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On the zero divisor graphs of Galois rings

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Abstract

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On the zero divisor graphs of Galois rings

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1. Introduction

In this section, we summarize some well known results on the zero divisor graphs of commutative rings.The definition of terms and standard notations can be obtained from [1, 4, 8]. The concept of a zero divisor graph was proposed by Ivstan Beck in [3]. According to him, $G(R) = R$ such that the distinct vertices u and v are adjacent iff $uv = 0$. Since the zero element is adjacent to every other vertex, the graph is connected with $\mathrm{diam}(G(R)) \leq 2$. He conjectured that the chromatic number coincides with the clique number of $G(R)$. Later, D.D. Anderson and M. Naseer in [1] provided an example where the clique number is strictly less than the chromatic number. D.F. Anderson and P.S. Livingston [2] simplified Beck's graph. They considered the non zero zero divisors as the vertices of the graph $\Gamma(R)$ and the adjacency concept is similar to Beck's. Amongst their major findings, the graph of commutative ring R is connected with diam $(\Gamma(R)) \leq 3$. S.B. Mulay in [6]introduced the zero divisor graph $\Gamma_E(R)$ of equivalent vertices. The graph $\Gamma_E(R)$ is connected with $\mathrm{diam}(\Gamma_E(R)) \leq 3$. This graph can be finite even if R is infinite, thus it is less "noisy" than its predecessors. S.P. Redmond in [7], originated a zero divisor graph with respect to an ideal I. The distinct vertices u and v are adjacent with $uv \in I$. In this paper, some graph theoretical properties of Galois rings have been determined.

2. Zero divisor graphs of Galois rings

A ring R is said to be Galois, if its subset of all the zero divisors (including zero) forms a principal ideal. It is an extension of the ring of integers modulo p^k (where p is a prime integer and k is a positive integer). The extension is usually represented by $\mathbb{Z}_{p^k}[x]/ < f(x) >$ where $f(x) \in \mathbb{Z}_{p^k}[x]$ is a monic polynomial of degree r and irreducible over $\mathbb{Z}_p$. It is denoted by $GR(p^{kr}, p^k)$. A Galois ring can be trivial depending on whether k or r equals to 1, otherwise it is nontrivial.
Let R_o be a Galois ring and $Z(R_o)$ be its subset of zero divisors (including zero). Then $Z(R_o)$

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is a unique maximal ideal of R_o and is therefore the Jacobson radical of R_o. A graph $\Gamma(R_o)$ is associated to R_o with vertices $Z(R_o)^* = Z(R_o) \setminus \{0\}$, the set of nonzero zero divisors of R_o. The graph theoretical properties are used to illuminate the structure of $Z(R_o)$. Two distinct vertices $u_1, u_2 \in Z(R_o)^*$ are adjacent if $u_1u_2 = 0$ and equivalent if $\mathrm{Ann}_{R_o}(u_1)=\mathrm{Ann}_{R_o}(u_2)$. The graph of equivalent vertices in R_o, $\Gamma_E(R_o)$ is simple with vertex set $Z(R_o)^*/\sim$ such that $[u_1], [u_2] \in Z(R_o)^*/\sim$, adjacent provided $u_1u_2 = 0$.
Let $u \in Z(R_o)^*$ and $s \neq 1$ be a unit in R_o. In $\Gamma(R_o)$, the vertices u and su are distinct and $[u] = [su]$ in $\Gamma_E(R_o)$. So $\Gamma_E(R^{'})$ is less noisy than $\Gamma(R)$, (see[5]).
In this paper, we investigate the connectedness of $\Gamma(R_o)$, $\Gamma_E(R_o)$ and determine the diameter, girth and binding number of $\Gamma(R_o)$ and $\Gamma_E(R_o)$.

