IMHOTEP

AFRICAN JOURNAL OF PURE AND APPLIED MATHEMATICS

Imhotep Mathematical Proceedings
Volume 2, Numéro 1, (2015),
pp. 71 – 75.

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Taiwo. O. Sangodapo,
Deborah. O. A Ajayi
Department of Mathematics,
University of Ibadan, Oyo State,
Nigeria.
toewuola77@gmail.com
adelaideajayi@yahoo.com

Abstract

This work seeks to investigates the equivalence of Topological Colored Tverberg Theorem and Colored Winding Number Conjecture on the plane.

Proceedings of the 4^{th} annual workshop on CRyptography, Algebra and Geometry (CRAG-4), 21- 25 July 2014, University of Dschang, Dschang, Cameroon.

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Imhotep Mathematical Proceedings

IMHOTEP - Math. Proc. **2**, No.1 (2015), 71–75
1608-9324/020071–70, DOI 13.1007/s00009-003-0000

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On The Topological Colored Tverberg Theorem and Colored Winding Number Conjecture On The Plane

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Abstract. This work seeks to investigates the equivalence of Topological Colored Tverberg Theorem and Colored Winding Number Conjecture on the plane.

Keywords. Simplex, Convex hulls, Partition, 2-Skeleton and Winding Number.

I. Introduction

Tverberg's theorem came into existence as a result of the generalization of Radon's theorem that concerns with the partition of four points in the plane into two disjoint subsets whose their intersection is nonempty [10]. This Tverberg's theorem asserts that, *for* $t \geq 2$ *and* $d = 2$*, any set of* $3(t-1)+1$ *points in* $\mathbb{R}^2$ *can be partitioned into* t *disjoint subsets* $A_1, A_2, ..., A_t$ *such that* $conv(A_1) \cap ... \cap conv(A_t) \neq \emptyset$. Such partition is called Tverberg partition and a point in the intersection of their convex hulls is called a Tverberg point.

Theorem I.1 (Topological Tverberg Theorem, 11). *For every continuous map*

$$f : \Delta^{3(t-1)} \longrightarrow \mathbb{R}^2$$

there is a Tverberg partition.

Theorem I.2 (Colored Tverberg's Theorem, 4). *Let* C_1, C_2, C_3 *be the sets (which are called color classes) of* t *points each in* $\mathbb{R}^2$*. Then, there is a partition of their union into* t *pairwise disjoint colorful sets* $F_1, F_2, ..., F_t$ *satisfying* $|F_i \cap C_j| \leq 1$ *for every* $i \in \{1, ..., t\}$*,* $j \in \{1, 2, 3\}$ *such that their convex hulls intersect, that is;* $\bigcap_{i=1}^{t} conv(F_i) \neq \emptyset$*. A family of such sub-collections* $F_1, F_2, ..., F_t$ *that contain at most one point from each color class* C_i *is called a rainbow t-partition.*

Theorem I.3 (Topological Colored Tverberg Theorem, 4). *Let* $d = 2$ *and* $t \geq 2$ *and* $N := 3(t-1)$*. Let* Δ^N *be an* N*-dimensional simplex with a partition of the vertex set into color*

Communication presentée au 4ème atelier annuel sur la CRyptographie, Algèbre et Géométrie (CRAG-4), 21 - 25 Juillet 2014, Université de Dschang, Dschang, Cameroun / Paper presented at the 4th annual workshop on CRyptography, Algebra and Geometry (CRAG-4), 21- 25 July 2014, University of Dschang, Dschang, Cameroon.

