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GLOBAL EXISTENCE FOR NONCONVEX THERMOELASTICITY

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Abstract. We prove global existence for a simplified model of one-dimensional thermoelasticity. The governing equations satisfy the balance of momentum and a modified energy balance. The application we wish to study by investigating this model are shape-memory alloys. They are a prominent example of solids undergoing structural phase transitions. A characteristic feature of these materials is that several crystalline variants are stable at low temperature. Consequently, the free energy considered here is nonconvex as a function of the deformation gradient for temperatures below a fixed threshold temperature. As a result of the nonconvexity of the free energy density, existence of weak solutions is not to be generally expected. We therefore show existence of a Young measure valued solution. The proof relies on vanishing capillarity.

1 Introduction

This paper deals with the thermomechanics of a one-dimensional, heat-conducting, elastic solid of constant density $\rho \equiv 1$. We show that a simplified variant of an associated initial-boundary value problem proves solvable globally in time. The model under consideration arises from the theory of solid-solid phase transitions. To take phase changes into account, we assume that the (Helmholtz) free energy density is nonconvex as a function of the deformation gradient below a critical temperature $\theta_c > 0$. Such an assumption permits the modeling of shape-memory alloys. At the same time, the nonconvexity, combined with nonlinear coupling in the system, is the main challenge for the mathematical analysis.
The isothermal model also has drawn a lot of attention over the past two decades. The main reason for this is that, in general, existence of solutions in the weak sense can not be expected. Instead, approximate solutions will typically show ever finer oscillations between the stable phases of the energy. The limit will therefore not be a function, but a probability measure, which describes the likelihood of finding a certain phase at a given point in the material body. These kinds of measures are called *Young measures*. They can be considered as generalized solutions. For further references see, e.g., [Tar79, Ped97]. In recent years, Young measures have been successfully applied to various dynamical problems. See for example [Sle91, KP94] for parabolic equations, [Dem97] for wave equations and [DST01, Rie03] for (isothermal) elasticity and certain hyperbolic-parabolic systems. Different regularizations of the equations of thermoelasticity have been studied to avoid the aforementioned nonuniqueness that is typically associated with Young measures.

i) Existence results for viscous regularizations in the one-dimensional situation were, among others, obtained in [CH94, RZ97, Wat00]. Results in several space dimensions can be found in [FD97] for the isothermal problem, and in [Zim04] for the full thermoviscoelastic problem.

ii) Alternatively (or in addition to a viscous regularization), capillarity-like regularizations have been studied. The capillarity can be seen as an interfacial energy, penalizing the formation of ever finer structures. Existence and uniqueness in the one-dimensional case is studied by Sprekels and Zheng [SZ89]. Pawłow and Żochowski consider a model with capillarity and a viscosity-like term in several space dimensions [PŻ02].

For classical thermoelasticity (i.e., convex energy), by contrast, global existence can so far only be proved for small initial data or partially linearized systems [JR00, Chapter 6]. Specifically, blow-up results prove, in general, the non-existence of global classical solutions for large data. Global weak solutions for the partially linearized system (5)–(6) with convex energy were obtained by Durek [JR00, Section 6.6].
We consider a modified system of thermoelasticity, given by Equations (3)–(4). We show that it is possible to obtain a global existence result for arbitrary large data, even for nonconvex energy densities. To show existence of solutions to (3)–(4), we first introduce an artificial capillarity in Equation (3). We prove the existence of solutions for this modified system. The convergence of these solutions when the capillarity approaches zero is shown. Finally, we demonstrate that the limit solves the unperturbed system (3)–(4), as the capillarity approaches zero.

The present analysis is merely a first step in the endeavor to describe the limit system of vanishing capillarity in nonconvex thermoelasticity. This problem has been open for discussion since the late 1980s [SZ89]. A drawback of the system (3)–(4) is that positivity of the absolute temperature can, in general, not be expected. We finally remark that the existence proof presented here relies crucially on the fact that the nonlinear term in the heat equation can be written as a time derivative. Therefore, the methods employed here can not be applied in a straightforward manner to the full system (1)–(2).