Theorem 2.1. *(See e.g [2]) Let R be a commutative ring (not necessarily Galois). Then $\Gamma(R)$ is finite iff R is finite or an integral domain.*

Remark 2.2. *From the above theorem, we notice that $\Gamma(R_o)$ is finite because R_o is finite.*

Remark 2.3. *If $R_o = GR(p^r, p)$, then $Z(R_o)^*$ is empty. It will therefore be of interest to characterize the zero divisor graphs of $R_o = GR(p^{kr}, p^k)$, where $k > 1, r \geq 1$*

Remark 2.4. *If p is prime, then $\Gamma(GR(p^2, p^2))$ is a complete graph on $p - 1$ vertices and $\Gamma_E(GR(p^2, p^2))$ is a single vertex. But $\Gamma(GR(p^3, p^3))$ is messy while $\Gamma_E(GR(p^3, p^3))$ is a single edge. Moreover, $\Gamma(GR(p^4, p^4))$ is even messier while $\Gamma_E(GR(p^4, p^4))$ is a star graph.*

Proposition 2.5. *Let $R_o = GR(p^k, p^k)$. Suppose α is a unit in R_o and $\mid \Gamma(R_o) \mid = p^l\alpha$, then the degree of $p^l\alpha$,*

$$deg(p^l\alpha) = \begin{cases} p^l - 1 \text{ if } 2l < k \\ \\ p^l - 2 \text{ if } 2l \geq k \end{cases}$$

Proof. Given that α is a unit in R_o, then $(\alpha, p^k) = 1$. So, all the vertices adjacent to $p^l\alpha$ in the set $pGR(p^k, p^k)$ of the graph $\Gamma(R_o)$ are the same vertices adjacent to p^l. Now, to find the degree of $p^l\alpha$, it suffices to find all the vertices adjacent to p^l. Let n be the number of vertices adjacent to p^l in $\Gamma(R_o)$. The first term in this sequence of vertices is p^{k-l} and the nth term is $p^{k-l} + (n-1)p^{k-l}$. Since the last term in the sequence is $p^k - p^{k-l}$, it easily follows that $p^{k-l} + (n-1)p^{k-l} = p^k - p^{k-l}$ leading to $n = p^l - 1$ if $p^{k-l} > p^l$ or $k > 2l$. If $p^{k-l} \leq p^l$ or $k \leq 2l$, then $p^{2l-k}(p^{k-l}) = p^l$ is adjacent to itself, so that $\deg(p^l) = p^l - 2$.

■

Proposition 2.6. *Given that $R_o = GR(p^{kr}, p^k)$ and α is a unit in R_o, then the degree of $p^{lr}\alpha$,*

$$deg(p^{lr}\alpha) = \begin{cases} p^{lr} - 1 \text{ if } 2l < k \\ \\ p^{lr} - 2 \text{ if } 2l \geq k \end{cases}$$

Proposition 2.7. *Let $R_o = GR(p^k, p^k)$, $k \geq 3$. Then,*

$$\Gamma(R_o) = \begin{cases} (p^{\frac{k}{2}} - 1) - partite \text{ if } k \text{ } is \text{ } even \\ \\ p^{\frac{k-1}{2}} - partite \text{ if } k \text{ } is \text{ } odd \end{cases}$$

Proof. Obviously $Z(R_o)^* = Z(R_o) - \{0\} = pR_o - \{0\}$.
Case (i): k is an even integer. We partition $Z(R_o)^*$ into the following subsets:

$$V_1 = Z(R_o)^* - \{j(p^{\frac{k}{2}}), 2 \leq j \leq p^{\frac{k}{2}} - 1\};$$

$$V_j = \{j(p^{\frac{k}{2}})\}, 2 \leq j \leq p^{\frac{k}{2}} - 1.$$

For $1 \leq i \leq p^{\frac{k}{2}} - 1$, $V_i \neq \emptyset$, V_1 does not contain adjacent vertices, each V_j is singleton,

$$V_1 \cap V_j = \emptyset, 2 \leq j \leq p^{\frac{k}{2}} - 1;$$

$$V_j \cap V_l = \emptyset, j \neq l, 2 \leq j, l \leq p^{\frac{k}{2}} - 1.$$

Finally

$$Z(R_o)^* = V_1 \cup (\cup_{j=2}^{p^{\frac{k}{2}}-1} V_j) = \cup_{i=1}^{p^{\frac{k}{2}}-1} V_i.$$