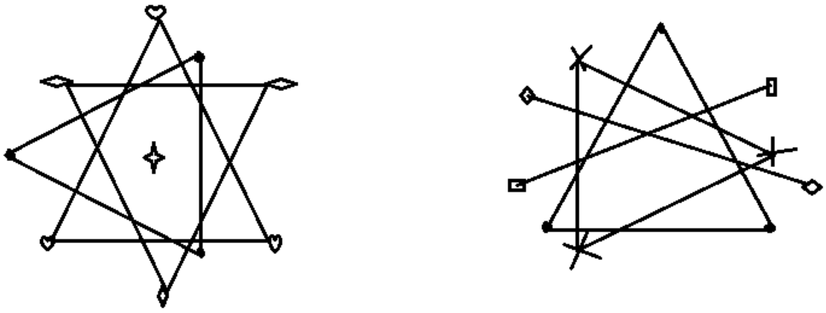


FIGURE 1. $t = 4$

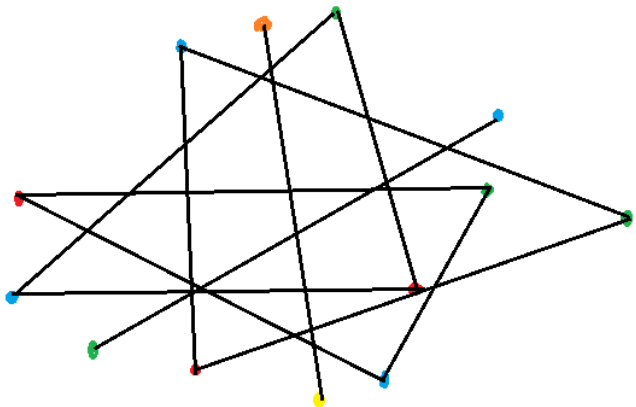


FIGURE 2. $t = 5$ $N = 13$

classes $\mathcal{C} = C_0 \cup C_1 \cup ... \cup C_{3(t-1)}$ *with* $|C_i| \leq t - 1$ *for all* i.
Then, for every continuous map

$$f : \Delta^N \longrightarrow \mathbb{R}^2$$

there are t *disjoint rainbow faces* $F_1, F_2, ..., F_t$ *of* Δ^N *whose images under* f *intersect, that is,* $|F_i \cap C_j| \leq 1$ *for every* $i \in \{1, ..., t\}$, $j \in \{0, ..., 3(t-1)\}$ *and* $\bigcap_{i=1}^{t} f(F_i) \neq \emptyset$. *The family of faces* $F_1, ..., F_t$ *is called a topological rainbow partition.*

Definition I.4 (Colored Winding Number w. r. t Colored Point). *Let* $f : (\mathbb{S}^1, \mathbb{Z}) \longrightarrow (\mathbb{R}^2 \backslash (cp), \mathbb{Z})$ *be a continuous map,* $(cp) \in \mathbb{R}^2 \backslash f(\mathbb{S}^1)$. *Then* f *induces a map* $f_* : \Pi_1(\mathbb{S}^1, \mathbb{Z}) \longrightarrow \Pi(\mathbb{R}^2 \backslash (cp), \mathbb{Z})$, *if* $cp \notin f(\mathbb{S}^1)$, *then colored winding number with respect to the colored point is defined by;* $CW(f, cp) := f_*([\alpha]) = [f \circ \alpha] \in \mathbb{Z}$; $[\alpha] \in \Pi_1(\mathbb{S}^1, \mathbb{Z})$.

Conjecture I.5 (Colored Winding Number Conjecture (C W N C) On The Plane). *Let* $d = 2$, $t \geq 2$, *prime and let* $\Delta^{3(t-1)}$ *be* $3(t-1)$*-dimensional simplex with a partition of the vertex*

