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Abstract

Study has been conducted on the action of a group on a set. Also, there is existing work on the action of a semigroup on a set (or even fuzzy set). This paper studies some properties of the action of a crisp group $G^{(1)}$ on a fuzzy set T_μ via the membership function $\mu_T : X \longrightarrow [0,1]$. It also seeks to establish some properties of this action in respect of the stabilizers of an element $t \in T_\mu$ among other things. In particular, it establishes that some sort of fuzzy middle cosets of a group G is a group action on a fuzzy subset. It also states and proves fuzzy version of orbit-stabilizer theorem.

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The Action of a Group on a Fuzzy Set via Fuzzy Membership Function

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Abstract. Study has been conducted on the action of a group on a set. Also, there is existing work on the action of a semigroup on a set (or even fuzzy set). This paper studies some properties of the action of a crisp group $G^{(1)}$ on a fuzzy set T_μ via the membership function $\mu_T : X \longrightarrow [0, 1]$. It also seeks to establish some properties of this action in respect of the stabilizers of an element $t \in T_\mu$ among other things. In particular, it establishes that some sort of fuzzy middle cosets of a group G is a group action on a fuzzy subset. It also states and proves fuzzy version of orbit-stabilizer theorem.

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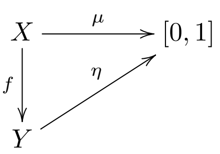
Keywords. Action, stabilizer, membership function, middle coset.

I. Introduction

[5] has shown that a collection $\Im$ of all fuzzy implications with the binary operation $\circledast$ is a monoid. Also, that the largest subgroup of $(\Im, \circledast)$ is the set S of all invertible elements. S is said to have partitioned $\Im$ and the action of S on $\Im$ is considered.

Furthermore, [4] noted that the function $\mu \longrightarrow [0, 1]$, can be denoted by $X^{(\mu)}$ and if $\mu(x) = a, \quad \forall x \in X$, we write $X^{(a)}$, and if X is crisp, $X^{(1)}$. Then, the fuzzy function $f : X^{(\mu)} \longrightarrow Y^{(\eta)}$ is a function $f : X \longrightarrow Y$ so that we have the **fuzzy triangle:**



in which case $\eta f \geq \mu$. The set of all fuzzy subsets of a set X, $\{X^{(\mu_i)}\}$, is called fuzzy power of X and is denoted **FSub**X. Then one can begin to think of category of fuzzy sets called **FSet**.

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II. Preliminaries

Definition II.1. *A fuzzy subset T_μ of a non-empty set X is a class of objects in X with the associated grade membership (or characteristic) function $\mu_T : X \longrightarrow [0,1]$ which assigns to every $x \in X$ a real value between 0 and 1.*

We shall henceforth denote the membership function simply by μ.

Definition II.2. *Let T_μ be a fuzzy subset of G. Then, T_μ is called a fuzzy subgroup of G if*
(i) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x) = \mu(x^{-1}) \quad \forall x, y \in T_\mu$

Definition II.3. *Let T_μ be a fuzzy subset of a group G. Then, for some $a \in G$, the set aT_μ characterized by the function $a\mu$ defined by $(a\mu)(x) = \mu(a^{-1}x)$ is called a fuzzy left coset.*

Definition II.4 (7). *Let T_μ be a fuzzy subset of a group G. Then, for some $a, b \in G$, the set $aT_\mu b$ characterized by the function $a\mu b$ defined by $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$ is called a fuzzy middle coset.*

Remark II.5 (7). *Note that fuzzy middle coset $aT_\mu a^{-1}$ is a fuzzy subgroup of G.*

Definition II.6 (1). *An action of a group G on a set T is a binary operation*

$$* : G \times T \longrightarrow T,$$

satisfying $\forall e, g_1, g_2 \in G$ and $\forall t \in T$
i. $e^*t = t$
ii. $g_1^*(g_2^*t) = (g_1g_2)^*t$.

Theorem II.7 (1). *Let a group act on a set T. Suppose that $G(t)$ is the stabilizer of t. Then for any $g_1, g_2 \in G$,*

$$g_1^*t = g_2^*t \Leftrightarrow g_2^{-1}g_1 \in G(t)$$

.