Thus $\Gamma(R_o)$ is $p^{\frac{k}{2}} - 1$ partite.
Case (ii): k is an odd integer. We partition $Z(R_o)^*$ into the following subsets:

$$V_1 = Z(R_o)^* - \{(j-1)(p^{\frac{k+1}{2}}), 2 \leq j \leq p^{\frac{k-1}{2}}\};$$

$$V_j = \{(j-1)(p^{\frac{k+1}{2}})\}, 2 \leq j \leq p^{\frac{k-1}{2}}$$

. For $1 \leq i \leq p^{\frac{k-1}{2}}$, $V_i \neq \emptyset$, V_1 does not contain adjacent vertices, each V_j is singleton,

$$V_1 \cap V_j = \phi, 2 \leq j \leq p^{\frac{k-1}{2}};$$

$$V_j \cap V_l = \emptyset, j \neq l, 2 \leq j, l \leq p^{\frac{k-1}{2}}.$$

Finally

$$Z(R_o)^* = V_1 \cup (\cup_{j=2}^{p^{\frac{k-1}{2}}} V_j) = \cup_{i=1}^{p^{\frac{k-1}{2}}} V_i.$$

Thus $\Gamma(R_o)$ is $p^{\frac{k-1}{2}}$ partite

■

Example 1.
Let $R_o = GR(8, 8)$. Then $Z(R_o)^* = \{2, 4, 6\}$. $V_1 = \{2, 6\}, V_2 = \{4\}$. So $\Gamma(R_o)$ is bipartite.
Example 2
Let $R_o = GR(81, 81)$. Then $Z(R_o)^* = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30,$
$36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75, 78\}$. $V_1 = \{3, 6, 9, 12, 15, 21, 24, 30,$
$33, 39, 42, 48, 51, 57, 60, 66, 69, 75, 78\}$, $V_2 = \{18\}$, $V_3 = \{27\}$, $V_4 = \{36\}$, $V_5 = \{45\}$, $V_6 = \{54\}$, $V_7 = \{63\}$, $V_8 = \{72\}$. So $\Gamma(R_o)$ is octapartite.

Proposition 2.8. *Suppose* $R_o = GR(p^k, p^k), k \geq 4$. *Then*
i)The diameter, $diam(\Gamma(R_o)) = 2$
ii) The girth, $gr(\Gamma(R_o)) = 3$
iii)The binding number,

$$b(\Gamma(R_o)) = \begin{cases} \frac{p^{\frac{k}{2}}-2}{p^{k-1}-p^{\frac{k}{2}}+1} & \text{if } k \text{ is even} \\ \\ \frac{p^{\frac{k-1}{2}}-1}{p^{k-1}-p^{\frac{k-1}{2}}} & \text{if } k \text{ is odd} \end{cases}$$

Proof. (i) and (ii) are clear. To prove (iii), we consider the two cases separately. If k is even, we consider

$$V_1 = Z(R_o)^* - \{j(p^{\frac{k}{2}}), 2 \leq j \leq p^{\frac{k}{2}} - 1\}$$

and

$$N(V_1) = \{j(p^{\frac{k}{2}}), 2 \leq j \leq p^{\frac{k}{2}} - 1\}.$$

Notice that

$$| V_1 |= p^{k-1} - p^{\frac{k}{2}} + 1$$

and

$$| N(V_1) |= p^{\frac{k}{2}} - 2$$

, so that

$$b(\Gamma(R_o)) = \frac{\mid N(V_1) \mid}{\mid V_1 \mid} = \frac{p^{k/2} - 2}{p^{k-1} - p^{\frac{k}{2}} + 1}.$$

If k is odd, we consider

$$V_1 = Z(R_O)^* - \{(j-1)(p^{\frac{k+1}{2}}), 2 \le j \le p^{\frac{k-1}{2}}\}$$

and

$$N(V_1) = \{(j-1)(p^{\frac{k+1}{2}}), 2 \le j \le p^{\frac{k-1}{2}}\}.$$

Notice that

$$\mid V_1 \mid = p^{k-1} - p^{\frac{k-1}{2}}$$

and

$$\mid N(V_1) \mid = p^{\frac{k-1}{2} - 1}$$

, so that

$$b(\Gamma(R_o)) = \frac{\mid N(V_1) \mid}{\mid V_1 \mid} = \frac{p^{\frac{k-1}{2}} - 1}{p^{k-1} - p^{\frac{k-1}{2}}}.$$

■

Remark 2.9. *Easy computations yield* $lim_{k \to \infty} b(\Gamma(R_o)) = 0$. *This implies that as* k *increases (whether odd or even), more vertices become non adjacent.*

Lemma 2.10. *Let* $R_o = GR(p^r, p)$, *then* $Z(R_o)^* = \emptyset$

Proof. Easy

■

Proposition 2.11. *Let* $R_o = GR(p^{2r}, p^2)$. *Then* $\Gamma(R_o) = K_{p^r - 1}$ *and* $\Gamma_E(R_o)$ *is a single vertex.*

