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**Abstract**

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# $n$-fold sub-implicative ideals and $n$-fold sub-commutative ideals of BCI-algebras

C. Lele, S. F. Tebu Tagne and M. Tonga

**Abstract.** The notions of n-fold sub-implicative ideals and n-fold sub-commutative ideals of BCI-algebras are introduced. We show that a nonempty subset of a BCI-algebra is an n-fold sub-implicative ideal if and only if it is both an n-fold sub-commutative ideal and an n-fold positive implicative ideal. We prove that any p-ideal is always an n-fold sub-implicative ideal and an n-fold sub-commutative ideal. We give a new characterization of n-fold positive implicative ideals of BCI-algebras. Moreover some other properties about n-fold sub-commutative ideals of BCI-algebras are given.

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**Keywords.** BCI-algebras, $n$-fold sub-implicative BCI-algebra, $n$-fold sub-commutative BCI-algebra, $n$-fold positive implicative BCI-algebra.

## I. Introduction

It is an important way to research the algebras by its ideals. The notion of sub-implicative and sub-commutative ideal(resp. positive implicative ideal) in BCI-algebra were introduced by Y. L. Liu and J. Meng [5] (resp. J. Meng and X. L. Xin [4]). There is a close relation among above-mentioned ideals. That is in a BCI-algebra X, a nonempty subset of X is a sub-implicative 12 ideal if and only if it is both a sub-commutative ideal and positive implicative ideal. Recently the notion of n-fold positive implicative and n-fold commutative ideals in BCI-algebra were

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introduced by C. Lele et al.( [1], [2]). In this note we establish the concepts of n-fold sub-implicative ideals and n-fold sub-commutative ideals in BCI-algebras and investigate some of their properties.

We recall in this section the definitions and results that will be used throughout the paper, most of the time without any further notice.

**Definition I.1.** [6] *An algebra $X = \langle X; \star, 0\rangle$ of type $< 2, 0 >$, is said to be a BCI-algebra if it satisfies the following conditions for all $x, y, z \in X$ :*

- *BCI1-* $((x \star y) \star (x \star z)) \star (z \star y) = 0$;
- *BCI2-* $x \star 0 = x$;
- *BCI3-* $x \star y = 0$ *and* $y \star x = 0$ *imply* $x = y$.

If a BCI-algebra X satisfies the condition $0 \star x = 0$ for all $x \in X$; then X is called a BCK-algebra. Hence, BCK-algebra form a subclass of BCI-algebra.

Let $n$ be a positive integer. Throughout this paper we appoint that $X := (X, \star, 0)$ denotes a BCI-algebra; $x \star y^n := (...((x \star y) \star y) \star ...) \star y$, in which $y$ occurs $n$ times; $x \star y^0 := x$ and $x \star \prod_{i=1}^{n} y_i$ denotes $(...((x \star y_1) \star y_2) \star ...) \star y_n$ where $x, y, y_i \in X$.

**Proposition I.2.** [6], [2] *On every BCI-algebra $X$, there is a natural order called the BCI-ordering defined by $x \leq y$ if and only if $x \star y = 0$. Under this order, the following axioms hold for all $x, y, z \in X$.*

$(b)$ *If $x \leq y$, then $x \star z \leq y \star z$ and $z \star y \leq z \star x$;*

$(c)$ $(x \star y) \star z = (x \star z) \star y$;

$(d)$ $x \star (x \star (x \star y)) = x \star y$;

$(e)$ $(x \star z) \star (y \star z) \leq x \star y$;

$(f)$ $0 \star (x \star y) = (0 \star x) \star (0 \star y)$;

$(g)$ $0 \star (x \star y) = 0 \star (0 \star (y \star x))$;

$(h)$ *If $x \leq y$, then $0 \star x = 0 \star y$;*

$(i)$ $x \star (x \star (x \star y))^n = x \star y^n$, *for all $n$.*

Throughout this note X always means a BCI-algebra without any specification.