Theorem II.8 (1). *Let a group G act on a set T. Then,*

$$g^*t = t' \Leftrightarrow t = g^{-1*}t \quad \forall g \in G \ \text{ and } \ t, t' \in T.$$

Theorem II.9 (8). *Let μ be a fuzzy subgroup of a group G. Then μ is a fuzzy normal subgroup if and only if the following are satisfied (and are equivalent):*
i $\mu(xyx^{-1}) = \mu(y)$
ii $\mu(xy) = \mu(yx)$
iii $x\mu = \mu x$

Definition II.10 (6). *Let μ be a fuzzy subgroup of a group G. If $\forall \phi \in Aut(G)$ we have $\mu(\phi(x)) \geq \mu(x), \ \forall x \in G$, we say that μ is a characteristic fuzzy subgroup. Furthermore, μ is a characteristic fuzzy subgroup if and only if $\mu(\phi(x)) = \mu(x), \ \forall x \in G$*

III. Action of a Group on a Fuzzy Set (or Fuzzy Subset)

One can look at **II.6** as an action $*_g : T \longrightarrow T$ defined by $*_g(t) = *(g, t) = g^*t = gt$

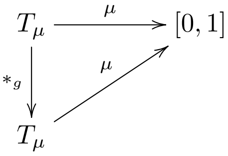
Definition III.1. *Let G be a group and T_μ any fuzzy subset such that μ is the characteristic function associated with T_μ. Then the action of G on T_μ is the operation*

$$* : G \times T_\mu \longrightarrow T_\mu,$$

satisfying $\forall e, g_1, g_2 \in G$ and $\forall t \in T_\mu$
i. $\mu(e^*t) = \mu(t) \Leftrightarrow e^*t = t$

ii. $\mu(g_1^*(g_2^*t)) = \mu((g_1g_2)^*t) \Leftrightarrow g_1^*(g_2^*t) = (g_1g_2)^*t$

Remarks III.2. *This definition shows that a group acts on a fuzzy subset if it acts on its underlying set. Indeed, this is analogous to the definition of the action of a semigroup on a fuzzy set by [4] since for* $G = G^{(\mu)} = G^{(1)}$ *and* $T_\mu = T^{(\mu)}$, *we have that for any* $g \in G$, $\mu(g^*t) \geq min\{\mu(g), \mu(t)\} = min\{1, \mu(t)\} = \mu(t)$ *so that* $\mu(g^*t) \geq \mu(t)$ *in which case we have a fuzzy triangle*



The characterization by equality above for a G-act is justified since if we assume that $g^*t = t$ *but* $\mu(g^*t) \neq \mu(t)$ *we have* $\mu(g^*t) \geq min\{\mu(g), \mu(t)\} = min\{1, \mu(t)\} = \mu(t)$. *Then we must choose that* $\mu(g^*t) > \mu(t)$. *If* $g = e$, *then* $\mu(t) < \mu(e^*t) = \mu(t)$ *which is a contradiction. Conversely, if* $\mu(g^*t) = \mu(t)$ *but* $g^*t \neq t$, *then for* $g = e$, $e^*t \neq t$ *which is also a contradiction.*

Example III.3. *Let* G *be a group and* $T = \{T_a : T_a = aT_\mu$ *for some* $a \in G\}$, *a collection of fuzzy left cosets in* G. *Define for* $g \in G$ *and any* $T_a \in T$ *by* $g^*T_a = T_{(ga)} = (ga)T_\mu = g'T_\mu$, *where* $g' = ga$. *Then* G *acts on* T $\forall e, g_1, g_2 \in G$ *and* $\forall t \in T$ *in the following way:*

i. $e^*T_a = T_a$. *By this, for* $t \in T_\mu$, *we have*

$$\mu((ea)^{-1}t) = \mu(a^{-1}t) = a\mu(t)$$

so that

$$e^*T_a = T_{(ea)} = (ea)T_\mu = aT_\mu = T_a$$

ii $g_1^*(g_2^*T_a) = (g_1g_2)^*T_a$. *By this, for* $t \in T_\mu$ *we have*

$$\mu((g_1g_2a)^{-1}t) = (g_1g_2a)\mu(t)$$

so that

$$g_1^*(g_2^*T_a) = g_1^*T_{(g_2a)} = T_{(g_1g_2a)} = (g_1g_2a)T_\mu = (g_1g_2)^*T_a$$

