Abstract

In this article, a certain new concept of connectedness in frames is introduced, namely, local connectedness with respect to. We show that whenever $h : L \longrightarrow M$ is a dense homomorphism with M locally connected with respect to h, then h preserves connectedness. (And this provides a "partial" converse to a result of Baboolal and Banaschewski.) Also, under the hypothesis, the right adjoint preserves pairwise disjoint joins.

Localconnectedness"withrespectto″

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1. Introduction. We introduce a new notion of local connectedness which we call " *local connectedness with respect to*" into the theory of frames. With it, it is shown that if $h : L \longrightarrow M$ is a dense homomorphism with M locally connected with respect to h then $h(u)$ is connected, whenever $u \in L$ is (*Lemma* 2.4). This provides a "partial" converse to a result of Baboolal and Banaschewski [1], namely, that a dense frame homomorphism $h : L \longrightarrow M$ reflects connectedness. Whereas it is known that under dense surjections, *uniform local connectedness with respect to* and *uniform local connectedness* coincide [6], we have not investigated conditions under which *local connectedness with respect to* is equivalent to *local connectedness.*

It is well-known that the right adjoint h_* of a frame homomorphism $L \longrightarrow M$ preserves arbitrary meets but not joins - not even disjoint binary joins. The second purpose then is to show that under the hypothesis of the previous paragraph, the right adjoint does preserve pairwise disjoint joins (*Proposition* 2.9). So, using a result of Baboolal and Banaschewski [1], it then follows that whenever M is locally connected with respect to h, a dense homomorphism $h : L \longrightarrow M$ preserves connectedness. Other results, which are of independent interest are also provided. (See *Proposition* 2.7 and 2.10.)

Recall that a *frame* is a complete lattice L in which the following *infinite distributive law* holds:

$$x \wedge \bigvee S = \bigvee_{s \in S} x \wedge s,$$

for any $S \subseteq L$. A *frame homomorpism* is one which preserves finitary meets (including the empty meet $\bigwedge 0 = e$) and arbitrary joins (including the empty join $\bigvee 0 = 0$).

We follow Baboolal and Banaschewski [1] and say an element $u \in L$ is *connected* if, whenever $u = x \vee y$ with $x \wedge y = 0$, then $x = 0$ or $y = 0$. The frame L is *locally connected* if

$$u = \bigvee_{x \in L} x \quad (x \textit{ is connected}),$$

for each $u \in L$. A *component* of an element $u \in L$ is a maximally connected $x \leq u$ ($x \in L$), that is, an element $x \in L$ is called a *component* of $u \in L$ if it is connected a connected $t \in L$ satisfies $x \leq t \leq u$ then $u = t$. Throughout the article, whenever x is a component of y our notation will be $x \leq_c y$. For properties of (local) connectedness, see [1] and [3], and for general knowledge on frames we refer to [5].

2. Local connectedness with respect to. According to Fox [4], a topological space X is locally connected in another topological space Y if there is a basis of Y such that $V \bigcap X$ is connected for every basic open set V. An example of a topological space which is not locally connected in another topological space Y is the following (also due to Fox): Let Y be the Cartesian plane, let $Y - X$ be the origin and the positive half of the real axis. Then X is not locally connected at any point of $Y - X$ except at the origin.

Now here is the point-free analogy of the notion "local connectedness in":

Definition 2.1. Given an onto frame homomorphism $h : L \longrightarrow M$, the frame M is said to be **locally connected with respect to** h if there is a basis B of L for which $h(b)$ is connected, for each $b \in B$.

Example 2.2. Given a uniformly locally connected frame L, let us denote its Banaschewski-Pultr uniform completion by CL. That is CL is the quotient $\mathfrak{R}L/L$ where $\mathfrak{R}L$ is the collection of all (uniformly) regular ideals on L. (For the construction of this completion, see [2].) It is known that the map γ_L :

$CL \longrightarrow L$ is a dense surjection. Since L is uniformly locally connected, so is CL. In particular, it follows from [6] that L is locally connected with respect to CL.

It is immediate that if M is locally connected with respect to h, then, for any $z \in M$, there is $x \in L$ with $z = h(x)$. Also, for the basis B of L,

$$x = \bigvee_{u \in B} u \quad (h(u) \textit{ is connected})$$

so that

$$z = \bigvee_{u \in B} h(u) \quad (h(u) \textit{ is connected}).$$

Thus, M is locally connected as well.

