Schiefcharaktere und reduzierte Kronecker-Produkte von Charakteren der Symmetrischen Gruppe

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Kurzzusammenfassung

In dieser Arbeit werden wir hauptsächlich Schiefcharaktere und zusätzlich im letzten Kapitel reduzierte Kronecker-Produkte untersuchen. Wir betrachten in dieser Arbeit ausschließlich Charaktere der symmetrischen Gruppen S_n über dem Körper $\mathbb{C}$. Schiefcharaktere sind Charaktere welche durch Schiefdiagramme beschrieben werden. Da das äußere Produkt von irreduziblen Charakteren bestimmten Schiefcharakteren entspricht, sind Resultate über Schiefcharaktere auch Resultate über das äußere Produkt. Des Weiteren gibt es starke Verbindungen zwischen Schiefcharakteren und schiefen Schur-Funktionen, Produkten von Schur-Funktionen sowie (wie in meiner Diplomarbeit gezeigt) Produkten von Schubert-Klassen. Ergebnisse über Schiefcharaktere können im Allgemeinen direkt in Ergebnisse in den anderen Gebieten übersetzt werden, welche wir jedoch in der Regel nicht noch einmal explizit erwähnen.

In Kapitel 1 geben wir eine Einführung zu den Charakteren der symmetrischen Gruppen (über $\mathbb{C}$) und schiefen Schur-Funktionen.

In Kapitel 2 leiten wir in Abschnitt 2.1 eine Formel für den Durchschnitt aller Diagramme von Komponenten eines gegebenen Schiefcharakters her. Wir beschreiben anschließend die Vereinigung aller Diagramme von Komponenten eines gegebenen äußeren Produktes oder speziellen Schiefcharakters. Dies liefert uns auch die Durfee-Größe in diesen Fällen. In Abschnitt 2.2 bestimmen wir für Schiefcharaktere die Komponenten mit maximalen diagonalen Hakenlängen. Wir erhalten einfache Formeln für die Hakenlängen sowie eine einfache Konstruktion für die Komponenten. Damit können wir Komponenten mit maximaler Durfee-Größe von äußeren Produkten und von speziellen Schiefcharakteren konstruieren. Dies gibt uns auch Bedingungen, die zwei Schiefdiagramme erfüllen müssen, damit sie den selben Schiefcharakter darstellen können.

In Kapitel 3 untersuchen wir wann zwei Schiefdiagramme den selben multiplizitätenfreien Schiefcharakter darstellen können. Wir werden sehen, dass dies nur möglich ist, wenn die Schiefdiagramme bis auf Rotation und Translation gleich sind oder wenn ein Schiefdiagramm eine Treppenpartition minus einer beliebigen anderen Partition ist, während das andere Schiefdiagramm dazu konjugiert ist.

In Kapitel 4 verallgemeinern wir die gestreckten Littlewood-Richardson-Koeffizienten $f(n) = c(n\lambda; n\mu, n\nu)$ zu $P(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$ und untersuchen diese Funktion. Dafür untersuchen wir die Anzahl der Komponenten und Konstituenten der Schiefcharaktere $[\mathcal{A}(n)] = [(n\lambda + \lambda')/(n\mu + \mu')]$.

In Kapitel 5 geben wir für Schiefcharaktere untere und obere Schranken für die Anzahl an Konstituenten, Komponenten und Paaren von Komponenten, dessen Diagramme sich um ein Kästchen unterscheiden, an. Wir klassifizieren schließlich alle Schiefcharaktere, welche maximal fünf Konstituenten haben, und alle, welche maximal fünf Komponenten haben.

In Kapitel 6 klassifizieren wir die reduzierten Kronecker-Produkte, die multiplizitätenfrei sind, und solche, welche maximal zehn Komponenten haben. Wir geben auch für die reduzierten Kronecker-Produkte untere Schranken für die Anzahl an Konstituenten, Komponenten und Paaren von Komponenten, dessen Diagramme sich um ein Kästchen unterscheiden, an. Weiterhin zeigen wir, dass zwei reduzierte Kronecker-Produkte nur dann gleich sein können, wenn ihre Faktoren übereinstimmen.

- Schlagwörter: Symmetrische Gruppe, Charaktere, Schur-Funktionen

Abstract

In this work we investigate mainly skew characters and in the last chapter reduced Kronecker products. Skew characters are characters of the symmetric group over the field $\mathbb{C}$ which correspond to skew diagrams (in this work the group is always the symmetric group while the field is $\mathbb{C}$). Outer products of irreducible characters correspond to certain skew characters, so results about skew characters are also results about outer products of irreducible characters. In fact, there is also a strong relation between skew characters and skew Schur functions, products of Schur functions and (as was pointed out in my Diploma thesis) products of Schubert classes. Results about skew characters can in general directly be transfered to results in the other mentioned contexts which we in general do not additionally point out.

In Chapter 1 we fix our notation and give an introduction to the characters of the symmetric group and symmetric functions.

In Chapter 2 we give in Section 2.1 formulas for the intersection of all diagrams of the components of a given skew character and the union of all diagrams of the components of a given outer product or certain kinds of skew characters. This gives us also the Durfee size of products and certain kinds of skew characters. In Section 2.2 we determine for skew characters the components having maximal principal hook lengths. We obtain an easy formula for those maximal principal hook lengths and an easy way to construct those characters. This gives us also a way to obtain components having maximal Durfee size for outer products and certain kinds of skew characters. We also get conditions for two skew diagrams to represent the same skew character.

In Chapter 3 we then show that two skew diagrams can represent the same multiplicity free skew character only in the trivial cases (rotation and translation) or if one skew diagram is a staircase partition minus an arbitrary partition and the other is the conjugated thereof.

In Chapter 4 we generalize the stretched LR coefficients $f(n) = c(n\lambda; n\mu, n\nu)$ to $P(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$ and investigate this function. For this we also analyze the number of components and constituents of the skew character $[\mathcal{A}(n)] = [(n\lambda + \lambda')/(n\mu + \mu')]$.

In Chapter 5 we determine those skew characters which have at most five constituents and those which have at most five components. For this a formula of Chapter 4 is crucial. We also give lower and upper bounds for the number of constituents, components and pairs of components, whose corresponding partitions differ by one box, of a given skew character.

Finally in Chapter 6 we determine those reduced Kronecker products which are multiplicity free and those which contain at most ten components. We also give lower bounds for the number of constituents, components and pairs of components, whose corresponding partitions differ by one box, of a given reduced Kronecker product. Furthermore, we prove that two reduced Kronecker products are equal only if they have the same factors.

- Keywords: Symmetric Group, Characters, Schur functions

Contents

CHAPTER 1

Introduction and Notation

In this chapter we will fix our notation and state known results. We want to keep this chapter rather short comparing to all the things which could be written about partitions, tableaux, representation theory in general and representation theory of the symmetric group in special, the Schubert calculus and the many appearances of the Littlewood Richardson coefficients. As we point out in Section 1.3 every result we obtain about characters is also a result about Schur functions.

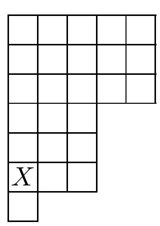
1.1. Partitions and tableaux

We mostly follow the standard notation in [**Sag**] or [**Sta**]. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a weakly decreasing sequence of non-negative integers where only finitely many of the λ_i are positive. We regard two partitions as the same if they differ only by the number of trailing zeros and call the positive λ_i the parts of λ. The length is the number of positive parts and we write $l(\lambda) = l$ for the length and $|\lambda| = \sum_i \lambda_i$ for the sum of the parts. Sometimes we will use the short notation $\lambda = (\lambda_1^{l_1}, \lambda_2^{l_2}, \ldots)$ which means that λ has l_1 times the part λ_1, l_2 times the part λ_2 and so forth. With a partition λ we associate a diagram, which we also denote by λ, containing λ_i left-justified boxes in the i-th row and we use matrix-style coordinates to refer to the boxes.

We will use δ_n always to refer to the staircase partition:

$$\delta_n = (n, n-1, n-2, \ldots, 2, 1).$$

We will call a partition λ with n different parts a $dp = n$ partition and write $dp(\lambda) = n$.

For example,  $= (5^3, 3^3, 1)$ is a $dp = 3$ partition with an X in position $(6, 1)$.

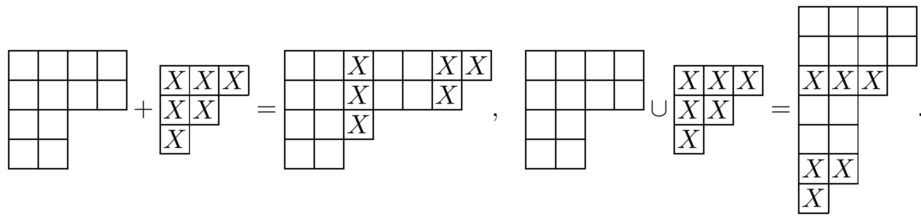
On the set of partitions we define the lexicographic order in the following way. For two partitions λ, μ we write $\lambda <_{\text{lex}} \mu$ if there is an i with $\lambda_i < \mu_i$ and for $j < i$ we have $\lambda_j = \mu_j$.

The conjugate λ^c of λ is the diagram which has λ_i boxes in the i-th column.

The sum $\mu + \nu = \lambda$ of two partitions μ, ν is defined by $\lambda_i = \mu_i + \nu_i$. The partition $\mu \cup \nu$ contains the parts of both μ and ν. These operations are conjugate to another:

$$(\mu + \nu)^c = \mu^c \cup \nu^c.$$

For example, we have:

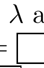
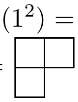
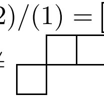


We sometimes say that for $\mu + \nu$ we insert the columns of ν into μ and for $\mu \cup \nu$ that we insert the rows of ν into μ. Note that this $+$ and $\cup$ introduce a partial order on the set of partitions and we say that a partition λ is larger than λ' if λ can be obtained from λ' by repeatedly using the operations $+, \cup$ with arbitrary partitions in any order. This should not be confused with the lexicographic order $<_{\text{lex}}$.

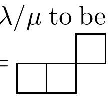
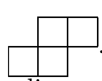
We define the intersection of two diagram $\mu \cap \nu = \lambda$ in the obvious way: $\lambda_i = \min(\mu_i, \nu_i)$.

The Durfee size $d(\lambda)$ of a partition λ is largest integer m such that the square (m^m) is contained in λ.

For $\mu \subseteq \lambda$, i.e. $\mu_i \leq \lambda_i$ for all i, we define the skew diagram λ/μ as the difference of the diagrams λ and μ defined as the difference of the set of the boxes. Rotation of λ/μ by $180°$ yields a skew diagram $(\lambda/\mu)^\circ$ which is well defined up to translation. We say that the skew diagram λ/μ is a proper skew diagram if neither λ/μ nor $(\lambda/\mu)^\circ$ is a partition. $|\lambda/\mu|$ is the number of boxes of λ/μ so $|\lambda/\mu| = |\lambda| - |\mu|$.

For two skew diagrams $\mathcal{A} = \lambda/\mu, \mathcal{B} = \lambda'/\mu'$ we define the operations $\mathcal{A} + \mathcal{B} = \alpha/\beta$ resp. $\mathcal{A} \cup \mathcal{B} = \alpha'/\beta'$ by $\alpha = \lambda + \lambda', \beta = \mu + \mu'$ resp. $\alpha' = \lambda \cup \lambda', \beta' = \mu \cup \mu'$. Clearly $\mathcal{A} + \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B}$ are then again skew diagrams. Usually we regard two skew diagrams as the same if they contain the same boxes but for this definition the underlying partitions λ, λ', μ and μ' are important because different choices for λ and μ would lead to different α/β. For example, we have $(2,1)/(1^2) = (2)/(1) =$ . But if we add in both cases (1^2) we would get: $(3,2)/(1^2) =$  $\neq$  $= (3,1)/(1)$. However, this will never cause any problem. We say that the skew diagram $\mathcal{A}^1$ is larger than $\mathcal{A}^2$ if there exist partitions $\lambda, \mu, \alpha, \beta$ such that up to translation $\lambda/\mu = \mathcal{A}^1, \alpha/\beta = \mathcal{A}^2$ and λ/μ can be obtained from α/β by repeatedly using the operations $+$ and $\cup$. So if $\mathcal{A}^1$ is larger than $\mathcal{A}^2$ then there exist skew diagrams $\mathcal{B}^i$ together with $\circ^i \in \{+, \cup\}$ such that:

$$\mathcal{A}^1 = \left(\cdots \left(\left(\mathcal{A}^2 \circ^1 \mathcal{B}^1 \right) \circ^2 \mathcal{B}^2 \right) \cdots \right) \circ^n \mathcal{B}^n.$$

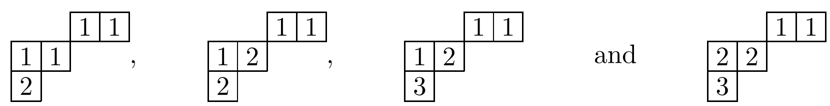
Notice that it is not enough that λ resp. μ are larger than α resp. β for λ/μ to be larger than α/β. For example, (2) is clearly larger than (1) but $(3,2)/(2) =$  is not larger than $(3,2)/(1) =$ .

A skew tableau T is a skew diagram in which the boxes are replaced by positive integers. We refer with $T(i,j)$ to the entry in box (i,j).

A standard (Young) tableau of shape λ/μ is a filling of λ/μ with the integers $1, 2, \ldots, |\lambda/\mu|$ such that the entries increase from left to right in each row and from top to bottom in each column. We denote the number of standard Young tableaux of shape λ/μ by $f^{\lambda/\mu}$.

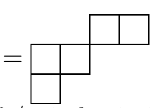
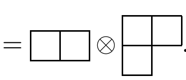
A semistandard tableau of shape λ/μ is a filling of λ/μ with positive integers such that the following inequalities hold for all (i,j) for which they are defined: $T(i,j) < T(i+1,j)$ and $T(i,j) \le T(i,j+1)$. So in a semistandard tableau the entries strictly increase among columns and weakly increase among rows. The content of a semistandard tableau T is $\nu = (\nu_1, \ldots)$ if the number of occurrences of the entry i in T is ν_i. The reverse row word of a tableau T is the sequence obtained by reading the entries of T from right to left and top to bottom starting at the first row. Such a sequence is said to be a lattice word if for all $i, n \ge 1$ the number of occurrences of i among the first n terms is at least the number of occurrences of $i+1$ among these terms. The Littlewood Richardson (LR) coefficient $c(\lambda;\mu,\nu)$ equals the number of semistandard tableaux of shape λ/μ with content ν such that the reverse row word is a lattice word. We will call those tableaux LR tableaux. The LR coefficients play an important role in different contexts as we will see later. We will refer to the above method of determining the LR coefficients as LR rule.

For example, the LR tableaux of shape $(4,2,1)/(2)$ are:



with content $(4,1), (3,2), (3,1^2), (2^2,1)$ respectively. So for $\lambda = (4,2,1), \mu = (2)$ we have $c(\lambda;\mu,\nu) = 1$ for $\nu \in \{(4,1),(3,2),(3,1^2),(2^2,1)\}$ and zero otherwise.

We say that a skew diagram $\mathcal{D}$ decays into the disconnected skew diagrams $\mathcal{A}$ and $\mathcal{B}$ if no box of $\mathcal{A}$ (viewed as boxes in $\mathcal{D}$) is in the same row or column as a box of $\mathcal{B}$. We write $\mathcal{D} = \mathcal{A} \otimes \mathcal{B}$ if $\mathcal{D}$ decays into $\mathcal{A}$ and $\mathcal{B}$. A skew diagram is connected if it does not decay.

So the above skew diagram $\mathcal{D} =$  decays into $\mathcal{D} =$ .

We say that a skew diagram λ/μ is basic if it doesn't contain empty rows or columns. So we have $\mu_i < \lambda_i$, $\mu_i \le \lambda_{i+1}$ for each $1 \le i \le l(\lambda)$. Note that empty rows and columns are irrelevant for LR fillings.

A hook is a partition which does not contain the subdiagram (2^2) and so is of the form $(r, 1^s)$. For each box (i,j) in a diagram λ we associate a hook starting at the box and going as far to the right and as far to the bottom as possible such that it still lies in λ. We define its arm respectively leg length as the number of boxes to the right resp. below of it in the same row resp. column. Furthermore, to a hook (or the box) we associate a ribbon which is the projection of the hook to the lower right border of λ. So the ribbon starts with the lower left box of the hook and follows the shape of λ to the upper right box of the hook. The hook length of a box is the sum of the arm and leg lengths plus 1 (for the box itself). Removing from λ the boxes of a certain hook corresponding to (i,j) and then shifting all the boxes which lie strictly to the right and strictly to the bottom of box (i,j) one box to the upper left yields a partition. The same partition is obtained by removing the corresponding ribbon. Suppose we choose the hook corresponding to the box

$(2,3)$ in $\lambda = (7,7,6,6,4,1)$. We then have

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
\ & \ & \ & \ & \ & \ & \ \\ \hline
\ & \ & X & X & X & X & X \\ \hline
\ & \ & X & \ & \ & \ \\ \cline{1-6}
\ & \ & X & \ & \ & \ \\ \cline{1-6}
\ & \ & X & \ \\ \cline{1-4}
\ \\ \cline{1-1}
\end{array}
\qquad \text{and} \qquad
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\ & \ & \ & \ & \ & \ & \ \\ \hline
\ & \ & \ & \ & \ & X & X \\ \hline
\ & \ & \ & \ & \ & X \\ \cline{1-6}
\ & \ & \ & X & X & X \\ \cline{1-6}
\ & \ & X & X \\ \cline{1-4}
\ \\ \cline{1-1}
\end{array}$$

where in the right diagram we marked the corresponding ribbon. Removing in the left diagram the boxes marked X and then shifting such that we have a partition yields the partition $(7,5,5,3,2,1)$, the same is obtained by removing the boxes marked X in the right diagram:

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
\ & \ & \ & \ & \ & \ & \ \\ \hline
\ & \ & \ & \ & \ \\ \cline{1-5}
\ & \ & \ & \ & \ \\ \cline{1-5}
\ & \ & \ \\ \cline{1-3}
\ & \ \\ \cline{1-2}
\ \\ \cline{1-1}
\end{array}\ .$$

There is a famous hook length formula which relates the number of standard Young tableaux f^λ of a given shape λ to the hook lengths which appear in λ. We have:

$$f^\lambda = \frac{|\lambda|!}{\prod(\text{hook length})}.$$

Obviously the number of standard Young tableaux with n boxes f_n is given by $f_n = \sum_{\lambda \vdash n} f^\lambda$.

For example, for $\lambda = (4^2, 2)$ we have the following hook lengths:

$$\begin{array}{|c|c|c|c|}
\hline
6 & 5 & 3 & 2 \\ \hline
5 & 4 & 2 & 1 \\ \hline
2 & 1 \\ \cline{1-2}
\end{array}$$

and so:

$$f^\lambda = \frac{10!}{6\cdot 5\cdot 3\cdot 2\cdot 5\cdot 4\cdot 2\cdot 1\cdot 2\cdot 1} = 9\cdot 7\cdot 4 = 252.$$

So there are 252 standard Young tableau of shape λ.

1.2. The symmetric groups and their characters

Since there are many (good) books on representation theory in general and the representation theory of the symmetric group in special (see for example [**Sag**], [**Sta**] or [**JK**]) we will keep this section rather short and state only the important facts.

The symmetric group S_n consists of all permutations σ of the set $\{1,\dots,n\}$, i.e. bijections from $\{1,\dots,n\}$ to itself. The group of bijections from any finite set Ω to itself is isomorphic to $S_{|\Omega|}$ and will be denoted S_Ω.

We let permutations act from the left and generally use the cycle notation. The cycle $(a,b,c,\dots e)$ maps a to b, b to c and so on and finally e to a. We say that a cycle is of length i if it contains i integers.

We can always write a permutation in cycle notation

$$\sigma = (1,\sigma 1,\dots,\sigma^{-1}1)(i,\sigma i,\dots \sigma^{-1} i)\cdots \in S_n$$

such that each each integer $1 \leq i \leq n$ appears exactly once. If there are m_i cycles of length i then we say the permutation is of cycle type $\lambda = (n^{m_n}, \ldots, 2^{m_2}, 1^{m_1})$ and have $\lambda \vdash n$. Permutations are in the same conjugacy class if and only if they have the same cycle type. So we can identify the conjugacy classes with the partitions. A transposition is a two cycle $(i\,j)$ and every permutation can be written as a product of transpositions. The sign function sign : $S_n \to \{\pm 1\}$ assigns a permutation σ the value $+1$ if σ is the product of an even number of transpositions and -1 if it is not.

A representation of a group G is a group homomorphism $\phi : G \to GL_m(F)$, so $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for $a, b \in G$, for some integer m and some field F. We always have the trivial representation $1 : a \mapsto 1$ and for $G = S_n, n \geq 2$ the sign representation sign which is different from 1 if the characteristic of F is not 2. In this work we always choose $G = S_n$ and $F = \mathbb{C}$. We have then the following nice situation: Every representation decays into a direct sum of irreducible representations. A representation $\phi : S_n \to GL_m$ is irreducible if there is no nontrivial subspaces of $\mathbb{C}^m$ which is invariant under the action of the S_n via ϕ, so if for all $\sigma \in S_n$ we have $\phi(\sigma)V \subseteq V$ for a subspace $V \subseteq \mathbb{C}^m$ then $V = 0$ or $V = \mathbb{C}^m$.

A character χ_ϕ is the trace of a representation ϕ:

$$\chi_\phi : S_n \to \mathbb{C}, \chi(\sigma) = \text{trace } \phi(\sigma).$$

Nearly everything interesting about the representations can be read off from the characters while the characters have many more nice properties. Characters corresponding to irreducible representations are called irreducible characters. If a representation is the direct sum of some irreducible representations then the corresponding character is the direct sum of the corresponding irreducible characters. The number of irreducible characters is the number of conjugacy classes and, therefore, for the S_n given by p_n, the number of partitions of n.

The irreducible characters $[\lambda]$ of the symmetric group S_n are naturally labeled by partitions $\lambda \vdash n$. They form a basis for the ring of class function (i.e. functions $f : S_n \to \mathbb{C}$ which are constant on the conjugacy classes) and so we can write every class function f or character χ of the symmetric group S_n in a unique way as $f = \sum_{\lambda \vdash n} a_\lambda[\lambda]$ resp. $\chi = \sum_{\lambda \vdash n} b_\lambda[\lambda]$ and have $b_\lambda \in \mathbb{N}_0$. We write $[\lambda] \in \chi$ if $b_\lambda > 0$ and in this case say that $[\lambda]$ is contained in χ. We say that $[\lambda]$ is a component of $\chi = \sum_{\lambda \vdash n} b_\lambda[\lambda]$ if $b_\lambda \neq 0$ and say that χ has $\sum_{b_\lambda \neq 0} 1$ components and $\sum_\lambda b_\lambda$ constituents. If χ has a components and b constituents we say that χ is of cc-type (a, b) and write $cc(\chi) = (a, b)$. We are sometimes sloppy when we talk about characters being contained in another character whereas we mean the irreducible characters being contained in the other character. The trivial character $1(\sigma) = 1$ is $[n]$ while the sign character sgn is given by $[1^n]$. The dimension of the irreducible character $[\lambda]$ is given by the number of standard young tableaux f^λ of shape λ for which is calculated by the hook formula (see the previous section).

We have an inner product in the ring of class function such that the irreducible characters form an orthonormal basis. We define in the ring of class functions of the S_n an inner product by

$$\langle \chi, \psi \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma)\overline{\psi(\sigma)},$$

where $\overline{a}$ is complex conjugation. Now $\langle [\mu], [\nu] \rangle = \delta_{\mu,\nu}$ and, therefore, if $\chi = \sum_\lambda c_\lambda[\lambda]$ then $c_\lambda = \langle \chi, [\lambda] \rangle$.

Since the conjugacy classes are indexed by partitions it makes sense to evaluate a character χ on a partition λ by defining $\chi(\lambda)$ as $\chi(\sigma)$ such that σ is of cycle type λ. There is a famous formula to evaluate the irreducible character $[\lambda]$ on the conjugacy class μ. Let μ contain some part a and let $\bar{\mu}$ be the partition obtained from μ by removing one part a. Then the Murnaghan Nakayama formula states the following

$$[\lambda](\mu) = \sum_x (-1)^{ll(x)} [\lambda \setminus x](\bar{\mu})$$

where the sum is over all hooks x of λ with hook length equal to a, $ll(x)$ is the leg length of x and $\lambda \backslash x$ is the partition obtained by removing the ribbon corresponding to x from λ. For example, let $\lambda = (5^2, 4, 3)$ and we want to evaluate $[\lambda]$ on the conjugacy class $\mu = \bar{\mu} \cup (3)$ which contains a part 3. We have the following hook lengths in λ

8	7	6	4	2
7	6	5	3	1
5	4	3	1	
3	2	1		

and so by the Murnaghan Nakayama rule:

$$[\lambda](\mu) = (-[5,3,3,3] - [5,5,2,2] + [5,5,4])\,(\bar{\mu}).$$

Since the characters are class functions we can use the ordinary product of functions and define the Kronecker product as

$$[\mu][\nu](\sigma) = [\mu](\sigma) \cdot [\nu](\sigma)$$

for all $\mu, \nu, \sigma \vdash n$. We can write this product as a sum of irreducible characters

$$[\mu][\nu] = \sum_\lambda g(\lambda, \mu, \nu)[\lambda]$$

which defines the Kronecker coefficients $g(\lambda, \mu, \nu) \in \mathbb{N}_0$. Using the scalar product we also have

$$g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\sigma \in S_n} [\mu](\sigma)[\nu](\sigma)\overline{[\lambda](\sigma)}$$

where $\overline{a}$ denotes complex conjugation and $\lambda, \mu, \nu \vdash n$. By the Murnaghan Nakayama Rule the character values are integers (so the complex conjugation doesn't matter) and so the Kronecker coefficient is symmetric in λ, μ, ν. Multiplication by the sign character conjugates the irreducible characters $[\lambda][1^n] = [\lambda^c]$ if $\lambda \vdash n$ which gives us the additional symmetry of the Kronecker coefficient:

$$g(\lambda, \mu, \nu) = g(\lambda^c, \mu, \nu^c).$$

We can identify $S_m \times S_n$ with the subgroup of S_{m+n} which permutes the first m and the last n integers independent of each other. So we can also define an outer product of two characters of different symmetric groups by inducing the obtained character from the subgroup $S_m \times S_n$ to S_{m+n}. Let $\mu \vdash m, \nu \vdash n$ then

$$[\mu] \times [\nu](\sigma, \rho) = [\mu](\sigma)[\nu](\rho)$$

is a character of $S_m \times S_n$. We then write $[\mu] \otimes [\nu]$ for the corresponding induced character of S_{m+n} which can again be written as a sum of irreducible characters:

$$[\mu] \otimes [\nu] = ([\mu] \times [\nu]) \uparrow_{S_m \times S_n}^{S_{m+n}} = \sum_\lambda c(\lambda; \mu, \nu)[\lambda].$$

Here the $c(\lambda;\mu,\nu)\in\mathbb{N}_0$ are again the famous Littlewood Richardson coefficients already defined in the previous section.

We can also define characters, called skew characters, corresponding to skew diagrams λ/μ:

$$[\lambda/\mu]=\sum_\nu c(\lambda;\mu,\nu)[\nu].$$

Dvir proved in his paper [**Dvir**] (Clausen and Meier proved in [**CM**] similar results) the following: The largest λ_1 of all $[\lambda]\in[\mu][\nu]$ is given by $\lambda_1=|\mu\cap\nu|$. Furthermore, for $\alpha=\mu\cap\nu$ the multiplicity of $[\beta]$ in $[\mu/\alpha][\nu/\alpha]$ and $[(|\alpha|,\beta)]$ in $[\mu][\nu]$ are the same. So there is a strong relation between skew characters and the Kronecker product.

We have the following symmetries for the LR coefficients and skew characters:

The translation symmetry gives $[\lambda/\mu]=[\alpha/\beta]$ if the skew diagrams of λ/μ and α/β are the same up to translation while rotation symmetry gives $[(\lambda/\mu)^\circ]=[\lambda/\mu]$. The conjugation symmetry $c(\lambda^c;\mu^c,\nu^c)=c(\lambda;\mu,\nu)$ is also well known and, furthermore, it is $c(\lambda;\mu,\nu)=c(\lambda;\nu,\mu)$.

If $\mathcal{D}=\mathcal{A}\otimes\mathcal{B}=\mathcal{C}$ then by translation symmetry $[\mathcal{D}]=[\mathcal{C}]$, so reordering $\mathcal{A},\mathcal{B}$ doesn't change the skew character. A skew character whose skew diagram $\mathcal{D}$ decays into disconnected (skew) diagrams $\mathcal{A},\mathcal{B}$ is equivalent to the product of the characters of the disconnected diagrams induced to a larger symmetric group. We have

$$[\mathcal{D}]=([\mathcal{A}]\times[\mathcal{B}])\uparrow_{S_n\times S_m}^{S_{n+m}}=:[\mathcal{A}]\otimes[\mathcal{B}]$$

with $|\mathcal{A}|=n,|\mathcal{B}|=m$. If $\mathcal{D}=\lambda/\mu$ and $\mathcal{A},\mathcal{B}$ are proper partitions α,β we have:

$$[\lambda/\mu]=\sum_\nu c(\lambda;\mu,\nu)[\nu]=\sum_\nu c(\nu;\alpha,\beta)[\nu]=[\alpha]\otimes[\beta].$$

We say that a character $\chi=\sum_\lambda c_\lambda[\lambda]$ is of cc-type (a,b) if χ has $a=\sum_{c_\lambda\neq 0}1$ components and $b=\sum_\lambda c_\lambda$ constituents. We then write $cc(\chi)=(a,b)$ and if $\chi=[\mathcal{A}]$ is a skew character we also write sometimes just $cc(\mathcal{A})$ instead of $cc([\mathcal{A}])$.

1.3. Symmetric functions

In this section we give a short introduction to symmetric functions based on Sagan's book [**Sag**].

We start with the ring of formal power series $\mathbb{C}[[x_1,x_2,\ldots]]=:\mathbb{C}[[x]]$ with infinitely many variables x_i. A weak composition $\omega=(\omega_1,\omega_2,\ldots)$ is an infinite sequence of non negative integers with only a finitely many ω_i nonzero. For a weak composition ω we set $x^\omega=x_1^{\omega_1}x_2^{\omega_2}\cdot$ and say that this monomial is of degree $n=\sum_i\omega_i\in\mathbb{N}_0$. $f(x)=\sum_{\omega^i}c_ix^{\omega_i}\in\mathbb{C}[[x]],c_i\neq 0$ is homogeneous of degree n if every monomial x^{ω_i} is of degree n, $\sum_j\omega_j^i=n$ for every i.

We let a permutation $\pi\in S_n$ act on $f(x)\in\mathbb{C}[[x]]$ in a natural way by $\pi f(x_1,x_2,x_3,\ldots)=f(x_{\pi 1},x_{\pi 2},x_{\pi 3},\ldots)$ with $\pi i=i$ for $i>n$.

Obviously the monomial symmetric functions m_λ, defined for partitions λ by $m_\lambda=\sum x^\omega$ where the sum is over all different weak compositions ω which ordered in a weakly decreasing order give the partition λ, is fixed under the action of π and is of degree $|\lambda|$.

We define the ring of symmetric functions Λ as the vector space spanned by all the monomial symmetric functions. Since we can multiply symmetric functions and get again a symmetric function this vector space is, in fact, a ring. The ring Λ

is graded $\Lambda = \bigoplus_{n\geq 0} \Lambda^n$ where Λ^n is the subspace spanned by all m_λ of degree n, $\lambda \vdash n$ (with $\Lambda^0 = \mathbb{C}$). So $\dim_{\mathbb{C}} \Lambda^n = p_n$ with p_n the number of partitions of n.

There are other important bases. We define the nth power symmetric function as

$$p_n = m_{(n)} = \sum_{i\geq 1} x_i^n,$$

the nth elementary symmetric function as

$$e_n = m_{(1^n)} = \sum_{i_1<\cdots<i_n} x_{i_1}\cdots x_{i_n}$$

and the nth complete homogeneous symmetric functions as

$$h_n = \sum_{\lambda\vdash n} m_\lambda = \sum_{i_1\leq\cdots\leq i_n} x_{i_1}\cdots x_{i_n},$$

with

$$p_0 = e_0 = h_0 = 1.$$

Furthermore, we set for a partition $\lambda \vdash n$:

$$p_\lambda = p_{\lambda_1}\cdots p_{\lambda_n}, \qquad e_\lambda = e_{\lambda_1}\cdots e_{\lambda_n}, \qquad h_\lambda = h_{\lambda_1}\cdots h_{\lambda_n}.$$

We have that all three sets $\{p_\lambda\}$, $\{e_\lambda\}$ and $\{h_\lambda\}$ are bases for Λ if λ runs over all partitions and bases for Λ^n if λ runs over all partitions of n.

The most important symmetric functions are the Schur functions s_λ. For a skew diagram λ/μ we define:

$$s_{\lambda/\mu} = \sum_{T \text{ SSYT of shape } \lambda/\mu} x^\nu$$

where the sum is over all semistandard Young tableaux T of shape λ/μ and ν is the content of T. The Schur functions $s_{\lambda/\mu}$ are symmetric functions (which is not obvious by definition) and the set $\{s_\lambda\}$ is a basis of Λ if λ runs over all partitions and a basis for Λ^n if λ runs over all partitions of n. The famous Jacobi-Trudi Determinant connects the Schur functions with the complete homogeneous respectively the elementary symmetric functions. For $\lambda = (\lambda_1, \cdots, \lambda_l)$ we have:

$$s_\lambda = |h_{\lambda_i - i+j}|$$

and

$$s_{\lambda^c} = |e_{\lambda_i - i+j}|$$

where both are $l\times l$ determinants and $h_n = e_n = 0$ for negative n. The Jacobi-Trudi determinant allows us to define Schur functions s_λ even if λ is a weak composition or has negative parts.

The product of two Schur functions can be decomposed in the Schur function basis and we get:

$$s_\mu s_\nu = \sum_\lambda c(\lambda;\mu,\nu) s_\lambda$$

where the $c(\lambda;\mu,\nu)$ are again the LR coefficients.

So the ordinary product of the Schur functions $s_\mu s_\nu$ corresponds to the outer product of irreducible characters $[\mu]\otimes[\nu] = \sum_\lambda c(\lambda;\mu,\nu)[\lambda]$. So everything said about the outer product of irreducible characters can be directly translated to the ordinary product of Schur functions.

Furthermore, decomposing the skew Schur function $s_{\lambda/\mu}$ in the Schur function basis we get:

$$s_{\lambda/\mu} = \sum_{\nu} c(\lambda;\mu,\nu) s_\nu$$

where the $c(\lambda;\mu,\nu)$ are also again the LR coefficients. Recall that

$$[\lambda/\mu] = \sum_{\nu} c(\lambda;\mu,\nu)[\nu]$$

so everything said about skew characters can also be directly translated to skew Schur functions.

We want to mention two more facts about the Schur functions (which we do not use in this work). First of all there is the characteristic map: Let R^n be the space of class functions of S_n. It is clear that $\dim R^n = p_n = \dim \Lambda^n$ so R^n and Λ^n are isomorphic vector spaces. For the usual inner product of R^n the irreducible characters form an orthogonal basis. We can define an inner product on Λ^n by

$$\langle s_\mu, s_\nu \rangle = \delta_{\mu,\nu}$$

with sesquilinear extension (complex conjugation in the second variable). Define now the map $\mathrm{ch}^n : R^n \to \Lambda^n$ by:

$$\mathrm{ch}^n(\chi) = \sum_{\lambda \vdash n} z_\lambda^{-1} \chi(\lambda) p_\lambda$$

where $z_\lambda = \prod_i i^{\lambda_i} \lambda_i!$ is the order of the centralizer of a permutation having cycle type λ.

We have $\mathrm{ch}^n([\lambda]) = s_\lambda$ and so ch^n is an isometry between R^n and Λ^n.

We close this section with the original definition of the Schur functions, given by Jacobi in 1841. For this we restrict Schur functions to finitely many variables $x_1, \ldots, x_l$ be setting all the other variables $x_i = 0$ for $i > l$. For a Schur function s_λ we assume that $l \geq l(\lambda)$.

For a weak composition $\mu = (\mu_1, \ldots, \mu_l)$ we define the alternant a_μ as:

$$a_\mu = \sum_{\pi \in S_l} (\mathrm{sgn}\ \pi) \pi x^\mu.$$

Note that a_μ is, therefore, also given by the determinant $a_\mu = \left| x_i^{\mu_j} \right|_{1 \leq i,j \leq l}$. So for the staircase partition $\delta = \delta_{l-1} = (l-1, l-2, \ldots, 1, 0)$ we have the Vandermonde determinant:

$$a_\delta = \prod_{1 \leq i < j \leq l} (x_i - x_j).$$

Then the original definition for the Schur function s_λ with $l \geq l(\lambda)$ variables is:

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}.$$

1.4. Schubert product

We define the Schubert product by restricting the ordinary outer product. We have $[\mu] \otimes [\nu] = \sum_\lambda c(\lambda;\mu,\nu)[\lambda]$ and restrict for the Schubert product $\star_{(k^l)}$ the sum to those partitions which lie in the rectangle (k^l):

$$[\mu] \star_{(k^l)} [\nu] := \sum_{\lambda \subseteq (k^l)} c(\lambda;\mu,\nu)[\lambda].$$

This corresponds to the product of two Schubert classes. In the cohomology ring $H^*(Gr(l,\mathbb{C}^n),\mathbb{Z})$ of the Grassmannian $Gr(l,\mathbb{C}^n)$ of l-dimensional subspaces of $\mathbb{C}^n$ the product of two Schubert classes σ_μ,σ_ν is given by:

$$\sigma_\mu \cdot \sigma_\nu = \sum_{\lambda\subseteq((n-l)^l)} c(\lambda;\mu,\nu)\sigma_\lambda.$$

Obviously for $k \geq \mu_1+\nu_1, l \geq l(\mu)+l(\nu)$ the Schubert product is just the ordinary outer product:

$$[\mu] \star_{(k^l)} [\nu] = [\mu] \otimes [\nu].$$

In my Diploma thesis [**Gut1**, Section4] we established a close connection between the Schubert product and skew characters. [**Gut1**, Theorem 4.2] states the following:

Theorem 1.4.1 ([**Gut1**, Theorem 4.2]). *Let λ,μ be partitions with $\mu\subseteq\lambda\subseteq(k^l)$ with some fixed integers k,l. Set $\bar\lambda=(k^l)/\lambda$.*

Then: The coefficient of $[\alpha]$ in $[\lambda/\mu]$ equals the coefficient of $[\bar\alpha]=[(k^l)/\alpha]$ in $[\mu]\star_{(k^l)}[\bar\lambda]$.

So results about the outer product $[\mu]\otimes[\nu]$ correspond to results about skew characters λ/μ with $\mu_1\leq\lambda_l, l(\mu)\leq l_1$ if $\lambda=(\lambda_1^{l_1},\lambda_{l_1+1},\ldots,\lambda_l)$ because then the Schubert product $\star_{(k^l)}$ is just the ordinary product $\otimes$. This are skew characters corresponding to the following picture:

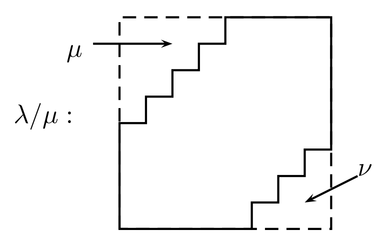


Figure 1. λ/μ

On the other hand, results on skew characters with skew diagram of the above type (or results on skew characters in general) give us results on outer products of irreducible characters.

We will use this correspondence extensively in Sections 2.1.2, 2.2.3 and 3.1.

1.5. Stretched Littlewood-Richardson coefficients

We have seen various appearances of the LR coefficients so far but there are still more.

Fulton gives in his paper [**Ful**] a good overview about other appearances of the LR coefficients.

One can construct objects called LR hives whose number is also given by the LR coefficients (see [**KT, KTW, KTT**]). Their size depends only on the largest length of the partitions involved, so this model is particular interesting if one analysis functions like $f(n)=c(n\lambda;n\mu,n\nu)$. This $f(n)$ is called stretched LR coefficient and is well analyzed. It is known that for $c(\lambda;\mu,\nu)\neq 0$ and $n\geq 0$ the stretched LR coefficient $f(n)=c(n\lambda;n\mu,n\nu)$ is given by a polynomial in n with $f(0)=1$.