Proof. Since $((Z(R_o))^2 = 0$, each zero divisor is adjacent to the other. But

$$\mid Z(R_o)^* \mid = p^r - 1$$

, so that $\Gamma(R_o)$ is complete on $p^r - 1$ vertices. That $\Gamma_E(R_o)$ is a single vertex follows from the fact that $\text{Ann}(Z(R_o)) = Z(R_o)$

■

Remark 2.12. *Let* $R_o = GR(p^{3r}, p^3)$. *Then* $\Gamma(R_o)$ *is noisy while* $\Gamma_E(R_o)$ *is a single edge.*

Proposition 2.13. *Given that* $R_o = GR(p^{kr}, p^k), k \ge 4$. *Then the clique number,*

$$\omega(\Gamma_E(R_o)) = \begin{cases} \frac{k}{2} \text{ if } k \textit{ is even} \\ \\ \frac{k+1}{2} \text{ if } k \textit{ is odd} \end{cases}$$

Proof. It suffices to find a maximal complete subgraph of $\Gamma_E(R_o)$. Let s be a unit in R_o.

Case I: k is even.
We show that $\Gamma_E(R_o)$ has a maximal complete subgraph S with vertices

$$\{[p^\iota s] = [p^\iota], \frac{k}{2} \le \iota \le k-1\}.$$

Suppose on the contrary that S is not maximal in $\Gamma_E(R_o)$ and that there exists a maximal complete subgraph $S^{'} \subset \Gamma_E(R_o)$ so that $S \subset S^{'}$. Without loss of generality, assume that $p^i \in S^{'}$ where

$$\begin{cases} 0 < i < \frac{k}{2}, \ k \text{ is even} \\ \\ 0 < i < \frac{k-1}{2}, \ k \text{ is odd} \end{cases}$$

then there exists some $j > i > 0$ so that

$$\begin{aligned} p^i.p^{k-1-j} &= p^{k-1+i-j} \\ &\neq 0. \end{aligned}$$

So $S^{'}$ is not complete, a contradiction.

Case II: k is odd.
Using similar steps as Case I above, we can show that $\Gamma_E(R_o)$ contains a maximal complete subgraph with vertices

$$\{[p^{\iota}s] = [p^{\iota}], \frac{k-1}{2} \leq \iota \leq k-1\}.$$

■

Proposition 2.14. *Let $R_o = GR(p^{kr}, p^k), k \geq 4$. Then*
i) the diameter,diam $(\Gamma_E(R_o)) = 2$
ii) the girth, $gr(\Gamma_E(R_o)) = 3$
iii)the binding number,

$$b(\Gamma_E(R_o)) = \begin{cases} \frac{k-4}{k} \text{ if } k \text{ is even} \\ \\ \frac{k-5}{k-1} \text{ if } k \text{ is odd} \end{cases}$$

Proof. (i) and (ii) are easy. To prove (iii), we consider the two cases separately. When k is even, then from Proposition 2 ,

$$\mid V_1 \mid = \frac{k}{2}$$

while

$$\mid N(V_1) \mid = \frac{k-4}{2},$$

so that

$$b(\Gamma(R_o)) = \frac{k-4}{k}.$$

Next, when k is odd,

$$\mid V_1 \mid = \frac{k-1}{2}$$

while

$$\mid N(V_1) \mid = \frac{k-5}{2}.$$

The result then easily follows.

■

Corollary 2.15. *Suppose $R_o = GR(p^{kr}, p^k)$ where $k \geq 4$. Then*

$$\Gamma_E(R_o) = \begin{cases} \frac{k}{2} \text{ partite if } k \text{ is even} \\ \\ \frac{k+1}{2} \text{ partite if } k \text{ is odd} \end{cases}$$

Proof. Case I: k is even.
$\Gamma_E(R_o)$ is partitioned into the following subsets.

$$V_1 = \{(Z(R_o))^\iota : 1 \leq \iota \leq \frac{k}{2}\}$$
$$V_i = \{(Z(R_o))^i\}, \frac{k}{2} < i \leq k-1.$$

For each i, $V_1 \cap V_i = \emptyset$ and the V_i are mutually disjoint. Moreover,

$$V_1 \cup (\cup_{i=k/2}^{k-1} V_i = \Gamma_E(R_o)$$

. The result then follows by counting the disjoint subsets of $\Gamma_E(R_o)$.

