\dot{x} + \frac{1}{k}x\widetilde{A})(kx^{k-1} + y) + x\left[\frac{k-1}{k}\widetilde{A}y + S_0(y) - x^2(kx^{k-1} + y)E + x^{k+1}Q\right] = 0.$$

Hence, there is a smooth function $M(x, y)$ such that:

$$\dot{x} + \frac{1}{k}x\widetilde{A} = xM, \tag{10.12}$$

and

$$\frac{k-1}{k}\widetilde{A}y + S_0(y) - x^2(kx^{k-1} + y)E + x^{k+1}Q = -(kx^{k-1} + y)M. \tag{10.13}$$

If we set $x = y = 0$ in the equation (10.13), then we get $S_0(0) = 0$. Hence $S_0(y) = yS(y)$. Thus

$$y\left[\frac{k-1}{k}\widetilde{A} + S(y) - x^2E + M\right] + x^{k-1}\left[-kx^2E + x^2Q + kM\right] = 0.$$

This implies to the existence of a smooth function $N(x, y)$ such that:

$$\frac{k-1}{k}\widetilde{A} + S(y) - x^2E + M = x^{k-1}N,$$

and

$$-kx^2E + x^2Q + kM = -yN.$$

If we substitute $\widetilde{A}$ and M from the last two equations in (10.12), we get:

$$\dot{x} = \frac{-1}{k-1}N(x^k + xy) + \frac{1}{k-1}xS(y) + x^3E - \frac{1}{k-1}x^3Q.$$

Now, we want to write the components of the vector field in terms of Q only. Differentiate $\int_0^x xb dx = x^2E$ with respect to x, we obtain : $b = 2E + xE_x$. Similarly, differentiate $\int_0^x k(k-1)x^kEdx = x^{k+1}Q$ with respect to x, we obtain:

$$E = \frac{k+1}{k(k-1)}Q + \frac{1}{k(k-1)}xQ_x, \quad \text{where} \quad \frac{\partial Q}{\partial x} = Q_x.$$

Differentiate $E = \frac{k+1}{k(k-1)}Q + \frac{1}{k(k-1)}xQ_x$ with respect to x, we get:

$$E_x = \frac{1}{k(k-1)}[(k+2)Q_x + xQ_{xx}], \quad \text{where} \quad Q_{xx} = \frac{\partial^2 Q_x}{\partial x^2}.$$

Hence, the function b can be written as:

$$b = \frac{1}{k(k-1)}\left[2(k+1)Q + (k+4)xQ_x + x^2Q_{xx}\right].$$

Substitute the functions E and b in the formulas of $\dot{x}$ and $\dot{y}$, we get:

$$\dot{x} = \frac{-1}{k-1}N(x^k + xy) + \frac{1}{k-1}xS + \frac{1}{k(k-1)}x^3Q + \frac{1}{k(k-1)}x^4Q_x,$$

and

$$\dot{y} = yS(y) + \frac{x}{k(k-1)}[2x^k + (k+1)xy]Q + \frac{x^2}{k(k-1)}[4x^k + (k+3)xy]Q_x + \frac{x^3}{k(k-1)}(x^k + xy)Q_{xx}.$$

For simplicity, set $\frac{-1}{k-1}N = \widetilde{N}$ and $\frac{1}{k(k-1)}Q = \widetilde{Q}$. Hence we get:

$$\dot{x} = \widetilde{N}(x^k + xy) + \frac{1}{k-1} x S_0 + x^3 \widetilde{Q} + x^4 \widetilde{Q}_x,$$

and

$$\dot{y} = yS(y) + x(2x^k + (k+1)xy)\widetilde{Q} + x^2(4x^k + (k+3)xy)\widetilde{Q}_x + x^3(x^k + xy)\widetilde{Q}_{xx}.$$

4. For the singularity $\widetilde{F}_4 : f = \frac{1}{4}x^4 + y^2 x$, we have:

$$A(x^3 + y^2) + 3x^2\dot{x} + 2y\dot{y} = 0,$$

where $\dot{y} = S_0(y) + \int\limits_0^x (x^3 + y^2)a(x, y)dx$. The $\dot{y}$ component can be represented as

$$\dot{y} = S(y) + (x^3 + y^2)xb(x, y) - x^4 E,$$

with an arbitrary function germ E, $xb = \int_0^x a dx$ and $\int\limits_0^x 3x^3 b(x, y)dx = x^4 E$.

Thus, the stationary algebra takes the form:

$$A(x^3 + y^2) + 3x^2\dot{x} + 2y\left[S_0(y) + (x^3 + y^2)xb - x^4 E\right] = 0.$$

This is equivalent to

$$\widetilde{A}(x^3 + y^2) + 3x^2\dot{x} + 2y[S_0(y) - x^4 E] = 0,$$

where $\widetilde{A} = A + 2yxb$, or equivalently

$$3x^2[\dot{x} + \frac{1}{3}\widetilde{A}x - \frac{2}{3}x^2 yE] + 2y[S_0(y) + \frac{1}{2}\widetilde{A}y] = 0.$$

This leads to the existence of a smooth function $M(x, y)$, such that:

$$\dot{x} + \frac{1}{3}\widetilde{A}x - \frac{2}{3}x^2 yE = 2yM \tag{10.14}$$

and

$$S_0(y) + \frac{1}{2}\widetilde{A}y = -3x^2 M. \tag{10.15}$$

If we set $x = y = 0$, in the equation (10.15), we get $S_0(0) = 0$. Thus $S_0(y) = yS(y)$. Hence, we have:

$$y[S(y) + \frac{1}{2}\widetilde{A}] + 3x^2 M = 0.$$

This means that there is a smooth function $N(x, y)$, such that: $S(y)+\frac{1}{2}\widetilde{A} = 3x^2 N$ and $M = -yN$. Substitute $\widetilde{A}$ and M from the last two equations in (10.14), we get:

$$\dot{x} = -2N(y^2 + x^3) + \frac{2}{3}xS(y) + \frac{2}{3}x^2 yE.$$

Now, differentiate $\int_0^x 3x^3 b dx = x^4 E$ with respect to x, we get: $b = \frac{4}{3}E + \frac{1}{3}xE_x$. Thus $\dot{y}$ component becomes:

$$\dot{y} = yS(y) + \frac{1}{3}x(x^3 + 4y^2)E + \frac{1}{3}x^2(x^3 + y^2)E_x.$$

∎

The classification of simple quasi strong classes is given in the following theorem.

Theorem 10.2.4 *The list of simple quasi projections of regular surfaces with boundaries in three space consists of the following normal forms of the projections $(x, y, z) \mapsto (y, z)$ of the germs at the origin of the graphs V of the functions $z = f(x, y)$ and the boundaries $g(x, y) = 0$:*

1. *For $\widetilde{A}_k : f = \frac{1}{k+1}x^{k+1} + yx, \quad k \geq 0$, the boundaries are the Arnold's simple boundary (with respect to the $w = 0$ boundary classes of curves $g(w, x) = 0$, where the coordinate $w = y + x^k$ vanish at the critical set of the projection of the surface:*

$$\begin{array}{ccccccccccc}
x & \leftarrow & w+x^2 & \leftarrow & w+x^3 & \leftarrow & \dots & \leftarrow & w+x^{k-1} & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & x^2 \pm w^2 & \leftarrow & xw+x^3 & \leftarrow & \dots & \leftarrow & xw+x^{k-1} & \leftarrow & xw+x^k \\
 & & \uparrow & & \uparrow & & & & & & \\
 & & x^2+w^3 & \leftarrow & x^3+w^2 & & & & & & \\
 & & \uparrow & & & & & & & & \\
 & & x^2 \pm w^4 & \leftarrow & x^2+w^5 & \leftarrow & \dots & \leftarrow & x^2 \pm w^n & \dots &
\end{array}$$

2. *For* $\widetilde{B}_k : f = \frac{1}{3}x^3 + y^k x, \quad k \geq 3,$

- *If* k *is odd*

$$\begin{array}{ccccccc}
x+y & \leftarrow & x+y^2 & \leftarrow & \dots & \leftarrow & x+y^{k-1} \\
\uparrow & & & & & & \\
y+x^2 & \leftarrow & y & & & &
\end{array}$$

- *If* k *is even*

$$\begin{array}{ccccccc}
x+y & \leftarrow & x+y^2 & \leftarrow & \dots & \leftarrow & x+y^{\frac{k}{2}-1} \\
\uparrow & & & & & & \\
y+x^2 & \leftarrow & y & & & &
\end{array}$$

3. *For* $\widetilde{C}_k : f = \frac{1}{k+1}x^{k+1} + x^2y, \quad k \geq 2,$

$$\begin{array}{llll} x+y & \leftarrow x+y^2 & \leftarrow \dots & \leftarrow x+y^n \dots \\ \uparrow & & & \\ y+x^2 & \leftarrow \dots & \leftarrow y+x^{k-2} & \end{array}$$

4. *And for* $\widetilde{F}_4 : f = x^4 + y^2x$*, there are only four simple classes*

$$\begin{array}{lll} x+y & \leftarrow y+x^2 & \leftarrow y+x^3 \\ \uparrow & & \\ x & & \end{array}$$

Remarks:

1. For $\widetilde{B}_2$, there are no simple boundary classes. So if the surface can be modified, then simple pairs correspond only to $\widetilde{A}_k$ classes.

2. Notice that, in the A_k case using $u = y + x^k$ instead of y reduces the stationary algebra to the algebra of the vector fields with the components:

$$\dot{u} = u\left(x\frac{k+1}{k}E + \frac{1}{k}x^2E_x + S - Mx^{k-1}\right),$$

and $\dot{x}$ as in (10.5). This is a subalgebra of vector fields tangent to the boundary $u = 0$. So to get the simple classes list we need to consider the splitting of standard boundary orbits into quasi-boundary ones.

Proof of Theorem 10.2.4

Knowing the stationary algebra the respective classifications are obtained by standard Arnold's spectral sequence method together with appropriate preliminary transformations.

We shall deal with semiquasi homogeneous function germs of the form $g = g_0 + \widetilde{g}$, with the principal quasi homogeneous part g_0 and $\widetilde{g}$ is a function germ of higher quasi degree, having the form

$$g(x,y) = a_1x + a_2y + b_1x^2 + b_2xy + b_3y^2 + \dots.$$

We consider successively all possible main quasi homogeneous parts of g ordered by increasing weights.

The quasi tangent space to the orbit at g takes the form:

$$TQS_g = gC(x,y) + \frac{\partial g}{\partial x}\dot{X} + \frac{\partial g}{\partial y}\dot{Y},$$

where C is an arbitrary function. The components of stationary algebra $(\dot{X}, \dot{Y})$ are described in Lemma 10.2.3.

The $\widetilde{A}_k$ case:

1) If $a_1 \neq 0$, then we may suppose that the main principal part is $g_0 = x$. Thus,

$$TQS_{g_0} = xA(x,y) + \frac{1}{k}x^2E(x,y) + \frac{1}{k}xS(y) - \frac{M(x,y)}{k}(y + x^k).$$

We obtain *mod* TQS_{g_0}: $x \equiv 0$ and $y + x^k \equiv 0$. The last equation is equivalent to $y \equiv 0$. So, the local algebra $\mathbf{C}_{x,y}/TQS_{g_0}$ is generated by the unit only. Hence, any function germ with the principal part $g_0 = x$ is equivalent to $\widetilde{g} = x$.