Furthermore, if this polynomial $f(n) \neq 0$ has an integer root $-t \in \mathbb{Z}$ then $t > 0$ and $f(n)$ also contains the factors $(n+i)$ for $1 \leq i \leq t$. Furthermore, there is a t such that $f(n) = g(n) \prod_{i=1}^{t}(n+i)$ with $g(n)$ a polynomial having no integer roots. So $f(n) \neq 0$ for some $n \geq 1$ implies $c(\lambda; \mu, \nu) \neq 0$ which is called the Saturation property (see [**KT**]).

We have $f(n) = 1$ for all or any $n \geq 1$ if and only if $f(1) = c(\lambda; \mu, \nu) = 1$. So $f(n)$ is either constant 1 or a polynomial of positive degree.

But by a paper of Etienne Rassart [**Ras**] the situation is even better: Suppose we only investigate those LR coefficients $c(\lambda; \mu, \nu)$ such that all partitions in the triple (λ, μ, ν) have at most k parts. Then this triple can be viewed as a subset of $\mathbb{R}^{3k}$. Since $c(\lambda; \mu, \nu) = 0$ unless $\mu, \nu \subseteq \lambda$ and $|\mu| + |\nu| = |\lambda|$ we can define a cone C_k^1 as the subset of $\mathbb{R}^{3k}$ containing those triples which obey these conditions. Furthermore, assuming that λ, μ, ν are partitions defines another cone $C_k^2 \subset \mathbb{R}^{3k}$. Finally the pullback of some other cones gives a complex whose intersection with the cones C_k^1 and C_k^2 defines the Littlewood-Richardson complex $\mathcal{LR}_k$ which contains all triples of partitions of length at most k having positive LR coefficient. This chamber complex decays into cones in which the LR coefficients are given by a polynomial in the variables $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_k$; the degree of this polynomial is at most $(k-1)(k-2)/2$ in the three sets of variables ([**Ras**, Theorem 4.1]).

Rassart also computes by polynomial interpolation for $\mathcal{LR}_3$ the polynomials associated to the maximal cones, which are 8-dimensional. For $\mathcal{LR}_3$ there are 18 maximal cones and as he told me in private communication for $\mathcal{LR}_4$ there are already thousands of cones.

We will generalize the stretched LR coefficients in Chapter 4.

CHAPTER 2

Shape of the components of skew characters

With the Littlewood-Richardson rule one has a good combinatorial tool for determining the components, together with their multiplicities, of a given skew character $[\lambda/\mu]$. But there is also a huge interest in knowing as much as possible about the components of skew characters in general.

For example, one would like to know when a skew character is homogeneous (i.e. has only one component) or irreducible (both only possible if the skew diagram is a partition or rotated partition, see for example [**BK99**]), when two skew diagrams represent the same skew character (problem unsolved but much work is done in determining sufficient or necessary conditions, see for example [**McN, MvW09a, RSW**]) or when the difference of two skew characters is again a character, so all components have non-negative multiplicity (also unsolved, see for example [**BM, McN, KWW, MvW09b**]). One would also like to know the number of components or constituents of a given skew character without having to expand the whole skew character by the LR rule (problem unsolved). Other questions are, for example, which skew characters only contain components with multiplicity 1 (classified in my Diploma-Thesis, see [**Gut1**]), what is the smallest or largest Durfee size of the partitions of the components of a skew character (unsolved) or what is the highest multiplicity of a component (unsolved).

There is also an interest in knowing certain components of skew characters. It is known, for example, that the lexicographic largest diagram of a component of a skew characters is obtained by reordering the columns of the corresponding skew diagram such that they form a partition. Furthermore, this component has multiplicity 1. Similarly the lexicographic smallest diagram of a component of a skew character also has multiplicity 1 and is obtained by reordering the parts of the corresponding skew diagram. These results and some other restrictions on the components follow directly from the LR rule.

In Section 2.1 we give in Remark 2.1.2 an easy combinatorial description for the intersection of the diagrams of all components of a skew character. For the ordinary outer product and the Schubert product as well as for a certain kind of skew characters we also describe the union of all diagrams of the components. This gives us upper and lower bounds for partitions α such that the corresponding character $[\alpha]$ can appear in a given outer product or Schubert product of two characters or in a skew character.

In Section 2.2 we construct for a given arbitrary skew character $[\mathcal{A}]$ all components whose partitions ν have maximal principal hook lengths, i.e. hook lengths of the boxes (i, i) of ν, among all components of $[\mathcal{A}]$. Furthermore, we show that these are also partitions with minimal Durfee size.

We use this to construct components of $[\mathcal{A}]$, together with their multiplicities, having maximal Durfee size for the cases when $\mathcal{A}$ decays into two partitions and for some special cases of $\mathcal{A}$. We also deduce necessary conditions for two skew diagrams to represent the same skew character.

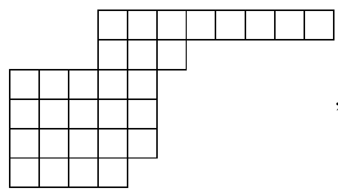
The results of Section 2.1 appeared in my paper [**Gut3**] while the results of Section 2.2 will appear in my paper [**Gut2**].

2.1. Base and cover partitions

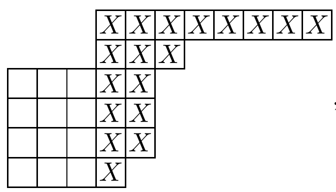
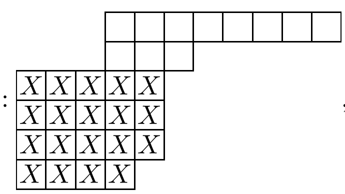
First of all we introduce some notation which we will need in this section.

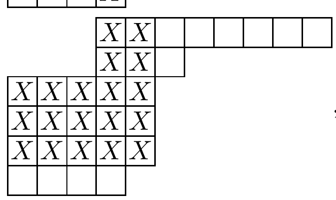
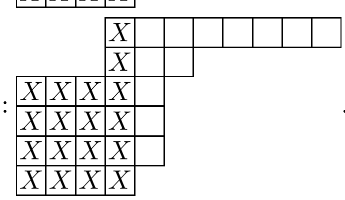
We say that a partition α is contained in $\mathcal{A}$ if there is a subdiagram of $\mathcal{A}$ which is α or α°.

For example, the skew diagram $\mathcal{A} = (11, 6, 5^3, 4)/(3^2)$,

 ,

contains the partitions $\alpha^1 = (8, 3, 2^3, 1), \alpha^2 = (5^3, 4), \alpha^3 = (5^3, 2^2), \alpha^4 = (4^4, 1^2)$:

α^1 :  , α^2 :  ,

α^3 :  , α^4 :  .

All other partitions contained in $\mathcal{A}$ are subdiagrams of some α^i.

For a skew diagram $\mathcal{A}$ we define the union partition $\Upsilon(\mathcal{A})$ as the union of all partitions α which are contained in $\mathcal{A}$. So:

$$\Upsilon(\mathcal{A})_i = \max(\alpha_i \mid \alpha \text{ is contained in } \mathcal{A}).$$

Since an arbitrary partition α is the union of the rectangles $((\alpha_i)^i)$ (for example, we have that $\alpha = (5, 3, 3, 2, 1)$ is the union of the five rectangles $(5^1), (3^2), (3^3), (2^4)$ and (1^5)), it is sufficient for Υ to restrict the union to all rectangles α which are contained in $\mathcal{A}$ but are not contained in a larger rectangle β contained in $\mathcal{A}$.

So in the above example we have $\Upsilon(\mathcal{A}) = (8, 5, 5, 4, 2, 1)$ and there are 6 of those maximal rectangles.

For a character χ we define the base partition $\mathcal{B}(\chi)$ as the intersection of all α with $[\alpha] \in \chi$. So:

$$\mathcal{B}(\chi)_i = \min(\alpha_i \mid [\alpha] \in \chi).$$

Similarly we define for a character χ the cover partition $\mathcal{C}(\chi)$ as the union of all α with $[\alpha] \in \chi$. So:

$$\mathcal{C}(\chi)_i = \max(\alpha_i \mid [\alpha] \in \chi).$$

Recall that the Durfee size $d(\lambda)$ of a partition λ is d if $(d^d) \subseteq \lambda$ is the largest square contained in λ. We define the Durfee size of a skew diagram $\mathcal{A}$ or skew character $[\mathcal{A}]$ as the biggest Durfee size of all partitions of the components of $[\mathcal{A}]$:

$$d(\mathcal{A}) = d([\mathcal{A}]) = \max(d(\nu) \mid [\nu] \in [\mathcal{A}]).$$

2.1.1. The base partition. In this subsection we give in Remark 2.1.2 an easy combinatorial description for the base partition $\mathcal{B}([\mathcal{A}])$ of a skew character $[\mathcal{A}]$ which also gives a description for the base partition of the ordinary product $\mathcal{B}([\mu] \otimes [\nu])$. We first prove that the base partition of a skew character is the union partition of the corresponding skew diagram.

THEOREM 2.1.1. *Let $\mathcal{A}$ be a skew diagram.*
Then: $\mathcal{B}([\mathcal{A}]) = \Upsilon(\mathcal{A})$.

PROOF. From the LR rule follows that if a rectangle (m^l) is contained in $\mathcal{A}$ then (m^l) is contained in every partition ν with $[\nu] \in [\mathcal{A}]$. So we have $\Upsilon \subseteq \mathcal{B} = \mathcal{B}([\mathcal{A}])$.

We will show by induction on the biggest length of a column contained in $\mathcal{A}$ that the lower bound for the rows of $\mathcal{B}$ is reached for some partitions α with $[\alpha] \in [\mathcal{A}]$.

Let us assume that the biggest length of a column in $\mathcal{A}$ is 1. Then $\mathcal{A}$ decomposes into disconnected rows and Υ is the biggest row contained in $\mathcal{A}$.

We have a character $[\alpha] \in [\mathcal{A}]$ such that α contains the parts of $\mathcal{A}$ and so $\alpha_1 = \Upsilon_1$ which gives $\Upsilon_1 = \mathcal{B}_1$.

On the other hand, if we place 1s into every box of $\mathcal{A}$ we obtain an LR tableau and so we have $[n] \in [\mathcal{A}]$ with $n = |\mathcal{A}|$ which gives us $\mathcal{B}_2 = 0$ and so $\Upsilon = \mathcal{B}$.

Let us now assume that $\Upsilon = \mathcal{B}$ holds for all skew diagrams which have columns of length not larger than $n-1$ and let $\mathcal{A}$ be a skew diagram which has one or more columns of length n.

Rearranging the parts of $\mathcal{A}$ gives again a partition α whose corresponding character satisfies $[\alpha] \in [\mathcal{A}]$ and $\alpha_1 = \Upsilon_1$ and so again $\Upsilon_1 = \mathcal{B}_1$.

We will now prove that $\Upsilon_i = \mathcal{B}_i$ holds also for $i \geq 2$.

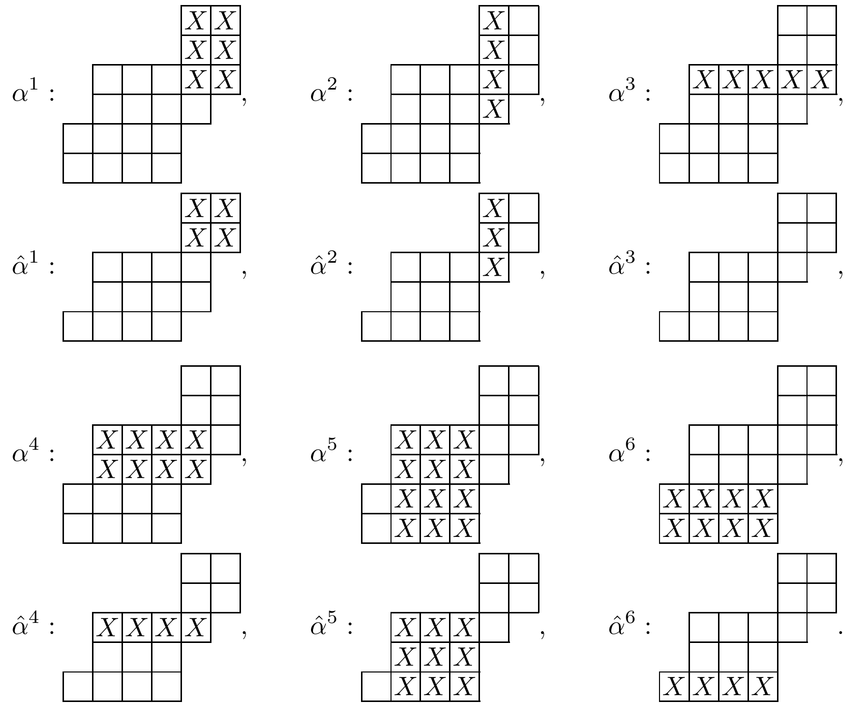
We have a $1-1$-relation between the LR tableaux A of shape $\mathcal{A}$ and 1s in the top boxes of every column and arbitrary LR tableaux D of shape $\mathcal{D}$, where $\mathcal{D}$ is the skew diagram $\mathcal{A}$ with the top boxes of every column removed, simply by removing all 1s from A and replacing the entry i in A with $i-1$.

For example, if we have $\mathcal{A} = (6^3, 5, 4^2)/(4^2, 1^2)$ and some arbitrary LR filling we get with the above construction:

$$\mathcal{A} = \begin{array}{cccccc} & & & & \square & \square \\ & & & & \square & \square \\ & \square & \square & \square & \square & \square \\ & \square & \square & \square & \square & \\ \square & \square & \square & \square & & \\ \square & \square & \square & \square & & \end{array}, \qquad A = \begin{array}{cccccc} & & & & 1 & 1 \\ & & & & 2 & 2 \\ & 1 & 1 & 1 & 3 & 3 \\ & 2 & 2 & 2 & 4 & \\ 1 & 3 & 3 & 5 & & \\ 2 & 4 & 4 & 6 & & \end{array} \longleftrightarrow D = \begin{array}{cccccc} & & & & 1 & 1 \\ & & & & 2 & 2 \\ & 1 & 1 & 1 & 3 & \\ & 2 & 2 & 4 & & \\ 1 & 3 & 3 & 5 & & \end{array}.$$

Removing the top boxes of each column from $\mathcal{A}$ reduces each of the maximal rectangles α^i by one row and gives $\hat{\alpha}^i$ which is then one of the maximal rectangles in $\mathcal{D}$. So we get $\Upsilon(\mathcal{A})_{i+1} = \Upsilon(\mathcal{D})_i$.

In the above example we have the following rectangles α^i in $\mathcal{A}$ and $\hat{\alpha}^i$ in $\mathcal{D}$:



Since the biggest length of columns in $\mathcal{D}$ is $n-1$ we have $\Upsilon(\mathcal{D}) = \mathcal{B}([\mathcal{D}])$. For $i \geq 1$ let $[\hat{\beta}^i] \in [\mathcal{D}]$ with $\hat{\beta}^i_i = \mathcal{B}([\mathcal{D}])_i$. Since the characters $[\hat{\beta}^i] \in [\mathcal{D}]$ correspond to characters $[\beta^i] \in [\mathcal{A}]$, with $\beta^i = (j, \hat{\beta}^i)$ and j equal to the number of columns in $\mathcal{A}$, we have in $[\mathcal{A}]$ characters $[\beta^i]$ with:

$$\beta^i_{i+1} = \hat{\beta}^i_i = \Upsilon(\mathcal{D})_i = \Upsilon(\mathcal{A})_{i+1}.$$

This gives $\Upsilon(\mathcal{A}) = \mathcal{B}([\mathcal{A}])$. □

The previous proof also gives us the following description for the base partition $\mathcal{B}([\mathcal{A}])$.

Remark 2.1.2. Let $\rho^i(\mathcal{A})$ be the skew diagram obtained from $\mathcal{A}$ by removing in every column the top $i-1$ boxes.

Then we have that the i-th part $\mathcal{B}_i$ of the base partition $\mathcal{B}$ is the maximal part of $\rho^i(\mathcal{A})$, and so for $\mathcal{A} = \lambda/\mu$ we have $\mathcal{B}_i = \max_j |\lambda_{i+j-1} - \mu_j|_+$ with $|x|_+ = \max(0, x)$.

Also $\rho^i(\mathcal{A})$ is the i-row overlap composition defined in [**RSW**]. There it was also proved that equality of $[\mathcal{A}^1] = [\mathcal{A}^2]$ for skew diagrams $\mathcal{A}^1, \mathcal{A}^2$ requires that $\rho^i(\mathcal{A}^1)$ and $\rho^i(\mathcal{A}^2)$ must have the same parts in the same quantity for every i. This follows easily from the $1-1$ correspondence used also in the proof of Theorem 2.1.1. In [**McN**] these $\rho^i(\mathcal{A})$ are used to get necessary conditions for positivity of $[\mathcal{A}^1] - [\mathcal{A}^2]$.

2.1.2. The cover partition. In this subsection we use Remark 2.1.2 and Theorem 1.4.1 to determine for the ordinary and the Schubert product of two characters and for some special skew characters the cover partition $\mathcal{C}$, which is the union of all partitions corresponding to the components of the product or the skew character.

We will use that for $k \geq \mu_1 + \nu_1, l \geq l(\mu) + l(\nu)$ the Schubert product $[\mu] \star_{(k^l)} [\nu]$ is the ordinary product $[\mu] \otimes [\nu]$.

Let us associate a skew diagram $\mathcal{A} = \big((k^l)/\mu\big)^\circ)/\nu$ to partitions μ, ν. Here for the Schubert product k, l are fixed by $[\mu] \star_{(k^l)} [\nu]$ and for the ordinary product chosen as $k = \mu_1 + \nu_1, l = l(\mu) + l(\nu)$. To obtain $\mathcal{A}$ we remove from the rectangle (k^l) the partition ν as usual and the partition μ rotated by 180° from the lower right corner (see Figure 1). We know by definition that if the box (i, j) is in the base

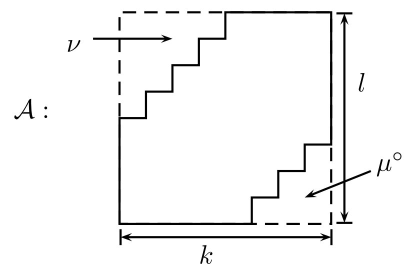


Figure 1. $\mathcal{A} = \big((k^l)/\mu\big)^\circ)/\nu$

partition $\mathcal{B}([\mathcal{A}])$ then it is also in every partition α with $[\alpha] \in [\mathcal{A}]$. So if we remove α from (k^l) to get $\bar{\alpha}$ this box will be removed every time and so by Theorem 1.4.1 there cannot be a partition β (which would be a rotated $\bar{\alpha}$) with $[\beta] \in [\mu] \star_{(k^l)} [\nu]$ containing the box $(k - i, l - j)$.

On the other hand, if the box (i, j) is not in the base partition then there is a partition α with $[\alpha] \in [\mathcal{A}]$ without this box. So if we now remove α from (k^l) to get $\bar{\alpha}$ this box is in $\bar{\alpha}$ and so there is a partition β (the rotated $\bar{\alpha}$) with $[\beta] \in [\mu] \star_{(k^l)} [\nu]$ which does contain the box $(k - i, l - j)$.

So we get the following theorem:

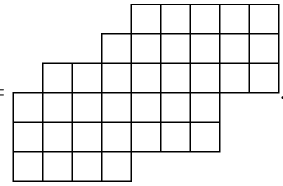
Theorem 2.1.3. *Let μ, ν be partitions and for fixed integers k, l define the skew diagram $\mathcal{A} = \big((k^l)/\mu\big)^\circ)/\nu$.*

Then: $\mathcal{C}([\mu] \star_{(k^l)} [\nu]) = \big((k^l)/\mathcal{B}([\mathcal{A}])\big)^\circ$ and $\mathcal{C}([\mu] \otimes [\nu]) = \big((k^l)/\mathcal{B}([\mathcal{A}])\big)^\circ$ (with $k = \mu_1 + \nu_1, l = l(\mu) + l(\nu)$).

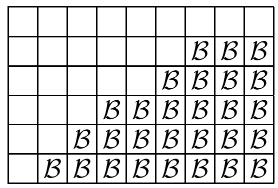
Remark 2.1.4. Since the Durfee size of the cover partition of the product is also the Durfee size of the product itself we can now easily read off the Durfee size of an arbitrary Schubert product (or ordinary product). In Subsection 2.2.3 we calculate the Durfee size only for those Schubert products which are ordinary products but give also characters $[\lambda] \in [\mu] \otimes [\nu]$ which have maximal Durfee size. Theorem 2.1.3 gives us the Durfee size but not characters which appear in the product.

We will give a short example and calculate for $\mu = (4, 3, 1)$ and $\nu = (5, 2, 2)$ the cover partition for the ordinary product $[\mu] \otimes [\nu]$ and the Schubert product $[\mu] \star_{(7^4)} [\nu]$.

In the case of the ordinary product we first construct the skew diagram

$$\mathcal{A}^1 = (9^3, 7^2, 4)/(4, 3, 1) =$$  .

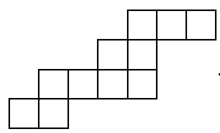
Using Remark 2.1.2 we get the base partition $\mathcal{B}([\mathcal{A}^1]) = (8, 7, 6, 4, 3)$ and so for the cover partition

$$\mathcal{C}([\mu] \otimes [\nu]) = ((9^6)/\mathcal{B}([\mathcal{A}^1]))^\circ =$$  $$= (9, 6, 5, 3, 2, 1)$$

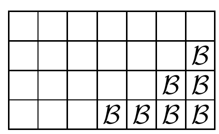
where the boxes labeled $\mathcal{B}$ form the partition $\mathcal{B}([\mathcal{A}^1])$.

Since we have $\mathcal{B}([\mu] \otimes [\nu]) = (5, 3, 2)$ we now have that if $[\alpha] \in [\mu] \otimes [\nu]$ then α has to satisfy $(5, 3, 2) \subseteq \alpha \subseteq (9, 6, 5, 3, 2, 1)$. These upper and lower bounds are strict in the sense that there are no better bounds.

In the case of the Schubert product the skew diagram is:

$$\mathcal{A}^2 = (7, 5, 5, 2)/(4, 3, 1) =$$  .

Using Remark 2.1.2 we get the base partition $\mathcal{B}([\mathcal{A}^2]) = (4, 2, 1)$ and so for the cover partition

$$\mathcal{C}([\mu] \star_{(7^4)} [\nu]) = ((7^4)/\mathcal{B}([\mathcal{A}^2]))^\circ =$$  $$= (7, 6, 5, 3)$$

where the boxes labeled $\mathcal{B}$ form the partition $\mathcal{B}([\mathcal{A}^2])$. From this we can also read off the Durfee size of the Schubert product as $d([\mu] \star_{(7^4)} [\nu]) = d(\mathcal{C}([\mathcal{A}^2])) = 3$.

Since we have $(5, 3, 2) = \mathcal{B}([\mu] \otimes [\nu]) \subseteq \mathcal{B}([\mu] \star_{(7^4)} [\nu])$ we now have that if $[\alpha] \in [\mu] \star_{(7^4)} [\nu]$ then α has to satisfy $(5, 3, 2) \subseteq \alpha \subseteq (7, 6, 5, 3)$. Here the lower bound is not strict and explicit calculations show $\mathcal{B}([\mu] \star_{(7^4)} [\nu]) = (5, 4, 2)$.

In the same way we can also construct the cover partition $\mathcal{C}$ of skew characters if we restrict the skew diagram in the way that the associated Schubert product is, in fact, an ordinary product. If the skew diagram does not satisfy the constraints of the following theorem then we would only get a trivial upper bound for the cover partition.

THEOREM 2.1.5. *Let $\mathcal{A} = \lambda/\mu$ be a skew diagram with $\lambda = (\lambda_1^n, \lambda_{n+1}, \ldots, \lambda_l)$. Let $\mu_1 \leq \lambda_l, l(\mu) \leq n$ and set $\bar{\lambda} = ((\lambda_1^l)/\lambda)^\circ$.*

Then: $\mathcal{C}([\mathcal{A}]) = \big(((\lambda_1)^l)/\mathcal{B}([\mu] \otimes [\bar{\lambda}])\big)^\circ$.

Remark 2.1.6. This again gives us the Durfee size of the skew character $[\lambda/\mu]$ if $\mu_1 \leq \lambda_l, l(\mu) \leq n$. In Subsection 2.2.3 we calculate the Durfee size only for those skew characters which additionally satisfy $\lambda_1 = l(\lambda)$ but give also characters

$[\nu] \in [\mathcal{A}]$ which have maximal Durfee size. Theorem 2.1.5 gives us the Durfee size but not characters which appear in the product.

2.2. Hook length partitions

We again fix at first our notation for this section.

A (proper) ribbon is a connected skew diagram which does not contain the subdiagram (2^2). A (disconnected) skew diagram which decomposes into ribbons will be called a weak ribbon.

For a ribbon $\mathcal{R}$, we define its arm resp. leg length as the number of columns resp. rows in $\mathcal{R}$ minus 1. The arm resp. leg length of a weak ribbon $\mathcal{R}$ is defined as the sum of the arm resp. leg lengths of the ribbons into which $\mathcal{R}$ decomposes, which is the number of columns resp. rows in $\mathcal{R}$ minus the number of ribbons into which $\mathcal{R}$ decomposes.

For each (connected) skew diagram $\mathcal{A}$ we define its first northwest ribbon $nw_1(\mathcal{A})$ as the subdiagram which starts in the lowest leftmost box traverses along the northwest border of $\mathcal{A}$ and ends in the box in the top right. To get the second northwest ribbon $nw_2(\mathcal{A})$ we remove nw_1 from $\mathcal{A}$ and repeat this process if $\mathcal{A}/nw_1$ is still connected. This we iterate to get $nw_i(\mathcal{A})$. For a disconnected skew diagram which decays into two or more skew diagrams $\mathcal{B}_j$ we define its northwest ribbons as the weak ribbons which contain the corresponding northwest ribbons of the $\mathcal{B}_j$. All the northwest ribbons together form the northwest ribbon decomposition. Furthermore, we define the northwest ribbon length partition $\pi_{nw}(\mathcal{A})$ associated to $\mathcal{A}$ as the partition where the ith row has as many boxes as the ith northwest ribbon $nw_i(\mathcal{A})$:

$$\pi_{nw}(\mathcal{A})_i = |nw_i(\mathcal{A})|\,.$$

The northwest ribbons are weak ribbons and only in some cases proper ribbons.

Example 2.2.1. We give the northwest ribbon decomposition for the skew diagrams $(10^2, 8^4, 5^2)/(5^4)$ and $(10^4, 8^2, 3^2)/(5^4)$ and label the boxes with i if they are contained in the i-th northwest ribbon nw_i:

$$\begin{array}{cccccccccc}
 & & & & & 1&1&1&1&1\\
 & & & & & 1&2&2&2&2\\
 & & & & & 1&2&3& & \\
 & & & & & 1&2&3& & \\
1&1&1&1&1&1&2&3& & \\
1&2&2&2&2&2&2&3& & \\
1&2&3&3&3& & & & & \\
1&2&3&4&4& & & & &
\end{array}\quad,\qquad
\begin{array}{cccccccccc}
 & & & & & 1&1&1&1&1\\
 & & & & & 1&2&2&2&2\\
 & & & & & 1&2&3&3&3\\
 & & & & & 1&2&3&4&4\\
1&1&1&1&1&1&2&3& & \\
1&2&2&2&2&2&2&3& & \\
1&2&3& & & & & & & \\
1&2&3& & & & & & &
\end{array}.$$

In both cases the third northwest ribbon nw_3 decays into two ribbons and $\pi_{nw} = (17, 15, 8, 2)$.

For a partition λ we define the ith principal hook length $hl_i(\lambda)$ as the hook length of the box (i, i) and the (principal) hook length partition as: $hl(\lambda) = (hl_1(\lambda), hl_2(\lambda), \ldots)$. So we have $hl_1(\lambda) = \lambda_1 + l(\lambda) - 1$.

For a skew diagram $\mathcal{A}$ we define its hook length partition $hl(\mathcal{A})$ as the lexicographic biggest hook length partition $hl(\lambda)$ of all λ with $[\lambda]$ appearing in $[\mathcal{A}]$:

$$hl(\mathcal{A}) = hl([\mathcal{A}]) = \max{}_{\mathrm{lex}}(hl(\nu) \,|\, [\nu] \in [\mathcal{A}]).$$

In a partition the ith northwest ribbon is the hook to the box (i, i) so for a partition λ we have $hl(\lambda) = \pi_{nw}(\lambda)$.

In this section we construct for a skew character $[\lambda/\mu] = \sum_\nu c(\lambda;\mu,\nu)[\nu]$ the ν with maximal principal hook lengths. From this we deduce the minimal Durfee size for characters in arbitrary skew characters and the maximal Durfee size for characters in products of characters and some special skew characters.

There has recently been much interest in the question of determining necessary or sufficient conditions for two skew diagrams $\mathcal{A},\mathcal{B}$ to have either $[\mathcal{A}]-[\mathcal{B}]$ positive or even $[\mathcal{A}]=[\mathcal{B}]$, see for example [**McN**],[**MvW09a**],[**RSW**]. In Subsection 2.2.4 we use the results from Subsection 2.2.2 to give necessary conditions for two skew diagrams $\mathcal{A}$ and $\mathcal{B}$ to represent the same skew character, i.e. $[\mathcal{A}]=[\mathcal{B}]$.

2.2.1. Maximal hook lengths in products of irreducible characters. In this subsection we study the case when $\mathcal{A}$ decomposes into two proper partitions α,β and show $hl(\mathcal{A})=\pi_{nw}(\mathcal{A})$. We will show that there are $2^{\min(d(\alpha),d(\beta))}$ partitions ν with $[\nu]\in[\mathcal{A}]$ and $hl(\nu)=hl(\mathcal{A})$ and show that each of those $[\nu]$ appears with multiplicity 1 in $[\mathcal{A}]$. In Remark 2.2.3 we will determine the exact shape of all these ν. Furthermore, we argue in Remark 2.2.5 that those $[\nu]$ with $hl(\nu)=hl(\mathcal{A})$ have minimal Durfee size of all $[\tilde{\nu}]\in[\mathcal{A}]$.

Theorem 2.2.2. *Let $\mathcal{A}$ be a skew diagram decomposing into the proper partitions α and β.*

Then $hl(\mathcal{A})=hl(\alpha)+hl(\beta)=\pi_{nw}(\mathcal{A})$.

Furthermore, there are $2^{\min(d(\alpha),d(\beta))}$ partitions ν with $hl(\nu)=hl(\mathcal{A})$ and $[\nu]$ appearing in $[\mathcal{A}]$. All these $[\nu]$ appear with multiplicity 1.

Proof. If $\mathcal{A}=\lambda/\mu$ decomposes into two partitions α and β we have $[\mathcal{A}]=[\alpha]\otimes[\beta]$. If we decompose $[\mathcal{A}]=\sum_\nu c(\lambda;\mu,\nu)[\nu]=\sum_\nu c(\nu;\alpha,\beta)[\nu]$ we have to create LR tableaux with shape ν/α and content β.

To create an LR tableau T with shape ν/α and content β and ν having the lexicographic biggest hook length partition of all $\bar{\nu}$ with $[\bar{\nu}]$ appearing in $[\mathcal{A}]$ we have to fill in T as many boxes as possible in the first row and column. Because of the LR conditions we can place only the entry 1 into the boxes of the first row and so we maximize ν_1 by placing all the β_1 entries 1 into the first row and get $\nu_1=\alpha_1+\beta_1$. Into the first column we can only place each entry once (but the entries 1 are used up already), so we get a maximized first column by filling the boxes with entries 2 to $l(\beta)$. So we have $l(\alpha)+l(\beta)-1=l(\nu)$ boxes in the first column and this gives us:

$$hl_1(\mathcal{A})=hl_1(\nu)=\alpha_1+\beta_1+l(\alpha)+l(\beta)-1-1=hl_1(\alpha)+hl_1(\beta).$$

Another way to obtain an LR tableau U with shape $\tilde{\nu}/\alpha$ and content β and $hl_1(\tilde{\nu})=hl_1(\nu)$ would be to place the entries 1 to $l(\beta)$ into the first column and the remaining β_1-1 entries 1 into the first row. Clearly these are the only ways to maximize $hl_1(\nu)$.

We show that this can be iterated by examining the filling of T (and U) which maximizes $hl_2(\nu)$. Without loss of generality we may assume $d(\beta)\le d(\alpha)$.

If $d(\beta)=1$ then we are finished and have no entries anymore to place in T (or U).

If $d(\beta)\ge 1$ then we have again two possibilities to maximize $hl_2(\nu)$ by either maximizing the second row or column. The fillings with maximized second row differs from the filling with maximized second column because the box $(2,2)$ belongs to α and so remains empty in T. We have only to show that both possibilities satisfy the LR conditions. Because there are only $\beta_2 - 1$ entries 2 left to place in T (or U) and there are either $\beta_1 \geq \beta_2 - 1$ (T) or $\beta_1 - 1 \geq \beta_2 - 1$ (U) entries 1 placed in the first row of T (or U) the LR conditions are satisfied for the entries 2 no matter where we place them. For the entries placed in the second column the LR lattice condition clearly is satisfied but we have to check that the entries are weakly increasing amongst the rows. If we assume that the box $(j, 2)$ is filled, its entry is $j - \alpha_2^c + 1$ ($+1$ if all remaining entries 2 are placed in the second row) with α_2^c the length of the second column of α. The box $(j, 1)$ is empty in the case $j \leq \alpha_1^c$ or otherwise has the entry $j - \alpha_1^c$ ($+1$ for T). By comparing the worst cases we have the condition: $j - \alpha_1^c + 1 \leq j - \alpha_2^c + 1$ which holds for all α. Furthermore, the maximized second row has no more than $\alpha_2 + \beta_2 - 1$ boxes which is not bigger than the number of boxes in the first row since there are $\alpha_1 + \beta_1$ (-1 for U) boxes in the first row, so ν is still a partition. The same reasoning applies to the columns.

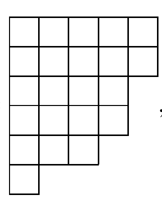
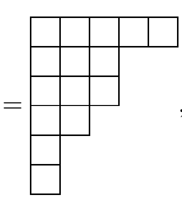
So all works well and we can iterate the process. In the end we had $d(\beta) = l(hl(\beta))$ choices to make, to either maximize the ith row or ith column for $i \leq d(\beta)$, and so get $2^{\min(d(\alpha), d(\beta))}$ different partitions ν. For each such ν, there is a unique (so the multiplicity of $[\nu]$ in $[\mathcal{A}]$ is 1) LR tableau T of shape ν/α and content β. $\square$

Remark 2.2.3. The proof tells us even more about the explicit form of the $[\nu]$ appearing in $[\mathcal{A}]$ with $hl(\nu) = hl(\mathcal{A})$.

Let γ be the partition such that the ith principal hook has as arm resp. leg length the sum of the arm resp. leg lengths of the ith principal hooks of α and β and containing the box (i, i) exactly if $i \leq \max(d(\alpha), d(\beta))$. Then γ is the intersection of all ν with maximal hook length partition and also the intersection of the ν where either all columns or all rows were maximized as described above.

From γ we can construct all partitions ν appearing in $\mathcal{A}$ with maximal hook length partition by adding for each $1 \leq j \leq \min(d(\alpha), d(\beta))$ a box to either the jth row or column of γ.

Example 2.2.4. If $\mathcal{A}$ decomposes into the partitions $\alpha = (5^2, 4^2, 3, 1)$ and $\beta = (5, 3^2, 2, 1^2)$, so

$\alpha =$  , $\beta =$  ,

the characters corresponding to the following partitions ν are the ones with maximal hook length partition in $[\mathcal{A}]$. In $\bar{\gamma}$ the unmarked boxes form the partition γ as in Remark 2.2.3 and the ν are obtained by choosing for each $i \in \{1, 2, 3\}$ exactly one box labeled i and add them to γ. T resp. U gives the actual LR filling with all rows resp. columns maximized:

$$\bar{\gamma} = \begin{ytableau} \, & \, & \, & \, & \, & \, & \, & \, & \, & 1 \\ \, & \, & \, & \, & \, & \, & 2 \\ \, & \, & \, & \, & 3 \\ \, & \, & \, & \, \\ \, & \, & \, \\ \, & \, & 3 \\ \, & \, \\ \, & 2 \\ \, \\ \, \\ \, \\ 1 \end{ytableau} \quad , \quad T = \begin{ytableau} \, & \, & \, & \, & \, & 1 & 1 & 1 & 1 & 1 \\ \, & \, & \, & \, & \, & 2 & 2 \\ \, & \, & \, & \, & 3 \\ \, & \, & \, & \, \\ \, & \, & \, \\ \, & 3 \\ 2 & 4 \\ 3 \\ 4 \\ 5 \\ 6 \end{ytableau} \quad , \quad U = \begin{ytableau} \, & \, & \, & \, & \, & 1 & 1 & 1 & 1 \\ \, & \, & \, & \, & \, & 2 \\ \, & \, & \, & \, \\ \, & \, & \, & \, \\ \, & \, & \, \\ \, & 2 & 3 \\ 1 & 3 \\ 2 & 4 \\ 3 \\ 4 \\ 5 \\ 6 \end{ytableau} \; .$$

For the hook length partitions we have:

$$hl(\nu) = (20, 11, 5, 1) = (10 + 10, 4 + 7, 1 + 4, 0 + 1) = hl(\alpha) + hl(\beta).$$

Remark 2.2.5. The maximum of the Durfee sizes of the partitions α and β is a lower bound for the minimal Durfee size of characters in $[\alpha] \otimes [\beta]$ and the $[\nu] \in [\alpha] \otimes [\beta]$ with $hl(\nu) = hl([\alpha] \otimes [\beta])$ have Durfee size $d(\nu) = \max(d(\alpha), d(\beta))$. So the characters $[\nu]$ with $hl(\nu) = hl([\alpha] \otimes [\beta])$ are characters with minimal Durfee size which is, therefore, given by $\max(d(\alpha), d(\beta))$.

2.2.2. Maximal hook lengths in skew characters. In this subsection we generalize Subsection 2.2.1 to the case when $\mathcal{A}$ is an arbitrary skew diagram and show that also in this case $hl(\mathcal{A}) = \pi_{nw}(\mathcal{A})$ is true. We will give an easy formula for the multiplicity of $[\nu]$ in $[\mathcal{A}]$ when ν is a partition with $hl(\nu) = hl(\mathcal{A})$. In an easy way similar to that in Subsection 2.2.1 we can construct all those ν explicitly. Furthermore, we show in Lemma 2.2.14 that also in this case the characters with maximal hook length partition have minimal Durfee size.

Lemma 2.2.6. *Let $\lambda, \nu, \bar{\lambda}, \bar{\nu}$ be partitions such that $\nu = (\lambda_1, \bar{\nu} + (1^{l-1})) = (\lambda_1, \bar{\nu}_1 + 1, \bar{\nu}_2 + 1, \ldots)$, $\lambda = (\lambda_1, \bar{\lambda} + (1^{l-1}))$ with $l = l(\lambda)$.*
Then for all μ:

$$c(\lambda; \mu, \nu) = c(\bar{\lambda}; \mu, \bar{\nu}).$$

Proof. We have $\bar{\lambda}/\bar{\nu} = \lambda/\nu$ and so $[\bar{\lambda}/\bar{\nu}] = [\lambda/\nu]$. This gives $c(\lambda; \mu, \nu) = c(\bar{\lambda}; \mu, \bar{\nu})$. □

Remark 2.2.7. So the LR fillings of $\bar{\lambda}/\bar{\nu}$ with content μ and the LR fillings of λ/ν with content μ are the same. If λ/μ is connected, this gives us an $1-1$-correspondence between the characters $[\bar{\nu}]$ in $[\bar{\lambda}/\mu]$ and the characters $[\nu]$ in $[\lambda/\mu]$ having maximal first principal hook length. In particular, we have $hl_1(\mathcal{A}) = \lambda_1 + l(\lambda) - 1$ if $\mathcal{A}$ is connected.

Lemma 2.2.8. *Let λ/μ be connected.*
If we remove $nw_1(\lambda/\mu)$ from λ/μ the remaining skew diagram is $\bar{\lambda}/\mu$. So $|nw_1(\mathcal{A})| = hl_1(\mathcal{A})$.