Case II: k is odd.
In this case, $\Gamma_E(R_o)$ is partitioned into the following subsets.

$$V_1 = \{(Z(R_o))^\iota : 1 \leq \iota \leq \frac{k-1}{2}\}$$
$$V_i = \{(Z(R_o))^i\}, \frac{k-1}{2} < i \leq k-1.$$

The rest of the proof involves steps similar to Case I with some slight modifications.

■

Proposition 2.16. *Let $R_o = GR(p^{kr}, p^k), k \geq 3$. Then*
i) the diameter,diam $(\Gamma(R_o)) = 2$
ii) the girth, $gr(\Gamma(R_o)) = 3$
iii) the binding number,

$$b(\Gamma(R_o)) = \begin{cases} \frac{p^{(\frac{k}{2})r}-2}{p^{(k-1)r}-p^{(\frac{k}{2})r}+1} & \text{if } k \text{ is even} \\ \\ \frac{p^{(\frac{k-1}{2})r}-1}{p^{(k-1)r}-p^{(\frac{k-1}{2})r}} & \text{if } k \text{ is odd} \end{cases}$$

Proof. (i) and (ii) are elementary. To prove (iii), begin with the case when k is even. Let $\epsilon_1, ..., \epsilon_r \in R_o$ with $\epsilon_1 = 1$ such that

$$\overline{\epsilon_1}, ..., \overline{\epsilon_r} \in R_o/pR_o$$

form a basis for R_o/pR_o regarded as a vector space over its prime subfield F_p. By the definition of V_1 in the previous Proposition,

$$N(V_1) = V_{\sum a_i \epsilon_i}.$$

So

$$\mid N(V_1) \mid = p^{(\frac{k}{2})r} - 2.$$

Also

$$\begin{aligned} \mid V_1 \mid &= \mid Z(R_o)^* \mid \\ &= \mid V_{\sum a_i \epsilon_i} \mid \\ &= p^{(k-1)r} - 1 - (p^{(k-1)r} - 2) \\ &= p^{(k-1)r} - p^{\frac{k}{2}r} + 1. \end{aligned}$$

Let k be odd. Then

$$\begin{aligned} \mid N(V_1) \mid &= \mid V_{\sum a_i \epsilon_i} \mid \\ &= p^{\frac{k-1}{2}r} - 1. \end{aligned}$$

Also

$$\begin{aligned} | V_1 | &=| Z(R_o)^* - X | \\ &=| Z(R_o)^* | - | X | \\ &= p^{(k-1)r} - 1 - (p^{(\frac{k-1}{2})r} - 1) \\ &= p^{(k-1)r} - p^{(\frac{k-1}{2})r} \end{aligned}$$

■

Proposition 2.17. *Let $R_o = GR(p^{kr}, p^k)$,where $k \geq 3$ and $r \in \mathbb{Z}^+$. Then*

$$\Gamma(R_o) = \begin{cases} p^{(\frac{k}{2})r} - 1 \text{ partite if } k \text{ is even} \\ \\ p^{(\frac{k-1}{2})r} \text{ partite if } k \text{ is odd} \end{cases}$$

Proof. Clearly

$$Z(R_o)^* = Z(R_o) \setminus \{0\} = pR_o \setminus \{0\}.$$

Let $\epsilon_1, ..., \epsilon_r \in R_o$ with $\epsilon_1 = 1$ such that

$$\overline{\epsilon_1}, ..., \overline{\epsilon_r} \in R_o/pR_o$$

form a basis for R_o/pR_o regarded as a vector space over its prime subfield F_p. We consider the two cases separately.

Case I:k is an even integer.
We partition $Z(R_o)^*$ into the following subsets.

$$X = \{a_i\epsilon_i, 1 \leq i \leq r, a_i \in \{0, j(p^{\frac{k}{2}})\}, 1 \leq j \leq p^{\frac{k}{2}} - 1\}$$

$$V_{\sum a_i\epsilon_i} = X \setminus \{0, p^{\frac{k}{2}}\}$$

$$V_1 = Z(R_o)^* \setminus V_{\sum a_i\epsilon_i}\}$$

For each $i = 1, ..., r$, $V_{\sum a_i\epsilon_i} \neq \emptyset$. Each of the $V_{\sum a_i\epsilon_i} \neq \emptyset$ contains no adjacent vertices, $V_1 \cap V_{\sum a_i\epsilon_i} = \emptyset$. The sets $V_{\sum a_i\epsilon_i}$ are mutually disjoint. Moreover $Z(R_o)^* = V_1 \cup (\cup_r V_{\sum a_i\epsilon_i})$. Thus $\Gamma(R_o)$ is $p^{(\frac{k}{2})r}$ partite.

