Topological entropy of minimal geodesics on surfaces

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June 18, 2007

Abstract

Let (M, g) be a compact Riemannian manifold of hyperbolic type, i.e M is a manifold admitting an another metric of strictly negative curvature. In this paper we study the geodesic flow $\tilde{\Phi}_t$ restricted to the set of geodesics which are minimal on the universal covering. In particular for surfaces we show that the topological entropy coincides with the volume entropy generalizing work of Freire and Mañé.

1 Introduction

Let (M, g) be a compact Riemannian manifold and X its universal Riemannian covering. In [24], Manning introduced the volume entropy $h(g)$ of M defined by:

$$h(g) := \lim_{r\to+\infty} \frac{1}{r} \log \mathrm{vol} B_r(p),$$

where $p \in X$ and $B_r(p)$ denotes the ball with center p and radius r. He proved that this limit exists and is independent of p. Let $h(\Phi_t)$ denotes the topological entropy of the geodesic flow Φ_t on the unit tangent bundle SM. Manning proved that the volume entropy is less or equal to the topological entropy. In the case of nonpositive curvature he showed that equality holds. Subsequently this was generalized by Freire and Mañé to metric without conjugate points (see [13] or [24]). Let $\tilde{S}M$ be the set of $v \in SM$ such that the lift of the geodesic c_v with $\dot{c}_v(0) = v$ is a globally minimizing geodesic. We denote by $\tilde{\phi}_t$ the restriction on $\tilde{S}M$ of the geodesic flow ϕ_t. In [19] Katok and Hasselblatt stated the following theorem:

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold, X be its universal Riemannian covering. Let $\tilde{S}X$ defined as follows:*

$$\tilde{S}X := \{v \in SX \mid c_v \text{ is a minimizing geodesic }\}$$

and $\tilde{S}M := d_p(\tilde{S}X)$, where $p : X \to M$ is the covering map. Let $\tilde{\phi}_t$ be the restriction to $\tilde{S}M$ of the geodesic flow ϕ_t. Then,

$$h(\tilde{\phi}_t) \geq h(g)$$

Let now (M,g) be a compact manifold of hyperbolic type. Then, there exists a metric of strictly negative curvature g_0 on M. The universal Riemannian covering X_0 of (M,g_0) is a Hadamard manifold satisfying $K_{X_0} \leq -k_0^2$ for some constant $k_0 > 0$. Hence X_0 and X are Gromov hyperbolic spaces. Therefore, the distance function on X is 4δ-convex. This help us to justify the following theorem.

Theorem 1.2. *Let (M,g) be a compact Riemannian manifold of hyperbolic type. There is some constant α_0 depending only on (M,g) such that:*

$$h(\tilde{\phi}_t, \alpha_0) \leq h(g).$$

We will use the notion of entropy expansiveness for the following proposition:

Proposition 1.3. *Let (M,g) be a compact Riemannian manifold of hyperbolic type. If $\tilde{\Phi}_t$ is h-expansive for some constant $\epsilon \geq \alpha_0$, we have:*

$$h(\tilde{\phi}_t) = h(\tilde{\phi}_t, \alpha_0) = h_g.$$

As an application, we proof the following result:

Theorem 1.4. *Let (M,g) be an orientable compact surface of genus ≥ 2. Let $\tilde{\phi}_t$ be the restriction to $\tilde{S}M$ of the geodesic flow ϕ_t. Then,*

$$h(\tilde{\phi}_t) = h(g).$$

This paper is organized as follows. In the first section we give a complete proof using the ideas provided by Katok and Hasselblatt that the topological entropy of the minimal geodesics is bounded below by the volume growth. In the second section we study topological entropy of minimal geodesics on manifolds of hyperbolic type and show that for surfaces the entropy equals the volume growth.

2 Topological entropy of minimal geodesics

Let (V,d) be a compact metric space and $\phi_t : V \to V$ be a continuous flow. For each $r > 0$, we define a new distance function

$$d_r(v,w) := \max_{0 \leq t \leq r} d(\phi_t(v), \phi_t(w)).$$

Let F be a subset of V. A set $Y \subset F$ is called a (d_r,ϵ)-separated set of F if for different points $y, y' \in Y$, $d_r(y,y') > \epsilon$. Let $s_r(F,\epsilon)$ denotes the maximal cardinality of a (d_r,ϵ)-separated set.

