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Error Estimates for a Regularization of a Class of Porous Medium Equations [1]

Koffi B. Fadimba

Abstract

In solving numerically the class of Porous Medium Equations (PME)

$$\frac{\partial S}{\partial t} + \nabla \cdot f(S)\mathbf{u} - \nabla \cdot k(S)\nabla S = Q(S)$$

with appropriate boundary and initial conditions, one often regularizes the problem, because of possible roughness in the numerical solution, due to the smallness or the vanishing, at some points, of the diffusion coefficient k.

We consider a regularization of the PME and establish convergence estimates for the first derivative, in time, of the regularized solutions, first for the case where the diffusion coefficient k vanishes at $S = 1$ and at $S = 0$. Next we give error estimates for the case where k does not vanish but is assumed small enough to perturb some numerical methods, though the theoretical solution of the PME might be sufficiently regular in this case.

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1 Introduction

We consider the equation

$$\frac{\partial}{\partial t}S + \nabla \cdot (f(S)\mathbf{u}) - \nabla \cdot (k(S)\nabla S) = Q(S) \qquad \text{on } \Omega \times (0, T]. \quad (1.1)$$

with the boundary conditions:

$$(f(S)\mathbf{u} - k(S)\nabla S) \cdot \mathbf{n} = q \quad \text{ on } \partial\Omega \times [0, T_0] \quad (1.2)$$

and the initial condition

$$S(x, 0) = S^0(x) \qquad \text{on } \Omega \quad (1.3)$$

with Ω a bounded domain of $\mathbf{R}^n$, $n = 1, 2, 3$, and where $0 \leq S^0(x) \leq 1$, for all $x \in \Omega$, $S^0 \in H^1(\Omega)$. For simplicity let $|\Omega| = 1$.

Equation (1.1) is called the saturation equation when k vanishes for $S = 0$ and for $S = 1$. This problem is obtained by modeling, for instance, flow through a porous medium (see e.g [1],[9],[5]). Here S is the saturation of the invading fluid, $\mathbf{u}$ is the Darcy velocity, k the conductivity of the medium. The flux, or fractional flow, function f defines the transport term, and Q defines the source and sink terms.

One difficulty in approximating numerically the solution to problem (1.1)–(1.3), lies in the fact that the diffusion coefficient k vanishes when the saturation is zero, or 1. In a past paper [7], analysis was done for the case where k vanishes, but this analysis concerned the solution itself. Here, we are interested in the first derivatives of the solution.

We make the following assumptions on the data of this problem.

$$k(s) \geq \begin{cases} c_1|s|^\mu & 0 \leq s \leq \alpha_1 \\ c_2 & \alpha_1 \leq s \leq \alpha_2 \\ c_3|1-s|^\mu & \alpha_2 \leq s \leq 1 \end{cases} \quad (1.4)$$

where $0 < \alpha_1 < \frac{1}{2} < \alpha_2 < 1$ are given. We assume $0 < \mu \leq 2$, and we set

$$K(\xi) = \int_0^\xi k(\tau)d\tau.$$

To get around some difficulties from direct numerical approximation, one often regularizes the problem in some way. Here we simply replace the degenerate problem by a nondegenerate one, and hope (in fact this is proved : see [7] for instance) that the solution to the new problem will approximate, in some space, the solution to the initial problem, as the perturbation parameter β tends to 0. We regularize by perturbing k as follows (as an example of regularization): we let $0 < \beta < \frac{1}{2}$ and

$$\begin{aligned} \delta_0 &= \min(k(\beta), k(1-\beta)), && (1.5)\\ \delta &= \min(k(\beta) - k(0), k(1-\beta) - k(1)) && (1.6) \end{aligned}$$

with k_β defined by

$$k_\beta(\xi) \begin{cases} = k(\xi) & k(\xi) > \delta \\ \frac{1}{2}\delta \leq k_\beta(\xi) \leq \delta & \text{otherwise.} \end{cases} \tag{1.7}$$

More generally, define k_β in such a way that k_β converges strongly to k as the perturbation parameter β tends to 0. In any case define K_β by $K_\beta(\xi) = \int_0^\xi k_\beta(\tau)d\tau$. Then the approximate problem for (1.1) is:

$$\frac{\partial}{\partial t}S_\beta + \nabla \cdot (f(S_\beta)\mathbf{u}) - \nabla \cdot (k_\beta(S_\beta)\nabla S_\beta) = Q(S_\beta) \text{ on } \Omega \times (0, T_0] \tag{1.8}$$

$$f(S_\beta)\mathbf{u} \cdot \mathbf{n} - \frac{\partial}{\partial \mathbf{n}} K_\beta(S_\beta) = q \text{ on } \partial\Omega \times [0, T_0] \tag{1.9}$$

$$S_\beta(x, 0) = S^0(x) \text{ on } \Omega. \tag{1.10}$$

Now the question is how well we approximate the initial problem by so doing. Many papers have dealt with this problem, in the degenerate case. José Carillo in [4], and Emile Rouvre and Gerard Gagneux in [13], study the same problem in general settings, with one degeneracy at 0, but do not regularize the problem. They are concerned directly with the existence of entropy solutions. M.E. Rose in [10] considers problem (1.1)–(1.3) with one degeneracy, without the transport term and in one dimension, and in [11, 12] the full advection-diffusion equation in one dimension. D.L. Smylie in his doctoral thesis [14] treats the pure parabolic case in several dimensions, and with two degeneracies. R.C Sharpley and K.B. Fadimba in [7] and [8], and K.B. Fadimba in [6] treat the full advection-diffusion equation in several dimensions with two degeneracies. Error analysis for S_β is performed in each of these papers.