A nonempty subset I of a BCI-algebra X is called an ideal of X if it satisfies

(i) $0 \in I$ ,

(ii) If $x \star y \in I$ and $y \in I$ imply $x \in I$

**Definition I.3.** [5] *A nonempty subset I of a BCI-algebra X is called a sub-implicative ideals if*

*(i)* $0 \in I$,

*(iii) If* $[(x \star (x \star y)) \star (y \star x)] \star z \in I$ *and* $z \in I$ *imply* $y \star (y \star x) \in I$ *for all* $x, y, z \in X$.

**Definition I.4.** [5] *A nonempty subset I of a BCI-algebra X is called a sub- commutative ideal if*

*(i)* $0 \in I$,

*(vi)* $(y \star (y \star (x \star (x \star y)))) \star z \in I$ *and* $z \in I$ *imply* $x \star (x \star y) \in I$ *for all* $x, y, z \in X$.

**Definition I.5.** [1]

*A nonempty subset I of a BCI-algebra X is called an n-fold positive implicative ideals if*

*(i)* $0 \in I$,

*(iii) If* $(x \star y^{n+1}) \star (0 \star y) \in I$ *implies* $x \star y^n \in I$ *for all* $x, y \in X$.

**Definition I.6.** [6] *A BCI-algebra is called commutative if for all* $x, y \in X$,

$y \star (y \star (x \star (x \star y))) = x \star (x \star y)$.

## II. n-fold Sub-Implicative Ideals and n-fold Sub-Commutative Ideals of BCI-Algebras

**Definition II.1.** *A nonempty subset I of a BCI-algebra X is called an n-fold sub-implicative ideal if*

*(i)* $0 \in I$,

*(iii) If* $[(x \star (x \star y)) \star (y \star x)^n] \star z \in I$ *and* $z \in I$ *imply* $y \star (y \star x)^n \in I$ *for all* $x, y, z \in X$.

**Example II.2.** *(1) Any ideal of a p-semisimple BCI-algebra* [6] *is an n-fold sub-implicative.*

*(2) Let* $X = \{0, 1, 2, 3, 4, 5\}$ *with the operation* $\star$ *defined by*

| $\star$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 3 | 3 | 3 |
| 1 | 1 | 0 | 1 | 3 | 3 | 3 |
| 2 | 2 | 2 | 0 | 3 | 3 | 3 |
| 3 | 3 | 3 | 3 | 0 | 0 | 0 |
| 4 | 4 | 3 | 4 | 1 | 0 | 0 |
| 5 | 5 | 3 | 5 | 1 | 1 | 0 |

*Then* $I = \{0, 1, 2\}$ *is an 2-fold sub-implicative ideal.*

**Proposition II.3.** *Let I be an ideal of X. Then I is an n-fold sub-implicative if and only if* $(x \star (x \star y)) \star (y \star x)^n \in I$ *implies* $y \star (y \star x)^n \in I$ *for all* $x, y \in X$.

**Proof:.** The proof is straightforward.

■

**Proposition II.4.** *Any n-fold sub-implicative ideal is an ideal, but the converse is not true.*

**Proof:.** The proof is straightforward

Consider the BCI-algebra $X$ whose Cayley's table is given by:

| $\star$ | 0 | 1 | 2 | $a$ | $b$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | $O$ | $a$ | $a$ |
| 1 | 1 | 0 | 0 | $a$ | $a$ |
| 2 | 2 | 2 | 0 | $b$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | 0 | 0 |
| $b$ | $b$ | $b$ | $a$ | 2 | 0 |

The Zero ideal of the BCI-algebra is not 2-fold sub-implicative since $(2\star(2\star1))\star(1\star2)^2 = 0 \in \{0\}$ and $1 \star (1 \star 2)^2 = 1 \notin \{0\}$.