2 Existence result

The solid will be identified with its reference configuration, which, by choosing the appropriate coordinates, can be assumed to be the unit interval $I := (0,1) \subset \mathbb{R}$. Let $T > 0$ be an arbitrary, but fixed time. The thermomechanical evolution of the body will be described in terms of the deformation field $u\colon I \times [0,T) \to \mathbb{R}$ and the absolute temperature field $\theta\colon I \times [0,T) \to \mathbb{R}$.
The equations of one-dimensional thermoelasticity (with Helmholtz free energy described by the Landau-Ginzburg model) are given by

$$u_{tt} - (\alpha_2 \theta u_x + \phi(u_x))_x = f, \tag{1}$$

$$\alpha_1 \theta_t - \kappa \theta_{xx} - \alpha_2 \theta u_x u_{tx} = g, \tag{2}$$

where $\kappa > 0$ is the heat conductivity, which will be assumed to be constant. The other terms appearing in (1)–(2) are explained below. The system (1)–(2) is to be equipped with initial conditions $u(\cdot,0) = u_0$, $u_t(\cdot,0) = u_1$ and $\theta(\cdot,0) = \theta_0$. We consider Dirichlet boundary conditions $u(0,\cdot) = u(1,\cdot) = 0$ in u, and Neumann boundary conditions $\theta_x(0,\cdot) = \theta_x(1,\cdot) = 0$ in θ.
The function ϕ is the isothermal part of the stress tensor, given by the derivative of the isothermal part of the stored energy density E. We assume

$$E(u_x,\theta) := \alpha_0 - \alpha_1 \theta \ln(\theta) + \frac{1}{2}\alpha_2 \theta u_x^2 - \frac{1}{2}\alpha_2 \theta_1 u_x^2 - \frac{1}{4}\alpha_4 u_x^4 + \frac{1}{6} u_x^6,$$

with $\alpha_0, \ldots, \alpha_4, \theta_1$ positive constants describing the properties of the thermoelastic material. In the following, we write $E(u_x,\theta) = \alpha_0 - \alpha_1 \theta \ln(\theta) + \frac{1}{2}\alpha_2 \theta u_x^2 + \Phi(u_x)$. Here, $\Phi(u_x)$ denotes the primitive of $\phi(u_x) = -\alpha_2 \theta_1 u_x - \alpha_4 u_x^3 + u_x^5$. The energy studied here is the classical Landau energy for martensitic phase transitions in one space dimension [Fal82]. (In the following we assume for simplicity $\alpha_0 = \cdots = \alpha_4 = \theta_1 = \kappa = 1$.) This energy has been chosen as the most relevant and frequently used one-dimensional energy for martensitic phase transitions. We point out that the linear dependence of the coupling part $\frac{1}{2}\alpha_2 \theta u_x^2$ in E on the temperature is not a simplification, but a widely accepted modeling assumption in the engineering literature.
It is not clear how to show that the temporal regularity of θ is high enough to prove existence of a solution of Equations (1)–(2). But as the temperature flow close to the critical temperature θ_c is of particular interest, we study the slightly modified system

$$u_{tt} - (\theta u_x + \phi(u_x))_x = f, \tag{3}$$

$$\theta_t - \theta_{xx} - \theta_c u_x u_{tx} = g. \tag{4}$$

Here, $\theta_c > 0$ is the *critical temperature*, that is, the temperature above which the noncaloric part of the free energy density E is convex. Another motivation for studying the system (3)–(4) is given by the following reasoning. Let us consider for a moment a nonconstant heat conductivity $\tilde{\kappa}(\theta) := \theta$ instead of κ. Suppose further that the caloric part of the energy E is given by $\alpha_0 - \frac{1}{2}\alpha_1\theta^2$. Then the heat equation without forcing (i.e., $g = 0$) reduces for $\alpha_0, \ldots, \alpha_4 = 1$ after a division by θ to

$$\theta_t - \theta_{xx} - \frac{\theta_x^2}{\theta} - u_x u_{tx} = 0.$$

This is, except for immaterial constants and the nonlinear lower order term $\frac{\theta_x^2}{\theta}$, identical to Equation (4) with $g = 0$.
The methods and results presented here are related to those obtained by Durek [JR00, Section 6.6], who considered the partially linearized system with convex energy given by

$$\begin{aligned} u_{tt} - (\phi(u_x))_x + \gamma\theta_x &= 0, && (5)\\ \delta\theta_t - \kappa\theta_{xx} + \gamma u_{tx} &= 0. && (6) \end{aligned}$$

with constant γ, δ, and κ. Her proof relies on vanishing viscosity and compensated compactness. The main difference between her approach and our work is that the system (3)–(4) satisfies an energy balance, and that we allow for nonconvex energy densities. Indeed, the proof of Lemma 3.3 reveals that, for $f = g = 0$, the conservation law