One main concern in this paper is the regularity of $S_{\beta t}$. A conjecture in [11] is that $S_{\beta t}$ is bounded independently of β, for the degenerate case. This is important since it improves the estimates in the error analyses for the finite element method applied to the problem. In [10] and [14], the analyses are performed assuming that this conjecture is true. Indeed the conjecture is true for $\mu = 1$ by [10].

In this paper, without proving the conjecture itself, we attempt to bring some improvement to the existing results on this matter. More precisely we show that $\|S_{\beta t} - S_t\|_{L^2((H^1)^*)}$ tends to 0 as $\beta \to 0$, in the degenerate case (condition (1.13)). We also show $\left\|\sqrt{k_\beta(S_\beta)}(S_{\beta t} - S_t)\right\|_{L^2(L^2)}$ tends to 0 as $\beta \to 0$, in the nondegenerate case (condition (1.12) or (1.11)),with possibly a_0 small), among other results. A drawback for the last result is that the constants appearing in the estimates depend on S_t, ∇S and S. For these reasons, we make the following assumptions.

$$\|S_t\|_{L^\infty(L^\infty)} + \|S\|_{L^\infty(L^\infty)} + \|\nabla S\|_{L^\infty(L^\infty)} \leq C. \tag{1.11}$$

Condition (1.11) is unlikely to be satisfied if we have a degenerate problem, i.e. if the coefficient k vanishes at some places. Hence the assumption

$$k(s) \geq a_0 \ \forall s \in [0, 1], \tag{1.12}$$

with a_0 possibly small. Nevertherless our first result (Theorem 3.1) is gotten under the degeneracy condition

$$k(0) = k(1) = 0. \tag{1.13}$$

The remaining of the paper is structured as follows. In Section 2, we give some preliminary results on the solution operator T. Also we give a summary of some previous results on the problem considered.

In Section 3, we prove our main result: $\|S_{\beta t}-S_t\|_{L^2((H^1)^*)}$ tends to 0 as β tends to 0. We also obtain in this section that $\left\|\sqrt{k_\beta(S_\beta)}(S_{\beta t}-S_t)\right\|_{L^2(L^2)}$ tends to 0 as $\beta \to 0$, but under the restriction (1.11).

In section 4, additional error estimates are given for $S_{\beta t}$, and convergence estimates are established for ∇S_β.

2 Preliminaries

2.1 The Solution Operator

The solution operator defined below is a very useful tool in the analysis of problems of parabolic nature, in the sense that it can help simplify somehow the analysis. In particular, it helps define an equivalent norm on $(H^1)^*$ more appropriate for the analysis of the P.D.E. than the usual dual norm. Here we give a summary of some of the properties of the solution operator and its discrete analogue. We refer to [14], [6], and also to [10, 11, 12] for more details. Denote

$$f_\Omega = \frac{1}{|\Omega|}\int_\Omega f dx = \frac{1}{|\Omega|}(f,1)$$

when this has a meaning. We consider $\omega \in H^1(\Omega)$ given by

$$\begin{cases} -\Delta\omega = f - f_\Omega & \text{in } \Omega \\ \frac{\partial \omega}{\partial n} = 0 & \text{on } \partial\Omega \\ \omega_\Omega = f_\Omega \end{cases} \tag{2.1}$$

Then ω is well-defined for a given $f \in (H^1)^*$. We define the Solution Operator T by

$$T(f) = \omega \text{ for all } f \in (H^1)^*$$

where ω is the unique solution to problem 2.1.

A weak formulation of Problem 2.1 is

$$\begin{cases} (\nabla Tf, \nabla\phi) = (f - f_\Omega, \phi) & \forall \phi \in H^1(\Omega) \\ (Tf)_\Omega = f_\Omega \end{cases} \tag{2.2}$$

If we take $\phi = Tg$ with $g \in (H^1)^*$, then we get

$$(\nabla Tf, \nabla Tg) = (f, Tg) - f_\Omega g_\Omega,$$

and if $g = f$ we get

$$(f, Tf) = \|\nabla Tf\|_{L^2}^2 + (f_\Omega)^2 \ \forall f \in (H^1)^*. \tag{2.3}$$

We have the following

Proposition 2.1 *The solution operator as defined by* (2.1) *is linear, symmetric in the sense* $(Tf, g) = (f, Tg)$ *for all* $f, g \in (H^1)^*$*,and positive definite in the sense that* $(Tf, f) > 0$ *if* $f \neq 0$*.*

This proposition allows us to define a new norm on $(H^1)^*$ (see [10],[12],[6] and [11]), by

$$\|u\|_{(H^1)^*} = (Tu, u)^{\frac{1}{2}} = (\|\nabla Tu\|_{L^2}^2 + (u_\Omega)^2)^{\frac{1}{2}} = \|Tu\|_{H^1}. \tag{2.4}$$

2.2 Previous Results

We summarize here some results from [7], which are still valid without condition (1.13) being necessarily satisfied [6], and which we need in the following analysis. Define

$$\gamma = \frac{2+\mu}{1+\mu} \tag{2.5}$$

i.e. γ is the conjugate complement of $2 + \mu$.