2) If $a_1 = 0$ and $a_2 \neq 0$, then let $g_0 = y$. Thus,

$$TQS_{g_0} = yA(x,y) + yS(y) + \frac{1}{k}x\left[x^k + (k+1)y\right]E(x,y) + \frac{1}{k}x^2(x^k + y)E_x(x,y).$$

We have *mod* TQS_{g_0}: $y \equiv 0$ and $x^{k+1} \equiv 0$. Hence, any function with principal part $g_0 = y$ can be reduced to the form

$$\widetilde{g} = y + \lambda_2 x^2 + \lambda_3 x^3 + \cdots + \lambda_k x^k.$$

Now, we may suppose that the principal part of the function germ $\widetilde{g}$ is $\widetilde{g}_0 = y + \lambda_i x^i$ where $2 \leq i \leq k$ and $\lambda_i \neq 0$ is the lowest non-zero element. Then,

$$
\begin{aligned}
TQS_{\tilde{g}_0} &= (y+\lambda_i x^i)A(x,y) + (i\lambda_i x^{i-1})\left[\frac{-M(x,y)}{k}(y+x^k) + \frac{1}{k}x^2E(x,y) + \frac{1}{k}xS(y)\right] \\
&+ yS(y) + \frac{1}{k}x(x^k+(k+1)y)E(x,y) + \frac{1}{k}x^2(x^k+y)E_x(x,y).
\end{aligned}
$$

Thus, we obtain *mod* $TQS_{\tilde{g}_0}$:

$$y \equiv -\lambda_i x^i, \tag{10.16}$$

$$\frac{i\lambda_i}{k}x^{i+1}E + (\frac{1}{k}x^{k+1} + \frac{k+1}{k}xy)E + (\frac{1}{k}x^{k+2} + \frac{1}{k}x^2y)E_x \equiv 0, \tag{10.17}$$

and

$$(y + \frac{i\lambda_i}{k}x^i)S(y) \equiv 0. \tag{10.18}$$

If we substitute y from the equation (10.16) into the equation (10.17), we get:

$$\left[\left(\frac{i\lambda_i}{k} - \frac{(k+1)\lambda_i}{k}\right)x^{i+1} + \frac{1}{k}x^{k+1}\right]E + \left(\frac{1}{k}x^{k+2} - \frac{\lambda_i}{k}x^{i+2}\right)E_x \equiv 0. \tag{10.19}$$

If $\lambda_i \neq 1$ when $i = k$, then the equation (10.19) yields that $x^{i+1} \equiv 0$ and $x^{k+1} \equiv 0$. By multiplying the equation (10.16) by y then by x we see that $xy \equiv 0$ and $y^2 \equiv 0$, respectively.

If $i \neq k$, then the equations (10.16) and (10.18) yield that $x^i \equiv 0$ and $y \equiv 0$. Thus, we conclude that $\tilde{g}$ is equivalent to $y + x^i$, $2 \leq i < k$.

Note that if $i = k$ then $\tilde{g}$ is non-simple. Also, note that $y + x^i \approx y + x^i + x^k$. Set $w = y + x^k$ where the coordinate w vanish at the critical set of the projection of the surface, then the class $y + x^i + x^k$ is right equivalent to the class $w + x^i$.

Thus, up to this stage, we get the following adjacencies:

$$x \leftarrow w + x^2 \leftarrow \cdots \leftarrow w + x^{k-1}.$$

3) If $a_1 = a_2 = 0$ but $b_1 \neq 0$, then we may consider $g_0 = x^2$. Thus

$$TQS_{g_0} = x^2 A(x,y) + 2x\left[\frac{-M(x,y)}{k}(y + x^k) + \frac{1}{k}x^2 E(x,y) + \frac{1}{k}xS(y)\right].$$

Thus, we get $mod\, TQS_{g_0}$, $x^2 \equiv 0$ and $xy \equiv 0$. Hence, the local algebra $\mathbf{C}_{x,y}/TQS_{g_0} \equiv H(y)$. It follows that g is equivalent to $\widetilde{g} = x^2 + h(y)$ where $h \in \mathcal{M}_y^2$.

Now consider $\widetilde{g}_0 = x^2 + \beta_i y^i$ where $i \geq 2$ and $\beta_i \neq 0$ is the lowest non-zero element of the function h. Then

$$\begin{aligned} TQS_{\widetilde{g}_0} &= (x^2 + \beta_i y^i)A(x,y) + 2x\left[\frac{M(x,y)}{k}(y + x^k) + \frac{1}{k}x^2 E(x,y) + \frac{1}{k}xS(y)\right] \\ &+ i\beta_i y^{i-1}\left[yS(y) + \frac{1}{k}x(x^k + (k+1)y)E(x,y) + \frac{1}{k}x^2(x^k + y)E_x(x,y)\right]. \end{aligned}$$

We have $mod\ TQS_{\widetilde{g}_0}$,

$$x^2 \equiv -\beta_i y^i, \tag{10.20}$$

$$xy + x^{k+1} \equiv 0, \tag{10.21}$$

$$\left[\frac{2}{k}x^3 + \frac{i\beta_i}{k}xy^{i-1}(x^k + (k+1)y)\right]E + \frac{i\beta_i}{k}x^2 y^{i-1}(x^k + y)E_x \equiv 0, \tag{10.22}$$

and

$$\frac{2}{k}x^2 S(y) + i\beta_i y^i S(y) \equiv 0. \tag{10.23}$$

The equation (10.21) can be written as

$$xy + x^2 \cdot x^{k-1} \equiv 0. \tag{10.24}$$

If we substitute x^2 from the equation (10.20) into the equation (10.24) we get $xy(1 - \beta_i y^{i-1} x^{k-2}) \equiv 0$. This yields that $xy \equiv 0$. Note that if $k = i + 1$, then we require that $\beta_i \neq 1$. However, $i \geq 2$. Hence, we get that $x^3 \equiv 0$ and $y^{i+1} \equiv 0$.

If $i \neq \frac{2}{k}$, then the equations (10.20) and (10.23) yield that $x^2 \equiv 0$ and $y^i \equiv 0$.

Hence, $\widetilde{g}$ is equivalent to $x^2 \pm y^i, i \geq 2$. Note that if $k = 2$, then $i = 1$ but $i \geq 2$.

Notice also that $x^2 \pm y^i \approx x^2 \pm (y + x^k)^i$. Set $w = y + x^k$, then right equivalence reduces the germ $x^2 \pm (y + x^k)^i$ to $x^2 \pm w^i$.

The adjacencies of the classes in this case as follows:

$$w + x^2 \leftarrow x^2 \pm w^2 \leftarrow x^2 + w^3 \leftarrow \cdots \leftarrow x^2 \pm w^n \ldots$$

4) If $a_1 = a_2 = b_1 = 0$ but $b_2 \neq 0$, then we may consider $g_0 = xy$. Thus,

$$\begin{aligned} TQS_{g_0} &= xyA(x,y) + y\left[\frac{M(x,y)}{k}(y + x^k) + \frac{1}{k}x^2E(x,y) + \frac{1}{k}xS(y)\right] \\ &+ x\left[yS(y) + \frac{1}{k}x(x^k + (k+1)y)E(x,y) + \frac{1}{k}x^2(x^k + y)E_x(x,y)\right]. \end{aligned}$$

Thus, We obtain *mod* TQS_{g_0}: $xy \equiv 0, y^2 \equiv 0$ and $x^{k+2} \equiv 0$. Hence, g is equivalent to $\widetilde{g} = xy + \alpha_3 x^3 + \cdots + \alpha_{k+1}x^{k+1}$.

Now we may suppose that $\widetilde{g}_0 = xy + \alpha_i x^i$ where $3 \leq i \leq k+1$ and $\alpha_i \neq 0$ is the lowest non-zero term. Then,

$$\begin{aligned} TQS_{\widetilde{g}_0} &= (xy + \alpha_i x^i)A(x,y) + (y + i\alpha_i x^{i-1})\left[\frac{M(x,y)}{k}(y + x^k) + \frac{1}{k}x^2E(x,y) + \frac{1}{k}xS(y)\right] \\ &+ x\left[yS(y) + \frac{1}{k}x(x^k + (k+1)y)E + \frac{1}{k}x^2(x^k + y)E_x(x,y)\right]. \end{aligned}$$

We have *mod* $TQS_{\widetilde{g}_0}$:

$$xy \equiv -\alpha_i x^i, \tag{10.25}$$

$$[\frac{1}{k}x^2(y + i\alpha_i x^{i-1}) + \frac{1}{k}x^2(x^k + (k+1)y)]E + \frac{1}{k}x^3(x^k + y)E_x \equiv 0, \tag{10.26}$$

$$y^2 + yx^k + i\alpha_i y x^{i-1} + i\alpha_i x^{k+i-1} \equiv 0, \tag{10.27}$$

and

$$[\frac{k+1}{k}xy + \frac{i\alpha_i}{k}x^i]S(y) \equiv 0. \tag{10.28}$$

If we substitute xy from the equation (10.25) into the equation (10.26), we get:

$$\left[\frac{(i-k-2)\alpha_i}{k}x^{i+1} + \frac{1}{k}x^{k+2}\right]E + \left[\frac{1}{k}x^{k+3} - \frac{\alpha_i}{k}x^{i+2}\right]E_x \equiv 0. \tag{10.29}$$

The relation (10.29) yields that $x^{i+1} \equiv 0$, if $i \neq k+1$(and $\alpha_i \neq 1$ when $i = k+1$). If we multiply the equation (10.25) by x and then by y, we get $x^2y \equiv 0$ and $xy^2 \equiv 0$, respectively.

Substitute xy from the equation (10.25) into the equation (10.27) to get:

$$y^2 + (i-1)\alpha_i x^{i+k-1} - i\alpha_i^2 x^{2i-2} \equiv 0. \tag{10.30}$$

The relation (10.30) yields that $y^2 \equiv 0$ as $x^{i+k-1} \equiv 0$ and $x^{2i-2} \equiv 0$. Now if $i \neq k+1$, then the equations (10.25) and (10.30) yield that $xy \equiv 0$ and $x^i \equiv 0$. Hence, $\widetilde{g}$ is equivalent to $xy + x^i$ where $3 \leq i < k+1$. Note that $xy + x^i$ can be written in the form $xw + x^i$, where $w = y + x^{k+1}$.

Also, note that if $i = k+1$, then $xy + \alpha_i x^i$ is non-simple.

The family $F_a = ax^2 \pm w^2 + x^3 + xy$ yields to the following adjacency

$$x^2 \pm w^2 \leftarrow xw + x^3.$$

On the other hand, the family $G_b = bw + xw + x^i$ gives the following adjacency

$$w + x^i \leftarrow xw + x^i, \quad \text{where} \quad 3 \leq i \leq k-1.$$

5) If $a_1 = a_2 = b_1 = b_2 = 0$ but $b_3 \neq 0$, then we may consider $g_0 = y^2$. Thus,

$$TQS_{g_0} = y^2A(x,y) + 2y\left[yS(y) + \frac{1}{k}x(x^k + (k+1)y)E(x,y) + \frac{1}{k}x^2(x^k + y)E_x(x,y)\right].$$

We have $mod\ TQS_{g_0}$: $y^2 \equiv 0$ and $yx^{k+1} \equiv 0$. Hence, g is equivalent to

$$\widetilde{g} = y^2 + \beta_2 yx^2 + \cdots + \beta_k yx^k + h(x), h \in \mathcal{M}_x^3.$$

Now suppose that $\widetilde{g} = y^2 + \beta_j yx^j + \cdots + \alpha_k yx^k + \alpha_i x^i + \ldots$, where $2 \le j \le k$, $i \ge 3$. Assume that $\alpha_3 \ne 0$ and consider $\widetilde{g}_0 = y^2 + \alpha_3 x^3$. Then, the tangent space at $\widetilde{g}_0$ is

$$\begin{aligned} TQS_{\widetilde{g}_0} &= (y^2 + \alpha_3 x^3)A(x,y) + 3\alpha_3 x^2 \left[\frac{M(x,y)}{k}(y + x^k) + \frac{1}{k}x^2 E(x,y) + \frac{1}{k}xS(y)\right] \\ &+ y\left[yS(y) + \frac{1}{k}x(x^k + (k+1)y)E + \frac{1}{k}x^2(x^k + y)E_x(x,y)\right]. \end{aligned}$$

Then, we obtain *mod* $TQS_{\widetilde{g}_0}$:

$$y^2 + \alpha_3 x^3 \equiv 0, \tag{10.31}$$

$$yx^2 + x^{k+2} \equiv 0, \tag{10.32}$$

$$\frac{4}{k}E + (\frac{1}{k}yx^{k+1} + \frac{k+1}{k}y^2x)E + (\frac{1}{k}yx^{k+2} + \frac{1}{k}y^2x^2)E_x \equiv 0, \tag{10.33}$$

and

$$\frac{1}{k}x^3S(y) + y^2S(y) \equiv 0. \tag{10.34}$$

If we substitute y^2 from the equation (10.31) and yx^2 from the equation (10.32) into the equation, we get $x^4 \equiv 0$ for $k \ge 2$ and $x^3 \equiv 0$ for $k = 1$. Hence, $y^3 \equiv 0$ and $yx^2 \equiv 0$. However, the equations (10.31) and (10.34) yield that $y^2 \equiv 0$ and $x^3 \equiv 0$. Thus, we conclude that $\widetilde{g}$ is equivalent to $G = y^2 + x^3$. Right equivalence yields that G can be reduced to the form $w^2 + x^3$, where $w = y + x^k$.

