Classification of regular maps of prime characteristic revisited: Avoiding the Gorenstein-Walter theorem

Journal Item

Classification of regular maps of prime characteristic revisited: Avoiding the Gorenstein-Walter theorem

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Abstract

Breda, Nedela and Širáň (2005) classified the regular maps on surfaces of Euler characteristic $-p$ for every prime p. This classification relies on three key theorems, each proved using the highly non-trivial characterisation of finite groups with dihedral Sylow 2-subgroups, due to D. Gorenstein and J.H. Walter (1965). Here we give new proofs of those three facts (and hence the entire classification) using somewhat more elementary group theory, using without referring to the Gorenstein-Walter theorem.

Keywords: Regular map; Automorphism group; Non-orientable surface; Euler characteristic.

1 Introduction

A *regular map* is a cellular embedding of a (finite) connected graph on a closed surface, such that its automorphism group acts regularly on flags of the map. (We recall that flags may be thought of as triangles on the carrier surface, with corners a vertex, the centre of an edge incident with the vertex, and the centre of a face incident with both the vertex and the edge.) In every such map, all vertices have the same valency, say, k, and all faces are bounded by closed walks of the same length, say, m, and then using the Schläfli symbol, we say that the map has *type* $\{m, k\}$ (although it is better to consider this as an ordered pair (m, k)). For foundations of the theory of regular maps we refer to the classic paper [8] and the more recent survey [23], and references therein.

It is well known that the automorphism group G of a regular map of type $\{m, k\}$ admits a partial presentation of the form

$$G = \langle\, x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^k, (zx)^m, \dots \rangle. \tag{1}$$

In (1), the involutions x, y, z are automorphisms that reflect a fixed flag in its three sides in such a way that the compositions $r = yz$ and $s = zx$ act as local rotations about the vertex and the centre of a face associated with the flag, respectively. Conversely, given a presentation of a finite group G as in (1), a regular map may be constructed by taking the right cosets of the subgroups $\langle x, y\rangle \cong C_2 \times C_2$, $\langle y, z\rangle \cong D_k$ and $\langle z, x\rangle \cong D_m$ as the edges, vertices and faces, with incidence of these map elements given by non-empty intersection of cosets. The group G then acts as the automorphism group of the map, by right multiplication of cosets. We will use this correspondence throughout, and represent regular maps by presentations of the form (1).

The subgroup $G^+ = \langle r, s\rangle = \langle yz, zx\rangle$ has index 1 or 2 in G, and has index 1 if and only if the surface carrying the regular map is non-orientable. For brevity, we will say that a map is orientable if and only if its carrier surface is orientable, and refer to the Euler characteristic of a map in the analogous way. If a regular map of type $\{m, k\}$ has automorphism group G, then it has $|G|/(2k)$ vertices, $|G|/4$ edges and $|G|/(2m)$ faces, and its Euler characteristic is $\chi = |G|(1/(2k) - 1/4 + 1/(2m)) = |G|(1/k + 1/m - 1/2)/2$.

All regular maps of Euler characteristic $\chi \geq 0$ have been known for several decades, but on the other hand, the problem of classification of regular maps with $\chi < 0$ has been largely left open. Apart from the lists [10] of all regular maps with $-600 \leq \chi \leq -1$ generated with the help of a computer, at the time of writing this note a full classification of regular maps on an infinite family of carrier surfaces was known only in the non-orientable case for $\chi \in \{-p, -p^2, -3p\}$ and in the orientable case for $\chi = -2p$. These classifications were obtained in [7], [14], [13] and [15], respectively. Also in [15] some of the work in [1] was extended to the general case, and a novel approach to the case $\chi = -p$ was introduced.

Our point of departure is the fact that *all* the classification results of [7, 14, 15] for non-orientable regular maps with $\chi \in \{-p, -p^2, -3p\}$ and orientable regular maps with $\chi = -2p$, for p prime, rely on a deep theorem about groups with dihedral Sylow 2-subgroups, by Gorenstein and Walter [17]. Although the original 160-page proof of the Gorenstein-Walter theorem in [17] was later supplanted by an alternative 25-page argument in [4, 3] using the theory of Brauer characters, the shorter proof still depends on a number of other substantial facts, including the Odd Order Theorem. This situation calls for more elementary proofs of the above classifications, given that they target a relatively narrow family of groups (generated by three involutions, two of which commute).

The classification of regular maps of Euler characteristic $\chi = -p$ for prime p in [7] is based on three key facts. These are restated below as Propositions 1 to 3, with their proofs in [7] depending on the Gorenstein-Walter theorem.

Proposition 1 [7, Proposition 5.1] *Let G be a group as given in* (1)*, with $G = \langle x, y, z\rangle = \langle r, s\rangle$ where $r = yz$ and $s = zx$, and with $|G| = 4km$ where k and m are odd and relatively prime, and $k, m \geq 3$. Then G is isomorphic to A_5.*

Proposition 2 [7, Proposition 5.2] *Let G be a group as given in* (1)*, with $G = \langle x, y, z\rangle = \langle r, s\rangle$ where $r = yz$ and $s = zx$, and with $|G| = 2km$ where k is odd, m is even, $k \geq 3$, $m \geq 4$, and $\gcd(k, m) = 1$. Then $G = \langle r\rangle\langle x, z\rangle \cong C_k\, D_8$, with k being a multiple of 3 and $m = 4$, and G admits a presentation obtained from* (1) *by adding one relator, as follows*:

$$G = \langle\, x, y, z \mid x^2, y^2, z^2, (xy)^2, r^k, s^m, (rs^{-1})^2 x\,\rangle. \tag{2}$$

Proposition 3 [7, Proposition 5.3] *Let G be a group as given in* (1)*, with $G = \langle x, y, z\rangle = \langle r, s\rangle$ where $r = yz$ and $s = zx$, and with $|G| = km$, where $k = 2j \geq 4$, and $m = 2\ell$ for odd $\ell \geq 3$, and* $\gcd(j, \ell) = 1$*. Then $G = \langle r^2, y\rangle\langle s^2, x\rangle \cong D_j \times D_\ell$, with $j = k/2$ odd, and G admits a presentation obtained from* (1) *by adding one relator, as follows*:

$$G = \langle\, x, y, z \mid x^2, y^2, z^2, (xy)^2, r^k, s^m, r^j s^\ell z \,\rangle. \tag{3}$$

In this paper, we give alternative proofs of Propositions 1 to 3, avoiding the Gorenstein-Walter theorem, and instead using somewhat more elementary group theory, including theorems that can be found in standard group theory texts such as [16, 21, 22].

