Boundary Conformal Field Theory Analysis of the H_3^+ Model

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Georg Wilhelm Friedrich Hegel, Einleitung zur Phänomenologie des Geistes

Zusammenfassung

Zentrales Thema dieser Arbeit sind Konsistenzbedingungen an maximal symmetrische Branen des H_3^+ Modells. Sie werden in Form sogenannter Shift Gleichungen hergeleitet und auf ihre Lösungen untersucht. Das Resultat sind explizite Ausdrücke für die Ein-Punkt Funktionen in den verschiedenen Bran-Hintergründen. Das Bran-Spektrum organisiert sich in kontinuierlichen und diskreten Serien.

Zunächst geben wir eine Einführung in die zweidimensionale konforme Feldtheorie (CFT) im Rahmen der Theorie von Vertexoperatoralgebren und ihren Moduln. Wir versuchen diesen Zugang an die speziellen Bedürfnisse des nichtrationalen H_3^+ Modells anzupassen.

Zu Beginn des zweiten Teils werden kurz die benötigten Analysetechniken für CFTen mit Rand bereitgestellt, darunter insbesondere die Cardy-Lewellen Klebebedingung, die den folgenden Konstruktionen wesentlich zu Grunde liegt. Danach führen wir in die Systematik der Bran-Lösungen ein der wir in dieser Arbeit folgen. Mit der Unterscheidung zwischen regulären und irregulären Ein-Punkt Funktionen schlagen wir ein neues, zusätzliches Ordnungskriterium für Bran-Lösungen vor. Weiter argumentieren wir, dass alle Isospin-Abhängigkeiten den Klebebedingungen unterworfen werden müssen. An dieser Stelle ist das auszuführende Programm skizziert und wir beginnen seine Umsetzung mit der Herleitung von neuen $1/2$-Shift Gleichungen, welche die vorher bekannten Gleichungen dieses Typs zu einer vollständige Liste für den Fall von AdS_2 Branen komplettieren.

Wir wenden uns dann den $b^{-2}/2$-Shift Gleichungen zu. Ihre Herleitung funktioniert nicht so direkt wie im vorhergehenden Fall: Der ursprüngliche Definitionsbereich einer bestimmten Zwei-Punkt Funktion (der CFT mit Rand) muss auf eine geeignete Region ausgedehnt werden. Dazu ist es unumgänglich, eine Fortsetzungsvorschrift anzunehmen. Der natürliche Kandidat ist analytische Fortsetzung. Wir demonstrieren, dass eine solche mit einigem Aufwand unter Benutzung verallgemeinerter hypergeometrischer Funktionen durchgeführt werden kann. Auf diese Weise gewinnen wir eine vollständige Liste von $b^{-2}/2$-Shift Gleichungen für AdS_2 Branen, untersuchen ihre Lösungen und lesen das Bran-Spektrum ab.

Nachfolgend rekapitulieren wir kurz die H_3^+/Liouville Korrespondenz und das Hosomichi-Ribault Proposal, welches Anlass zu unserer nächsten Konstruktion gibt. Sie realisiert das Hosomichi-Ribault Proposal, welches eine von obiger Annahme abweichende Fortsetzungsvorschrift vorschlägt, explizit im H_3^+ Modell. Wir zeigen, dass mit unserer Konstruktion wiederum sinnvolle $b^{-2}/2$-Shift Gleichungen hergeleitet werden können und diskutieren deren Lösungen. Das resultierende Spektrum von AdS_2-Branen ist dem obigen analog. Abschließend werden beide Zugänge verglichen. Wir skizzieren ein mögliches Unterscheidungskriterium und spekulieren, wie sich unsere Resultate auf eine gewisse Klasse nichtkompakter nichtrationaler CFTen verallgemeinern könnten.

Schlagworte: Nichtrationale Konforme Feldtheorie, H_3^+ Modell, D-Branen

Abstract

The central topic of this thesis is the study of consistency conditions for the maximally symmetric branes of the H_3^+ model. It is carried out by deriving constraints in the form of so-called shift equations and analysing their solutions. This results in explicit expressions for the one point functions in the various brane backgrounds. The brane spectrum becomes organized in certain continuous and discrete series.

In the first part, we give an introduction to two dimensional conformal field theory (CFT) in the framework of vertex operator algebras and their modules. As this approach has been developed along with rational CFT, we pay attention to adapt it to the special needs of the nonrational H_3^+ model.

Part two deals with boundary CFT only. We start with a review of some basic techniques of boundary CFT and the Cardy-Lewellen sewing relations that will be at the heart of all following constructions. Afterwards, we introduce the systematics of brane solutions that we are going to follow. With the distinction between regular and irregular one point functions, we propose a new additional pattern according to which the brane solutions must be organized. We argue that all isospin dependencies must be subjected to the sewing constraints. At this point, the programme to be carried out is established and we are ready to derive the missing $1/2$-shift equations for the various types of AdS_2 branes in order to make the list of this kind of equation complete.

Then we address the $b^{-2}/2$-shift equations. It turns out that their derivation is not straightforward: One needs to extend the initial region of definition of a certain (boundary CFT) two point function to a suitable patch. Therefore, a continuation prescription has to be assumed. The most natural candidate is analytic continuation. We show that it can be carried out, although it is rather technical and involves the use of certain generalized hypergeometric functions in two variables. In this way, we derive a complete set of $b^{-2}/2$-shift equations for AdS_2 branes, study their solutions and extract the resulting brane spectrum.