Notice that the family $F_a = axw + w^2 + x^3$ provides the adjacency $xw + x^3 \leftarrow x^3 + w^2$. On the other hand, the family

$$F_t = (x\cos t + w\sin t)^3 + (-x\sin t + w\cos t)^2$$

gives the the adjacency $x^2 + w^3 \leftarrow x^3 + w^2$.

Consider now the quasi homogeneous function $H = y^2 + \beta_2 yx^2 + \alpha x^4$ with respect weights $w_x = \frac{1}{4}$ and $w_y = \frac{1}{2}$. Then, comparing the dimension of H with the dimension of the quasi homogeneous functions of quasi degree 1 with respect weights, given above, in the tangent space at H yields that g is non-simple.

The $\widetilde{B}_k$ case:

Let $g(x, y) = a_1x + a_2y + b_1x^2 + b_2xy + b_3y^2 + \dots$.

1) If $a_1 \neq 0$ and $a_2 \neq 0$, then we may consider $g_0 = x + ay$. Thus,

$$\begin{aligned} TQS_{g_0} &= (x+ay)A + \left\{N(x^2+y^k) + \frac{k}{2}xS(y) + \frac{k}{2}x^2y^{k-1}E\right\} \\ &+ a\left\{yS(y) + \frac{1}{2}x(x^2+3y^k)E + \frac{1}{2}x^2(x^2+y^k)E_x\right\}. \end{aligned}$$

Then, we obtain *mod* TQS_{g_0}:

$$x \equiv -ay, \tag{10.35}$$

$$x^2 + y^k \equiv 0, \tag{10.36}$$

and

$$\frac{k}{2}xS(y) + ayS(y) \equiv 0. \tag{10.37}$$

If we substitute x from the equation (10.35) into the equation (10.36), we see that $y^2 \equiv 0$. Hence, $xy \equiv 0$ and $x^2 \equiv 0$.

If $k \neq 2$, then the equations (10.35) and (10.37) yield that $x \equiv 0$ and $y \equiv 0$. Hence, we conclude that: g is equivalent to $x + y$.

If $k = 2$, then $g \approx x + ay$.

2) If $a_1 \neq 0$ and $a_2 = 0$, then we may suppose that $g_0 = x$. Then

$$TQS_{g_0} = xA + N(x^2+y^k) + \frac{k}{2}xS(y) + \frac{k}{2}x^2y^{k-1}E.$$

We have *mod* T_{g_0}: $x \equiv 0$ and $y^k \equiv 0$. Hence g is equivalent to $\widetilde{g} = x + \alpha_2y^2 + \alpha_3y^3 + \dots + \alpha_{k-1}y^{k-1}$.

Now suppose that $\widetilde{g}_0 = x + \alpha_i y^i$, with $\alpha_i \neq 0$ is the lowest non-zero term and $2 \leq i \leq k-1$. Then,

$$\begin{aligned} TQS_{\widetilde{g}_0} &= (x+\alpha_i y^i)A + \left\{N(x^2+y^k) + \frac{k}{2}xS(y) + \frac{k}{2}x^2y^{k-1}E\right\} \\ &+ i\alpha_i y^{i-1}\left\{yS(y) + \frac{1}{2}x(x^2+3y^k)E + \frac{1}{2}x^2(x^2+y^k)E_x\right\}. \end{aligned}$$

Thus, we get *mod* $TQS_{\widetilde{g}_0}$:

$$x \equiv -\alpha_i y^i, \tag{10.38}$$

$$x^2 + y^k \equiv 0, \tag{10.39}$$

and

$$\frac{k}{2}xS(y) + i\alpha_i y^i S(y) \equiv 0. \tag{10.40}$$

If we substitute x from the equation (10.38) into the equation (10.39), we get $y^{2i} \equiv 0$ and $y^k \equiv 0$. Hence $x^2 \equiv 0$ and $xy^i \equiv 0$. Also, if $k \neq 2i$, then the equations (10.38) and (10.40) yield that $x \equiv 0$ and $y^i \equiv 0$. Hence, $\widetilde{g}$ is equivalent to $x + y^i$ where $2 \leq i \leq k-1$ and $k \neq 2i$. Thus, we get the following adjacencies

$$x + y \leftarrow x + y^2 \leftarrow \cdots \leftarrow x + y^{k-1}.$$

If $k = 2i$, then we obtain the following adjacencies

$$x + y \leftarrow x + y^2 \leftarrow \cdots \leftarrow x + y^{\frac{k}{2}-1}.$$

3) If $a_1 = 0$ and $a_2 \neq 0$, then we may suppose that $g_0 = y$.

$$TQS_{g_0} = yA + yS(y) + \frac{1}{2}x(x^2+3y^k)E + \frac{1}{2}x^2(x^2+y^k)E_x.$$

Then clearly,: $y \equiv 0$ and $x^3 \equiv 0$ *mod* TQS_{g_0} . Hence, g is equivalent to $\widetilde{g} = y + \alpha x^2$. Now consider the tangent space to the orbit $\widetilde{g}$,

$$\begin{aligned} TQS_{\widetilde{g}} &= (y+\alpha x^2)A + 2\alpha x\left\{N(x^2+y^k)+\frac{k}{2}xS(y)+\frac{k}{2}x^2y^{k-1}E\right\} \\ &+ \left\{yS(y)+\frac{1}{2}x(x^2+3y^k)E+\frac{1}{2}x^2(x^2+y^k)E_x\right\}. \end{aligned}$$

Then, we obtain *mod* $TQS_{\widetilde{g}}$:

$$y \equiv -\alpha x^2, \tag{10.41}$$

$$x^3 + xy^k \equiv 0, \tag{10.42}$$

and

$$\alpha k x^2 S(y) + yS(y) \equiv 0. \tag{10.43}$$

If we substitute y from the equation (10.41) into the equation (10.42), then we see that $x^3 \equiv 0$. Hence, $xy \equiv 0$ and $y^2 \equiv 0$. The equations (10.41) and (10.43) yield that $x^2 \equiv 0$ and $y \equiv 0$. Hence the class $\widetilde{g}$ is simple and $\widetilde{g}$ is equivalent to $y + x^2$.

4) If $a_1 = a_2 = 0$, then similar calculation as before shows that any function $g(x, y)$ with first jet $J_0^1(g)$ being zero is non-simple.

The $\widetilde{C}_k$ case:

Let $g(x, y) = a_1x + a_2y + b_1x^2 + \ldots$

1) If $a_1 \neq 0$, then we may suppose that $g_0 = x$. Thus,

$$TQSg_0 = xA(x,y) + \widetilde{N}(x,y)(x^k+xy) + \frac{1}{k-1}xS(y) + x^3\widetilde{Q}(x,y) + x^4\widetilde{Q}_x(x,y).$$

Clearly $\mathbf{C}_{x,y}/TQS_{g_0} \cong \mathbf{C}_y$. Therefore, the germ g is equivalent to $\widetilde{g} = x + h(y)$, where $h \in \mathcal{M}_y$.

Now, assume that $h = c_my^m + \widetilde{h}(y)$ where $\widetilde{h} \in \mathcal{M}_y^{m+1}$ and $c_m \neq 0$ is the lowest degree monomial in h. Consider $\widetilde{g}_0 = x + c_my^m$. Thus,

$$\begin{aligned} TQS_{\widetilde{g}_0} &= (x+c_m y^m)A(x,y) + (x^k+xy)N(x,y) + \frac{1}{k-1}xS(y) + x^3Q(x,y) + x^4Q_x \\ &+ c_m y^{m-1}\left\{yS(y) + x[2x^k+(k+1)xy]Q(x,y) + x^2[4x^k+(k+3)xy]Q_x(x,y)\right. \\ &+ \left. x^3(x^k+xy)Q_{xx}(x,y)\right\}. \end{aligned}$$

Thus, we obtain *mod* $TQS_{\widetilde{g}_0}$:

$$x \equiv -c_m y^m, \tag{10.44}$$

$$x^k + xy \equiv 0, \tag{10.45}$$

and

$$\frac{1}{k-1}xS(y) + mc_m y^m S(y) \equiv 0. \tag{10.46}$$

Substitute x from the equation (10.44) into the equation (10.45), then we get:

$$(-c_m)^k y^{mk} - c_m y^{m+1} \equiv 0. \tag{10.47}$$

We distinguish, the following cases:

i.If $m \neq 1$ and $k \neq 2$, then $TQS_{\widetilde{g}_0}$ contains y^{km} and y^{m+1}. Multiply the equation (10.44) by y and then by x, we see that $TQS_{\widetilde{g}_0}$ contains also xy and x^2, respectively.

ii. If $m = 1$ and $k = 2$, then y^2 belongs to $TQS_{\widetilde{g}_0}$ if $c_m \neq 1$. Hence, $TQS_{\widetilde{g}_0}$ contains also xy and x^2.

The equations (10.44) and (10.46) yield that, $TQS_{\widetilde{g}_0}$ contains y^m and x, if $k \neq 2$ and $m \neq 1$. Hence, the germ $\widetilde{g}$ is equivalent to $x + y^m$. If $k = 2$ and $m = 1$, then $\widetilde{g}$ is equivalent to the non simple germ $x + ay^2$.

2) If $a_1 = 0$ but $a_2 \neq 0$, then we may suppose that $g_0 = y$. Thus,

$$TQS_{g_0} = yA(x,y) + yS(y) + x[2x^k+(k+1)xy]Q(x,y) + x^2[4x^k+(k+3)xy]Q_x + x^3(x^k+xy)Q_{xx}.$$

Hence, we obtain *mod* TQS_{g_0}: $y \equiv 0$ and $x^{k+1} \equiv 0$. Therefore, the germ g is equivalent to $\widetilde{g} = y + \alpha_2 x^2 + \alpha_3 x^3 + \cdots + \alpha_k x^k$.

Now, consider $\widetilde{g}_0 = y + \alpha_i x^i$, with $\alpha_i \neq 0$ the lowest non-zero term and $2 \leq i \leq k$. Thus,

$$\begin{aligned} TQS_{\widetilde{g}_0} &= (y+\alpha_i x^i)A(x,y) + i\alpha_i x^{i-1}\left\{(x^k+xy)N(x,y) + \frac{1}{k-1}xS(y) + x^3Q(x,y) + x^4Q_x\right\} \\ &+ yS(y) + x[2x^k+(k+1)xy]Q(x,y) + x^2[4x^k+(k+3)xy]Q_x + x^3(x^k+xy)Q_{xx}. \end{aligned}$$

We have *mod* $TQS_{\widetilde{g}_0}$:

$$y \equiv -\alpha_i x^i, \tag{10.48}$$

$$\left(i\alpha_i x^{i+2} + 2x^{k+1} + (k+1)x^2y\right)Q + \left(i\alpha_i x^{i+3} + 4x^{k+2} + (k+3)x^3y\right)Q_x + (x^{k+3} + x^4y)Q_{xx} \equiv 0, \tag{10.49}$$

and

$$\frac{i\alpha_i}{k-1}x^iS(y) + yS(y) \equiv 0. \tag{10.50}$$

If we substitute y from the equation (10.48) in the equation (10.50), then we see that $x^{k+1} \equiv 0$ and $x^{i+2} \equiv 0$ and $\alpha_i \neq 1$ when $i = k-1$. Hence, $x^2y \equiv 0$ and $y^2 \equiv 0$.

If $i \neq k-1$, then the equations (10.48) and (10.50) yield that $y \equiv 0$ and $x^i \equiv 0$. Multiply the equation (10.48) by x and then substitute y from the equation (10.50) in the new relation, we see that $x^{i+1} \equiv 0$, if $i \neq k-1$.