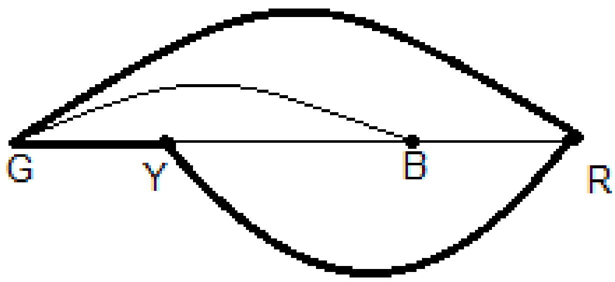
set into color classes $\mathcal{C} = C_o \cup C_1 \cup C_2 \cup C_3$ with $|C_j| \leq t-1 \ \forall \ j = 0,1,2,3$. Then, for every continuous map $f : \Delta_1^{3(t-1)} \longrightarrow \mathbb{R}^2$ there are t disjoint rainbow faces $r\sigma_1, ..., r\sigma_t$ (i.e $|\sigma_i \cap C_j| \leq 1$) of 2-skeleton of the $3(t-1)$ simplex $\Delta_2^{3(t-1)}$ and a colored point $cp \in \mathbb{R}^2$ such that for each $r\sigma_i$ one of the following holds;
(i) $dim(r\sigma_i) \leq 1$ and $cp \in f(r\sigma_i)$
(ii) $dim(r\sigma_i) \leq 2$ and either $cp \in f(\partial r\sigma_i)$ or $cp \notin f(\partial r\sigma_i)$ and $CW(f|_{\partial r\sigma_i}, cp) \neq 0$.
Such a set of t disjoint rainbow faces $S = \{\sigma_1, ..., r\sigma_r\}$ is called colored winding partition (c .w.p) and cp is called the colored winding point.

Remark I.6. *For every continuous map $f : \Delta_1^{3(t-1)} \longrightarrow \mathbb{R}^2$, we can choose t disjoint at most 2- dimensional rainbow faces $r\sigma^2$ of the $3(t-1)$-dimensional simplex $\Delta^{3(t-1)}$ together with a colored point $cp \in \mathbb{R}^2$ such that for every rainbow face either*
(i) the colored point is in the image of the rainbow faces or
(ii) the image of the boundary of the rainbow face winds around the colored point.

Note I.7. *Colored Winding Number Conjecture (C W N C) on the plane is a drawing of complete graph $\mathbb{K}_{3t-2}$. If the drawing is in general position, colored winding number conjecture says; in the drawing of the complete graph $\mathbb{K}_{3t-2}$, either $t-1$ rainbow triangles wind around one vertex or $t-2$ rainbow triangles wind around the intersection of two edges with the triangles, edges and vertices being of different colors.*

Example 1. *$t = 2, \mathbb{K}_4$, $t = 3$ $\mathbb{K}_7$ and $t = 4$ $\mathbb{K}_{10}$ respectively.*
KEY: G = green, Y = yellow, B = blue, P = purple and R = red.

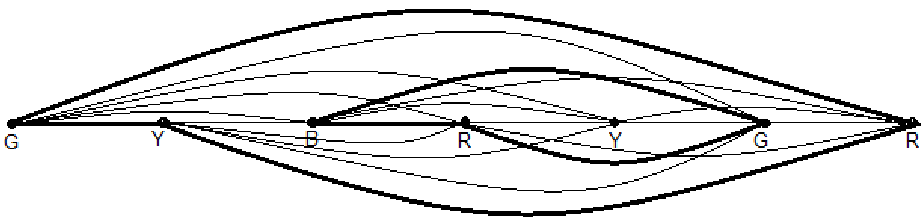
The thick edges in the drawings of $\mathbb{K}_4, \mathbb{K}_7, \mathbb{K}_{10}$ form the winding partitions and their winding points are 3rd, 5th and 7th vertices respectively.