Lemma 2.3. For any dense frame homomorphism $h : L \longrightarrow M$ with M locally connected with respect to h, the frame L is locally connected.
Proof. For any $x \in L$,

$$x = \bigvee_{u \in B} u \quad (h(u) \textit{ is connected}).$$

Since h reflects connectedness [1], the result follows. □

Recall (cf. e. g. [4]) that a frame homomorphism $h : L \longrightarrow M$ has a right adjoint $h_* : M \to L$ given by

$$h_*(x) = \bigvee_{y \in B} y \quad (h(y) \leq x).$$

If h is onto then $hh_*(x) = x$. The homomorphism h is said to be *dense* whenever $h(a) = 0$ implies $a = 0$, or equivalently, h is dense whenever $h_*(0) = 0$.

In [1], it was shown that a dense homomorphism reflects connectedness, and, that a dense onto homomorphism whose right adjoint preserves disjoint binary joins preserves connectedness. Here is a partial converse of the former statement:

Lemma 2.4. Let $h : L \longrightarrow M$ be a dense frame homomorphism with M locally connected with respect to h. Then, if $u \in L$ is connected, so is $h(u)$.
Proof. Let $h(u) = x \vee y$ but $x \wedge y = 0$. Find a basis B of L such that

$$u = \bigvee_{w \in B} w \quad (h(w) \textit{ is connected}).$$

Then each w is connected. Now,

$$h(u) = x \vee y = \bigvee_{w \in B} h(w) \quad (h(w) \text{ is connected}),$$

which implies

$$h(w) \leq x \vee y.$$

Since $x \wedge y = 0$ with $h(w)$ is connected, we must have

$$h(w) \leq x \quad \text{or} \quad h(w) \leq y; \quad \text{i.e.,} \quad w \leq h_*(x) \quad \text{or} \quad w \leq h_*(y).$$

Taking joins over all such w yields

$$\begin{aligned} u &= \bigvee_{w \in B} w \quad (h(w) \text{ is connected}) \\ &\leq h_*(x) \vee h_*(y) \quad \text{and} \quad h_*(x) \wedge h_*(y) = 0. \end{aligned}$$

Connectedness of u implies that $u \leq h_*(x)$, say. Then $h(u) \leq x$, so $x \leq h(u)$ gives $h(u) = x$, and

$$y = y \wedge (x \vee y) = y \wedge k(u) = y \wedge x = 0,$$

which proves that $h(u)$ is connected.□

In the following proposition, we characterize components in L in relation to those in M, for $h : L \longrightarrow M$; it is shown that the components x of y in M are exactly those for which the components of $h_*(y)$ are $h_*(x)$ in L, whenever M is locally connected with respect to h.

Lemma 2.5. For any dense homomorphism $h : L \longrightarrow M$ with M locally connected with respect to h,

$$x \leq_c y \quad \text{in} \quad M \quad \Longrightarrow \quad h_*(x) \leq_c h_*(y) \quad \text{in} \quad L.$$

Proof. Take $x \leq_c y$ in M. Since h reflects connectedness and $h \circ h_*(x) = x$ is connected, $h_*(x)$ is connected. Now, if

$$h_*(x) \leq t \leq h_*(y) \quad \text{with} \quad t \quad \text{connected},$$

then $x \leq h(t) \leq y$. By *Lemma* 2.4, $h(t)$ is connected, so $x = h(t)$. Now,

$$t \leq h_* \circ h(t) = h_*(x)$$

and then $t = h_*(x)$; thus $h_*(x) \leq_c h_*(y)$. □

Lemma 2.6. For any dense homomorphism $h : L \longrightarrow M$ with M locally connected with respect to h,

$$h_*(x) \leq_c h_*(y) \quad in \quad L \quad \Longrightarrow \quad x \leq_c y \quad in \quad M.$$

Proof. Certainly, $x \leq y$ in M and x is connected by *Lemma* 2.4. Suppose then that

$$x \leq w \leq y \quad with \quad w \quad connected.$$

Then

$$h_*(x) \leq h_*(w) \leq h_*(y) \quad and \quad h_*(w) \quad is\ connected.$$

Now, $h_*(x) \leq_c h_*(y)$ implies that $h_*(x) = h_*(w)$, so $x = w$. □

Combining these results we have the following characterization

Proposition 2.7. For any dense homomorphism $h : L \longrightarrow M$ with M locally connected with respect to h, the components in M are precisely those that are components in L under the right adjoint $h_* : M \longrightarrow L$ in L. □

Remark. Observe also that under the hypothesis of the proposition if $z \leq_c h_*(a)$ then $h(z)$ is connected. So, if $h(z) \leq u \leq a$ with u connected then $z \leq h_*(u) \leq h_*(a)$ and $h_*(u)$ is connected. So, $z \leq_c h_*(a)$ implies $z = h_*(u)$ or $h(z) = u$, i.e., $h(z) \leq_c a$. Consequently, if $z \leq_c h_*(a)$, then $h(z) \leq_c a$ and $z = h_*h(z)$.