Proof. The skew diagram $\bar{\lambda}/\mu$ consists of those boxes (i, j) that are in λ/μ such that the box $(i-1, j-1)$ is also in λ/μ. The same is true for the skew diagram obtained by removing the first northwest ribbon of λ/μ. Furthermore, we have $|nw_1(\mathcal{A})| = \lambda_1 + l(l) - 1$ and so $|nw_1(\mathcal{A})| = hl_1(\mathcal{A})$. □

Example 2.2.9. If we take as example $\lambda = (6, 2, 5, 3^2, 2^2)$ and $\mu = (3^2, 2, 1)$ then:

$$\lambda/\mu = \begin{array}{|c|c|c|c|c|c|}
\hline
\mu & \mu & \mu & & & \\ \hline
\mu & \mu & \mu & & & \\ \hline
\mu & \mu & & & & \multicolumn{1}{c}{} \\ \cline{1-5}
\mu & & & \multicolumn{3}{c}{} \\ \cline{1-3}
 & & & \multicolumn{3}{c}{} \\ \cline{1-3}
 & & \multicolumn{4}{c}{} \\ \cline{1-2}
 & & \multicolumn{4}{c}{} \\ \cline{1-2}
\end{array},$$

$$(\lambda/h)/\mu = \begin{array}{|c|c|c|c|c|c|}
\hline
h & h & h & h & h & h \\ \hline
h & \mu & \mu & \mu & & \\ \hline
h & \mu & \mu & \mu & & \multicolumn{1}{c}{} \\ \cline{1-5}
h & \mu & \mu & \multicolumn{3}{c}{} \\ \cline{1-3}
h & \mu & & \multicolumn{3}{c}{} \\ \cline{1-3}
h & & \multicolumn{4}{c}{} \\ \cline{1-2}
h & & \multicolumn{4}{c}{} \\ \cline{1-2}
\end{array} = \begin{array}{|c|c|c|c|c|c|}
\hline
\mu & \mu & \mu & 1 & 1 & 1 \\ \hline
\mu & \mu & \mu & 1 & & \\ \hline
\mu & \mu & 1 & 1 & & \multicolumn{1}{c}{} \\ \cline{1-5}
\mu & 1 & 1 & \multicolumn{3}{c}{} \\ \cline{1-3}
1 & 1 & & \multicolumn{3}{c}{} \\ \cline{1-3}
1 & & \multicolumn{4}{c}{} \\ \cline{1-2}
1 & & \multicolumn{4}{c}{} \\ \cline{1-2}
\end{array} = (\lambda/\mu)/\{nw_1(\lambda/\mu)\}.$$

Here h marks the boxes in $h = (\lambda_1, 1^{l(\lambda)-1})$ and 1 the boxes in $nw_1(\lambda/\mu)$.

THEOREM 2.2.10. *Let $\mathcal{A} = \lambda/\mu$ be a skew diagram.*
Then $hl(\mathcal{A}) = \pi_{nw}(\mathcal{A})$.

PROOF. We prove this by induction on the length of $\pi_{nw}(\mathcal{A})$ and the number of proper ribbons into which the first northwest ribbon decays. Obviously this is true for the empty skew diagram, $l(\pi_{nw}(\mathcal{A})) = 0$, but we also prove this for $l(\pi_{nw}(\mathcal{A})) = 1$.

For $l(\pi_{nw}(\mathcal{A})) = 1$, $\mathcal{A}$ is either a ribbon or a weak ribbon. Suppose $\mathcal{A}$ is a proper ribbon. In this case we can use Lemma 2.2.8 to get $hl_1(\mathcal{A}) = \pi_{nw}(\mathcal{A})_1$. Furthermore, $\pi_{nw}(\mathcal{A})_2 = 0$ because of the correspondence given in Remark 2.2.7.

So now suppose that the claim holds if $l(\pi_{nw}(\mathcal{A})) = 1$ and $\mathcal{A}$ decays into $j-1$ proper ribbons. Suppose now that $\mathcal{A}$ decays into j proper ribbons $\mathcal{B}_i$ and so

$$[\mathcal{A}] = [\mathcal{B}_1] \otimes [\mathcal{B}_2] \otimes \cdots \otimes [\mathcal{B}_j] = [\mathcal{B}_1] \otimes ([\mathcal{B}_2] \otimes \cdots \otimes [\mathcal{B}_j]).$$

By induction we know, that $hl(\mathcal{B}_1) = \pi_{nw}(\mathcal{B}_1)$ and $hl(\mathcal{C}) = \pi_{nw}(\mathcal{C})$ with $\mathcal{C} = \bigotimes_{2 \le i \le j} \mathcal{B}_i$.

We decompose $[\mathcal{B}_1] = \sum_a [\nu_a]$ and $[\mathcal{C}] = \sum_b [\xi_b]$ into sums of irreducible characters $[\nu_a]$ resp. $[\xi_b]$ with ν_a and ξ_b proper partitions. We now have $[\mathcal{A}] = \sum_{a,b} [\nu_a] \otimes [\xi_b]$. By Theorem 2.2.2 we have $hl([\nu_a] \otimes [\xi_b]) = hl(\nu_a) + hl(\xi_b)$. So $hl(\mathcal{A}) = \max_{a,b}\big(hl(\nu_a) + hl(\xi_b)\big)$ but since ν_a and ξ_b are independent we have:

$$hl(\mathcal{A}) = \max_{a,b}\big(hl(\nu_a) + hl(\xi_b)\big) = \max_a hl(\nu_a) + \max_b hl(\xi_b) = hl(\mathcal{B}_1) + hl(\mathcal{C}).$$

But by induction $hl(\mathcal{B}_1) = \pi_{nw}(\mathcal{B}_1)$ and $hl(\mathcal{C}) = \pi_{nw}(\mathcal{C})$ and so in total $hl(\mathcal{A}) = \pi_{nw}(\mathcal{B}_1) + \pi_{nw}(\mathcal{C})$. But by the definitions of the northwest ribbons and π_{nw} we have $\pi_{nw}(\mathcal{A}) = \pi_{nw}(\mathcal{B}_1) + \pi_{nw}(\mathcal{C})$ if $\mathcal{A} = \mathcal{B}_1 \otimes \mathcal{C}$ as in this case. This gives finally $hl(\mathcal{A}) = \pi_{nw}(\mathcal{A})$ for $l(\pi_{nw}(\mathcal{A})) = 1$.

So let us assume that the claim holds for $l(\pi_{nw}(\mathcal{A})) = i - 1$. Suppose now that $l(\pi_{nw}(\mathcal{A})) = i$.

Let $\mathcal{A}$ be connected. Lemma 2.2.6 tells us that $|nw_1(\mathcal{A})| = hl_1(\mathcal{A})$. Since $(\mathcal{A})/nw_1(\mathcal{A})$ has $i-1$ northwest ribbons we can use induction and the $1-1$-correspondence given in Remark 2.2.7 and the claim holds true.

Let us now assume that the claim holds true if $l(\pi_{nw}(\mathcal{A})) = i$ and $\mathcal{A}$ decays into $j-1$ disconnected skew diagrams. Suppose $\mathcal{A}$ decays into j disconnected skew diagrams $\mathcal{B}_i$. We can use the same argument as in the weak ribbon case (in the above argument we never used the fact that the $\mathcal{B}_i$ are ribbons) to get $hl(\mathcal{A}) = \pi_{nw}(\mathcal{A})$. □

Remark 2.2.11. The above proof tells us also the exact shape of the ν with $hl(\nu) = hl(\mathcal{A})$ and $[\nu] \in [\mathcal{A}]$. Again, as in Remark 2.2.3 we construct a partition γ such that the box (i,i) has the same arm resp. leg length as $nw_i(\mathcal{A})$ and the box (i,i) is in γ if $\pi_{nw}(\mathcal{A})_i \neq 0$. Let $nw_j(\mathcal{A})$ decay into k_j disconnected ribbons. To obtain the partitions ν we have to add for each j $k_j - 1$ boxes to the jth row or column in γ. The number of ways to obtain ν from γ by adding these boxes is then the multiplicity with which $[\nu]$ appears in $[\mathcal{A}]$.

Proof. In the proof of Theorem 2.2.10 we have seen how to get the characters $[\nu] \in [\mathcal{A}]$ with $hl(\nu) = hl(\mathcal{A})$. To construct these ν we have to maximize the first principal hook length and then the second and so on. Suppose we want to maximize the j-th principal hook length while the 1-st to $j-1$-th principal hook lengths of ν are maximal. If the j-th northwest ribbon decays into k_j proper ribbons then the skew diagram obtained by removing the first $j-1$ northwest ribbons of $\mathcal{A}$ also decays into k_j disconnected skew diagrams $\mathcal{B}_i$. We have then to calculate all the products of characters in different $[\mathcal{B}_i]$ having maximal hook length partitions. But from Remark 2.2.3 we know how to maximize the first hook length of partitions whose corresponding character is in this product (and so maximize the j-th hook length of partitions whose corresponding character is in $[\mathcal{A}]$). We have to construct the hook having as arm resp. leg length the sum of the arm resp. leg lengths of the first principal hooks of the partitions whose corresponding characters are multiplied and then add a box to either the first row or column and then iterate this. □

Since there are $\binom{a}{b}$ ways to choose from a boxes b boxes and put them into a row and put the other $a-b$ boxes into a column we get the following Lemma:

Lemma 2.2.12. *Let $nw_i(\lambda/\mu)$ decay into k_i disconnected ribbons.*

Then there are $\prod_i k_i$ different characters $[\nu] \in [\lambda/\mu]$ with $hl(\nu) = hl(\lambda/\mu)$ and we have

$$c(\lambda;\mu,\nu) = \prod_i \binom{k_i - 1}{\alpha_i}$$

where α_i is the number of boxes placed in the i-th row in the construction of ν from γ (as in Remark 2.2.11).

We give an example in which two of the $[\nu] \in [\mathcal{A}]$ have multiplicity 2. We take $\mathcal{A} = (8^2, 7, 4, 3^2)/(4, 3, 2)$. We have the following northwest ribbon decomposition, where the boxes are labeled i if they are in $nw_i(\mathcal{A})$. We notice that $k_1 = 1, k_2 = 3, k_3 = 2$ and so expect 6 different characters ν_i with $hl(\nu_i) = hl(\mathcal{A})$ and the value 2 as highest multiplicity of one of these characters.

$$\begin{array}{cccccccc}
 & & & & 1 & 1 & 1 & 1 \\
 & & & 1 & 1 & 2 & 2 & 2 \\
 & & 1 & 1 & 2 & 2 & 3 & \\
1 & 1 & 1 & 2 & & & & \\
1 & 2 & 2 & & & & & \\
1 & 2 & 3 & & & & &
\end{array}.$$

If we follow the proof from Theorem 2.2.10, we remove the first northwest ribbon and obtain $\mathcal{B} = (7,6,3,2,2)/(4,3,2)$, which decomposes into $\mathcal{C}_1 = (2^2), \mathcal{C}_2 = (1)$ and $\mathcal{C}_3 = (4,3)/(1)$. To calculate μ with $[\mu] \in [\mathcal{B}]$ and $hl(\mu) = hl(\mathcal{B})$ we have to multiply the characters with the maximal hook length partitions in the $[\mathcal{C}_i]$, but for $i = 1, 2$ the $[\mathcal{C}_i]$ are already those characters. To obtain the characters with maximal hook length partition in $[\mathcal{C}_3]$ we can use Theorem 2.2.10 again. Removing the first northwest ribbon of $\mathcal{C}_3$ gives the partition (1) so the character corresponding to $\alpha = (4,2)$ is the only one with maximal hook length partition in $[\mathcal{C}_3]$.

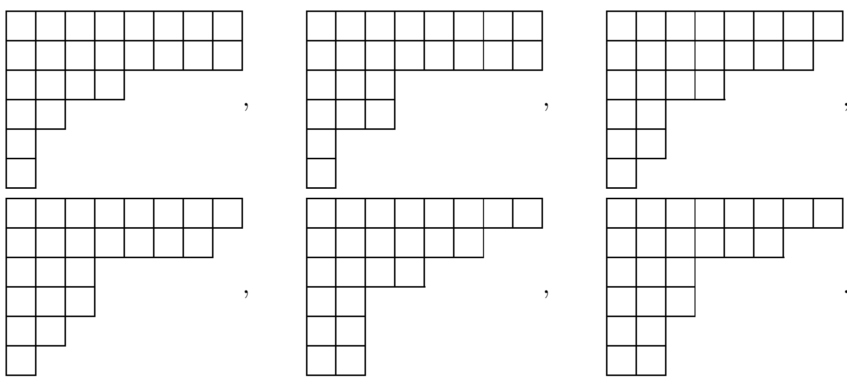
So we have to calculate the product knowing that we are only interested in those $[\mu]$ with $hl(\mu) = hl(\mathcal{B})$. So we first multiply $[1]$ with $[2^2]$ which gives us $[1] \otimes [2,2] = [3,2] + [2,2,1]$ where we had the choice to maximize the first row or column. To obtain the $[\mu]$ we now multiply both with $[4,2]$ and obtain:

$$[3,2] \otimes [4,2] = [7,3,1] + [7,2,2] + [6,3,1,1] + [6,2,2,1] + 14 \text{ other characters}$$

$$[2,2,1] \otimes [4,2] = [6,3,1,1] + [6,2,2,1] + [5,3,1,1,1] + [5,2,2,1,1] + 10 \text{ other characters}$$

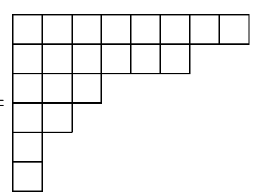
where we had the choice to maximize the first row or column and the second row or column. We also see that $[6,3,1,1]$ and $[6,2,2,1]$ appear with multiplicity 2 in $[\mathcal{B}]$, because we maximized once the first row and once the first column.

This tells us that the characters $[\nu] \in [\mathcal{A}]$ with $hl(\nu) = hl(\mathcal{A})$ are those corresponding to the following partitions:



Here $[8,7,4,2,2,1]$ and $[8,7,3,3,2,1]$ appear with multiplicity 2. Furthermore, since $[\mathcal{B}]$ has 25 different irreducible characters, we know that in $[\mathcal{A}]$ there are 25 different irreducible characters $[\xi]$ with $hl_1(\xi) = hl_1(\mathcal{A})$.

If we would follow the construction in Remark 2.2.11 we construct first γ with

$\gamma =$ 

and then add $k_1 - 1 = 0$ boxes to the first row or column, $k_2 - 1 = 2$ boxes to the second row or column and $k_3 - 1 = 1$ boxes to the third row or column with the same result.

We now want to show that the minimal Durfee size of all $[\mu] \in [\mathcal{A}]$ is $l(hl(\mathcal{A}))$. For this we need the following:

LEMMA 2.2.13. *Let $\mathcal{A}$ be a skew diagram and set $H(i,j) = a$ if the box (i,j) belongs to $nw_a(\mathcal{A})$.*

If $H(i+1,j+1) > 1$ then $H(i+1,j+1) = H(i,j)+1$.

PROOF. Let $H(i+1,j+1) = b$ and let $\mathcal{B}$ be the skew diagram where the first $b-2$ northwest ribbons are removed from $\mathcal{A}$. If the box (i,j) is not in $\mathcal{B}$ then the box $(i+1,j+1)$ would belong to the $b-1$th northwest ribbon of $\mathcal{A}$ which is not the case. So (i,j) is in $\mathcal{B}$ and so must belong to the $b-1$th northwest ribbon of $\mathcal{A}$ and so we have $H(i,j) = b-1 = H(i+1,j+1)-1$. $\square$

LEMMA 2.2.14. *Let $[\mathcal{A}]$ be a skew character.*

The $[\nu] \in [\mathcal{A}]$ with $hl(\nu) = hl(\mathcal{A})$ have minimal Durfee size of all $[\mu] \in [\mathcal{A}]$. In particular, the minimal Durfee size of a character $[\mu] \in [\mathcal{A}]$ is $l(hl(\mathcal{A}))$.

PROOF. Set $h = l(hl(\mathcal{A}))$. The previous lemma tells us that if a box belongs to $nw_h(\mathcal{A})$ then it lies in the southeastern corner of a square (h^h) which lies completely in $\mathcal{A}$. But if the square (h^h) lies in $\mathcal{A}$ then the square (h^h) lies also in all partitions μ whose corresponding character appears in $[\mathcal{A}]$ and so h is a lower bound for the Durfee size of a character $[\mu] \in [\mathcal{A}]$. But since the $[\nu] \in [\mathcal{A}]$ with $hl(\nu) = hl(\mathcal{A})$ have Durfee size $d(\nu) = h$ the claim holds true. $\square$

2.2.3. Maximal Durfee sizes in skew characters. In this subsection we use Theorem 2.2.10 and Theorem 1.4.1 to construct for a product of two characters and for some special skew characters some characters with maximal Durfee size.

We will use that for $k \geq \mu_1+\nu_1, l \geq l(\mu)+l(\nu)$ the Schubert-Product $[\mu] \star_{(k^l)} [\nu]$ is the ordinary product $[\mu] \otimes [\nu]$.

Let us associate a skew diagram $\mathcal{A} = ((m^m)/\alpha)^\circ)/\beta$ to partitions α, β with $m = \max(\alpha_1+\beta_1, l(\alpha)+l(\beta))$. To obtain $\mathcal{A}$ we remove from the square (m^m) the partition β as usual and the partition α rotated by $180°$ from the lower right corner.

Theorem 1.4.1 tells us that characters $[\nu]$ in $[\mathcal{A}]$ correspond to characters $[(m^m)/\nu]$ in the product $[\alpha] \otimes [\beta]$. So characters with maximal Durfee size in the product $[\alpha] \otimes [\beta]$ correspond to the characters with minimal Durfee size in its associated skew character $[\mathcal{A}]$. But from the previous subsection we know some characters with minimal Durfee size.

So from Theorem 2.2.10 we get:

LEMMA 2.2.15. *Let α, β be partitions, $m = \max(\alpha_1+\beta_1, l(\alpha)+l(\beta)), \mathcal{A} = ((m^m)/\alpha)^\circ/\beta$.*

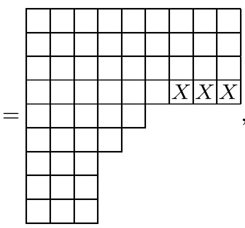
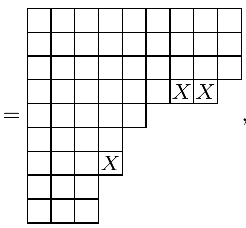
Then for the product $[\alpha] \otimes [\beta]$:

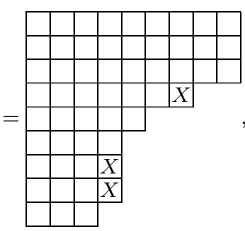
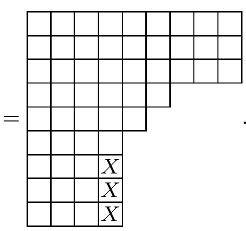
1. *$d([\alpha] \otimes [\beta]) = m - l(hl(\mathcal{A}))$.*
2. *Let $nw_i(\mathcal{A})$ decompose into k_i disconnected ribbons. Then there are at least $\prod_i k_i$ different characters with maximal Durfee size in $[\alpha] \otimes [\beta]$ and the highest multiplicity of a character with maximal Durfee size is at least $\prod_i \binom{k_i-1}{\lfloor \frac{k_i-1}{2} \rfloor}$.*
3. *If $[\nu] \in [\mathcal{A}]$ with $hl(\nu) = hl(\mathcal{A})$ then $[\bar{\nu}] = [(m^m)/\nu] \in [\alpha] \otimes [\beta]$ has maximal Durfee size in $[\alpha] \otimes [\beta]$. (By Remark 2.2.11 we know how to construct those ν explicitly.)*

Example 2.2.16. If we want to know for $\alpha = (5^2, 3^2, 2), \beta = (4,3,1^2)$ some characters with maximal Durfee size in the product $[\alpha] \otimes [\beta]$ we first construct the associated skew diagram $\mathcal{A}$

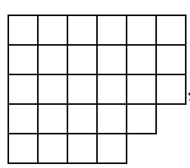
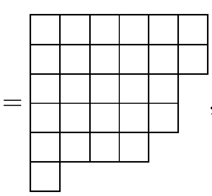
$$\mathcal{A} = \begin{array}{ccccccccc}
 & & & & & 1 & 1 & 1 & 1 \\
 & & & & & 1 & 2 & 2 & 2 \\
 & & & 1 & 1 & 1 & 2 & 3 & 3 \\
 & & & 1 & 2 & 2 & 2 & 3 & 4 \\
 & & 1 & 1 & 2 & 3 & 3 & 3 & 4 \\
1 & 1 & 1 & 2 & 2 & 3 & 4 & 4 & \\
1 & 2 & 2 & 2 & 3 & 3 & 4 & 5 & \\
1 & 2 & 3 & 3 & 3 & 4 & & & \\
1 & 2 & 3 & 4 & 4 & & & &
\end{array}$$

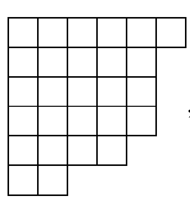
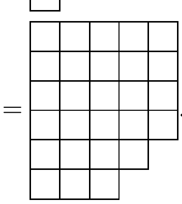
where the boxes have the entry i if they belong to $nw_i(\mathcal{A})$. By Remark 2.2.11 we have the following partitions with maximal principal hook length partition in $[\mathcal{A}]$:

$\nu_1 =$  , $\nu_2 =$  ,

$\nu_3 =$  , $\nu_4 =$  .

The empty boxes in the ν_i form the partition γ from Remark 2.2.11. By Lemma 2.2.12 the multiplicities of ν_1 and ν_4 are 1 and the multiplicities of ν_2 and ν_3 are 3. So we have in $[\alpha] \otimes [\beta]$ the characters $[\bar{\nu}_i] = [(9^9)/\nu_i]$ with maximal Durfee size and corresponding multiplicities:

$\bar{\nu}_1 =$  , $\bar{\nu}_2 =$  ,

$\bar{\nu}_3 =$  , $\bar{\nu}_4 =$  .

But there are many more characters with maximal Durfee size and the characters $[7, 6, 5, 4, 3, 1^2]$, $[7, 5^2, 4, 3, 2, 1]$, $[7, 6, 5, 4, 3, 2]$ and $[7, 6, 4^2, 3, 2, 1]$ all appear with the highest multiplicity which is 13.

For a skew diagram λ/μ with $\lambda = (\lambda_1^k, \lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_l)$, Theorem 1.4.1 together with Theorem 2.2.2 and Remark 2.2.5 gives us some characters $[\alpha] \in [\lambda/\mu]$ with maximal Durfee size if $\lambda_1 = l, k \geq l(\mu)$ and $\mu_1 \leq \lambda_l$.

So we get the following:

LEMMA 2.2.17. *Let λ, μ be partitions with $\lambda = (\lambda_1^k, \lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_l), \lambda_1 = l, k \geq l(\mu)$ and $\mu_1 \leq \lambda_l$. We set $\bar{\lambda} = (l^l)/\lambda$. Then:*

(1) $d(\lambda/\mu) = l - \max(d(\mu), d(\bar{\lambda}))$.
(2) *There are at least $2^{\min(d(\mu),d(\bar{\lambda}))}$ different characters with maximal Durfee size in $[\lambda/\mu]$ and at least $2^{\min(d(\mu),d(\bar{\lambda}))}$ of them appear with multiplicity* 1.
(3) *If $[\alpha] \in [\mu] \otimes [\bar{\lambda}]$ with $hl(\alpha) = hl([\mu] \otimes [\bar{\lambda}])$ then $[\bar{\alpha}] = [(m^m)/\alpha] \in [\lambda/\mu]$ has maximal Durfee size in $[\lambda/\mu]$.*

2.2.4. Necessary conditions for equality of skew characters. The results of Subsection 2.2.2 give us the following conditions for two skew diagrams $\mathcal{A}, \mathcal{B}$ to represent the same skew characters.

THEOREM 2.2.18. *Let $\mathcal{A}, \mathcal{B}$ be skew diagrams.*
If $[\mathcal{A}] = [\mathcal{B}]$ then the following holds true:

(1) $\pi_{nw}(\mathcal{A}) = \pi_{nw}(\mathcal{B})$.
(2) *For every i the numbers of ribbons into which $nw_i(\mathcal{A})$ and $nw_i(\mathcal{B})$ decompose are the same.*
(3) *For every i the arm resp. leg length of $nw_i(\mathcal{A})$ and $nw_i(\mathcal{B})$ are the same.*
(4) *For every i, if we remove the first i northwest ribbons from $\mathcal{A}$ resp. $\mathcal{B}$ to get $\tilde{\mathcal{A}}$ resp. $\tilde{\mathcal{B}}$ then $[\tilde{\mathcal{A}}] = [\tilde{\mathcal{B}}]$.*

We want to use Theorem 2.2.18 to check if the skew diagrams

$$\mathcal{A} = (10^2, 8^4, 5^2)/(5^4) \qquad \text{and} \qquad \mathcal{B} = (10^4, 8^2, 3^2)/(5^4)$$

given in Example 2.2.1 give rise to the same skew character. We label the boxes contained in nw_i with i and have the following situation:

$$\mathcal{A} = \begin{array}{cccccccccc} &&&&& 1&1&1&1&1 \\ &&&&& 1&2&2&2&2 \\ &&&&& 1&2&3&& \\ &&&&& 1&2&3&& \\ 1&1&1&1&1&1&2&3&& \\ 1&2&2&2&2&2&2&3&& \\ 1&2&3&3&3&&&&& \\ 1&2&3&4&4&&&&& \end{array}, \qquad \mathcal{B} = \begin{array}{cccccccccc} &&&&& 1&1&1&1&1 \\ &&&&& 1&2&2&2&2 \\ &&&&& 1&2&3&3&3 \\ &&&&& 1&2&3&4&4 \\ 1&1&1&1&1&1&2&3&& \\ 1&2&2&2&2&2&2&3&& \\ 1&2&3&&&&&&& \\ 1&2&3&&&&&&& \end{array}.$$

We see that the parts $1-3$ of Theorem 2.2.18 hold true and investigate 4. If we remove the first three northwest ribbons the remaining skew diagram is in both cases the partition (2). If we remove only the first two northwest ribbons we get $\tilde{\mathcal{A}} = (4^4, 3^2)/(3^4)$ and $\tilde{\mathcal{B}} = (4^2, 2^2, 1^2)/(1^4)$. We have

$$[\tilde{\mathcal{A}}] = [4^2, 1^2] + [4, 3, 1^3] + [3^2, 1^4],$$
$$[\tilde{\mathcal{B}}] = [4^2, 1^2] + [4, 3, 1^3] + [3^2, 1^4] + [4, 3, 2, 1] + [3^2, 2^2] + [3^2, 2, 1^2]$$

and so $[\mathcal{A}] \neq [\mathcal{B}]$.

The decomposition of $[\tilde{\mathcal{A}}]$ and $[\tilde{\mathcal{B}}]$ gives us the following partitions whose characters appear with multiplicity 1 in both $[\mathcal{A}]$ and $[\mathcal{B}]$:

$$\begin{ytableau} \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & X & X & X & X \\ \ & \ & X & X & X & X \\ \ & \ & X \\ \ & \ & X \\ \ & \ \\ \ & \ \end{ytableau}\ , \quad \begin{ytableau} \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & X & X & X & X \\ \ & \ & X & X & X \\ \ & \ & X \\ \ & \ & X \\ \ & \ & X \\ \ & \ \end{ytableau}\ , \quad \begin{ytableau} \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & X & X & X \\ \ & \ & X & X & X \\ \ & \ & X \\ \ & \ & X \\ \ & \ & X \\ \ & \ & X \end{ytableau}$$

and the partitions whose corresponding characters appear with multiplicity 1 in $[\mathcal{B}]$ but not in $[\mathcal{A}]$:

$$\begin{ytableau} \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & X & X & X & X \\ \ & \ & X & X & X \\ \ & \ & X & X \\ \ & \ & X \\ \ & \ \\ \ & \ \end{ytableau}\ , \quad \begin{ytableau} \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & X & X & X \\ \ & \ & X & X & X \\ \ & \ & X & X \\ \ & \ & X & X \\ \ & \ \\ \ & \ \end{ytableau}\ , \quad \begin{ytableau} \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \ & \ & X & X & X \\ \ & \ & X & X & X \\ \ & \ & X & X \\ \ & \ & X \\ \ & \ & X \\ \ & \ \end{ytableau}\ .$$

These are all characters appearing in $[\mathcal{A}]$ or $[\mathcal{B}]$ with maximal first and second principal hook length.

CHAPTER 3

Equality of multiplicity free skew characters

The question under which circumstances two different skew diagrams λ/μ and α/β give rise to the same skew character $[\lambda/\mu] = [\alpha/\beta]$ has lately received much attention and is by Theorem 1.4.1 equivalent to the question under which circumstances two products of Schubert classes $\sigma_{\alpha_1} \cdot \sigma_{\alpha_2}$ and $\sigma_{\beta_1} \cdot \sigma_{\beta_2}$ are equal.

Trivial cases for equality of skew characters $[\lambda/\mu] = [\alpha/\beta]$ are given if the skew diagrams λ/μ and α/β are the same up to translation or rotation.

In [**RSW**] and later in [**MvW09a**] a method for constructing skew diagrams with nontrivial equality of their corresponding skew characters was presented. The fact that for a staircase partition $\lambda = (l, l-1, \dots, 2, 1)$ the skew diagram λ/μ and its conjugate $(\lambda/\mu)^c$ give rise to the same skew character was proved in [**RSW**, Theorem 7.32].

On the other hand, new necessary conditions for two skew diagrams λ/μ and α/β to give rise to the same skew character have been given recently in [**McN**] and also in Subsection 2.2.4.

Using algebraic arguments Reiner et al. showed in [**RSW**] that equality of $[\lambda/\mu] = [\alpha/\beta]$ when λ/μ decays into partitions is only possible in the trivial cases.

We will examine in this chapter the case when λ/μ and α/β are connected skew diagrams and $[\lambda/\mu] = [\alpha/\beta]$ is multiplicity free. We will see that the only nontrivial case for $[\lambda/\mu] = [\alpha/\beta]$ is when $\lambda = \alpha$ is a staircase partition and additionally $\lambda/\mu = (\alpha/\beta)^c$.

The results of this chapter appeared in my paper [**Gut4**].

3.1. Notation and preliminary

In the next section we will prove the main theorem of this chapter:

THEOREM 3.1.1. *Let λ/μ and α/β be (connected or decaying) basic skew diagrams and $[\lambda/\mu] = [\alpha/\beta]$ multiplicity free.*

Then up to translation or rotation

- $\lambda/\mu = \alpha/\beta$
- *or $\lambda = \alpha = (l, l-1, \dots, 2, 1)$ and the skew diagrams are conjugate of each other $\lambda/\mu = (\alpha/\beta)^c$.*

Here translation or rotation does not require the entire skew diagram to be translated or rotated, but also allows translation or rotation of the skew diagrams into which λ/μ decays if it decays.

Before we prove this theorem we will fix our notation for this chapter and remind respectively prove some general lemmas.

Let λ/μ be a basic skew diagram. We can define two paths on λ/μ. The inner path starts in the lower left corner with an upward segment, follows the shape of μ

and ends with a segment to right in the upper right corner. The outer path starts in the lower left corner with a segment to the right, follows the shape of λ and ends with an upward segment in the upper right corner.

With s_{in} we refer to the length of the shortest straight segment of the inner lattice path while s_{out} is the length of the shortest straight segment of the outer lattice path.

In my Diploma thesis [**Gut1**] I classified the skew diagrams λ/μ whose corresponding skew character $[\lambda/\mu]$ is multiplicity free which means that in the decomposition $[\lambda/\mu] = \sum c(\lambda;\mu,\nu)[\nu]$ all coefficients $c(\lambda;\mu,\nu)$ are either 0 or 1. For this I used the following theorem which will be generalized in Lemma 4.1.1:

THEOREM 3.1.2 ([**Gut1**, Theorem 3.1]). *Let λ, μ, ν be partitions and $a, b \geq 0$ integers. Then:*

$$c(\lambda;\mu,\nu) \leq c(\lambda + (1^{a+b}); \mu + (1^a), \nu + (1^b)).$$

The classification of the multiplicity free skew characters is then the following:

THEOREM 3.1.3 ([**Gut1**, Theorem 3.8]). *Let λ/μ be a basic skew diagram which is neither a partition nor a rotated partition. Then $[\lambda/\mu]$ is multiplicity free if and only if up to rotation of λ/μ μ is a rectangle and additionally one of the following conditions holds:*

(1) $s_{in} = 1$,
(2) $s_{in} = 2$, $dp(\lambda) = 3$,
(3) $dp(\lambda) = 3$, $s_{out} = 1$,
(4) $dp(\lambda) = 2$.

Recall that $dp(\lambda)$ is the number of different parts in λ.

Reiner et al. proved in [**RSW**, Section 6] the following, using the Jacobi-Trudi determinant but no LR combinatorics:

LEMMA 3.1.4 ([**RSW**, Section 6]). *Let $\mathcal{A}^1$ and $\mathcal{A}^2$ be skew diagrams and $\mathcal{A}^1$ decay into partitions. Let $[\mathcal{A}^1] = [\mathcal{A}^2]$.*

Then the equality is trivial i.e. up to translation or rotation $\mathcal{A}^1$ and $\mathcal{A}^2$ are the same.

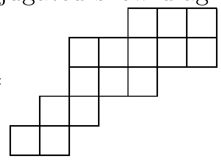
Here again translation or rotation does not require the entire skew diagram to be translated or rotated but also includes the case when the partitions into which $\mathcal{A}^1$ decays are translated or rotated independent of each other.

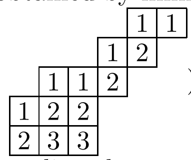
We will use this lemma always to argue that $\mathcal{A}^1 = \gamma \otimes \delta$ with partitions γ, δ and $[\mathcal{A}^1] = [\mathcal{A}^2]$ requires $\mathcal{A}^2 = \gamma \otimes \delta$.

To exclude the trivial cases for $[\lambda/\mu] = [\alpha/\beta]$ we will in the following always assume that μ is a rectangle and both λ/μ and α/β are basic skew diagrams. Because of the previous Lemma 3.1.4 we may furthermore assume that λ/μ (and so α/β) is connected.

Furthermore, we will use the following notation:
$\lambda = (\lambda_1^{l_1}, \lambda_2^{l_2}, \ldots)$ with $l = l(\lambda) = \sum_i l_i$ and $\lambda_i > \lambda_{i+1}$, $\mu = (\mu_1^m)$,
$\alpha = (\alpha_1^{a_1}, \alpha_2^{a_2}, \ldots)$ with $a = l(\alpha) = \sum_i a_i$ and $\alpha_i > \alpha_{i+1}$, $\beta = (\beta_1^b)$.
Before we begin with the proofs we state some additional facts.

Recall that the parts of a skew diagram $\mathcal{A} = \gamma/\delta$ are the numbers $\gamma_i - \delta_i$ $(1 \leq i \leq l(\gamma))$ and so are the number of boxes in the rows of $\mathcal{A}$. Furthermore, the heights of a skew diagram $\mathcal{A}$ are the number of boxes in the columns of $\mathcal{A}$ and so are the parts of the conjugated skew diagram. For example, the skew diagram

$\mathcal{A} = (7^2, 5, 3, 2)/(4, 2^2, 1) =$  has the parts $3, 5, 3, 2, 2$ and heights $1, 2, 3, 2, 3, 2, 2$.

It is known, that skew diagrams which give rise to the same skew character have to have the same parts and heights in the same quantity. This follows from the fact, that the LR tableau of the skew diagram $\mathcal{A}$ obtained by filling every column with the entries 1 to the height of the column (e.g. 

) has as content ν the lexicographic biggest partition whose corresponding character $[\nu]$ appears in the decomposition of $[\mathcal{A}]$. Clearly this is the partition obtained by reordering the heights of $\mathcal{A}$ to form a partition. So skew diagrams which give rise to the same skew character have to have the same heights in the same quantity and by conjugation the same holds true for the parts.

In the following we will assume that $[\lambda/\mu] = [\alpha/\beta]$. Since we need the same number of rows and columns in λ/μ and α/β we need $\lambda_1 = \alpha_1$ and $a = l$ if we assume $[\lambda/\mu] = [\alpha/\beta]$.

We used the following $1-1$ relation already in our proof of Theorem 2.1.1. If we remove from an LR tableau of shape λ/μ containing λ_1 entries 1 all the boxes with entry 1 and replace every entry $i > 1$ by $i-1$ we obtain an LR tableau of a shape which is obtained from λ/μ by removing the top box from every column. This gives us a $1-1$ relation between the characters $[\nu] \in [\lambda/\mu]$ with maximal first part $\nu_1 = \lambda_1$ and arbitrary characters $[\xi] \in [\hat{\lambda}/\mu]$ with $\hat{\lambda}/\mu$ the skew diagram obtained by removing the top boxes in every column of λ/μ so $\hat{\lambda} = (\lambda_1^{l_1-1}, \lambda_2^{l_2}, \ldots)$ and the $1-1$ relation is given by $\xi_i = \nu_{i+1}$. So we have the following lemma:

Lemma 3.1.5. *Let $[\lambda/\mu] = [\alpha/\beta]$.*

Let $\hat{\lambda}/\mu$ (resp. $\hat{\alpha}/\beta$) be the skew diagram obtained from λ/μ (resp. α/β) by removing the top $i \geq 1$ boxes from every column of λ/μ (resp. α/β). Let $\tilde{\lambda}/\mu$ (resp. $\tilde{\alpha}/\beta$) be the skew diagram obtained from λ/μ (resp. α/β) by removing in every row of λ/μ (resp. α/β) the left $i \geq 1$ boxes.

Then $[\hat{\lambda}/\mu] = [\hat{\alpha}/\beta]$ and $[\tilde{\lambda}/\mu] = [\tilde{\alpha}/\beta]$.

Proof. $[\hat{\lambda}/\mu] = [\hat{\alpha}/\beta]$ follows from the above $1-1$ relation. $[\tilde{\lambda}/\mu] = [\tilde{\alpha}/\beta]$ follows then by conjugation symmetry. □

In most cases we remove the boxes until $\hat{\lambda}/\mu$ decays into two disconnected skew diagrams so that we can use Lemma 3.1.4.

If we say in the following that we remove l_j resp. λ_i boxes from the top resp. left of λ/μ we always mean that we remove in every column l_j resp. in every row λ_i boxes.

For $k \geq \mu_1 + \nu_1, l \geq l(\mu) + l(\nu)$ the Schubert product $[\mu] \star_{(k^l)} [\nu]$ is the ordinary product $[\mu] \otimes [\nu]$.

LEMMA 3.1.6. *Let* $[\mathcal{A}^1] = [\mathcal{A}^2]$ *with* $\mathcal{A}^1 = \gamma/\delta$ *being an arbitrary skew diagram having a part* γ_1 *and a height* $l(\gamma)$ *(see Figure 1).*
Then $\mathcal{A}^1 = \mathcal{A}^2$ *or* $\mathcal{A}^1 = (\mathcal{A}^2)^\circ$.

PROOF. Since $\mathcal{A}^1$ and $\mathcal{A}^2$ have the same heights and parts we have also the part γ^1 and the height $l(\gamma)$ in $\mathcal{A}^2$. The lemma follows now from Lemma 3.1.4 and Theorem 1.4.1.

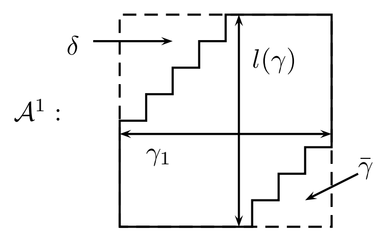


Figure 1. Lemma 3.1.6: $\mathcal{A}^1$

□

3.2. Proof of the main theorem

The proof of Theorem 3.1.1 is arranged as follows. We first prove that $[\lambda/\mu] = [\alpha/\beta]$ and $dp(\lambda) = 2$ requires $\lambda/\mu = \alpha/\beta$ or $\lambda/\mu = (\alpha/\beta)^\circ$. Next we prove that $[\lambda/\mu] = [\alpha/\beta]$ and $dp(\lambda) = 3$ requires $dp(\alpha) = 3$. Then we prove that $\lambda/\mu = \alpha/\beta$ is required if both λ and α are $dp = 3$ partitions and $[\lambda/\mu] = [\alpha/\beta]$.

Then we examine the cases when both λ and α have more than 3 different parts and $[\lambda/\mu] = [\alpha/\beta]$ is multiplicity free.

LEMMA 3.2.1. *Let* $[\lambda/\mu] = [\alpha/\beta]$ *and* λ *be a* $dp = 2$ *partition.*
Then α *is also a* $dp = 2$ *partition and* $\alpha/\beta = \lambda/\mu$ *or* $\alpha/\beta = (\lambda/\mu)^\circ$.

PROOF. We will assume that α has $n \geq 2$ different parts, $dp(\alpha) = n \geq 2$, and show first that $n = 2$.

Case 1: $\lambda_2 > \mu_1$, $l_1 > m$. This case is covered by Lemma 3.1.6.

So we have either *Case 2:* $l_1 \leq m$ or *Case 3:* $\lambda_2 \leq \mu_1$. We cannot have both at the same time without λ/μ decaying into two disconnected rectangles. It is sufficient to examine only the *Case 2* since *Case 3* follows then by conjugation symmetry.

Case 2: $l_1 \leq m, \lambda_2 > \mu_1$. By comparing the heights of length l and the parts λ_1 in λ/μ and α/β we get $a_1 \leq b, \alpha_n > \beta_1$.