Example III.4. *Let* $T = \{T_\mu\}$ *be the collection of fuzzy subsets in a group* G *and for any* $g \in G$, *where* μ *is a membership function, we define* $g^*T_\mu = gT_\mu g^{-1}$. *Then* G *acts on* T *in the following way:*

i. *For* $e \in G$, *we have that* $e^*T_\mu = eT_\mu e^{-1} = T_\mu$. *By this, for* $t \in T_\mu$, $\mu(e^*t) = \mu(ete^{-1}) = \mu(t)$.
ii. *For* $a, b \in G$, $b^*(a^*T_\mu) = (ba)T_\mu(ba)^{(-1)} = (ba)^*T_\mu$, *whence we have that for* $t \in T_\mu$, $\mu(b^*(a^*t)) = \mu(b^*(ata^{-1}) = \mu((ba)t(ba)^{-1}) = \mu((ba)^*t)$

Remark III.5. *Note that, by the action of* G *defined in* **III.4** *any fuzzy middle coset of the form* $a_0^{-1}\mu a_0$ *is an action of the group on the fuzzy subset. This is from the fact that* $\mu((ba)t(ba)^{-1}) = (a_0^{-1}\mu a_0)(t)$ *for* $t \in T_\mu$.

From the foregoing, we can then define the set $G(t)$ *of stabilizers of an element* t *in a fuzzy set* T_μ *via the characteristic function* μ

Definition III.6. *An element* $g \in G$ *stabilizes* $t \in T_\mu$ *if* $\mu(g^*t) = \mu(t)$. *The collection of such* g *is denoted* $G(t)$.

Lemma III.7. *Let* G *act on the set* T_μ *for* $a, b \in G$. *Then that the set* $G(t)$ *is a group, indeed a subgroup of* G.

Proof. Let $g_1, g^{-1} \in G(t)$. Then

$$\mu(g_1^*(g^{-1*}t)) = \mu((g_1 g^{-1})^* t) \text{ by III.1(ii).}$$

But

$$\mu(g_1^*(g^{-1*}t)) = \mu(g_1^* t) = \mu(t) \text{by III.6.}$$

Hence,

$$\mu((g_1 g^{-1})^* t) = \mu(g_1^*(g^{-1*}t)) = \mu(t).$$

Thus, it implies that $g_1 g^{-1} \in G(t)$, in which case $G(t)$ is a group.

■

Remarks III.8. *From* **III.7***, applying Lagrange theorem, it can be seen that* $|G(t)| \mid |G|$ *and* $|G(t)| = |gG(t)| = |G(t)g|$. *Also, that* $\mu(g^*t) = \mu(t') \Leftrightarrow \mu(g^{-1*}t') = \mu(t)$ *is also trivial. Since*

$$\mu(g^*t) = \mu(t') \Leftrightarrow g^*t = t' \Leftrightarrow t = g^{-1*}t' \Leftrightarrow \mu(g^{-1*}t') = \mu(t) \text{ by II.8}$$

Proposition III.9. *If a relation* $t \sim t'$ *is defined by* $\mu(g^*t) = \mu(t')$ *for some* $g \in G$ *which acts on* T_μ *and a fixed* $t \in T_\mu$, *then* $T_{\mu\sim}$ *is an equivalence class and* $\sim$ *is an equivalence relation.*

Proof. $t \sim t$, since $e \in G$ and G acts on T such that $\mu(e^*t) = \mu(t)$. The relation is *reflexive.*
It is also *symmetric* since if $t \sim t'$, for $g, g^{-1} \in G$, we can have $\mu(g^*t) = \mu(t') \Leftrightarrow \mu(g^{-1*}t') = \mu(t)$. This implies that $t' \sim t$ by III.7.
The relation is also *transitive.* If $t \sim t'$ and $t' \sim t''$, we have $g, g_1 \in G$ such that $\mu(g^*t) = \mu(t') \Leftrightarrow g^*t = t'$ and $\mu(g_1^*t') = \mu(t'')$. We can find such $g_2 = g_1 g \in G$ such that $\mu(t'') = \mu(g_1^* g^* t) = \mu((g_1 g)^* t) = \mu(g_2^* t)$, whence we conclude that $t \sim t''$.