We make the following assumption.

$$C^*|f(b) - f(a)|^2 \leq (K(b) - K(a))(b - a) \tag{2.6}$$

for all $0 \leq a \leq b \leq 1$, and for some positive constant C^*.

Assumption (2.6) is verified under condition (1.4) for the nondegenerate case. In the degenerate case we need the additional hypothesis:

$$f'(0) = f'(1) = 0 \tag{2.7}$$

We will be using (2.6) in the form

$$C^*\|f(u)-f(v)\|^2_{L^2(\Omega)} \le (K(u)-K(v),\, u-v) \tag{2.8}$$

A consequence of (2.6) is

$$|f'(\xi)| \le C\sqrt{k(\xi)} \tag{2.9}$$

These inequalities remain true when k is replaced by k_β. Also

$$C^{**}\|u-v\|^{2+\mu}_{L^{2+\mu}} \le (K(u)-K(v),\, u-v) \tag{2.10}$$

for all $u, v \in L^{2+\mu}$. Inequality (2.10) is also true when K is replaced by K_β. Given that K is Lipschitz, we have:

$$\begin{aligned}(K(u)-K(v))^2_\Omega &\le \|K(u)-K(v)\|^2_{L^2}\\ &\le \|k\|_\infty (K(u)-K(v),\, u-v).\end{aligned} \tag{2.11}$$

The next Theorem gives the error estimates for $S-S_\beta$, when the initial problem (1.1)–(1.3) is replaced by the regularized nondegenerate problem (1.8)–(1.10).

Theorem 2.1 *Assume that hypothesis* (2.6) *hold. Let S_β be the solution to* (1.8)–(1.10)*, and S the solution to* (1.1)–(1.3)*, then*

$$\sup_{0\le t\le T_0} \|S_\beta - S\|^2_{(H^1)^*} + \eta\int_0^{T_0} (K_\beta(S_\beta)-K_\beta(S),\, S_\beta - S)(\tau)d\tau \le C(\beta\delta)^\gamma \tag{2.12}$$

$$\|K_\beta(S_\beta)-K_(S)\|^2_{L^2(L^2)} \le C(\beta\delta)^\gamma. \tag{2.13}$$

and

$$\|S_\beta - S\|^{2+\mu}_{L^{2+\mu}(L^{2+\mu})} \le C(\beta\delta)^\gamma. \tag{2.14}$$

The following regularity results are found in [7].

Lemma 2.1 *If S_β is the solution to* (1.8)–(1.10), *then*

$$\left\| \frac{\partial S_\beta}{\partial t} \right\|_{L^\infty(0,T_0,L^1(\Omega))} \leq C. \tag{2.15}$$

Lemma 2.2 *Under the conditions of Lemma* 2.1, *we have*

$$\|S_\beta\|^2_{L^\infty(L^2)} + \left\| \sqrt{k_\beta(S_\beta)} \nabla S_\beta \right\|^2_{L^2(L^2)} \leq C \cdot T_0 + \|S^0\|^2_{L^2}. \tag{2.16}$$

Lemma 2.3 *Under the hypothesis of Lemma* 2.1, *we have*

$$\left\| \sqrt{k_\beta(S_\beta)} S_{\beta t} \right\|^2_{L^2(L^2)} + \|\nabla K_\beta(S_\beta)\|^2_{L^\infty(L^2)} \leq C. \tag{2.17}$$

The above Lemmas help to prove the following.

Theorem 2.2 *Assume the hypotheses of Theorem* 2.1 *hold. Then*

$$\|S_{\beta t}\|^\gamma_{L^\gamma(L^\gamma)} \leq C\beta^{-\frac{\mu}{1+\mu}} \tag{2.18}$$

and

$$\left\| \frac{\partial S_\beta}{\partial t} + \nabla \cdot f(S_\beta)\mathbf{u} \right\|^\gamma_{L^\gamma(L^\gamma)} \leq C\beta^{-\frac{\mu}{1+\mu}}. \tag{2.19}$$

Finally we have the following

Theorem 2.3 *Assume $\mu \geq 1$. Then*

$$\sup_{0 \leq t \leq T_0} (K(S_\beta) - K(S),\, S_\beta - S) + \eta \|\nabla(K_\beta(S_\beta) - K(S))\|^2_{L^2(L^2)} \leq C\delta^{\frac{1}{\mu}} \tag{2.20}$$

$$\|S_\beta - S\|_{L^\infty(L^{2+\mu})} \leq C\delta^{\frac{1}{2+\mu}\frac{1}{\mu}} \tag{2.21}$$

$$\|K(S_\beta) - K(S)\|_{L^2(0,T_0,H^1)} \leq C\delta^{\frac{1}{2\mu}}. \tag{2.22}$$

3 Error Analysis for $S_{\beta t}$

In this section we establish our main results. We show that $\|S_{\beta t} - S_t\|_{L^2((H^1)^*)}$ and $\|\sqrt{k_\beta(S_\beta)}(S_{\beta t} - S_t)\|_{L^2(L^2)}$ (for $0 < \mu < 2$) tend to 0 as $\beta \to 0$.