Therefore, we conclude that $\widetilde{g}$ is equivalent to $y + x^i$ where $2 \leq i \leq k-2$. The adjacencies between the classes is given as follows:

$$\begin{array}{llll} x+y & \leftarrow x+y^2 & \leftarrow \ \dots & \leftarrow x+y^n\dots \\ \uparrow & & & \\ y+x^2 & \leftarrow \ \dots & \leftarrow y+x^{k-2} & \end{array}$$

The $\widetilde{F}_4$ case:

Let $g(x,y) = a_1x + a_2y + \ldots$

1) If $a_1 \neq 0$ and $a_2 \neq 0$, then we may consider $g_0 = x + \alpha y$. Thus,

$$\begin{aligned} TQS_{g_0} &= (x+\alpha y)A + N(x^3+y^2) + \frac{2}{3}xS(y) + \frac{2}{3}x^2yE \\ &+ \alpha\left\{yS(y) + \frac{1}{3}x(x^3+4y^2)E + \frac{1}{3}x^2(x^3+y^2)E_x\right\}. \end{aligned}$$

Hence, we obtain *mod* TQS_{g_0}:

$$x \equiv -\alpha y, \tag{10.51}$$

$$x^3S(y) + y^2S(y) \equiv 0, \tag{10.52}$$

and

$$\frac{2}{3}x + \alpha y \equiv 0. \tag{10.53}$$

If we substitute x from the equation (10.51) into the equation (10.52) we see that $y^2 \equiv 0$ and $x^3 \equiv 0$. This yields that $xy \equiv 0$ and $x^2 \equiv 0$. Now, the equations (10.51) and (10.53) yield that $x \equiv 0$ and $y \equiv 0$. Hence, after normalization α we conclude that g is equivalent to $x + y$.

2) If $a_2 = 0$ but $a_1 \neq 0$, then we may suppose that $g_0 = x$. Thus,

$$T_{g_0} = xA + N(x^3+y^2) + \frac{2}{3}xS(y) + \frac{2}{3}x^2yE.$$

Clearly, the tangent space contains x and y^2. Hence, g is equivalent to x.

3) If $a_1 = 0$ but $a_2 \neq 0$, then we may suppose that $g_0 = y$. Thus,

$$TQS_{g_0} = yA + yS(y) + \frac{1}{3}x(x^3+4y^2)E + \frac{1}{3}x^2(x^3+y^2)E_x.$$

We have *mode* TQS_{g_0}: $y \equiv 0$ and $x^4 \equiv 0$. Hence, g is equivalent to $\widetilde{g} = y + \alpha_2x^2 + \alpha_3x^3$.

Now, consider $\widetilde{g}_0 = y + \alpha_ix^i$, where $\alpha_i \neq 0$, $i = 2, 3$. Then,

$$\begin{aligned} TQS_{\tilde{g}_0} &= (y+\alpha_i x^i)A(x,y) + i\alpha_i x^{i-1}\left\{N(x,y)(x^3+y^2) + \frac{2}{3}xS(y) + \frac{2}{3}x^2yE(x,y)\right\} \\ &+ \left\{yS(y) + \frac{1}{3}x(x^3+4y^2)E(x,y) + \frac{1}{3}x^2(x^3+y^2)E_x(x,y)\right\}. \end{aligned}$$

If we substiute $y \equiv -\alpha_i x^i$ in the local algebra $\mathbf{C}_{x,y}/TQS_{\tilde{g}_0}$, we see that

$$\mathbf{C}_{x,y}/TQS_{\tilde{g}_0} \cong \mathbf{C}_x/\widetilde{T}QS_{\tilde{g}_0} \cong \mathbb{R}\{1, x, x^{i-1}\}$$

where

$$\begin{aligned} \widetilde{T}QS_{\tilde{g}_0} &= i\alpha_i x^{i-1}\left\{\widetilde{N}(x)(x^3+\alpha^2x^{2i}) + \frac{2}{3}xS(x^i) + \frac{-2\alpha_i}{3}x^{2+i}\widetilde{E}(x)\right\} \\ &+ \left\{yS(x^i) + \frac{1}{3}x(x^3+4y^2)\widetilde{E}(x) + \frac{1}{3}x^2(x^3+4\alpha_i^2x^{2i})\widetilde{E}_x(x)\right\}. \end{aligned}$$

Hence, $\tilde{g}_0$ is equivalent to $y + x^i$.

4) If the function has zero first jet $J_0^1(g)$, then similar calculations as in the previous cases show that g is non-simple.

Thus, the complete list of simple classes is described in the following diagram

$$\begin{array}{lll} x+y & \leftarrow\ y+x^2 & \leftarrow\ y+x^3 \\ \uparrow & & \\ x & & \end{array}$$

10.3 The weak equivalence relation

Even more rough equivalence relation which may help to understand some invariants of the singularities of quasi-projections with boundaries is defined in this section. However we outline here only the details of the definition, no classification results are given.

Definition 10.3.1 Two hypersurfaces with boundaries $G_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$ and $G_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$ are called pseudo-weak equivalent if there exists a diffeomorphism $\theta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that:

1. $G_1 = MG_2 \circ \theta$, where $M = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, where $a, b, c \in \mathbf{C}_{x,y}$ with $a(0)c(0) \neq 0$.
2. The differential of θ preserves the direction of the projection only at boundary points which are also critical points of the projection of V_1.

Remarks:

1. We call this equivalence **weak** because we preserve the direction at critical points of the projection which belongs to the boundary only.
2. Denote by $G = \begin{pmatrix} f \\ g \end{pmatrix}$ the pair which define the the variety V with the boundary B.
3. Denote by J_G the ideal generated by f, g and $\frac{\partial f}{\partial x}$.
4. Denote by $Rad(J_G)$ the ideal consisting of function germs $h(x, y)$ whose certain power h^m belongs to the ideal J_G.
5. Denote by $IRad(J_G)$ the module of function germs g such that $\frac{\partial g}{\partial x} \in Rad(J_G)$.
6. Denote by IJ_G the integral of the ideal J_G; it consists of all functions germs h such that $\frac{\partial h}{\partial x} \in J_G$. To be explicit

$$IJ_G = \{A + \int_0^x (fB + gC + \frac{\partial f}{\partial x} D)dx : A \in \mathbf{C}_y, B, C, D \in \mathbf{C}_{x,y}\}.$$

Proposition 10.3.1 *The tangent space TPW_G is given by the formula*

$$TPW_G = \begin{pmatrix} TPW_f \\ TPW_g \end{pmatrix} = \begin{pmatrix} fA + f_x\dot{X} + \sum_{i=1}^{n-1} f_{y_i}\dot{Y}_i \\ fB + gC + g_x\dot{X} + \sum_{i=1}^{n-1} g_{y_i}\dot{Y}_i \end{pmatrix}$$

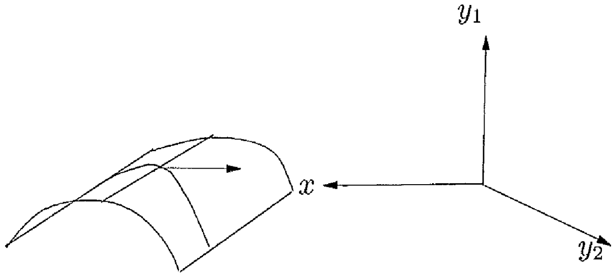

Figure 10.2: Weak Quasi Projection with boundary .

where $A, B, C, \dot{X} \in \mathbf{C}_{x,y}, \dot{Y}_i \in IRad(J_G)$.

Proof. The proof is similar to the proof of proposition 9.1.1. ∎

The improved definition is given as follows:

Definition 10.3.2 Two hypersurfaces with boundaries $G_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$ and $G_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$ are called QW-equivalent if there exists a family of diffeomorphisms $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ continuously and piecewise smoothly depending on parameter $t \in [1, 2]$ and a family of matrices

$$M_t = \begin{pmatrix} a_t & 0 \\ b_t & c_t \end{pmatrix},$$

where $a_t, b_t, c_t \in \mathbf{C}_{x,y,t}$, $a_t(0)c_t(0) \neq 0$,$M_2 = I_2$ and $\theta_2 = id$ and a family of pairs $G_t = \begin{pmatrix} f_t \\ g_t \end{pmatrix}$, with $t \in [1, 2]$, such that: For any $t \in [1, 2]$ we have $G_t = M_t G_2 \circ \theta_t$ and the vector field $V = (\dot{X}, \dot{Y}_i)$ generated by θ_t is of the following form: $\dot{X} \in \mathbf{C}_{x,y}$ and $\dot{Y}_i \in IJ_{G_t}$.

Chapter 11

Quasi projection of graphs of mappings

In this chapter we classify simple classes of quasi projection of graphs of two different type of mappings. The idea is similar to the one discussed in chapter 9.

11.1 Quasi projection of graphs of parametrized plane curve germs

Assume that $(C,0) \subset \mathbb{R}^2$ is a germ of a curve in the plane. There are two approaches: either consider its defining equation $f = 0$ where $f : (\mathbb{R}^2, 0) \to \mathbb{R}$ or its parameterization $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0); t \mapsto (y = \alpha(t), z = \beta(t))$. Standard Arnold singularities $A_k, D_k, E_s; s = 6, 7, 8$ describe the simple classifications of the curves $f = 0$ [1]. J.W.Bruce, T.J, Gaffney [14], L.Rudolph [28] and V. Arnold [4] considered the classifications of simple singularities of parameterization curves up to $\mathcal{A}$ equivalence relation [14, 4].

We will consider a parameterized curve $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0); t \mapsto (y = \alpha(t), z = \beta(t))$. Consider its graph $\Omega = \{(t, y, z) : y = \alpha(t), z = \beta(t)\} \subset \mathbb{R} \times \mathbb{R}^2$. Consider the projection $\pi : (t, y, z) \mapsto (y, z)$.

Definition 11.1.1 Two graphs $\Omega_i; i = 1, 2$ corresponding to the parameterized curves $\gamma_i : t \mapsto (y = \alpha_i(t), z = \beta_i(t))$ are called pseudo equivalent, if there exists a diffeomorphism $\Theta : (\mathbb{R} \times \mathbb{R}^2, 0) \to (\mathbb{R} \times \mathbb{R}^2, 0)$ such that $\Theta(\Omega_1) = \Omega_2$ and the differential of Θ preserve the direction of the projection π at the critical points of the projection.

Let P_Ω be an equivalence class of the germ Ω and call it a pseudo orbit of Ω. If we differentiate all deformations of Ω upon a parameter, we get the tangent space TP_Ω.

Proposition 11.1.1 *The tangent space of the orbit P_Ω at γ is given as follows:*

$$TP_\Omega = \begin{pmatrix} \frac{\partial \alpha(t)}{\partial t} A(t) \\ \frac{\partial \beta(t)}{\partial t} A(t) \end{pmatrix} + \begin{pmatrix} \dot{Y} \\ \dot{Z} \end{pmatrix},$$

where the components $\dot{Y}$ and $\dot{Z}$ satisfy

$$\frac{\partial \dot{Y}}{\partial t}, \frac{\partial \dot{Z}}{\partial t} \in Rad\left\{y - \alpha(t), z - \beta(t), \frac{\partial \alpha(t)}{\partial t}, \frac{\partial \beta(t)}{\partial t}\right\};$$

and $A \in \mathbf{C}_t$.

Proof. Let the vector field $V = \dot{T}\frac{\partial}{\partial t} + \dot{Y}\frac{\partial}{\partial y} + \dot{Z}\frac{\partial}{\partial z}$ be phase flow generated by the diffeomorphism $\Theta : (t, y, z) \mapsto (T, Y, Z)$. Then Θ preserve the direction $\frac{\partial}{\partial t}$ along a trajectory if the following is satisfied: $[V, \frac{\partial}{\partial t}] = c\frac{\partial}{\partial t}$ with some factor c. This gives $\frac{\partial \dot{Y}}{\partial t} = 0$ and $\frac{\partial \dot{Z}}{\partial t} = 0$. From the definition we want this properties hold only at the critical points of the projection. This means that:

$$\frac{\partial \dot{Y}}{\partial t}, \frac{\partial \dot{Z}}{\partial t} \in Rad\left\{y - \alpha(t), z - \beta(t), \frac{\partial \alpha(t)}{\partial t}, \frac{\partial \beta(t)}{\partial t}\right\}.$$

∎

Remark: Integrating by parts gives :

$$\Pi_1 = \int_0^t (y - \alpha)A(t)dt = (y - \alpha)\int_0^t A(t)dt - \int_0^t \frac{\partial \alpha}{\partial t}(\int_0^t A(t))dt.$$

Restricting to the curve by substituting $y = \alpha(t)$ gives

$$\Pi_1 = \int_0^t t\frac{\partial\alpha}{\partial t}\widetilde{A}dt,$$

with $t\widetilde{A} = \int_0^t A(t)dt$. Similarly we can deduce that:

$$\Pi_2 = \int_0^t (z-\beta)Bdt,$$

yields that:

$$\Pi_2 = \int_0^t t\frac{\partial\beta}{\partial t}\widetilde{B}dt,$$

where $\widetilde{B} \in \mathbf{C}_t$.