$$\int_I \left[\frac{1}{2}(u_t(\tau))^2 + \Phi(u_x(\tau)) + \frac{1}{\theta_c}(\theta(\tau))^2\right] \mathrm{d}x + \frac{1}{\theta_c}\int_0^\tau \int_I (\theta_x)^2 \,\mathrm{d}x\,\mathrm{d}t = \int_I \left[\frac{1}{2}(u_t(0))^2 + \Phi(u_x(0)) + \frac{1}{\theta_c}(\theta(0))^2\right] \mathrm{d}x \qquad (7)$$

holds for every $\tau \in [0, T)$.
In order to define solutions for the system (3)–(4), we use the concept of Young measures. A *Young measure* (or *parameterized measure*) is a family of probability measures $(\nu_x)_{x\in\Omega}$ on $\mathbb{R}^N$ associated with a sequence of measurable functions $f_j \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ such that for any continuous function $\phi \colon \mathbb{R}^N \to \mathbb{R}$ the function

$$\overline{\phi}(x) := \int_{\mathbb{R}^N} \phi(F)\, d\nu_x(F) =: \langle \phi, \nu_x \rangle$$

is measurable, and for every sequence $(f_j)_j$ converging weakly in $L^p(\Omega)$, we have

$$(\phi(f_j))_j \rightharpoonup \overline{\phi} \text{ in } L^p(\Omega).$$

We can think of the Young measure as a one-point statistic for the sequence $(f_j)_j$. Namely, ν_x describes (in a certain sense that can be made mathematically precise) the probability distribution of the values of the sequence $(f_j)_j$ at $x \in \Omega$ [Ped97].
A *gradient Young measure* is a Young measure generated by a sequence $(f_j)_j$ with $f_j = \nabla u_j$, where $u_j \in W^{1,2}(\Omega)$. In this case we often write $\nabla u_j \rightharpoonup \nu$. Whenever $(\nabla u_j)_j$ is uniformly bounded (e.g., in $L^2(\Omega)$), there exists a Young measure ν such that $\nabla u_j \rightharpoonup \nu$, see, e.g., [Tar79]. Introductions to Young measures and their applications can be found, e.g., in [Ped97].

Definition 2.1 (Young measure solutions) *Let $T \geq 0$, let*

$$\begin{aligned} u &\in W^{1,\infty}((0,T),L^2(I)) \cap L^\infty((0,T),W^{1,6}(I)), \\ \theta &\in W^{1,\infty}((0,T),L^1(I)) \cap L^2((0,T),H^1(I)), \end{aligned}$$

and let ν be a gradient Young measure with $\langle Id, \nu\rangle = u_x$ a.e.. Then (u,ν,θ) is a Young measure solution of Equations (3)–(4) *if for all $\xi, \zeta \in H_0^1((0,T),H_0^1(I))$*

$$\int_0^T \int_I [u_t\xi_t - u_x\theta\xi_x - \langle\phi,\nu\rangle\xi_x] \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_I f\xi \,\mathrm{d}x\,\mathrm{d}t, \tag{8}$$

$$\int_0^T \int_I \left[\theta\,\zeta_t - \theta_x\zeta_x - \frac{1}{2}\theta_c\langle|\cdot|^2,\nu\rangle\zeta_t\right] \mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_I g\zeta \,\mathrm{d}x\,\mathrm{d}t, \tag{9}$$

and if $u(\cdot,0) = u_0$, $u_t(\cdot,0) = u_1$, $\theta(\cdot,0) = \theta_0$, $u(0,\cdot) = u(1,\cdot) = 0$, $\theta_x(0,\cdot) = \theta_x(1,\cdot) = 0$.

We observe that a sufficiently smooth solution in the sense of Definition 2.1 satisfies (3)–(4) in the weak sense.

Remark 2.1 *Let (u,ν,θ) be a Young measure solution. Let u and θ be twice continuously differentiable and let $\nu := \delta_{u_x}$. Then u and θ satisfy* (3) *and* (4) *in the weak sense.*

The main result of this paper is existence of a Young measure solution to (3)–(4). The precise statement is given in the following theorem.