For this purpose we subtract equation (1.1) from equation (1.8) to get

$$S_{\beta t} - S_t + \nabla\cdot(f(S_\beta) - f(S))\mathbf{u} - \Delta(K_\beta(S_\beta) - K(S)) = Q(S_\beta) - Q(S). \quad (3.1)$$

To simplify we assume from now on that $Q \equiv 0$, and $q \equiv 0$.

3.1 Analysis in $L^2((H^1)^*)$

We have the following Theorem.

Theorem 3.1 *Assume conditions* (1.4), (1.13), (2.6) *and* (2.7) *hold. Also assume* $f \in C^2([0,1])$. *In addition let* $0 < \mu \leq 2$. *Then*

$$\|S_{\beta t} - S_t\|^2_{L^2((H^1)^*)} + \sup_{0\leq t\leq T_0}(K_\beta(S_\beta) - K_\beta(S), S_\beta - S) \leq C\delta^{\alpha_0}, \quad (3.2)$$

where $\alpha_0 = \frac{1}{\mu}$, *and* C *independent of* β.

Before giving the proof of the Theorem, we deduce from the above results the following known results.

Corollary 3.1 *Under the assumptions of Theorem* 3.1, *we have*

$$\|K_\beta(S_\beta) - K(S)\|_{L^\infty(L^2)} \leq C\delta^{\frac{\alpha_0}{2}} \quad (3.3)$$

$$\|S_\beta - S\|_{L^\infty(L^{2+\mu})} \leq C\delta^{\frac{\alpha_0}{2+\mu}} \quad (3.4)$$

$$\|f(S_\beta) - f(S)\|_{L^\infty(L^2)} \leq C\delta^{\frac{\alpha_0}{2}}. \quad (3.5)$$

Estimate (3.3) comes from (3.2) and (2.11). Estimate (3.4) is a consequence of (3.2) and (2.10). Finally, estimate (3.5) results from (3.2) and condition (2.6).

Now we prove Theorem 3.1.

Proof.
First notice that under assumption (1.13) we get that $\delta_0 = \delta$. We go from identity (3.1). Multiply each side by $T(S_{\beta t} - S_t)$, integrate over the domain Ω (assume $|\Omega| = 1$), use the Divergence Theorem, and the boundary conditions (1.2) and (1.9) to get

$$\begin{aligned} &\|S_{\beta t} - S_t\|^2_{(H^1)^*} - ((f(S_\beta) - f(S))\mathbf{u}, \nabla T(S_{\beta t} - S_t)) \\ &\quad + (\nabla(K_\beta(S_\beta) - K(S)), \nabla T(S_{\beta t} - S_t)) = 0, \end{aligned} \tag{3.6}$$

where we have made use of (2.4). By (2.2) the last term on the left hand side of (3.6) can be rewritten as

$$\begin{aligned} &(\nabla(K_\beta(S_\beta) - K(S)), \nabla T(S_{\beta t} - S_t)) \\ &\quad = (S_{\beta t} - S_t - (S_{\beta t} - S_t)_\Omega, K_\beta(S_\beta) - K(S)). \end{aligned} \tag{3.7}$$

Now

$$(S_{\beta t} - S_t)_\Omega = ((S_\beta - S)_\Omega)_t = 0.$$

Next use the product rule on the second term of the right side of (3.7) and substitute in (3.6) to get

$$\begin{aligned} &\|S_{\beta t} - S_t\|^2_{(H^1)^*} + \frac{d}{dt}(K_\beta(S_\beta) - K(S), S_\beta - S) \le \|(f(S_\beta) - f(S))\mathbf{u}\|^2_{L^2} \\ &\quad + \frac{1}{4}\|\nabla T(S_{\beta t} - S_t)\|^2_{L^2} + |(K_\beta(S_\beta)_t - K(S)_t, (S_\beta - S))|. \end{aligned} \tag{3.8}$$