In a following interlude we review the H_3^+/Liouville correspondence and explain the Hosomichi-Ribault continuity proposal, which motivates our next construction. Its purpose is the explicit realization of the Hosomichi-Ribault proposal within the H_3^+ model. As this proposal suggests a continuation prescription that differs from the above, one needs to study its implications for the brane solutions. Based on our explicit realization, we show that sensible $b^{-2}/2$-shift equations can be extracted from the Hosomichi-Ribault proposal and we study their solutions and the corresponding brane spectrum. The two approaches are finally compared. We outline a possible demarcation criterion that still has to be worked out and speculate about how our results might generalize to a certain class of noncompact nonrational CFTs.

Keywords: Nonrational Conformal Field Theory, H_3^+ Model, D-Branes

Contents

1 Introduction

This thesis is mainly concerned with the derivation of certain constraints in noncompact nonrational two dimensional boundary conformal field theory and the important question how the strength of these constraints depends on the different assumptions one can make. The interest in these kinds of questions stems from the study of branes in string theory. Two dimensional conformal field theory (CFT)[1] is the natural language[2] of string theory and branes are its monopole-like non-perturbative states. Their effect on the CFT description is to introduce a boundary on the two dimensional worldsheet of the string. As there are numerous motivations for studying strings in curved noncompact target spaces, and the associated CFTs are the noncompact nonrational ones, this connects directly to the primary concern of this thesis.

The uses of CFT are not restricted to string theory alone. It also appears in the study of critical phenomena in statistical physics, supplies quantum field theorists with a wide range of exactly solvable models and in some cases, its techniques even reach out into the realm of more general integrable models. Moreover, mainly through developments in string theory, connections to higher dimensional gauge theory and complex algebraic geometry (e.g. mirror symmetry, Verlinde formula) have been uncovered. Several axiomatic approaches to CFT have even made it an object of mathematical interest by itself, generalizing well-known and revealing new structures and connections to pure maths.

For this reason, it has become customary to study CFT in its own right. This will be our viewpoint as well. Although we motivate and introduce the specific model that our study is based on, the H_3^+ model, from the viewpoint of a string theory in curved spacetime with non-vanishing NSNS two form, our main interest will be solely in the CFT properties of the model. We therefore give a non-technical overview of CFT, its developments, connections to other fields and its various distinguished classes in a second part of the introduction. It is in this context that we explain the importance of studying the H_3^+ model.

Finally, we give an outline of our thesis with short descriptions of the contents to be found in the various chapters and appendices.

[1] Talking about CFT in this thesis, we always mean two dimensional CFT.
[2] natural from the worldsheet perspective

1.1 String Theory

String theory grew out of an attempt to understand the strong interactions using dual resonance models in the 1960s. While this approach failed with regard to its original purpose, it left behind the famous Veneziano amplitude and the realization that it can be obtained from a theory of strings, naturally formulated as a two dimensional nonlinear sigma model[3]. This model is conformally and Weyl invariant and therefore, two dimensional CFT established itself as the natural language of string theory.

As a tribute to the hadronic resonances they once tried to describe, dual resonance models incorporate higher spins gently and with ease. So it came that, when quantum chromodynamics (QCD) was established as the preferable theory of quarks and their strong interactions, string theory was not immediately dead. The massless spin two excitation of the closed string was realized to provide a candidate graviton. In addition, due to its history as a model of quarks being bound together, the string was always thought to possess gauge theory degrees of freedom at its ends, so-called Chan-Paton factors. Therefore, one could hope that string theory might provide a unified and consistent picture of all forces, gauge and gravity.

But this bosonic string had its problems: Alongside with the graviton, it also has a tachyon in its spectrum and is therefore immediately inconsistent. Also, when it gives a unification of forces, what about the particles then? A purely bosonic string cannot account for fermionic matter. The birth of supersymmetry helped to overcome these problems: The superstring even enforces the incorporation of fermions, that is matter, and projects out the tachyon in a natural way. The only missing ingredient to make it a candidate for a fully consistent unified theory of all matter and forces was to show that it is free of anomalies. When Green and Schwarz [1, 2] finally showed that anomalies could be cancelled for a restricted set of string theories, this must have felt like a revolution indeed. Today, the Green-Schwarz anomaly cancellation mechanism is still celebratedly credited as the "first superstring revolution".

One peculiarity of string theory is that it is most easily formulated in a preferred number of flat spacetime dimensions, which is called the critical dimension. As this dimension is $d = 26$ for bosonic and $d = 10$ for supersymmetric strings, one needs to compactify parts of space in order to make contact with the usual four dimensions. Once, there were hopes that only one consistent compactification would exist which would reproduce in a unique way all aspects of our four dimensional physics, including coupling constants and other parameters. But these are shattered dreams. The various compactification procedures

[3] Indeed, this model shows asymptotic freedom, predicts a confining potential and reproduces the Regge trajectories. Thus, the imagination of two quarks being "glued" together by a string might appear plausible. However, many other features of the strong interactions are not captured by dual amplitudes.

one can think of lead to an incredibly vast and arbitrary "landscape" of consistent possibilities. Yet, this should not be a reason to despair, as there are possibilities to circumvent the critical dimension and avoid compactification. Moving away from the critical dimension in flat spacetime, things get harder because one needs to include the Liouville mode into the theory. This Liouville sector has been avoided in the past, because it was not tractable. Actually, Liouville theory belongs to the noncompact nonrational CFTs (see also section 1.2), which are our concern in this thesis. In recent years, there has been a lot of progress in the understanding of quantum Liouville theory, so that it might be possible to explore strings in the flat background away from the critical dimension now.