Lemma 2.8. Let $h : L \longrightarrow M$ be a dense homomorphism with M locally connected with respect to h. Then $h_* : M \longrightarrow L$ preserves disjoint binary joins.

Proof.

Pick x, $y \in M$ with $x \wedge y = 0$, and a basis B of L such that

$$h_*(x \vee y) = \bigvee_{s \in B} s \quad (h(s)\ isconnected).$$

Then, for each $s \in B$,

$$h(s) \leq h \circ h_*(x \vee y) = x \vee y.$$

Now, connectedness of $h(s)$ together with $x \wedge y = 0$ ensures that

$$h(s) \leq x \quad or \quad h(s) \leq y, \quad that\ is, \quad s \leq h_*(x) \quad or \quad s \leq h_*(y),$$

which then implies

$$s \leq h_*(x) \vee h_*(y)$$

Now, taking joins over all such s, we have

$$\begin{aligned} h_*(x \vee y) &= \bigvee_{s \in B} s \quad (h(s) \textit{ is connected}) \\ &\leq h_*(x) \vee h_*(y). \end{aligned}$$

so that $h_*(x \vee y) = h_*(x) \vee h_*(y)$. □

In general, we have

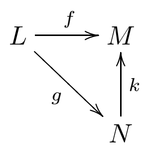
Proposition 2.9. Under the hypothesis of *Lemma* 2.8,

$$h_*(\bigvee_{u_i \in M} u_i) = \bigvee_{u_i \in M} h_*(u_i),$$

whenever the $u_i \in M$ are pairwise disjoint.□

Note that *Lemma 2.4* can now be rediscovered from this result and *Lemma* 1.8 of Baboolal and Banaschewski [1].

Proposition 2.10. In the following commutative triangle



L and M are arbitrary frames, f and g are frame homomorphisms, N is a locally connected frame and k is an onto frame homomorphism which preserves connectedness. Then $k(t) \leq_c f(u)$, whenever $t \leq_c g(u)$, for some $u \in L$.
Proof. For any $u \in L$, we have

$$g(u) = \bigvee_{c_i \in N} c_i \quad (c_i \leq_c g(u)).$$

Now $t \leq_c g(u)$ implies that $t = c_j$, for some j. Also,

$$f(u) = \bigvee_{c_i \in N} k(c_i) \quad (c_i \leq_c g(u))$$

and $k(t) = k(c_j)$, for this j. Then $k(t) \leq f(u)$. By hypothesis, $k(t)$ is connected. Suppose then that

$$k(t) \leq w \leq f(u), \quad \textit{and} \quad w \in M \textit{ is connected.}$$

Then

$$w \leq \bigvee_{c_i \in N} k(c_i) \quad (c_i \leq_c g(u)).$$

We observe that $k(c_i) \wedge k(c_j) = 0$, for any $i \neq j$. Since w is connected, we must have $w \leq k(c_i)$, for some i. Thus

$$k(t) = k(c_j) \leq w \leq k(c_i),$$

for these $i \neq j$. But $k(c_i) \wedge k(c_j) = 0$, so $k(c_j) \leq k(c_i)$ holds only if $k(c_i) = k(c_j)$. Therefore $w = k(c_j) = k(c_i) = k(t)$, thus $k(t) \leq_c f(u)$. □

References

[1] Baboolal, D and Banaschewski, B.: *Compactification and local connectedness of frames*, J. Pure & Appl. Algebra 70 (1991), $3 - 16$.

[2] Banaschewski, B and Pultr, A.: *Samuel compactification and completion of uniform frames*, Math. Proc. Camb. Phil. Soc 108 (1990), 63 - 78.

[3] Chen, X.: *On local connectedness of frames*, J. Pure & Appl. Algebra 79 (1992), $35 - 43$.

[4] Fox, R. G.: Covering spaces with singularities, *Algebraic Geometry* & *topology*: A Symposium in Honour of S. Lefschetz, edited by R. H. Fox et al, Princeton University Press, Princeton, New Jersey, 1957, 243 - 257.

[5] Johnstone, P. T.: *Stone spaces*, Cambridge University Press (Cambridge), 1982.

[6] Siweya, H. J.: On relative uniform local connectedness, (in Press, Journal of Transfigural Mathematics, 1999).

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