As usual the radical behaves badly when the ideal depends on parameters. So the previous definitions can be improved by replacing the radical by the ideal itself.

Definition 11.1.2 Two graphs $\Omega_i, i = 1, 2$ of the parameterized curves $\gamma_i : t \mapsto (y = \alpha_i, z = \beta_i), i = 1, 2$ are called quasi equivalent if there exists a family of parameterized curves $\Omega_\varepsilon = \{(t, y, z) : y = \alpha_\varepsilon, z = \beta_\varepsilon\}$ which continuously depends on parameters $\varepsilon \in [1, 2]$ and a family of diffeomorphisms $\Theta_\varepsilon : (t, y, z) \mapsto (T_\varepsilon, Y_\varepsilon, Z_\varepsilon)$ continuously and piecewise smooth depending on parameter $\varepsilon \in [1, 2]$ such that $\Theta_\varepsilon(\Omega_1) = \Omega_\varepsilon$ where $\Theta_1 = id$ and the components $\dot{Y}$ and $\dot{Z}$ of the vector field generated by Θ_ε satisfy the following:

$$\frac{\partial \dot{Y}_\varepsilon}{\partial t}, \frac{\partial \dot{Z}_\varepsilon}{\partial t} \in \left\{\frac{\partial\alpha_\varepsilon(t)}{\partial t}, \frac{\partial\beta_\varepsilon(t)}{\partial t}\right\}.$$

The classification of simple quasi projection classes is given in the following theorem.

Theorem 11.1.2 *Any simple projection of a graph of parametrized curve $\gamma : t \mapsto (\alpha(t), \beta(t))$, with respect to quasi equivalence is equivalent to the graph of the curve $\widetilde{\gamma} : t \mapsto (\pm t^k, 0)$ for some $k \geq 1$. The remaining germs form a subset of infinite codimension in the space of germs.*

Proof of Theorem 11.1.2

Let $\gamma : t \mapsto \left(\alpha(t) = a_k t^k + a_{k+1} t^{k+1} + \ldots, \beta(t) = b_s t^s + b_{s+1} t^{s+1} + \ldots\right)$.

Up to permutation between α and β, suppose that $k < s$ (if $k = s$, then by left transformation, γ can be reduced to the form

$$\widetilde{\gamma} : t \mapsto \left(\widetilde{\alpha}(t) = \tilde{a}_l t^l + \tilde{a}_{l+1} t^{l+1} + \ldots, \beta(t) = b_s t^s + b_{s+1} t^{s+1} + \ldots\right),$$

with $l < s$.)

By right equivalence γ can be reduced to the form:

$$\gamma_1 : t \mapsto \left(\alpha_1(t) = \pm t^k, \beta_1(t) = b_s t^s + b_{s+1} t^{s+1} + \ldots\right).$$

Take the family, $\gamma_\epsilon : t \mapsto \left(\alpha_\epsilon(t) = \pm t^k, (\beta_\epsilon(t) = \epsilon(b_s t^s + b_{s+1} t^{s+1} + \ldots)\right)$, with $\epsilon \in [0,1]$. Then, the respective homological equation is

$$\begin{pmatrix} 0 \\ b_s t^s + b_{s+1} t^{s+1} + \ldots \end{pmatrix} = \begin{pmatrix} \pm k t^{k-1} A \\ \epsilon(s b_s t^{s-1} + (s+1) b_{s+1} t^s + \ldots) A \end{pmatrix} + \begin{pmatrix} \dot{Y} \\ \dot{Z} \end{pmatrix}.$$

Note here that:

$$\dot{Y} = Y(\alpha_\epsilon, \beta_\epsilon) + \int_0^t \left[\pm k t^{k-1} B_1 + \epsilon(s b_s t^{s-1} + (s+1) b_{s+1} t^s B_2)\right] dt.$$

Hence:

$$\dot{Y} = Y(\alpha_\epsilon, \beta_\epsilon) + t^k C_1 + \epsilon(\tilde{b}_s t^s + \ldots) C_2.$$

Similarly, we see that:

$$\dot{Z} = Z(\alpha_\epsilon, \beta_\epsilon) + t^k C_3 + \epsilon(\tilde{b}_s t^s + \ldots) C_4.,$$

for some functions A, C_1, C_2, C_3 and $C_4 \in \mathbf{C}_t$.

Thus, the homological equation is solvable for $\epsilon = 0$ by setting : $A = C_1 = C_2 = C_4 = Y(\alpha_\epsilon, \beta_\epsilon) = Z(\alpha_\epsilon, \beta_\epsilon) = 0$ and $C_3 = b_s t^{s-k} + b_{s+1} t^{s+1-k} \ldots$.

For $\epsilon \in (0,1]$, by setting: $A = C_1 = C_2 = C_3 = Y(\alpha_\epsilon, \beta_\epsilon) = Z(\alpha_\epsilon, \beta_\epsilon) = 0$ and

$$C_4 = \frac{b_s + b_{s+1}t + \dots}{\epsilon(\widetilde{b}_s + \widetilde{b}_{s+1}t + \dots)}.$$

The theorem is proven.

Remark: We get the same results if the direction of the projection is preserved on the whole space curve. In this case:

$$\frac{\partial \dot{Y}}{\partial t}, \frac{\partial \dot{Z}}{\partial t} \in \left\{ t\frac{\partial \alpha(t)}{\partial t}, t\frac{\partial \beta(t)}{\partial t} \right\}.$$

Thus, similarly to the proof of the previous theorem we see that:

$$\dot{Z} = \int_0^t \left[\pm k t^k B_1 + \epsilon(s b_s t^s + (s+1) b_{s+1} t^{s+1}) \right] dt.$$

Hence:

$$\dot{Z} = Z(\alpha_\epsilon, \beta_\epsilon) + t^{k+1} B_1 + \epsilon(\tilde{b}_s t^{s+1} + \dots) B_2.$$

Similarly:

$$\dot{Y} = Y(\alpha_\epsilon, \beta_\epsilon) + t^{k+1} B_3 + \epsilon(\tilde{b}_s t^{s+1} + \dots) B_4.$$

for some functions A, B_1, B_2, B_3 and $B_4 \in \mathbf{C}_t$. Therefore, the homological equation is solvable for $\epsilon = 0$ by setting: $A = B_3 = B_2 = B_4 = Y(\alpha_\epsilon, \beta_\epsilon) = Z(\alpha_\epsilon, \beta_\epsilon) = 0$ and $B_1 = b_s t^{s-k-1} + b_{s+1} t^{s+1-k-1} \dots$. Note her that $s \geq k+1$.

For $\epsilon \in (0,1]$, set: $Z(\alpha_\epsilon, \beta_\epsilon) \equiv a_0 \beta_\epsilon$ and $A = B_1 = B_2 = B_3 = B_4 = Y(\alpha_\epsilon, \beta_\epsilon) = 0$. Hence, the homological equation is solvable by taking: $a_0 \in \mathbb{R}$ such that $a_0 \epsilon = 1$.

11.2 Quasi projections of graph mappings germs $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$

Consider a germ of C^∞-smooth mapping $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$; $(u, v) \mapsto (z = f(u, v), w = g(u, v))$. Let $\Gamma = \{(u, v, z, w) : z = f(u, v), w = g(u, v)\} \subset \mathbb{R}^2 \times \mathbb{R}^2$ be the graph of the mapping F and consider the projection $\pi : (u, v, z, w) \mapsto (z, w)$.

Definition 11.2.1 Two projections of graphs $\Gamma_i, i = 1, 2$ which correspond to the mappings $F_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$; $(u, v) \mapsto (z = f_i(u, v), w = g_i(u, v))$, $i = 1, 2$ are called quasi equivalent if there exists a diffeomorphism $\Phi : (\mathbb{R}^2 \times \mathbb{R}^2) \to (\mathbb{R}^2 \times \mathbb{R}^2)$, such that $\Phi(\Gamma_1) = \Gamma_2$ and the differential of Φ preserve the direction of the projection only at the points which lie on the graph.

Remark: The difference between standard $\mathcal{A}-$ classification of mappings and quasi classification of mappings is in the multiple points sets. Assume that two points m_1 and m_2 on the graph lie on the same fibre and therefore they are mapped to the same image. Then, this property persist for the $\mathcal{A}-$ equivalent mappings. However, this is not the case for quasi projection equivalence as the points m_1 and m_2 might be mapped to different fibres and hence they are mapped to different images. The quasi equivalence only preserves the direction field of the projection at all points of the graph.

Some possible applications are pointed out in the conclusion chapter.

Denote by Q_Γ the equivalence class of a germ Γ and call it a pseudo orbit of Γ. If we differentiate all deformations of Γ upon a parameter, we get the tangent space TQ_Γ.

Proposition 11.2.1 *The tangent space TQ_Γ is given by the formula*

$$TQ_\Gamma = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \begin{pmatrix} \dot{U} \\ \dot{V} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot{Z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{W}$$

where

$$\frac{\partial \dot{Z}}{\partial u} = \alpha_1 \frac{\partial f}{\partial u} + \beta_1 \frac{\partial g}{\partial u} \quad and \quad \frac{\partial \dot{Z}}{\partial v} = \alpha_1 \frac{\partial f}{\partial v} + \beta_1 \frac{\partial g}{\partial v};$$

$$\frac{\partial \dot{W}}{\partial u} = \alpha_2 \frac{\partial f}{\partial u} + \beta_2 \frac{\partial g}{\partial u} \quad and \quad \frac{\partial \dot{W}}{\partial v} = \alpha_2 \frac{\partial f}{\partial v} + \beta_2 \frac{\partial g}{\partial v};$$

for arbitrary smooth functions α_i, β_i and $\dot{U}, \dot{V} \in \mathbf{C}_{u,v}$.

Proof. Let

$$\Phi_t : \mathbb{R}^4 \to \mathbb{R}^4; (u,v,z,w) \mapsto (U_t(u,v,z,w), V_t(u,v,z,w), Z_t(u,v,z,w), W_t(u,v,z,w))$$

be a family of diffeomorphisms. Let $a_1 = \frac{\partial}{\partial u}$ and $a_2 = \frac{\partial}{\partial v}$ be the basis of vector space $\mathbb{R}^2$. Then the family of differentials Φ_t^* preserve the directions of the projection if the following relations are satisfied:

$$\Phi_t^*(a_1) = \lambda_1(t)a_1 + \lambda_2(t)a_2,$$

and

$$\Phi_t^*(a_2) = \lambda_3(t)a_1 + \lambda_4(t)a_2.$$

Differentiate these relations with respect to t and substitute $t = 0$, we get (respectively):

$$[V, a_1] = \widetilde{\lambda}_1 a_1 + \widetilde{\lambda}_2 a_2,$$

and

$$[V, a_2] = \widetilde{\lambda}_1 a_3 + \widetilde{\lambda}_4 a_2.$$

Here $V = \dot{U}\frac{\partial}{\partial u} + \dot{V}\frac{\partial}{\partial v} + \dot{Z}\frac{\partial}{\partial z} + \dot{W}\frac{\partial}{\partial w}$. These relations are equivalent to (respectively):

$$-\left(\frac{\partial \dot{U}}{\partial u}\frac{\partial}{\partial u} + \frac{\partial \dot{V}}{\partial u}\frac{\partial}{\partial v} + \frac{\partial \dot{Z}}{\partial u}\frac{\partial}{\partial z} + \frac{\partial \dot{W}}{\partial u}\frac{\partial}{\partial w}\right) = \widetilde{\lambda}_1 a_1 + \widetilde{\lambda}_2 a_2,$$

and

$$-\left(\frac{\partial \dot{U}}{\partial v}\frac{\partial}{\partial u} + \frac{\partial \dot{V}}{\partial v}\frac{\partial}{\partial v} + \frac{\partial \dot{Z}}{\partial v}\frac{\partial}{\partial z} + \frac{\partial \dot{W}}{\partial v}\frac{\partial}{\partial w}\right) = \widetilde{\lambda}_3 a_1 + \widetilde{\lambda}_4 a_2.$$

Hence we see that: $\frac{\partial \dot{Z}}{\partial u} = \frac{\partial \dot{W}}{\partial u} = \frac{\partial \dot{Z}}{\partial v} = \frac{\partial \dot{W}}{\partial v} = 0$.

