



Citation for published version:

Tsai, TL & Dawes, JHP 2011, 'Dynamics near a periodically forced robust heteroclinic cycle', *Journal of Physics: Conference Series*, vol. 286, no. 1, 012057. <https://doi.org/10.1088/1742-6596/286/1/012057>

DOI:

[10.1088/1742-6596/286/1/012057](https://doi.org/10.1088/1742-6596/286/1/012057)

Publication date:

2011

Document Version

Peer reviewed version

[Link to publication](#)

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Dynamics near a periodically forced robust heteroclinic cycle

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Abstract. In this article we discuss, with a combination of analytical and numerical results, a canonical set of differential equations with a robust heteroclinic cycle, subjected to time-periodic forcing. We find that three distinct dynamical regimes exist, depending on the ratio of the contracting and expanding eigenvalues at the equilibria on the heteroclinic cycle which exists in the absence of forcing. By reducing the dynamics to that of a two dimensional map we show how frequency locking and complex dynamics arise.

1. Introduction

Nonlinear differential equations used to model competitive or cooperative interactions are widely recognised to be capable of generating complex dynamics. In many cases, such differential equations also contain subspaces of the solution phase space that are flow-invariant. Such invariant subspaces arise, for example, in the presence of symmetry or through modelling assumptions. The existence of invariant subspaces further produces generic, and ‘natural’ kinds of dynamics that are not at all generic in the absence of invariant subspaces. One of these is heteroclinic cycling, which arises robustly in the presence of invariant subspaces.

A *heteroclinic cycle* is a collection of flow-invariant sets $\{\xi_1, \dots, \xi_n\}$ and connecting orbits $\{\gamma_1(t), \dots, \gamma_n(t)\}$ whose α - and ω -limit sets satisfy $\alpha(\gamma_i) = \xi_i$ and $\omega(\gamma_i) = \xi_{(i \bmod n)+1}$. A heteroclinic cycle is said to be *robust* if, for every $1 \leq i \leq n$ there exists an invariant subspace P_i such that

- the connecting orbit γ_i is contained in P_i ,
- $\xi_{(i \bmod n)+1}$ is a sink within P_i .

The flow-invariant sets ξ_1, \dots, ξ_n may of course be periodic orbits or chaotic invariant sets. For simplicity we will here consider the ξ_i to be equilibria, and the invariant subspace P_i to be two-dimensional. The presence of invariant subspaces gives rise to a natural division of the eigenvalues of the Jacobian matrix at ξ_i into four classes: radial eigenvalues r_{ij} are those whose eigenvector lies in $L_i \equiv P_{i-1} \cap P_i$; expanding eigenvalues $e_{ij} > 0$ correspond to $P_i \setminus L_i$; contracting eigenvalues $-c_{ij} < 0$ correspond to $P_{i-1} \setminus L_i$. All other eigenvalues are denoted transverse (t_{ij}). Stability of the robust heteroclinic cycle (RHC) can be ensured when the local contraction dominates the expansion near each equilibrium, as given by the following result due to Krupa and Melbourne [1].

Theorem. Asymptotic stability is guaranteed if

$$t_i < 0 \quad \forall i \quad \text{and} \quad \prod_{i=1}^n \min(c_i, e_i - t_i) > \prod_{i=1}^n e_i$$

where, at ξ_i , $t_i = \max_j \{Re t_{ij}\} < 0$ and $c_i = \min_j \{Re c_{ij}\} > 0$ are the real parts of the weakest transverse and contracting eigenvalues, and $e_i = \max_j \{Re e_{ij}\} > 0$ is the real part of the strongest expanding eigenvalue.

It should be noted that in many cases necessary and sufficient conditions for asymptotic stability differ from the above inequalities [2, 3]. Moreover, useful notions of stability that are weaker than asymptotic stability exist, for example ‘essential asymptotic stability’ [4, 5].

Turning to applications, there is a substantial literature on RHCs that has developed principally in two areas. The first of these applies methods developed in equivariant bifurcation theory [6, 7, 5] to fluid mechanical problems [8] and coupled oscillator networks [9, 10]. The second part of the literature is in evolutionary game theory and mathematical biology, including population dynamics [11, 12] and neuroscience [13]. The review article by M. Krupa [14] contains further details and many literature references.