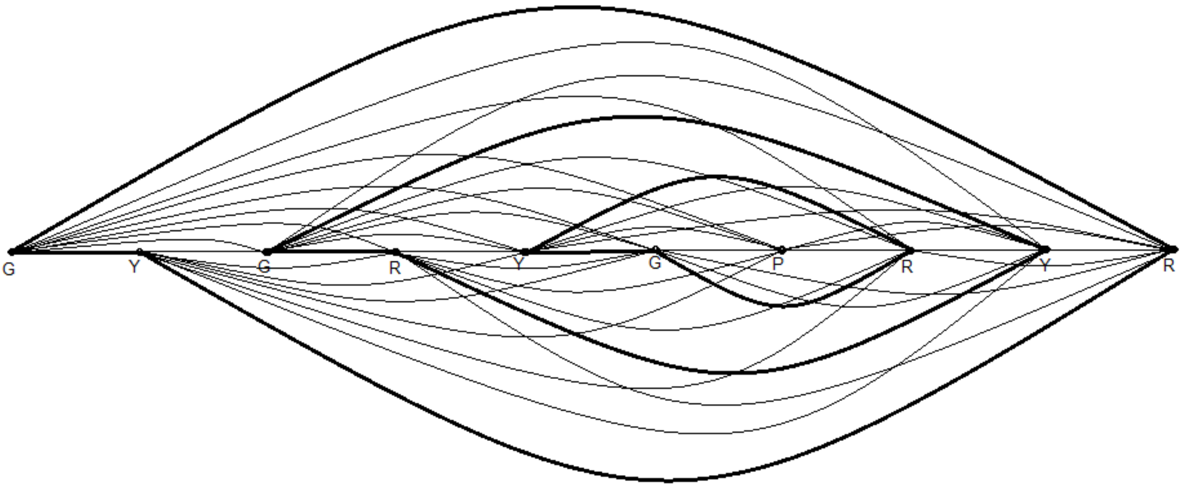


Conjecture I.8 (2-Skeleton Conjecture). *For every continuous map*

$$f : \Delta_2^{3(t-1)} \longrightarrow \mathbb{R}^2$$

there is a colored Tverberg partition.





Definition I.9 (Colored Winding Partition, colored winding point). *Let $t \geq 2$, be positive integer and let $\Delta^{3(t-1)}$ be $3(t-1)$-dimensional simplex with its vertex set being partitioned into color classes $\mathcal{C} = C_o \cup C_1 \cup C_2 \cup C_3$ with $|C_j| \leq t-1$ $\forall$ $j = 0, 1, 2, 3,$ $f : \Delta_1^{3(t-1)} \longrightarrow \mathbb{R}^2$ be a continuous map, $S = \{r\sigma_1, ..., r\sigma_t\}$ be a set of t disjoint rainbow faces of $\Delta_2^{3(t-1)}$ and $cp \in \mathbb{R}^2$. Then, S is called a colored winding partition for f if;*
(i) the colored point is in the image of the rainbow faces or
(ii) the image of the boundary of the rainbow face winds around the colored point.
Any colored point $cp \in \mathbb{R}^2$ is called colored winding point.

Colored Winding Number Conjecture (Equivalent Version): For every continuous map

$$f : \Delta_1^{3(t-1)} \longrightarrow \mathbb{R}^2$$

there is a colored winding partition.

II. Statement of the Problem

Theorem II.1. *The 2-Skeleton Conjecture is equivalent to the Topological Colored Tverberg Theorem.*

Theorem II.2. *The Colored Winding Number Conjecture implies the 2-Skeleton Conjecture.*

Theorem II.3. *Topological Colored Tverberg Theorem is equivalent to Colored Winding Number Conjecture.*

III. Methodology of the Proof

The following steps give the summary of the proof:
(i) Topological Colored Tverberg Theorem implies 2-Skeleton Conjecture for piecewise linear maps in general position.
(ii) 2-Skeleton Conjecture for piecewise linear maps in general position implies 2-Skeleton Conjecture for continuous maps.
(iii) 2-Skeleton Conjecture for continuous maps implies Colored Winding Number Conjecture for continuous maps.
(iv) Colored Winding Number Conjecture for continuous maps implies 2-Skeleton Conjecture for continuous maps.
(v) 2-Skeleton Conjecture for continuous maps implies Topological Colored Tverberg Theorem.

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Taiwo. O. Sangodapo and Deborah. O. A Ajayi
Department of Mathematics, University of Ibadan, Oyo State, Nigeria.
e-mail: `toewuola77@gmail.com`
e-mail: `adelaideajayi@yahoo.com`