The tools used in our proof of Proposition 1 are based on a selection of auxiliary results from [22] and [2]. Our approaches to Propositions 2 and 3 are based on solubility of the corresponding group G, and on the observation made in [7] that the odd-order Sylow subgroups of G are cyclic and the Sylow 2-subgroups of G are dihedral. Such groups are called *almost Sylow-cyclic* in [14]. The soluble almost Sylow-cyclic groups were classified by Zassenhaus [25], and a somewhat more accessible form of the classification was given by Wolf [24]. Rather than going through the lists of groups given in [25] and [24], however, we take inspiration from [14] and adopt as the main tool in our elementary proof of Propositions 2 and 3 the analysis of the Fitting subgroup of G.

Recall that the Fitting subgroup F of a group H is the (unique) largest nilpotent normal subgroup of H, or equivalently, the product of all nilpotent normal subgroups of H. Clearly F is a characteristic subgroup of H. Moreover, if H is soluble, then the centraliser on F in H and the centre of F coincide, so that $C_H(F) = Z(F) \subseteq F$ (see [21, 5.4.4], for example). This equality and inclusion imply that conjugation of F by elements of G induces a group homomorphism from G into $\mathrm{Aut}(F)$, with kernel contained in F.

We will present our elementary proofs of Propositions 1 to 3 in Sections 2 to 4, and we make further comment on our approach in Section 5. Besides the main aim of this paper (namely to prove the classification [7] of regular maps of negative prime Euler characteristic in a more elementary way), we also wish to promote some of the methods available for dealing with finite groups which have not been used before or have been used with limited success before in the study of regular maps, in the hope that this may be beneficial for others working in this field.

2 Proof of Proposition 1

In this section, $G = \langle x, y, z\rangle$ as in (1), with also $G = G^+ = \langle r, s\rangle = \langle yz, zx\rangle$, and $|G| = 4km$ where k and m are odd and relatively prime, and $k, m \geq 3$.

We begin by observing that the group G is perfect, meaning that $G = G'$ (the commutator subgroup of G); indeed since $G = G^+$, the group G is generated by elements r and s of odd coprime orders k and m with $(rs)^2 = 1$, which implies that its abelianisation G/G' is trivial. Similarly, we note that the order of the intersection of $\langle y, z\rangle$ and $\langle z, x\rangle$ divides $\gcd(|\langle y, z\rangle|, |\langle z, x\rangle|) = \gcd(|D_k|, |D_m|) = 2$ and therefore $\langle y, z\rangle \cap \langle z, x\rangle = \langle z\rangle \cong C_2$. Also we note that $r^z = zy = r^{-1}$ and $s^z = xz = s^{-1}$. Next, $\langle x, y\rangle \cong C_2 \times C_2$ is a Sylow 2-subgroup of G, and by Sylow theory G contains also a Sylow 2-subgroup of the form $\langle t, z\rangle \cong C_2 \times C_2$ for some involution $t \in G$.

We continue with a series of lemmas.

Lemma 4 *Every element of G can be uniquely expressed in the form $r^a t^i z^j s^b$, for some quadruple (a, b, i, j) of integers such that $0 \le a < k$, $0 \le b < m$, and $0 \le i, j \le 1$.*

Proof. As $|G| = 4km$, all we have to do is prove that the elements of the given form are distinct in G.

To do this, first we note that the elements of the form $r^a s^b$ are all distinct, since the cyclic subgroups generated by r and s have orders k and m and have trivial intersection. Next, if $z = r^a s^b$ for some (a, b), then conjugation by z implies that $r^a s^b = (r^a s^b)^z = r^{-a} s^{-b}$, and so $1 = r^{2a} s^{2b}$ and hence $a = b = 0$, which gives $z = r^a s^b = 1$, contradiction. It now follows easily that the elements of the form $r^a z^j s^b$ (with $i = 0$) are all distinct.

Similarly, if $t = r^a z^j s^b$ for some (a, b, j), then conjugation by z implies that $r^a z^j s^b = t = t^z = r^{-a} z^j s^{-b}$ and therefore $a \equiv -a \bmod j$ and $b \equiv -b \bmod m$, which gives $a = b = 0$ and so $t = z^j$, contradiction. It follows that the elements of the form $r^a t z^j s^b$ (with $i = 1$) are all distinct from the $2km$ elements of the form $r^a z^j s^b$ (with $i = 0$), because if $r^a t z^j s^b = r^{a'} z^{j'} s^{b'}$ then $t = r^{a'-a} z^{j'} s^{b'-b} z^{-j} = r^{a'-a} z^{j'} s^{b'-b}$ or $r^{a'-a} z^{j'+j} s^{b-b'}$, depending on whether $j = 0$ or 1.