In this article we examine the effect of an external time-periodic forcing on the dynamics near a RHC. While it is well-known that a constant symmetry-breaking perturbation typically breaks the heteroclinic cycle and generates a single periodic orbit which lies close to the remaining invariant subspaces, the effect of an external time-periodic forcing term has not previously been systematically studied. In this article we continue our earlier analysis [15] and identify and explain the existence of three distinct regimes as the eigenvalue ratio $\delta \equiv c/e > 1$ decreases towards unity. Our study is crucial to advancing the detailed understanding of the dynamics that arises when RHCs are coupled together [16, 17], since a second RHC acts similarly to an external periodic forcing. A full paper containing details of the results is currently in preparation [18].

The remainder of this article is laid out as follows. Section 2 introduces the model dynamical system. In section 2.1 we describe the systematic reduction of the dynamics to those of a two-dimensional map. Section 3 describes the dynamics at large δ where frequency locking does not occur (referred to as region I), and at intermediate δ where frequency locking appears (region II). In section 4 we reduce our return map to the well-known continuous-time ODEs for the forced damped pendulum and hence explain the bistability between frequency locking and quasiperiodic dynamics that arises when δ is very close to unity (region III).

2. Canonical example and ODE dynamics

Consider the vector field in \mathbb{R}^3 given by the differential equations

$$\dot{x} = x(1 - X - cy + ez) + \gamma(1 - x) \sin^2 \omega t, \quad (1)$$

$$\dot{y} = y(1 - X - cz + ex), \quad (2)$$

$$\dot{z} = z(1 - X - cx + ey), \quad (3)$$

where $X \equiv x + y + z$ and, for $\gamma = 0$, $c, e > 0$ are the absolute values of the contracting and expanding eigenvalues at the equilibrium points $\xi_1 \equiv (1, 0, 0)$, $\xi_2 \equiv (0, 1, 0)$ and $\xi_3 \equiv (0, 0, 1)$ which lie on a RHC. If $c > e$ (i.e. $\delta > 1$) then the above theorem due to Krupa and Melbourne implies that the RHC is asymptotically stable. System (1) - (3) with $\gamma = 0$ was analysed first by May and Leonard [19] as a model of competitive Lotka–Volterra type interactions between three populations. Independently [20], it was derived in the weakly nonlinear analysis of the Küppers–Lortz instability of convection rolls in rotating Rayleigh–Bénard convection. A proof

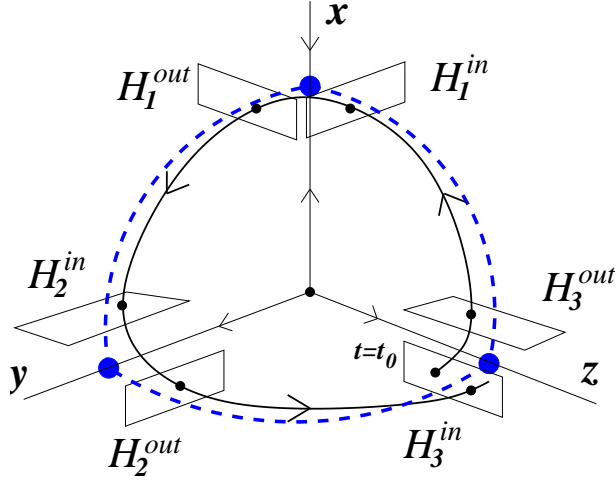


Figure 1. Robust heteroclinic cycle (thick dashed line) for (1) - (3) when $\gamma = 0$, and transverse planes H_i^{in} and H_i^{out} used in the return map construction for $\gamma > 0$.

of the existence of the RHC in (1) - (3) when $c, e > 0$ was given by Guckenheimer and Holmes [21]. Space does not permit a complete description of the dynamics of (1) - (3), save to remark that if $ce < 0$ then additional equilibria exist that destroy the connecting orbits on the RHC.