Theorem 2.2 (Global Existence Theorem) *Suppose $f, g \in L^2((0,T),L^2(I))$ and u_0, $u_1 \in H_0^1(I)$, $\theta_0 \in H^1(I)$. Under these assumptions, a Young measure solution to* (3)–(4) *exists.*

The idea of the proof of Theorem 2.2 is to consider the Equations (3)–(4) with an additional capillarity term of order ε. Similar equations were studied in [SZ89]. (The results obtained there carry over to the technically simpler case of Neumann boundary conditions $\theta_x(0,\cdot) = \theta_x(1,\cdot) = 0$.) The aim of Section 3 is to derive an existence result and a priori estimates for the regularized equations following the ideas of [SZ89, BS98]. In Section 4, we study the limit of these solutions as $\varepsilon \to 0$ and prove convergence to a Young measure solution of the system (3)–(4).
We remark that it seems to be difficult to obtain our results by using a viscous regularization, although this idea works well in the isothermal case. The reason is that the additional nonlinear term εu_{xt}^2 in the heat equation derived from the viscous regularization cannot be easily controlled as the viscosity goes to zero.

3 Existence for the regularized equations

In this section, we prove existence and uniqueness of the regularized problem

$$u_{tt}^\varepsilon - (u_x^\varepsilon\theta^\varepsilon)_x - (\phi(u_x^\varepsilon))_x + \varepsilon u_{xxxx}^\varepsilon = f, \tag{10}$$

$$\theta_t^\varepsilon - \theta_{xx}^\varepsilon - \theta_c u_x^\varepsilon u_{xt}^\varepsilon = g, \tag{11}$$

with initial and boundary conditions specified below.
We first state a local existence theorem.

Theorem 3.1 *Consider the system* (10)–(11)*, equipped with initial conditions* $u^\varepsilon(\cdot,0) = u_0$*,* $u_t^\varepsilon(\cdot,0) = u_1$*,* $\theta^\varepsilon(\cdot,0) = \theta_0$*, and boundary conditions* $u^\varepsilon(0,\cdot) = u^\varepsilon(1,\cdot) = 0$*,* $u_{xx}^\varepsilon(0,\cdot) = u_{xx}^\varepsilon(1,\cdot) = 0$*,* $\theta_x^\varepsilon(0,\cdot) = \theta_x^\varepsilon(1,\cdot) = 0$*. Suppose*

$$\begin{aligned} f &\in H^1((0,T),H^1(I)), \quad f_{tt} \in L^2((0,T),L^2(I)), && (12)\\ g &\in L^2((0,T),H^2(I)) \cap H^1((0,T),H^1(I)), && (13)\\ u_0 &\in H_E^5(I) := \{u \in H^5(I) \mid u(0) = u_{xx}(0) = u(1) = u_{xx}(1) = 0\}, && (14)\\ u_1 &\in H_E^4(I) := \{u \in H^4(I) \mid u(0) = u_{xx}(0) = u(1) = u_{xx}(1) = 0\}, && (15)\\ \theta_0 &\in H^3(I). && (16) \end{aligned}$$

Then a $T > 0$ *exists such that this initial-boundary value problem has a unique solution in* $[0,T)$*.*

Proof: The proof relies on the fact that (10) can be seen as a perturbed one-dimensional plate equation. Existence can be proved by deriving the necessary a priori estimates that allow an application of Tikhonov's fixed point theorem. We refrain from presenting the computations here and remark that the methods presented in [BS98] apply, with minor modifications, to the system (3)–(4). □

The main theorem of this section is given as follows.

Theorem 3.2 *Consider the system* (10)–(11)*, equipped with initial conditions* $u^\varepsilon(\cdot,0) = u_0$*,* $u_t^\varepsilon(\cdot,0) = u_1$*,* $\theta^\varepsilon(\cdot,0) = \theta_0$*, and boundary conditions* $u^\varepsilon(0,\cdot) = u^\varepsilon(1,\cdot) = 0$*,* $u_{xx}^\varepsilon(0,\cdot) = u_{xx}^\varepsilon(1,\cdot) = 0$*,* $\theta_x^\varepsilon(0,\cdot) = \theta_x^\varepsilon(1,\cdot) = 0$*. Suppose that the regularity assumptions* (12)–(16) *hold. Then the system* (10)–(11) *has a unique classical solution* $(u^\varepsilon,\theta^\varepsilon)$ *on* $I \times [0,T]$*, where* $T < \infty$ *is arbitrary. The functions* $u_{xxxx}^\varepsilon, u_{xxt}^\varepsilon, u_{tt}^\varepsilon, \theta_t^\varepsilon, \theta_{xx}^\varepsilon$ *all belong to* $\mathcal{C}^{\alpha,\frac{\alpha}{2}}$*, for some* $\alpha \in (0,1)$*.*

We start by deriving the following lemma, which provides an a priori energy estimate.