A set $Z \subset V$ is called a (d_r,ϵ)-spanning set of F if for each $y \in F$, there exists $z \in Z$ such that $d_r(y,z) \leq \epsilon$. Let $t_r(F,\epsilon)$ denotes the minimal cardinality of (d_r,ϵ)-spanning set. It is easy to see that for all $\epsilon > 0$,

$$t_r(F,\epsilon) \leq s_r(F,\epsilon) \leq t_r(F,\frac{\epsilon}{2}).$$

Furthermore

$$h(\phi_t, F, \epsilon) := \overline{\lim_{r\to\infty}} \frac{1}{r} \log s_r(F,\epsilon) = \overline{\lim_{r\to\infty}} \frac{1}{r} \log t_r(F,\epsilon).$$

(see [27] or [28] for details).

Definition 2.1. *The topological entropy $h(\phi_t)$ of the flow $\phi_t : V \to V$ is defined by:*

$$h(\phi_t) := \lim_{\epsilon \to 0} h(\phi_t, V, \epsilon).$$

In the sequel we will need the following lemma which is similar to Lemma 2.1 in [3]

Lemma 2.2. *Let (V, d) be a compact metric space, $\phi^t : V \to V$ a continuous flow and A be a subset of V. Then for all sequences $0 = t_0 < t_1 < \cdots < t_k = t$ and $\delta > 0$*

$$\prod_{i=1}^{k} s_{t_i - t_{i-1}}(\phi^{t_{i-1}} A, \delta) \geq s_t(A, 2\delta),$$

where $s_r(B, \delta)$ is the maximal cardinality of a (d_r, δ)-separating set of B.

Proof. Let $L = E_t(A, 2\delta)$ be a maximal $(d_t, 2\delta)$-separating set of A and for each $i \in \{1, \ldots, k\}$ let $L_i = E_{t_i - t_{i-1}}(\phi^{t_{i-1}} A, \delta)$ be a maximal $(d_{t_i - t_{i-1}}, \delta)$-separated set of $\phi^{t_{i-1}}(A)$.

For each $(x_1, \ldots, x_k) \in L_1 \times \cdots \times L_k$ consider the set

$$B(x_1, \ldots, x_k) = \{z \in L \mid d(\phi^{s + t_{i-1}} z, f^s x_i) \leq \delta, 0 \leq s \leq t_i - t_{i-1}\}.$$

On the other hand, the set L is $(d_t, 2\delta)$-separated and, therefore, the triangle inequality implies that the cardinality of each $B(x_1, \ldots, x_k)$ is at most 1.

Therefore,

$$S_t(A, 2\delta) = \mathrm{card} L \leq \prod_{i=1}^{k} \mathrm{card} L_i = \prod_{i=1}^{k} \mathrm{card} S_{t_i - t_{i-1}}(\phi^{t_{i-1}} A, \delta)$$

□

We need the following Theorem stated in the book of Katok and Hasselblatt on the topological entropy of minimal geodesic on Riemannian manifolds. Even though the main ideas of the proof they provide is correct, it contains some inaccuracy and a mistake. Therefore and for the convenience of the reader we will provide a complete proof of the result.

Theorem 2.3. *Let (M, g) be a compact Riemannian manifold, X be its universal Riemannian covering. Let $\tilde{S}X$ defined as follows:*

$$\tilde{S}X := \{v \in SX \mid c_v \text{ is a minimizing geodesic } \}$$

and $\tilde{S}M := d_p(\tilde{S}X)$, where $p : X \to M$ is the covering map.

Let $\tilde{\phi}_t$ be the restriction to $\tilde{S}M$ of the geodesic flow ϕ_t. Then,

$$h(\tilde{\phi}_t) \geq h_g$$

Proof. Fix $x \in X$, $T, \delta > 0$, and a maximal 3δ-separated set N in the annulus $B(x, (1+\delta)T)\ B(x, T)$. If $K_T := \sup_{y \in M} \mathrm{vol}(B(y, 3\delta T))$ then

$$\#N \geq \frac{1}{K_T}(\mathrm{vol}(B(x, T)) - \mathrm{vol}(B(x, (1-\delta)T))) \geq e^{(h_g)(1-3\delta)T}$$

for sufficient large T, where h_g is the volume entropy of the manifold M.