From the definition, this means that these derivatives belong to the radical of the set defining the graph Γ. This means that

$$\frac{\partial \dot{Z}}{\partial u}, \frac{\partial \dot{W}}{\partial u}, \frac{\partial \dot{Z}}{\partial v}, \frac{\partial \dot{W}}{\partial v} \in Rad\{z - f(u,v), w - g(u,v)\}.$$

Notice that $Rad\{z - f(u,v), w - g(u,v)\} = \{z - f(u,v), w - g(u,v)\}$.

Let $\frac{\partial \dot{Z}}{\partial u} = (z - f)\alpha + (w - g)\beta$, with $\alpha, \beta \in \mathbf{C}_{u,v,z,w}$.

By H'Adamard Lemma we can always write

$$\dot{Z} = \dot{Z}_0(u,v) + (z - f)\alpha_1 + (w - g)\beta_1 + \psi \quad \text{where} \quad \psi \in \{z - f, w - g\}^2 \quad (*).$$

Differentiation (*) with respect to u then restricting $\frac{\partial \dot{Z}}{\partial u}$ to the surface by setting $z = f(u,v)$ and $w = g(u,v)$ gives $\frac{\partial \dot{Z}}{\partial u} = \frac{\partial \dot{Z}_0}{\partial u} - \frac{\partial f}{\partial u}\alpha_1 - \frac{\partial g}{\partial u}\beta_1$. On the other hand, we have $\frac{\partial \dot{Z}}{\partial u} = 0$. Hence, we obtain $\frac{\partial \dot{Z}_0}{\partial u} = \frac{\partial f}{\partial u}\alpha_1 + \frac{\partial g}{\partial u}\beta_1$. Similar arguments yield that $\frac{\partial \dot{Z}_0}{\partial v} = \frac{\partial f}{\partial v}\alpha_1 + \frac{\partial g}{\partial v}\beta_1$.

The same conclusions hold for $\dot{W}$.

∎

The classification of simple quasi projection classes is given in the following theorems.

Theorem 11.2.2 *If the mapping $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is of corank 1 and the graph of f is simple with respect to quasi projection then the projection of the graph of F is quasi equivalent to the graph of one of the following mappings: $G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0); (u, v) \mapsto (g, v)$ where g is listed as follows:*

$$\widetilde{A}_k : g = v^{k+1} + uv; \quad k \geq 0,$$

$$\widetilde{B}_k : g = v^3 + u^k v; \quad k \geq 2,$$

$$\widetilde{C}_k : g = v^{k+1} + v^2 u; \quad k \geq 2,$$

$$\widetilde{F}_4 : g = v^4 + u^2 v.$$

Remark: Notice that this list coincides with the list in theorem 9.4.2 of simple quasi projections singularities of regular surfaces embedded into three space.

Proof of Theorem 11.2.2.

It is enough to notice that the tangent space at F with respect to quasi projection along two dimensional fibre coincide with tangent space with respect to quasi projection along one dimensional fibre.

By standard right equivalence one can reduce the mapping $F : (u,v) \mapsto (z = f(u,v), w = g(u,v))$ to the form $F : (u,v) \mapsto (z = f^*(u,v), w = v)$.

Consider the deformations $F_t : (u,v) \mapsto (z = f_t^*(u,v), w = v)$ of the mapping F. Then, the tangent space at F_t with quasi projection along two dimensional fibre is given as follows:

$$\begin{pmatrix} -\frac{\partial f_t^*}{\partial t} \\ 0 \end{pmatrix} = \begin{pmatrix} h_1(z - f_t^*(u,v)) + h_2(w-v) \\ h_3(z - f_t^*(u,v)) + h_4(w-v) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_t^*}{\partial u} A + \frac{\partial f_t^*}{\partial v} B \\ -B \end{pmatrix} + \begin{pmatrix} \dot{Z} \\ \dot{W} \end{pmatrix},$$

where $h_1, h_2, h_3, h_4 \in \mathbf{C}_{u,v,w,z}$ and $\dot{Z}$(similarly $\dot{W}$) takes the forms:

$$\frac{\partial \dot{Z}}{\partial u} = a_1(z - f_t^*(u,v)) + a_2(w-v) \quad \text{and} \quad \frac{\partial \dot{Z}}{\partial v} = a_3(z - f_t^*(u,v)) + a_4(w-v).$$

By integration by parts, this is equivalent to:

$$\dot{Z} = \int_0^u x\frac{\partial f_t^*}{\partial u} A_1(u,v,w,z)du + C_1(z,w) + \alpha_1(u,v,w,z)(z - f_t^*(u,v)) + \beta_1(u,v,w,z)(w-v), \tag{11.1}$$

and

$$\dot{Z} = v^2 A_2(u,v,z,w) + C_2(z,w) + \alpha_2(u,v,w,z)(z - f_t^*(u,v)) + \beta_2(u,v,w,z)(w-v). \tag{11.2}$$

Restrict the equation (11.2) to $w = v$ and differentiate it with respect to u to get:

$$\frac{\partial \dot{Z}}{\partial u} = v^2 \widetilde{A}_2(u,v,z) + \widetilde{\alpha}_2(u,v,z)(z - f_t^*(u,v)) + \frac{\partial f_t^*}{\partial u}\widetilde{\beta}_2(u,v,z). \tag{11.3}$$

Similarly, restrict the equation (11.1) to $w = v$ and differentiate it with respect to u to get:

$$\frac{\partial \dot{Z}}{\partial u} = u\frac{\partial f_t^*}{\partial u}\widetilde{A}_3(u,v,z) + \widetilde{\alpha}_3(u,v,z)(z - f_t^*(u,v)) + \frac{\partial f_t^*}{\partial u}\widetilde{\beta}_3(u,v,z). \tag{11.4}$$

The equation 11.4 is equivalent to:

$$\frac{\partial \dot{Z}}{\partial u} = \frac{\partial f_t^*}{\partial u}\widetilde{A}_4(u,v,z) + \widetilde{\alpha}_3(u,v,z)(z - f_t^*(u,v)). \tag{11.5}$$

The intersection of the relations 11.4 and 11.6 yields that:

$$\frac{\partial \dot{Z}}{\partial u} = \frac{\partial f_t^*}{\partial u}\widetilde{A}_5(u,v,z) + \widetilde{\alpha}_4(u,v,z)(z - f(u,v)). \tag{11.6}$$

Restrict the last equation to $z = f$ to get:

$$\frac{\partial \dot{Z}}{\partial u} = \frac{\partial f_t^*}{\partial u}\widetilde{A}_5(u,v,z). \tag{11.7}$$

Similar argument shows that:

$$\frac{\partial \dot{W}}{\partial u} = \frac{\partial f_t^*}{\partial u}\widetilde{B}_5(u,v,z). \tag{11.8}$$

Now the second row of the homological equation after restricting $z = f_t^*$ and $w = v$ gives:

$$B = \dot{W}.$$

Thus the first row of the homological equation takes the form:

$$\frac{\partial f_t^*}{\partial t} = \frac{\partial f_t^*}{\partial u}A(u,v,z) + \frac{\partial f_t^*}{\partial v}\dot{W} + \dot{Z}$$

where $\frac{\partial \dot{W}}{\partial u} = \frac{\partial f_t^*}{\partial u}\widetilde{B}_5(u,v,z)$ and $\frac{\partial \dot{Z}}{\partial u} = \frac{\partial f_t^*}{\partial u}\widetilde{A}_5(u,v,z)$.

The last tangent space coincides with the tangent space of the quasi-projection of the regular surfaces $z - f_t^* = 0$ to $(z, v = w)$-plane. Hence the list of simple classes in this case is the same list of the simple quasi-projection classifications of the regular surfaces $z - f_t^* = 0$ to (z, w)-plane in this case which was considered before. The theorem is proven.

Moreover,

Theorem 11.2.3 *If the mapping $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is of corank 2, and the projection of the graph is simple with respect to quasi projection. Then, the projection of the graph of F is quasi projection equivalent to the graph of the following mappings:*

$$G : (u, v) \mapsto (u^2 \pm v^2, uv).$$

Proof of Theorem 11.2.3.

The table of adjacencies of the second jets of the mapping $F : (u, v) \mapsto (z = f(u, v), w = g(u, v))$ up to right-left equivalence of corank 2 is given as follows:

$$I : (u^2 \pm v^2, uv) \leftarrow II : (uv, u^2) \leftarrow III : (u^2 \pm v^2, 0) \leftarrow V : (u^2, 0) \leftarrow IV : (0, 0).$$

Consider separately these cases.

The case (I):

Let $F : (u, v) \mapsto (z = u^2 \pm v^2 + f^*(u, v), w = uv + g^*(u, v))$ where $f^*, g^* \in \mathcal{M}^3_{u,v}$. By standard right equivalence, one can reduce F to the form:

$$\widetilde{F} : (u, v) \mapsto (z = u^2 \pm v^2), w = uv + \widetilde{g}(u, v)) \quad \text{where} \quad \widetilde{g} \in \mathcal{M}^3_{u,v}.$$

Take the deformation:

$$F_t : (u, v) \mapsto (z = u^2 \pm v^2, w = uv + t\widetilde{g}(u, v)), \quad \text{where} \quad t \in [0, 1].$$

We want to show that all F_t are quasi equivalent.

The homological equation takes the form:

$$\begin{pmatrix} 0 \\ -\widetilde{g}(u,v) \end{pmatrix} = \begin{pmatrix} 2uA + 2vB \\ (v + t\frac{\partial \widetilde{g}}{\partial u})A + (u + t\frac{\partial \widetilde{g}}{\partial v})B \end{pmatrix} + \begin{pmatrix} \dot{Z} \\ \dot{W} \end{pmatrix}$$

where

$$\frac{\partial \dot{Z}}{\partial u}, \frac{\partial \dot{W}}{\partial u} \in \left\{ u, (v + t\frac{\partial \widetilde{g}}{\partial u}) \right\},$$

$$\frac{\partial \dot{Z}}{\partial v}, \frac{\partial \dot{W}}{\partial v} \in \left\{ v, (u + t\frac{\partial \widetilde{g}}{\partial v}) \right\}.$$

Consider the first row of the homological equation and solve it for A and B. Thus, we have

$$2uA + 2vB + \dot{Z} = 0.$$

Solve this equation first at $t = 0$. In this case $\dot{Z}$ (and $\dot{W}$) takes the form:

$$\dot{Z} = h_1(u) + h_2(v) + u^2v^2h_3(u,v) + h_4(u^2 \pm v^2, uv), \tag{11.9}$$

where $h_1 \in \mathcal{M}_u^3, h_2 \in \mathcal{M}_v^3, h_3 \in \mathbf{C}_{u,v}$ and $h_4 \in \mathbf{C}_{f,g}$. Note that if $h_4 \in \mathcal{M}_{f,g}^2$ then $h_4 \subset h_1(u) + h_2(v) + u^2v^2h_3(u,v)$. Hence, the equation (11.9) takes the form:

$$2uA + 2vB + h_1(u) + h_2(v) + u^2v^2h_3(u,v) + a_0 + a_1uv + a_2(u^2 \pm v^2) = 0,$$

or equivalently,

$$u[2A + \widetilde{h}_1(u) + uv^2h_3(u,v) + a_1v + a_2u] + v[2B + \widetilde{h}_2(v) \pm a_2v] + a_0 = 0.$$

where $\widetilde{h}_1 \in \mathcal{M}_u^2, \widetilde{h}_2 \in \mathcal{M}_v^2$ and $a_0, a_1, a_2 \in \mathbb{R}$.