Lemma 3.3 *The following energy estimate holds, with a constant* C *independent of* $\varepsilon > 0$*:*

$$\sup_{t\in[0,T]} \int_I \left[(u_t^\varepsilon)^2 + \Phi(u_x^\varepsilon) + \frac{1}{\theta_c}(\theta^\varepsilon)^2 + \varepsilon(u_{xx}^\varepsilon)^2\right] \mathrm{d}x + \frac{1}{\theta_c}\int_0^T \int_I (\theta_x^\varepsilon)^2 \,\mathrm{d}x\,\mathrm{d}t < C. \tag{17}$$

Proof: A multiplication of (10) by u_t^ε, followed by integration over space and time, yields

$$\begin{aligned} &\frac{1}{2}\int_I (u_t^\varepsilon(t))^2 \,\mathrm{d}x + \int_I \Phi(u_x^\varepsilon)(t)\,\mathrm{d}x + \frac{\varepsilon}{2}\int_I (u_{xx}^\varepsilon(t))^2 \,\mathrm{d}x && (18)\\ &= -\int_0^t \int_I \theta^\varepsilon u_x^\varepsilon u_{xt}^\varepsilon \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_I (u_t^\varepsilon(0))^2 \,\mathrm{d}x \\ &\quad + \int_I \Phi(u_x^\varepsilon)(0)\,\mathrm{d}x + \frac{\varepsilon}{2}\int_I (u_{xx}^\varepsilon(0))^2 \,\mathrm{d}x + \int_0^t \int_I f u_t^\varepsilon \,\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

for almost every t in $[0,T]$. Since $\theta_c > 0$, it is possible to multiply (11) by $\frac{\theta^\varepsilon}{\theta_c}$. Integrating over space and time and using the Neumann boundary condition for θ^ε, we obtain

$$\begin{aligned} \frac{1}{\theta_c}\int_I (\theta^\varepsilon(t))^2 \,\mathrm{d}x + \frac{1}{\theta_c}\int_0^t \int_I (\theta_x^\varepsilon)^2 \,\mathrm{d}x\,\mathrm{d}s &= \int_0^t \int_I \theta^\varepsilon u_x^\varepsilon u_{xt}^\varepsilon \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \frac{1}{\theta_c}\int_I (\theta^\varepsilon(0))^2 \,\mathrm{d}x + \frac{1}{\theta_c}\int_0^t \int_I g\theta^\varepsilon \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{19}$$

Addition of equations (18) and (19), together with Young's inequality for the last terms on the right hand side of Equations (18) and (19), gives the desired result. □

From the bound on u^ε_{xx} in Lemma 3.3, the crucial estimate

$$\|u^\varepsilon_x\|_{L^\infty(L^\infty)} < C \tag{20}$$

follows via Sobolev embedding. Similarly, we get

$$\int_0^T ||\theta^\varepsilon(t)||^2_{L^\infty}\,\mathrm{d}t \le C.$$

Moreover we derive from Lemma 3.3

$$\begin{aligned}\sup_{0\le t\le T}\int_I |\theta^\varepsilon(t)|\,\mathrm{d}x \;&\le\; \sup_{0\le t\le T}\int_I \max\left(|\theta^\varepsilon(t)|,1\right)\,\mathrm{d}x \;\le\; \sup_{0\le t\le T}\int_I \max\left(|\theta^\varepsilon(t)|^2,1\right)\,\mathrm{d}x\\ &\le\; \sup_{0\le t\le T}\int_I |\theta^\varepsilon(t)|^2\,\mathrm{d}x \;+\; |I| \;\le\; C.\end{aligned}$$

Having obtained these crucial bounds, one can proceed as in [SZ89]. Indeed, a careful examination of [SZ89] reveals that the estimates obtained there (Lemma 2.6–Lemma 2.8) can be derived with straightforward modifications for the system under consideration. In particular, the modification of the boundary conditions and the lack of a lower bound on the temperature for the system under consideration do not alter the argumentation substantially. Therefore, we refrain from reproducing the corresponding estimates and refer the reader to the work by Sprekels and Zheng [SZ89].