Another possibility to circumvent the critical dimension is to use a curved target space, that is to switch on gravity and put our strings in a curved background. Perturbatively, if one desires to decouple the Liouville sector, this still needs the critical dimension. But in a non-perturbatively exact background, strings exist in dimensions other than the critical one. This can easily be understood if one realizes that the curved background introduces new parameters in the theory, for example a curvature radius. In exact backgrounds, this affects the central charge of the string theory and one can therefore dispose of more parameters than just the dimensionality of spacetime in order to set the total central charge to zero. Examples for such models are provided by the WZNW models with compact target, but also the noncompact nonrational $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset CFT and the H_3^+ model (see also section 1.2) show such behaviour. The latter models are of course distinguished, because they are more realistic: The spacetime that we observe is noncompact. Also, for the description of time dependent phenomena, noncompact targets are inevitable, as time is noncompact.

Having seen a "first superstring revolution", one may suspect that string theory has possibly undergone a "second revolution". This is indeed true. Its subject is the discovery of Dp-branes, which are non-perturbative states in the string theory and serve as sources for RR fields [3]. In contrast to the string, they are not one dimensional objects, but p-dimensional submanifolds in d-dimensional target space ($p \le d$). The ends of open strings are restricted to move in Dp-brane worldvolumes only. Thus, Dp-branes are naturally thought of as setting boundary conditions for open strings. In CFT language, their inclusion amounts to the introduction of a boundary on the two dimensional worldsheet. The Dp-brane vacua can be analysed with the methods of boundary CFT (see section 4).

Putting Dp-branes to good use, one has yet another option to circumvent the compactification problem. This time, it is not by avoiding the critical dimension, but rather by thinking of our observable spacetime as such a p-dimensional submanifold. The picture is supported by the reinterpretation of the old Chan-Paton factors that followed: As open string ends are confined to the Dp-brane worldvolume and it is them who carry the gauge theory degrees of freedom, we simply find a gauge theory (for example, after some engineering, the standard model) confined to the brane, which therefore plays the rôle of our observable universe.

Rising up in string theory, these p-dimensional gauge theories should have a description in terms of two dimensional conformal field theory. Some works have dealt with such maps between a two dimensional CFT on a particular Riemann curve to p-dimensional gauge theory (see for example the quite recent [4]), but there is certainly much more potential here.

Another rather theoretical development that Dp-branes have brought us are the several Anti-de-Sitter (AdS) to CFT (AdS/CFT) duality conjectures. In their strongest form they state that string theory on certain AdS spacetimes has a dual description in terms of a conformal field theory which can be thought of as living on the conformal boundary of the AdS spacetime. To learn more about these dualities, it is necessary to be able to study string theory in those AdS backgrounds, which are again noncompact and curved, meaning that their corresponding CFTs are of the noncompact nonrational type.

1.2 Two Dimensional Conformal Field Theory and Mathematical Physics

CFT originally arose in the attempt to understand universality classes of critical phenomena in statistical physics [5]. But it is also of great (and sometimes central) importance for other branches of physics: The worldsheet formulation of string theory gives rise to a CFT and it is thus an appropriate language for string theory. Even more, regarding research in quantum fied theory, CFTs provide examples of exactly solvable quantum field theories, due to their infinitely many conserved charges. This also makes them prototypes of integrable models.

CFT and Mathematics

Some results in CFT have triggered or at least influenced new developments in mathematics, in particular complex algebraic geometry. At other times, new mathematical developments have given directions to research done in CFT. Prominent examples for this interplay of CFT and mathematics include the Verlinde formula [6] (for connections to algebraic geometry and the first strict proof, see [7, 8]; for a treatment in the context of vertex operator algebras, see [9, 10]), Calabi-Yau geometry and mirror symmetry (Gepner models [11], the Greene-Plesser construction [12], toric geometry [13]), stochastic (Schramm-) Löwner evolution (SLE) [14, 15, 16, 17], the correspondence between ordinary differential equations (ODE) and integrable models (IM) (ODE/IM correspondence) [18, 19] and the Geometric Langlands Programme [20, 21, 22, 23, 24].

The probably most intimate connection between CFT and mathematics, however, is still the development of the theory of vertex operator algebras (VOAs)[4]

[4] Borcherds introduced the central notion of vertex algebras (VAs). It was later slightly extended to that of VOAs by Frenkel, Lepowsky and Meurman.

by Borcherds [25] and Frenkel, Lepowsky and Meurman [26]. Indeed, the VOA construction makes CFT[5] an axiomatized framework and thus an object of mathematical interest of its own. The theory of VOAs and their modules shows a very rich mathematical structure, generalizing at the same time the theory of Lie groups as well as that of commutative associative algebras and the representations of these objects [27, 28]. It has lead to the construction of a natural representation, the moonshine module by Frenkel, Lepowsky and Meurman [26], for the Fisher-Griess Monster, which is the largest sporadic finite simple group. Their construction is essentially the construction of a CFT, the so-called Monster CFT.

It is worth mentioning that other axiomatizations of CFT have been proposed, most notably by Segal who put forward a categorical approach to CFT and by Gaberdiel and Goddard who like to use a structure called meromorphic conformal field theory [29]. The latter also uses vertex operators and in some respects reminds of the VOAs mentioned above. Yet, compared to Borcherds work, it is much more adapted to the way CFT is used in physics. The approach that we have chosen in order to introduce the basic CFT notions and concepts in chapter 2 mainly follows the theory of vertex operator algebras, presented in a manner which is suitable for physics. Some additional ideas are taken from Goddard and Gaberdiel, while the spirit remains that of the original operator approach à la Belavin, Polyakov and Zamolodchikov [5].