■

**Theorem II.5.** *Any n-fold sub-implicative ideal is an n-fold positive implicative ideal, but the converse does not hold*

**Proof:.**

Assume that I is an n-fold sub-implicative ideal of X. It follows from Proposition II.4 that I is an ideal. In order to prove that I is an n-fold positive implicative ideal from Definition II.4 (ii) it suffice to show that if $(x \star y^{n+1}) \star (0 \star y) \in I$ implies $x \star y^n \in I$ for all $x, y \in X$. By Proposition II.3 , for any $u, v \in X$, we have $(u \star (u \star v)) \star (v \star u)^n \in I$ implies $v \star (v \star u)^n \in I$. Substituting $y \star x$ for $u$ and $y$ for $v$ then

$$\begin{aligned}
[(y \star x) \star ((y \star x) \star y)] \star (y \star (y \star x))^n &= [(y \star x) \star (0 \star x)] \star (y \star (y \star x))^n \\
&= ((y \star (\star(y \star (y \star x))^n)) \star x) \star (0 \star x) \\
&= ((y \star x)^n \star x) \star (0 \star x) \\
&= ((y \star x)^{n+1} \star (0 \star x)
\end{aligned}$$

Hence, if $[(y\star x)\star((y\star x)\star y)]\star(y\star(y\star x))^n = ((y\star x)^{n+1}\star(0\star x) \in I$, then $y\star(y\star(y\star x)^n) \in I$ ie $y \star x^n \in I$ Thus we prove that I is an n-fold positive implicative ideal.

Consider the BCI-algebra $X$ whose Cayley's table is given by:

| $\star$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $c$ |
| $a$ | $a$ | 0 | 0 | $c$ |
| $b$ | $b$ | $b$ | 0 | $c$ |
| $c$ | $c$ | $c$ | $c$ | 0 |

$I = \{0\}$ is an ideal of X, but not an 2-fold sub-implicative ideal of X since $(b \star (b \star a)) \star (a \star b)^2 = (b \star b) \star 0^2 = 0 \in I$, but $a \star (a \star b)^2 = a \star 0 = a \notin I$. By routine calculations we know that $\{0\}$ is an 2-fold positive implicative ideal. This completes the proof.

■

**Lemma II.6.** *([5], Theorem 2.7) An ideal I of a BCI-algebra X is a p-ideal of X if and only if $0 \star (0 \star x) \in I$ implies $x \in I$.*

Nothing that, any p-ideal is an ideal.

**Theorem II.7.** *Any p-ideal is an n-fold sub-implicative ideal, but the converse does not hold.*

**Proof:.**

Suppose that I is a p-ideal. Then I is an ideal. Let $x, y \in X$ such that $(x \star (x \star y)) \star (y \star x)^n \in I$. We can show that $y \star (y \star x)^n \in I$. It suffices to show that $0 \star (0 \star a) \in I$ where $a = y \star (y \star x)^n$

$(x \star (x \star y)) \star (y \star x)^n \leq y \star (y \star x)^n$ implies $0 \star [(x \star (x \star y)) \star (y \star x)^n] = 0 \star [y \star (y \star x)^n]$.

Hence

$$\{0 \star [0 \star (y \star (y \star x)^n)]\} \star \{(x \star (x \star y)) \star (y \star x)^n\}$$
$$= \{0 \star (0 \star [(x \star (x \star y)) \star (y \star x)^n]\} \star \{(x \star (x \star y)) \star (y \star x)^n\} = 0 \in I$$

we have $0 \star [0 \star (y \star (y \star x)^n)] \in I$. By Lemma II.6, $(y \star (y \star x)^n) \in I$. It means that I is an n-fold sub-implicative ideal.

The last half part is shown by the Example II.2. We have know that zero ideal is 2-fold sub-implicative ideal of X, but it is not a p-ideal since $0 \star (0 \star 1) = 0 \in \{0\}$, but $1 \notin \{0\}$. This completes the proof

■

Next we introduce the notion of sub-commutative ideals of BCI-algebras.