Case II:k is an odd integer.
We partition $Z(R_o)^*$ into the following subsets.

$$X = \{a_i\epsilon_i, 1 \leq i \leq r, a_i \in \{0, (j-1)(p^{\frac{k+1}{2}})\}, 1 \leq j \leq p^{\frac{k-1}{2}}\}$$

$$V_{\sum a_i\epsilon_i} = X \setminus \{0\}$$

$$V_1 = Z(R_o)^* \setminus X.$$

The rest of the proof is similar to the previous case with slight modifications.

■

3. Automorphisms of zero divisor graphs of Galois rings

Consider the integer $k \geq 2$ and a positive integer r. Distinct ring automorphisms of $GR(p^{kr}, p^k)$ induce distinct graph automorphisms of $\Gamma(GR(p^{kr}, p^k))$, because $GR(p^{kr}, p^k)$ is a finite ring which is not a field, (see [2]). A graph automorphism, f of a graph $\Gamma(R_o)$ is a bijection $f : \Gamma \to \Gamma$ which preserves adjacency. The set Aut(Γ) of all graph automorphisms of Γ forms a group under the usual composition of functions, (see [2]). If $\mid \Gamma \mid= p^k$, then in the obvious way, Aut(Γ) is isomorphic to a subgroup of S_{p^k} and clearly Aut$(K_{p^k}) \cong S_{p^k}$. Infact, for a graph Γ of order p^k, Aut$(\Gamma) \cong S_{p^k}$ iff $\Gamma = K_{p^k}$. Now, by restricting each $f \in$Aut(R_o) to $Z(R_o)^*$ we obtain a natural group homomorphism ϕ :Aut$(R_o) \to$Aut$(\Gamma(R_o))$.

Theorem 3.1. *(See [2]) Let $R_o = GR(p^{kr}, p^k)$ and let $f \in Aut(R_o)$. If $f(x) = x, \forall x \in Z(R_o)$, then $f = 1_{R_o}$. Thus, ϕ :$Aut(R_o) \to Aut(\Gamma(R_o))$ is a monomorphism.*

Now, consider $R_o = GR(p^k, p^k)$. For $p^k \geq 4$, $k \neq 1$, let

$$X = \{j \in \mathbb{Z} \mid 1 < j < p^k, j \mid p^k\}.$$

For each, $j \in X$, let

$$V_j = \{l \in \mathbb{Z} \mid 1 < l < p^k, (l, p^k) = j\}.$$

Note that

$$Z(GR(p^k, p^k))^*$$

is the disjoint union of V_js. Notice that two vertices have the same degrees iff they are in the same V_d (See [2]).

Proposition 3.2. *Consider the integer $k \geq 2$. Then*

$$\mid Aut(\Gamma(p^{kr}, p^k)) \mid= \begin{cases} \prod_{l=2}^{k}(2^{(k-l)r})! \text{ if } p = 2 \\ \\ \prod_{l=2}^{k}(p^{(k-l)r}(p^r - 1))! \text{ if } p \neq 2 \end{cases}$$