■

The collection of all such equivalence classes is denoted t^G *as in the crisp case. We can now state and prove a theorem analogous to the orbit-stabilizer theorem that we have in the crisp case.*

Theorem III.10. *If* G *acts on* T_μ *and* $t_1 \in t^G$, *then* $G(t_1) = gG(t)g^{-1}$ *if and only if* $\mu((g^{-1}g'g)^*t) = \mu(t)$ *for some* $g \in G(t)$ *and any* $g' \in G(t_1)$.

Proof. Since $t_1 \in t^G$, there is a $g \in G$ such that $\mu(g^*t) = \mu(t_1) \Leftrightarrow g^*t = t_1$. Then
$G(t_1) = \{g' \in G : \mu(g'^*t_1) = \mu(t_1)\}$
$= \{g' \in G : \mu(g'^*(g^*t)) = \mu(t_1)\}$
$= \{g' \in G : \mu(g'^*(g^*t)) = \mu(g^*t)\}$ since g' stabilizes $t_1 = g^*t$
$= \{g' \in G : \mu(g^{-1*}g'^*(g^*t)) = \mu(t) \Leftrightarrow (g^{-1*}g'^*g^*) \in G(t) \Leftrightarrow g' \in gG(t)g^{-1}\}$ by III.6 and III.8
$= gG(t)g^{-1}$

■

Remarks III.11. *The direct consequence of this is that* $g_1, g_2 \in gG(t)$ *if and only if* $g_1G(t) = g_2G(t)$. *Hence, two different elements of* G *act on* t *the same way if and only if they belong to the same left coset of* $G(t)$. *For if we suppose that*

$$\mu(g_1^*t) = \mu(t) \Leftrightarrow g_1^*t = t$$

and we let

$$g_1^*t = g_2^*t,$$

then

$$t = (g_1^{-1}g_2)^*t \Leftrightarrow \mu(t) = \mu((g_1^{-1}g_2)^*t),$$

so that

$$g_1^{-1}g_2 \in G(t) \text{ by III.6}$$

and

$$g_2 \in g_1G(t).$$

Alternatively, if we let

$$g_1G(t) = g_2G(t)$$

but

$$g_2 \notin g_1G(t),$$

then we have

$$g_2^*t = t' \neq t.$$

But

$$g_1 \in g_1G(t)$$

so that

$$t = g_1^*t = g_2^*t = t'.$$

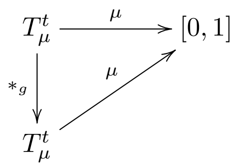
This is a contradiction! Hence,

$$g_2 \in g_1G(t).$$

IV. Conclusion

It can be seen clearly that a set G acts on a fuzzy set if it already acts on its underlying set. Besides, if we replace the fuzzy subset T_μ by a t-level subgroup T_μ^t so that the action $G \times T_\mu^t \longrightarrow T_\mu^t$ induces a homomorphism $\phi_g : G \longrightarrow Aut(T_\mu^t)$, then in the action defined in **III.1** and **III.2**, μ is a characteristic fuzzy subgroup as in [6].

Hence, we can characterize the action so defined in the following way: **A fuzzy subgroup μ of a group G is a characteristic fuzzy subgroup if and only if the group G acts on its level subgroup T_μ^t** by **II.10**. If $G = T_\mu^t$ is a fuzzy (level) subgroup which acts on itself, we have the fuzzy triangle:



so that $\mu(x_ix_k) \geq \mu(x_i) \wedge \mu(x_k)$ defines a product semigroup on the collection $\{\mu(x_i)\}$. In actual sense, $\mu(x_ix_k) = \mu(x_i) \wedge \mu(x_k)$ since the problem of T_μ^t is now reduced to a problem of crisp group (a trivial subgroup of G), though a level subgroup of μ, where μ is a fuzzy subgroup of G.

Also, as opposed to what happens in classical group theory that two left cosets are either identical or disjoint, one can have two fuzzy cosets which are neither disjoint nor identical as shown by **Example 1.2.6 of [7]**. But from **III.8** it may still be expected to have the equation $n = [G : G(t)]$ for some $n \in \mathbf{Z}$ and $n = |t^G|$. This is because $G(t)$ is a subgroup of G and Lagrange theorem holds.

This whole idea can become more general if instead of $[0, 1]$ a lattice L or a partially ordered set is taken.

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