4 Vanishing capillarity

We investigate system (3)–(4) as the limit of the system (10)–(11) studied in the previous section, as $\varepsilon \to 0$. In the limit case the assumptions (12)–(16) on the initial conditions and the conditions on the right hand sides f and g appear to be unnecessarily strong. We approximate them by functions with more natural regularities. We choose $f^\varepsilon \to f$ in $L^2(L^2)$, $g^\varepsilon \to g$ in $L^2(L^2)$, $u^\varepsilon_0 \to u_0$ in $H^1_0(I)$, $u^\varepsilon_1 \to u_1$ in $H^1_0(I)$ and $\theta^\varepsilon_0 \to \theta_0$ in $H^1(I)$, such that f^ε, g^ε, u^ε_0, u^ε_1, θ^ε_0 satisfy (12)–(16). The existence of such sequences f^ε, g^ε and θ^ε_0 is obvious. To construct approximating sequences u^ε_0 and u^ε_1 that satisfy the boundary conditions $(u^\varepsilon_0)_{xx}(0) = (u^\varepsilon_0)_{xx}(1) = 0$ and $(u^\varepsilon_1)_{xx}(0) = (u^\varepsilon_1)_{xx}(1) = 0$ requires a small technical trick which we illustrate for u_0.

First, we approximate $u_0 \in H^1_0(I)$ by smooth functions $u^\gamma_0 \in C^\infty(I)_0$ such that $u^\gamma_0 \to u_0$ in $H^1_0(I)$. Then we define

$$u^{\gamma,\delta}_0(x) := \begin{cases} (u^\gamma_0)_x(0)\cdot x & \text{for } x \in (0,\delta)\\ u^\gamma_0(x) & \text{for } x \in (2\delta, 1-2\delta)\\ (u^\gamma_0)_x(1)\cdot(1-x) & \text{for } x \in (1-\delta,1)\end{cases},$$

and interpolate smoothly the remaining values. The function $u^{\gamma,\delta}_0$ is smooth and satisfies $(u^{\gamma,\delta}_0)_{xx}(0) = (u^{\gamma,\delta}_0)_{xx}(1) = 0$. Moreover, $u^{\gamma,\delta}_0 \to u^\gamma_0$ in $H^1_0(I)$ as $\delta \to 0$. Taking a diagonal sequence of $u^{\gamma,\delta}_0$ we find a sequence u^ε_0 with the desired properties.

Now let $(u^\varepsilon, \theta^\varepsilon)$ be solutions of (10)–(11) for $\varepsilon > 0$ and consider the limit of vanishing capillarity. The coupling between elasticity and heat equation in thermoelasticity poses some technical difficulties that can so far only be solved in one space dimension.
By Lemma 3.3, we can extract subsequences of u^ε and θ^ε (not relabeled) such that $u^\varepsilon \rightharpoonup u$ in $L^\infty(W^{1,6}) \cap W^{1,\infty}(L^2)$, u_x^ε generates a Young measure ν and $\theta^\varepsilon \rightharpoonup \theta$ in $L^2(H^1) \cap L^\infty(L^1)$. We consider the weak formulation of the regularized system,

$$\begin{aligned} \int_0^T \int_I [u_t^\varepsilon \xi_t - u_x^\varepsilon \theta^\varepsilon \xi_x - \phi(u_x^\varepsilon)\xi_x \\ -\varepsilon u_{xx}^\varepsilon \xi_{xx}] \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^T \int_I f^\varepsilon \xi \,\mathrm{d}x\,\mathrm{d}t, \qquad (21) \\ \int_0^T \int_I \left[\theta^\varepsilon \zeta_t - \theta_x^\varepsilon \zeta_x - \frac{1}{2}\theta_c (u_x^\varepsilon)^2 \zeta_t\right] \mathrm{d}x\,\mathrm{d}t &= -\int_0^T \int_I g^\varepsilon \zeta \,\mathrm{d}x\,\mathrm{d}t, \qquad (22) \end{aligned}$$

for $\xi, \zeta \in \mathcal{C}_0^\infty(I \times (0,T))$ and study the limit of (21):
By the weak convergence of u_t^ε, we have

$$\int_0^T \int_I u_t^\varepsilon \xi_t \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_I u_t \xi_t \,\mathrm{d}x\,\mathrm{d}t.$$