Proof. Case I: $r = 1$.
Let $p = 2$. Set $X = \{2, 4, ..., 2^{k-1}\}$. Then $V_2 = \{2t \mid t \text{ is odd}\}$, $V_4 = \{4t \mid t \text{ is odd}\}$,...,$V_{2^{k-1}} = \{2^{k-1}t \mid t \text{ is odd}\}$. Upon counting, $\mid V_2 \mid= 2^{k-2}$, $\mid V_4 \mid= 2^{k-3}$ and continuing in a similar manner, $\mid V_{2^{k-2}} \mid= 2$, and $\mid V_{2^{k-1}} \mid= 1$. So Aut$(\Gamma(GR(2^k, 2^k))) \cong \prod_{l=2}^{k} S_{2^{k-l}}$ and the result easily follows.
Let $p \neq 2$. Set $X = \{p, p^2, ..., p^{k-1}\}$. Then $V_p = \{pt \mid (t, p) = 1\}$, $V_{p^2} = \{p^2t \mid (t, p^2) = 1\}$,...,$V_{p^{k-1}} = \{p^{k-1}t \mid (t, p^{k-1}) = 1\}$. Upon counting, $\mid V_p \mid= p^{k-2}(p - 1)$, $\mid V_{p^2} \mid= p^{k-3}(p - 1)$ and continuing in a similar manner, $\mid V_{p^{k-1}} \mid= p - 1$. So Aut$(\Gamma(GR(p^k, p^k))) \cong \prod_{l=2}^{k} S_{p^{k-l}(p-1)}$ and the result follows immediately.
Case II: $r > 1$.
Let $\epsilon_1, ..., \epsilon_r \in R_o$ with $\epsilon_1 = 1$ such that $\overline{\epsilon_1}, ..., \overline{\epsilon_r} \in R_o/Z(R_o)$ form a basis for $R_o/Z(R_o)$ regarded as a vector space over its prime subfield $GF(p)$. For $p = 2$, let $X = \{2, 4, ..., 2^{k-1}\}$ and $V_{\sum_{a_i \epsilon_i}}$ where $a_i \in X$ be the disjoint vertices, then clearly Aut$(\Gamma(GR(2^{kr}, 2^k))) = S_{2^{(k-l)r}}$. The steps are similar for the case when p is odd.

■

4. A class of finite rings

Let R_o be the Galois ring of the form $GR(p^{kr}, p^k)$. For each $i = 1, ..., h$, let $u_i \in Z(R_o)$ such that U is an h dimensional R_o- module generated by $u_1, ..., u_h$ so that $R = R_o \oplus U$ is an additive group. On this group, define multiplication by the following relations:

: (i)If $k = 1, 2$ then $pu_i = u_iu_j = u_ju_i = 0$, $u_ir_o = r_ou_i$.

: (ii)If $k > 3$ then $p^{k-1}u_i = 0$, $u_i u_j = p^2\gamma_{ij}, {u_i}^k = {u_i}^{k-1}u_j = u_i {u_j}^{k-1} = 0, u_i r_o = r_o u_i$. where $r_o, \gamma_{ij} \in R_o$, $1 \leq i,j \leq h$, p is a prime integer, n and r are positive integers. Moreover if $u_i|_U$, then the additive order of u_i is p.

It can be shown that the multiplication turns the additive group into a commutative ring with identity. The structure of the Von Neumann regular elements of the ring is well known. We present some results on the structure its zero divisors.

Proposition 4.1. *Let R be a ring constructed in this section. If $Z(R) = pR_o \oplus U$; $ann(Z(R_o)) = p^{n-1}R_o \oplus U$; $J^{n-1} = p^{n-1}R_o$.*
If
i) $x \in ann(J)$ then $deg(x) =\mid J \mid -2$
ii)$y \in Z(R)$ but $y \in R - ann(Z(R))$, then $deg(y) =\mid ann(Z(R)) \mid -1$

Proposition 4.2. *Let R be a ring constructed in this section. Then*
i)

$$\Gamma(R) = \begin{cases} p^{(\frac{k}{2}+h)r} - 1 \text{ partite if } k \text{ is even} \\ \\ p^{(\frac{k-1+2h}{2})r} \text{ partite if } k \text{ is odd} \end{cases}$$

ii)$diam(\Gamma(R)) = 2$
iii) $gr(\Gamma(R)) = 3$
iv)

$$b(\Gamma(R)) = \begin{cases} \frac{p^{(\frac{k}{2}+h)r}-2}{p^{(k-1+h)r}-p^{(\frac{k}{2}+h)r}} \text{ if } k \text{ is even} \\ \\ \frac{p^{(\frac{k-1}{2}+h)r}-1}{p^{(k-1+h)r}-p^{(\frac{k-1}{2}+h)r}} \text{ if } k \text{ is odd} \end{cases}$$

5. Conclusion

This study reveals, that it is possible to generalize the graph theoretical properties of Galois rings. The zero divisor graphs of Galois rings are symmetrical as confirmed from their automorphisms. It would be interesting to investigate, whether these properties extend to the zero divisor graphs of the idealizations of the Galois rings.

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