It remains to show that the elements of the form $r^a t z^j s^b$ (with $i = 1$) are all distinct. If that is not the case, then in a similar way to the previous argument we find that $t = r^a t z^j s^b$ for some $(a, b, j) \ne (0, 0, 0)$. Rewriting $t = r^a t z^j s^b$ gives $t r^{-a} t = z^j s^b$, but now $t r^{-a} t$ is a conjugate of a power of r and hence its order divides k, while the order of $z^j s^b$ divides $|\langle s, z\rangle| = |\langle x, z\rangle| = |D_m| = 2m$, and then since $\gcd(k, 2m) = 1$ we conclude that both elements are trivial and so $a = b = j = 0$, contradiction. This completes the proof. $\square$

Lemma 5 *Every two involutions in G are conjugate in G.*

Proof. This is a special case of Thompson's transfer lemma (as in [22, Theorem 4.3.8]). Let u and v be two non-conjugate involutions in G. Then $gug^{-1} \notin \langle v\rangle$ for any $g \in G$, so $\langle v\rangle gu \ne \langle v\rangle g$ for every $g \in G$, which implies that right multiplication by the involution u induces a fixed-point free permutation on the space $(G : \langle v\rangle)$ of $2km$ right cosets of $\langle v\rangle$. This permutation is a product of km transpositions, and hence is odd. It follows that G has an index 2 subgroup, consisting of the elements inducing even permutations, but then G cannot be perfect, contradiction. $\square$

Lemma 6 *Every Sylow 2-subgroup of G is self-centralising. Moreover, every two distinct Sylow 2-subgroups of G have trivial intersection, and G contains no elements of even order greater than 2.*

Proof. It suffices to show that the centraliser of every involution in G is a Sylow 2-subgroup of G, and by Lemma 5 and Sylow theory, we need only do this for the involution z. So let $z' \in C_G(z)$. Then by Lemma 4 we have $z' = r^a t^i z^j s^b$ for some (a, b, i, j), and conjugation by z gives $r^a t^i z^j s^b = z' = (z')^z = r^{-a} t^i z^j s^{-b}$, and then by Lemma 4 we have $a = b = 0$, so $z' = t^i z^j$. Thus $z' \in \langle t, z\rangle$, which is a Sylow 2-subgroup of G. The rest follows easily. $\square$

Lemma 7 *The normaliser $N_G(S)$ of a Sylow 2-subgroup S of G is isomorphic to A_4, having the form $S \rtimes \langle g \rangle$ for some element g of order 3 in G.*

Proof. By conjugacy of Sylow subgroups, we may assume without loss of generality that $S = \langle x, y \rangle$, and then by Lemma 6 we know that $C_G(S) = S$. Also by Lemma 5 we know that the elements x and y are conjugate in G. Moreover, by Burnside's fusion control lemma (as given in [22, Theorem 4.3.7]), we find that x and y are also conjugate in $N_G(S)$, and hence $S = C_G(S)$ is a proper subgroup of $N = N_G(S)$. Next, $N/S = N_G(S)/C_G(S)$ is isomorphic to a subgroup of $\mathrm{Aut}(S) \cong \mathrm{Aut}(C_2 \times C_2) \cong S_3$ and so $|N:S| \le 6$, but $|S| = 4$ while $|G|$ is not divisible by 8, so $|N:S|$ cannot be even, and therefore $|N:S| = 3$ and $|N| = 3|S| = 12$. It follows that N contains an element g of order 3, conjugation by which cyclically permutes the three non-trivial elements of S, and the rest follows easily. □

The next few lemmas follow the discussion in the introduction of [2], and provide further insight into the structure of the group G.

Lemma 8 *The number of involutions in G is $|G|/4$, and every right coset of a Sylow 2-subgroup S of G not contained in $N_G(S)$ contains exactly one involution.*

Proof. The number of Sylow 2-subgroups of G is $|G : N_G(S)| = |G|/12$ by Lemma 8. Each of them contains three involutions, and any two of them have trivial intersection, by Lemma 6. It follows that the number of involutions in G is $3|G:N_G(S)| = |G|/4$.

Now let $g \in G \backslash N_G(S)$, and suppose that the coset Sg contains two involutions, say ug and vg, for distinct elements $u, v \in S \cong C_2 \times C_2$. Then since $(ug)^2 = 1 = (vg)^2$, we have $ugu = g^{-1}$ and $vgv = g^{-1}$, so uv commutes with g, and hence uvg has order 6 in G, contrary to the conclusion of Lemma 6. It follows that every right coset Sg with $g \notin N_G(S)$ contains at most one involution. On the other hand, in $N_G(S)$ itself there is just one right coset of S containing involutions, namely the trivial coset S, which contains three involutions. Hence the remaining $(|G| - 12)/4$ right cosets of S in G contain the other $|G|/4 - 3$ involutions, which (by the earlier observation) implies that they contain exactly one each. □

Lemma 9 *Let S be a Sylow 2-subgroup of G and let $N = N_G(S)$. For every involution $u \in G \backslash N$ we have $N \cap N^u \cong C_3$, and conjugation by u inverts every element of $N \cap N^u$. Moreover, the centraliser in G of any element $g \in N_G(S)$ of order 3 is $\langle g \rangle$.*

Proof. By Lemma 7, we know that $N = S \rtimes \langle g \rangle \cong A_4$ for some $g \in N$ of order 3. Now let u be an involution in $G \backslash N$. Then also $gu \notin N$, and hence by Lemma 8 the coset Sgu contains exactly one involution, say τgu for some $\tau \in S$. Then because $(\tau gu)^2 = 1$ we find that $u(\tau g)u = (\tau g)^{-1}$, and then because $\tau g \in N \backslash S$ it follows that τg has order 3, and lies in $N \cap N^u$. On the other hand, all involutions in N lie in S, but $S^u \neq S$ (since $u \notin N$) and so S^u has trivial intersection with S (Lemma 6) and therefore $N \cap N^u$ contains no involutions. Thus $N \cap N^u = \langle \tau g \rangle \cong C_3$, with u inverting τg.