2.1. Return map

When $0 < \gamma \ll 1$ we analyse the dynamics by assuming that (after transients) trajectories continue to pass repeatedly through neighbourhoods of the three equilibria in turn, and converge to some (possibly chaotic) invariant set. The standard approach is to construct and analyse a Poincaré return map that gives the coordinates of successive intersections of a trajectory with a transverse surface, say H_3^{in} , near one of the equilibria, see figure 1. Between H_i^{in} and H_i^{out} we use the linearisation of (1) - (3) to integrate along trajectories and construct a ‘local map’. Between H_i^{out} and $H_{(i \bmod 3)+1}^{in}$ we estimate the leading order behaviour of trajectories near the unstable manifold $W^u(\xi_i)$ and so construct a ‘global map’. In both cases we consider the perturbation amplitude γ to be asymptotically small, and we keep only the leading order terms in γ . Composition of local and global maps results in a return map defined (at leading order) on two variables: the x -coordinate on H_3^{in} and the time t_n at which the trajectory hits H_3^{in} :

$$x_{n+1} = x_n^d + \gamma \mu_2 (1 - a_1 \cos 2\omega g_n - b_1 \sin 2\omega g_n) + \gamma f(x_n, t_n), \quad (4)$$

$$t_{n+1} = t_n + \mu_3 - \xi \log x_n - \gamma \frac{\xi}{2ex_n} (1 - a_2 \cos 2\omega t_n + b_2 \sin 2\omega t_n), \quad (5)$$

where we have kept terms up to $O(\gamma)$ only, $g_n \equiv t_n + \mu_3 - \xi_1 \log x_n$ and $d \equiv (c/e)^3$. We further define the parameters $\xi = (c^2 + ce + e^2)/e^3$, $a_1 = c^2/(c^2 + 4\omega^2)$, $a_2 = e^2/(e^2 + 4\omega^2)$, $b_1 = 2c\omega/(c^2 + 4\omega^2)$ and $b_2 = 2e\omega/(e^2 + 4\omega^2)$. The global maps cannot be computed analytically and lead to the introduction of the unknown function $f(x_n, t_n)$ into (4). For a suitable choice of $f(x_n, t_n)$ [15] the dynamics of the return map agree quantitatively extremely well with those of the ODEs (1) - (3) over a wide range of the parameters c , e and γ ; this is illustrated in figure 2. Figure 2 shows a sequence of intervals in ω in which stable, frequency-locked periodic orbits exist, separated by intervals of more complicated (quasiperiodic or chaotic) dynamics. In figure 2(a), it is interesting to note that for $\omega < 0.05$, the periodic orbit undergoes a period-doubling bifurcation within each frequency-locking interval; this bifurcation behaviour occurs in

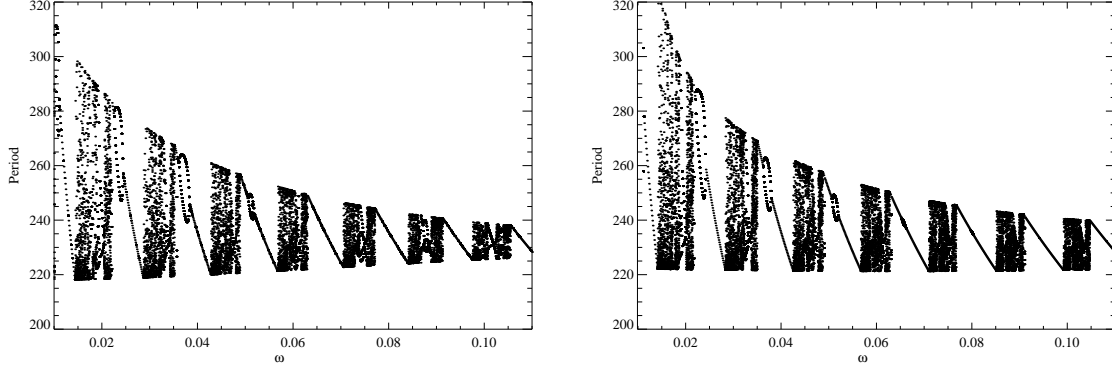


Figure 2. (a) Period T of trajectories of the ODEs (1) - (3) returning to H_3^{in} , as a function of driving frequency ω . (b) Dynamics of the return map (4) - (5), where the ‘period’ is $t_{n+1} - t_n$. Note the detailed agreement, including the regions of frequency locking in between complex dynamics (this corresponds to region II). $c = 0.25$, $e = 0.2$, $\gamma = 10^{-6}$.