**Definition II.8.** *A nonempty subset I of a BCI-algebra X is called an n-fold sub- commutative ideal if*

*(i)* $0 \in I$,

*(vi)* $[y \star (y \star (x \star (x \star y)^n))] \star z \in I$ *and* $z \in I$ *imply* $x \star (x \star y)^n \in I$ *for all* $x, y, z \in X$.

**Example II.9.** *1) Any ideal of a p-semisimple BCI-algebra is a sub-commutative ideal*

*(2) Consider the BCI-algebra X whose Cayley's table is given by:*

| $\star$ | 0 | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | $c$ | $c$ | $a$ |
| 1 | 1 | 0 | $c$ | $c$ | $a$ |
| $a$ | $a$ | $a$ | 0 | 0 | $c$ |
| $b$ | $b$ | $a$ | 1 | 0 | $c$ |
| $c$ | $c$ | $c$ | $a$ | $a$ | 0 |

*It is easy to that $I = \{0, 1\}$ is an 3-fold sub-commutative ideal.*

The following theorems are similar to the cases of n-fold sub-implicative ideals

**Proposition II.10.** *An ideal $I$ of $X$ is n-fold sub-commutative if and only if $y \star (y \star (x \star (x \star y)^n)) \in I$; we have $x \star (x \star y)^n \in I$ for all $x, y \in X$.*

**Proposition II.11.** *Any n-fold sub-commutative ideal is an ideal, but the converse does not hold.*

**Proof:.**

The proof is straightforward.

We consider example in proof of Proposition II.4, the zero ideal of the BCI-algebra X is not 2-fold sub-commutative because $2 \star (2 \star (1 \star (1 \star 2)^2)) = 0 \in \{0\}$ but $1 \star (1 \star 2)^2 = 1 \notin \{0\}$.

■

**Lemma II.12.** *Let $X$ be a BCI-algebra. If $(x \star (x \star y)) \star (y \star x)^n = y \star (y \star x)^n$ then $(x \star (x \star y)) \star (y \star x)^{n+1} = y \star (y \star x)^{n+1}$, for all $x, y \in X$.*

**Proof:.**

Assume that $(x \star (x \star y)) \star (y \star x)^n = y \star (y \star x)^n$, for all $x, y \in X$.

$$\begin{aligned} (x \star (x \star y)) \star (y \star x)^{n+1} = [(x \star (x \star y)) \star (y \star x)^n] \star (y \star x) &= (y \star (y \star x)^n) \star (y \star x) \\ &= y \star (y \star x)^{n+1} \end{aligned}$$

■

By above lemma, we have the following corollary

**Corollary II.13.** *In an implicative BCI-algebra every ideal is an n-fold sub-implicative ideal.*

**Theorem II.14.** *Let $f$ be a homomorphism of BCI-algebra $X$ into a BCI-algebra $Y$*

*(i) If $f$ is an onto and $A$ is an n-fold sub-implicative ideal of X, then $f(A)$ is an n-fold sub-implicative ideal of Y.*

*(ii) If B is an n-fold sub-implicative ideal of Y, then $f^{-1}(B)$ is an n-fold sub-implicative ideal of Y.*

*(iii) If X is implicative BCI-algebra, then kerf is an n-fold sub-implicative ideal of X.*

**Proof:.** (i) Let A be an n-fold sub-implicative ideal of X. Clearly $0 \in f(A)$. Let $a, b \in Y$ such that $(a \star (a \star b)) \star (b \star a)^n \in f(A)$. Since $f$ is onto, there exits $x, y \in X$ such that $f(x) = a$; $f(y) = b$ and $(f(x) \star (f(x) \star f(y))) \star (f(y) \star f(x))^n = f[(x \star (x \star y)) \star (y \star x)^n] \in f(A)$, we have $x \star (x \star y)) \star (y \star x)^n \in A$. Since A is an n-fold sub-implicative ideal, then $y \star (y \star x)^n \in A$ and hence $f(y \star (y \star x)^n) = f(y) \star (f(y) \star f(x))^n = b \star (b \star a)^n \in f(A)$ . We have f (A) is an n-fold sub-implicative ideal of Y.