It remains to show that every subgroup $\langle g\rangle$ of order 3 in N of order 3 is self-centralising in G. This is true in $N \cong A_4$, so let us assume that $gh = hg$ for some $h \in G\backslash N$. Then by Lemma 8 the coset Sh contains one involution, say $u = \tau h$ where $\tau \in S$, and then since also $\tau^2 = 1$ it follows that $g\tau u = gh = hg = \tau ug$, and therefore $g^\tau = \tau g\tau = ugu = g^u$. Now g^τ is an element of N of order 3, and hence so is g^u. But $g^u \neq g^{\pm 1}$, for otherwise $g^\tau = g^{\pm 1}$, which is impossible inside $\langle \tau, g\rangle = N \cong A_4$. Hence $g^u = vg^{\pm 1}$ for some non-trivial element $v \in S$. Now $g = (vg^{\pm 1})^u = v^u(g^u)^{\pm 1}$, and as $g \in N$ and $g^u \in N$, we have $v^u \in N$, from which it follows that $N^u = \langle v, g\rangle^u = \langle v^u, g^u\rangle = N$, so u normalises N. By Sylow theory, however, $N_G(N) = N_G(N_G(S)) = N_G(S) = N$, and so this gives $u \in N$, contradiction. Hence there exists no such h, and therefore $C_G(g) = \langle g\rangle$. □

We are ready to finish the proof of Proposition 1, by showing that $G \cong A_5$.

Let N be the normaliser of some Sylow 2-subgroup S of G. Observe that $N \neq G$, because S is conjugate to $\langle x, y\rangle$, which is not normal in G. We now show that there are exactly 12 involutions in G lying outside N. By Lemma 9, for every involution $u \in G\backslash N$ the intersection $N \cap N^u$ is isomorphic to C_3, and each of its elements is inverted under conjugation by u. It u and v are any two such involutions, then $N \cap N^u = N \cap N^v$ if and only if uv centralises $J = N \cap N^u$ $(= N \cap N^v) \cong C_3$, and by the last part of Lemma 9, this happens if and only if uv is an element of J. Hence the number of involutions of G lying outside N is equal to three times the number of subgroups of order 3 in N, namely $3 \cdot 4 = 12$. A further three involutions lie inside N, and so G has exactly 15 involutions. On the other hand, by Lemma 8 the number of involutions in G is equal to $|G|/4$, and so $|G| = 60$. Finally, since G is perfect, it follows that $G \cong A_5$.

3 Proof of Proposition 2

In this section, $G = \langle x, y, z\rangle$ as in (1), with also $G = G^+ = \langle r, s\rangle = \langle yz, zx\rangle$, and $|G| = 2km$ where k is odd, m is even, $k \geq 3$, $m \geq 4$, and $\gcd(k, m) = 1$. Our aim is to show that k is divisible by 3 while $m = 4$, and that a complete presentation of G is obtained by adding the relator $(rs^{-1})^2x$ to (1).

Here we note that the commutator subgroup $G' = [G, G]$ contains $[z, x] = (zx)^2 = s^2$ and $[y, z] = (yz)^2 = r^2$, and then since r has odd order k, it follows that G' contains the subgroup of $\langle r, s\rangle = G$ generated by r and s^2, which has index 1 or 2 in G. This implies that $[r, s^2] \neq 1$, for otherwise $|\langle r, s^2\rangle| = o(r)o(s^2) = km/2$ and then $|G| \leq 2|\langle r, s^2\rangle| = km < 2km$. Hence also G' cannot be abelian.

Next, the assumptions on the order of G and its generators imply that G is expressible as the product $\langle r\rangle\langle x, z\rangle$ of a cyclic subgroup and a dihedral subgroup, which implies that G is soluble, by a relatively easy theorem of Huppert [18]. Then furthermore, by the remarks near the end of Section 1, the Fitting subgroup F of G satisfies $C_G(F) = Z(F) \leq F$, and G itself is almost Sylow-cyclic. Also by definition F is nilpotent, and hence is a direct product of its Sylow subgroups, and it follows from what we know about $|G| = 2km$ that

$F = F_1 \times F_2$ where F_1 is a cyclic group of odd order and F_2 is a 2-group or trivial. We will proceed by determining whether or not F_2 (and hence F) is cyclic.

First, suppose that F_2 is cyclic. Then also F is cyclic, so $\mathrm{Aut}(F)$ is abelian, and $F \le C_G(F)$, which then implies that $F = C_G(F)$, and so $G/F = G/C_G(F)$ is isomorphic to a subgroup of $\mathrm{Aut}(F)$ and hence is abelian. But then $G' \le F$ and so G' is abelian, contradiction. Hence F_2 is not cyclic.

Next, we note that the 2-subgroup F_2 is characteristic in F and hence normal in G, and so is contained in every Sylow 2-subgroup of G. Also since $|G| = 2km$ with k odd, every Sylow 2-subgroup of $\langle x, z\rangle \cong D_m$ is a Sylow 2-subgroup of G, and therefore F_2 is a non-cyclic normal subgroup of $\langle x, z\rangle = \langle x, s\rangle$. It follows that F_2 is either $\langle x, z\rangle$, or $\langle z, s^2\rangle$, or $\langle x, s^2\rangle$, of order $2m$, m or m, respectively. If $F_2 = \langle x, z\rangle$ or $\langle z, s^2\rangle$, however, then G/F_2 is generated by F_2y or by $\{F_2x, F_2y\}$, and so $|G|$ divides $2|F_2| = 4m$ or $4|F_2| = 4m$, respectively, but then $|G|$ is not divisible by k, contradiction. Thus $F_2 = \langle x, s^2\rangle$ of order m, with $G/F_2 = \langle F_2y, F_2z\rangle \cong \langle y, z\rangle \cong D_k$ of order $2k$.

Now consider conjugation of F_2 by the involution y. If $(s^2)^y \in \langle s^2\rangle$, then $(s^2)^y = s^{2j}$ for some j for which $j^2 \equiv 1 \bmod m/2$, and conjugation by z gives $(s^2)^r = (s^2)^{yz} = (s^{2j})^z = s^{-2j}$. It follows that $(s^2)^{r^2} = (s^2)^{(-j)^2} = (s^2)^{j^2} = s^2$, so $[r^2, s^2] = 1$, and then since r has odd order k, we find that $[r, s^2] = 1$, contradiction. Thus $(s^2)^y$ is an element of $\langle x, s^2\rangle \cong D_{m/2}$ not contained in the cyclic subgroup $\langle s^2\rangle \cong C_{m/2}$. In particular, $(s^2)^y$ must be an involution, which implies that $m = 4$, and hence also $|F_2| = 4$, with $F_4 = \{1, x, s^2, xs^2\}$.