Hide the second term of the right side of (3.8) in the left side, by (2.4). Hölder Inequality on the last term then yields

$$\begin{aligned} &\frac{1}{2}\|S_{\beta t} - S_t\|^2_{(H^1)^*} + \frac{d}{dt}(K_\beta(S_\beta) - K(S), S_\beta - S) \\ &\le C\{f(S_\beta) - f(S)\|^2_{L^2} + \|K_\beta(S_\beta)_t - K(S)_t\|_{L^\gamma}\|S_\beta - S\|_{L^{2+\mu}}\} \end{aligned} \tag{3.9}$$

with $\gamma = \frac{2+\mu}{1+\mu}$. Integrating in time over $[0, T_0]$, and using again Hölder, we obtain:

$$\|S_{\beta t} - S_t\|^2_{L^2((H^1)^*)} + \sigma \sup_{0\le t\le T_0} (K_\beta(S_\beta) - K_\beta(S), S_\beta - S)$$

$$\le C\{\|f(S_\beta) - f(S)\|^2_{L^2(L^2)} + \|K_\beta(S_\beta)_t - K(S)_t\|_{L^\gamma(L^\gamma)}\|S_\beta - S\|_{L^{2+\mu}(L^{2+\mu})}\} + \sup_{0\le t\le T_0} |(K_\beta(S) - K(S), S_\beta - S)|. \quad (3.10)$$

Thus referring back to Theorem 2.1, condition (2.6), estimate (2.17), and the definition of K_β, we get

$$\|S_{\beta t} - S_t\|^2_{L^2((H^1)^*)} + \sigma \sup_{0\le t\le T_0} (K_\beta(S_\beta) - K_\beta(S), S_\beta - S) \le C\delta^{\alpha_0}, \quad (3.11)$$

where

$$\alpha_0 = \frac{1}{\mu}. \quad (3.12)$$

This ends the proof of the Theorem. □

Note : If we make the following hypotheses, as made in many papers [10, 11, 14] :

$$k(s) + k(1-s) \le C\beta^\mu, \quad (3.13)$$

then

$$\delta^{\alpha_0} \asymp \beta.$$

3.2 Analysis in $L^2(L^2)$

From now on, we assume that condition (1.12) holds in place of condition (1.13), unless otherwise stated.

Now we show that $\left\|\sqrt{k_\beta(S_\beta)}(S_{\beta t} - S_t)\right\|_{L^2(L^2)} \to 0$ as $\beta \to 0$. We go from (3.1). Multiply each side of (3.1) by $K_\beta(S_\beta)_t - K(S)_t$, integrate over Ω, use the Divergence Theorem, and the boundary conditions (1.2) and (1.9), to get

$$(S_{\beta t} - S_t, K_\beta(S_\beta)_t - K(S)_t) + (\nabla(K_\beta(S_\beta) - K(S)), \nabla(K_\beta(S_\beta)_t - K(S)_t)$$

$$= ((f(S_\beta) - f(S))\mathbf{u}, \nabla(K_\beta(S_\beta)_t - K(S)_t)). \tag{3.14}$$

We split the first term on the left side of (3.14) into 2 terms:

$$(S_{\beta t} - S_t, K_\beta(S_\beta)_t - K(S)_t) = (S_{\beta t} - S_t, k_\beta(S_\beta)(S_{\beta t} - S_t))$$

$$+(S_{\beta t} - S_t, (k_\beta(S_\beta) - k(S))S_t) \tag{3.15}$$

so that (3.14) becomes

$$(S_{\beta t} - S_t, k_\beta(S_\beta)(S_{\beta t} - S_t)) + \frac{1}{2}\frac{d}{dt}\|\nabla(K_\beta(S_\beta) - K(S))\|_{L^2}^2$$

$$= ((f(S_\beta) - f(S))\mathbf{u}, \nabla(K_\beta(S_\beta)_t - K(S)_t)) + (S_{\beta t} - S_t, (k_\beta(S_\beta) - k(S))S_t). \tag{3.16}$$

Next, using the product rule on the first term of the rightside of (3.16) we have

$$\left\|\sqrt{k_\beta(S_\beta)}(S_{\beta t} - S_t)\right\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla(K_\beta(S_\beta) - K(S))\|_{L^2}^2$$

$$= \frac{d}{dt}((f(S_\beta) - f(S))\mathbf{u}, \nabla(K_\beta(S_\beta) - K(S)))$$

$$+(((f(S_\beta) - f(S))\mathbf{u})_t, \nabla(K_\beta(S_\beta) - K(S))) + (S_{\beta t} - S_t, (k_\beta(S_\beta) - k(S))S_t). \tag{3.17}$$

By the product rule

$$((f(S_\beta) - f(S))\mathbf{u})_t = ((f'(S_\beta) - f'(S))S_t\mathbf{u})$$
$$+f'(S_\beta)(S_{\beta t} - S_t)\mathbf{u} + (f(S_\beta) - f(S))\mathbf{u}_t. \tag{3.18}$$

The second term of the right side of (3.18) gives

$$(f'(S_\beta)(S_{\beta t} - S_t)\mathbf{u}, \nabla(K_\beta(S_\beta) - K(S)))$$

$$\leq \frac{1}{4}\|f'(S_\beta)(S_{\beta t} - S_t)\|_{L^2}^2 + C\|\nabla(K_\beta(S_\beta) - K(S))\|_{L^2}^2$$

$$\leq \frac{1}{4}\left\|\sqrt{k_\beta(S_\beta)}(S_{\beta t}-S_t)\right\|_{L^2}^2 + C\|\nabla(K_\beta(S_\beta)-K(S))\|_{L^2}^2 \tag{3.19}$$

where we have used the fact that $|f'(\sigma)| \leq C\sqrt{k_\beta(\sigma)}$ (see (2.9)). Since $f \in C^2([0,1])$, f' is Lipschitz. Thus the first term of the right side of (3.18) gives

$$\begin{aligned}((f'(S_\beta)-f'(S))S_t\mathbf{u}, \nabla(K_\beta(S_\beta)-K(S))) &\leq C\|S_t\mathbf{u}\|_{L^\infty(L^\infty)}\|S_\beta - S\|_{L^2}^2 \\ &+\|\nabla(K_\beta(S_\beta)-K(S))\|_{L^2}^2 \end{aligned}\tag{3.20}$$