(ii) Let B be an n-fold sub-implicative ideal of $Y$ , since $f(0) = 0$, $0 \in f^{-1}(B)$. Let $((x \star (x \star y)) \star (y \star x)^n) \in f^{-1}(B)$ for all $x, y \in X$, then

$f[((x \star (x \star y)) \star (y \star x)^n] = ((f(x) \star (f(x) \star f(y))) \star (f(y) \star f(x))^n \in B$. Since B is an n-fold sub- implicative ideal, we have $f(y) \star (f(y) \star f(x))^n = f(y \star (y \star x)^n) \in B$, and hence $y \star (y \star x)^n) \in f^{-1}(B)$, then $f^{-1}(B)$ is an n-fold sub-implicative ideal.

(iii) Let $x, y \in X$ be such that $(x \star (x \star y)) \star (y \star x)^n \in kerf$ , then $f[(x \star (x \star y)) \star (y \star x)^n] = 0$, since $f$ is homomorphism we have

$f[(x \star (x \star y)) \star (y \star x)^n] = ((f(x) \star (f(x) \star f(y))) \star (f(y) \star f(x))^n) = 0$, but X is implicative BCI-algebra, then $(x \star (x \star y)) \star (y \star x) = y \star (y \star x)$; by Lemma II.12

$(x \star (x \star y)) \star (y \star x)^n = y \star (y \star x)^n$ therefore $f(y \star (y \star x)^n) = 0$, ie. $y \star (y \star x)^n \in Kerf$. Then kerf is an n-fold sub-implicative ideal of X.

■

**Theorem II.15.** *Any n-fold sub-implicative ideal is an n-fold sub-commutative ideal, but the converse does not hold*

**Proof:.** Suppose that I is an n-fold sub-implicative ideal of X. By Proposition II.4 I is an ideal. To prove that I is an n-fold sub-commutative ideal from Proposition II.10, it suffices to show that if $y \star (y \star (x \star (x \star y)^n)) \in I$ ; then $x \star (x \star y)^n \in I$ for all $x, y \in X$.

By Proposition II.3 , for any $u, v \in X$, we have $(u \star (u \star v)) \star (v \star u)^n \in I$ implies $v \star (v \star u)^n \in I$. Substituting $y$ for $u$ and $x$ for $v$ then

$$\begin{aligned}
&[(y \star (y \star x)) \star (x \star y))^n] \star [y \star (y \star (x \star (x \star y)^n))] \\
&= [(y \star (y \star (y \star (x \star (x \star y)^n)))) \star (y \star x)] \star (x \star y)^n \\
&= [(y \star (x \star (x \star y)^n)) \star (y \star x)] \star (x \star y)^n \\
&= ((y \star (y \star x)) \star (x \star (x \star y)^n)) \star (x \star y)^n \\
&\leq (x \star (x \star (x \star y)^n)) \star (x \star y)^n = 0 \in I
\end{aligned}$$

Hence, if $y \star (y \star (x \star (x \star y)^n)) \in I$, then $(y \star (y \star x)) \star (x \star y))^n \in I$, and so $x \star (x \star y)^n \in I$. Therefore I is an sub-commutative ideal.

We consider Example II.9 part (2), We have known that $\{0\}$ is a 1-fold sub-commutative ideal of X, but not a 1-fold sub-implicative ideal of X since $(1 \star (1 \star b)) \star (b \star 1) = 0 \in \{0\}$, but $b \star (b \star 1) = 1 \notin \{0\}$. The proof is complete.