Next, $(s^2)^y \in \{1, x, s^2, xs^2\} \setminus \{1, s^2\}$, but $(s^2)^y \ne x$ because y centralises x $(\ne s^2)$, and therefore $(s^2)^y = xs^2$. Equivalently, $(s^2)^r = (s^2)^{yz} = (xs^2)^z = zxs^2z = s^4x = x$. The last relation (namely $(s^2)^r = x$) leads us to consider the effect of conjugation by r on the three non-trivial elements x, s^2 and xs^2 of F_2. From $xy = rs$ we see that $r^{-1}xy = s$, and then $x^r = r^{-1}xr = r^{-1}xyz = sz = s^2x$. It follows that conjugation by r induces the 3-cycle (x, xs^2, s^2) on the non-trivial elements of $F_2 \cong C_2 \times C_2$. In particular, k must be a multiple of 3, and we have $r^{-3}xr^3 = x$. This may be rewritten as $(r^3)^x = r^3$, and hence $sr^3s^{-1} = (r^3)^{xz} = (r^3)^z = r^{-3}$, giving the relation $sr^3s^{-1} = r^{-3}$ which appears in the original statement of Proposition 5.2 in [7]. It also implies that $r^{-2}s = r(r^{-3}s) = r(sr^3) = (rs)r^3 = s^{-1}r^{-1}r^3 = s^{-1}r^2$ which, when applied to $r^{-1}s^2r = x$, gives $x = r(r^{-2}s)sr = r(s^{-1}r^2)r^{-1}s^{-1} = (rs^{-1})^2$, and hence the extra relator $(rs^{-1})^2x$ in the statement of Proposition 2.

It remains to prove that the presentation given in the statement of Proposition 2 defines a group of order $2km$. To do this we may consider the more general group U with presentation $\langle\, x, y, z \mid x^2, y^2, z^2, (xy)^2, (xz)^4, (yzxz)^2x \,\rangle$.

In this group U we have $x = (rs^{-1})^2$, and hence $s^{-2} = (xz)^2 = (x^z)x = (rs^{-1})^2(r^{-1}s)^2$, and then since $(rs)^2 = s^4 = 1$, it follows that

$$\begin{aligned} 1 &= rs^{-2}s^2r^{-1} = r(rs^{-1})^2(r^{-1}s)^2s^2r^{-1} = r^2s^{-1}r(s^{-1}r^{-1})sr^{-1}s^3r^{-1} \\ &= r^2s^{-1}r^2s^2(r^{-1}s^{-1}r^{-1}) = r^2s^{-1}r^2s^3 = (r^2s^{-1})^2 = r^2(rsr)r^2s^{-1} = r^3sr^3s^{-1}. \end{aligned}$$

This also gives $(r^{-3})^x = (r^3)^{zx} = (r^3)^s = r^{-3}$, and hence $[x, r^3] = 1$. Thus $N = \langle r^3\rangle$ is a normal subgroup of G of order dividing $k/3$, with $r^3 = (yz)^3$ centralised by x and inverted under conjugation by each of y and z.

An easy computation in Magma [5] shows that the quotient U/N (obtained from U by adding the relation $(yz)^3 = 1$) is isomorphic to S_4, of order 24, which is the automorphism group of a regular embedding of the complete graph K_4 in the projective plane. Moreover, by Reidemeister-Schreier theory (as explained in [20] and implemented as the `Rewrite` command in Magma [5]), the subgroup N is free of rank 1 (and hence infinite cyclic). It follows that for any positive integer j we can factor out the normal subgroup generated by r^{3j}, to obtain a quotient of order $24j = 2km$ where $k = 3j$ (and $m = 4$), with the required presentation.

The resulting family of maps is also the same as the one in [12, Example 3.1].

4 Proof of Proposition 3

In this section, $G = \langle x, y, z\rangle$ as in (1), with also $G = G^+ = \langle r, s\rangle = \langle yz, zx\rangle$, and $|G| = km$, where $k = 2j \geq 4$, and $m = 2\ell$ for odd $\ell \geq 3$, and $\gcd(j, \ell) = 1$. Our aim is to show that $G = \langle r^2, y\rangle\langle s^2, x\rangle \cong D_j \times D_\ell$, with $j = k/2$ odd, and that G admits a presentation obtained from (1) by adding the single relator $r^j s^\ell z$.

We note again that the commutator subgroup $G' = [G, G]$ contains $[z, x] = (zx)^2 = s^2$ and $[y, z] = (yz)^2 = r^2$. Also the assumptions on the order of G and its generators imply that $G = \langle r\rangle\langle z, x\rangle$ and $G = \langle y, z\rangle\langle s\rangle$, and from either of these, Huppert's theorem [18] tells us that G is soluble. On the other hand, G is not abelian, since yz has order $k > 2$.

Now the subgroups $\langle r\rangle$ and $\langle z, x\rangle$ have orders k and $2m$ and their product G has order km, so they intersect in a subgroup of order 2, generated by the involution $u = r^{k/2} = r^j$, which must be either $(zx)^\ell = s^\ell$ or $z(zx)^t = zs^t$ for some t. If $u = r^j = s^\ell$, then the involution u is central in $\langle r, s\rangle = G$, and hence lies in every subgroup of exponent 2 in a dihedral Sylow 2-subgroup of G, and in particular, lies in $\langle x, y\rangle$. But x and y are not central (since $m = o(zx) > 2$ and $k = o(yz) > 2$), so $u = xy = yx$. This gives $yx = u = r^j = (yz)^j$ and therefore $r^{j-1} = (yz)^{j-1} = y(yx)z = xz = s^{-1}$, so r^{j-1} has order 2ℓ, which is impossible since the order of r divides $k = 2j$, and $\gcd(j, \ell) = 1$. Hence $r^j = u = zs^t$ for some t. Moreover, this implies that $s^t = zr^j$, which is an involution in $\langle z, r\rangle = \langle y, z\rangle$, so $t = m/2 = \ell$ and $zr^j = s^\ell = s^{-\ell}$, which gives us the relation $r^j s^\ell z = 1$.