Consider $y \in N$ and a minimizing geodesic c_y joining x and y. If y_1 and y_2 are two distinct elements of N, and $p_i = c_{y_i}(T)$, we have

$$d(p_1, p_2) \geq d(y_1, y_2) - d(y_1, p_1) - d(y_2, p_2) \geq \delta T.$$

Thus the set

$$S := \{\dot{c}_y(0) \mid y \in N\}$$

is $(d_T, \delta T)$-separated in SX. Let us assume that δT is at least as big as the injectivity radius inj (M) of M. and consider the projections $\gamma_y := \pi \circ c_y$. Then the set

$$\{\dot{\gamma}_y(0) \mid y \in N\}$$

is $(d_T, \frac{1}{2}inj(M))$-separated in SM.

Consider the set

$$V(x, N, T) := \{\dot{\gamma}_y(t) \mid y \in N, t \in [\sqrt{T}, T - \sqrt{T}]\} = \bigcup_{\sqrt{T} \leq t \leq T - \sqrt{T}} \phi^t S$$

This is a subset of

$$\mathcal{M}_T := \{v \in SM \mid \exists s \in [-T+\sqrt{T}, -\sqrt{T}], \text{ such that } c_v \text{ is minimal on } [s, s+T]\},$$

since for each $v = \dot{\gamma}_y(t) \in V(x, N, T)$ the geodesic c_v is minimal on $[-t, T-t]$ and $-t \in [-T + \sqrt{T}, -\sqrt{T}]$. Note, that for each $v \in \mathcal{M}_T$ the geodesic arc $c_v : [-\sqrt{T}, \sqrt{T}] \to SM$ is minimal. Therefore,

$$\bigcap_{T>0} \mathcal{M}_T = \tilde{S}M.$$

Let $s_t(A, \epsilon)$ be the maximal cardinality of a (d_t, ϵ)- separated set of A. Then by lemma 3.2

$$s_{T-\sqrt{T}}(\phi^{\sqrt{T}} S, \rho) \cdot s_{\sqrt{T}}(S, \rho) \geq s_T(S, 2\rho) \geq e^{h(g)(1-3\delta)T}$$

where $\rho = 1/4$inj (M)

Let $h(\phi^t)$ be the topological entropy of the unrestricted geodesic flow. Then there exists $T_0 > 0$ such that for all $T > T_0$

$$s_{\sqrt{T}}(S, \rho) \leq s_{\sqrt{T}}(SM, \rho) \leq e^{2h(\phi^t)\sqrt{T}}$$

Therefore, choosing T such that $\sqrt{T} \geq \frac{2h(\phi^t)}{h(g)\delta}$, we obtain

$$s_{T-\sqrt{T}}(\phi^{\sqrt{T}} S, \rho) \leq e^{h(g)T(1-4\delta)}$$

Choose $t_0 \in (0, T - 2\sqrt{T}]$ and let $m \in \mathbb{N}$ and $k \in [0, t_0)$ be such that $T - 2\sqrt{T} = mt_0 + k$. For $j \in \{0, \ldots, m-1\}$ consider the sets

$$L_j := \phi^{jt_0} \phi^{\sqrt{T}} S = \{\dot{\gamma}_y(\sqrt{T} + jt_0) \mid y \in N\}$$

By the considerations above they are all subsets of $\mathcal{M}_T$. For $0 \leq j \leq m-1$ let l_j be the maximal cardinality of a $(d_{t_0}, \rho/2)$-separated subset of L_j . Applying Lemma 3.2 again, we obtain

$$l_0 \cdot l_1 \cdot \ldots \cdot l_{m-1} \geq s_{T-\sqrt{T}}(\phi^{\sqrt{T}} S, \rho) \geq \ e^{h(g)T(1-4\delta)}.$$

Then, for some $j \in \{0, 1, \ldots, m-1\}$ we must have

$$l_j \geq= e^{\frac{h(g)(1-4\delta)T}{m}} = e^{h(g)(1-4\delta)(t_0 + \frac{k+2\sqrt{T}}{m})} \geq e^{(h_g)(1-4\delta)t_0}$$

Thus for all T large enough the set $\mathcal{M}_T$ and hence $\tilde{S}M$ contains a $(d_{t_0}, \frac{inj(M)}{8})$-separated set of cardinality at least $e^{(h(g)(1-4\delta)t_0}$. This implies that for all $\delta > 0$ the estimate $h(\tilde{\phi_t}) \geq h_g - 4\delta$ holds. □

3 Manifolds of hyperbolic type

Definition 3.1. *A Riemannian manifold (M, g) is called of hyperbolic type if there exists another Riemannian metric g_0 of strictly negative curvature (background metric) which is Lipschitz equivalent to g.*

Note that, in dimension 2, a manifold M is of hyperbolic type if and only if its genus ≥ 2.