The third term on the right side of (3.18) gives the bound

$$\begin{aligned}|((f(S_\beta)-f(S))\mathbf{u}_t, \nabla(K_\beta(S_\beta)-K(S)))| &\leq \frac{1}{2}\|(f(S_\beta)-f(S))\mathbf{u}_t\|_{L^2}^2 \\ &+\frac{1}{2}\|\nabla(K_\beta(S_\beta)-K(S))\|_{L^2}^2 \end{aligned}\tag{3.21}$$

Finally the last term on the right side of (3.17) is bounded as follows:

$$\begin{aligned}(S_{\beta t}-S_t, (k_\beta(S_\beta)-k(S))S_t) &\leq \frac{1}{4}\|\sqrt{k_\beta(S_\beta)}(S_{\beta t}-S_t)\|_{L^2}^2 \\ &+C\|S_t\|_{L^\infty(L^\infty)}\left\|\frac{k_\beta(S_\beta)-k(S)}{\sqrt{k_\beta(S_\beta)}}\right\|_{L^2}^2. \end{aligned}\tag{3.22}$$

We can hide the first term on the right side of (3.22) in the left side of (3.17). The last term is treated as follows:

$$\begin{aligned}&\left\|\frac{k_\beta(S_\beta)-k(S)}{\sqrt{k_\beta(S_\beta)}}\right\|_{L^2}^2 \\ &\quad\leq \left\|\frac{k_\beta(S_\beta)-k_\beta(S)}{\sqrt{\delta_0}}\right\|_{L^2}^2 + \left\|\frac{k_\beta(S)-k(S)}{\sqrt{\delta_0}}\right\|_{L^2}^2 \\ &\quad\leq \left\|\frac{k_\beta(S_\beta)-k_\beta(S)}{\sqrt{\delta_0}}\right\|_{L^2}^2 + C\frac{\delta^2}{\delta_0} \end{aligned}\tag{3.23}$$

where we use the definition of k_β (see (1.7)). By the Lipschitz nature of k_β, we have

$$\left\| \frac{k_\beta(S_\beta) - k_\beta(S)}{\sqrt{\delta_0}} \right\|_{L^2} \leq \|k_\beta\|_{L^\infty} \delta_0^{-\frac{1}{2}} \|S_\beta - S\|_{L^2}. \tag{3.24}$$

Now going back to estimate (3.17), substituting and hiding the appropriate terms from the above analysis, we obtain

$$\begin{aligned} \frac{1}{2} \left\| \sqrt{k_\beta(S_\beta)}(S_{\beta t} - S_t) \right\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt} \|\nabla(K_\beta(S_\beta) - K(S))\|_{L^2}^2 \\ \leq \frac{d}{dt}((f(S_\beta) - f(S))\mathbf{u}, \nabla(K_\beta(S_\beta) - K(S))) \\ + C\|S_t \mathbf{u}\|_{L^\infty(L^\infty)} \{ \|S_\beta - S\|_{L^2}^2 + \|(f(S_\beta) - f(S))\mathbf{u}\|_{L^2}^2 \\ + \delta_0^{-1} \|S_\beta - S\|_{L^2}^2 + \frac{\delta^2}{\delta_0} \} + C\|\nabla(K_\beta(S_\beta) - K(S))\|_{L^2}^2. \end{aligned} \tag{3.25}$$

Using Grönwall Lemma yields:

$$\begin{aligned} \left\| \sqrt{k_\beta(S_\beta)}(S_{\beta t} - S_t) \right\|_{L^2(L^2)}^2 + \eta \|\nabla(K_\beta(S_\beta) - K(S))\|_{L^\infty(L^2)}^2 \\ \leq C \sup_{0 \leq t \leq T_0} |((f(S_\beta) - f(S))\mathbf{u}, \nabla(K_\beta(S_\beta) - K(S)))| \\ + C\|S_t \mathbf{u}\|_{L^\infty(L^\infty)} \{ \|S_\beta - S\|_{L^2(L^2)}^2 + \|f(S_\beta) - f(S)\|_{L^2(L^2)}^2 \\ + \delta_0^{-1} \|S_\beta - S\|_{L^2(L^2)}^2 + \frac{\delta^2}{\delta_0} T_0 + \|\nabla(K_\beta(S^0) - K(S^0))\|_{L^2}^2 \} \end{aligned} \tag{3.26}$$

where S^0 is as in (1.3). The first term on the right side of (3.26) is bounded as follows:

$$\begin{aligned} \sup_{0 \leq t \leq T_0} |((f(S_\beta) - f(S))\mathbf{u}, \nabla(K_\beta(S_\beta) - K(S)))| \\ \leq C(\mathbf{u}) \|f(S_\beta) - f(S)\|_{L^\infty(L^2)}^2 + \eta/2 \|\nabla(K_\beta(S_\beta) - K(S))\|_{L^\infty(L^2)}^2 \end{aligned} \tag{3.27}$$

so that the second term of the right side of (3.27) can be hidden in the left side of (3.26). In [7] (page 46) it was shown that

$$\|\nabla(K_\beta(S^0) - K(S^0))\|_{L^2} \leq C\delta^{\frac{1}{2}} \left\| \sqrt{k_\beta(S^0)} \nabla S^0 \right\|_{L^2} \tag{3.28}$$

so that the last term on the right hand side of (3.26) is bounded by $C\delta$. Finally using Theorems 2.1, 2.2, Corollary 3.1 for the second term on the right side of (3.26), taking $\sigma = \frac{\eta}{2}$, we see that we have just proved the following.

Theorem 3.2 *Assume conditions* (1.4), (1.11), (1.12), *and* (2.6) *hold. Also assume* $f \in C^2([0,1])$ *and that* k *is Lipschitz. Then we have*

$$\left\| \sqrt{k_\beta(S_\beta)}(S_{\beta t} - S_t) \right\|^2_{L^2(L^2)} + \sigma \|\nabla(K_\beta(S_\beta) - K(S))\|^2_{L^\infty(L^2)} \leq C\delta_1 \tag{3.29}$$

where

$$\delta_1 = \frac{\delta^{\frac{2}{\mu}} + \delta^2 + \delta_0 \delta}{\delta_0}. \tag{3.30}$$

Before deriving in the next section additional error estimates for $S_{\beta t}$ and convergence results for ∇S_β, we make some comments on the above two theorems. First notice that the results in Theorem 3.1 are valid even for the degenerate case, that is when (1.13) holds. Next, Theorems 3.2 and 3.1 give new results at least if we are referring back to [6], [7], [10], [11], [12] and [14]. Namely,

$$\|S_{\beta t} - S_t\|_{L^2((H^1)^*)} \leq C\delta^{\frac{\alpha_0}{2}}, \tag{3.31}$$

$$\left\| \sqrt{k_\beta(S_\beta)}(S_{\beta t} - S_t) \right\|_{L^2(L^2)} \leq C\delta_1^{\frac{1}{2}}, \tag{3.32}$$

$$\|S_\beta - S\|_{L^\infty(L^{2+\mu})} \leq C\delta^{\frac{1}{\mu(2+\mu)}} \tag{3.33}$$

$$\|K_\beta(S_\beta) - K(S)\|_{L^\infty(H^1)} \leq C\delta_1^{\frac{1}{2}}. \tag{3.34}$$

δ_1 being defined by (3.30) . Inequality (3.33) is obtained from Theorem 3.1 and inequality (2.10). Thus Theorem 3.1 is an alternative way for obtaining (3.33), the other one being Theorem 2.3.

4 Additional Error Estimates for $S_{\beta t}$ and Convergence Results for ∇S_β.

4.1 Additional Estimates .

With the help of Theorem 3.2 we prove that $K_\beta(S_\beta)_t - K(S)_t$ tends to 0 as β tends to 0, but under condition (1.11).

Lemma 4.1 *Under the hypotheses of Theorem* 3.2*, we have*

$$\|K_\beta(S_\beta)_t - K(S)_t\|_{L^2(L^2)} \leq C(\|\mathbf{u}\|_{L^\infty(L^\infty)}, \|S_t\|_{L^\infty(L^\infty)})\delta_1^{\frac{1}{2}}. \tag{4.1}$$

Proof.
The proof is straightforward since

$$\begin{aligned}\|K_\beta(S_\beta)_t - K(S)_t\|_{L^2(L^2)} &\leq \|k_\beta(S_\beta)(S_{\beta t} - S_t\|_{L^2(L^2)} \\ &+\|(k_\beta(S_\beta) - k(S))S_t\|_{L^2(L^2)}.\end{aligned} \tag{4.2}$$

Theorem 3.2 then does the rest. □

4.2 Convergence Results for ∇S_β.

As for $S_{\beta t}$ we prove that, for $0 < \mu < 2$, $\sqrt{k_\beta(S_\beta)}\nabla(S_\beta - S) \to 0$ as $\beta \to 0$.

Theorem 4.1 *Under the hypotheses of Theorem* 3.2*, we have*

$$\|S_\beta - S\|^2_{L^\infty(L^2)} + \sigma \left\|\sqrt{k_\beta(S_\beta)}\nabla(S_\beta - S)\right\|^2_{L^2(L^2)} \leq C\delta_1, \tag{4.3}$$

where $C = C(\|\nabla S\|_{L^\infty(L^\infty)})$, and where δ_1 is defined by (3.30).