This yields that:

$$A = vC(u,v) + \widetilde{h}_1(u) + uv^2h_3(u,v) + a_1v + a_2u, \quad B = -uC(u,v) + \widetilde{h}_2(v) \pm a_2v,$$

and $a_0 = 0$.

Thus, the second row of the homological equation takes the form:

$$-g(u,v) = vA + uB + \dot{W},$$

where A and B as above and $\dot{W}$ takes the form:

$$\dot{W} = \hat{h}_1(u) + \hat{h}_2(v) + u^2v^2\hat{h}_3(u,v) + \tilde{a}_0 + \tilde{a}_1uv + \tilde{a}_2(u^2 \pm v^2),$$

Here $\hat{h}_1 \in \mathcal{M}_u^3, \hat{h}_2 \in \mathcal{M}_v^3, \hat{h}_3 \in \mathbf{C}_{u,v}$ and $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2 \in \mathbb{R}$.

Hence, clearly the homological equation is solvable at $t = 0$. In fact similar arguments solve the homological equation for any $t \in [0,1]$.

The case (II) is non-simple. Suppose that $H : (u,v) \mapsto (uv, \frac{u^2}{2} + g(u,v))$ with $g \in \mathcal{M}_{u,v}^3$. Assume that g_t with $t \in [0,1]$ consists of quasi-equivalent projections. Then the homological equation takes the form:

$$\begin{pmatrix} 0 \\ -\frac{\partial g_t}{\partial t} \end{pmatrix} = \begin{pmatrix} vA + uB \\ (2u + \frac{\partial g_t}{\partial u})A + (\frac{\partial g_t}{\partial v})B \end{pmatrix} + \begin{pmatrix} \dot{Z} \\ \dot{W} \end{pmatrix}$$

Solving the following relations for $\dot{Z}$

$$\frac{\partial \dot{Z}}{\partial u} = \alpha_1 v + \beta_1(u + \frac{\partial g_t}{\partial u}) \quad \text{and} \quad \frac{\partial \dot{Z}}{\partial v} = \alpha_1 u + \beta_1 \frac{\partial g_t}{\partial v}$$

we see that $\dot{Z} = C_1(uv, \frac{u^2}{2} + g_t(u,v))$. Similarly, we get $\dot{W} = C_2(uv, \frac{u^2}{2} + g_t(u,v))$.

Notice that H'Adamard Lemma leads that

$$\dot{Z} = C_1(0,0) + uvP_1(uv, \frac{u^2}{2} + g_t(u,v)) + (\frac{u^2}{2} + g_t(u,v))P_2(uv, \frac{u^2}{2} + g_t(u,v)).$$

Solve the first row of the homological equation for A and B. We get

$$A = uK(u,v) - uP_1 - v^2\varphi_2 \text{ and } B = -vK - \frac{u}{2}P_2,$$

for arbitrary function $K \in \mathcal{M}_{u,v}$.

Let $\widetilde{g} = \frac{u^2}{2} + \alpha_t v^3$ be the lowest quasi homogenous part of g_t. Computations shows that the second row of the tangent space does not contain the term vu^2. This means that the mapping H is non-simple with respect to quasi projection.

The case (III) is also is non-simple as it is adjacent to the case (II). However, we give detailed arguments with different approach. In fact, let $F : (u,v) \mapsto (z = u^2 \pm v^2, w = g)$ where $g \in \mathcal{M}^3_{u,v}$. Consider the third jet of the mapping F. Let $J^3(g) = a_1u^3 + a_2uv^2 + a_3u^2v + a_4v^3$. The homological equation takes the form:

$$\begin{pmatrix} 0 \\ -J^3(g) \end{pmatrix} = \begin{pmatrix} uA + vB \\ \frac{\partial J^3(g)}{\partial u}A + \frac{\partial J^3(g)}{\partial v}B \end{pmatrix} + \begin{pmatrix} \dot{Z} \\ \dot{W} \end{pmatrix},$$

where $\frac{\partial \dot{Z}}{\partial u}, \frac{\partial \dot{W}}{\partial u} \in \{u^2, 3a_1u^3 + a_2uv^2 + 2a_3u^2v\}$ and $\frac{\partial \dot{Z}}{\partial v}, \frac{\partial \dot{W}}{\partial v} \in \{v^2, 2a_2uv^2 + a_3u^2v + 3a_4v^3\}$.

Let $\dot{Z} = c_1u^3 + c_2u^2v + c_3uv^2 + c_4v^3$. Then, $\frac{\partial \dot{Z}}{\partial u} = 3c_1u^2 + 2c_2uv + c_3v^2$. On the other hand, we have $\frac{\partial \dot{Z}}{\partial u} = u^2\alpha_1 + (3a_1u^3 + a_2uv^2 + 2a_3u^2v)\alpha_2$. Comparing the last two equations, we see that : $3c_1 = \alpha_1, \alpha_2 = c_2 = c_3 = 0$.

Similarly, $\frac{\partial \dot{Z}}{\partial v} = c_2u^2 + 2c_3uv + 3c_4v^2$. On the other hand, we have $\frac{\partial \dot{Z}}{\partial v} = v^2\beta_1 + (2a_2uv^2 + a_3u^2v + 3a_4v^3)\beta_2$. Comparing the last two equations, we see that :

$$3c_4 = \beta_1, \beta_2 = c_2 = c_3 = 0.$$

Thus, $\dot{Z} = \widetilde{c}_1u^3 + \widetilde{c}_2v^3$. Similarly, $\dot{W} = \widetilde{c}_3u^3 + \widetilde{c}_4v^3$. Hence, the first row of the homological equations becomes:

$$uA + vB + \widetilde{c}_1u^3 + \widetilde{c}_2v^3 = 0.$$

If we solve the previous equation for A and B, we get:

$$A = vd(u,v) - \widetilde{c}_1u^2 \quad \text{and} \quad B = -ud(u,v) - \widetilde{c}_2v^2, \quad \text{where} \quad d \in \mathbf{C}_{u,v}$$

The cubic terms in the second row of the tangent space are obtained from:

$$\frac{\partial J^3(g)}{\partial u}vd_0 - \frac{\partial J^3(g)}{\partial v}ud_0 \quad \text{and} \quad \widetilde{c}_3u^3 + \widetilde{c}_4v^3, \quad \text{where} \quad d_0 \in \mathbb{R},$$

which form a subspace of dimension 3. Hence, cubic terms can not belong to finitely many orbit.

Other cases are adjacent to the non-simple case (III). The theorem is proven.

Chapter 12

Conclusion

Here we recall the main results of the thesis with some final comments.

In chapters 2-6 we have classified simple singularities with respect to quasi border and quasi flag equivalences. There are much more simple classes than for the standard equivalence.

The classification theorems of simple classes are the following

Theorem: A simple quasi boundary singularity class on the boundary $(y = 0)$ is a class of a stabilizations of one of the following germs:

1. $B_k : \pm x_1^2 \pm y^k, \quad k \geq 2 \quad k;$
2. $F_{k,m} : \pm(y \pm x_1^k)^2 \pm x_1^m, \quad 2 \leq k < m \quad k + m - 1.$

Theorem: Let the germ $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, be simple with respect to the quasi corner equivalence. Then the following is true:

• If f_2 is a non-degenerate form, then f is quasi corner equivalent to Morse function $\mathcal{B}_2 : \pm x^2 \pm y^2 + \sum\limits_{i=1}^{n-2} \pm z_i^2$.

• If f_2 is degenerate form of corank 1 then f is stably quasi corner equivalent to one of the following simple classes:

1. $\mathcal{B}_m : \pm(x \pm y)^2 \pm y^m, \quad m \geq 3, \quad m + 1;$

2. $\mathcal{F}_{k,m} : \pm(x \pm y^k)^2 \pm y^m$, $m > k \geq 2$, $k + m$;

3. $\mathcal{H}_{m,n,k} : \pm(x \pm z_1^m)^2 \pm (y \pm z_1^n)^2 \pm z_1^k$, where $k > n \geq m, \geq 2$ $m + n + k - 1$.

Theorem: Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, be simple with respect to the quasi cusp equivalence. Then, either f_2 is a non-degenerate form and f is quasi cusp equivalent to $\mathcal{L}_2 : \pm x^2 \pm y^2 + \sum_{i=1}^{n-2} \pm z_i^2$ or f_2 is degenerate form of corank 1 and f is stably quasi cusp equivalent to one of the following simple classes:

1. $\mathcal{L}_k : \pm x^2 \pm y^k$, $k \geq 3$; $k + 1$.
2. $\mathcal{M}_k : \pm y^2 \pm x^k$, $k \geq 3$, when $s = 3$; $k + 2$.
3. $\mathcal{M}_3 : \pm y^2 + x^3$, when $s \geq 4$; 5.
4. $\mathcal{N}_{2,2,3} : \pm(x \pm z_1^2)^2 \pm (y \pm z_1^2)^2 \pm z_1^3$, when $s \geq 3$; 7.
5. $\mathcal{N}_{m,2,k} : \pm(x \pm z_1^m)^2 \pm (y \pm z_1^2)^2 \pm z_1^k$, $k \geq m > 2$, when $s = 3$; $m + k + 3$.

Theorem: Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, be simple with respect to the quasi cone equivalence. Then, either f_2 is a non-degenerate form and hence f is quasi cone equivalent to $\mathcal{P}_2 : \pm x^2 \pm y^2 \pm z^2 + \sum_{i=1}^{n-3} \pm w_i^2$ or f_2 is degenerate form of corank 1 and hence f is stably quasi cone equivalent to one of the following simple classes (up to a possible permutation of x, y coordinates and up to the addition with a quadratic form in some extra variables):

1. $\mathcal{P}_k :$ $\pm x^2 \pm y^2 \pm z^k$, $k \geq 3$; k.
2. $\mathcal{O}_m : \pm z^2 \pm (x - y)^2 \pm y^m$, $m \geq 3$; $m + 2$.
3. $\mathcal{S}_{k,m} :$ $\pm z^2 \pm (x \pm y^k)^2 \pm y^m$, $m > k \geq 2$; $k + m + 1$.
4. $\mathcal{Y}_l :$ $\pm x^2 \pm y^2 \pm z^2 \pm w^l$, $l \geq 3$; $4l - 4$.
5. $\mathcal{W}_{k,l} :$ $\pm x^2 \pm y^2 \pm (z + w_1^k)^2 \pm z^l$, $l > k$; $3k + l - 1$.
6. $\mathcal{Q}_{m,l} :$ $\pm(x \pm w_1^m)^2 \pm (y \mp w_1^m)^2 \pm z^2 \pm w_1^l$, $l > m \geq 2$; $3m + l - 1$.

7. $\mathcal{V}_{m,n,l}: \quad \pm(x+w_1^m)^2 \pm y^2 \pm (z+w_1^n)^2 \pm w_1^l, \quad l > n > m; \qquad 2m+n+l-1.$

Theorem: A simple (with respect to quasi **complete** flag equivalence) function germ $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ with a critical point at the origin is quasi complete flag equivalent (up to addition with a quadratic form in some extra variables x) either to the germ of one of the classes :

1. $\mathbb{B}_k : \pm z^2 \pm y^k, \qquad k \geq 2 \qquad k+1,$
2. $\mathbb{F}_{k,m} : \pm(y \pm z^k) \pm y^m \qquad m > k, m \geq 3, k \geq 2 \qquad m+k,$
3. $\mathbb{H}_{m,n,k} : \pm(y \pm x_1^m)^2 \pm (z \pm x_1^n)^2 \pm x_1^k, \quad k > n \geq m \qquad m+n+k-1,$

and therefore has corank 1 of the second differential.