Definition 3.2. *Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A map $\Phi : X_1 \longrightarrow X_2$ is called a quasi-isometric map, if there exist constants $A > 1$ and $\alpha > 0$ with:*

$$\frac{1}{A} d_1(x, y) - \alpha \leq d_2(\Phi(x), \Phi(y)) \leq A d_1(x, y) + \alpha \quad \forall x, y \in X_1.$$

In a metric space X, a quasi-geodesic (resp. quasi-geodesic ray) is a quasi-isometric map $\Phi : \mathbb{R} \longrightarrow X$ (resp. $\Phi : \mathbb{R}^+ \longrightarrow X$).

Definition 3.3. *Let (X, d) be a metric space and $A, B \subset X$. The Hausdorff distance of A and B is defined as follow:*

$$d_H(A, B) = \inf\{r \mid A \subset T_r(B), B \subset T_r(A)\},$$

where

$$T_r(A) := \{x \in X \mid d(x, A) \leq r\}.$$

Theorem 3.4. *(Morse Lemma) Let (X_0, g_0) be a Hadamard manifold with sectional curvature $K_{X_0} \leq -k_0 < 0$ for some constant $k_0 > 0$. Then for each quasi-geodesic (resp. quasi-geodesic ray) $c : \mathbb{R} \longrightarrow X$ (resp. $c : \mathbb{R}^+ \longrightarrow X$), there exists a geodesic (resp. geodesic ray) $\gamma : \mathbb{R} \longrightarrow X$ (resp. $\gamma : \mathbb{R}^+ \longrightarrow X$) such that $d(c(t), \gamma(t)) \leq r_0$ (resp. $d(c(\mathbb{R}^+), \gamma(\mathbb{R}^+)) \leq r_0$); r_0 depends only on A, α and k_0.*

Proof. (see [20]) □

Definition 3.5. *A function $f : \mathbb{R} \to \mathbb{R}$ is called k-convex if for all $x, y \in \mathbb{R}$, and $t \in [0, 1]$,*

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y) + k.$$

Lemma 3.6. *Let (X, g) be a Hadamard manifold of hyperbolic type and $c_1 : [0, a] \to X$ and $c_2 : [0, b] \to X$ two minimizing geodesic arcs such that $c_1(0) = c_2(0) = 0$. Then there exists a constant α_0 depending only on (X, g) and the background metric g_0 such that*

$$d(c_1(ta), c_2(tb)) \leq t d(c_1(a), c_2(b)) + \alpha_0$$

for all $t \in [0, 1]$.

Carlos: Could you give a proof of this. In the book of Coornaert etc. they require that the geodesics are globally minimizing and not only on an intervall.The proof should follow from the Morse lemma and the fact that up to constants a triangle is a tripod.

Proof. □

Using this Lemma one obtains:

Proposition 3.7. *(See [7], [8] or [9]) Let (X, g) be a Hadamard manifold of hyperbolic type and $c_1, c_2 : [a, b] \to X$ be two minimizing geodesic arcs. Then the function*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto d(c_1(t), c_2(t)) \end{aligned}$$

is α-convex.

Proof. **Is an easy application of the lemma above using 2 triangles. Proof it!** □

Now we like to proof that in the class of manifolds of hyperbolic type the topological entropy of the minimal geodesic flow is equal to the volume growth. Let (M, g) be a Riemannian manifold of hyperbolic type and let g_0 be a second metric of negative curvature (background metric). Denote by SM and SX the unit tangent bundles of M and its universal cover X with respect to the metric g. Let $\mathcal{F}$ be a fundamental domain in X of diameter a, ϵ a small positive number. For some large r consider the set $F_r := \{z \in X \mid r - a \leq d_g(z, \mathcal{F}) \leq r\}$ which for each $x \in \mathcal{F}$ is contained in the ball $B(x, r + a)$. Let F_r^ϵ be some maximal (d, ϵ)-separated subset of F_r, where d is the metric induced by the Riemannian metric g. Then

$$\# F_r^\epsilon \leq C_\epsilon \text{vol} B(x, r + a + \frac{\epsilon}{2}),$$

where $C_\epsilon = \frac{1}{\inf\limits_{y \in M} \text{vol} B(y, \frac{\epsilon}{2})}$