Classes of CFTs

In the more physics oriented literature, several interesting classes of CFTs have been identified:[6] Self-dual and self-dual extremal, rational, logarithmic, nonrational and noncompact nonrational CFTs. Let us introduce them briefly.

Self-Dual and Self-Dual Extremal CFTs

Self-dual CFTs are modular invariant theories which nevertheless use one chirality only. They are therefore truly meromorphic. Their central charges must be multiples of twenty-four, $c = 24k$ with $k \in \mathbb{Z}_{>0}$. For $k = 1$, they have been classified by Schellekens [30].

For a self-dual extremal CFT, an additional requirement concerning the conformal weights of primary fields must be met: The lowest conformal weight of primary fields other than the identity must be $h = k + 1$, if the central charge is $c = 24k$ as above. Interestingly, the Monster CFT by Frenkel, Lepowsky and Meurman [26] is an example of such a CFT. The class of self-dual extremal CFTs is of interest to mathematicians, because of further expected connections to sporadic

[5] at least certain classes of CFTs

[6] The members of this list can have nonempty intersections. We have not included superconformal field theories (SCFTs).

finite simple groups, codes and lattices [31]. Recently, interest was also revived in the physics community, due to an article by Witten [32], which conjectures relations to the quantization of three dimensional gravity with negative cosmological constant.

Rational CFTs

A particularly well-understood class of CFTs is constituted by the so-called rational CFTs (RCFTs). By this term one usually understands CFTs which are consistent using only a finite number of irreducible representations of their underlying symmetry algebra. Sometimes, in the case of logarithmic CFT (see next subsection), this is mildly relaxed to also allow indecomposable representations [33, 34]. The most important examples of RCFTs are the minimal models, the compact WZNW models at integer level and (in the mildly generalized sense) the logarithmic $c_{p,q}$ models. The motivation to call these CFTs “rational” comes from the fact that their central charges and the conformal weights of their primary fields are in fact rational numbers.

An outstanding problem one still hopes to solve is the classification of all RCFTs. Those with central charge $c = 1$ are classified. The $c = 1$ models obtained from a free boson on a torus and its orbifolds (also called gaussian $c = 1$ CFTs) have been studied in [35]; A classification of the rational models at $c = 1$ (unitary and non-unitary) was then almost achieved in [36, 37] and completed through the usage of $\mathcal{W}$-algebras [38] with surprising consequences for their moduli space [39]. Concerning central charge $c < 1$, one knows that there is only a discrete series of unitary rational models, the so-called unitary minimal models [40]. They are contained in the series of minimal models [5], which can also be non-unitary and are RCFTs with $c < 1$. But this list of RCFTs with central charge $c < 1$ is presumably not complete. Results on the classification of $c > 1$ RCFTs are also quite rare.

All RCFTs are believed to be WZNW cosets $\hat{\mathfrak{g}}/\hat{\mathfrak{h}}$ with semi-simple Lie algebra $\hat{\mathfrak{g}}$ and a gauged subalgebra $\hat{\mathfrak{h}}$. The known minimal models at $c < 1$ fit into this scheme, since they can be obtained as $[\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{su}}(2)_1] / \widehat{\mathfrak{su}}(2)_{k+1}$ WZNW coset models, with fractional levels k being allowed for the non-unitary ones. The first step in a systematic classification of RCFTs would therefore be the classification of all (ungauged) WZNW models. This would yield a list of models with $c \geq 1$. But already this problem still seems to be much too complex. Some special cases could have been dealt with however: All $\widehat{\mathfrak{su}}(2)_k$ WZNW models were classified by Cappelli, Itzykson and Zuber [41] and an A-D-E pattern was found. Gannon has worked on a generalization of their proof and could classify the $\widehat{\mathfrak{su}}(3)_k$ WZNW models [42] as well as $\widehat{\mathfrak{su}}(n)_2$ and $\widehat{\mathfrak{su}}(n)_3$ [43]. Together with Walton, he also found some results on diagonal cosets [44], but that appears to be all for now[7].

[7] There are some more results on rational SCFTs. For instance, Cappelli has classified the $\mathcal{N} = 1$ and

In practice, the formulation of RCFTs as WZNW models is very powerful, because it makes available some general concepts associated to affine Lie algebras. These are technologies like, for instance, the Knizhnik-Zamolodchikov equations, affine singular vectors, integrable representations, the Kac-Walton formula, the depth rule and level-rank duality. RCFTs are undoubtedly the class of CFTs that is by far best understood.

Logarithmic CFTs

Logarithmic CFT (LCFT) was discovered by Gurarie in [47], see also [48, 49] for an overview. Its developement has started in the physical literature, but by now it has also become an object of mathematical study within the theory of vertex operator algebras [50, 51]. Connections to supergroup WZNW models are also very interesting [52]. In LCFT, correlation functions can show logarithmic divergencies and the underlying symmetry algebra may be represented by operators with a Jordan cell structure, which means that not all representations are completely reducible.

The most prominent representative of LCFT is certainly the triplet model [53, 54], which has been shown to constitute a (mild generalization of) rational CFT [34], if one allows for indecomposable representations in the definition of rationality, as suggested by Flohr [33]. The triplet model is the simplest in a whole series of LCFTs with central charges $c_{p,1}$ and Flohr has shown that all of them possess the mildly generalized rationality property just alluded to. Further evidence that these so-called $c_{p,1}$ models are rational in a generalized sense is provided by the existence of fermionic character formulae [55]. The Verlinde formula has also been generalized in the context of these models [56, 57].