Proof.
The argument goes as in the proofs of Theorems 3.1 and 3.2. We go from equality (3.1), which we multiply by $S_\beta - S$, assuming that $S_{\beta t} - S_t \in$

$L^2(L^2)$, integrate the result over Ω, use the Divergence Theorem together with the boundary conditions (1.2) and (1.9). We then get:

$$\frac{1}{2}\frac{d}{dt}\|S_\beta - S\|_{L^2}^2 + \left\|\sqrt{k_\beta(S_\beta)}\nabla(S_\beta - S)\right\|_{L^2}^2 = ((f(S_\beta) - f(S))\mathbf{u}, \nabla(S_\beta - S))$$
$$+((k(S) - k(S_\beta))\nabla S, \nabla(S_\beta - S)). \tag{4.4}$$

We can rewrite and bound the terms on the right side of (4.3) as follows:

$$((f(S_\beta) - f(S))\mathbf{u}, \nabla(S_\beta - S)) = \left(\frac{(f(S_\beta) - f(S))\mathbf{u}}{\sqrt{k_\beta(S_\beta)}}\ , \sqrt{k_\beta(S_\beta)}\nabla(S_\beta - S)\right)$$

$$\leq C\left\|\frac{(f(S_\beta) - f(S))\mathbf{u}}{\sqrt{k_\beta(S_\beta)}}\right\|_{L^2}^2 + \frac{1}{4}\left\|\sqrt{k_\beta(S_\beta)}\nabla(S_\beta - S)\right\|_{L^2}^2. \tag{4.5}$$

and

$$|((k_\beta(S_\beta) - k(S))\nabla S, \nabla(S_\beta - S))|$$
$$= \left|\left(\frac{k_\beta(S_\beta) - k(S)}{\sqrt{k_\beta(S_\beta)}}\nabla S, \sqrt{k_\beta(S_\beta)}\nabla(S_\beta - S)\right)\right|$$
$$\leq C\left\|\frac{k_\beta(S_\beta) - k(S)}{\sqrt{k_\beta(S_\beta)}}\nabla S\right\|_{L^2}^2 + \frac{1}{4}\left\|\sqrt{k_\beta(S_\beta)}\nabla(S_\beta - S))\right\|_{L^2}^2. \tag{4.6}$$

Now we see the second terms of the right sides of (4.5) and (4.6) can be hidden in the left side of (4.4). It remains to treat the corresponding first terms. But recall that k and k_β are assumed Lipschitz in this analysis, and that we have $|k_\beta(s) - k(s)| \leq C\delta$, by definition of k_β. Integrate in time (4.4) over $[0, T_0]$, use the fact that $S_\beta(x, 0) = S(x, 0) = S^0(x)$ to get

$$\|S_\beta - S\|_{L^\infty(L^2)}^2 + \sigma\left\|\sqrt{k_\beta(S_\beta)}\nabla(S_\beta - S)\right\|_{L^2(L^2)}^2$$
$$\leq C\delta_0^{-1}\|f(S_\beta) - f(S)\|_{L^2(L^2)}^2$$

$$+C\|\nabla S\|^2_{L^\infty(L^\infty)}\{\|S_\beta - S\|^2_{L^2(L^2)}\delta_0^{-1} + \delta T_0\}. \tag{4.7}$$

Finally, use (2.14) (since $2 < 2 + \mu$), with condition (2.6) to see that the right hand side of estimate (4.7) is bounded by $C(\|\nabla S\|_{L^\infty(L^\infty)})\delta_1$, which proves the Theorem. □

5 Conclusion.

The main result in this paper is Theorem 3.1 which is established for the degenerate problem i.e. equation (1.1) satisfying condition (1.13), and which was not given in our previous paper [7], or in any other paper of our knowledge. But the estimate on the error $S_{\beta t} - S_t$ is gotten in the negative space $(H^1(\Omega))^*$. We wish we had it in a more regular space. The other results in this paper come from an attempt to get the same result in the space $L^2(\Omega)$ for the degenerate problem. But we are doubly handicapped: first we obtain an estimate for $\sqrt{k(S_{\beta t})}(S_{\beta t} - S)$ (with some weight), and secondly we impose a stronger condition on the solution (condition (1.11)).

We notice that if condition (1.13) is satisfied then we have

$$\delta_0 = \delta$$

so that δ_1 becomes

$$\delta_1 \le C\delta^{\lambda_0}$$

where $\lambda_0 = \min\left(1, \frac{2-\mu}{\mu}\right)$. But then (1.11) and (1.13) are unlikely to be compatible, though the constants appearing in the various estimates remain independent of β, the regularization parameter. So the open problem is the following. Assuming that (1.13) holds and that $\sqrt{k(S_\beta)}(S_{\beta t} - S_t) \in L^2(L^2)$, for all β sufficiently small, do we have

$$\sqrt{k(S_\beta)}(S_{\beta t} - S_t) \to 0$$

in $L^2(L^2)$, as $\beta \to 0$?

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Institute for Scientific Computation
Texas A & M University
College Station, TX, USA
and
Université du Bénin, Lomé, Togo

Current address :
Department of Mathematics
University of Rhode Island
Kingston, RI 02881, USA
E-mails : fadimba@math.uri.edu
kfadimba@hotmail.com