■

**Theorem II.16.** *Any p-ideal is an n-fold sub-commutative ideal, but the converse does not hold.*

**Proof:.**

Assume that I is a p-ideal. Then I is an ideal. Now we show that if $y \star (y \star (x \star (x \star y)^n)) \in I$ implies $x \star (x \star y)^n \in I$ for all $x, y \in X$. since $y \star (y \star (x \star (x \star y)^n)) \leq x \star (x \star y)^n)$, we have $0 \star (y \star (y \star (x \star (x \star y)^n))) = 0 \star (x \star (x \star y)^n)$

If $y \star (y \star (x \star (x \star y)^n)) \in I$, because

$$(0 \star (0 \star (x \star (x \star y))^n)) \star [y \star (y \star (x \star (x \star y)^n))]$$
$$= (0 \star (y \star (y \star (x \star (x \star y)^n)))) \star (0 \star (x \star (x \star y)^n))$$
$$= (0 \star (x \star (x \star y)^n)) \star (0 \star (x \star (x \star y)^n)) = 0 \in I$$

we have $0 \star (0 \star (x \star (x \star y))^n) \in I$. By Lemma II.6 $x \star (x \star y))^n) \in I$. By Proposition II.10 I is an n-fold sub-commutative ideal.

To show the converse is not true, we see Example II.2. We have known that the zero ideal is 2-fold sub-commutative, but it is not a p-ideal since $0 \star (0 \star 1) = 0 \in \{0\}$, but $1 \notin \{0\}$.

The proof is complete.

■

Next we give a new characterization of an n-fold positive implicative ideals in BCI-algebras.

**Lemma II.17.** *In any BCI-algebra X; we have*

$$x \star y^n = (x \star y) \star ((x \star y) \star (x \star y^n))$$

**Proof:.** Since $a \star (a \star b) \leq b$, $\forall a, b \in X$; we have $(x \star y) \star ((x \star y) \star (x \star y^n)) \leq x \star y^n$

In addition

$$\begin{aligned} ((x \star y^n) \star [(x \star y) \star ((x \star y) \star (x \star y^n))] &= [(x \star y) \star ((x \star y) \star ((x \star y) \star (x \star y^n)))] \star y^{n-1} \\ &= ((x \star y) \star (x \star y^n) \star y^{n-1} \\ &= (x \star y^n) \star (x \star y^n) = 0 \end{aligned}$$

Hence $x \star y^n \leq (x \star y) \star ((x \star y) \star (x \star y^n))$. Therfore $x \star y^n = (x \star y) \star ((x \star y) \star (x \star y^n))$

■

**Theorem II.18.** *An ideal I of a BCI-algebra is n-fold positive implicative if and only if it satisfies*

$(x \star (x \star y)) \star (y \star x)^n \in I$ *implies* $x \star (x \star (y \star (y \star x)^n)) \in I$, *for all* $x, y \in X$

**Proof:.** Sufficiency. Assume that $(a\star(a\star b))\star(b\star a)^n \in I$ implies $a\star(a\star(b\star(b\star a)^n)) \in I$, for all $a, b \in X$. Let $(x \star y^{n+1}) \star (0 \star y) \in I$. Substituting $x \star y$ for $a$ and $x$ for $b$, we obtain

$$((x\star y)\star((x\star y)\star x))\star(x\star(x\star y))^n) = ((x\star(x\star(x\star y))^n)\star y)\star(0\star y) = ((x\star y^n)\star y)\star(0\star y) \in I.$$

Thus $I \ni (x \star y) \star [(x \star y) \star (x \star (x \star (x \star y))^n] = (x \star y) \star ((x \star y) \star (x \star y^n)) = x \star y^n$ by Lemma II.17.

Hence $x \star y^n \in I$, therefore $I$ is an n-fold positive implicative ideal.

Necessity. Assume that I is an n-fold positive implicative ideal of X. ie $(a\star b^{n+1})\star(0\star b) \in I$ implies $a \star b^n \in I$, for all $a, b \in X$.