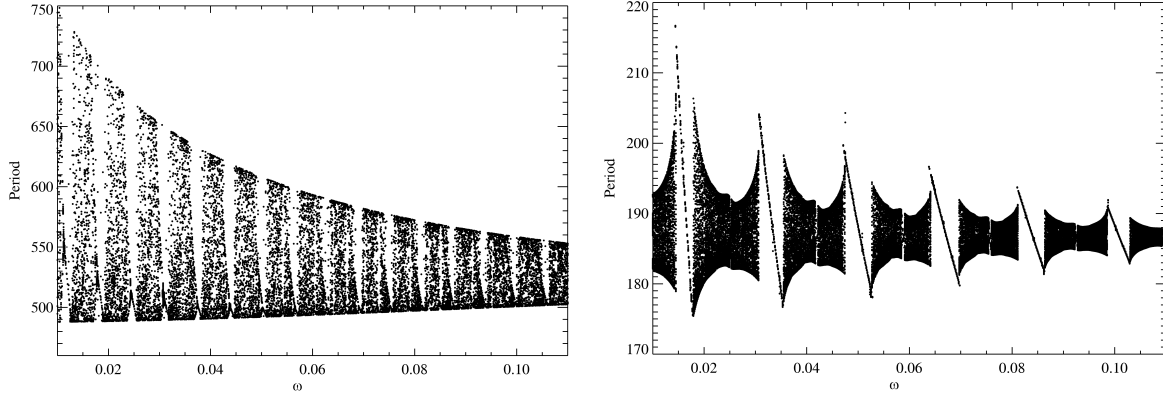


Figure 3. Period T of trajectories of the ODEs (1) - (3) returning to H_3^{in} , as a function of driving frequency ω . (a) $c = 0.5$ showing the disappearance of the intervals in ω of frequency locking at $\delta = 5/2$ (region I); (b) $c = 0.205$ showing the emergence of bistability between frequency-locking (slanted curve sections) and quasiperiodic dynamics (black regions), i.e. region III. Other parameters: $e = 0.2$, $\gamma = 10^{-6}$.

the map (4) - (5) as well [15]. It is also interesting to note that the position of the frequency-locked periodic orbit (which corresponds to a fixed point in the map) coincides exactly with one of the extrema of the ranges of periods for oscillations in the regions of complicated dynamics. Figure 3 illustrates the ODE dynamics for larger and smaller values of c (regions I and III respectively); the map dynamics at these parameter values also agree quantitatively with the ODE dynamics.

3. Regions I and II: complex dynamics and frequency locking

The system (1) - (3) was investigated by Rabinovich et al. [22] who noted the existence of intervals in ω containing frequency-locked periodic orbits, and presented an *ad hoc* construction of a circle map that was claimed to capture the system dynamics. In contrast, our results [15]

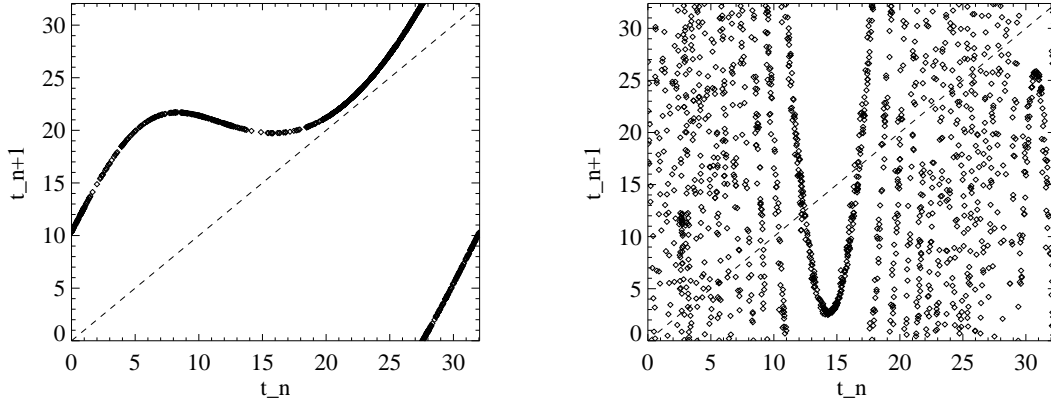


Figure 4. Plots of $t_{n+1} \bmod \pi/\omega$ against $t_n \bmod \pi/\omega$ for the ODEs (1) - (3). (a) One-dimensional circle-map-like dynamics for $c = 0.25$, $\omega = 0.098$ near the end of a frequency-locking interval in ω (corresponding to region II, same values of c and e as in figure 2); (b) complex dynamics for $c = 1.5$, $\omega = 0.097$ (corresponding to region I). Other parameters: $e = 0.2$, $\gamma = 10^{-6}$.