Since $\lambda_2 > \mu_1$ we have $\lambda_2 - \mu_1$ times the height l in λ/μ and so need $\lambda_2 - \mu_1 = \alpha_n - \beta_1$.

If we remove in λ/μ the left $\lambda_2 - \mu_1$ boxes the remaining skew diagram is:

$$\tilde{\lambda}/\mu = ((\underbrace{\lambda_1 - \mu_1 - (\lambda_2 - \mu_1)}_{=\lambda_1 - \lambda_2})^{l_1}) \otimes ((\underbrace{\lambda_2 - \lambda_2 + \mu_1}_{=\mu_1})^{l-m}).$$

If we remove in α/β the left $\lambda_2 - \mu_1 = \alpha_n - \beta_1 < \alpha_n$ boxes the remaining skew diagram has to decay by Lemmas 3.1.4 and 3.1.5 but decays only for $b \geq l - a_n$ and is then:

$$\tilde{\alpha}/\beta = ((\lambda_1 - \alpha_n)^{a_1}, (\alpha_2 - \alpha_n)^{a_2}, \ldots (\alpha_{n-1} - \alpha_n)^{a_{n-1}}) \otimes ((\beta_1)^{l-b}).$$

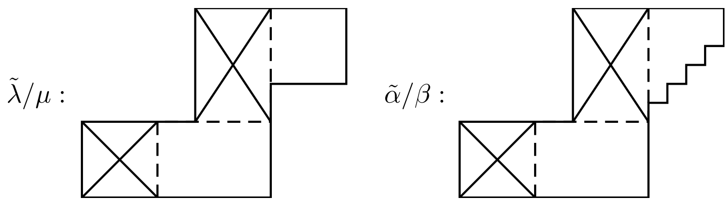


Figure 2. *Case 2:* $\tilde{\lambda}/\mu$ and $\tilde{\alpha}/\beta$

By Lemmas 3.1.4 and 3.1.5 $\tilde{\lambda}/\mu$ and $\tilde{\alpha}/\beta$ have to be related by translation or rotation if $[\tilde{\lambda}/\mu] = [\tilde{\alpha}/\beta]$ and so we have $[\lambda/\mu] \neq [\alpha/\beta]$ for $n \geq 3$.

For $n = 2$ we have:

$$\tilde{\alpha}/\beta = ((\lambda_1 - \alpha_2)^{a_1}) \otimes (\beta_1^{l-b}).$$

By Lemmas 3.1.4 and 3.1.5 we have either

$$\textit{Case 2.1:} \qquad (\lambda_1 - \lambda_2)^{l_1} = (\lambda_1 - \alpha_2)^{a_1}, \qquad \mu_1^{l-m} = \beta_1^{l-b}$$

or

$$\textit{Case 2.2:} \qquad (\lambda_1 - \lambda_2)^{l_1} = \beta_1^{l-b}, \qquad \mu_1^{l-m} = (\lambda_1 - \alpha_2)^{a_1}.$$

In *Case 2.1* we immediately get $\lambda/\mu = \alpha/\beta$.

In *Case 2.2* we have

$$\lambda_1 - \lambda_2 = \beta_1, \qquad \lambda_1 - \alpha_2 = \mu_1, \qquad b = l - l_1, \qquad l - a_1 = m$$

which means $\lambda/\mu = (\alpha/\beta)^\circ$. □

Lemma 3.2.2. *Let λ be a $dp = 3$ partition and $[\lambda/\mu] = [\alpha/\beta]$. Then $dp(\alpha) = 3$.*

Proof. By Lemma 3.2.1 α cannot have only 2 different parts. So we will assume, that $dp(\alpha) = n \geq 4$ and show that we get contradictions.

Case 1: $l_1 \leq m$, $\lambda_3 \leq \mu_1$. Since we do not have parts λ_1 or heights l in λ/μ we also have $a_1 \leq b$ and $\alpha_n \leq \beta_1$.

For λ/μ to be connected we need $\lambda_2 > \mu_1$ and $l_1 + l_2 > m$.

The cases *Case 1.1:* $\lambda_3 \leq \lambda_2 - \mu_1$ and *Case 1.2:* $l_1 \leq l_1 + l_2 - m$ are related by conjugation symmetry so it is sufficient to consider only *Case 1.1*.

Case 1.1: $\lambda_3 \leq \lambda_2 - \mu_1$. If we remove in λ/μ λ_3 boxes from the left and in total $l \cdot \lambda_3$ boxes the remaining skew diagram is $((\lambda_1 - \lambda_3)^{l_1}, (\lambda_2 - \lambda_3)^{l_2})/\mu$ and is not a partition and decays (in the case $\lambda_2 - \mu_1 = \lambda_3$) only after λ_3 boxes from the left are removed and is connected if only $\lambda_3 - 1$ boxes are removed from the left.

If we remove in α/β λ_3 boxes from the left the remaining skew diagram $\tilde{\alpha}/\beta$ has to be by Lemma 3.1.5 and Lemma 3.2.1 a $dp = 2$ partition with a rectangle removed. This cannot be obtained. The best we can obtain is the following. If we demand that $\tilde{\alpha}/\beta$ is a $dp = 2$ partition with or without another partition removed and demand that removing λ_3 boxes from the left removes in total $l \cdot \lambda_3$ boxes we need $n = 4, \alpha_4 = \lambda_3$ as well as either $b = a_1, \alpha_1 - \beta_1 - \lambda_3 = 0$ or $b = a_1 + a_2, \alpha_2 - \beta_1 - \lambda_3 = 0$ (see Figure 3. The case $b = a_1 + a_2 + a_3$, $\alpha_3 - \beta_1 - \lambda_3 = 0$ can be excluded, because in this case either α/β decays or has a height l.). In both cases removing $\lambda_3 - 1$ boxes from the left of α/β gives a decaying skew diagram.

Case 1.3: $\lambda_3 > \lambda_2 - \mu_1$, $l_1 > l_1 + l_2 - m$.

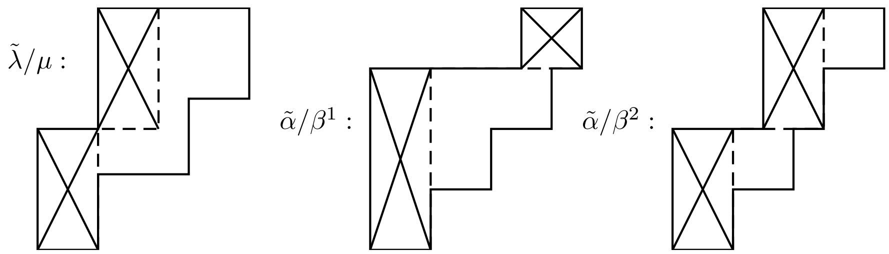


Figure 3. *Case 1.1*: $\tilde{\lambda}/\mu$ and $\tilde{\alpha}/\beta^i$

If we remove in λ/μ $\lambda_2 - \mu_1$ boxes from the left and in total $l \cdot (\lambda_2 - \mu_1)$ boxes the remaining skew diagram $\tilde{\lambda}/\mu$ decays into a rectangle and a $dp = 2$ partition. If we remove only $\lambda_2 - \mu_1 - 1$ boxes from the left the remaining skew diagram does not decay. We have (see Figure 4):

$$\tilde{\lambda}/\mu = ((\lambda_1 - \lambda_2)^{l_1}) \otimes (\mu_1^{l_1+l_2-m}, (\lambda_3 - \lambda_2 + \mu_1)^{l_3}). \tag{3.2.1}$$

If we remove in λ/μ $l_1 + l_2 - m$ boxes from the top which are in total $\lambda_1 \cdot (l_1 + l_2 - m)$ boxes the remaining skew diagram $\hat{\lambda}/\mu$ decays also into a rectangle and a $dp = 2$ partition. Again removing only $l_1 + l_2 - m - 1$ boxes from the top gives a connected skew diagram. We have (see Figure 4):

$$\hat{\lambda}/\mu = ((\lambda_1 - \mu_1)^{m-l_2}, (\lambda_2 - \mu_1)^{l_2}) \otimes (\lambda_3^{l_3}). \tag{3.2.2}$$

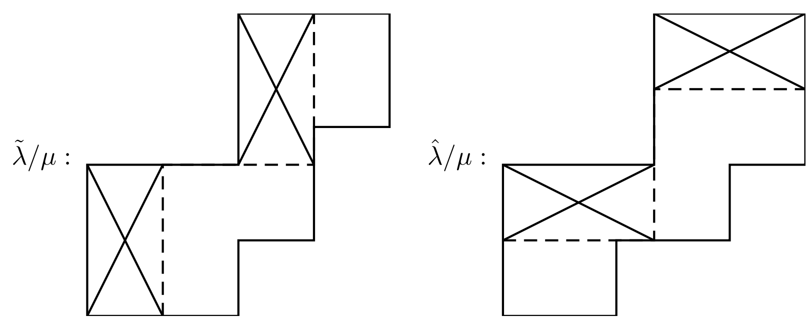


Figure 4. *Case 1.3*: $\tilde{\lambda}/\mu$ and $\hat{\lambda}/\mu$

If we remove in α/β $\lambda_2 - \mu_1$ boxes from the left the remaining skew diagram has by Lemma 3.1.5 also to decay into a rectangle and a $dp = 2$ partition. To obtain this and remove in total $l \cdot (\lambda_2 - \mu_1)$ boxes and also be in the situation that after removing $\lambda_2 - \mu_1 - 1$ boxes from the left the remaining skew diagram $\tilde{\alpha}/\beta$ does not decay we need $n = 4$, $\alpha_4 = \lambda_2 - \mu_1$ as well as either $a_1 \leq b < a_1 + a_2$, $\alpha_2 - \beta_1 = \lambda_2 - \mu_1$ or $a_1 + a_2 \leq b < a_1 + a_2 + a_3$, $\alpha_3 - \beta_1 = \lambda_2 - \mu_1$ (see Figure 5).

With the same reasoning the skew diagram $\hat{\alpha}/\beta$ obtained by removing $l_1 + l_2 - m$ boxes from the top of α/β must decay into a rectangle and a skew diagram and we need $n = 4$, $a_1 = l_1 + l_2 - m$ as well as either $\alpha_3 \leq \beta_1 < \alpha_2$, $a_1 + a_2 - b = l_1 + l_2 - m$ or $\alpha_4 \leq \beta_1 < \alpha_3$, $a_1 + a_2 + a_3 - b = l_1 + l_2 - m$ (see Figure 6).

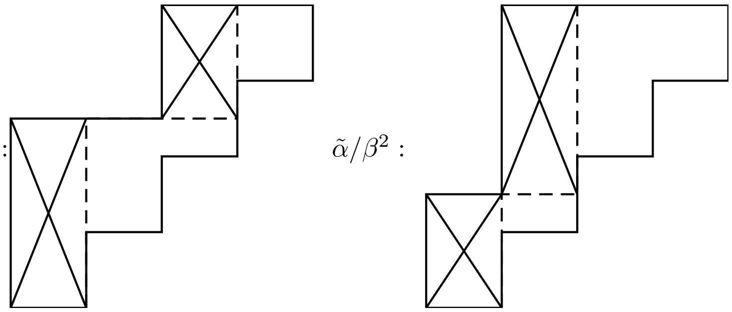


Figure 5. *Case 1.3*: The two cases of $\tilde{\alpha}/\beta^i$

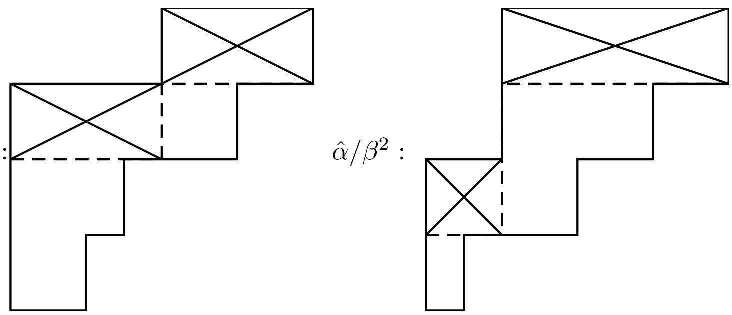


Figure 6. *Case 1.3*: The two cases of $\hat{\alpha}/\beta^i$

By Lemmas 3.1.4 and 3.1.5 the skew diagrams $\hat{\lambda}/\mu$ resp. $\tilde{\lambda}/\mu$ and $\hat{\alpha}/\beta$ resp. $\tilde{\alpha}/\beta$ have to decay into the same partitions. So we get the following:

In the *Case 1.3.1:* $a_1 \leq b < a_1 + a_2$, $\alpha_2 - \beta_1 = \lambda_2 - \mu_1$ we have:

$$\tilde{\alpha}/\beta^1 = ((\lambda_1 - \alpha_2)^{a_1}) \otimes ((\underbrace{\alpha_2 - \lambda_2 + \mu_1}_{=\beta_1})^{a_1+a_2-b}, (\alpha_3 - \alpha_2 + \beta_1)^{a_3}).$$

Comparing with (3.2.1) gives:

$$\alpha_2 = \lambda_2, \qquad \beta_1 = \mu_1, \qquad \alpha_3 = \lambda_3. \tag{3.2.3}$$

In the *Case 1.3.2:* $a_1 + a_2 \leq b < a_1 + a_2 + a_3$, $\alpha_3 - \beta_1 = \lambda_2 - \mu_1$ we have:

$$\tilde{\alpha}/\beta^2 = ((\lambda_1 - \alpha_3)^{a_1}, (\alpha_2 - \lambda_2 - \beta_1 + \mu_1)^{a_2}) \otimes (\beta_1^{a_1+a_2+a_3-b}).$$

Comparing with (3.2.1) gives:

$$\alpha_3 = \lambda_1 - \mu_1, \qquad \alpha_3 - \beta_1 = \lambda_2 - \mu_1, \qquad \alpha_2 = \lambda_3 + \beta_1. \tag{3.2.4}$$

In the *Case 1.3.x.1:* $\alpha_3 \leq \beta_1 < \alpha_2$, $a_1 + a_2 - b = l_1 + l_2 - m$ we have:

$$\hat{\alpha}/\beta^1 = ((\alpha_2 - \beta_1)^{a_2}) \otimes (\alpha_3^{a_3}, \alpha_4^{a_4}).$$

Comparing with (3.2.2) gives:

$$\alpha_3 \leq \beta_1, \qquad \alpha_2 = \lambda_3 + \beta_1, \qquad \alpha_4 = \lambda_2 - \mu_1. \tag{3.2.5}$$

In the *Case 1.3.x.2:* $\alpha_4 \leq \beta_1 < \alpha_3$, $a_1 + a_2 + a_3 - b = l_1 + l_2 - m$ we have:

$$\hat{\alpha}/\beta^2 = ((\alpha_2 - \beta_1)^{a_2}, (\alpha_3 - \beta_1)^{a_3}) \otimes (\alpha_4^{a_4}).$$

Comparing with (3.2.2) gives:

$$\alpha_2 = \lambda_1 - \mu_1 + \beta_1, \qquad \alpha_3 = \lambda_2 + \beta_1 - \mu_1, \qquad \alpha_4 = \lambda_3. \tag{3.2.6}$$

From the equations of *Case 1.3.1.1* ((3.2.3) and (3.2.5)) follows $\alpha_3 + \beta_1 = \lambda_3 + \beta_1 = \alpha_2 = \lambda_2 = \alpha_4 + \mu_1 = \alpha_4 + \beta_1$, but $\alpha_3 \neq \alpha_4$.

From the equations of *Case 1.3.1.2* ((3.2.3) and (3.2.6)) follows $\alpha_3 = \lambda_3 = \alpha_4$, but $\alpha_3 \neq \alpha_4$.

From the equations of *Case 1.3.2.1* ((3.2.4) and (3.2.5)) follows $0 \geq \alpha_3 - \beta_1 = \lambda_2 - \mu_1$, but since we are in the case $l_1 \leq m$ and $\lambda_3 \leq \mu_1$, λ/μ decays for $\lambda_2 - \mu_1 \leq 0$.

From the equations of *Case 1.3.2.2* ((3.2.4) and (3.2.6)) follows $\alpha_3 = \lambda_1 - \mu_1 = \alpha_2 - \beta_1 = \lambda_3 = \alpha_4$, but $\alpha_4 \neq \alpha_3$.

These contradictions finish *Case 1*.

Case 2: $l_1 > m$, $\lambda_3 \leq \mu_1$ and *Case 3:* $l_1 \leq m$, $\lambda_3 > \mu_1$ are related by conjugation symmetry so it is sufficient to consider only *Case 2*.

Case 2: $l_1 > m$, $\lambda_3 \leq \mu_1$. Since we have the part λ_1 $l_1 - m$ times in λ/μ we need $l_1 - m = a_1 - b$. Removing $l_1 - m$ boxes from the top of λ/μ gives a connected skew diagram $\hat{\lambda}/\mu$ for $\lambda_2 > \mu_1$ with

$$\hat{\lambda}/\mu = (\lambda_1^m, \lambda_2^{l_2}, \lambda_3^{l_3})/\mu$$

or a decaying skew diagram $\hat{\lambda}/\mu$ for $\lambda_2 \leq \mu_1$ with

$$\hat{\lambda}/\mu = ((\lambda_1 - \mu_1)^m) \otimes (\lambda_2^{l_2}, \lambda_3^{l_3}).$$

For $\lambda_2 > \mu_1$ removing $l_1 - m = a_1 - b$ boxes from the top of α/β must yield a connected skew diagram $\hat{\alpha}/\beta$ and we get:

$$\hat{\alpha}/\beta = (\alpha_1^b, \alpha_2^{a_2}, \alpha_3^{a_3}, \alpha_4^{a_4}, \ldots)/\beta.$$

Since we are now in *Case 1* with $l_1 \leq m$, $\lambda_3 \leq \mu_1$ we get from the above $[\lambda/\mu] \neq [\alpha/\beta]$.

For $\lambda_2 \leq \mu_1$ removing $l_1 - m = a_1 - b$ boxes from the top of α/β must yield a decaying skew diagram $\hat{\alpha}/\beta$ and we get:

$$\hat{\lambda}/\mu = ((\alpha_1 - \beta_1)^b) \otimes (\alpha_2^{a_2}, \alpha_3^{a_3}, \alpha_4^{a_4}, \ldots).$$

Lemma 3.1.4 gives $[\lambda/\mu] \neq [\alpha/\beta]$.

Case 4: $l_1 > m$, $\lambda_3 > \mu_1$ is covered by Lemma 3.1.6.

□

Lemma 3.2.3. *Let λ, α be $dp = 3$ partitions and $[\lambda/\mu] = [\alpha/\beta]$. Then $\lambda/\mu = \alpha/\beta$.*

Proof. *Case 1:* $l_1 \leq m$, $\lambda_3 \leq \mu_1$. Since we do not have heights l and parts λ_1 in λ/μ we also need $a_1 \leq b$, $\alpha_3 \leq \beta_1$.

For λ/μ to be connected we need $\lambda_2 > \mu_1$, $l_1 + l_2 > m$.

Case 1.1: $l_1 > l_1 + l_2 - m$ and *Case 1.2:* $\lambda_3 > \lambda_2 - \mu_1$ are related by conjugation symmetry so it is sufficient to consider only *Case 1.1*.

Case 1.1: $l_1 > l_1 + l_2 - m$.

Removing $l_1 + l_2 - m$ boxes from the top of λ/μ gives:

$$\hat{\lambda}/\mu = (\lambda_3^{l_3}) \otimes ((\lambda_1 - \mu_1)^{m - l_2}, (\lambda_2 - \mu_1)^{l_2}).$$

If we remove only l_1+l_2-m-1 boxes from the top of λ/μ the remaining skew diagram does not decay, so we need $l_1+l_2-m = a_1+a_2-b$, $b > a_2$ and removing a_1+a_2-b boxes from the top of α/β gives then:

$$\hat{\alpha}/\beta = (\alpha_3^{a_3}) \otimes ((\lambda_1-\beta_1)^{b-a_2}, (\alpha_2-\beta_1)^{a_2}).$$

This gives $\lambda/\mu = \alpha/\beta$.

Case 1.3: $l_1 \leq l_1+l_2-m \Leftrightarrow m \leq l_2$, $\lambda_3 \leq \lambda_2-\mu_1$.

Removing l_1 boxes from the top of λ/μ removes in total $l_1 \cdot \lambda_1$ boxes and gives a skew diagram which either decays into two disconnected rectangles (for $l_1 = l_1+l_2-m$) or is $(\lambda_2^{l_2}, \lambda_3^{l_3})/\mu$ (for $l_1 < l_1+l_2-m$). We need $a_1 \leq l_1$ because removing l_1 boxes from the top of α/β must either yield a decaying skew diagram or a $dp = 2$ partition with a rectangle removed. But since removing the top l_1 boxes of α/β has to remove in total $l_1 \cdot \lambda_1$ boxes we need also $a_1 \geq l_1$ and so $a_1 = l_1$.

Removing l_1+l_2-m-1 boxes from the top of λ/μ yields a connected skew diagram but removing the top l_1+l_2-m boxes yields:

$$\hat{\lambda}/\mu = (\lambda_3^{l_3}) \otimes ((\lambda_2-\mu_1)^m).$$

Since α/β must also decay after removing l_1+l_2-m boxes from the top but not after removing less boxes we get $a_1+a_2-b = l_1+l_2-m$ and so $a_2-b = l_2-m$.

In an analogous way by removing λ_3 resp. $\lambda_2-\mu_1$ boxes from the left we get $\alpha_3 = \lambda_3$ and $\alpha_2-\beta_1 = \lambda_2-\mu_1$.

We will first examine the *Case 1.3.1:* $\mu_1 \neq \beta_1$ and show that this gives $[\lambda/\mu] \neq [\alpha/\beta]$. This covers by conjugation symmetry also the *Case 1.3.2:* $m \neq b$.

For the following construction we only need:

$$\alpha_2-\beta_1 = \lambda_2-\mu_1, \qquad a_1 = l_1, \qquad m \leq l_2, \qquad b \leq a_2. \tag{3.2.7}$$

Without loss of generality we may also assume that $\mu_1 < \beta_1$ and set $\beta_1 = \mu_1+n$ which gives $\alpha_2 = \lambda_2+n$.

Case 1.3.1.1: $\lambda_1-(\lambda_2+n)+1 \leq \mu_1$.

We can in λ write entries 1 to l_1 into the columns λ_2+n to λ_1 which are in total $\lambda_1-(\lambda_2+n)+1$ columns (see Figure 7). If we place the remaining entries so that they obey the LR rule we get an LR filling of λ with content μ where the box (l_1, λ_2+n) is filled. So there is a character $[\nu]$ in $[\lambda/\mu]$ with ν not containing the box (l_1, λ_2+n).

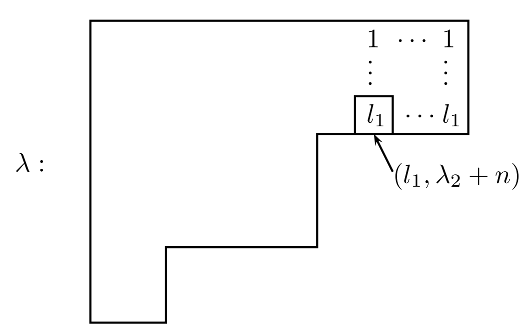


Figure 7. LR filling of λ in the *Case 1.3.1.1*

In α there are $a_2 \geq b$ boxes below the box $(l_1, \lambda_2 + n) = (a_1, \alpha_2)$ so in every LR filling of α with content β the box $(l_1, \lambda_2 + n)$ remains empty. So for every character $[\nu]$ in $[\alpha/\beta]$, ν contains a box in position $(l_1, \lambda_2 + n)$. This gives $[\lambda/\mu] \neq [\alpha/\beta]$.

Case 1.3.1.2: $\lambda_1 - (\lambda_2 + n) \geq \mu_1 \Leftrightarrow \lambda_1 - \lambda_2 \geq \mu_1 + n = \beta_1$.

Let us now construct an LR filling of α with content β which leaves the box $(l_1 + 1, \lambda_2 + 1)$ empty and so gives $[\lambda/\mu] \neq [\alpha/\beta]$.

Place the entries b into the the rows l and $l_1 + a_2$ with at least one b in row l (see Figure 8). Now place for $1 < i < b$ the i above $i + 1$ if possible. For $b - i = a_3$ we cannot place the entry i above the entry $i + 1$ unless there are $\alpha_2 - \alpha_3$ entries b in row $l_1 + a_2$. If there are less than $\alpha_2 - \alpha_3$ entries b in row $l_1 + a_2$ we place the entry i in row $l_1 + a_2$ directly left to the b. If we now place $\beta_1 - 1$ entries 1 into the right $\beta_1 - 1$ columns we only get to column $\lambda_1 - (\beta_1 - 1) + 1 = \lambda_1 - \beta_1 + 2 \geq \lambda_2 + 2$. If we now place the remaining 1 atop of one of the entries 2 which are in row $l_1 + a_2 + a_3 - b + 2$ (in one of the columns having in row $a_1 + a_2$ an entry $b - a_3$ instead of an entry b) then the 1 is placed in row $l_1 + a_2 + a_3 - b + 1 \geq l_1 + 2$ where we used $a_2 \geq b$ and $a_3 \geq 1$. Also the highest position of a 2 in this filling is in row $l_1 + a_2 - b + 2$ and so below row $l_1 + 2$. So the box $(l_1 + 1, \lambda_2 + 1)$ is empty in this LR filling. So we have a character $[\nu]$ in $[\alpha/\beta]$ with ν having a box in position $(l_1 + 1, \lambda_2 + 1)$. But since $(l_1 + 1, \lambda_2 + 1)$ is not in λ there is no character $[\nu]$ in $[\lambda/\mu]$ with ν containing a box in position $(l_1 + 1, \lambda_2 + 1)$ and so $[\lambda/\mu] \neq [\alpha/\beta]$.

α :

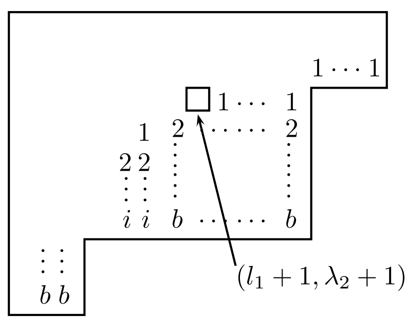


Figure 8. LR filling of α in the *Case 1.3.1.2*

Case 1.3.3: $\mu_1 = \beta_1$, $m = b$.

Since we have $a_2 - b = l_2 - m$ and $m = b$ we have also $a_2 = l_2$ and from this follows $l_3 = l - l_1 - l_2 = l - a_1 - a_2 = a_3$.

Also from $\alpha_2 - \beta_1 = \lambda_2 - \mu_1$ and $\beta_1 = \mu_1$ follows $\alpha_2 = \lambda_2$.

Together with $l_1 = a_1$ and $\lambda_3 = \alpha_3$, which we proved above, this gives the desired $\lambda/\mu = \alpha/\beta$ and so finishes the case $l_1 \leq m$, $\lambda_3 \leq \mu_1$.

Case 2: $l_1 > m$, $\lambda_3 \leq \mu_1$ and *Case 3:* $l_1 \leq m$, $\lambda_3 > \mu_1$ are related by conjugation symmetry so it is sufficient to consider only *Case 2*.

Case 2: $l_1 > m$, $\lambda_3 \leq \mu_1$.

Since we have $l_1 - m$ times the part λ_1 in λ/μ we need $l_1 - m = a_1 - b$.

The skew diagram $\hat{\lambda}/\mu$ obtained after removing $l_1 - m$ boxes from the top of λ/μ decays for $\lambda_2 \leq \mu_1$ but is connected for $\lambda_2 > \mu_1$. So we have $\lambda_2 \leq \mu_1$ if and only if $\alpha_2 \leq \beta_1$.

Case 2.1: $\lambda_2 \leq \mu_1$.

After removing $l_1 - m = a_1 - b$ boxes from the top of λ/μ and α/β we have the skew diagrams

$$\hat{\lambda}/\mu = ((\lambda_1 - \mu_1)^m) \otimes (\lambda_2^{l_2}, \lambda_3^{l_3})$$

and

$$\hat{\alpha}/\beta = ((\lambda_1 - \beta_1)^b) \otimes (\alpha_2^{a_2}, \alpha_3^{a_3})$$

which gives by Lemma 3.1.4 $\lambda/\mu = \alpha/\beta$.

Case 2.2: $\lambda_2 > \mu_1$.

Removing $l_1 - m = a_1 - b$ boxes from the top of λ/μ and α/β gives

$$\hat{\lambda}/\mu = (\lambda_1^m, \lambda_2^{l_2}, \lambda_3^{l_3})/(\mu_1^m)$$

and

$$\hat{\alpha}/\beta = (\lambda_1^b, \alpha_2^{a_2}, \alpha_3^{a_3})/(\beta_1^b).$$

Using the result of *Case 1:* $l_1 \leq m$, $\lambda_3 \leq \mu_1$ gives $\alpha/\beta = \lambda/\mu$ and finishes *Case 2.*

Case 4: $l_1 > m$, $\lambda_3 > \mu_1$ is covered by Lemma 3.1.6. □

We will now compare the multiplicity free skew characters $[\lambda/\mu]$ and $[\alpha/\beta]$ when both λ and α have more than 4 different parts and assume for the following lemmas that $dp(\lambda), dp(\alpha) \geq 4$.

There are 4 cases when $[\lambda/\mu]$ is multiplicity free and λ has more than 4 different parts and we will compare them against each other (see Figure 9).

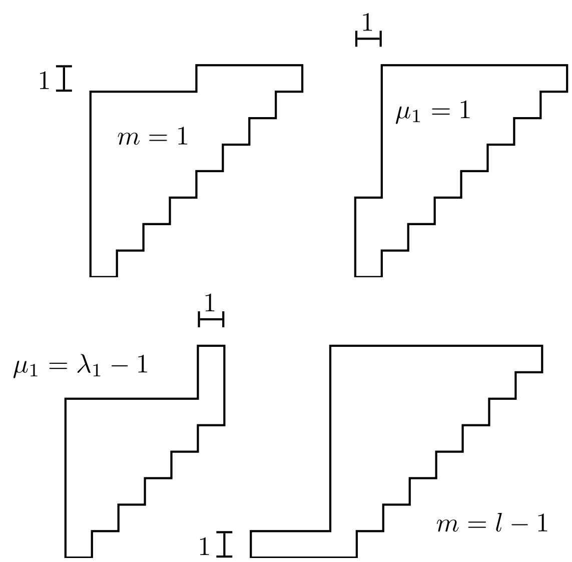


Figure 9. The four multiplicity free cases with λ a $dp = n > 3$ partition

The first lemma covers by conjugation symmetry also the case when $\mu_1 = \beta_1 = 1$.

Lemma 3.2.4. *Let λ/μ and α/β be skew diagrams with $m = b = 1$ and $[\lambda/\mu] = [\alpha/\beta]$.*

Then $\lambda/\mu = \alpha/\beta$.

PROOF. We have in λ/μ $l_1 - 1$ times the part λ_1. Since we need in α/β the part λ_1 also $a_1 - 1$ times we get $a_1 = l_1$.

If we remove l_1 boxes from the top of λ/μ we either get a connected skew diagram for $\lambda_2 > \mu_1$, $l_2 > 1$ or $\lambda_3 > \mu_1$, a disconnected skew diagram for $\lambda_2 > \mu_1 \geq \lambda_3$, $l_2 = 1$ or a partition for $\lambda_2 \leq \mu_1$ (see Figure 10). Obviously the same must apply for $\hat{\alpha}/\beta$ if $[\lambda/\mu] = [\alpha/\beta]$.

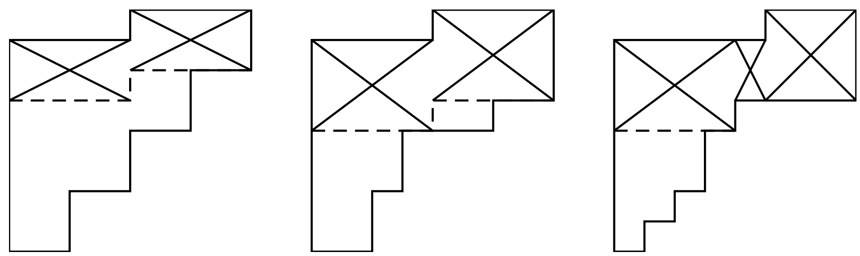


Figure 10. Lemma 3.2.4: The three cases for $\hat{\lambda}/\mu$

Suppose we get a connected skew diagram $\hat{\lambda}/\mu$ with $\hat{\lambda}$ having less than 4 different parts. Then we can use Lemmas 3.2.2 and 3.2.3 to get $\lambda/\mu = \alpha/\beta$.

If $\hat{\lambda}/\mu$ is connected but $\hat{\lambda}$ has 4 or more different parts we can iterate this process until we reach the case where $\hat{\lambda}$ has less than 4 different parts or $\hat{\lambda}/\mu$ is either a disconnected skew diagram or a partition.

Suppose $\hat{\lambda}/\mu$ is a partition. If we remove only $l_1 - 1$ boxes from the top of λ/μ and α/β to get $\hat{\lambda}/\mu^\star$ and $\hat{\alpha}/\beta^\star$ we get:

$$\hat{\lambda}/\mu^\star = (\lambda_1 - \mu_1) \otimes (\lambda_2^{l_2}, \lambda_3^{l_3}, \lambda_4^{l_4}, \ldots)$$

and

$$\hat{\alpha}/\beta^\star = (\lambda_1 - \beta_1) \otimes (\alpha_2^{a_2}, \alpha_3^{a_3}, \alpha_4^{a_4}, \ldots).$$

This gives $\lambda/\mu = \alpha/\beta$.

So now suppose $\hat{\lambda}/\mu$ is a disconnected skew diagram. Then we have:

$$\hat{\lambda}/\mu = (\lambda_2 - \mu_1) \otimes (\lambda_3^{l_3}, \lambda_4^{l_4}, \ldots)$$

and

$$\hat{\alpha}/\beta = (\alpha_2 - \beta_1) \otimes (\alpha_3^{a_3}, \alpha_4^{a_4}, \ldots).$$

This gives $\lambda_i^{l_i} = \alpha_i^{a_i}$ for $i \geq 3$ and $\lambda_2 - \mu_1 = \alpha_2 - \beta_1$.

For $\mu_1 = \beta_1$ we get $\lambda_2 = \alpha_2$ and since we have in this case also $l_2 = a_2 = 1$ we get $\lambda/\mu = \alpha/\beta$.

For $\mu_1 \neq \beta_1$ we can use the construction of LR fillings following equation (3.2.7) (page 45) and get $[\lambda/\mu] \neq [\alpha/\beta]$. □

LEMMA 3.2.5. *Let* λ/μ *and* α/β *be skew diagrams with* $\mu \neq (1) \neq \beta$, $m = 1 = \beta_1$ *and* $[\lambda/\mu] = [\alpha/\beta]$.

Then $\lambda = (l, l-1, l-2, \ldots, 2, 1)$ *is a staircase partition and* λ/μ *is the conjugate of* α/β, $\alpha/\beta = \lambda/\mu^c$.

PROOF. Suppose $l_1 > 1$. Then we have the part λ_1 $l_1 - 1$ times in λ/μ and so need $a_1 = b + l_1 - 1$. The smallest height in λ/μ is either l_1 (in the case $\lambda_2 \geq \mu_1$) or $l_1 - 1$ (for $\lambda_2 < \mu_1$). The smallest height in α/β is either $a_1 = b + l_1 - 1 > l_1$ or $l - b$. For $\lambda_2 \geq \mu_1$ this gives $l - b = l_1$ and so $l = b + l_1 = a_1 + 1$ which means that α has only 2 different parts. For $\lambda_2 < \mu_1$ this gives $l - b = l_1 - 1$ and so $l = l_1 + b - 1 = a_1$ which means that α is a rectangle.

So we have $l_1 = 1$ and since λ/μ does not decay we need $\lambda_2 > \mu_1$.

If we remove the top box of λ/μ and α/β we get a connected skew diagram for either $l_2 > 1$ or for $\lambda_3 > \mu_1$. Suppose we are in this case then if the new skew diagrams have less than 4 different parts we can use Lemma 3.2.3 which then gives that $[\lambda/\mu] \neq [\alpha/\beta]$. If the new skew diagram has 4 or more different parts we get $\lambda_2 = \alpha_2$, because λ_2 is the number of columns in the new skew diagram, and so $a_1 = 1$. Since we have the part $\lambda_1 - 1$ in α/β we need $\lambda_2 = \lambda_1 - 1$, because we need a the part $\lambda_1 - 1$ also in λ/μ. We can repeat the above argument until the skew diagram we obtain after removing the top box decays.

So we now assume that the skew diagrams $\hat{\lambda}/\mu$ and $\hat{\alpha}/\beta$ obtained after removing the top box of λ/μ and α/β decay and so we need $l_2 = 1$, $\lambda_3 \leq \mu_1$. If $\hat{\lambda}/\mu$ decays we have:

$$\hat{\lambda}/\mu = (\lambda_2 - \mu_1) \otimes (\lambda_3^{l_3}, \lambda_4^{l_4}, \ldots).$$

Since $\hat{\alpha}/\beta$ must also decay we need $\alpha_n = 1$ and $a = b + a_n + 1$ if α/β has n different parts. We then have:

$$\hat{\alpha}/\beta = (1^{a_n}) \otimes (\underbrace{(\alpha_1 - 1)^{a_1 - 1}}_{\text{for } a_1 > 1}, (\alpha_2 - 1)^{a_2}, (\alpha_3 - 1)^{a_3}, \ldots).$$

Since we have height 1 in λ/μ and the smallest height in α/β is either a_1 or $l - b = a_n + 1 > 1$ we need $a_1 = 1$.

Comparing $\hat{\lambda}/\mu$ with $\hat{\alpha}/\beta$ gives $\lambda_2 - \mu_1 = 1$, $a_n = 1$, $\lambda_i^{l_i} = (\alpha_{i-1} - 1)^{a_{i-1}}$ for $i = 3, \ldots, n$. Since we have again the part $\lambda_1 - 1$ in α/β we need $\lambda_2 = \lambda_1 - 1$ and so $\mu_1 = \lambda_1 - 2$.

Suppose we have $l_i = a_i = 1$ and $\alpha_i = \lambda_i$ for $1 \leq i < p$ and fixed $p \geq 2$. This holds true for $p = 2$. Since $l_{i+1} = a_i$ for $i \geq 2$ and $l_2 = 1$ we have also $l_p = 1$.

Suppose we have $\lambda_p > \lambda_{p+1} + 1$ where $\lambda_{p+1} = 0$ is allowed.

We now construct an LR filling of λ with content μ. We place in λ entries 1 into every column of λ but not into the columns $\lambda_{p+1}+1$ and $\lambda_{p+1}+2$ and so we get an LR filling which leaves the box $(p, \lambda_{p+1} + 2)$ empty (see Figure 11). Thus there is a character $[\nu] \in [\lambda/\mu]$ with ν containing a box in position $(p, \lambda_{p+1} + 2)$. Since $\alpha_p = \lambda_{p+1}+1$ the box $(p, \lambda_{p+1}+2)$ is not in α and so there is no character $[\nu] \in [\alpha/\beta]$ with ν containing a box in position $(p, \lambda_{p+1} + 2)$. This gives $[\lambda/\mu] \neq [\alpha/\beta]$. So we need $\lambda_p = \lambda_{p+1} + 1 = \alpha_p$.

So now suppose we have $a_p > 1$.

Placing in α the entries into the rows 1 to $p-1$ and $p+2$ to l gives an LR filling which leaves the box $(p+1, \alpha_p)$ empty (see Figure 11) and so we have a character $[\nu] \in [\alpha/\beta]$ with ν containing a box in position $(p+1, \alpha_p)$. Since $l_p = 1$ the $p+1$th row in λ has only $\lambda_{p+1} < \lambda_p = \alpha_p$ boxes and so there is no character $[\nu] \in [\lambda/\mu]$ with ν containing a box in position $(p+1, \alpha_p)$ and so $[\lambda/\mu] \neq [\alpha/\beta]$ for $a_p > 1$ and so we need $a_p = 1$.

It now follows by induction that $\alpha = \lambda$ has to be a staircase partition $\lambda = (l, l-1, \ldots, 2, 1)$ which then also gives $b = \mu_1$ and so $\lambda/\mu = (\alpha/\beta)^c$. $\square$

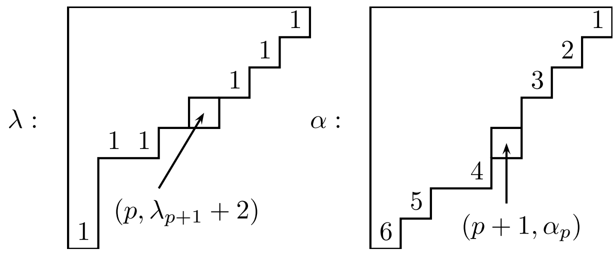


Figure 11. Lemma 3.2.5: LR fillings of λ and α

The following lemma covers by conjugation also the case when $\mu_1 = 1, b = a - 1 = l - 1$.