LCFT has turned out to be more than just a theoretical curiosity: It plays an important rôle in the understanding of certain critical phenomena. Examples include the fractional quantum Hall effect [58, 59], two dimensional turbulence [60] as well as two dimensional percolation [61]. Moreover, connections of LCFT to Seiberg-Witten theory have been noted [62], see also [63, 64], which prospectively opens up the possibility for LCFT to play a rôle in topics of fundamental mathematics, as for example the classification of four-manifolds [65].

Nonrational and Noncompact Nonrational CFTs

While being very well-behaved and providing many tools for their solution, the RCFT models discussed above are certainly not the most generic CFTs. Typically, one will have to deal with an infinity of representations, that is with nonrational CFT. If the infinite set of occuring representations is even continuous, one talks about noncompact nonrational CFT, because a continuous spectrum is usually

Gannon the $\mathcal{N} = 2$ minimal models [45, 46].

associated to noncompact directions in a target space or, respectively, to an underlying non–compact symmetry group. Very little is known about nonrational CFTs up to now. With this thesis, we make a contribution to their exploration.

Since an underlying symmetry usually facilitates the analysis of a theory, the noncompact nonrational CFTs with an underlying noncompact symmetry are the most natural starting point for a study of nonrational CFT, although it is a priori not clear, how the inevitable continuous spectra might affect their feasibility. For the time being, this is precisely the status of the exploration of nonrational CFT: Besides Liouville theory (the "minimal model of nonrational CFT"), some concrete models with noncompact symmetry are studied, most importantly the $SL(2,\mathbb{R})/U(1)$ coset theory and the H_3^+ model, which has an $SL(2,\mathbb{C})$ symmetry. We introduce it in much detail in chapter 3.

On general grounds, one expects many new features and difficulties in the study of noncompact nonrational CFTs. As the continuous representations do not have a highest or lowest weight, there are no singular vectors in the continuous current algebra representations. However, singular vectors are one of the central tools in the analysis of RCFTs. They allow for an algebraic determination of the fusion rules and imply powerful differential equations on correlation functions. In addition, such successful concepts as that of integrable representations and the depth rule, well-known from RCFT, break down when it comes to continuous representations. Furthermore, the operator product expansion (OPE) of two operators may contain operators that do not correspond to normalizable states of the theory, but rather to non-normalizable ones. Such non-normalizable states can usually be defined in a distributional sense on suitable subspaces of the normalizable states [66], but when non-normalizable states appear in intermediate channels, it is no longer a priori ensured that all expressions can be evaluated: For example, certain scalar products (the Shapovalov form) may simply not be defined. In such cases, it is not clear whether one can still maintain an operator state correspondence, which is again a central notion of common CFT.

The problems of non-normalizable states can be resolved in the H_3^+ model, mainly due to its remarkable analyticity properties [66]. It is these analyticity properties that make the model feasible in the current state of the art. Indeed, an analytic continuation in the representation labels, known in the literature as Teschner's trick, brings back the degenerate representations that possess singular vectors and thus allows the use of some RCFT techniques in this nonrational model. In particular, the conformal bootstrap becomes feasible and many explicit results can be obtained. For the purpose of our analysis, we shall make use of these extraordinary properties of the H_3^+ model. Interestingly, the problem we are going to discuss in this thesis also involves questions of analyticity. This is however not analyticity in the representation labels, but rather in the H_3^+ isospin coordinate.

To conclude our overview of nonrational CFT, let us just state that this topic certainly still remains in his childhood and a lot of new territory is to be discovered here. Current studies (including ours) concentrate on some extraordinarily well-behaved prototypes that are so gentle to allow restricted use of certain RCFT techniques. In the present state of the art this is all that can be done.

1.3 Topics and Outline of this Thesis

Objectives

As already said, our original contributions concentrate on the boundary CFT of the H_3^+ model. We have two principle goals: Firstly, to get a full and systematic overview of the maximally symmetric D-branes in the H_3^+ model [67]. For this purpose, we introduce the distinguishing notions of regular and irregular D-branes (see section 5.2.1) and stress the importance of the distinction between discrete and continuous ones (section 5.2.4). Our second goal is a detailed study of the applicability of the Cardy-Lewellen factorization constraints in this noncompact nonrational boundary CFT [68, 69]. All of this material is contained in part II of this thesis.

In part I, we have allowed ourselves to give a very extensive introduction to CFT within the framework of vertex operator algebras and their modules (chapter 2). This language is applied to the H_3^+ model in chapter 3. Our motivation for doing this, besides a general preference for rigour and the emergence of rich structures from minimal assumptions, is the following: The vertex operator algebra approach has been developed along with rational CFT. What we like to demonstrate is that the H_3^+ model, which constitutes a nonrational CFT, basically still fits into this framework, if one relaxes some (not very many) of the original requirements. What we cannot capture in this generality is the analytic continuation of the H_3^+ model to non-physical states. This is really a novelty which is not accommodated by the established vertex operator approach. Also, the general OPE is problematic, but that is actually already the case for RCFT.

An expert reader who is primarily interested in our original work will probably skip the first part and use it for reference only. We give a guide to minimal reading at the end of the outline (which itself is at the very end of this chapter). For now, let us come back to the objectives of our original work.