This relation gives not only $z = r^j s^\ell$ and $z = z^{-1} = s^{-\ell} r^{-j} = s^\ell r^j$, but also $x = zs = r^j s^{\ell+1}$ and $y = rz = r^{j+1} s^\ell$. Substituting these into $xy = yx$ and cancelling left and right gives $ss^\ell r^j r = rs^\ell r^j s$, and then since $z = r^j s^\ell = s^\ell r^j$ we obtain $szr = rzs$, from which it follows that $1 = (szr)^{-1} rzs = r^{-1} zs^{-1} rzs = r^{-1} sr^{-1} s = (r^{-1}s)^2$. The resulting relation $(r^{-1}s)^2 = 1$ appears in the statement of Proposition 5.3 of [7]. It can also be rewritten as $1 = r^{-1}sr^{-1}s = r^{-1}s^2 s^{-1} r^{-1} s^{-1} s^2 = r^{-1}s^2 r s^2$ and hence it gives $(s^2)^r = s^{-2}$ and therefore $(s^2)^y = (s^2)^{rz} = (s^{-2})^z = s^2$, so $[y, s^2] = 1$. Similarly $(r^2)^s = r^{-2}$ and $[x, r^2] = 1$, and from either $(s^2)^r = s^{-2}$ or $(r^2)^s = r^{-2}$ it follows that $[r^2, s^2] = 1$.

We continue as in the previous section, by noting that G is almost Sylow-cyclic, and considering the Fitting subgroup F of G. Again we find that $F = F_1 \times F_2$, where F_1 is cyclic of odd order (a consequence of the fact that G is almost Sylow-cyclic), and F_2 is a 2-group, possibly trivial. Once more we consider two cases.

Case (1): Suppose F_2 is cyclic. Then just as before, also F is cyclic and $F = C_G(F)$, and so $G/F = G/C_G(F)$ is isomorphic to a subgroup of $\mathrm{Aut}(F)$ and hence is abelian, so $\langle r^2, s^2\rangle \le G' \le F$. In particular, since r^2 and s^2 have relatively prime orders j and ℓ, it follows that $j\ell$ divides $|F|$.

Now if F_2 is non-trivial, then it contains a unique involution v, and then since F_2 is normal in G, this involution is central in G. Again it follows that v lies in every subgroup isomorphic to $C_2 \times C_2$, and in particular, $v \in \langle x, y\rangle$. Also $v \notin \{x, y\}$ as before, so $v = xy = rs$, but then centrality of v gives $r(rs) = (rs)r$ and hence $rs = sr$, which implies that G is abelian, contradiction. Thus F_2 is trivial, and F has odd order.

In particular, $k/2 = j$ must be odd. It now follows from the relations $[x, y] = [x, r^2] = [y, s^2] = 1$ and the oddness and coprimality of $\ell = m/2$ and $j = k/2$ that G is the direct product of its dihedral subgroups $\langle r^2, y\rangle \cong D_j$ and $\langle s^2, x\rangle \cong D_\ell$, as required. Note here that $z = r^j s^\ell = r^{j-1} rs s^{\ell-1} = (r^2)^{(j-1)/2} yx (s^2)^{(\ell-1)/2}$.

Case (2): Suppose F_2 is not cyclic. Then F_2 must be dihedral. Since ℓ is odd, every Sylow 2-subgroup of $\langle y, z\rangle \cong D_k$ is a Sylow 2-subgroup of G, and hence we find that F_2 is a normal subgroup of $\langle y, z\rangle$, so must be $\langle y, z\rangle$, $\langle z, r^2\rangle$ or $\langle y, r^2\rangle$. If $F_2 = \langle y, z\rangle$ or $\langle z, r^2\rangle$, however, then G/F_2 is generated by F_2x or by $\{F_2x, F_2y\}$, and so $|G|$ divides $2|F_2| = 4k$ or $4|F_2| = 4k$, respectively, but then $|G|$ is not divisible by $m/2 = \ell$, contradiction. Thus $F_2 = \langle y, r^2\rangle$, of order j, which must be a power of 2. It follows that the quotient G/F_2 has order $|G|/|F_2| = (km)/j = 2m$, and is generated by $\{F_2x, F_2z\} = \{F_2s, F_2z\}$. But now consider the relation $r^j s^\ell z = 1$. As j is even, $r^j \in \langle r^2\rangle \in F_2$ and so $F_2 = F_2 r^j s^\ell z = F_2 s^\ell z$, which gives $F_2 z = F_2 s^\ell$. Thus G/F_2 is cyclic, generated by $F_2 s$, and its order divides m, contradiction. Hence this case is impossible.

To complete the proof, we show that the presentation of G resulting from Case (1) above, namely $\langle\, x, y, z \mid x^2, y^2, z^2, (xy)^2, r^k, s^m, r^j s^\ell z \,\rangle$, with $j = k/2$ and $m = \ell/2$ odd and coprime, determines a group of order km. This can be approached in a number of ways.

By what we found in the fourth paragraph of this section, the relation $r^j s^\ell z = 1$ implies that $[x, r^2] = [y, s^2] = [r^2, s^2] = 1$, and hence the dihedral subgroups $\langle r^2, y\rangle$ and $\langle s^2, x\rangle$ commute with each other. Also from the relations $(rs)^2 = (r^{-1}s)^2 = 1$ and $r^j s^\ell z = 1$ it follows that $G = \langle r\rangle\langle s\rangle$ with $\langle r\rangle \cap \langle s\rangle = \{1\}$, and so $|G| = |\langle r\rangle|\,|\langle s\rangle| = km$. Alternatively, one may write G as the product of its dihedral subgroups $\langle r, z\rangle$ and $\langle s, z\rangle$, which intersect in the subgroup $\langle r^j, s^\ell\rangle = \{1, z, r^j, s^\ell\}$ isomorphic to $C_2 \times C_2$, giving $|G| = (2k)(2m)/4 = km$.