Lemma 3.2.6. *Let λ/μ and α/β be skew diagrams with $m = 1, \beta_1 = \alpha_1 - 1 = \lambda_1 - 1$, $b > 1$.*
Then $[\lambda/\mu] \neq [\alpha/\beta]$.

Proof. We have $a_1 > b$ since otherwise α/β would decay.
Removing $a_1 - b$ boxes from the top of α/β gives $\hat{\alpha}/\beta$ which decays into

$$\hat{\alpha}/\beta = (1^b) \otimes (\alpha_2^{a_2}, \alpha_3^{a_3}, \alpha_4^{a_4}, \ldots).$$

Since $\mu = (\mu_1)$ we have that if λ/μ decays after $a_1 - b$ boxes are removed from the top and in total $(a_1 - b) \cdot \lambda_1$ boxes it decays into

$$\hat{\lambda}/\mu = (\lambda_1 - \mu_1) \otimes (\lambda_2^{l_2}, \lambda_3^{l_3}, \lambda_4^{l_4}, \ldots).$$

Since $b > 1$ we get $[\lambda/\mu] \neq [\alpha/\beta]$. □

The following lemma covers by conjugation also the case when $\mu_1 = 1, \beta_1 = \alpha_1 - 1 = \lambda_1 - 1$.

Lemma 3.2.7. *Let λ/μ and α/β be skew diagrams with $\mu_1 > 1 = m, b = l - 1 = a - 1$.*
Then $[\lambda/\mu] \neq [\alpha/\beta]$.

Proof. If we remove the top boxes from α/β we get a partition $\hat{\alpha}/\beta$.
If the skew diagram $\hat{\lambda}/\mu$ obtained after removing the top box of every column in λ/μ is a partition we have $l_1 = 1$ and $\lambda_2 \leq \mu_1$. Thus λ/μ decays. □

The following lemma covers by conjugation also the case when $m = b = l - 1$.

Lemma 3.2.8. *Let λ/μ and α/β be skew diagrams with $\mu_1 = \beta_1 = \lambda_1 - 1$ and $[\lambda/\mu] = [\alpha/\beta]$.*
Then $\lambda/\mu = \alpha/\beta$.

Proof. Since λ/μ and α/β are connected we need $l_1 > m$, $a_1 > b$ and because of the part λ_1 which, therefore, exists in λ/μ and α/β we have $l_1 - m = a_1 - b$. Removing the top $l_1 - m$ boxes of λ/μ gives:

$$\hat{\lambda}/\mu = (1^m) \otimes (\lambda_2^{l_2}, \lambda_3^{l_3}, \lambda_4^{l_4} \ldots)$$

and removing the top $l_1 - m = a_1 - b$ boxes of α/β gives:

$$\hat{\alpha}/\beta = (1^b) \otimes (\alpha_2^{a_2}, \alpha_3^{a_3}, \alpha_4^{a_4} \ldots).$$

This gives $\lambda/\mu = \alpha/\beta$. □

LEMMA 3.2.9. *Let λ/μ and α/β be skew diagrams with $m > 1, \mu_1 = \lambda_1 - 1, b = l - 1$.*

Then $[\lambda/\mu] \neq [\alpha/\beta]$.

PROOF. Removing the top box of every column of α/β gives a partition $\hat{\alpha}/\beta$. Since $\mu = ((\lambda_1 - 1)^m)$ with $l_1 > m > 1$ removing the top boxes of every column of λ/μ can only give a partition if $\lambda_2 = 0$. $\square$

Since the previous 6 lemmas cover all cases when $[\lambda/\mu]$ and $[\alpha/\beta]$ are multiplicity free skew characters with both λ and α having 4 or more different parts this proves together with the previous lemmas the Theorem 3.1.1.

CHAPTER 4

Generalized stretched Littlewood-Richardson coefficients

Some recent research was concerned with the behaviour of the stretched LR coefficients $f(n) = c(n\lambda; n\mu, n\nu)$ (see [**KT**],[**KTW**],[**Buch**],[**DW**],[**KTT**],[**Ras**] and Section 1.5 where we summarized some of the results). It is known that this function is a polynomial and, therefore, is either constant or increases without bound (for example by Lemma 4.1.1). It is constant if and only if $c(\lambda; \mu, \nu) = 1$ (see [**KTW**]).

In my Diploma thesis I used the fact that we get information about $[\lambda/\mu]$ by analyzing $[\alpha/\beta]$ if λ/μ is larger than α/β (see Theorem 3.1.2 and Lemma 4.1.1). Since there are many nice results about $c(n\lambda; n\mu, n\nu)$ and the triple $(n\lambda, n\mu, n\nu)$ is obtained by adding the triple (λ, μ, ν) repeatedly to itself, natural questions that arise are: what can be said about the skew character $[(n\lambda + \lambda')/(n\mu + \mu')]$, corresponding to repeatedly adding the skew diagram λ/μ to λ'/μ', and how does the LR coefficient $c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$, corresponding to repeatedly adding the triple (λ, μ, ν) to another triple (λ', μ', ν'), behave?

In this chapter we prove in Lemma 4.1.1 that $c(\lambda+\lambda'; \mu+\mu', \nu+\nu') \geq c(\lambda; \mu, \nu)$ for $c(\lambda'; \mu', \nu') \neq 0$. We then investigate in Section 4.2 $Q(n) = Q^{\lambda,\mu}_{\lambda',\mu'}(n) = \sum_\nu c(n\lambda + \lambda'; n\mu + \mu', \nu)$ as a function of $n \in \mathbb{N}_0$ and show that $Q(n)$ is bounded above if and only if λ/μ is a partition or rotated partition. $Q(n)$ counts the number of LR tableaux of shape $(n\lambda+\lambda')/(n\mu+\mu')$ or the total number of irreducible characters (i.e. the number of constituents) in the skew character $[(n\lambda+\lambda')/(n\mu+\mu')]$.

Furthermore, we investigate in Section 4.3 the function $P(n) = P^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$ as a function of $n \in \mathbb{N}_0$. So the stretched LR coefficient $f(n)$ is the special case of $P(n)$ with $\lambda' = \mu' = \nu' = 0$.

The results of this chapter will appear in [**Gut5**].

4.1. Preliminaries: $c(\lambda + \lambda'; \mu + \mu', \nu + \nu') \geq c(\lambda; \mu, \nu)$

It is known (see [**Zel**]) that the triples of partitions with non-zero LR coefficient form an additive semigroup. We can generalize this and Theorem 3.1.2 of my diploma thesis to the following:

Lemma 4.1.1. *Let $\lambda, \mu, \nu, \lambda', \mu', \nu'$ be partitions with $c(\lambda; \mu, \nu), c(\lambda'; \mu', \nu') \neq 0$. Then:*

$$c(\lambda; \mu, \nu) \leq c(\lambda + \lambda'; \mu + \mu', \nu + \nu')$$

and by conjugation:

$$c(\lambda; \mu, \nu) \leq c(\lambda \cup \lambda'; \mu \cup \mu', \nu \cup \nu').$$

Proof. Let $\mathcal{A}$ be a fixed LR tableau of shape λ'/μ' with content ν'. Let A_j be the multiset of the entries in the jth row of $\mathcal{A}$.

For any LR tableau $\mathcal{C}^i$ of shape λ/μ and content ν we let C^i_j be the multiset of the entries in the jth row of $\mathcal{C}^i$.

We can now define for every $\mathcal{C}^i$ a tableau $\mathcal{D}^i$ of shape $(\lambda+\lambda')/(\mu+\mu')$ with content $\nu+\nu'$ by placing into row j the entries of $A_j \cup C^i_j$ in weakly increasing order. Because the columns of $\mathcal{A}$ and $\mathcal{C}^i$ are strictly increasing, also the columns of $\mathcal{D}^i$ are strictly increasing. It is also clear that the tableau word is a lattice word because it can be divided into two subsequences (corresponding to the entries in $\mathcal{D}^i$ having their origin either in $\mathcal{A}$ or $\mathcal{C}^i$) which are both lattice words. So the $\mathcal{D}^i$ are, in fact, LR tableaux.

Suppose we have $\mathcal{D}^i = \mathcal{D}^l$. Then we know from the construction that the multiset of the entries in the jth row of $\mathcal{D}^i$ is $A_j \cup C^i_j$ while the multiset of the entries in the jth row of $\mathcal{D}^l$ is $A_j \cup C^l_j$. This gives us $C^i_j = C^l_j$ for all j and since an LR tableau of a given shape is uniquely determined by the content of its row it follows that $\mathcal{C}^i = \mathcal{C}^l$. So we have that different LR tableaux of shape λ/μ with content ν give different LR tableaux of shape $(\lambda+\lambda')/(\mu+\mu')$ with content $\nu+\nu'$ and so:

$$c(\lambda;\mu,\nu) \leq c(\lambda+\lambda';\mu+\mu',\nu+\nu').$$

□

Remark 4.1.2. In the hive model (which we do not use in this work) the proof is also easy. Choose one LR hive corresponding to the triple (λ',μ',ν') and add this hive to all the LR hives corresponding to (λ,μ,ν). It is easy to see that all the new hives are different LR hives corresponding to $(\lambda+\lambda',\mu+\mu',\nu+\nu')$.

Remark 4.1.3. It is known that $f(n) = c(n\lambda;n\mu,n\nu)$ is a polynomial which is constant if and only if $c(\lambda;\mu,\nu)=1$ (see [**KT**],[**KTW**]). Suppose λ,μ,ν are chosen in such a way that $f(n)$ is not constant then we know that

$$F(n) = c(n\lambda+\lambda';n\mu+\mu',n\nu+\nu')$$

increases without bound if $c(\lambda';\mu',\nu') \neq 0$.

4.2. Behaviour of $Q(n) = \sum_\nu c(n\lambda+\lambda';n\mu+\mu',\nu)$

We always assume in this section $\mu \subseteq \lambda$ and $\mu' \subseteq \lambda'$, define $Q^{\lambda;\mu}_{\lambda',\mu'}(n) = \sum_\nu c(n\lambda+\lambda';n\mu+\mu',\nu)$ and write simply $Q(n)$ if $\lambda,\mu,\lambda',\mu'$ are known from the context.

Lemma 4.2.1. *Let λ/μ be a proper skew diagram. Then $Q^{\lambda;\mu}_{\lambda',\mu'}(n)$ increases without bound. Furthermore,*

$$\sum_{\substack{\nu \\ c(n\lambda+\lambda';n\mu+\mu',\nu)\neq 0}} 1 \to \infty \qquad \text{for } n\to\infty.$$

So both the number of constituents and the number of components of the skew character $[(n\lambda+\lambda')/(n\mu+\mu')]$ increase without bound if λ/μ is a proper skew diagram.

Proof. Since λ/μ is a proper skew diagram it is larger than the skew diagram $(2,1)/(1)$, which means λ/μ can be obtained from $(2,1)/(1)$ by repeatedly using the operations $+,\cup$ together with skew diagrams $\mathcal{B}^i$. Since $\mathcal{B}^i$ is a skew diagram there exists at least one component $[\nu^i] \neq 0$ of $[\mathcal{B}^i]$. But we have clearly $\alpha \circ^i \nu^i \neq \beta \circ^i \nu^i$

for $\alpha \neq \beta$ and $\circ^i \in \{+, \cup\}$. This argument together with Lemma 4.1.1 gives: $\sum_\nu c(\lambda; \mu, \nu) \geq \sum_\nu c((2,1); (1), \nu)$. By the same argument we also get:

$$\sum_\nu c(n\lambda + \lambda'; n\mu + \mu', \nu) \geq \sum_\nu c(n\lambda; n\mu, \nu) \geq \sum_\nu c(n(2,1); n(1), \nu).$$

It is easy to see that $\sum_\nu c(n(2,1); n(1), \nu) = n + 1$ because an LR tableau of shape $(2,1)/(1)$ contains n entries 1 in row 1 and i $(0 \leq i \leq n)$ entries 1 as well as $n - i$ entries 2 in row 2 and for each such i there is exactly one LR tableau. So $Q^{\lambda,\mu}_{\lambda',\mu'}(n)$ increases without bound.

Furthermore, since the number of components of $[n(2,1)/n(1)]$ is $n + 1$ the number of components of $[n\lambda + \lambda'/n\mu + \mu']$ is also at least $n + 1$ and so

$$\sum_{\substack{\nu \\ c(n\lambda+\lambda'; n\mu+\mu', \nu) \neq 0}} 1 \geq n + 1.$$

□

LEMMA 4.2.2. *Let λ/μ be a partition or rotated partition.*

Then there exists an m with $Q^{\lambda,\mu}_{\lambda',\mu'}(n) = Q^{\lambda,\mu}_{\lambda',\mu'}(m)$ for $n \geq m$.

Furthermore, if $\lambda = (\alpha_1^{a_1}, \alpha_2, \alpha_3, \ldots \alpha_k), \alpha_k \neq 0$, $\mu = (\alpha_1^{a_1 - 1})$ and λ'/μ' basic we can choose

$$m = \left\lceil \max_{\substack{1 \leq j \leq k \\ \alpha_j > \alpha_{j+1}}} \Big(\frac{\lambda'_1 - \lambda'_{a_j} + \lambda'_{a_j+1} + \mu'_{a_1} - \mu'_{a_1-1}}{\alpha_j - \alpha_{j+1}} \Big) \right\rceil$$

with $a_j = a_1 - 1 + j, \alpha_{k+1} = 0$ (for $a_1 = 1$ set $\mu'_0 = \lambda'_1$) which then also gives $Q^{\lambda,\mu}_{\lambda',\mu'}(m) > Q^{\lambda,\mu}_{\lambda',\mu'}(m-1) > \ldots > Q^{\lambda,\mu}_{\lambda',\mu'}(0)$.

These inequalities are also satisfied in the general case if we choose the smallest m satisfying $Q^{\lambda,\mu}_{\lambda',\mu'}(n) = Q^{\lambda,\mu}_{\lambda',\mu'}(m)$ for $n \geq m$.

PROOF. We look at the skew diagram $\mathcal{A}(n) = (n\lambda + \lambda')/(n\mu + \mu')$.

By rotation symmetry we may assume that λ/μ is a partition instead of a rotated partition.

Let $a_1 > a_2 > \ldots > a_k$ be the non-empty rows of λ/μ. If we have $\lambda_i = \mu_i > \lambda_{i+1}$ for some $i \neq a_1, ..., a_k$ and choose n big enough then $\mathcal{A}(n)$ decays into a skew diagram $\mathcal{A}^{up}$ containing the upper i rows and a skew diagram $\mathcal{A}_{lo}$ containing the rows below row i. If we increase n even more then the skew diagrams $\mathcal{A}^{up}$ and $\mathcal{A}_{lo}$ are translated relative to one another which is irrelevant for the skew character $[\mathcal{A}(n)]$. So if there are some $i \neq a_1, ..., a_k$ with $\lambda_i = \mu_i > \lambda_{i+1}$ we may choose n large enough so that for each such i $\mathcal{A}(n)$ decays into an upper skew diagram and a lower skew diagram. Instead of looking at this situation we may then investigate the case that $\lambda'/\mu' = \mathcal{A}(n)$ for an n large enough and have no $i \neq a_1, ..., a_k$ with $\lambda_i = \mu_i > \lambda_{i+1}$. So we may assume that $\mu_i = \lambda_i = \lambda_{a_1}$ for $i < a_1$ and $\mu_i = \lambda_i = \mu_{a_k}$ for $a_k < i \leq l(\mu)$ (and since λ/μ is a partition we also have $\mu_{a_1} = \mu_{a_k}$.). If $\mu_{\alpha_1} > 0$ there is for the same reason as above an n such that $\mathcal{A}(n)$ decays into skew diagrams containing the upper $l(\mu)$ rows and the rows below row $l(\mu)$ and increasing n translates these skew diagrams relative to another so we may assume that $\mu_{\alpha_1} = 0$.

So we have without loss of generality $\lambda = (\alpha_1^{a_1}, \alpha_2, \alpha_3, \ldots \alpha_k), \alpha_k \neq 0$ (not necessarily $\alpha_i \neq \alpha_{i+1}$) and $\mu = (\alpha_1^{a_1 - 1})$. To prove $Q(n) = Q(m)$ for $n \geq m$ we have to construct an m such that removing in an LR tableau of shape $\mathcal{A}(n)$ from the row a_i with $1 \le i \le k$ the entry i $(n-m)\alpha_i$ times and translating the upper $a_1 - 1$ rows $(n-m)\alpha_1$ boxes to the left yields an LR tableau of shape $\mathcal{A}(m)$.

By our choice of λ and μ the number N of non-empty columns among the upper $a_1 - 1$ rows of $\mathcal{A}(n)$ is independent of n (we have $N \le \lambda'_1 - \mu'_{a_1-1}$ and may by translation symmetry assume equality, set $\mu'_0 = \lambda'_1$ for $a_1 = 1$). So the number of entries 1 among the upper a_1 rows of an LR filling of $\mathcal{A}(n)$ is at most N. So for $1 \le i \le k$ there are at most N entries larger i in row a_i of an LR filling of $\mathcal{A}(n)$. Furthermore, the number of entries smaller i in row a_i is at most $\mu'_{a_1} - \mu'_{a_i}$, also independent of n. On the other hand, there are in row a_i of $\mathcal{A}(n)$ $\lambda'_{a_i} - \mu'_{a_i} + n\alpha_i$ boxes. So the number of entries i in row a_i of an LR filling of $\mathcal{A}(n)$ is at least

$$\lambda'_{a_i} - \mu'_{a_i} + n\alpha_i - N - (\mu'_{a_1} - \mu'_{a_i}) = \lambda'_{a_i} - \mu'_{a_1} - N + n\alpha_i.$$

Obviously if $\lambda'_{a_k} - \mu'_{a_1} - N + n\alpha_k \ge 0$ then also $\lambda'_{a_i} - \mu'_{a_1} - N + n\alpha_i \ge 0$ for every $1 \le i \le k$. So for

$$n > n' \ge \frac{\mu'_{a_1} + N - \lambda'_{a_k}}{\alpha_k} \tag{4.2.1}$$

there are at least $(n - n')\alpha_i$ entries i in row a_i of every LR tableau of shape $\mathcal{A}(n)$.

We have to investigate the j $(1 \le j \le k)$ with $\alpha_j > \alpha_{j+1}$ (for example $j = k$). Removing in an LR tableau α_i times the entry i from row a_i removes more entries j than $j+1$ so the new tableau can violate the lattice word condition even if there are enough entries i to remove. As calculated above the number of entries j in row a_j of an LR tableau of shape $\mathcal{A}(n)$ is at least: $\lambda'_{a_j} - \mu'_{a_1} - N + n\alpha_j$. Furthermore, the number of entries $j+1$ in an LR tableau of shape $\mathcal{A}(n)$ below row j is at most $\lambda'_{a_j+1} + n\alpha_{j+1}$ since there are only so many columns below row a_j. So for

$$\lambda'_{a_j} - \mu'_{a_1} - N + n\alpha_j \ge \lambda'_{a_j+1} + n\alpha_{j+1}$$

the number of entries j in row a_j is at least as large as the number of entries $j+1$ below row a_j in every LR tableau of shape $\mathcal{A}(n)$. We can solve the above inequality and get:

$$n \ge \frac{\lambda'_{a_j+1} - \lambda'_{a_j} + \mu'_{a_1} + N}{\alpha_j - \alpha_{j+1}}.$$

Since we have $\alpha_k > 0 = \alpha_{k+1}$ setting $j = k$ gives

$$\frac{\lambda'_{a_k+1} - \lambda'_{a_k} + \mu'_{a_1} + N}{\alpha_k} \ge \frac{-\lambda'_{a_k} + \mu'_{a_1} + N}{\alpha_k}$$

which is the right hand side of inequality (4.2.1).

Let us set

$$m = \left\lceil \max_{\substack{1 \le j \le k \\ \alpha_j > \alpha_{j+1}}} \left(\frac{\lambda'_1 - \lambda'_{a_j} + \lambda'_{a_j+1} + \mu'_{a_1} - \mu'_{a_1-1}}{\alpha_j - \alpha_{j+1}} \right) \right\rceil$$

where $\lceil x \rceil$ denotes as usual the smallest integer larger or equal to x.

Then we know for $n \ge m$ from our reasonings above that every LR tableau $\mathcal{C}_n$ of shape $\mathcal{A}(n)$ contains at least $(n-m)\alpha_i$ entries i in row a_i $(1 \le i \le k)$ and, furthermore, removing $(n-m)\alpha_i$ entries i from every row a_i $(1 \le i \le k)$ and translating the upper $a_1 - 1$ rows $(n-m)\alpha_1$ boxes to the left yields a tableau $\mathcal{C}_m$ which contains (for those j with $\alpha_j > \alpha_{j+1}$) in row a_j at least as much entries j as there are entries $j+1$ below row a_j. So the tableau $\mathcal{C}_m$ satisfies the lattice word condition. Furthermore, the entries in the rows increase weakly from left to right.

We have to check that the entries in the columns are strictly increasing from top to bottom which is not trivial because we remove more entries j from row a_j than entries $j+1$ from row a_j+1 if $\alpha_j > \alpha_{j+1}$. But our condition on m ensures that in $\mathcal{C}_m$ there is an entry smaller $j+1$ above every entry in row a_j+1 so there is no problem for the entries weakly larger than $j+1$ in row a_j+1. But the entries in $\mathcal{C}_m$ in row a_j+1 which are smaller than $j+1$ have an entry smaller than itself in the box directly above itself because $\mathcal{C}_n$ is semistandard. So $\mathcal{C}_m$ has to be, in fact, an LR tableau. So every LR tableau of shape $\mathcal{A}(n)$ is obtained from an LR tableau of shape $\mathcal{A}(m)$ by adding $(n-m)\alpha_i$ entries to row a_i $(1 \le i \le k)$ and translating the above $a_1 - 1$ rows $(n-m)\alpha_1$ boxes to the right. So for $n \ge m$ we have $Q(n) = Q(m)$.

We now have to prove that $Q^{\lambda,\mu}_{\lambda',\mu'}(m) > Q^{\lambda,\mu}_{\lambda',\mu'}(m-1) > \ldots > Q^{\lambda,\mu}_{\lambda',\mu'}(0)$ if λ'/μ' is basic.

For $n < \frac{\lambda'_1 - \lambda'_{a_k} + \mu'_{a_1} - \mu'_{a_1-1}}{\alpha_k}$ we can construct an LR tableau of shape $\mathcal{A}(n)$ containing less than α_k entries k in row a_k. This gives $Q(n) > Q(n-1)$.

So now suppose $\frac{\lambda'_1 - \lambda'_{a_k} + \mu'_{a_1} - \mu'_{a_1-1}}{\alpha_k} \le n < \frac{\lambda'_1 - \lambda'_{a_j} + \lambda'_{a_j+1} + \mu'_{a_1} - \mu'_{a_1-1}}{\alpha_j - \alpha_{j+1}}$ for some $1 \le j \le k$ with $\alpha_j > \alpha_{j+1}$. We can construct an LR tableau C_n of shape $\mathcal{A}(n)$ with the following conditions:

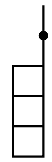
- There are $\lambda'_1 - \mu'_{a_1-1}$ entries 1 in the upper $a_1 - 1$ rows of C_n (this is possible because λ'/μ' is basic).
- There are $\lambda'_1 - \mu'_{a_1-1}$ entries j in the upper $a_j - 1$ rows of C_n.
- There are $\lambda'_1 - \mu'_{a_1-1}$ entries $j+1$ in row a_j (the lower bound on n ensures that there are enough boxes in row a_j).
- There are $x \ge \alpha_j$ entries j in row a_j and x entries $j+1$ below row a_j (the upper bound on n ensures that there are at least x columns below row a_j in which we can write the entry $j+1$).
- There is no entry j below row a_j.

So we have an LR tableau C_n and removing from every row a_i α_i entries i and translating the upper $a_1 - 1$ rows α_1 boxes to the left yields a tableau C_{n-1} which contains more entries $j+1$ than entries j and so is no LR tableau. This gives $Q(n) > Q(n-1)$.

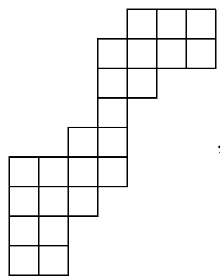
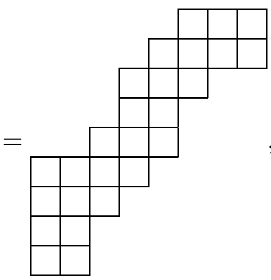
This proves $Q^{\lambda,\mu}_{\lambda',\mu'}(m) > Q^{\lambda,\mu}_{\lambda',\mu'}(m-1) > \ldots > Q^{\lambda,\mu}_{\lambda',\mu'}(0)$ in the case $\lambda = (\alpha_1^{a_1}, \alpha_2, \alpha_3, \ldots \alpha_k), \mu = (\alpha_1^{a_1-1})$.

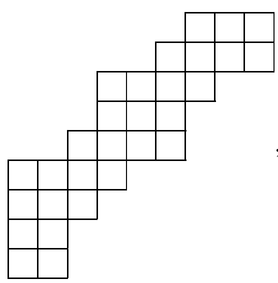
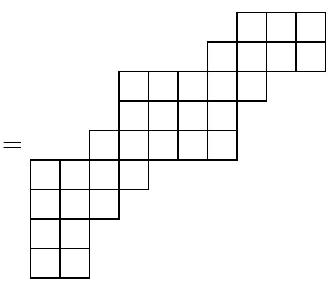
In the more general case there can be i with $\mu_i = \lambda_i > \lambda_{i+1}$ and $\mu'_i < \lambda'_{i+1}$ (so the rows i and $i+1$ of $\mathcal{A}(0) = \lambda'/\mu'$ are connected). We notice that for $n < \frac{\lambda'_{i+1} - \mu'_i}{\mu_i - \lambda_{i+1}}$ we can construct an LR tableau C_n of shape $\mathcal{A}(n)$ containing in row $i+1$ $\mu'_i - \mu'_{i+1} + n(\mu_i - \mu_{i+1})$ times the entry 1. Furthermore, we notice that no LR tableau of shape $\mathcal{A}(n-1)$ can contain $\mu'_i - \mu'_{i+1} + n(\mu_i - \mu_{i+1}) - (\lambda_{i+1} - \mu_{i+1})$ entries 1 in row row $i+1$ because there are not enough boxes in row $i+1$ without a box directly atop. So we again have $Q(n) > Q(n-1)$ for these n and for the other n we can specialize to the above case with $\lambda = (\alpha_1^{a_1}, \alpha_2, \alpha_3, \ldots \alpha_k)$, $\mu = (\alpha_1^{a_1-1})$. □

Example 4.2.3. Let $\lambda' = (7^2, 5, 4^3, 3, 2^2), \mu' = (4, 3^3, 2), \lambda = (1^5), \mu = (1^2)$. So

$\lambda/\mu =$ 

and

$\mathcal{A}(0) = \lambda'/\mu' =$  , $\mathcal{A}(1) =$  ,

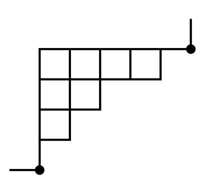
$\mathcal{A}(2) =$  , $\mathcal{A}(3) =$ 

and by Lemma 4.2.2 we have for $n \geq m = 7$: $Q(n) = Q(7) > Q(6) > \ldots > Q(0)$.

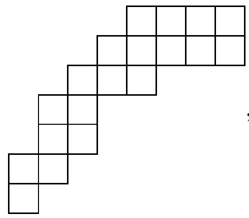
And, in fact, we have:

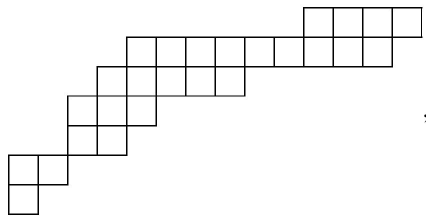
$Q(0)$	$Q(1)$	$Q(2)$	$Q(3)$	$Q(4)$	$Q(5)$	$Q(6)$	$Q(n \geq 7)$
2184	26.421	92.030	172.795	229.660	254.420	260.761	261.512

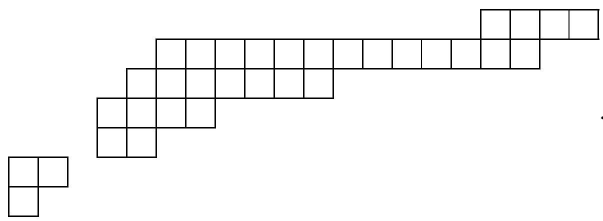
Example 4.2.4. Let $\lambda = (6,5,3,2,1), \mu = (6,1^4), \lambda' = (8^2,5,3^2,2,1)$ and $\mu' = (4,3,2,1^2)$. So

$\lambda/\mu =$ 

and

$\mathcal{A}(0) = \lambda'/\mu' =$  ,

$\mathcal{A}(1) =$  ,

$\mathcal{A}(2) =$  .

By Lemma 4.2.2 there exists an m with $Q(n) = Q(m)$ for $n \geq m$ but we cannot use the given formula. For $n = 0$ the skew diagram $\mathcal{A}(n)$ is connected, for $1 \leq n < 4$ $\mathcal{A}(n)$ decays into 2 skew diagrams, one containing the upper 5 rows and one the rows below row 5. For $4 \leq n$ the skew diagram decays into 3 skew diagrams, one containing the topmost row, one containing the rows 2 to 5 and one containing the rows below. Deleting the empty columns in $\mathcal{A}(4)$ and ignoring the parts of λ/μ which only translate the disconnected skew diagrams we can now use the formula on $\widetilde{\mathcal{A}(4)} = (29, 25, 14, 8, 4, 2, 1)/(25, 4, 3, 2, 2)$ and $\widetilde{\lambda/\mu} = (4, 4, 2, 1)/(4)$ which gives $\widetilde{m} = 4$. So in total we have for $n \geq m = 8 = 4 + \widetilde{m}$: $Q(n) = Q(8) > Q(7) > \ldots > Q(0)$.

And, in fact, we have:

$Q(0)$	$Q(1)$	$Q(2)$	$Q(3)$	$Q(4)$	$Q(5)$	$Q(6)$	$Q(7)$	$Q(n \geq 8)$
910	18.271	38.016	49.635	54.176	55.480	55.826	55.889	55.895.

4.3. Behaviour of $P(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$

For $c(\lambda; \mu, \nu), c(\lambda'; \mu', \nu') \neq 0$ we define $P^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$ and write simply $P(n)$ if $\lambda, \mu, \nu, \lambda', \mu', \nu'$ are known from the context.

LEMMA 4.3.1. *Let $c(\lambda; \mu, \nu) = 1$, $c(\lambda'; \mu', \nu') > 0$. Let one of $\lambda/\mu, \lambda/\nu$ or $\left((\lambda_1)^{l(\lambda)}/\mu\right)^\circ/\nu$ be a partition or a rotated partition.*

Then there exists an m with $P^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}(n) = P^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}(m)$ for $n \geq m$.

PROOF. This follows directly from Lemma 4.2.2. $\square$

Remark 4.3.2. We can use the formula in Lemma 4.2.2 to obtain an m with $P(n) = P(m)$ for $n \geq m$ but the m obtained by the formula in Lemma 4.2.2 doesn't have to be minimal.

Many calculations suggest that Lemma 4.3.1 can be generalized:

CONJECTURE 4.3.3. *Let* $f(n) = c(n\lambda; n\mu, n\nu)$ *be a polynomial of degree* d. *Let* $c(\lambda'; \mu', \nu') \neq 0$.

Then there exists a polynomial $g(n)$ *of degree* d *and an integer* m *such that* $P^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}(n) = g(n)$ *for* $n \geq m$.

In particular, for $c(\lambda; \mu, \nu) = 1$ *there exists an integer* m *with* $P(n) = P(m)$ *for* $n \geq m$.

Example 4.3.4. Let $\lambda = (6, 5, 4, 3^2, 1), \mu = (5, 3, 2, 1), \nu = (5, 3, 2, 1)$ then

$$c(n\lambda; n\mu, n\nu) = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)(2n^2+5n+7)}{840}$$

is of degree 7.

Let $\lambda' = (9^3, 7, 3^4, 2, 1), \mu' = (7^2, 3, 2^3, 1^2), \nu' = (8, 5, 3^2, 2^2, 1)$. We then have

n :	0	1	2	3	4	$n \geq 5$
$P(n)$:	39	30.920	509.202	3.101.626	12.098.348	$g(n)$
$g(n)$:	55.407	50.333	513.782	3.102.223	12.098.382	$g(n)$

with

$$g(n) = \frac{1}{360}\big(8490n^7 + 214.525n^6 + 1.664.232n^5 + 5.835.910n^4 + 904.140n^3 + 8.621.725n^2 - 19.075.662n + 19.946.520\big).$$

(We checked $P(n) = g(n)$ for $5 \leq n \leq 17$ by computer.)

Remark 4.3.5. At least some part of the Conjecture 4.3.3 follows from the work of Etienne Rassart [**Ras**] describing the LR chamber complex $\mathcal{LR}_k$. The chamber complex $\mathcal{LR}_k$ decays into a finite number of cones in which the LR coefficients are given by a polynomial in the variables λ_i, μ_i, ν_i (one polynomial for each cone). Thus the triple $(n\lambda + \lambda', n\mu + \mu', n\nu + \nu')$ has to stay in one of the cones for large n and then $c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$ is given by a polynomial. However, if $(n\lambda, n\mu, n\nu)$ lies on the wall between two cones then the polynomials, in the variables $\lambda_i, \dots$, giving the LR coefficient of $(n\lambda + \lambda', n\mu + \mu', n\nu + \nu')$ and $(n\lambda, n\mu, n\nu)$ can and probably will be different. But even if the polynomials are the same, let us call it $p = p(\lambda_i, \ldots)$, $g(n)$ and $f(n)$ could be of different degree because there might occur cancellation of higher order terms in determining $f(n)$ from p which do not vanish in determining $g(n)$.

So from the work of Rassart [**Ras**] follows only the existence of $g(n)$ and m but not that $g(n)$ has degree d.

We will say that a triple of partitions (λ, μ, ν) is larger than another triple (λ', μ', ν') if there exist triples $(\lambda^i, \mu^i, \nu^i)$ with $c(\lambda^i; \mu^i, \nu^i) \neq 0$ and for every i $\circ^i \in \{+, \cup\}$ with:

$$\lambda = \big(\cdots\big(\big(\lambda' \circ^1 \lambda^1\big) \circ^2 \lambda^2\big) \cdots\big) \circ^n \lambda^n,$$
$$\mu = \big(\cdots\big(\big(\mu' \circ^1 \mu^1\big) \circ^2 \mu^2\big) \cdots\big) \circ^n \mu^n,$$
$$\nu = \big(\cdots\big(\big(\nu' \circ^1 \nu^1\big) \circ^2 \nu^2\big) \cdots\big) \circ^n \nu^n.$$

LEMMA 4.3.6. *Let* $f(n) = c(n\lambda; n\mu, n\nu)$ *be a polynomial of degree* d. *Let a multiple of the triple* (λ, μ, ν) *be larger than the triple* (λ', μ', ν').

Then there exists a polynomial $g(n)$ of degree d and an integer m such that $P^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}(n) = g(n)$ for $n \geq m$.

Proof. Choose k such that $(k\lambda, k\mu, k\nu)$ is larger than (λ', μ', ν').

We then know from Remark 4.3.5 that there exists a polynomial $g(n)$ and an integer m such that $P(n) = g(n)$ for $n \geq m$. Suppose in the following that $n \geq m$. We now have $g(n) \geq f(n)$ by Lemma 4.1.1. But since $(k\lambda, k\mu, k\nu)$ is larger than (λ', μ', ν') we also have $f(k+n) \geq g(n)$, also by Lemma 4.1.1. Since both $f(n)$ and $f(k+n)$ have degree d and $f(k+n) \geq g(n) \geq f(n)$ it follows that also $g(n)$ has to be of degree d. $\square$

CHAPTER 5

The number of components and constituents of skew characters

As mentioned before it is a natural question to ask how many components and constituents a skew character $[\lambda/\mu]$ contains.

In this chapter we will give a lower bound for the number of components and constituents in a skew character $[\lambda/\mu]$ depending on the number of different parts of λ and μ (Theorem 5.2.9). This lower bound will be the number of partitions resp. the number of standard Young diagrams for some number n depending on the number of different parts.

Furthermore, we will give a lower bound for the number of pairs $([\nu^1],[\nu^2])$ of components of $[\lambda/\mu]$ such that ν^1 and ν^2 differ only by one box (Theorem 5.2.7). This lower bound will also depend on the number of different parts of λ and μ and is related to the number of partitions of some integer n when there are two different kinds of 1's and 2's which can be used for the partitions. For this we give an easy bijection between partitions of n with two different kinds of 1's and 2's to pairs of partitions of $n+2$ which differ by only one box (Lemma 5.2.3).

We then determine all skew characters which contain at most 5 components or constituents respectively (Theorem 5.3.1). We also list explicitly all the skew characters which have 2 or 3 components in Remarks 5.3.3 and 5.3.4.

This chapter will be used in an upcoming paper [**BvW**] by Christine Bessenrodt and Stephanie van Willigenburg which classifies those Kronecker products of irreducible characters of the symmetric group which only contain 3 or 4 components. Questions regarding the number of components of Kronecker products of the symmetric and alternating groups [**BK99**] and of Kronecker products of spin characters of the double covers of the symmetric groups [**BK01**] have been asked and to a certain degree answered before.

The results of this chapter will appear in [**Gut7**].

5.1. Notation

We say that a skew diagram $\mathcal{A}$ or skew character $[\mathcal{A}] = [\lambda/\mu] = \sum_\nu c(\lambda;\mu,\nu)$ is of cc-type (a,b) if $[\mathcal{A}]$ has $a = \sum_{c(\lambda;\mu,\nu)\neq 0} 1$ components and $b = \sum_\nu c(\lambda;\mu,\nu)$ constituents. We then also write $cc(\mathcal{A}) = (a,b)$ or $cc([\mathcal{A}]) = (a,b)$. Note that always $a \leq b$ so there is no way of confusing the order. Furthermore, we say that $\mathcal{A}$ with $cc(\mathcal{A}) = (a,b)$ has cc-type at least (c,d) if $a \geq c$ and $b \geq d$.

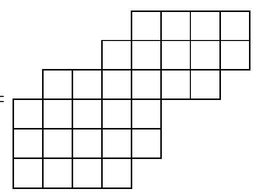
For the classification of those skew characters which contain at most five components and those which contain at most five constituents we have to refine the description of the inner and outer path used in the classification of multiplicity free skew characters. On page 37 after Theorem 3.1.1 we defined for a basic skew diagram λ/μ two paths. The inner path starts in the lower left corner with an upward segment, follows the shape of μ and ends with a segment to right in the upper right corner. The outer path starts in the lower left corner with a segment to the right, follows the shape of λ and ends with an upward segment in the upper right corner.

We define an inner-horizontal- (resp. vertical-)k-step as one horizontal (resp. vertical) segment of the inner path which traverses exactly k boxes. We define outer-horizontal- (resp. vertical-)k-steps in the obvious way and say that λ/μ contains a horizontal (resp. vertical) k-step if there is an inner-horizontal- (resp. vertical-)k-step or an outer-horizontal-(resp. vertical-)k-step.

We mostly use the short notation h for horizontal, ih for inner horizontal, ov for outer vertical and so on. We also write $ih(\lambda/\mu) = (a_1, a_2, \ldots)$ if λ/μ has a_1 ih-1-steps, a_2 ih-2-steps and so on and define $h(\lambda/\mu), v(\lambda/\mu), \ldots$ analog. So $ih(\lambda/\mu)_1$ is the number of inner-horizontal-1-steps.

We say that the inner-k-steps are of type $a_1 + a_2 + a_3 + \ldots$ if there are a_1 inner-k-steps which are connected, a_2 other inner-k-steps which are also connected and so on. The type of outer steps is defined accordingly. So if the inner path is $4, 3, 2, \underline{1, 1, 1}, 3, 4, 5, 6, 3, 2, \underline{1, 1, 1}, 2, 3, \underline{1}, 2, \underline{1, 1}, 2, \underline{1, 1, 1, 1}$ then the inner-1-steps are of type $3 + 3 + 1 + 2 + 4$. Since the order doesn't matter the type of the i-k-steps is a partition of $i(\lambda/\mu)_k$.