Cardy-Lewellen Factorization Constraints

In works by Cardy and Lewellen [70, 71] it was established that the structure data of rational boundary CFTs are highly constrained by certain “cutting and sewing relations”, commonly called Cardy-Lewellen constraints or, interchangeably, factorization constraints. We set out to study the rôle of these constraints in the noncompact nonrational H_3^+ boundary CFT. Here, the explicit derivation of the constraints becomes feasible because of the remarkable analytic structure of the model. Using Teschner's trick in a two point function, a degenerate field label can be reached, what means that the space of its conformal blocks becomes finite dimensional. Additionally, the two point function under consideration satisfies Knizhnik-Zamolodchikov equations from the current algebra symmetry of the model. From these pieces of information, the two point function can be computed explicitly. The Cardy-Lewellen constraint is then implemented by taking a limit where the two point function factorizes into a product of two one point functions.

In boundary CFT, the one point functions are important structure data and carry all information about the boundary states (the D-branes). They are actually fixed to great extent by boundary Ward identities. Their only remaining degree of freedom is the so-called one point amplitude. This is an interesting object to study, because it describes the coupling of a closed string in the bulk to a D-brane. Accordingly, it must depend on the properties of these two objects. Seeing that closed strings are characterized by an $\mathfrak{sl}(2,\mathbb{C})$-'spin' label j and branes are labelled by a complex parameter α, a one point amplitude is denoted $A(j|\alpha)$.[8] In the sequel, when talking about a brane solution, we actually mean a solution for the one point amplitude.

Implementing the Cardy-Lewellen constraint in the way described above results in a so-called shift equation. Its nature is to relate the one point amplitude for some string label j to a sum of one point amplitudes taken at shifted string labels like e.g. $j \pm 1/2$ (the shift is given by the represenation label of the degenerate field). Usually these constraints can be solved and the one point functions obtained. However, a solution will generically not exist for arbitrary boundary conditions, but restrictions will apply. By the same token, the labels j of strings that do couple consistently are expected to be constrained.

This approach has been pursued before, most significantly in [72]. Yet, only the simplest case, which uses a degenerate field with $\mathfrak{sl}(2,\mathbb{C})$-'spin' label $j = 1/2$ and from which a $1/2$-shift equation descends, has been treated. The solution to only this one shift equation is however not unique (for example, multiplication with an arbitrary $1/2$-periodic function again yields a solution). Therefore, a further shift equation would be desirable. The natural candidate from which to derive that second factorization constraint is the boundary two point function involving the next simple degenerate field which has $\mathfrak{sl}(2,\mathbb{C})$ label $j = b^{-2}/2$.

For that degenerate field however there are some difficulties in constructing the two point function in a region of the (u,z)-plane[9] that also covers the domain in which the factorization limit is to be taken. While a solution to the Knizhnik-Zamolodchikov equation can be given in the region $z < u$, it was unclear how

[8] It can also depend on some more data, see section 5.2.1.

[9] The conformal fields of the H_3^+ theory depend on two complex variables: A space-time coordinate z and an internal variable u - see chapter 3. The real (u,z)-plane we talk about here, is the plane spanned by the real-valued crossing ratios formed from internal (u) and space-time (z) positions of the fields in the boundary CFT correlator.

it could be continued to the patch $u < z$, which is the patch relevant to the factorization limit. In particular, a suitable continuation prescription is needed here.

These problems were resolved in our works [68, 69] and we were able to derive the desired $b^{-2}/2$-shift equation for the first time. Yet, new questions arise from our findings: Two different continuation prescriptions are possible and it remains unclear which one is preferable. This question is presumably linked to a conceptual question in nonrational CFT, namely the question whether the Cardy-Lewellen constraints remain fully intact. Let us explain this point in a little more detail here.

The first continuation prescription one naturally thinks of is analytic continuation of the boundary two point function, since, in its initial domain $z < u$, it is in fact an analytic function of both variables (u, z). To be precise, it is an Appell function of the first kind (see appendix C; functions of Appell type will play a central rôle throughout this thesis). We analysed this prescription and the shift equation it leads to in [69]. There is nothing unusual about the Cardy-Lewellen constraints within this approach.

Now, general H_3^+ boundary correlators were studied in a work by Hosomichi and Ribault [73] by making use of a mapping to Liouville theory. This mapping constitutes a very remarkable correspondence between Liouville theory and the H_3^+ model. It was established for the bulk theories in [74] and then generalized to the boundary CFTs in [73]. In formulating this mapping for the boundary theories, it was necessary to distinguish between two non-overlapping regimes, the so-called bulk and boundary regime. Crucially, the mapping breaks down at the interface of the two regimes and has to be supplemented by a continuation prescription that determines how correlators behave when moving from one regime into the other. The situation is very reminiscient of what we said above about the domain of the boundary two point function. Indeed, the two patches $z < u$ and $u < z$ correspond to bulk and boundary regime, respectively. Now, the proposal which has been put forward in [73] is that correlators should have a finite limit and be continuous at the interface of the two regimes. This is the continuity assumption of [73] that we refer to as the Hosomichi-Ribault proposal. It is of course a weaker requirement than the analyticity that we assume in [69]. Based on their assumption, the authors of [73] expect a weakening of the Cardy-Lewellen factorization constraint in the sense that the boundary two point function will cease to be fully determined in the bulk regime. Based on this scenario, one can then speculate that a unique $b^{-2}/2$-shift equation can in principle not be derived under the continuity assumption of Hosomichi and Ribault.