are based on a careful asymptotic reduction of the ODEs to the two-dimensional return map above. As discussed in [15], and illustrated in figure 4, numerical results obtained both for the ODEs (1) - (3) and for the map (4) - (5) indicate that the dynamics might be that of a circle map when δ is moderately close to unity (figure 4a), but is certainly not circle-map-like when δ is large (figure 4b). We distinguish between the region of complex dynamics that exists when $\delta \gg 1$ (region I) and the range of δ for which the dynamics contains substantial intervals of frequency-locking and therefore might be circle-map-like (referred to as region II). To determine whether or not the asymptotic dynamics is one-dimensional, we appeal to the ‘Annulus Principle’ due to Afraimovich et al. [23], reported also in Afraimovich and Hsu [24]:

Theorem (Annulus Principle). Suppose that the 2D map

$$x_{n+1} = F(x_n, t_n), \quad t_{n+1} = t_n + G(x_n, t_n) \pmod{2\pi},$$

maps an annulus $\mathcal{A} \equiv \{(x, t) : c_1 < x < c_2\}$ into itself, and also satisfies the four conditions

$$|(1 + G_t)^{-1}| < \infty, \quad 1 - |(1 + G_t)^{-1}| |F_x| > 2|(1 + G_t)^{-1}| \sqrt{|G_x| |F_t|}, \quad (6)$$

$$|F_x| < 1, \quad 1 + |(1 + G_t)^{-1}| |F_x| < 2|(1 + G_t)^{-1}|, \quad (7)$$

then the maximal attractor in \mathcal{A} is an invariant circle which is the graph of a smooth 2π -periodic function $x = h(t)$.

A direct analytic computation of the range of parameters c , e , ω under which the Annulus Principle holds for (4) - (5) becomes extremely messy. Instead we resort to a simplification of (4) - (5) that should be valid for δ near unity and $\omega \ll 1$ and which preserves the essential qualitative features of the dynamics. With these assumptions, we obtain the simplified map

$$x_{n+1} = x_n^d + \gamma\mu_2[1 - \sqrt{a_2} \cos(2\omega t_n)], \quad (8)$$

$$t_{n+1} = t_n + \mu_3 - \xi \log x_{n+1} \pmod{\pi/\omega}. \quad (9)$$

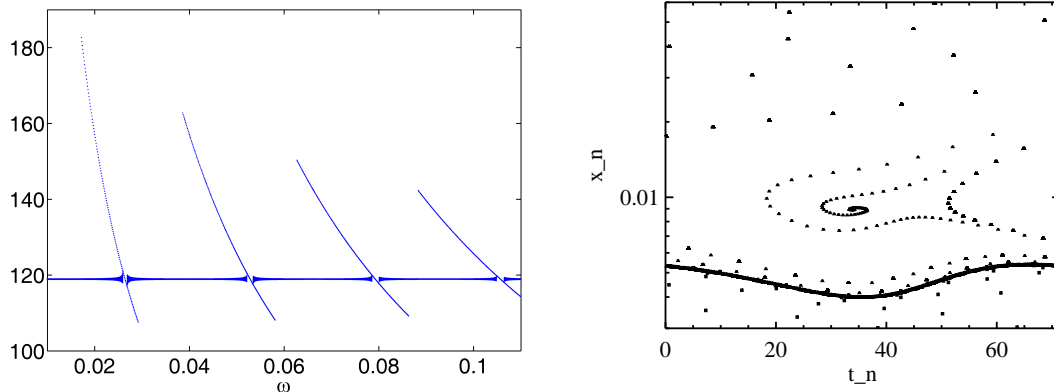


Figure 5. Dynamics in region III. (a) Period T of frequency-locked and quasiperiodic orbits in (8) - (9) returning to H_3^{in} , as a function of driving frequency ω . $c = 0.2001$, $e = 0.2$, $\gamma = 2 \times 10^{-5}$. (b) Dynamics of iterates of (4) - (5) in the (t, x) plane showing stable coexistence of frequency locked state (stable focus) and the invariant curve. $c = 0.20215$, $e = 0.2$, $\omega = 0.0439772$, $\gamma = 10^{-5}$.