Ultimately, however, these arguments depend on confirming the orders of various elements or subgroups, and perhaps the best way is to consider the more general group U with presentation $\langle\, x, y, z \mid x^2, y^2, z^2, (xy)^2, [x, (yz)^2], [y, (xz)^2] \,\rangle$.

In this group U, the elements $r^2 = (yz)^2$ and $s^2 = (zx)^2$ generate a normal subgroup N of index 8 with quotient $U/N \cong C_2 \times C_2 \times C_2$. Under conjugation, we have

$$(r^2, s^2)^x = (r^2, s^{-2}),\ \ (r^2, s^2)^y = (r^{-2}, s^2)\ \text{ and }\ (r^2, s^2)^z = (r^{-2}, s^{-2}).$$

Moreover, by Reidemeister-Schreier theory (as used in the previous section), the subgroup N is free abelian of rank 2. Hence for any positive integers j and ℓ we can factor out the normal subgroup $N^{(j,\ell)}$ generated by r^{2j} and $s^{2\ell}$ and obtain a quotient $U/N^{(j,\ell)}$ of order $8j\ell$.

In this quotient, if j and ℓ are odd then the element $r^j s^\ell z$ is a central involution. To prove this, we first note that $[r^j, s^\ell] = 1$, since the relations $(s^2)^r = s^{-2}$ and $(r^2)^s = r^{-2}$ found earlier give $s^\ell r^j = ss^{\ell-1}r^{j-1}r = sr^{j-1}s^{\ell-1}r = r^{1-j}srs^{1-\ell} = r^{1-j}r^{-1}s^{-1}s^{1-\ell} = r^j s^\ell$, from which it follows that $(r^j s^\ell z)^2 = r^j s^\ell r^{-j} s^{-\ell} = 1$. Then also

$$(r^j s^\ell z)^x = (r^{j-1} r s^\ell z)^x = r^{j-1} r^x s^{-\ell} z^x = r^j (r^{-1} r^x s^{-\ell}) s^{-2} z = r^j (zyxyzxs^{-2}) s^\ell z = r^j s^\ell z,$$
$$(r^j s^\ell z)^y = (r^j s^{\ell+1} s^{-1} z)^y = r^{-j} s^{\ell+1} (s^{-1} z)^y = r^j s^\ell (sys^{-1}zy) = r^j s^\ell z (xy)^2 = r^j s^\ell z,$$

and

$$(r^j s^\ell z)^z = r^{-j} s^{-\ell} z = r^j s^\ell z,$$

Hence we can factor out $\langle r^j s^\ell z \rangle$ and obtain a quotient of order $4j\ell = km$, which has the presentation we want.

5 Remarks

The proof of the classification of regular maps of negative prime Euler characteristic $-p$ in [7] proceeds as follows:

First, by the list given in [11] of all regular maps on non-orientable surfaces of genus 4 to 30 available at the time of publication of [7], one may suppose that $p \geq 29$. Next, for a regular map with Euler characteristic $\chi = -p$ and type $\{m, k\}$ given by a group G as in (1), the Euler formula gives $-p = \chi = (1/(2k) - 1/4 + 1/(2m))|G|$, or equivalently, $|G| = 4kmp/(km - 2k - 2m)$. By considering Sylow subgroups of G and a few other elementary arguments, one can conclude that p does not divide $|G|$. Hence $km - 2k - 2m$ must be a multiple of p, and further arguments give $|G| = tkm$ where $t \in \{1, 2, 4\}$.

These three possibilities for t translate easily to the three hypotheses on G given in Propositions 1 to 3. The corresponding Propositions 5.1 to 5.3 in [7] were proved with the help of the Gorenstein-Walter theorem. Finally, the equation $km - 2k - 2m = 4p/t$ is used to enumerate the ways G can be presented as in the conclusion of Proposition 2 and 3, up to isomorphism and duality.

It is interesting to note that our more elementary proofs of Propositions 1 to 3 occupy just about double the space taken up by the original proofs of Propositions 5.1 to 5.3 in [7], but of course the latter were proved with the help of the highly non-elementary Gorenstein-Walter theorem. In the remaining part of this paper, we make a few more observations about aspects of our proofs.

There are other (but perhaps less elementary) ways of proving Proposition 1 without invoking the Gorenstein-Walter theorem [17]. For example, having established Lemma 6 one could use Burnside's theorem [9] on groups containing no elements of even order other than involutions, among which the only perfect groups are those isomorphic to $SL(2, 2^\ell)$ for $\ell \geq 2$, and among those the only one with Sylow 2-subgroup of order 4 is $SL(2, 4) \cong A_5$. In the same vein, another option for proving Lemma 7 is to use Burnside's theorem on normal complements (as given in [16, Theorem 7.4.3]), again in combination with the fact that G is perfect.

Another (but much less elementary) way of avoiding [17] in proving Proposition 1 is to show that G is simple, which we do at the end of this section. Then after Lemma 6, we may invoke Theorem 15.2.1 from [16], which states that a simple group with a self-centralising Sylow 2-subgroup of order 4 is isomorphic to $PSL(2,q)$ for some $q \equiv 3$ or 5 mod 8. (This also follows from the Brauer-Suzuki-Wang theorem [6]; see [2] for a shorter proof.) Still another option would be to proceed from Lemma 6 by applying Theorem 15.2.5 of [16], which states that if G is a simple group with a Sylow 2-subgroup S of order 4 such that $S = C_G(g)$ for every $g \in S$, then G is isomorphic to A_5. (The proof of the latter theorem in [16] is based on permutation group theory developed by Frobenius and Zassenhaus.)