We have analysed this question in [68]. There, we were able to continue the boundary two point function to the region $u < z$ according to the Hosomichi-Ribault proposal. We found that it is indeed not fully determined by the continuity assumption. Yet surprisingly, the ambiguity resides in a part which is irrelevant to taking the factorization limit. Consequently, this limit can still be taken in a meaningful way, resulting again in a unique $b^{-2}/2$-shift equation. This is the outcome of our investigation [68].

We should say here, that the $b^{-2}/2$-shift equations that we derived differ slightly depending on the continuation prescription that is used. Nonetheless, the spectrum of consistent branes that we derive fits well with the expectations drawn from Cardy's work [75] in both cases. It is merely the regularity behaviour of the consistently allowed one point functions that changes when passing from one prescription to the other. This leads us to our second theme: The introduction of regular and irregular brane solutions. Let us explain the issues about this regularity behaviour now.

Regular and Irregular Branes and Systematics of Maximally Symmetric Branes

To begin with, we account briefly for the different kinds of brane solutions that are found in the existing literature. In [72], the authors showed that there are two classes of branes: AdS_2 and S^2 branes. They derived one shift equation for each class and also proposed solutions. Afterwards, [76] enlarged the picture and introduced the so-called $AdS_2^{(d)}$ branes, (d) standing for *discrete*. The author of [76] was guided by some relation between the ZZ and FZZT branes of Liouville theory that, in the spirit of the Liouville/H_3^+ correspondence of [74], was carried over to the AdS_2 branes of [72]. However, we like to point out that these new branes can also be understood as arising from the following difference in the derivation of the shift equation: The degenerate field is always expanded in terms of boundary fields, using its bulk-boundary OPE. Now, assuming a discrete open string spectrum on the brane, the occuring bulk-boundary OPE coefficient that corresponds to propagation of the identity in the open string channel can be identified with the one point amplitude. Hence, the two point function factorizes into a product of two one point functions. On the other hand, assuming a continuous open string spectrum, the above identification is lost. Instead, the two point function becomes a product of a one point function and a residue of the bulk-boundary OPE coefficient corresponding to the identity propagation. This is explained in [77] and we review it in section 5.2.4. The first case results in the $AdS_2^{(d)}$, whereas the second case leads to the $AdS_2^{(c)}$ shift equations, (c) standing for *continuous*. This treatment can always be applied, no matter what gluing condition we are using. This has actually been recognized, but not fully exploited, by the authors of [78].

Besides this scheme, that we think should be employed more systematically, there is another pattern that has not been taken much care of up to now. In [78], a solution to the boundary conformal Ward identities for the one point function, that is everywhere regular in the internal variable u, was proposed. Opposed to this solution, [72, 76] and [79] use a one point function that is not everywhere regular. While both solutions are correct (see section 5.2), we find that they give rise to slightly different shift equations (see sections 5.5.1 to 5.5.4 and 6.1.1 to 6.1.4 in case of the discrete as well as 5.6.1 to 5.6.3 and 6.2.1 to 6.2.4 for the continuous branes). The modifications that arise for the regular dependence opposed to the irregular one, change the qualitative behaviour of possible solutions significantly. Consequently, not only should one distinguish between continuous and discrete, but also between regular and irregular D-brane solutions. In [69] we could demonstrate that consistent non-trivial solutions for regular discrete AdS_2 branes do indeed exist.

Summary of our Achievements

We give for the first time a systematic treatment of all types of AdS_2 branes in the H_3^+ model. In particular, we carefully pay attention to the patterns discrete and continuous as well as regular and irregular. Introduction of the latter notions was proposed by us in [69]. For all of these branes, we derive two independent shift equations corresponding to degenerate fields $1/2$ and $b^{-2}/2$, respectively. Using these, we can fix the solutions for the one point amplitudes uniquely and determine the spectrum of AdS_2 branes. In the derivation of the $b^{-2}/2$-shift equations, a continuation prescription needs to be chosen. We motivate and discuss two different prescriptions and analyse and compare their consequences. A remarkable feature is that the Hosomichi-Ribault continuity assumption leaves the two point function partially undetermined. We can show that this does however not weaken the Cardy-Lewellen constraint. Therefore, a sensible $b^{-2}/2$-shift equation, that we derive explicitly, exists also in this approach. Since the resulting pictures are both acceptable, we have to leave the question which of the two prescriptions is the preferred one an open problem; yet, we outline and speculate about a possible demarcation criterion in the conclusion.

Outline of this Thesis

We start with an introductory chapter, chapter 2, on CFT and vertex operator algebras. Our aim is to describe in general the construction of a CFT within an approach that does not only encompass RCFT, but also at least the H_3^+ model. All notions and techniques that are needed for an understanding of the H_3^+ model, and in particular for our work, are introduced here. Then, in chapter 3, we introduce the H_3^+ model along the lines of chapter 2 and summarize those results that are indispensable for our following analysis. This is the content of part I. The following part II is solely concerned with boundary CFT. In chapter 4, we give an introduction to the basic techniques of boundary CFT and review the Cardy-Lewellen constraints. Chapter 5 recapitulates the results about the boundary H_3^+ model that were already known before our work and introduces the systematics that we will follow in considering the brane solutions, particularly the pattern regular/irregular. It reviews the different gluing maps, symmetries and equivalences of branes and discusses the derivation of the $1/2$-shift equations. Due to our systematic application of the patterns discrete/continuous and regular/irregular, some of the $1/2$-shift equations here are new, although their derivation was in principle and conceptually not problematic (unlike it was for the $b^{-2}/2$-shift equations). In the next chapter, chapter 6, we give the details of our derivation of the $b^{-2}/2$-shift equations using analytic continuation. The consequences for the D-brane solutions and the brane spectrum are also discussed. Afterwards, we need to insert a short intermezzo in chapter 7, where we review the H_3^+/Liouville correspondence (for bulk and boundary CFT) and describe the Hosomichi-Ribault proposal. Chapter 8 is then concerned with the derivation of $b^{-2}/2$-shift equations according to the continuity assumption. The strength and validity of the Cardy-Lewellen constraint is examined in detail. Again, the consequences for the D-brane solutions and the brane spectrum are investigated. Finally, we summarize our results and give some future directions in the conclusion, chapter 9. The appendices contain some mathematical and technical backgrounds. Appendix A gives an overview of the general representation theory of locally compact groups with a special focus on $SL(2,\mathbb{C})$ and $SL(2,\mathbb{R})$. The method of induced representations is treated in much detail. Appendix B briefly summarizes the different isospin bases that are commonly used for H_3^+ primary fields. In appendix C, we give an account of the first Appell function and some related special functions (the ordinary hypergeometric and the second Horn function) and assemble some formulae that we need in our work. For the second Horn function, we give a new generalized series representation for a special (degenerate) case that is crucial for our calculations.