Theorem: A simple (with respect to quasi **non-complete** flag equivalence) function $f : (\mathbb{R}^n, 0) \mapsto \mathbb{R}$ with a critical point at the origin is quasi non-complete flag equivalent to the germ of one of the classes (up to permutation y and z, and stabilization in x),

1. $g(y,z); \quad g \in A_k : y^2 + z^{k+1}; k \geq 1, \quad D_k : z^2y + y^{k-1}; k \geq 4,$
   $E_6 : z^3 + y^4, \quad E_7 : z^3 + zy^3, \quad E_8 : z^3 + y^5,$
2. $\mathbf{H}_{m,n,k} : \pm(y \pm x_1^m)^2 \pm (z \pm x_1^n)^2 \pm x_1^k, k \geq m \geq n.$

In all theorem, the orbit codimension in the space of germs is shown in the right column.

The main application of these classes is the classification of simple and stable projections of Lagrangian submanifolds with borders. Explictly,

Theorem: A germ L, Γ is stable if and only if its arbitrary generating family is versal with respect to quasi border equivalence and addition with functions in parameters. Any stable and simple projection of Lagrangian submanifold with a border is symplectically equivalent to the projection determined by the generating families which are quasi border reduced-versal deformations of the simple quasi border classes.

Also, we have found algebraic invariant of the border orbit in terms of local pairs consists of local algebra and ideal in it.

Proposition: If f_1 and f_2 are quasi border equivalent, then their local pairs are isomorphic.

The classification of simple quasi border singularities has nice description in terms of associated local pairs. In particular, the normal forms of simple classes yield the following

Proposition: The associated local pair of the simple quasi boundary singularity B_k is Q_{A_k}, I_1. The associated local pair of the simple quasi boundary singularity $F_{m,k}$ is Q_{A_k}, I_m. The associated local triple of the simple quasi corner singularity $\mathcal{H}_{m,n,k}$ consisting of the local algebra and two ideals corresponding to two sides $x = 0,\ y = 0$ of the corner is Q_{A_k}, I_n, I_m. For $n = 1$ we get the triple of $\mathcal{F}_{m,k}$, and for $n = 1, m = 1$ we get the triple of $\mathcal{B}_k$.

In contrast to the standard classes, quasi bifurcation diagrams and caustics for simple classes consist of several components. The formulas of versal deformations provide the explicit description of the bifurcation diagrams and caustics of quasi boundary, quasi corner and quasi complete flag singularities. They are described as follows.

Proposition:

1. The bifurcation diagram of B_2 in (λ_0, λ_1)-plane is a smooth curve and a distinguished point on it. The bifurcation diagram of $B_3 \subset \mathbb{R}^3$ is a cuspidal cylinder and a line in it which is tangent to the ridge. In general, the hypersurface component of the bifurcation diagram for B_k series is a product of generalized swallow tail and a line. The second component is the maximal smooth submanifold passing through the vertex of the generalized swallow tail times a line.

2. The B_3 caustic is the union of two tangent lines, for B_4 this is a seimicubic cylinder and a plane (the configuration is isomorphic to the discriminant of the standard C_3 boundary singularity).

3. The caustic of $F_{2,2}$ is the union of Whitney umbrella which is the second component, and a smooth tangent surface which is the caustic of the A_2 singularity.

4. The caustic of singularity B_k is a union of cylinder over generalized swallow tail (with one-dimensional generator) and a smooth hypersurface having smooth $k-3$-dimensional intersection with the first component.

5. The caustic of $F_{k,l}$ singularity is a union of a cylinder over a generalized swallow tail of type A_l and an image of Morin stable mapping (generalized Whitney umbrella) being the set of common zeros of two polynomials of degree l and k.

Proposition:

1. The bifurcation diagram of $\mathcal{B}_2$ is a smooth surfaces with two transversal lines in it . The first component of the bifurcation diagram of $\mathcal{B}_3$ is a product of a cusp and a plane in $\mathbb{R}^4$. Two other components are smooth surfaces inside the first one. They are tangent to the cuspidal ridge.

   All three components of the caustic of $\mathcal{B}_3$ are smooth pairwise tangent surfaces in 3-space.

2. The caustic of $\mathcal{B}_k$ is a union of a cylinder over a generalized swallow tail and two smooth hypersurfaces tangent to the first component.

3. The caustic of $\mathcal{F}_{k,m}$ is a union of a cylinder over a generalized swallow tail, a smooth hypersurface and a generalized Whitney umbrella multiplied by a line. In particular, the caustics of $\mathcal{F}_{2,3}$ is union of two smooth hypersurface in $\mathbb{R}^4$ and Whitney umbrella multiplied by a line.

4. The caustic of $\mathcal{H}_{k,m,n}$ is a union of a cylinder over a generalized swallow tail and two generalized Whitney umbrellas of respective dimensions.

Proposition:

1. The first component of the bifurcation diagram (caustics) of any simple quasi flag complete singularity is a cylinder over the generalized swallowtail.

2. In particular, the first component of the bifurcation diagram of the class $\pm z^2 \pm y^3$ is product of a cusp and a plane in $\mathbb{R}^4$. The second component is a smooth surface inside the first one. The third component is a line inside the second

component. The second and third components are tangent to the cuspidal edge.

3. The first and second components of the class $\pm z^2 \pm y^3$ are smooth tangent surfaces in $\mathbb{R}^3$ and their intersection is exactly.

4. The caustics of $\pm z^2 \pm y^k, k \geq 3$ is a union of a cylinder over a generalized swallowtail and smooth hypersurafaces and $k-2$-dimnestional space. The second and third components are tangent to the first one.

5. The caustics of the class $\pm(y \pm z^k)^2 \pm z^m$ is a union of a cylinder over a generalized swallowtail and a generalized Whitney umbrella times a line and a $(k+m-3)$- dimensional space.

6. The caustics of the class $\pm(y \pm x_1^m)^2 \pm (z \pm x_1^n)^2 \pm x_1^k$ is a union of a cylinder over a generalized swallowtail and a generalized Whitney umbrella and intersection of two generalized Whitney umbrellas.

Besides being wavefronts and caustics of Lagrange submanifold projections with borders, these objects appear as bifurcation diagrams of function depending on parameters in various problems with inequality constraints.

The further study of similar non-standard equivalence relation and its comparison with standard one will give extra information on the nature of singularity classes in various optimization problems and problems in variations theory with constraints, on the adjacencies of respective singularity classes and on their invariants

In chapters 9-11, we have dealt with a different equivalence relation in projectionn theory. It is non-standard relation and called **quasi projection**. The pseudo and quasi equivalences of two hypersurafes implies the existences of diffeomorphis which transfer one hypersuraces to the other one and also posses the following properties: it preserves the tangency of the vector field or preserve the direction of the vector belonging to the hypersurafes.

In addition to the equivalence relation which was described in [38], we have introduced and classified two different types in this diecrion.

Firstly, we classified simple singularities of projections to a plane of surfaces embedded into three-space and equipped with a boundary. The classification result is stated in the following theorem:

Theorem: The list of simple quasi projections of regular surfaces with boundaries in three space consists of the following normal forms of the projections $(x, y, z) \mapsto (y, z)$ of the germs at the origin of the graphs V of the functions $z = f(x, y)$ and the boundaries $g(x, y) = 0$:

1. For $\widetilde{A}_k : f = \frac{1}{k+1}x^{k+1} + yx, \quad k \geq 0$, the boundaries are the Arnold's simple boundary (with respect to the $w = 0$ boundary classes of curves $g(w, x) = 0$, where the coordinate $w = y + x^k$ vanish at the critical set of the projection of the surface:

$$\begin{array}{ccccccccccc}
x & \leftarrow & w+x^2 & \leftarrow & w+x^3 & \leftarrow & \dots & \leftarrow & w+x^{k-1} & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & x^2 \pm w^2 & \leftarrow & xw+x^3 & \leftarrow & \dots & \leftarrow & xw+x^{k-1} & \leftarrow & xw+x^{k-1} \\
 & & \uparrow & & \uparrow & & & & & & \\
 & & x^2+w^3 & \leftarrow & x^3+w^2 & & & & & & \\
 & & \uparrow & & & & & & & & \\
 & & x^2 \pm w^4 & \leftarrow & x^2+w^5 & \leftarrow & \dots & \leftarrow & x^2 \pm w^n \;\dots & &
\end{array}$$

2. For $\widetilde{B}_k : f = \frac{1}{3}x^3 + y^k x, \quad k \geq 3$,

• If k is odd

$$\begin{array}{ccccccc}
x+y & \leftarrow & x+y^2 & \leftarrow & \dots & \leftarrow & x+y^{k-1} \\
\uparrow & & & & & & \\
y+x^2 & \leftarrow & y & & & &
\end{array}$$

• If k is even

$$\begin{array}{ccccccc}
x+y & \leftarrow & x+y^2 & \leftarrow & \dots & \leftarrow & x+y^{\frac{k}{2}-1} \\
\uparrow & & & & & & \\
y+x^2 & \leftarrow & y & & & &
\end{array}$$

3. For $\widetilde{C}_k : f = \frac{1}{k+1}x^{k+1} + x^2y, \quad k \geq 2,$

$$\begin{array}{llll} x+y & \leftarrow x+y^2 & \leftarrow \dots & \leftarrow x+y^n\dots \\ \uparrow & & & \\ y+x^2 & \leftarrow \dots & \leftarrow y+x^{k-2} & \end{array}$$

4. And for $\widetilde{F}_4 : f = x^4 + y^2x$, there are only four simple classes

$$\begin{array}{lll} x+y & \leftarrow y+x^2 & \leftarrow y+x^3 \\ \uparrow & & \\ x & & \end{array}$$

Finally, we classified simple classes of quasi projection of graphs of two different type of mappings:

1. Quasi projections of graphs of parameterized plane curve germs.

Theorem: Any simple projection of a graph of parametrized curve $\gamma : t \mapsto (\alpha(t), \beta(t))$, with respect to quasi equivalence is equivalent to the graph of the curve $\widetilde{\gamma} : t \mapsto (\pm t^k, 0)$ for some $k \geq 1$. The remaining germs form a subset of infinite codimension in the space of germs.

2. Quasi projections of graph mappings germs $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

Theorem: If the mapping $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is of corank 1 and the graph of f is simple with respect to quasi projection then the projection of the graph of F is quasi equivalent to the graph of one of the following mappings: $G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0); (u, v) \mapsto (u, g)$ where g is listed as follows:

$$\widetilde{A}_k : g = v^{k+1} + uv; \quad k \geq 0,$$

$$\widetilde{B}_k : g = v^3 + u^k v; \quad k \geq 2,$$

$$\widetilde{C}_k : g = v^{k+1} + v^2 u; \quad k \geq 2,$$

$$\widetilde{F}_4 : g = v^4 + u^2 v.$$

If the mapping $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is of corank 2, and the projection of the graph is simple with respect to quasi projection. Then, the projection of the graph of F is quasi projection equivalent to the graph of of the following mappings:

$$G : (u, v) \mapsto (u^2 \pm v^2, uv).$$

One of the interesting application for quasi projection equivalence relation is used in partial differential equations (PDE) with boundary value problems.

Consider the characteristics method solving the simplest Cauchy problem for first order linear PDE: $\sum a_i(x) \frac{\partial u}{\partial x_i} = 0$, where $u(x)$ is unknown function with $x \in \mathbb{R}^n$ and $a_i(x)$ are given functions. The problem includes the boundary hypersurface $S \subset \mathbb{R}^n$ and the boundary values $U|_S = U_0$. Generically the characteristic vector field $v = a_i \frac{\partial}{\partial x_i}$ is tangent to S at some points which are called characteristic. Outside the set K of characteristic points , the problem has a unique local solution. So the geometry of the set K is essential feature of the problem. If we rectify the vector field getting say $\frac{\partial}{\partial x_1}$, then the problem to classify K is exactly to find critical points of the projection of S along parallel rays.

Our pseudo (or quasi) equivalence preserves critical locus and is even better than the standard one. The main difference with standard one is that the pseudo (or quasi) equivalence does not preserve the sets of points in the same fibre (multiple singular points on one fibre can go to different fibres).

Similarly in many other complicated PDE boundary value problems, mainly in continuum mechanics, the generalisation of Neumann boundary condition is used. The derivative of the unknown function is taken along a given vector field (for Neumann this is normals to the boundary surface). Again, the locus of the points on the surface where the vector field is not transversal to the surface is of importance.

The further research which goes beyond the present thesis might be related to the interesting question on the topology of the bifurcation diagrams of the given classes and to the applications of our classification of projections to the above mentioned boundary value problems.

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