So we have for

$\lambda/\mu =$ 

$$iv(\lambda/\mu) = (3, 0, 1), \qquad ih(\lambda/\mu) = (2, 1, 0, 1), \qquad ov(\lambda/\mu) = (2, 2),$$
$$oh(\lambda/\mu) = (2, 1, 0, 1), \qquad i(\lambda/\mu) = (5, 1, 1, 1), \qquad o(\lambda/\mu) = (4, 3, 0, 1),$$
$$v(\lambda/\mu) = (5, 2, 1), \qquad h(\lambda/\mu) = (4, 2, 0, 2).$$

The i-1-steps are of type $3 + 2$ and the o-1-steps are of type $2 + 2$.

5.2. Bounds for the cc-type and certain pairs of components

For the following proofs we use Lemma 4.1.1 which said that

$$c(\lambda; \mu, \nu) \leq c(\lambda + \lambda'; \mu + \mu', \nu + \nu')$$

(and by conjugation $c(\lambda; \mu, \nu) \leq c(\lambda \cup \lambda'; \mu \cup \mu', \nu \cup \nu')$) for $\lambda, \mu, \nu, \lambda', \mu', \nu'$ partitions with $c(\lambda'; \mu', \nu') \neq 0$.

Remark 5.2.1. Note that $\lambda^1 + \nu \neq \lambda^2 + \nu$ for $\lambda^1 \neq \lambda^2$ so this lemma tells us that adding a skew diagram $\mathcal{B}$ to a skew diagram $\mathcal{A}$ weakly increases the number of components and constituents of $[\mathcal{A}+\mathcal{B}]$ compared to $[\mathcal{A}]$ (or $[\mathcal{B}]$). By conjugation the same applies to the row wise addition of two skew diagrams $\mathcal{A} \cup \mathcal{B}$. This allows us to consider small examples of $[\mathcal{A}]$ to give a lower bound on the number of components and constituents of larger $[\mathcal{A}']$ if $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by successively adding, column or row wise, $\mathcal{B}^i$ for some skew diagrams $\mathcal{B}^i$.

Recall that we say that $\mathcal{A}^1$ is larger than $\mathcal{A}^2$ if there exists partitions $\lambda, \mu, \alpha, \beta$ such that up to translation we have $\lambda/\mu = \mathcal{A}^1, \alpha/\beta = \mathcal{A}^2$ and λ/μ is larger than α/β. Note that this doesn't cause any problem since no matter which λ/μ with $\lambda/\mu = \mathcal{A}^1$ we choose the skew character $[\lambda/\mu]$ stays the same. Since we are only interested in informations about the skew characters we can and will always assume that both skew diagrams involved are basic skew diagrams. This will always be enough to get the desired information about the skew characters.

We will sometimes say that a skew character χ^1 is larger than another skew character χ^2. With this we mean that there exists skew diagrams $\mathcal{A}^1, \mathcal{A}^2$ with $\chi^1 = [\mathcal{A}^1]$ and $\chi^2 = [\mathcal{A}^2]$ such that $\mathcal{A}^1$ is larger than $\mathcal{A}^2$.

Also in my Diploma thesis [**Gut1**] I gave a proof for the following well known lemma:

LEMMA 5.2.2 ([**Gut1**, Lemma 3.3]). *Let $\lambda = (\lambda_1^{l_1}, \ldots, \lambda_j^{l_j}), \mu = (\mu_1, \ldots, \mu_m), \nu$ be partitions.*

(1) *If $m \leq l_i$ for some $1 \leq i \leq j$ then for all $n \geq 0$:*
$$c(\lambda; \mu, \nu) = c(\lambda \cup (\lambda_i^n); \mu, \nu \cup (\lambda_i^n)).$$
(2) *If $\mu_1 \leq \lambda_i - \lambda_{i+1}$ (as usual $\lambda_{j+1} = 0$) for some $1 \leq i \leq j$ then let $r_i = \sum_{a=1}^{i} l_a$ and for all $n \geq 0$:*
$$c(\lambda; \mu, \nu) = c(\lambda + (n^{r_i}); \mu, \nu + (n^{r_i})).$$

This lemma tells us that it is in some cases sufficient to look at small skew characters to get the cc-type of larger skew characters.

We will use the following notation also in the remaining part of this chapter.

We let $\bar{p}_n$ denote the number of partitions of n with two different kinds of 1's and 2's.

Let g_n denote the number of unordered pairs (ν^1, ν^2) of partitions of n with $\left|\nu^1 \cap \nu^2\right| = n - 1$. So g_n counts the pairs of partitions of n which differ only by one box.

LEMMA 5.2.3. *Then $\bar{p}_n = g_{n+2}$ for all n.*

PROOF. We give a bijection of partitions of n with two different kinds of 1's and 2's to pairs (ν^1, ν^2) of partitions of $n+2$ which differ only by one box. We may assume that ν^1 is lexicographically larger than ν^2.

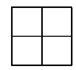
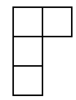
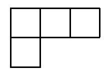
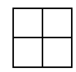
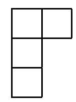
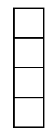
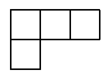
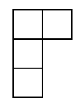
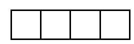
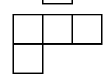
Suppose the two kinds of 1's are the usual 1 and the other be $1'$ and the two kinds of 2's are 2 and $2'$. Let $\bar{\lambda}$ be such a partition of n and let λ denote the partition formed by the usual parts of $\bar{\lambda}$. Furthermore, let n_1 denote the number of $1'$ in $\bar{\lambda}$ and n_2 denote the number of $2'$ in $\bar{\lambda}$. So $\bar{\lambda} = \lambda \cup (2'^{n_2}, 1'^{n_1})$.

For a partition $\bar{\lambda}$ now define the bijection by setting:
$$\nu^1 = \lambda \cup (n_1 + n_2 + 2, n_2), \nu^2 = \lambda \cup (n_1 + n_2 + 1, n_2 + 1).$$

Now obviously ν^1 is lexicographic larger than ν^2 and both partitions differ only by one box. Furthermore, different $\bar{\lambda}$ correspond to different triples (λ, n_1, n_2) and so give different pairs (ν^1, ν^2).

Finally the inverse map is obtained as follows: If ν^1 and ν^2 differ by only one box (and ν^1 is lexicographic larger than ν^2) , then ν^2 is obtained from ν^1 by removing a box in one row and placing it in a lower row. Let all the other rows form λ then the two rows which are different are of the form $(a+1)$ and (b) in ν^1 and (a) and $(b+1)$ in ν^2 for $a \geq b \geq 0$. Now $a+1 > b+1$ since otherwise $\nu^1 = \nu^2$. So to exclude this case we may instead assume that the rows are $(c+2)$ and (b) in ν^1 and $(c+1)$ and $(b+1)$ in ν^2 for $c \geq b \geq 0$. Setting $n_1 = c - b$ and $n_2 = b$ gives the inverse map. $\square$

Example 5.2.4. We have $\bar{p}_2 = 5$ and there is the following correspondence given by the above bijection.

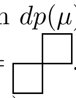
$\bar{\lambda}$	λ	ν^1	ν^2
2	2		
$2'$	0		
$1+1$	$1+1$		
$1+1'$	1		
$1'+1'$	0		

Remark 5.2.5. Lemma 5.2.3 is useful because one sees directly that the generating function for $\bar{p}_n$ is given by:

$$\sum_{i\geq 0} \bar{p}_i x^i = \frac{1}{(1-x)(1-x^2)} \prod_{i\geq 1} \frac{1}{1-x^i}.$$

LEMMA 5.2.6. *Let λ/μ be a basic skew diagram with $dp(\lambda) > dp(\mu) \geq n-1 \geq 1$. Then λ/μ is larger than δ_n/δ_{n-1} (with $\delta_n = (n, n-1, n-2, \dots, 2, 1)$).*

PROOF. We will prove this lemma only for the cases $n = 2, 3$. It should then be obvious that it is true for all cases.

Assume $n = 2$ then $dp(\mu) \geq 1$ and $dp(\lambda) \geq 2$. We will show that then λ/μ is larger than $(2,1)/(1) =$ .

Let $\lambda = (\lambda_1, \dots, \lambda_l), \mu = (\mu_1, \dots \mu_m)$ with $\lambda_l, \mu_m \geq 1$. Then we have $\lambda_1 > \lambda_l, \mu_1$ and $m < l$ because λ/μ is basic.

Then the skew diagram $(\lambda_1, \lambda_l)/(\mu_1)$ is larger than $(2,1)/(1)$:

$$(\lambda_1, \lambda_l) = (2,1) + (\lambda_1 - 2, \lambda_l - 1), \qquad (\mu_1) = (1) + (\mu_1 - 1)$$

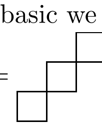
and $\mathcal{A} = (\lambda_1 - 2, \lambda_l - 1)/(\mu_1 - 1)$ is a skew diagram.

Now λ/μ is larger than $(\lambda_1, \lambda_l)/(\mu_1)$:

$$\lambda = (\lambda_1, \lambda_l) \cup (\lambda_2, \lambda_3, \dots, \lambda_{l-1}), \qquad \mu = (\mu_1) \cup (\mu_2, \mu_3, \dots, \mu_m)$$

and $\mathcal{B} = (\lambda_2, \lambda_3, \dots \lambda_{l-1})/(\mu_2, \mu_3, \dots, \mu_m)$ is a skew diagram.

So λ/μ is larger than $(2,1)/(1)$.

Now assume $n = 3$, so $dp(\mu) \geq 2$ and $dp(\lambda) \geq 3$. Because λ/μ is basic we have $\lambda_m > \mu_m$. We will show that then λ/μ is larger than $(3,2,1)/(2,1) =$ .

If $\lambda_m < \lambda_1$ then we have

$$(\lambda_1, \lambda_m, \lambda_l) = (3,2,1) + (\lambda_1 - 3, \lambda_m - 2, \lambda_l - 1),$$
$$(\mu_1, \mu_m) = (2,1) + (\mu_1 - 2, \mu_m - 1)$$

and $\mathcal{A} = (\lambda_1 - 3, \lambda_m - 2, \lambda_l - 1)/(\mu_1 - 2, \mu_m - 1)$ is a skew diagram. Obviously

$$\lambda = (\lambda_1, \lambda_m, \lambda_l) \cup (\lambda_2, \dots, \lambda_{m-1}, \lambda_{m+1}, \dots \lambda_{l-1}),$$
$$\mu = (\mu_1, \mu_m) \cup (\mu_2, \dots, \mu_{m-1}).$$

Set $\mathcal{B} = (\lambda_2, \dots, \lambda_{m-1}, \lambda_{m+1}, \dots \lambda_{l-1})/(\mu_2, \dots, \mu_{m-1})$ then since $\mathcal{B}$ consists of rows of λ/μ $\mathcal{B}$ is also a skew diagram. So λ/μ is larger than δ_3/δ_2.

If $\lambda_m = \lambda_1$ then choose i such that λ_i is the largest part of λ smaller than λ_1, so $\lambda_1 > \lambda_i > \lambda_l$. Then we have $(3, 3, 2, 1) = (3, 2, 1) \cup (3)$ and

$$(\lambda_1, \lambda_1, \lambda_i, \lambda_l) = (3, 3, 2, 1) + (\lambda_1 - 3, \lambda_1 - 3, \lambda_i - 2, \lambda_l - 1),$$
$$(\mu_1, \mu_m) = (2, 1) + (\mu_1 - 2, \mu_m - 1)$$

and $\mathcal{A} = (\lambda_1 - 3, \lambda_1 - 3, \lambda_i - 2, \lambda_l - 1)/(\mu_1 - 2, \mu_m - 1)$ is a skew diagram. Obviously

$$\lambda = (\lambda_1, \lambda_1, \lambda_i, \lambda_l) \cup (\lambda_3, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots \lambda_{l-1}),$$
$$\mu = (\mu_1, \mu_m) \cup (\mu_2, \dots, \mu_{m-1}).$$

Set $\mathcal{B} = (\lambda_3, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots \lambda_{l-1})/(\mu_2, \dots, \mu_{m-1})$ then since $\mathcal{B}$ again consists of rows of λ/μ (λ contains the part λ_1 at least m times while μ only has m parts) $\mathcal{B}$ is again a skew diagram. So also in this case λ/μ is larger than δ_3/δ_2. □

In the following theorem the condition $dp(\lambda) \geq dp(\mu) + 1 \geq 2$ only makes sure that λ/μ is neither a partition nor a rotated partition but constrains λ/μ not in any other way. The case that λ/μ is a partition α or rotated partition α° is uninteresting for the theorem because then $[\lambda/\mu] = [\alpha]$ is irreducible.

Theorem 5.2.7. *Let λ/μ be a basic skew diagram with $dp(\lambda) \geq n = dp(\mu) + 1 \geq 2$.*

Then $[\lambda/\mu] = \sum_\nu c(\lambda; \mu, \nu)[\nu]$ contains at least g_n characters $[\nu^1], [\nu^2]$ whose corresponding diagrams differ only by one box, i.e. there are ν^1, ν^2 with $|\nu^1 \cap \nu^2| = |\nu^1| - 1 = |\nu^2| - 1$ and $c(\lambda; \mu, \nu^1), c(\lambda; \mu, \nu^2) \neq 0$ (with g_n as in Lemma 5.2.3).

Furthermore, if $\lambda = (\lambda_1, \dots, \lambda_l), \mu = (\mu_1, \dots \mu_m)$ with $\lambda_l, \mu_m \geq 1$ set $\mathcal{A} = (\lambda_1 - 2, \lambda_l - 1)/(\mu_1 - 1)$ and $\mathcal{B} = (\lambda_2, \lambda_3, \dots \lambda_{l-1})/(\mu_2, \mu_3, \dots, \mu_m)$ with $[\mathcal{A}]$ having a components and $[\mathcal{B}]$ having b components. Then there are at least $\max(a, b)$ of those pairs ν^1, ν^2.

Proof. We first show there are at least $\max(a, b)$ pairs ν^1, ν^2.

We can deduce this part of the theorem from the fact that $[(2, 1)/(1)] = [2] + [1^2]$ contains two characters whose corresponding diagrams differ only by one box.

In the proof of Lemma 5.2.6 we explicitly showed how to obtain λ/μ from $(2, 1)/(1)$.

The skew diagram $(\lambda_1, \lambda_l)/(\mu_1)$ is larger than $(2, 1)/(1)$:

$$(\lambda_1, \lambda_l) = (2, 1) + (\lambda_1 - 2, \lambda_l - 1), \qquad (\mu_1) = (1) + (\mu_1 - 1)$$

and $\mathcal{A} = (\lambda_1 - 2, \lambda_l - 1)/(\mu_1 - 1)$ is a skew diagram. Let α be a partition such that $[\alpha]$ appears in $[\mathcal{A}]$, so $c((\lambda_1 - 2, \lambda_l - 1); (\mu_1 - 1), \alpha) \neq 0$.

Then by Lemma 4.1.1 $[\alpha + (1^2)]$ and $[\alpha + (2)]$ both appear in $[(\lambda_1, \lambda_l)/(\mu_1)]$ and, furthermore, $\alpha + (1^2) \cap \alpha + (2) = \alpha + (1)$ so $\alpha + (1^2)$ and $\alpha + (2)$ differ by only one box.

Now λ/μ is larger than $(\lambda_1, \lambda_l)/(\mu_1)$:

$$\lambda = (\lambda_1, \lambda_l) \cup (\lambda_2, \lambda_3, \dots, \lambda_{l-1}), \qquad \mu = (\mu_1) \cup (\mu_2, \mu_3, \dots, \mu_m)$$

and $\mathcal{B} = (\lambda_2, \lambda_3, \dots \lambda_{l-1})/(\mu_2, \mu_3, \dots, \mu_m)$ is a skew diagram. Let β be a partition such that $[\beta]$ appears in $[\mathcal{B}]$.

Then by Lemma 4.1.1 $[(\alpha + (1^2)) \cup \beta]$ and $[(\alpha + (2)) \cup \beta]$ both appear in $[\lambda/\mu]$ and

$$\big((\alpha + (1^2)) \cup \beta\big) \cap ((\alpha + (2)) \cup \beta) = (\alpha + (1)) \cup \beta$$

so $\nu^1 = (\alpha + (1^2)) \cup \beta$ and $\nu^2 = (\alpha + (2)) \cup \beta$ differ only by one box.

Furthermore, notice that a different choice for α or β yields a different pair ν^1, ν^2. This proves that there are at least $\max(a, b)$ pairs ν^1, ν^2.

Now we will prove that there are also at least g_n pairs ν^1, ν^2.

As an easy consequence of the LR rule we have

$$[\delta_n/\delta_{n-1}] = [\underbrace{(1) \otimes (1) \otimes \cdots \otimes (1)}_{n\text{-times}}] = [1]^n = \sum_{\lambda \vdash n} f^\lambda [\lambda]$$

where f^λ is the number of standard Young tableaux of shape λ, in particular, all irreducible characters of S_n appear in $[\delta_n/\delta_{n-1}]$. So by definition of g_n $[\delta_n/\delta_{n-1}]$ contains g_n characters $[\alpha], [\beta]$ whose corresponding diagrams differ only by one box.

By Lemma 5.2.6 λ/μ is larger than δ_n/δ_{n-1}, so there exist skew diagrams $\mathcal{B}^i$ such that λ/μ is obtained from δ_n/δ_{n-1} by using the operations $+, \cup$ together with the $\mathcal{B}^i$. Let $\circ^i$ be either $+$ or $\cup$ then:

$$\lambda/\mu = ((\delta_n/\delta_{n-1} \circ^1 \mathcal{B}^1) \circ^2 \mathcal{B}^2) \cdots \circ^j \mathcal{B}^j.$$

Choose $[\alpha^i]$ contained in $[\mathcal{B}^i]$ and $[\bar\nu^1], [\bar\nu^2]$ contained in $[\delta_n/\delta_{n-1}]$ with $\left|\bar\nu^1 \cap \bar\nu^2\right| = n - 1$. Set

$$\nu^1 = ((\bar\nu^1 \circ^1 \alpha^1) \circ^2 \alpha^2) \cdots \circ^j \alpha^j, \qquad \nu^2 = ((\bar\nu^2 \circ^1 \alpha^1) \circ^2 \alpha^2) \cdots \circ^j \alpha^j$$

then by Lemma 4.1.1 both $[\nu^1], [\nu^2]$ appear in $[\lambda/\mu]$ and, furthermore, $\left|\nu^1 \cap \nu^2\right| = \left|\nu^1\right| - 1$. Finally a different choice of $\bar\nu^1, \bar\nu^2$ gives different ν^1, ν^2 (for fixed $(\alpha^i, \circ^i)$) and there are by definition g_n choices for $\bar\nu^1, \bar\nu^2$. $\square$

Remark 5.2.8. In my diploma thesis I proofed using Theorem 3.1.2 that skew characters which are not irreducible contain at least one pair of irreducible characters whose corresponding skew diagrams differ by only one box. Obviously the above theorem is an improvement.

Theorem 5.2.9. *Let λ/μ be a basic skew diagram with $dp(\lambda) \geq n = dp(\mu) + 1$. Then $cc(\lambda/\mu)$ is at least (p_n, f_n) where p_n is the number of partitions of n and f_n the number of standard Young tableaux with n boxes.*

Proof. Let $\delta_n = (n, n-1, n-2, \dots, 2, 1)$ then, as already mentioned above, as an easy consequence of the LR rule we have

$$[\delta_n/\delta_{n-1}] = [\underbrace{(1) \otimes (1) \otimes \cdots \otimes (1)}_{n\text{-times}}] = [1]^n = \sum_{\lambda \vdash n} f^\lambda [\lambda]$$

where f^λ is the number of standard Young tableaux of shape λ. So we have $cc(\delta_n/\delta_{n-1}) = (p_n, f_n)$. Since $dp(\lambda) \geq n, dp(\mu) = n - 1$ λ/μ is larger than δ_n/δ_{n-1} by Lemma 5.2.6 and so $cc(\lambda/\mu)$ is at least (p_n, f_n). $\square$

Remark 5.2.10. In the On-Line Encyclopedia of Integer Sequences [**OEIS**] $g_n = \bar{p}_{n-2}$ has the id: A000097, p_n has the id: A000041 and f_n has the id: A000085. Their first terms are:

$n:$	1	2	3	4	5	6	7	8	9	10	11	12	13
$g_n:$	0	1	2	5	9	17	28	47	73	114	170	253	365
$p_n:$	1	2	3	5	7	11	15	22	30	42	56	77	101
$f_n:$	1	2	4	10	26	76	232	764	2620	9496	35696	140152	568504

Lemma 5.2.11. *Let α, β be partitions with $dp(\alpha) \geq dp(\beta) = n$. Then $[\alpha] \otimes [\beta]$ has cc-type at least (p_{n+1}, f_{n+1}) and contains g_{n+1} pairs of components $([\nu^1], [\nu^2])$ such that their corresponding partitions differ only by one box.*

Proof. This follows directly from the previous theorems by setting $\lambda/\mu = \alpha \otimes \beta^\circ$ because then $dp(\lambda) = dp(\alpha) + 1, dp(\mu) = dp(\beta)$. $\square$

Lemma 5.2.12. *Let λ/μ be a skew diagram with $|\lambda/\mu| = n$.*
Then $[\lambda/\mu]$ contains at most

- *g_n pairs $[\nu^1], [\nu^2]$ such that $\left|\nu^1 \cap \nu^2\right| = n - 1$.*
- *p_n components.*
- *$\min(f_n, p_n f^\mu, p_n f^{\bar{\lambda}})$ constituents (with $\bar{\lambda} = (\lambda_1 - \lambda_l, \lambda_1 - \lambda_{l-1}, \ldots, \lambda_1 - \lambda_3, \lambda_1 - \lambda_2, 0)$).*

Proof. The first two statements are trivial, because there are not more irreducible characters of S_n.

For the third statement notice, that λ/μ is smaller than δ_n/δ_{n-1} which gives by Lemma 4.1.1: $c(\lambda;\mu,\nu) \leq c(\delta_n;\delta_{n-1},\nu) = f^\nu$. Since the LR coefficient is symmetric in μ and ν we also have $c(\lambda;\mu,\nu) \leq f^\mu$ and by rotation symmetry: $c(\lambda;\mu,\nu) \leq f^{\bar{\lambda}}$.

So for the number of constituents of $[\lambda/\mu]$:

$$\begin{aligned}
\sum_\nu c(\lambda;\mu,\nu) &\leq \sum_\nu f^\nu = f_n, \\
\sum_\nu c(\lambda;\mu,\nu) &= \sum_{\nu \vdash n} c(\lambda;\mu,\nu) \leq \sum_{\nu \vdash n} f^\mu = p_n f^\mu, \\
\sum_\nu c(\lambda;\mu,\nu) &= \sum_{\nu \vdash n} c(\lambda;\mu,\nu) \leq \sum_{\nu \vdash n} f^{\bar{\lambda}} = p_n f^{\bar{\lambda}}.
\end{aligned}$$

Notice that all three bounds are reached for $\lambda/\mu = \delta_n/\delta_{n-1}$. $\square$

5.3. Skew characters containing few constituents or components

The classification of skew diagrams λ/μ whose corresponding skew character $[\lambda/\mu]$ has at most five constituents or components is the following:

Theorem 5.3.1. *Let λ/μ be a basic skew diagram.*
$[\lambda/\mu]$ contains at most 5 constituents (so contains at most 5 characters) if and only if λ/μ satisfies up to rotation and/or conjugation one of the following:

- *λ/μ is a partition (1 constituent)*
- *$dp(\lambda) = 2, dp(\mu) = 1$, $v(\lambda/\mu)_1 \geq 1$ and*
  - *$h(\lambda/\mu)_1 \geq 1$ (2 constituents)*
  - *$h(\lambda/\mu)_2 \geq 1$ (3 constituents)*
  - *$h(\lambda/\mu)_3 \geq 1$ (4 constituents)*
  - *$h(\lambda/\mu)_4 \geq 1$ (5 constituents)*

- *$dp(\lambda) = 3, dp(\mu) = 1$ and*
  - *$i(\lambda/\mu)_1 \geq 2$ (3 constituents)*
  - *$iv(\lambda/\mu)_1 = 1, oh(\lambda/\mu)_1 = 2$ (4 constituents)*
  - *$iv(\lambda/\mu)_1 = 1, ih(\lambda/\mu)_2 = 1, oh(\lambda/\mu)_1 = 1$ (5 constituents)*
  - *$o(\lambda/\mu)_1 = 4$ and the o-1-steps are of type $2 + 2$ (5 constituents)*
- *$dp(\lambda) = 3, dp(\mu) = 2$ and*
  - *$\lambda/\mu = (1) \otimes (1) \otimes (1)$ (4 constituents)*
  - *$i(\lambda/\mu)_1 = 5, o(\lambda/\mu)_1 = 5$ and the o-1-steps are of type $4 + 1$ or $3 + 2$ (5 constituents)*
- *$dp(\lambda) = 4, dp(\mu) = 1$ and $i(\lambda/\mu)_1 = 2$ (4 constituents)*
- *$dp(\lambda) = 5, dp(\mu) = 1$ and $i(\lambda/\mu)_1 = 2$ (5 constituents)*

$[\lambda/\mu]$ contains at most 5 components (so contains at most 5 different characters) if λ/μ satisfies up to rotation and/or conjugation one of the following:

- *λ/μ is a partition (1 component)*
- *$dp(\lambda) = 2, dp(\mu) = 1$, $v(\lambda/\mu)_1 \geq 1$ and*
  - *$h(\lambda/\mu)_1 \geq 1$ (2 components)*
  - *$h(\lambda/\mu)_2 \geq 1$ (3 components)*
  - *$h(\lambda/\mu)_3 \geq 1$ (4 components)*
  - *$h(\lambda/\mu)_4 \geq 1$ (5 components)*
- *$dp(\lambda) = 3, dp(\mu) = 1$ and*
  - *$i(\lambda/\mu)_1 \geq 2$ (3 components)*
  - *$iv(\lambda/\mu)_1 = 1, oh(\lambda/\mu)_1 = 2$ (4 components)*
  - *$iv(\lambda/\mu)_1 = 1, ih(\lambda/\mu)_2 = 1, oh(\lambda/\mu)_1 = 1$ (5 components)*
  - *$o(\lambda/\mu)_1 = 4$ and the o-1-steps are of type $2 + 2$ (5 components)*
- *$dp(\lambda) = 3, dp(\mu) = 2$ and*
  - *$\lambda/\mu = (1) \otimes (1) \otimes (1)$ (3 components)*
  - *$\lambda/\mu = (1) \otimes (2) \otimes (2)$ (5 components)*
  - *$i(\lambda/\mu)_1 = 5, o(\lambda/\mu)_1 = 5$ and the o-1-steps are of type $4 + 1$ or $3 + 2$ (4 components)*
  - *$i(\lambda/\mu)_1 = 5, o(\lambda/\mu)_1 = 5$ and the o-1-steps are of type 5 (5 components)*
  - *$\lambda/\mu = (1) \otimes \mathcal{A}$ with $\mathcal{A} = (\alpha_1^{a_1}, \alpha_2^{a_2})/(\beta_1^{b_1})$ being a basic skew diagram with $i(\mathcal{A})_1 = 2$ and $\mathcal{A}$ containing o-1-steps of type 2 or 3 (5 components)*
- *$dp(\lambda) = 4, dp(\mu) = 1$ and $i(\lambda/\mu)_1 = 2$ (4 components)*
- *$dp(\lambda) = 4, dp(\mu) = 2$ and $\lambda/\mu = (1) \otimes (1) \otimes (2, 1)$ (5 components)*
- *$dp(\lambda) = 4, dp(\mu) = 3$ and $\lambda/\mu = (1) \otimes (1) \otimes (1) \otimes (1)$ (5 components)*
- *$dp(\lambda) = 5, dp(\mu) = 1$ and $i(\lambda/\mu)_1 = 2$ (5 components)*

Remark 5.3.2. Clearly for $dp(\lambda) = 2, dp(\mu) = 1$ with $v(\lambda/\mu)_1 = 1, h(\lambda/\mu)_3 = 1$ then $[\lambda/\mu]$ has exactly 4 components only if there are no h-1- or h-2-steps.

Furthermore, note that conjugation corresponds simply to exchanging vertical and horizontal steps and by rotation symmetry we may assume that $dp(\lambda) > dp(\mu)$.

Before we prove this theorem in the next section we will now list explicitly the skew characters containing 2 and 3 constituents and components. Here $\alpha, \beta, \gamma, a, b, c$ are arbitrary non-negative integers such that all characters appearing on the left hand side correspond to partitions. For example, $[(\alpha + 1)^{a+1}, \alpha^b, \beta] + [(\alpha + 1)^a, \alpha^{b+1}, \beta + 1] = [((\alpha + 1)^{a+1}, \alpha^{b+1})/(\alpha - \beta)]$ implies that $\alpha \geq \beta + 1$ since otherwise $[(\alpha+1)^a, \alpha^{b+1}, \beta+1]$ would not correspond to a partition. So in this example $\beta, a, b \geq 0$ and $\alpha \geq \beta+1$. Choosing the minimal values $\alpha = 1, \beta = a = b = 0$ gives $[2] + [1^2] = [(2,1)/(1)]$.

Remark 5.3.3. The skew characters with 2 constituents and components are:

- $[\alpha+1, \alpha^a] + [\alpha^{a+1}, 1] = [1] \otimes [\alpha^{a+1}]$
- $[\alpha, 1^{a+1}] + [\alpha+1, 1^a] = [(\alpha+1, 1^{a+1})/(1)] = [1^{a+1}] \otimes [\alpha]$
- $[(\alpha+1)^{a+1}, \alpha^b, \beta] + [(\alpha+1)^a, \alpha^{b+1}, \beta+1] = [((\alpha+1)^{a+1}, \alpha^{b+1})/(\alpha-\beta)]$
- $[(\alpha+1)^a, \alpha, (\beta+1)^{b+1}] + [(\alpha+1)^{a+1}, (\beta+1)^b, \beta] =$ $[((\alpha+1)^{a+1}, (\beta+1)^{b+1})/(1)]$
- $[\alpha^{a+1}, \beta+1, \beta^b] + [\alpha+1, \alpha^a, \beta^{b+1}] = [((\alpha+1)^{a+1}, (\beta+1)^{b+1})/(1^{a+b+1})]$
- $[\alpha^a, \beta^{b+1}, 1] + [\alpha^a, \beta+1, \beta^b] = [(\alpha^{a+1}, \beta^{b+1})/(\alpha-1)]$
- $[\alpha, (\beta+1)^{a+1}, \beta^b] + [\alpha+1, (\beta+1)^a, \beta^{b+1}] = [(\alpha+1, (\beta+1)^{a+b+1})/(1^{b+1})]$
- $[\alpha^a, \beta, 1^{b+1}] + [\alpha^a, \beta+1, 1^b] = [(\alpha^{a+1}, 1^{b+1})/(\alpha-\beta)]$

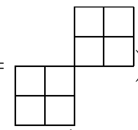
Remark 5.3.4. The skew characters with 3 constituents and components are:

- $[\alpha+1, \alpha^a, \beta^{b+1}] + [\alpha^{a+1}, \beta+1, \beta^b] + [\alpha^{a+1}, \beta^{b+1}, 1] = [1] \otimes [\alpha^{a+1}, \beta^{b+1}]$
- $[\alpha+2, \alpha^a] + [\alpha+1, \alpha^a, 1] + [\alpha^{a+1}, 2] = [2] \otimes [\alpha^{a+1}]$
- $[(\alpha+1)^2, \alpha^a] + [\alpha+1, \alpha^{a+1}, 1] + [\alpha^{a+2}, 1^2] = [1^2] \otimes [\alpha^{a+2}]$
- $[\alpha+2, 2^a] + [\alpha+1, 2^a, 1] + [\alpha, 2^{a+1}] = [(\alpha+2, 2^{a+1})/(2)] = [\alpha] \otimes [2^{a+1}]$
- $[\alpha^2, 1^{a+2}] + [\alpha+1, \alpha, 1^{a+1}] + [(\alpha+1)^2, 1^a] = [((\alpha+1)^2, 1^{a+2})/(1^2)] =$ $[\alpha^2] \otimes [1^{a+2}]$
- $[(\alpha+2)^{a+1}, \alpha^b, \beta] + [(\alpha+2)^a, \alpha+1, \alpha^b, \beta+1] + [(\alpha+2)^a, \alpha^{b+1}, \beta+2] =$ $[((\alpha+2)^{a+1}, \alpha^{b+1})/(\alpha-\beta)]$
- $[(\alpha+1)^a, \alpha^{b+2}, (\beta+1)^2] + [(\alpha+1)^{a+1}, \alpha^{b+1}, \beta+1, \beta] + [(\alpha+1)^{a+2}, \alpha^b, \beta^2] =$ $[((\alpha+1)^{a+b+2}, (\beta+1)^2)/(1^{b+2})]$
- $[\alpha^a, \beta, 2^{b+1}] + [\alpha^a, \beta+1, 2^b, 1] + [\alpha^{a+1}, \beta+2, 2^b] = [(\alpha^{a+1}, 2^{b+1})/(\alpha-\beta)]$
- $[\alpha^2, (\beta+1)^{a+2}, \beta^b] + [\alpha+1, \alpha, (\beta+1)^{a+1}, \beta^{b+1}] + [(\alpha+1)^2, (\beta+1)^a, \beta^{b+2}] =$ $[((\alpha+1)^2, (\beta+1)^{a+b+2})/(1^{b+2})]$
- $[(\alpha+2)^{a+1}, (\beta+2)^b, \beta] + [(\alpha+2)^a, \alpha+1, (\beta+2)^b, \beta+1] +$ $[(\alpha+2)^a, \alpha, (\beta+2)^{b+1}] = [((\alpha+2)^{a+1}, (\beta+2)^{b+1})/(2)]$
- $[(\alpha+1)^a, \alpha^2, (\beta+1)^{b+2}] + [(\alpha+1)^{a+1}, \alpha, (\beta+1)^{b+1}, \beta] +$ $[(\alpha+1)^{a+2}, (\beta+1)^b, \beta^2] = [((\alpha+1)^{a+2}, (\beta+1)^{b+2})/(1^2)]$
- $[\alpha^a, \beta^{b+1}, 2] + [\alpha^a, \beta+1, \beta^b, 1] + [\alpha^a, \beta+2, \beta^b] = [(\alpha^{a+1}, \beta^{b+1})/(\alpha-2)]$
- $[\alpha^{a+2}, (\beta+1)^2, \beta^b] + [\alpha+1, \alpha^{a+1}, \beta+1, \beta^{b+1}] + [(\alpha+1)^2, \alpha^a, \beta^{b+2}] =$ $[((\alpha+1)^{a+2}, (\beta+1)^{b+2})/(1^{a+b+2})]$
- $[\alpha+2, \alpha^a, \beta^{b+1}] + [\alpha+1, \alpha^a, \beta+1, \beta^b] + [\alpha^{a+1}, \beta+2, \beta^b] =$ $[(\alpha+2, \alpha^{a+b+1})/((\alpha-\beta)^{b+1})]$
- $[\alpha^a, \beta^{b+2}, 1^2] + [\alpha^a, \beta+1, \beta^{b+1}, 1] + [\alpha^a, (\beta+1)^2, \beta^b] =$ $[(\alpha^{a+b+2}, 1^2)/((\alpha-\beta)^{b+2})]$
- $[\alpha+2, (\beta+2)^a, \beta^{b+1}] + [\alpha+1, (\beta+2)^a, \beta+1, \beta^b] + [\alpha, (\beta+2)^{a+1}, \beta^b] =$ $[(\alpha+2, (\beta+2)^{a+b+1})/(2^{b+1})]$
- $[\alpha^a, \beta^2, 1^{b+2}] + [\alpha^a, \beta+1, \beta, 1^{b+1}] + [\alpha^a, (\beta+1)^2, 1^b] =$ $[(\alpha^{a+2}, 1^{b+2})/((\alpha-\beta)^2)]$
- $[(\alpha+1)^a, \alpha, (\beta+1)^{b+1}, (\gamma+1)^{c+1}] + [(\alpha+1)^{a+1}, (\beta+1)^b, \beta, (\gamma+1)^{c+1}] +$ $[(\alpha+1)^{a+1}, (\beta+1)^{b+1}, (\gamma+1)^c, \gamma] =$ $[((\alpha+1)^{a+1}, (\beta+1)^{b+1}, (\gamma+1)^{c+1})/(1)]$
- $[\alpha^a, \beta^{b+1}, \gamma^{c+1}, 1] + [\alpha^a, \beta^{b+1}, \gamma+1, \gamma^c] + [\alpha^a, \beta+1, \beta^b, \gamma^{c+1}] =$ $[(\alpha^{a+1}, \beta^{b+1}, \gamma^{c+1})/(\alpha-1)]$

- $[\alpha^{a+1}, \beta^{b+1}, \gamma+1, \gamma^c] + [\alpha^{a+1}, \beta+1, \beta^b, \gamma^{c+1}] + [\alpha+1, \alpha^a, \beta^{b+1}, \gamma^{c+1}] = [((\alpha+1)^{a+1}, (\beta+1)^{b+1}, (\gamma+1)^{c+1})/(1^{a+b+c+2})]$

We will now prove Theorem 5.3.1 by proving the cases in the following lemmas. We will assume that $\lambda = (\lambda_1^{l_1}, \lambda_2^{l_2}, \ldots)$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. The same applies to $\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots)$. Furthermore, set $l(\lambda) = l = \sum l_i, l(\mu) = m = \sum m_i$.

Lemma 5.3.5. *Let $dp(\lambda) = 2, dp(\mu) = 1$ and λ/μ be a basic skew diagram. Then $[\lambda/\mu]$ contains 5 or less components or constituents only in the cases of Theorem 5.3.1.*

Proof. We have $cc\big((4^2, 2^2)/(2^2) = (2^2) \otimes (2^2) =$  $) = (6, 6)$ and so λ/μ contains at least 6 components (and so at least 6 constituents) unless at least one of $\lambda_1 - \lambda_2, \lambda_2, \mu_1, \lambda_1 - \mu_1, l_1, l_2, m$ or $l - m$ has value 1. So by Lemma 4.1.1 at least one step in λ/μ is a 1-step and by conjugation we may assume that $v(\lambda/\mu)_1 \geq 1$.

Furthermore, we have $cc((n) \otimes (n)) = (n+1, n+1)$ for every $n \geq 1$. So if all horizontal-steps are at least n-steps for some $n \geq 1$ ($h(\lambda/\mu)_i = 0$ for $i < n$) then $[\lambda/\mu]$ contains at least $n+1$ constituents.

On the other hand, by rotation symmetry we may assume that $iv(\lambda/\mu)_1 \geq 1$, so either $m = 1$ or $l - m = 1$. For $m = 1$ we may by Lemma 5.2.2 assume that $l_1 = l_2 = 1$ and that $\lambda_2 \leq \mu_1$. So we may assume that $\lambda/\mu = (n_1) \otimes (n_2)$ which has cc-type $(n_1 + 1, n_1 + 1)$ for $n_1 \leq n_2$. One checks easily that n_1 is the value of the smallest h-step in λ/μ.

In the case $l - m = 1$ we may assume by Lemma 5.2.2 that $\lambda_2 \leq \mu_1$ so $\lambda/\mu = (\lambda_2) \otimes ((\lambda_1 - \mu_1)^{l_1})$ decays and by reordering this is the case $m = 1$ again. □

Lemma 5.3.6. *Let $dp(\lambda) = 3, dp(\mu) = 1$ and λ/μ be a basic skew diagram. Then $[\lambda/\mu]$ contains 5 or less components or constituents only in the cases of Theorem 5.3.1.*

Proof. Let us first check that the cases of Theorem 5.3.1 have the given cc-type.

For arbitrary λ with $dp(\lambda) \geq 3$, $i(\lambda/\mu)_1 = 2$ is possible only in the cases: $\mu = (1), \mu = (\lambda_1 - 1)$ or $\mu = (1^{l-1})$. From the LR rule it follows directly that $\lambda/(1)$ has cc-type $(dp(\lambda), dp(\lambda))$ so in the case of this lemma $cc(\lambda/\mu) = (3, 3)$. For $\mu = (\lambda_1 - 1)$ we may by Lemma 5.2.2 assume that for all $i \geq 1$ we have $l_i = 1$. So λ/μ decays: $\lambda/\mu = (1) \otimes (\lambda_2, \lambda_3, \ldots)$ and again by an easy consequence of the LR rule we have $cc(\lambda/\mu) = (dp(\lambda), dp(\lambda))$ so in the case of this lemma again $cc(\lambda/\mu) = (3, 3)$. For $\mu = (1^{l-1})$ we may by Lemma 5.2.2 assume that $\lambda_l = 1$ so λ/μ decays and by reordering the parts we have again the case $\mu = (\lambda_1 - 1)$. We will not repeat this argument in the Lemmas 5.3.8 and 5.3.11.