Our next comment concerns the presentation appearing as a conclusion of Proposition 2 in connection with the relator $r^3sr^3s^{-1}$ derived in the proof. If no reference to the involutions x, y, z in the presentation is made, the group $\langle r, s \mid r^k, s^4, (rs)^2, r^3sr^3s^{-1}\rangle$ for $k = 3j$ and $m = 4$ has also order $2km$, arguing again by normality of $\langle r^3\rangle$. This group is the orientation-preserving automorphism group of a canonical orientable double cover of the non-orientable map associated with the group G from Proposition 2. On the other hand, however, the group presented (using $r = yz$ and $s = zx$) in the form $\langle x, y, z \mid x^2, y^2, z^2, (xy)^2, r^k, s^4, (rs)^2, r^3sr^3s^{-1}\rangle$ has order $4km$ and is the full automorphism group of the canonical orientable double cover mentioned above. This shows that to specify the group G from the statement of Proposition 2 we need to add to the second presentation a relator expressing one of the involutions x, y, z in terms of r, s, such as the relator $(rs^{-1})^2x$ derived in the proof.

We continue with a similar comment on the concluding presentation in Proposition 3. If no reference to x, y, z in the presentation $\langle r, s \mid r^{2j}, s^{2\ell}, (rs)^2, (r^{-1}s)^2\rangle$ is made (which includes omission of the relator $r^js^\ell z$), the resulting group has order km for $k = 2j$, $m = 2\ell$ and is the orientation-preserving automorphism group of a canonical orientable double cover of the non-orientable map determined by the group G from Proposition 3. At the same time, the presentation $\langle x, y, z \mid x^2, y^2, z^2, (xy)^2, r^k, s^m, (r^{-1}s)^2\rangle$ determines the full automorphism group of the canonical double cover and has order $2km$. Here one sees that it is necessary to add the relator $r^js^\ell z$ to the last presentation; note that this relator implies that $G = \langle x, y, z\rangle = \langle r, s\rangle$ and, as we saw in the proof of Proposition 3, it also implies the relator $(r^{-1}s)^2$ appearing in the original statement of Proposition 5.3 of [7].

Finally, as promised earlier, we give the following, which shows directly that a group G satisfying the hypotheses of Proposition 1 is simple.

Proposition 10 *Let G be a group as given in* (1)*, with $G = \langle x, y, z\rangle = \langle r, s\rangle$ where $r = yz$ and $s = zx$, and with $|G| = 4km$ where k and m are odd and relatively prime, and $k, m \geq 3$. Then G is a non-abelian simple group.*

Proof. First G is generated by elements r and s of odd coprime orders k and m with $(rs)^2 = 1$, so its abelianisation G/G' is trivial. Hence G is perfect, and therefore insoluble.

Now let T be a minimal normal subgroup of G. Then T is either elementary abelian, or isomorphic to a direct product of isomorphic non-abelian simple groups. We will show that $T = G$. Then since T is minimal normal in G, it will follow that G itself is simple, and since G is perfect, also non-abelian.

Let a and b be the smallest positive integers such that $r^a \in T$ and $s^b \in T$, with orders k/a and m/b, respectively. Since these two orders are relatively prime, we have $|T| = (k/a)(m/b)c$ for some positive integer c. Similarly, G/T has two cyclic subgroups $\langle Tr\rangle$ and $\langle Ts\rangle$ of coprime orders a and b, and so we have $|G/T| = dab$ for some positive integer d, and then from $4km = |G| = |G/T|\,|T|$ we deduce that $cd = 4$ and hence that $d \in \{1, 2, 4\}$, so $|G/T| \in \{ab, 2ab, 4ab\}$. We consider these three cases in turn.

Case (1): Suppose $|G/T| = ab$. This is odd, and so $|T|$ is a multiple of 4 and hence T contains a Sylow 2-subgroup of G. But T is normal in G, and hence contains every Sylow 2-subgroup of G, and in particular, T contains $\langle x, y\rangle$. But then $G/T = \langle zT\rangle$, and as $|G/T|$ is odd, it follows that $G = T$, as required.

Case (2): Suppose $|G/T| = 2ab$. Then $|T| = 2km/(ab)$, so T contains an involution, and since T is normal in G, it must contain an involution from each Sylow 2-subgroup of G. In particular, T contains one of the elements x, y and xy from $\langle x, y\rangle$. Now if $x \in T$ or $y \in T$ then $G/T = \langle yT, zT\rangle$ or $\langle xT, zT\rangle$, and in both cases G/T is dihedral (of order $2ab \ge 2$), but then G cannot be perfect, contradiction. On the other hand, if $xy = rs \in T$, then $rT = s^{-1}T$, and as $G = \langle r, s\rangle$ it follows that $G/T = \langle rT\rangle = \langle s^{-1}T\rangle$ is a cyclic group of order dividing $\gcd(k, m) = 1$, another contradiction. Hence this case is impossible.

Case (3): Suppose $|G/T| = 4ab$. Then $|T| = km/(ab)$, and so T is a product $\langle r^a\rangle\langle s^b\rangle$ of two cyclic subgroups of orders k/a and m/b. By Îto's Theorem [19], the derived subgroup $T' = [T, T]$ is abelian, and so T is soluble and hence T itself is abelian. In particular, T is the direct product of $A = \langle r^a\rangle$ and $B = \langle s^b\rangle$, which have coprime orders k/a and m/b and hence are characteristic in T and therefore normal in G. By minimality of T, one of them must be trivial, say the latter, in which case $T = A = \langle r^a\rangle$. But now $G/C_G(T)$ is isomorphic to a subgroup of $\mathrm{Aut}(T) \cong \mathrm{Aut}(C_{k/a})$ which is abelian, and since G is perfect it follows that $G = C_G(T)$, so T is central in G. On the other hand, conjugation by z inverts the generator r^a of T (of odd order k/a), contradiction. Hence this case is impossible too.

We conclude that $G = T$ and is a simple group, as claimed. □

Acknowledgements

The first author acknowledges support from the N.Z. Marsden Fund, project UOA1626. The second author acknowledges support from the APVV Research Grants 15-0220 and 17-0428, and the VEGA Research Grants 1/0142/17 and 1/0238/19.

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