Let $\mathcal{F}^\epsilon$ be some maximal ϵ-separated set of $\mathcal{F}$. For any $y \in \mathcal{F}^\epsilon$, $z \in F_r^\epsilon$, let γ_{yz} be the geodesic with respect to the background metric g_0 joining y to z. Choose for each such y, z a vector $v = v_{yz} \in S_y X$ such that $c_v : [0, \infty) \to \mathbb{R}$ is a minimizing g- geodesic with $c_v(+\infty) = \gamma_{yz}(+\infty)$. Let P_r be the set of all such vectors, i.e.

$$P_r := \{v_{yz} \mid y \in \mathcal{F}^\epsilon, z \in F_r^\epsilon\}$$

Lemma 3.8. *There exists a constant $\beta > 0$ such that for each minimizing geodesic $c : [0, \infty) \to \mathbb{R}$ with $c(0) \in \mathcal{F}$ there exists $v \in P_r$ such that*

$$d(c_v(t), c(t)) \leq \beta$$

for all $t \in [0, r]$. In particular, if $\tilde{S}\mathcal{F}$ is the set of the unit tangent vectors in $\mathcal{F}$ corresponding to minimal geodesics

$$\bigcup_{v \in P_r} B_{d_r}(v, \beta) \supset \tilde{S}\mathcal{F}$$

Proof. Let $c : [0, \infty) \to \mathbb{R}$ be a minimal geodesic with $c(0) \in \mathcal{F}$. This implies that $c(r)$ is contained in F_r. Since $\mathcal{F}^\epsilon$ and F_r^ϵ are maximal (d, ϵ)-separated sets, there exists $x \in \mathcal{F}^\epsilon$ and $z \in F_r^\epsilon$ such that $d(c(0), y)$ and $d(c(r), z)$ are at not bigger than ϵ. Consider the vector $v = v_{yz} \in P_r$. Since the function $f : [0, r] \to \mathbb{R}$ with $f(t) = d(c(t), c_v(t))$ is α- convex and $f(0) \leq \epsilon$ as well as $f(r) \leq \epsilon$ the proof follows for $\beta = 2\epsilon + \alpha$. □

Now we study the cardinality of separated sets of minimal geodesics on an infinitesimal scale. For that we have to restrict ourself to surfaces.

Lemma 3.9. *Let M be a surface and $\mathcal{F}$ as above. Then for all $\delta > 0$ there exists α_0 such that for all (d_r, δ)- separated sets E of $\tilde{B}_{d_r}(v, \beta) := \tilde{S}\mathcal{F} \cap B_{d_r}(v, \beta)$ we have that*

$$\# E \leq r\alpha_0$$

Proof. **The following proof does not work without modification. The reason is that minimal geodesics might intersect once and therefore the balls in the construction below might intersect. Can you give a modification?** For a given small $\delta > 0$ and some $n \in \mathbb{N}$, let $E \subset Z_\epsilon(v)$ be a (n, δ)-separated set (with respect to $\tilde{\phi}_t$), where $v \in \tilde{S}M$. Let w_k and w_{k+1} be two distinct elements of E. Then, there is $t_k \in [0, n+1]$ such that $l_k := d(c_{w_k}(t_k), c_{w_{k+1}}(t_k)) \geq \delta$. Let γ be the minimizing geodesic satisfying $\gamma(0) = c_{w_k(t_k)}$ and $\gamma(l_k) = c_{w_{k+1}}(t_k)$. Let us put $P_k := \gamma(\frac{\delta}{5})$. Since w_k and w_{k+1} are elements of $Z_\epsilon(v)$, $d(c_{w_k}(t), c_{w_{k+1}}(t)) \leq 2\epsilon \quad \forall t \in \mathbb{R}$. Let $B(P_k, \frac{\delta}{5})$ denote the geodesic ball of radius $\frac{\delta}{5}$ about P_k.