Let us check now the case that $iv(\lambda/\mu)_1 = 1, oh(\lambda/\mu)_1 = 2$ and so $cc(\lambda/\mu) = (4, 4)$. There are two possibilities for μ: $\mu = (\mu_1)$ or $\mu = (\mu_1^{l-1})$ with $1 < \mu_1 < \lambda_1 - 1$ (otherwise this would be the case $i(\lambda/\mu)_1 = 2$).

For the case $\mu = (\mu_1)$ it is by Lemma 5.2.2 enough to check the cases with $l_1 = l_2 = l_3 = 1$. So exactly 2 of the values $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3$ and λ_3 equal to 1 while the third value is larger than 1. Let a, b be the rows of λ which have exactly one box more than the following row and c the other row. Then the characters $[\nu]$ in $[\lambda/\mu]$ are obtained by determining which boxes are filled with entry 1. Because we can choose the rows a and b only once but the other row as often as we like we get the following four choices:

(1) row a and row b filled,
(2) row a filled and row b empty,
(3) row a empty and row b filled,
(4) row a and row b empty.

So the cc-type is $(4,4)$ in this case.

In the case $\mu = (\mu_1^{l-1})$ we have either $\lambda_3 = 1$ or $\lambda_3 > 1$. If $\lambda_3 = 1$ this is the case $i(\lambda/\mu)_1 = 2$ because λ/μ is basic. So suppose $\lambda_3 > 1$ and so $\lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$. Because of Lemma 5.2.2 we may assume that $\lambda_3 \leq \mu_1$ and so λ/μ decays: $\lambda/\mu = (\lambda_3) \otimes (2^{l_1}, 1^{l_2})$. It follows directly from the LR rule that $cc(\lambda/\mu) = (4,4)$ since $\lambda_3 > 1$ (one could also use the result for $\mu = (\mu_1)$ by reordering λ/μ).

Let us check now the case $iv(\lambda/\mu)_1 = 1, ih(\lambda/\mu)_2 = 1, oh(\lambda/\mu)_1 = 1$ and so $cc(\lambda/\mu) = (5,5)$.

There are the following possibilities for μ: $\mu = (2)$, $\mu = (\lambda_1 - 2)$, $\mu = (2^{l-1})$ and $\mu = ((\lambda_1 - 2)^{l-1})$ while in the lase case $oh(\lambda/\mu)_1 = 2$ and so is one of the previous cases.

Note that because of the previous cases $ih(\lambda/\mu)_1 = 0, oh(\lambda/\mu)_1 = 1$.

For the cases $\mu = (2)$ let a be the row in λ which has one box more than the successive row and let b and c be the rows which have 2 or more boxes more than the successive row. Filling the boxes in rows a, b, c gives us characters in $[\lambda/\mu]$ and there are the following five cases:

(1) fill 1 box in row a and 1 box in row b
(2) fill 1 box in row a and 1 box in row c
(3) fill 2 boxes in row b
(4) fill 1 box in row b and 1 box in row c
(5) fill 2 boxes in row c

So the cc-type is $(5,5)$ in this case.

In the case $\mu = (\lambda_1 - 2)$ we also have (with the same notation as in the case $\mu = (2)$) to fill the boxes in rows a, b, c but this time have to leave exactly 2 of the boxes, which could be filled, empty instead of filling them. So we have the same choices as above if we instead of filling the boxes we leave them empty.

For the case $\mu = (2^{l-1})$ we have $\lambda_3 > 1$ because otherwise λ/μ would not be basic. Because of Lemma 5.2.2 we may assume $\lambda_3 = 2$ and $\lambda_1 = 5$ (it is one of $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_3$ equal to 1 and the other can assumed to be 2 by Lemma 5.2.2). So $\lambda/\mu = (2) \otimes (3^{l_1}, 2^{l_2})$ or $\lambda/\mu = (2) \otimes (3^{l_1}, 1^{l_2})$ and both cases are of cc-type $(5,5)$ by the LR rule.

Let us check now the case that λ/μ has 4 o-1-steps of type $2+2$ and so cc-type $(5,5)$. Because of the previous cases we may assume $i(\lambda/\mu)_1 = 0$.

There are 3 possibilities: $l_1 = l_3 = \lambda_1 - \lambda_2 = \lambda_3 = 1$ and $l_1 = l_3 = \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$ which is conjugate to $l_2 = l_3 = \lambda_1 - \lambda_2 = \lambda_3 = 1$ so we only have to check the first 2.

In the case $l_1 = l_3 = \lambda_1 - \lambda_2 = \lambda_3 = 1$ there are the LR fillings as in Fig. 1 and 2 (with $n^1 = n - 1 = l(\mu)$).

All of them are possible and different because $\lambda_1 - \mu_1, \mu_1, m, l - m \geq 2$.

In the case $l_1 = l_3 = \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$ there are the LR fillings as in Fig. 3 and 4 (with $n^1 = n - 1 = l(\mu)$).

Again all of them are possible and different because $\lambda_1 - \mu_1, \mu_1, m, l - m \geq 2$.

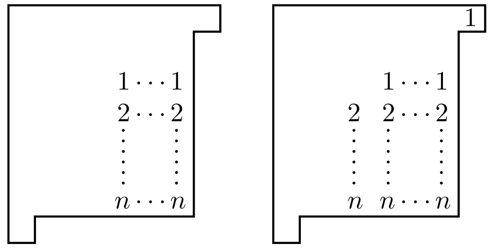


Figure 1. Lemma 5.3.6: $l_1 = l_3 = \lambda_1 - \lambda_2 = \lambda_3 = 1$, part 1

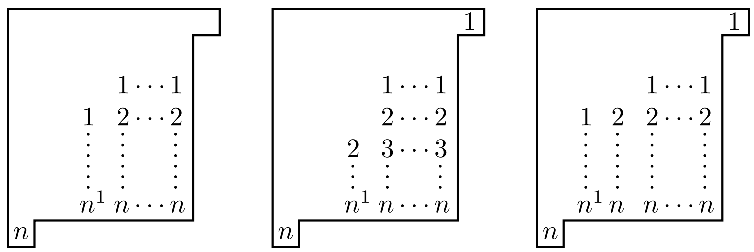


Figure 2. Lemma 5.3.6: $l_1 = l_3 = \lambda_1 - \lambda_2 = \lambda_3 = 1$, part 2

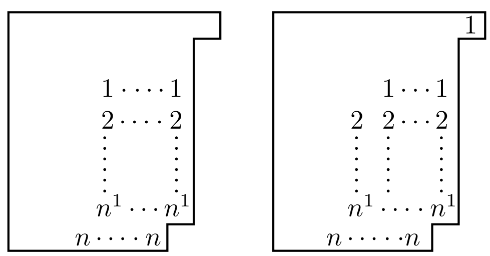


Figure 3. Lemma 5.3.6: $l_1 = l_3 = \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$, part 1

We will now prove that the mentioned cases are the only ones with 5 or less components and constituents.

There are no basic skew diagrams λ/μ with $dp(\lambda) = 3, dp(\mu) = 1$ and $i(\lambda/\mu)_1 \geq 3$. So suppose $i(\lambda/\mu)_1 = 1$ then we may assume by conjugation symmetry that this is an iv-1-step. So we have either $\mu = (\mu_1)$ or $\mu = (\mu_1^{l-1})$.

Suppose $\mu = (\mu_1)$. Suppose $oh(\lambda/\mu)_1 = 0$ then λ/μ is larger than one of the following skew diagrams: $(6,4,2)/(2)$, $(6,4,2)/(3)$ and $(6,4,2)/(4)$. And we have $cc((6,4,2)/(2)) = (6,6)$, $cc((6,4,2)/(3)) = (7,7)$ and $cc((6,4,2)/(4)) = (6,6)$, so $cc(\lambda/\mu)$ is also at least $(6,6)$.

So now suppose $ih(\lambda/\mu)_1 = ih(\lambda/\mu)_2 = 0$ and $oh(\lambda/\mu)_1 \leq 1$ then λ/μ is larger than one of the following skew diagrams: $(6,4,2)/(3)$, $(6,5,3)/(3)$,$(6,4,2)/(3)$,

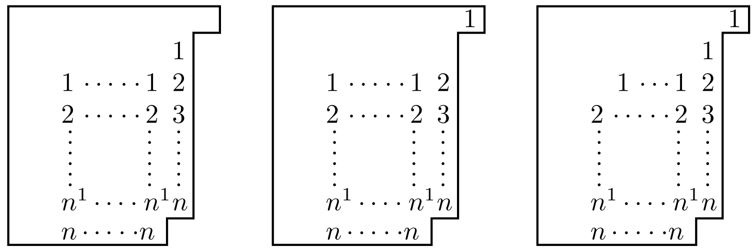
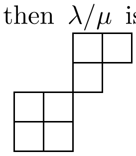
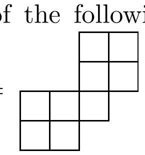
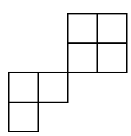


Figure 4. Lemma 5.3.6: $l_1 = l_3 = \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$, part 2

$(6,4,3)/(3)$, $(6,3,2)/(3)$, $(6,4,1)/(3)$, $(6,3,1)/(3)$. Now $cc((6,4,2)/(3)) = (7,7)$ while the other 6 have cc-type $(6,6)$, so $cc(\lambda/\mu)$ is also at least $(6,6)$.

In the case $\mu = (\mu_1^{l-1})$ we may by Lemma 5.2.2 assume that $\lambda_3 = \mu_1$ (λ/μ is basic). And so λ/μ decays: $\lambda/\mu = (\lambda_3) \otimes ((\lambda_1 - \lambda_3)^{l_1}, (\lambda_2 - \lambda_3)^{l_2})$. So we have $[\lambda/\mu] = [\lambda_3] \otimes [(\lambda_1 - \lambda_3)^{l_1}, (\lambda_2 - \lambda_3)^{l_2}] = [(\lambda_1, (\lambda_1 - \lambda_3)^{l_1}, (\lambda_2 - \lambda_3)^{l_2})/(\lambda_1 - \lambda_3)]$ and so this is the above case $\mu = (\mu_1)$.

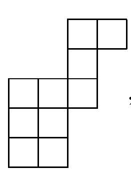
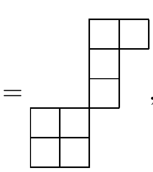
Suppose now $i(\lambda/\mu)_1 = 0$. If $o(\lambda/\mu)_1 \geq 5$ then it would follow $i(\lambda/\mu)_1 \geq 1$. So we need to check the cases that λ/μ has 4 o-1-steps of type 4, $3+1$ or $2+1+1$ and the cases $o(\lambda/\mu)_1 < 4$. There are no basic skew diagram with o-1-steps of type $3+1$ or $2+1+1$ without i-1-steps.

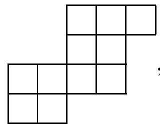
If the o-1-steps are of type 4 then λ/μ is larger than one of the following skew diagrams: $(4,3,2,2)/(2,2) =$ , $(4,4,3,2)/(2,2) =$  or $(4,4,2,1)/(2,2) =$  and all have cc-type $(6,6)$. So $cc(\lambda/\mu)$ is also at least $(6,6)$.

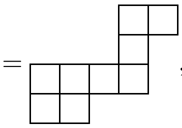
Suppose now $o(\lambda/\mu)_1 \leq 3$. For each skew diagram $\mathcal{A}$ with $o(\mathcal{A})_1 < 3$ there is a skew diagram $\mathcal{B}$ with $o(\mathcal{B})_1 = 3$ which is smaller than $\mathcal{A}$ so we need to check only the skew diagrams with 3 o-1-steps. Furthermore, each skew diagrams with 3 o-1-steps of type 3 is larger than one of the skew diagrams with o-1-steps of type 4. So we need only check the skew diagrams with o-1-steps of type $2+1$ or $1+1+1$. One easily checks that there are no basic skew diagrams without i-1-steps and o-1-steps of type $1+1+1$.

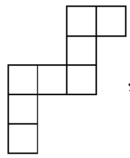
So suppose λ/μ has o-1-steps of type $2+1$.

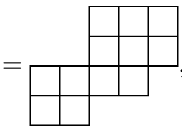
The smallest basic skew diagrams with o-1-steps of type $2+1$ without i-1-steps are up to conjugation the following:

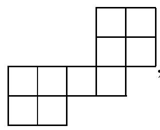
$(4,3,3,2,2)/(2,2) =$ , $(4,3,3,2,2)/(2,2,2) =$ ,

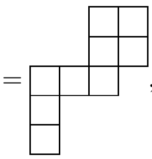
$(5,4,4,2)/(2,2) =$  ,

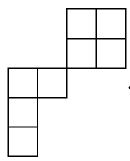
$(5,4,4,2)/(3,3) =$  ,

$(4,3,3,1,1)/(2,2) =$  ,

$(5,5,4,2)/(2,2) =$  ,

$(5,5,4,2)/(3,3) =$  ,

$(4,4,3,1,1)/(2,2) =$  ,

$(4,4,2,1,1)/(2,2) =$  .

All of them are of cc-type $(7,7)$ except $(5,4,4,2)/(3,3)$ which has cc-type $(6,6)$. $\square$

LEMMA 5.3.7. *Let $dp(\lambda) = 3, dp(\mu) = 2$ and λ/μ be a basic skew diagram. Then $[\lambda/\mu]$ contains* 5 *or less components or constituents only in the cases of Theorem 5.3.1.*

PROOF. Let us first check that the cases of Theorem 5.3.1 have the given cc-type.

We have $cc((1) \otimes (1) \otimes (1)) = (3,4)$, $cc((1) \otimes (2) \otimes (2)) = (5,7)$.

Let us now check that skew diagrams with 5 i-1-steps and 5 o-1-steps of type $4+1$ or $3+2$ are of cc-type $(4,5)$.

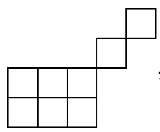
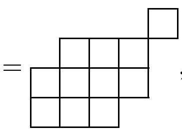
Suppose the o-1-steps are of type $4+1$. By conjugation we may assume that $l_1 = l_2 = l_3 = \lambda_3 = \lambda_2 - \lambda_3 = 1$. This forces $\lambda/\mu = (1) \otimes (1) \otimes (n)$ with $n \geq 2$ and we have $cc(\lambda/\mu) = (4,5)$ by the LR rule.

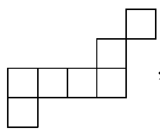
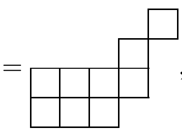
Suppose now that the o-1-steps are of type 3+2. By conjugation we may assume that $l_1 = l_2 = l_3 = \lambda_3 = \lambda_1 - \lambda_2 = 1$. This forces $\lambda/\mu = (1) \otimes (n) \otimes (1)$ with $n \geq 2$ and $cc(\lambda/\mu) = (4,5)$ or $\lambda/\mu = (1) \otimes ((n, n-1)/(1))$ with $n \geq 3$. In the second case we may by Lemma 5.2.2 assume that $n = 3$ and we have $cc((1) \otimes ((3,2)/(1))) = (4,5)$.

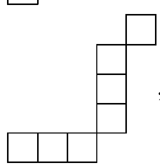
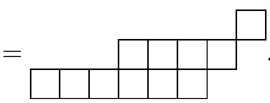
Let us now check that skew diagrams λ/μ with 5 i-1-steps and 5 o-1-steps of type 5 have 5 components and 6 constituents unless the rotated skew diagram λ/μ° is in one of the above cases. We may assume by conjugation that $l_1 = l_2 = l_3 = \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$. Since the i-1-steps of λ/μ are not of type 4+1 or 3+2 we have $\mu = (2,1)$. We may, therefore, assume by Lemma 5.2.2 that $\lambda/\mu = (4,3,2)/(2,1)$ and have $cc((4,3,2)/(2,1)) = (5,6)$.

Let us now check the decaying skew diagrams $\lambda/\mu = (1) \otimes \mathcal{A}$ such that $\mathcal{A} = (\alpha_1^{a_1}, \alpha_2^{a_2})/(\beta_1^{b_1})$ is a basic skew diagram which contains 2 i-1-steps and o-1-steps of type 2 or 3. It is seen easily that there are up to rotation and conjugation only 6 possibilities for $\mathcal{A}$:

$$\lambda/\mu^1 = (1) \otimes (1) \otimes (n^m), \qquad \lambda/\mu^2 = (1) \otimes (n^m, n-1)/(1),$$
$$\lambda/\mu^3 = (1) \otimes (n^m, 1)/(n-1), \qquad \lambda/\mu^4 = (1) \otimes (n^m, n-1)/(n-1),$$
$$\lambda/\mu^5 = (1) \otimes (n) \otimes (1^m), \qquad \lambda/\mu^6 = (1) \otimes (n, n-1)/(m),$$

$\lambda/\mu^1 =$  , $\lambda/\mu^2 =$  ,

$\lambda/\mu^3 =$  , $\lambda/\mu^4 =$  ,

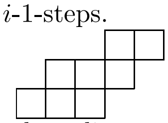
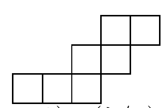
$\lambda/\mu^5 =$  , $\lambda/\mu^6 =$  .

To check that all are of cc-type $(5, 6)$ we may assume by Lemma 5.2.2 that $m = 2$ and then increase n until the the cc-type doesn't change anymore. Lemma 4.2.2 tells us that the cc-type has to become constant and then doesn't change anymore.

We will now prove that the mentioned cases are the only ones with 5 or less components and constituents. We already proved that in the cases when $[\lambda/\mu]$ has 5 components that then there are 6 constituents. By rotation symmetry we may assume that $i(\lambda/\mu)_1 \leq o(\lambda/\mu)_1$.

The cases of Theorem 5.3.1 cover all cases when $o(\lambda/\mu)_1 = 6$ or $i(\lambda/\mu)_1 = o(\lambda/\mu)_1 = 5$.

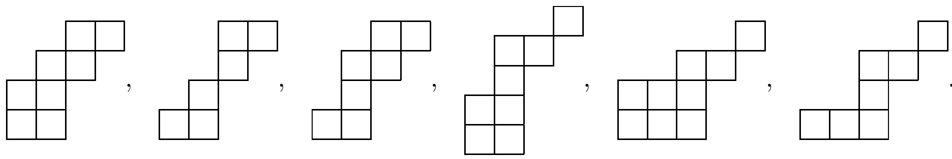
So suppose $o(\lambda/\mu)_1 = 5 > i(\lambda/\mu)_1$. By conjugation symmetry we may assume that $l_1 = l_2 = l_3 = 1$ so there are 3 possibilities for λ: $\lambda = (n, n-1, n-2)$, $\lambda = (n, n-1, 1)$ or $\lambda = (n, 2, 1)$ while in the last case there are no basic skew diagrams with less than 5 i-1-steps.

Both $(5, 4, 3)/(3, 1) =$  and $(5, 4, 3)/(3, 2) =$  have cc-type $(6, 7)$ and all basic skew diagrams with $\lambda = (n, n-1, n-2)$, $i(\lambda/\mu)_1 < 5$, λ/μ not decaying into $(1) \otimes \mathcal{A}$ with $\mathcal{A}$ having 2 i-1-steps and o-1-steps of type 2 or 3, are larger than one of these.

In the case $\lambda = (n, n-1, 1)$ λ/μ decays into $(1) \otimes \mathcal{A}$ with $\mathcal{A}$ having 2 i-1-steps and o-1-steps of type 3 which is of cc-type $(5, 6)$ as checked above.

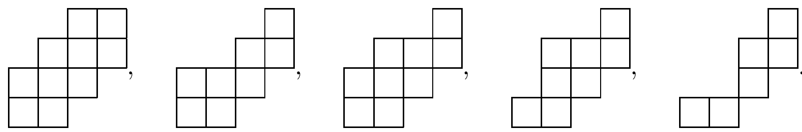
Suppose now $o(\lambda/\mu)_1 = 4 \geq i(\lambda/\mu)_1$ and λ/μ does not decay into $(1) \otimes \mathcal{A}$ with $\mathcal{A}$ having 2 i-1-steps and o-1-steps of type 2. Then up to conjugation λ is one of the following: $(\lambda_1, \lambda_1 - 1, (\lambda_1 - 2)^{l_3})$, $(\lambda_1^{l_1}, \lambda_1 - 1, \lambda_1 - 2)$, $(\lambda_1, \lambda_1 - 1, \lambda_3)$, $(\lambda_1, \lambda_1 - 1, 1^{l_3})$, $(\lambda_1, \lambda_2, \lambda_2 - 1)$, $(\lambda_1, (\lambda_1 - 1)^{l_2}, \lambda_1 - 2)$, $(\lambda_1, (\lambda_1 - 1)^{l_2}, 1)$, $(\lambda_1, \lambda_2, 1)$ (the case $\lambda = (\lambda_1, 2, 1^{l_3})$ is not possible, because this forces $\lambda/\mu = (1) \otimes (n) \otimes (1^m)$ which is one of the cases above with cc-type $(5, 6)$). The λ_i, l_i are to be chosen such that $o(\lambda/\mu)_1 = 4$, so in the first case $\lambda_1 - 2, l_3 \geq 2$.

If $\lambda = (\lambda_1, \lambda_1 - 1, (\lambda_1 - 2)^{l_3})$ then all skew diagrams (which are possible under the above conditions) are larger than one of



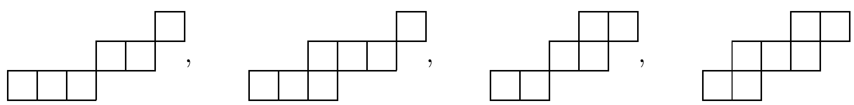
The first two skew diagrams have cc-type $(7, 8)$ while the other 4 have cc-type $(6, 7)$.

If $\lambda = (\lambda_1^{l_1}, \lambda_1 - 1, \lambda_1 - 2)$ then all skew diagrams are larger than one of the following or their conjugate

(5.3.1) 

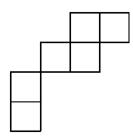
The first two skew diagrams have cc-type $(7, 8)$ while the other 3 have cc-type $(6, 7)$.

If $\lambda = (\lambda_1, \lambda_1 - 1, \lambda_3)$ then all skew diagrams are larger than one of



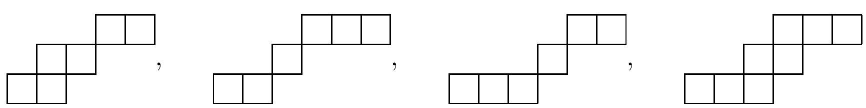
which are all of cc-type $(6, 8)$.

If $\lambda = (\lambda_1, \lambda_1 - 1, 1^{l_3})$ then $\mu = (m, 1)$ and all skew diagrams are larger than



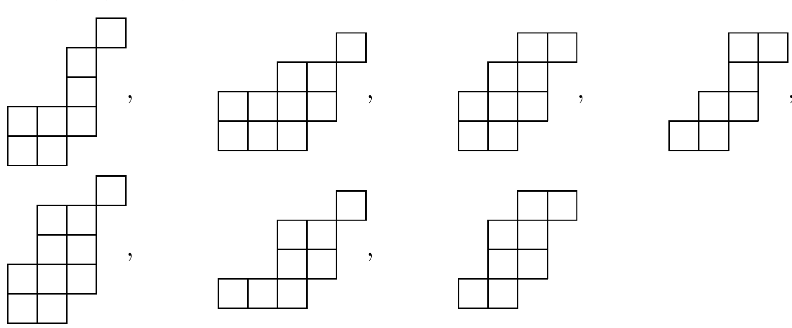
which has cc-type $(6, 7)$.

If $\lambda = (\lambda_1, \lambda_2, \lambda_2 - 1)$ then all skew diagrams are larger than one of



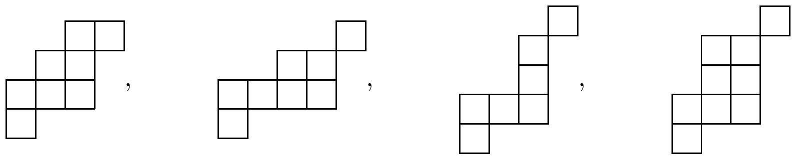
which are all of cc-type $(6, 8)$ except the last which is of cc-type $(8, 10)$ (the last one is also larger than $(5, 4, 3)/(3, 2)$ having 5 o-1-steps and cc-type $(6, 7)$ as checked above).

If $\lambda = (\lambda_1, (\lambda_1 - 1)^{l_2}, \lambda_1 - 2)$ then all skew diagrams are larger than one of



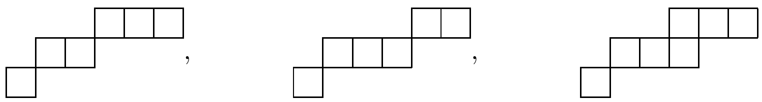
which are all of cc-type $(6, 7)$.

If $\lambda = (\lambda_1, (\lambda_1 - 1)^{l_2}, 1)$ then up to conjugation all skew diagrams are larger than one of



which are all of cc-type $(6, 7)$.

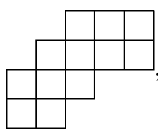
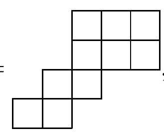
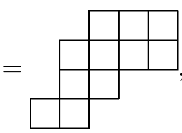
If $\lambda = (\lambda_1, \lambda_2, 1)$ then $\mu = (m, 1)$ and all skew diagrams are larger than one of

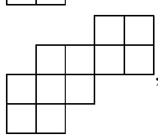
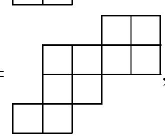
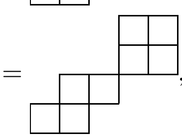


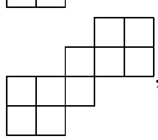
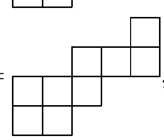
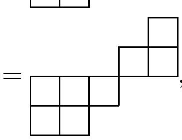
which are all of cc-type $(6, 8)$.

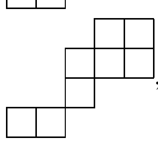
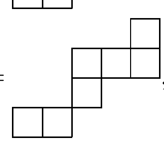
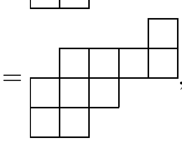
This finishes the case $o(\lambda/\mu)_1 = 4 \geq i(\lambda/\mu)_1$.

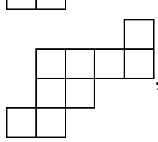
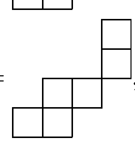
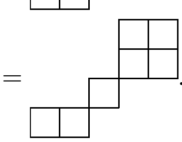
We will now check the skew diagrams λ/μ with $i(\lambda/\mu)_1 \leq o(\lambda/\mu)_1 \leq 3$. For this we again will only check a few skew diagrams such that all other with $o(\lambda/\mu)_1 \leq 3$ are larger than the checked ones. Usually those checked ones will be larger than one of the skew diagrams with $o(\lambda/\mu)_1 = 4$ and cc-type at least $(6, 7)$. For example, if we assume that $o(\lambda/\mu)_1 = 3 \geq i(\lambda/\mu)_1$ with $l_2 = l_3 = \lambda_2 - \lambda_3 = 1$, then λ/μ is larger than one of the following skew diagrams:

$A^1 =$ , $A^2 =$ , $A^3 =$ ,

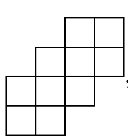
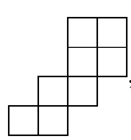
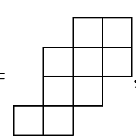
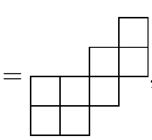
$A^4 =$ , $A^5 =$ , $A^6 =$ ,

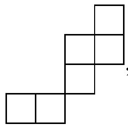
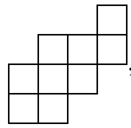
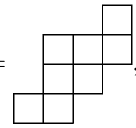
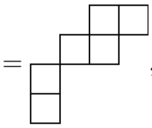
$A^7 =$ , $A^8 =$ , $A^9 =$ ,

$A^{10} =$ , $A^{11} =$ , $A^{12} =$ ,

$A^{13} =$ , $A^{14} =$ , $A^{15} =$ .

Let

$B^1 =$ , $B^2 =$ , $B^3 =$ , $B^4 =$ ,

$B^5 =$ , $B^6 =$ , $B^7 =$ , $B^8 =$ ,

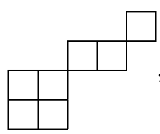
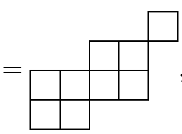
where the B^i are the skew diagrams from equation (5.3.1) or their conjugate and so have all cc-type at least $(6, 7)$.

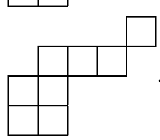
Now:

$$\begin{aligned}
A^1 &= B^1 + (1^2), & A^2 &= B^2 + (1^2), & A^3 &= B^3 + (1^2),\\
A^4 &= B^1 + (1^2)/(1), & A^5 &= B^3 + (1^2)/(1), & A^6 &= B^2 + (1^2)/(1^2),\\
A^7 &= B^4 + (1^2), & A^8 &= B^4 + (1^2)/(1), & A^9 &= B^4 + (1^2)/(1^2),\\
A^{10} &= B^5 + (1^2), & A^{11} &= B^5 + (1^2)/(1), & A^{12} &= B^6 + (1^2)/(1),\\
A^{13} &= B^7 + (1^2)/(1).
\end{aligned}$$

Furthermore, A^{14} is just a reordering of B^8, $A^{14} = (1^2) \otimes (3, 2)/(1) = B^8$. A^{15} is not larger than any of the skew diagrams λ/μ listed above with $o(\lambda/\mu)_1 = 4$ but has cc-type $(7, 9)$. So all skew diagrams λ/μ with $o(\lambda/\mu)_1 = 3 \geq i(\lambda/\mu)_1$ and $l_2 = l_3 = \lambda_2 - \lambda_3 = 1$ have at least cc-type $(6, 7)$.

Checking also the other cases with $o(\lambda/\mu)_1 = 3 \geq i(\lambda/\mu)_1$ we obtain the result, that all those skew diagrams λ/μ are larger than one of the above skew diagrams having 4 o-1-steps and cc-type at least $(6, 7)$ (or their conjugate, rotation or rearrangement) or are larger than one of the following three skew diagrams:

$(1) \otimes (2) \otimes (2^2) =$ , $(1) \otimes ((4^2, 2)/(2)) =$ ,

$(1) \otimes ((4, 2^2)/(1)) =$ .

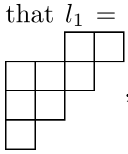
The first two skew diagrams have cc-type $(7, 9)$ and the last one has cc-type $(6, 7)$. Now all skew diagrams λ/μ with $o(\lambda/\mu)_1 < 3$ are larger than one of the skew diagrams having 3 o-1-steps.

So all skew diagrams with $o(\lambda/\mu)_1, i(\lambda/\mu)_1 < 4$ have cc-type at least $(6, 7)$. This finishes the proof. □

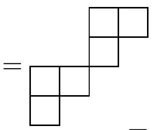
Lemma 5.3.8. *Let $dp(\lambda) = 4, dp(\mu) = 1$ and λ/μ be a basic skew diagram. Then $[\lambda/\mu]$ contains* 5 *or less components or constituents if and only if $i(\lambda/\mu)_1 = 2$.*

Proof. It was argued in Lemma 5.3.6 that $cc(\lambda/\mu) = (4, 4)$ for $i(\lambda/\mu)_1 = 2$.

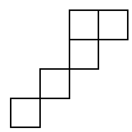
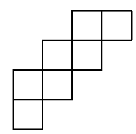
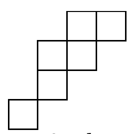
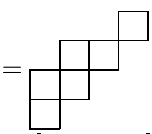
So now suppose $i(\lambda/\mu)_1 = 1$, then we may by conjugation symmetry assume $ih(\lambda/\mu)_1 = 1$ and we have to consider the two cases $\mu = (\mu_1)$ and $\mu = (\mu_1^{l-1})$. Suppose first that $\mu = (\mu_1)$ with $\mu_1 \neq 1, \lambda_1 - 1$. Then we may by Lemma 5.2.2 assume that $l_1 = l_2 = l_3 = l_4 = 1$ and so obviously λ/μ is larger than $(4, 3, 2, 1)/(2) =$

, which has cc-type $(6, 6)$, so also λ/μ has cc-type at least $(6, 6)$.

In the case $\mu = (\mu_1^{l-1})$ we may by Lemma 5.2.2 assume that $\lambda_4 \leq \mu_1$ and since λ/μ is basic we have $\lambda_4 = \mu_1, l_4 = 1$. So λ/μ decays $\lambda/\mu = (\mu_1) \otimes ((\lambda_1 - \mu_1)^{l_1}, (\lambda_2 - \mu_1)^{l_2}, (\lambda_3 - \mu_1)^{l_3})$ and by reordering the parts we may also write $\lambda/\mu = (\lambda_1, (\lambda_1 - \mu_1)^{l_1}, (\lambda_2 - \mu_1)^{l_2}, (\lambda_3 - \mu_1)^{l_3})/(\lambda_1 - \mu_1)$ which means that this is the above case $\mu = (\mu_1)$.

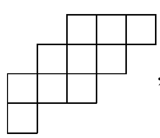
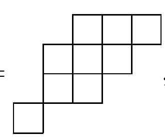
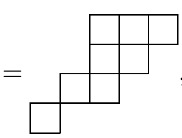
Suppose now $i(\lambda/\mu)_1 = 0$. Then λ/μ is larger than $(4, 3, 2, 1)/(2^2) =$ 
which has cc-type $(7, 8)$ and so λ/μ also has cc-type at least $(7, 8)$. □

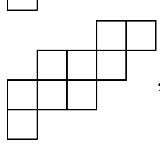
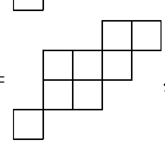
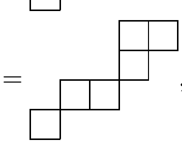
Lemma 5.3.9. *Let $dp(\lambda) = 4, dp(\mu) = 2$ and λ/μ be a basic skew diagram. Then $[\lambda/\mu]$ contains 5 or less components or constituents only in the cases of Theorem 5.3.1.*

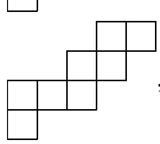
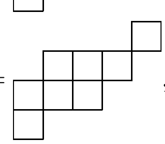
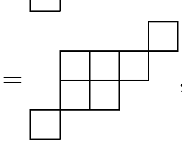
Proof. If $\lambda/\mu = (1) \otimes (1) \otimes (2, 1) =$  then $cc(\lambda/\mu) = (5, 8)$. The other cases with $o(\lambda/\mu)_1 = 8$ and so $\lambda = (4, 3, 2, 1)$ are $A^1 =$  with cc-type $(7, 11)$ and $A^2 =$  resp. $A^3 =$  with cc-type $(7, 9)$.

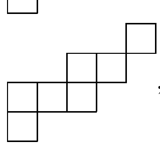
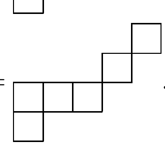
Suppose now $o(\lambda/\mu)_1 < 8$, then it is larger than one skew diagram which has 7 o-1-steps. Because of conjugation symmetry it is enough to check the skew diagrams with $ov(\lambda/\mu)_1 = 4, oh(\lambda/\mu)_1 = 3$. We will as an example check the skew diagrams with $\lambda = (\lambda_3 + 2, \lambda_3 + 1, \lambda_3, 1), \lambda_3 - 1 \geq 2$.

All skew diagrams λ/μ with $\lambda = (\lambda_3 + 2, \lambda_3 + 1, \lambda_3, 1)$ are larger than one of the following skew diagrams:

$B^1 =$ , $B^2 =$ , $B^3 =$ ,

$B^4 =$ , $B^5 =$ , $B^6 =$ ,

$B^7 =$ , $B^8 =$ , $B^9 =$ ,

$B^{10} =$ , $B^{11} =$ .

The skew diagrams B^1, B^4, B^7 are larger than A^1, the skew diagrams B^2, B^5 are larger than A^2 and the skew diagrams B^8, B^{10} are larger than A^3 and so those B^i have at least cc-type $(7, 9)$.

Let $\lambda = (5, 4, 3, 1)$ then $cc(\lambda/\mu)$ is

- $(6, 9)$ for $\mu = (4, 1^2)$ and $\mu = (4, 3)$
- $(7, 10)$ for $\mu = (2^2, 1)$
- $(8, 10)$ for $\mu = (2, 1^2)$ and $\mu = (4, 1)$
- $(8, 12)$ for $\mu = (3^2, 1)$ and $\mu = (4, 2)$
- $(9, 13)$ for $\mu = (2, 1)$ and $\mu = (3, 1^2)$
- $(10, 15)$ for $\mu = (3, 1)$
- $(10, 16)$ for $\mu = (3, 2)$

So λ/μ has cc-type at least $(6, 9)$ if $\lambda = (\lambda_3 + 2, \lambda_3 + 1, \lambda_3, 1), \lambda_3 - 1 \geq 2$.

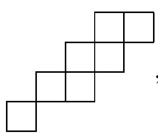
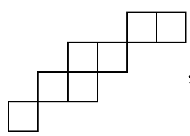
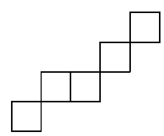
Checking also the other three cases with $o(\lambda/\mu)_1 = 7$ finishes the proof. □

Lemma 5.3.10. *Let $dp(\lambda) = 4, dp(\mu) = 3$ and λ/μ be a basic skew diagram. Then $[\lambda/\mu]$ contains 5 or less components or constituents only in the cases of Theorem 5.3.1.*

Proof. We have $cc((1) \otimes (1) \otimes (1) \otimes (1)) = (5, 10)$. This is the only possibility for $o(\lambda/\mu)_1 = 8$.

Suppose $o(\lambda/\mu)_1 < 8$, then λ/μ is larger than one skew diagram which has 7 o-1-steps. Again as in Lemma 5.3.9 because of conjugation symmetry it is enough to check the skew diagrams with $ov(\lambda/\mu)_1 = 4, oh(\lambda/\mu)_1 = 3$. We will again as an example check the skew diagrams with $\lambda = (\lambda_3 + 2, \lambda_3 + 1, \lambda_3, 1)$.

All skew diagrams λ/μ with $\lambda = (\lambda_3 + 2, \lambda_3 + 1, \lambda_3, 1)$ are larger than one of the following skew diagrams:

 ,  , 

which have cc-type $(9, 18), (8, 15)$ and $(6, 13)$ (in this order).

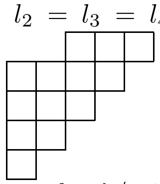
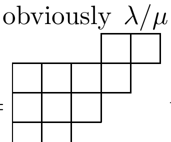
So λ/μ has cc-type at least $(6, 13)$ if $\lambda = (\lambda_3 + 2, \lambda_3 + 1, \lambda_3, 1)$.

Checking also the other cases with $o(\lambda/\mu)_1 = 7$ finishes the proof. □

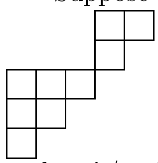
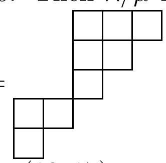
Lemma 5.3.11. *Let $dp(\lambda) = 5, dp(\mu) = 1$ and λ/μ be a basic skew diagram. Then $[\lambda/\mu]$ contains 5 or less components or constituents only in the cases of Theorem 5.3.1.*

Proof. It was argued in Lemma 5.3.6 that $cc(\lambda/\mu) = (5, 5)$ for $i(\lambda/\mu)_1 = 2$.

So suppose $i(\lambda/\mu)_1 = 1$, then we may by conjugation symmetry assume that $ih(\lambda/\mu)_1 = 1$ and we have to consider the two cases $\mu = (\mu_1)$ and $\mu = (\mu_1^{l-1})$. Suppose first that $\mu = (\mu_1)$ with $\mu_1 \neq 1, \lambda_1 - 1$. Then we may by Lemma 5.2.2 assume that $l_1 = l_2 = l_3 = l_4 = l_5 = 1$ and so obviously λ/μ is larger than

$(5, 4, 3, 2, 1)/(2) =$  or $(5, 4, 3, 2, 1)/(3) =$  which both are of cc-type $(10, 10)$, so also λ/μ is of cc-type at least $(10, 10)$.

In the case $\mu = (\mu_1^{l-1})$ we may by Lemma 5.2.2 assume that $\lambda_5 \leq \mu_1$ and since λ/μ is basic we have $\lambda_5 = \mu_1, l_5 = 1$. So λ/μ decays $\lambda/\mu = (\mu_1) \otimes ((\lambda_1 - \mu_1)^{l_1}, (\lambda_2 - \mu_1)^{l_2}, (\lambda_3 - \mu_1)^{l_3}, (\lambda_4 - \mu_1)^{l_4})$ and by reordering the parts we may also write $\lambda/\mu = (\lambda_1, (\lambda_1 - \mu_1)^{l_1}, (\lambda_2 - \mu_1)^{l_2}, (\lambda_3 - \mu_1)^{l_3}, (\lambda_4 - \mu_1)^{l_4})/(\lambda_1 - \mu_1)$ which means that this is the above case $\mu = (\mu_1)$.