To prove convergence of the second term we use the Div-Curl-Lemma (see, e.g., [Tar79]): Let $a^\varepsilon := (\theta^\varepsilon, 0)$, then $\operatorname{div} a^\varepsilon = \theta_x^\varepsilon$ is bounded in $L^2(I)$ and hence compact in $H^{-1}(I)$. Moreover, $b^\varepsilon := (u_x^\varepsilon, u_t^\varepsilon)$ satisfies $\operatorname{curl} b^\varepsilon = \partial_t u_x^\varepsilon - \partial_x u_t^\varepsilon = 0$. Thus, in the sense of distributions, the inner product $a^\varepsilon b^\varepsilon$ converges to the inner product ab, so $\theta^\varepsilon u_x^\varepsilon \to \theta u_x$. Since on the other hand $u_x^\varepsilon \theta^\varepsilon$ is bounded in $L^1(L^1)$, we deduce for a subsequence that

$$\int_0^T \int_I u_x^\varepsilon \theta^\varepsilon \xi_x \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_I u_x \theta \xi_x \,\mathrm{d}x\,\mathrm{d}t.$$

The definition of Young measures results for the third term of (21) in

$$\int_0^T \int_I \phi(u_x^\varepsilon)\xi_x \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_I \langle \phi, \nu \rangle \xi_x \,\mathrm{d}x\,\mathrm{d}t.$$

The Cauchy-Schwarz inequality together with the a priori estimate for u_{xx}^ε as stated in Lemma 3.3 gives

$$\int_0^T \int_I \varepsilon u_{xx}^\varepsilon \xi_{xx} \,\mathrm{d}x\,\mathrm{d}t \le \left(\varepsilon^2 \int_0^T ||u_{xx}^\varepsilon||^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T ||\xi_{xx}||^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \to 0.$$

The convergence of the right hand side in (21) is obvious, since $f^\varepsilon \to f$ in $L^2(L^2)$. Thus, we have obtained the limit of the elasticity equation (21).
Similarly, we can consider the limit of (22): By the weak convergence of θ^ε and θ_x^ε, the first two terms converge to $\int\int \theta \zeta_t \,\mathrm{d}x\,\mathrm{d}t$ and $-\int\int \theta_x \zeta_x \,\mathrm{d}x\,\mathrm{d}t$, respectively. The convergence of the third term follows (as in the corresponding term of the elasticity equation) from the definition of the Young measure. Finally, the right hand side converges since $g^\varepsilon \to g$ in $L^2(L^2)$.

We now pass to the limit $\varepsilon \to 0$ in Equations (21)–(22). We just have to combine the limits of the individual terms obtained behind Equation (22). The limiting equations for u, ν and θ read

$$\begin{aligned}
\int_0^T \int_I [u_t \xi_t - u_x \theta \xi_x - \langle \phi, \nu \rangle \xi_x] \, \mathrm{d}x \, \mathrm{d}t &= -\int_0^T \int_I f \xi \, \mathrm{d}x \, \mathrm{d}t, \\
\int_0^T \int_I \left[\theta \, \zeta_t - \theta_x \zeta_x - \frac{1}{2} \theta_c \langle | \cdot |^2, \nu \rangle \zeta_t \right] \mathrm{d}x \, \mathrm{d}t &= -\int_0^T \int_I g \zeta \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

The equality $\langle Id, \nu \rangle = u_x$ a.e. is the only remaining claim to prove. However, it follows immediately from the fact that ν is generated by u_x^ε and $u_x^\varepsilon \rightharpoonup u_x$. Thus, we have proved Theorem 2.2. □

5 Open problems

It seems interesting to generalize the results presented here to several space dimensions. In the one-dimensional situation, we build on a well-established existence theory for regularized equations of thermoelasticity with nonconvex energy. In more than one space dimension, we are only aware of relatively few results. In [PŻ02], a regularized model with viscosity and capillarity is studied. A model with a purely viscous regularization is investigated in [Zim04]. For equations of thermoviscoelasticity without a capillarity-like regularization, the situation is particularly subtle in several space dimensions, cf. [Zim04] for a discussion.

A further natural extension would be to study the original system (1)–(2) instead of (3)–(4).

Numerical analysis of nonconvex problems with Young measure valued solutions is an active area of research. See, e.g., [CR04] for an application to elastodynamics. An extension to the field of nonconvex thermoelasticity is likely to be an interesting, but challenging subject.

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