We have to prove that $(y \star (y \star x)) \star (x \star y)^n \in I$ implies $y \star (y \star (x \star (x \star y)^n)) \in I$, for all $x, y \in X$. On one hand by Lemma II.17

$$\begin{aligned} a \star b^n &= (a \star b) \star ((a \star b) \star (a \star b^n))) \\ &= (a \star b) \star ((a \star b) \star (a \star (a \star (a \star b^n)))) \\ &= y \star (y \star (x \star (x \star y)^n)) \end{aligned}$$

where $a$ is substituting by $x$ and $a \star b$ by $y$

And the other hand

$$\begin{aligned} (a \star b^{n+1}) \star (0 \star b) &= ((a \star b^n) \star ((a \star b) \star a)) \star b \\ &= ((a \star (a \star (a \star b))^n) \star ((a \star b) \star a)) \star b \\ &= ((a \star b) \star ((a \star b) \star a)) \star (a \star (a \star b))^n \\ &= ((y \star (y \star x)) \star (x \star y)^n \end{aligned}$$

where $a$ is substituting by $x$ and $a \star b$ by $y$

Hence $(y \star (y \star x)) \star (x \star y)^n = (a \star b^{n+1}) \star (0 \star b) \in I$ where $a$ is substituting by $x$ and $a \star b$ by $y$, therefore $a \star b^n = y \star (y \star (x \star (x \star y)^n)) \in I$.

Thus we show that if $(y \star (y \star x)) \star (x \star y)^n \in I$ then $y \star (y \star (x \star (x \star y)^n)) \in I$. The proof is complete.

■

In the following theorem is a relation in BCI-algebras setting among n-fold positive implicative ideals, n-fold sub-commutative ideals and n-fold sub-implicative ideals, which is similar to the case of BCK-algebras.

**Theorem II.19.** *Let I be a nonempty subset of a BCI-algebra X. Then I is n-fold sub-implicative iff it both n-fold positive implicative and n-fold sub-commutative.*

**Proof:.** Necessity. By Theorem II.5 and TheoremII.15.

Sufficiency. Assume that $X$ is both n-fold positive implicative and n-fold sub-commutative. By Proposition II.11 I is an ideal. Let $x, y \in X$ be such that $(x \star (x \star y)) \star (y \star x)^n \in I$.

By Theorem II.18, we have $x \star (x \star (y \star (y \star x)^n)) \in I$ and by sub-commutatively $y \star (y \star x)^n \in I$

It follows from Proposition II.3 that I is an n-fold sub-implicative ideal. This proof is complete.

■

## III. Other Properties of n-fold Sub-commutative Ideals

In this section, we shall study some other properties about n-fold sub-commutative ideals.

**Lemma III.1.** *Let $X$ be a BCI-algebra. If $x\star(x\star y)^n = y\star(y\star(x\star(x\star y)^n))$ then $x\star(x\star y)^{n+1} = y \star (y \star (x \star (x \star y)^{n+1}))$*

**Proof:.** Assume that $x \star (x \star y)^n) = y \star (y \star (x \star (x \star y)^n))$. Then

$$\begin{aligned} x \star (x \star y)^{n+1}) &= (x \star (x \star y)^n) \star (x \star y) \\ &= (y \star (y \star (x \star (x \star y)^n))) \star (x \star y) \end{aligned}$$

We can show that

$(y \star (y \star (x \star (x \star y)^n))) \star (x \star y) = (y \star (y \star (x \star (x \star y)^{n+1})))$

On one hand

$$\begin{aligned} ((y \star (y \star (x \star (x \star y)^n))) \star (x \star y)) \star ((y \star (y \star (x \star (x \star y)^{n+1})))) &= \\ ((y \star (y \star (x \star (x \star y)^n))) \star ((y \star (y \star (x \star (x \star y)^{n+1})))) \star (x \star y)) &\leq \\ ((y \star (x \star (x \star y)^{n+1})) \star (y \star (x \star (x \star y)^n))) \star (x \star y) &\leq \\ (((x \star (x \star y)^n)) \star (x \star (x \star y)^{n+1})) \star (x \star y) &= \\ (((x \star (x \star y)^n)) \star (x \star y)) \star (x \star (x \star y)^{n+1})) = 0 & \end{aligned}$$