Such a simplified map was discussed by Afraimovich, Hsu and Lin [25] as a model for the dynamics of a time-periodically forced system that is very similar to ours (the change of variable $y_n = x_n^d$ brings the simplified system (8) - (9) into their form). These authors refer to (8) - (9) as a ‘dissipative separatrix map’ since it corresponds closely to well-known and much-studied return maps near separatrices in perturbed Hamiltonian systems when $d = 1$. It is straightforward to check the inequalities in (6) - (7) for the dissipative separatrix map (8) - (9). Intriguingly, it turns out that the first of the four inequalities, (6a), is not satisfied in the limit $\delta \rightarrow 1^+$.

The qualitative nature of the conclusion is surprising; an attracting invariant curve $x = h(t)$ (and therefore one-dimensional dynamics) exists (numerically) for large enough $\delta > 1$, but we cannot guarantee that the dynamics remains one-dimensional as $\delta \rightarrow 1^+$. Further numerical results are consistent with this conclusion: for δ just above unity additional stable invariant sets exist - this region of bistability defines region III.

4. Region III: bistability between frequency locking and quasiperiodicity.

For δ close to unity we observe that there is bistability between frequency locking and quasiperiodic motion (region III), see figure 2(b). As $\delta \rightarrow 1^+$ this bistability becomes much more pronounced, see figure 5(a). The typical dynamics of iterates of (8) - (9) in the (t, x) phase space, near one of the frequencies ω where the frequency-locked state has a period close to that of the quasiperiodic dynamics (i.e. $T \approx 119$ in figure 5a) are shown in figure 5(b). Two fixed points exist away from the invariant curve; one saddle point and one stable focus. As usual, the stable manifold of the saddle provides the boundary dividing the basin of attraction of the invariant curve from that of the stable focus.

In the double limit in which $0 < \epsilon := \delta - 1 \ll 1$ and $\gamma \ll 1$ we find that (8) - (9) can be simplified to a second-order ODE for a rescaled time variable $s \sim t_{n+1} - t_n$; we obtain the canonical equation for a forced damped pendulum [26, 27]

$$\ddot{s} + \eta^{-1} \dot{s} - \sqrt{a_2} \cos s = \lambda,$$

where η and λ are combinations of the parameters in (8) - (9). If $|\lambda| > 1$ the only invariant set is a stable periodic orbit (the invariant curve corresponding to quasiperiodic dynamics here). If

$|\lambda| < 1$ and η^{-1} is large (i.e. at large $\epsilon > 0$) then only equilibria (frequency-locked solutions) exist; thus there are intervals in ω within which frequency locking occurs, but where the periodic orbit does not, see figure 2; this is region II. If $|\lambda| < 1$ and η^{-1} is small (i.e. at small ϵ) then equilibria coexist with the periodic orbit, as in figure 5; this bistability explains region III.

5. Summary

We have analysed the dynamics near a time-periodically forced robust heteroclinic cycle. There are three distinct regions for the dynamics; in region I the eigenvalue ratio $\delta \gg 1$ and frequency-locking does not occur. In region II, for intermediate values of δ , the 2D map can be reduced to a (non-invertible) circle map and frequency locking intervals exist. In region III where δ is close to unity, bistability exists and hence the dynamics is no longer that of a circle map: instead the 2D map can be reduced to the continuous time dynamics of a forced damped pendulum.

Since the forcing term in (1) - (3) has no particular nongeneric features we conjecture that the use of other forcing terms would lead to qualitatively similar dynamical regimes. Work in progress also indicates that similar dynamical regimes exist in systems of coupled RHCs.

Acknowledgments

TLT acknowledges past financial support from the Government of Taiwan and from the Cambridge Overseas Trust. JHPD is supported by the Royal Society through a University Research Fellowship.

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