<u>**1.case**</u> $B(P_k, \frac{\delta}{5}) \bigcap c_{w_{k+1}}(]t_k; +\infty[) \neq \emptyset$

Let $s > 0$ such that $c_{w_{k+1}}(t_k + s) \in B(P_k, \frac{\delta}{5})$. Then:

$$\begin{aligned} d(c_{w_{k+1}}(0)), c_{w_{k+1}}(t_k + s) &\leq d(c_{w_{k+1}}(0), c_{w_k}(0)) + \\ &\quad + d(c_{w_k}(0), c_{w_k}(t_k)) + \frac{\delta}{5}, \quad \text{hence,} \\ t_k + s &\leq d(c_{w_{k+1}}(0), c_{w_k}(0)) + t_k + \frac{\delta}{5}. \quad \text{Therefore,} \\ d(c_{w_{k+1}}(0), c_{w_k}(0)) &\geq s - \frac{\delta}{5}. \end{aligned}$$

Otherwise,

$$\begin{aligned} d(c_{w_{k+1}}(t_k)), c_{w_{k+1}}(t_k + s) &\geq d(c_{w_{k+1}}(t_k), c_{w_k}(t_k)) - d(c_{w_k}(t_k), c_{w_{k+1}}(t_k + s)). \\ \text{Then,} \quad s &\geq \delta - \frac{\delta}{5}. \end{aligned}$$

$$\text{Therefore,} \quad d(c_{w_{k+1}}(0), c_{w_k}(0)) \geq \frac{\delta}{5}.$$

<u>**2.case**</u> $B(P_k, \frac{\delta}{5}) \bigcap c_{w_{k+1}}([0, t_k]) \neq \emptyset$

Let $s > 0$ such that $c_{w_{k+1}}(t_k - s) \in B(P_k, \frac{\delta}{5})$. In the same way as above, we obtain:

$$d(c_{w_{k+1}}(0), c_{w_k}(0)) \geq \frac{\delta}{5}.$$

<u>**Rest of the proof of lemma 4.4**</u> Since $t_k > 0$, the ball $B(p_k, \frac{\delta}{5})$ lies in the ϵ-tubular neighborhood $T_{\epsilon,v,n}$ of the geodesic segment

$$\{c_v(t) \text{ such that } t \in [-1; n+1]\}.$$

Let know $\mathcal{P}_{n,\delta}$ denote the number of disjoint geodesic balls of radius $\frac{\delta}{5}$ lying in $T_{\epsilon,v,n}$. Since $dim(M) = 2$, there is a constant $C > 0$ such that the volume of $T_{\epsilon,v,n}$ is smaller than $C\epsilon(n+2)$. Then,

$$\begin{aligned} \inf_{p \in Z_\epsilon(v)} \text{vol}(B(p_k, \frac{\delta}{5}))\mathcal{P}_{n,\delta} &\leq C\epsilon(n+2). \quad \text{Hence,} \\ \mathcal{P}_{n,\delta} &\leq C'_\delta \epsilon(n+2), \end{aligned}$$

where C'_δ is a constant depending on δ. Moreover, if $\mathcal{P}'_{n,\delta}$ is the number of $w_k \in Z_\epsilon(v)$ such that $d(c_{w_{k+1}}(0), c_{w_k}(0)) \geq \frac{\delta}{5}$, we have $\#\mathcal{P}'_{n,\delta} \leq \frac{10\epsilon}{\delta}$. Then, $\#E \leq \mathcal{P}_{n,\delta} + \mathcal{P}'_{n,\delta} \leq C'_\delta \times \epsilon \times (n+2) + \frac{10\epsilon}{\delta}$. Hence, if $s(n, \delta, Z_\epsilon(v))$ denotes the maximal cardinality of a (n, δ)-separated subset of $Z_\epsilon(v)$, we obtain,

$$s(n, \delta, Z_\epsilon(v)) \leq C'_\delta \epsilon(n+2) + \frac{10\epsilon}{\delta}.$$

Therefore,

$$\overline{\lim_{n \to +\infty}} (n)^{-1} \log s(n, \delta, Z_\epsilon(v)) = 0$$

This implies that $h(\tilde{\phi}_t, \epsilon) = 0$. □

Corollary 3.10. *For each $\delta \geq 0$ there exists a δ-separated set L of $\tilde{S}\mathcal{F}$ such that*

$$\#L \leq \#P_r r\alpha_0$$

From Proposition 4.3 and the previous Corollary, we obtain the following result:

Theorem 3.11. *Let (M, g) be an compact surface of genus ≥ 2, X be its universal Riemannian covering. Let $\tilde{S}X$ defined as follows:*

$$\tilde{S}X := \{v \in SX \mid c_v \textit{ is a minimizing geodesic } \}$$

and $\tilde{S}M := d_p(\tilde{S}X)$, where $p : X \to M$ is the covering map.

Let $\tilde{\phi}_t$ be the restriction to $\tilde{S}M$ of the geodesic flow ϕ_t. Then,

$$h(\tilde{\phi}_t) = h(g).$$

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