Suppose now that $i(\lambda/\mu)_1 = 0$. Then λ/μ is larger than $(5,4,3,2,1)/(3^2) =$  or $(5,4,3,2,1)/(2^3) =$  which are both of cc-type $(13,17)$ and so λ/μ also has cc-type at least $(13,17)$. □

Lemma 5.3.12. *Let $dp(\lambda) = 5, dp(\mu) \geq 2$ and λ/μ be a basic skew diagram. Then $[\lambda/\mu]$ contains more than 5 components and constituents.*

Proof. All skew diagrams are larger than one of the skew diagrams λ/μ with $\lambda = (5,4,3,2,1)$ and μ contained in λ. So we only have to check those skew diagrams.

For $dp(\mu) = 2$ and $\lambda = (5,4,3,2,1)$ the cc-type of λ/μ is at least $(8,14)$ (obtained for $\mu = (2^3,1)$, it's conjugate $\mu = (4,3)$, and $\mu = (4,1^3)$; these are the cases $\lambda/\mu = (1) \otimes (1) \otimes (3,2,1))$ and at most $(17,27)$ (obtained for $\mu = (2,1^2)$ and it's conjugate $\mu = (3,1)$) respectively $(16,32)$ (obtained for $\mu = (2^2,1)$, it's conjugate $\mu = (3,2)$, and $\mu = (3,1^2)$).

For $dp(\mu) = 3$ and $\lambda = (5,4,3,2,1)$ the cc-type of λ/μ is at least $(9,24)$ (obtained for the cases $\lambda/\mu = (1) \otimes (1) \otimes (1) \otimes (2,1))$ and at most $(16,51)$ (obtained for $\mu = (3,2,1)$).

Finally for $dp(\mu) = 4$ and $\lambda = (5,4,3,2,1)$ we have $cc((1)\otimes(1)\otimes(1)\otimes(1)\otimes(1)) = (7,26)$.

So if $dp(\lambda) = 5, dp(\mu) \geq 2$ then λ/μ has at least 7 components and at least 14 constituents. □

Lemma 5.3.13. *Let $dp(\lambda) \geq 6, dp(\mu) \geq 1$ and λ/μ be a basic skew diagram. Then $[\lambda/\mu]$ contains more than 5 components and constituents.*

Proof. If $dp(\mu) \geq 2$ then λ/μ is larger than one skew diagram with $dp(\lambda) = 5, dp(\mu) \geq 2$ which have cc-type at least $(7,14)$ by Lemma 5.3.12.

If $dp(\mu) = 1$ then it is checked using Stembridge's SF-package [**Ste**] that for $\lambda = (6,5,4,3,2,1)$ λ/μ has cc-type at least $(6,6)$ (obtained for $dp(\lambda) = 6$ with $\mu = (1), \mu = (\lambda_1 - 1)$ and $\mu = (1^{l-1})$) and at most $(39,63)$ (obtained for $\mu = (3^2)$ and it's conjugate $\mu = (2^3)$). An arbitrary basic skew diagram λ/μ with $dp(\lambda) \geq 6, dp(\mu) = 1$ is larger than one of the skew diagrams λ'/μ' with $\lambda' = (6,5,4,3,2,1), dp(\mu') = 1$ and so also has cc-type at least $(6,6)$. □

This proofs Theorem 5.3.1.

CHAPTER 6

Reduced Kronecker products

Since the irreducible representations of the symmetric groups are to a certain degree well understood and easily to construct the next problem which comes to mind is understanding the Kronecker product $[\lambda][\mu]$ of two irreducible characters of the same symmetric group (for example, see work of Murnaghan of the late 1930s and 1950s: [**Mur37**], [**Mur38**],[**Mur55a**], [**Mur55b**]). One of the main open problems in representation theory is finding a good combinatorial description for the product or their coefficients like one has with the LR coefficients for the outer product of two irreducible characters. Only for some special shapes of λ, μ combinatorial interpretations of the Kronecker product $[\lambda][\mu]$ have been found (see [**GR**], [**Rem89**], [**Rem92**], [**RW**], [**Ros**], [**BOR08**]). For current research regarding a general combinatorial description of the Kronecker coefficients see [**Cas**] and [**AV**].

As mentioned already in the introduction (Section 1.2) the Kronecker product is also related to skew characters by works of Dvir ([**Dvir**]) and Clausen and Meier ([**CM**]).

6.1. Introduction to reduced Kronecker products

We will now look at a nice stabilizing property of the Kronecker product which was observed already by Murnaghan. We will demonstrate it with an example:

$$\begin{aligned}
[21^2][22] &= [31] \qquad + \ [21^2],\\
[31^2][32] &= [41] + [32] + 2[31^2] \qquad + \ [221] + [21^3],\\
[41^2][42] &= [51] + [42] + 2[41^2] + [33] + 2[321] + [31^3] \qquad + [221^2],\\
[51^2][52] &= [61] + [52] + 2[51^2] + [43] + 2[421] + [41^3] + [331] + [321^2],\\
[61^2][62] &= [71] + [62] + 2[61^2] + [53] + 2[521] + [51^3] + [431] + [421^2],\\
[*1^2][*2] &= [*1] + [*2] + 2[*1^2] + [*3] + 2[*21] + [*1^3] + [*31] + [*21^2].
\end{aligned}$$

Murnaghan showed that this product always stabilizes if one increases the first part. For a partition λ we define the sequence $\lambda[n] = (n - |\lambda|, \lambda_1, \lambda_2, \ldots)$ which is a weak composition of n for $n \geq |\lambda|$ and a partition of n for $n \geq |\lambda| + \lambda_1$.

The stabilizing property of the Kronecker product then means that for all partitions λ, μ there is an integer n such that for all $k \geq 0$ and all partitions ν we have:

$$g(\lambda[n], \mu[n], \nu[n]) = g(\lambda[n+k], \mu[n+k], \nu[n+k]). \tag{6.1.1}$$

So it makes sense to define the reduced Kronecker coefficients

$$\bar{g}(\lambda, \mu, \nu) = \lim_{n \to \infty} g(\lambda[n], \mu[n], \nu[n])$$

which is by Murnaghan's Theorem well defined.

As said before, the Jacobi-Trudi determinant

$$s_\lambda = \det(h_{\lambda_i} + i - j)_{1\le i,j\le n}$$

allows us to define the Schur function $s_{\lambda[n]}$ even for the case that $\lambda[n]$ is not a partition. Nevertheless, we have either $s_{\lambda[n]} = \pm s_{\lambda'}$ for some partition λ' or $s_{\lambda[n]} = 0$. We set $[\lambda[n]] = \pm[\lambda']$ if $s_{\lambda[n]} = \pm s_{\lambda'}$ and $[\lambda[n]] = 0$ if $s_{\lambda[n]} = 0$.

Murnaghan's Theorem is then the following (see also [**Thi**]):

Theorem 6.1.1 (Murnaghan,[**Mur37, Mur38, Mur55a**]). *For all $n \ge 0$ we have:*

$$[\lambda[n]][\mu[n]] = \sum_\nu \bar{g}(\lambda,\mu,\nu)[\nu[n]].$$

At first, this result seems to be surprising because we see that the decomposition of $[21^2][22]$ and $[*1^2][*2]$ are different. But we have by the Jacobi-Trudi determinant:

$$[13] = -[22], \qquad [121] = 0, \qquad [031] = -[211], \qquad [0211] = -[1111],$$

and so:

$$\begin{aligned}[21^2][22] &= [31] + [22] + 2[21^2] + [13] + 2[121] + [1^4] + [031] + [021^2]\\ &= [31] + [22] + 2[21^2] - [22] + 0 + [1^4] - [21^2] - [1^4]\\ &= [31] + [21^2].\end{aligned}$$

Murnaghan's Theorem inspires the following. Let $\{[\lambda]_\bullet | \lambda \text{ a partition}\}$ be a basis for a $\mathbb{C}$ vector space. We then define a product on these basis vectors which can be linearly extended to the full vector space:

$$[\lambda]_\bullet \star [\mu]_\bullet := \sum_\nu \bar{g}(\lambda,\mu,\nu)[\nu]_\bullet.$$

We call this the reduced Kronecker product. The connection to the usual Kronecker product should be obvious by Theorem 6.1.1. In fact, one could set $[\lambda]_\bullet$ formally as $[\lambda]_\bullet = \sum_{n\in\mathbb{N}}[\lambda[n]]$ with $\star$ the usual Kronecker product (this works because $[\lambda[n]][\mu[m]] = 0$ for $n \ne m$). We will call these $[\lambda]_\bullet$ irreducible characters even if they are not traces of representations.

In this notation our example from above simply reads:

$$[1^2]_\bullet \star [2]_\bullet = [1]_\bullet + [2]_\bullet + 2[1^2]_\bullet + [3]_\bullet + 2[21]_\bullet + [1^3]_\bullet + [31]_\bullet + [21^2]_\bullet.$$

This particular product can already be found in [**Mur38**, Number 20].

The reduced Kronecker coefficients has some nice properties. If $|\lambda| = |\mu| + |\nu|$ then $\bar{g}(\lambda,\mu,\nu) = c(\lambda;\mu,\nu)$. In [**BOR09**] Briand et al. showed the following:

Theorem 6.1.2 ([**BOR09**, Theorem 1.2]). *For arbitrary partitions λ,μ the smallest n which can be chosen for equation* (6.1.1) *is $n = |\lambda| + |\mu| + \lambda_1 + \mu_1$.*

In other words, the coefficients in $[\lambda[n]][\mu[n]]$ stabilize for $n = |\lambda| + |\mu| + \lambda_1 + \mu_1$.

So we can calculate $[\lambda]_\bullet \star [\mu]_\bullet$ with Stembridge's Maple Package [**Ste**] by calculating $[\lambda[n]][\mu[n]]$ for $n = |\lambda| + |\mu| + \lambda_1 + \mu_1$.

Furthermore, it is conjectured by Klyachko [**Kly**, Conjecture 6.2.4] and Kirillov [**Kir**, Conjecture 2.33] that the reduced Kronecker coefficients satisfy, like the LR coefficients, the Saturation property. So it is conjectured that if $\bar{g}(n\lambda, n\mu, n\mu) \ne 0$ for some $n \ge 1$ then $\bar{g}(\lambda,\mu,\mu) \ne 0$. Obviously this property holds if $|\lambda| = |\mu| + |\nu|$.

There are many examples which show that the ordinary Kronecker coefficients do not satisfy the saturation conjecture. For example, we have:

$$g((n,n),(n,n),(n,n)) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} .$$

We will use the following lemma in our later proofs. It can be found in [**Mur55b**, page 1098] and [**Thi**, page 217]:

Lemma 6.1.3 ([**Mur55b**, page 1098]).

$$[1]_\bullet \star [\lambda]_\bullet = dp(\lambda)[\lambda]_\bullet + \sum_\mu [\mu]_\bullet$$

where the sum is over all partitions μ different from λ which can be obtained from λ by adding a box, deleting a box or first deleting and then adding a box.

In particular, $[\lambda]_\bullet$ has multiplicity $dp(\lambda)$ and all other characters have multiplicity 0 or 1.

6.2. Preliminaries: $\bar{g}(\lambda+\lambda',\mu+\mu',\nu+\nu')\geq\bar{g}(\lambda,\mu,\nu)$

In the next sections we classify the reduced Kronecker products which are multiplicity free and those which contain less than 10 components. We also give lower bounds for the number of constituents and components of a given reduced Kronecker product and for those pairs of components, whose corresponding partitions differ by only one box. We also show that equality of reduced Kronecker products is possible only if the factors are the same.

The results of this chapter will appear in [**Gut6**].

Christandl et al. proved in [**CHM**, Theorem 3.1] that $g(\lambda+\lambda',\mu+\mu',\nu+\nu')\neq 0$ for $g(\lambda,\mu,\nu), g(\lambda',\mu',\nu')\neq 0$.

In [**Man**] Manivel extends this result and states the following lemma in the proof of his Theorem 1.

Lemma 6.2.1 ([**Man**]). *Let both $g(\lambda,\mu,\nu), g(\lambda',\mu',\nu')\neq 0$.*
Then

$$g(\lambda+\lambda',\mu+\mu',\nu+\nu')\geq g(\lambda,\mu,\nu)$$

and by conjugation:

$$g(\lambda\cup\lambda',\mu+\mu',\nu\cup\nu')\geq g(\lambda,\mu,\nu).$$

We will now prove that the reduced Kronecker coefficients also obey the same property:

Lemma 6.2.2. *Let $m\in\mathbb{N}$ such that $g(\lambda'[m],\mu'[m],\nu'[m])\neq 0$.*
Then

$$\bar{g}(\lambda+\lambda',\mu+\mu',\nu+\nu')\geq\bar{g}(\lambda,\mu,\nu)$$

and

$$\bar{g}(\lambda\cup(\lambda'[m]),\mu+\mu',\nu\cup(\nu'[m]))\geq\bar{g}(\lambda,\mu,\nu).$$

Proof. We have

$$\begin{aligned}\bar{g}(\lambda+\lambda',\mu+\mu',\nu+\nu') &= g((\lambda+\lambda')[n+m],(\mu+\mu')[n+m],(\nu+\nu')[n+m]) \\ &= g(\lambda[n]+\lambda'[m],\mu[n]+\mu'[m],\nu[n]+\nu'[m]) \\ &\geq g(\lambda[n],\mu[n],\nu[n]) = \bar{g}(\lambda,\mu,\nu)\end{aligned}$$

for n large enough and by Lemma 6.2.1.

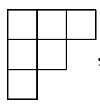
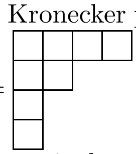
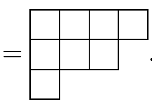
For n large enough we also get by using Lemma 6.2.1:

$$\begin{aligned}\bar{g}(\lambda \cup (\lambda'[m]), \mu + \mu', \nu \cup (\nu'[m])) &= g(\lambda[n] \cup \lambda'[m], \mu[n] + \mu'[m], \nu[n] \cup \nu'[m]) \\ &\geq g(\lambda[n], \mu[n], \nu[n]) = \bar{g}(\lambda, \mu, \nu).\end{aligned}$$

□

Remark 6.2.3. Let $\lambda' \vdash n$, then obviously we have $\bar{g}(\lambda', \emptyset, \lambda') = 1, g(\lambda', (n), \lambda') = 1$. So we have $\bar{g}(\lambda+\lambda', \mu, \nu+\lambda') \geq \bar{g}(\lambda, \mu, \nu)$ and also $\bar{g}(\lambda\cup\lambda', \mu, \nu\cup\lambda') \geq \bar{g}(\lambda, \mu, \nu)$.

This property allows us to get informations about the product $[\lambda]_\bullet \star [\mu]_\bullet$ if we know the product $[\lambda']_\bullet \star [\mu']_\bullet$ and λ is larger than λ' and μ is larger than μ'.

This is not the case for the ordinary Kronecker product! For example, let $\lambda' = \mu' = (3, 2, 1) =$ , $\lambda = (4, 2, 1, 1) =$  and $\mu = (4, 3, 1) =$ .

So we have $\lambda = \lambda' + (1) \cup (1)$ (in this particular case $+$ and $\cup$ commute which is not true in general) and $\mu = \mu' + (1^2)$. So we know by applying Lemma 6.2.2 three times that if $[\nu]_\bullet$ appears in $[\lambda']_\bullet \star [\mu']_\bullet$ with multiplicity c then $[(\nu+(1^2)+(1))\cup(1)]_\bullet$ appears in $[\lambda]_\bullet \star [\mu]_\bullet$ with multiplicity at least c:

$$\begin{aligned}\bar{g}(\lambda', \mu', \nu) &\leq \bar{g}(\lambda' + (1), \mu', \nu + (1)) \\ &\leq \bar{g}(\lambda' + (1), \underbrace{\mu' + (1^2)}_{\mu}, \nu + \underbrace{(1) + (1^2)}_{(2,1)}) \\ &\leq \bar{g}(\underbrace{\lambda' + (1) \cup (1)}_{\lambda}, \mu, (\nu + (2, 1)) \cup (1)).\end{aligned}$$

On the other hand, the only way to get μ from μ' is by adding $+(1^2)$ and this has to be done directly, not in two steps. So in our case with $\mu = (4, 3, 1) = \mu' + (1^2)$ we can deduce something about $[\lambda][\mu]$ from $[\lambda'][\mu']$ using Lemma 6.2.1 only if $\lambda \in \{\lambda' + (2), \lambda' \cup (2), \lambda' + (1^2), \lambda' \cup (1^2)\}$ which is not the case.

Remark 6.2.4. Again as in Remark 5.2.1 we have the following: We have $\lambda + \nu \neq \mu + \nu$ and $\lambda \cup \nu \neq \mu \cup \nu$ for $\lambda \neq \mu$ and arbitrary ν. So if $[\lambda']_\bullet \star [\mu']_\bullet$ has cc-type (a', b') and λ is larger than λ' and μ is larger than μ'. If $[\lambda]_\bullet \star [\mu]_\bullet$ has cc-type (a, b) then it is $a \geq a'$ and $b \geq b'$.

6.3. Multiplicity free reduced Kronecker products

Using Lemma 6.2.2 and Remark 6.2.4 it is now easy to classify the multiplicity free reduced Kronecker products.

THEOREM 6.3.1. *Let λ, μ be partitions. Then $[\lambda]_\bullet \star [\mu]_\bullet$ is multiplicity free if and only if up to exchanging λ and μ we are in one of the following two cases:*

(1) $\lambda = \emptyset$, $\quad\mu$ *arbitrary*
(2) $\lambda = (1)$, $\quad\mu = (\alpha^a)$ *is a rectangle*

PROOF. Obviously for $\lambda = \emptyset$ we have the trivial product $[\emptyset]_\bullet \star [\mu]_\bullet = [\mu]_\bullet$ which is multiplicity free.

By Lemma 6.1.3 $[1]_\bullet \star [\mu]_\bullet$ is multiplicity free if and only if $dp(\mu) \leq 1$, which is the case only for μ a rectangle.

So now suppose that neither λ nor μ is (1). Then λ and μ are larger than (2) or (1^2). So by Lemma 6.2.2 and Remark 6.2.3 it is enough to check if the following products have multiplicity: $[2]_\bullet \star [2]_\bullet$, $[2]_\bullet \star [1^2]_\bullet$ and $[1^2]_\bullet \star [1^2]_\bullet$. We already know that $\bar{g}(1^2, 2, 1^2) = 2$ from our example and so both $[2]_\bullet \star [1^2]_\bullet$ and $[1^2]_\bullet \star [1^2]_\bullet$ have multiplicity. Using Stembridge's Maple package to calculate $[2[n]][2[n]]$ for $n = 8$ gives $\bar{g}(2, 2, 2) = 2$ which proves that also $[2]_\bullet \star [2]_\bullet$ has multiplicity. $\square$

6.4. On the components and equality of reduced Kronecker products

In this section we will at first classify those reduced Kronecker products which contain less than 10 components and then give lower bounds for the number of components, constituents and pairs of components, whose corresponding partitions differ by one box. Finally we give a short argument that two reduced Kronecker products can be equal only if their factors are the same.

THEOREM 6.4.1. *Let λ, μ be partitions. Then $[\lambda]_\bullet \star [\mu]_\bullet$ has less than 10 components if and only if up to exchanging λ and μ we are in one of the following cases:*

(1) *$\lambda = \emptyset$, μ arbitrary (cc-type $(1, 1)$)*
(2) *$\lambda = (1)$, $\mu = (\alpha^a)$ is rectangle. In this case we have*
   (a) *cc-type $(4, 4)$ if $\mu = (1)$*
   (b) *cc-type $(5, 5)$ if either $\alpha = 1$ or $a = 1$*
   (c) *cc-type $(6, 6)$ if $\alpha, a \geq 2$*
(3) *$\lambda = (1)$, $\mu = (\alpha^a, \beta^b)$ is a fat hook and we have, furthermore,*
   (a) *$\mu = (2, 1)$ (cc-type $(8, 9)$)*
   (b) *3 of the values $\alpha - \beta, \beta, a, b$ are equal to 1 (cc-type $(9, 10)$)*
(4) *$\lambda = (1^2)$, $\mu = (2)$ (cc-type $(8, 10)$)*

PROOF. We have already seen that $[1^2]_\bullet \star [2]_\bullet$ has the given cc-type.

To check that the other cases are true one simply uses the known formula for $[1]_\bullet \star [\lambda]_\bullet$.

For $\lambda = (1), \mu = (\alpha^a)$ we have by Lemma 6.1.3:

$$\begin{aligned}[1]_\bullet \star [\alpha^a]_\bullet =& [\alpha^a]_\bullet + [\alpha^{a-1}]_\bullet + [\alpha^a, 1]_\bullet + [\alpha^{a-1}, \alpha - 1]_\bullet \\ &+ [\alpha + 1, \alpha^{a-2}, \alpha - 1]_\bullet + [\alpha^{a-1}, \alpha - 1, 1]_\bullet\end{aligned}$$

where the last component appears only for $\alpha \geq 2$ and the penultimate only for $a \geq 2$.

For $\lambda = (1), \mu = (\alpha^a, \beta^b)$ we have by Lemma 6.1.3:

$$\begin{aligned}[1]_\bullet \star [\alpha^a, \beta^b]_\bullet =& 2[\alpha^a, \beta^b]_\bullet + [\alpha + 1, \alpha^{a-1}, \beta^b]_\bullet + [\alpha^a, \beta + 1, \beta^{b-1}]_\bullet + [\alpha^a, \beta^b, 1]_\bullet \\ &+ [\alpha + 1, \alpha^{a-2}, \alpha - 1, \beta^b]_\bullet + [\alpha^{a-1}, \alpha - 1, \beta + 1, \beta^{b-1}]_\bullet \\ &+ [\alpha^{a-1}, \alpha - 1, \beta^b, 1]_\bullet + [\alpha + 1, \alpha^{a-1}, \beta^{b-1}, \beta - 1]_\bullet \\ &+ [\alpha^a, \beta + 1, \beta^{b-2}, \beta - 1]_\bullet + [\alpha^a, \beta^{b-1}, \beta - 1, 1]_\bullet \\ &+ [\alpha^{a-1}, \alpha - 1, \beta^b]_\bullet + [\alpha^a, \beta^{b-1}, \beta - 1]_\bullet\end{aligned}$$

where the 5th characters appears only for $a \geq 2$, the 6th only for $\alpha - \beta \geq 2$, the 9th only for $b \geq 2$ and the 10th only for $\beta \geq 2$.

We will check now that all other products have at least 10 components. From the formula above we can see that if $\lambda = (1)$ and $\mu = (\alpha^a, \beta^b)$ and none of the additional conditions given in the theorem is satisfied we have at least 10 components.

Suppose now that $\lambda = (1)$ and $dp(\mu) \geq 3$. Then μ is larger than or equal to $(3,2,1)$. Since $[1]_\bullet \star [321]_\bullet$ has cc-type $(14,16)$ we know by Lemma 6.2.2 and Remark 6.2.4 that $[1]_\bullet \star [\mu]_\bullet$ has at least 14 components.

So suppose now that both λ, μ are different from (1) and we are not in the situation $[1^2]_\bullet \star [2]_\bullet$.

If both $\lambda_1, \mu_1 \geq 2$ then both λ, μ are larger than or equal to (2). It is

$$[2]_\bullet \star [2]_\bullet = [\emptyset]_\bullet + [1]_\bullet + 2[2]_\bullet + [1^2]_\bullet + [3]_\bullet + 2[21]_\bullet + [1^3]_\bullet + [4]_\bullet + [31]_\bullet + [2^2]_\bullet$$

and so $[2]_\bullet \star [2]_\bullet$ has cc-type $(10,12)$ (this product can already be found in [**Mur38**, Number 19]). It follows by Remark 6.2.4 that $[\lambda]_\bullet \star [\mu]_\bullet$ also has at least 10 components.

If both $l(\lambda), l(\mu) \geq 2$ then both λ, μ are larger than or equal to (1^2). It is

$$\begin{aligned}[1^2]_\bullet \star [1^2]_\bullet = &[\emptyset]_\bullet + [1]_\bullet + 2[2]_\bullet + [1^2]_\bullet + [3]_\bullet \\ &+ 2[21]_\bullet + [1^3]_\bullet + [2^2]_\bullet + [21^2]_\bullet + [1^4]_\bullet\end{aligned}$$

and so $[1^2]_\bullet \star [1^2]_\bullet$ also has cc-type $(10,12)$ (this product can already be found in [**Mur38**, Number 29]). It follows by Remark 6.2.4 that $[\lambda]_\bullet \star [\mu]_\bullet$ also has at least 10 components.

So we may now suppose that $\lambda = (\lambda_1)$ and $\mu = (1^m)$ with $\lambda_1, m \geq 2$ and at least one of $\lambda_1, m \geq 3$. Using Stembridge's Maple package [**Ste**] we check that $[2]_\bullet \star [1^3]_\bullet$ has cc-type $(10,13)$ and $[3]_\bullet \star [1^2]_\bullet$ has cc-type $(11,13)$ (these products can also be found in [**Mur38**, Numbers 23 and 30]). It follows by Remark 6.2.4 that $[\lambda]_\bullet \star [\mu]_\bullet$ also has at least 10 components. □

In the reduced Kronecker products appear characters whose corresponding partitions are of different size. So in the following if we say that two partitions λ, μ differ by one box we mean that $|\lambda \cap \mu| = \max(|\lambda|, |\mu|) - 1$. So μ can be obtained from λ by deleting a box, adding a box or first deleting and then adding a box.

As in Chapter 5 we let p_n denote the number of partitions of n, f_n denote the number of standard Young tableaux of size n and g_n denote the number of pairs of partitions of n which differ by one box. Their first terms were listed in Remark 5.2.10.

THEOREM 6.4.2. *Let λ, μ be partitions with $dp(\lambda) = n \geq dp(\mu) = m \geq 1$. Then $[\lambda]_\bullet \star [\mu]_\bullet$ contains at least*

- $n^2 + 1 + \max(p_{m+1}, n+1)$ *components,*
- $n^2 + n + \max(f_{m+1}, n+1)$ *constituents,*
- $n^3 + n + 1 + \max\left(g_{m+1}, \frac{1}{2}(n+1)n\right)$ *pairs of components $([\nu^1]_\bullet, [\nu^2]_\bullet)$ such that their corresponding partitions ν^1, ν^2 differ by only one box.*

PROOF. We first investigate the product $[1]_\bullet \star [\delta_n]_\bullet$ which is well known by Lemma 6.1.3. We have:

$$[1]_\bullet \star [\delta_n]_\bullet = n[\delta_n]_\bullet + \sum_\nu [\nu]_\bullet$$

where the sum is over all partitions ν different from δ_n which can be obtained from δ_n by adding a box, deleting a box or first deleting and then adding a box.

We label the characters appearing in $[1]_\bullet \star [\delta_n]_\bullet$ by the four ways in which they can be obtained. Let a denote $[\delta_n]_\bullet$, b label those characters whose partitions are obtained from δ_n by adding a box, c those which are obtained by deleting a box and d those which are obtained by first deleting and then adding a box.

So we have 1 character labeled a with multiplicity n, $n+1$ labeled b, n labeled c and $n(n-1)$ labeled d.

Now λ is larger than δ_n and μ is larger than (1). Using Remarks 6.2.3 and 6.2.4 we know that $[\lambda]_\bullet \star [\mu]_\bullet$ has at least $1+n+1+n+n(n-1) = n^2+1+(n+1)$ components and $n+n+1+n+n(n-1) = n^2+n+(n+1)$ constituents. The partitions of characters labeled b have size $|\delta_n|+|(1)|$. So in $[\lambda]_\bullet \star [\mu]_\bullet$ there are also at least $n+1$ characters whose corresponding partitions are of size $|\lambda|+|\mu|$. But we have also $\bar{g}(\lambda,\mu,\nu) = c(\nu;\mu,\lambda)$ for $|\nu| = |\mu|+|\lambda|$ so we can deduce from Lemma 5.2.11 that there are also at least p_{m+1} components and f_{m+1} constituents in $[\lambda]_\bullet \star [\mu]_\bullet$ whose corresponding partitions are of size $|\mu|+|\lambda|$. So we have in total at least $n^2+1+\max(p_{m+1}, n+1)$ components and $n^2+n+\max(f_{m+1}, n+1)$ constituents.

We now prove the lower bound for the number of pairs of partitions which differ by one box.

We have in $[1]_\bullet \star [\delta_n]_\bullet$ the following number of pairs of components, whose diagrams differ by one box (where type (a,b) means that one character is labeled a and the other b):

Type :	(a,a)	(a,b)	(a,c)	(a,d)	(b,b)
Number :	0	$n+1$	n	$n(n-1)$	$\frac{1}{2}(n+1)(n)$
Type :	(b,c)	(b,d)	(c,c)	(c,d)	(d,d)
Number :	0	$n(n-1)$	$\frac{1}{2}n(n-1)$	$n(n-1)$	$n(n-1)(n-\frac{5}{2})$.

The first four numbers are clear since all diagrams labeled b, c or d differ from δ_n by only one box. The numbers for type (b,b) and type (c,c) are also clear since all diagrams of characters of type b differ from one another only by the additional box. The same goes for all characters of type c. Furthermore, the diagrams of characters of type b have two more boxes than those of type c hence the 0 for type (b,c).

We now look at pairs of type (b,d). Suppose the partition α corresponds to a character of type b and β to a character of type d and α and β differ by only one box. If $\alpha = \delta_n + (1)$ (so the additional box of α is in the first row) then there are $n-1$ partitions β with character of type d (β is obtained from α by deleting a box in a row other than the first row). The same goes if $\alpha = \delta_n \cup (1)$. So now suppose α is neither $\delta_n + (1)$ nor $\delta_n \cup (1)$. We have $n-1$ of those α and in each case there are $n-2$ β whose corresponding character is of type d such that α and β differ by one box. So we have in total:

$$2(n-1) + (n-1)(n-2) = n(n-1).$$

For type (c,d) we can choose one of the n characters of type c say with corresponding partition α. There are $n-1$ places in which we can add a box to α to obtain a partition β whose corresponding character is of type d. So we have $n(n-1)$ of those pairs.

Let us now check the number of pairs of type (d,d). Let α and β be partitions with corresponding character of type d which differ by one box. For α we can choose any of the $n(n-1)$ partitions. Suppose α is obtained from δ_n by deleting the box A and then adding the box B. For β we can then choose any of the $n-2$ partitions which are obtained from δ_n by also deleting A and then adding a box different from B or any of the $n-3$ partitions which are obtained from δ_n by deleting a box different from A (such that the box B can be added afterwards) and then adding the box B. Since we count each pair only once we get a factor $\frac{1}{2}$ and so in total:

$$\frac{1}{2}n(n-1)(n-2+n-3) = n(n-1)(n-\frac{5}{2}).$$

Adding all the number of pairs except pairs of type (b,b) we get:

$$n+1+n+n^2-n+n^2-n+\frac{1}{2}n^2-\frac{1}{2}n+n^2-n+n^3-\frac{7}{2}n^2+\frac{5}{2}n =$$
$$n^3+n+1.$$

Now λ is still larger than δ_n and μ is larger than (1). Using again Remarks 6.2.3 and 6.2.4 and by the same argument as in Theorem 5.2.7 different pairs of components, whose corresponding partitions differ by one box, of $[1]_\bullet \star [\delta_n]_\bullet$ give different pairs in $[\mu]_\bullet \star [\lambda]_\bullet$. Notice that the pairs of type (b,b) correspond to pairs of components, whose partitions have as size the sum of the sizes of (1) and δ_n. So the number of pairs of components of $[\mu]_\bullet \star [\lambda]_\bullet$, whose partitions are of size $|\mu|+|\lambda|$ and which differ by one box, is at least $\frac{1}{2}n(n-1)$, the number of pairs of type (b,b). But since we have $\bar{g}(\lambda,\mu,\nu) = c(\nu;\mu,\lambda)$ if $|\nu| = |\mu|+|\lambda|$ we know by Lemma 5.2.11 that the number of those pairs is also at least g_{m+1}.

This gives in total

$$n^3+n+1+\max\left(g_{m+1}, \frac{1}{2}(n+1)n\right)$$

pairs of components, whose corresponding partitions differ by only one box. □

The used property $\bar{g}(\lambda,\mu,\nu) = c(\nu;\mu,\lambda)$ if $|\nu| = |\mu|+|\lambda|$ answers also the question of equality of reduced Kronecker products:

LEMMA 6.4.3. *Let $\lambda, \lambda', \mu, \mu'$ be partitions with $[\lambda]_\bullet \star [\mu]_\bullet = [\lambda']_\bullet \star [\mu']_\bullet$. Then either $\lambda = \lambda', \mu = \mu'$ or $\lambda = \mu', \mu = \lambda'$.*

PROOF. This follows directly from $\bar{g}(\lambda,\mu,\nu) = c(\nu;\mu,\lambda)$ if $|\nu| = |\mu|+|\lambda|$ and Lemma 3.1.4 which stated the result of the Lemma for the outer ordinary product. □

Remark 6.4.4. Note again that the situation for the ordinary Kronecker product is not as nearly as nice as for the reduced Kronecker product. We can't determine anything about the number of components or constituents of the ordinary Kronecker product $[\lambda][\mu]$ by analyzing the corresponding ordinary Kronecker product:

$$[\frac{n(n+1)}{2}-1,1][\delta_n] = (n-1)[\delta_n] + \sum_\nu [\nu]$$

where the sum is over all partitions ν different from δ_n which can be obtained from δ_n by first deleting and then adding a box. As already mentioned in Remark 6.2.3 only in some cases we get informations about the product $[\lambda][\mu]$ even if $dp(\lambda) = n$ (so λ is larger than δ_n) and μ is larger than $(\frac{n(n+1)}{2}-1,1)$ from facts about the product $[\frac{n(n+1)}{2}-1,1][\delta_n]$.

Furthermore, for the ordinary Kronecker product we have $[\lambda][\mu] = [\lambda^c][\mu^c]$ which we have already seen in the examples does not hold for the reduced Kronecker product, and by Lemma 6.4.3 never holds (unless λ, μ are symmetric or $\lambda = \mu^c$).

CHAPTER 7

Summary of the results

In the preceding chapters we have proven the following results.

We proved in Theorem 2.1.1 that the base partition of a skew character $\mathcal{B}([\lambda/\mu])$ is the union partition of the corresponding skew diagram $\Upsilon(\lambda/\mu)$, and furthermore, that we have $\mathcal{B}_i = \max_j |\lambda_{i+j-1} - \mu_j|_+$ with $|x|_+ = \max(0, x)$ (Remark 2.1.2).

From the base partition we obtain in some cases the cover partition $\mathcal{C}(\chi)$ via the skew character - Schubert product correspondence. For products we define the skew diagram $\mathcal{A} = \big((k^l)/\mu\big)^\circ)/\nu$ and then have $\mathcal{C}([\mu] \star_{(k^l)} [\nu]) = \big((k^l)/\mathcal{B}([\mathcal{A}])\big)^\circ$ (k, l arbitrary) and $\mathcal{C}([\mu] \otimes [\nu]) = \big((k^l)/\mathcal{B}([\mathcal{A}])\big)^\circ$ ($k = \mu_1 + \nu_1, l = l(\mu) + l(\nu)$) (Theorem 2.1.3). If a skew diagram $\mathcal{A} = \lambda/\mu$ with $\lambda = (\lambda_1^n, \lambda_{n+1}, \ldots, \lambda_l)$ satisfies $\mu_1 \leq \lambda_l, l(\mu) \leq n$ then we obtain the cover partition of the corresponding skew character $[\mathcal{A}]$ as $\mathcal{C}([\mathcal{A}]) = \big(((\lambda_1)^l)/\mathcal{B}([\mu] \otimes [\bar{\lambda}])\big)^\circ$ with $\bar{\lambda} = ((\lambda_1^l)/\lambda)^\circ$ (Theorem 2.1.5).

We showed that the principal hook length partition $hl([\mathcal{A}])$ of a skew character is given by the northwest ribbon length partition $\pi_{nw}(\mathcal{A})$ of the corresponding skew diagram, $hl([\mathcal{A}]) = \pi_{nw}(\mathcal{A})$ (Theorems 2.2.2 and 2.2.10). For $\mathcal{A} = \alpha \otimes \beta$ we proved that there are $2^{\min(d(\alpha), d(\beta))}$ partitions ν with $[\nu] \in [\mathcal{A}]$ and $hl(\nu) = hl([\mathcal{A}])$ and that the corresponding characters all have multiplicity 1 (Theorem 2.2.2) and minimal Durfee size (Remark 2.2.5). In Remark 2.2.3 we determined the exact shape of all these ν. For arbitrary skew diagrams $\mathcal{A}$ we gave in Remark 2.2.11 an explicit (and easy) description of the irreducible characters $[\nu]$ having maximal principal hook length. If $nw_i(\lambda/\mu)$ decays into k_i disconnected ribbons then we have $\prod_i k_i$ of those ν and an easy formula for their multiplicities (Lemma 2.2.12). Furthermore, those ν have minimal Durfee size of all partitions whose characters appear in $[\mathcal{A}]$ (Lemma 2.2.14). Using again the skew character - Schubert product correspondence we obtain a formula for the Durfee size, components having maximal Durfee size, a lower bound for their number and their multiplicities for the ordinary product (Lemma 2.2.15) and some skew characters (Lemma 2.2.17).

Using the above results we obtained necessary conditions for the equality of two skew characters (Theorem 2.2.18).

We proved in Lemma 3.1.6 if λ/μ is of some special shape that $[\lambda/\mu] = [\alpha/\beta]$ is possible only in trivial case (λ/μ and α/β are the same up to translation and rotation). In Section 3.2 we proved that if $[\lambda/\mu] = [\alpha/\beta]$ is multiplicity free that then this equality is trivial or $\lambda = \delta_n$ is a staircase partition and λ/μ and α/β are conjugate of each other (Theorem 3.1.1).

We shoved that $c(\lambda + \lambda'; \mu + \mu', \nu + \nu') \geq c(\lambda; \mu, \nu)$ if we have $c(\lambda'; \mu', \nu') \neq 0$ (Lemma 4.1.1). The function $Q(n) = \sum_\nu c(n\lambda + \lambda'; n\mu + \mu', \nu)$ increases without bound if and only if λ/μ is a proper skew diagram (Lemmas 4.2.1 and 4.2.2). This is also a result about the number of constituents and components of the skew characters $[(n\lambda + \lambda')/(n\mu + \mu')]$. Furthermore, if λ/μ is a partition or rotated partition then we showed that $Q(n)$ strictly increases till it gets constant and then stays constant. Furthermore, we gave an explicit formula for the smallest m such that $Q(n) = Q(m)$ for $n \geq m$ (also Lemma 4.2.2). So the stretched LR coefficient $c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$ is in some cases also bounded above (Lemma 4.3.1). Furthermore, we proved that in some cases there is an integer m and a polynomial $g(n)$ with the same degree as the polynomial $c(n\lambda; n\mu, n\nu)$ such that $g(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$ for $n \geq m$ (Lemma 4.3.6).

We showed that if λ/μ is a basic skew diagram with $dp(\lambda) \geq n = dp(\mu) + 1$ that then $[\lambda/\mu]$ has at least p_n components, f_n constituents (Theorem 5.2.9) and contains at least g_n pairs of components $([\nu^1], [\nu^2])$ such that ν^1 and ν^2 differ by one box (Theorem 5.2.7), with p_n the number of partitions of n, f_n the number of standard Young tableau with n boxes and g_n the number of pairs of partitions of n which differ by one box. For this we used that λ/μ is larger than δ_n/δ_{n-1} (Lemma 5.2.6). We gave a bijection between the pairs of partitions of n which differ by one box and partitions of $n-2$ with two different kinds of 1 and 2. (Lemma 5.2.3). The results about skew characters clearly give us also results about the product $[\alpha] \otimes [\beta]$ (Lemma 5.2.11).

Furthermore, we classified all skew characters containing at most five components and those containing at most five constituents (Theorem 5.3.1) and listed explicitly all skew characters having two or three components (Remarks 5.3.3 and 5.3.4).

We showed that also the reduced Kronecker coefficients satisfy $\bar{g}(\lambda + \lambda', \mu + \mu', \nu + \nu') \geq \bar{g}(\lambda, \mu, \nu)$ if there is an m with $g(\lambda'[m], \mu'[m], \nu'[m]) \neq 0$ (Lemma 6.2.2). We then classified the reduced Kronecker products which are multiplicity free (Theorem 6.3.1) and those containing at most ten components (Theorem 6.4.1). We then also gave lower bounds for the number of components, constituents and pairs of components, whose corresponding partitions differ by one box, of a reduced Kronecker product (Theorem 6.4.2). Finally we gave a short proof that equality of reduced Kronecker products is only possible if the factors are the same (Lemma 6.4.3).

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