On the other hand

$$\begin{aligned} (y \star (y \star (x \star (x \star y)^{n+1}))) \star ((y \star (y \star (x \star (x \star y)^n))) \star (x \star y)) &= \\ (y \star ((y \star (y \star (x \star (x \star y)^n))) \star (x \star y))) \star (y \star (x \star (x \star y)^{n+1})) &\leq \\ (x \star (x \star y)^{n+1}) \star (y \star (y \star (x \star (x \star y)^n)) \star (x \star y)) &= \\ ((x \star (x \star y)^n) \star (x \star y)) \star (y \star (y \star (x \star (x \star y)^n)) \star (x \star y)) &= \\ ((x \star (x \star y)^n) \star (y \star (y \star (x \star (x \star y)^n))) = 0 & \end{aligned}$$

■

**Theorem III.2.** *Let $X$ be a BCI-algebra. Then $X$ is commutative if and only if $x \star (x \star y)^n) = y \star (y \star (x \star (x \star y)^n))$*

**Proof:.** See Lemma III.1.

■

**Theorem III.3.** *If X is a commutative BCI-algebra, then every ideal of X is n-fold sub-commutative.*

**Proof:.** See Theorem III.2 and Proposition II.10

■

**Corollary III.4.** *The BCK-part $B(X) = \{x \in X/0 \star x = 0\}$ is an n-fold sub-implicative and n-fold sub-commutative ideal.*

**Theorem III.5.** *Let X be a BCI-algebra and I an ideal. If the quotient algebra $\langle X/I, \star_I, I_0\rangle$ (see* [1]*, remark 5.17) is commutative, then I is an n-fold sub-commutative ideal. Conversely, if I is an n-fold sub-commutative ideal containing B(X); then $X/I$ is a commutative BCI-algebra.*

**Proof:.** If $\langle X/I, \star_I, I_0\rangle$ is a commutative BCI- algebra, for any $x, y \in X$, by Theorem III.2, we have

$I_y \star_I (I_y \star_I (I_x \star_I (I_x \star_I I_y)^n)) = I_x \star_I (I_x \star_I I_y)^n$. That is

$I_{y\star(y\star(x\star(x\star y)^n))} = I_{x\star(x\star y)^n}$.

Hence $(x \star (x \star y)^n) \star (y \star (y \star (x \star (x \star y)^n))) \in I$. Now if $y \star (y \star (x \star (x \star y)^n)) \in I$, then $x \star (x \star y)^n) \in I$. By Proposition II.10, I is an n-fold sub-commutative ideal of X.

Conversely, assume that I is an n-fold sub-commutative ideal containing B(X).

Since $y\star(y\star(x\star(x\star y)^n)) \leq x\star(x\star y)^n$, we have $(y\star(y\star(x\star(x\star y)^n)))\star(y\star(y\star(x\star(x\star y)^n))) \leq (x \star (x \star y)^n) \star (y \star (y \star (x \star (x \star y)^n)))$,

that is $0 \leq (x\star(x\star y)^n)\star(y\star(y\star(x\star(x\star y)^n)))$. hence $x\star(x\star y)^n)\star(y\star(y\star(x\star(x\star y)^n)) \in B(X) \subset I$.

On the other hand, it is clear that $y \star (y \star (x \star (x \star y)^n)) \leq x \star (x \star y)^n$. Then $(y \star (y \star (x \star (x \star y)^n))) \star (x \star (x \star y)^n) = 0 \in I$. Hence we obtain

$I_{y\star(y\star(x\star(x\star y)^n))} = I_{x\star(x\star y)^n}$. That is

$I_y \star_I (I_y \star_I (I_x \star_I (I_x \star_I I_y)^n)) = I_x \star_I (I_x \star_I I_y)^n$.

By Theorem III.2 $X/I$ is commutative. The proof is complete.

■

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