For an expert reader who is primarily interested in our original work, let us give a guide to minimal reading: Part I can mainly be skipped. It is recommended to go quickly through section 3.2 in order to remind oneself of some basic H_3^+ model conventions. At some points it may become necessary to have a look at section 2.7. Moving on to part II, one should in any case start with section 4.2 on the Cardy-Lewellen constraints, as this is at the very heart of our study. Then, chapters 5, 6 and 8 can be studied thoroughly. They represent the core of our original work. Before starting with chapter 8, one will at least read section 7.5. Finally, a summary and discussion of our results are found in chapter 9. In the course of this reading, one will at least need to refer to sections 3.5 and 4.9 for some needed formulae.

Part I

The Bulk Theory

»B versteht das System der Reihe« heißt doch nicht einfach: B fällt die Formel »$a_n = \ldots$« ein! Denn es ist sehr wohl denkbar, dass ihm die Formel einfällt und er doch nicht versteht.

Ludwig Wittgenstein, Philosophische Untersuchungen Teil 1

2 Chiral Vertex Operators and Conformal Field Theory

The approach to CFT we are going to present here is mainly based on the theory of vertex operator algebras [27], but brought into a form which is more convenient for physics (for example, formal calculus is practically not needed). Some ideas have also been taken from Gaberdiel and Goddard [29, 80], but let us stress that the viewpoint we are taking is contrary to their meromorphic CFT framework, in that we do not define the theory from its meromorphic correlation functions, but focus on a space of states and the operator algebra entirely. For us, the correlation functions are the objects to be extracted. This is very much in the spirit of the good old operator approach by Belavin, Polyakov and Zamolodchikov [5].

2.1 The Worldsheet

We start with a two dimensional lorentzian manifold with local coordinates (t, x) and the topology of a cylinder[1], i.e. $(t, x) \in \mathbb{R} \times S^1$. Usually one goes over to *light cone coordinates* (x^+, x^-), whith $x^\pm = t \pm x$. Ultimately, we want to introduce fields that are defined on the space we are just describing. A field that only depends on x^+ (x^-) will then be called *chiral* (*antichiral*).

For the complex structure that we now introduce, it is essential to make a Wick rotation $t \mapsto \mathrm{i}t$ and thus reach euclidean signature. The cylinder can then be mapped to the complex plane by the exponential map

$$x^\pm \mapsto e^{-\mathrm{i}x^\pm} \equiv \left\{ \begin{array}{c} z \\ \bar{z} \end{array} \right. . \tag{2.1}$$

Note that under this map, time ordering becomes radial ordering, since $t_1 < t_2$ implies $|z_1| < |z_2|$. *Chiral* (*antichiral*) fields are now those that depend only on z ($\bar{z}$). A more general field will depend on both coordinates $(z, \bar{z})$ with the additional restriction that $\bar{z} = z^*$, $*$ denoting complex conjugation. Yet, those more general fields we shall be interested in, the so-called *conformal fields* (see section 2.8), will be built from chiral and antichiral fields. It is therefore most convenient to study the chiral and antichiral parts separately. Going over to the

[1]Regarded as a string worldsheet, this just means we are dealing with closed strings.

more general conformal fields is then easily done by merging the chiral halves and taking the two dimensional cut $\bar{z} = z^*$.

Having talked much about fields already, it is now high time to introduce these objects properly. For this purpose, we start from even more special objects, called chiral vertex operators, from which chiral fields will be build later[2], in section 2.4.1.

2.2 Chiral Vertex Operators

2.2.1 Vertex Algebras

In order to define a vertex algebra, we first need a vector space $\mathcal{V}$ which is $\mathbb{Z}$-graded and obeys the following *grading restrictions*:

$$\mathcal{V} = \coprod_{h\in\mathbb{Z}} \mathcal{V}_h\,, \quad \dim \mathcal{V}_h < \infty \ \ \forall\, h \in \mathbb{Z}\,, \quad \dim \mathcal{V}_0 = 1\,, \quad \mathcal{V}_h = 0 \ \text{ for } \ h < 0\,. \tag{2.2}$$

We fix an element $\Omega \in \mathcal{V}_0$ that generates all of $\mathcal{V}_0$ and call it the *vacuum*. Using the natural pairing, we obtain linear functionals on $\mathcal{V}$ and the $\mathbb{Z}$-graded dual space

$$\mathcal{V}^* = \coprod_{h\in\mathbb{Z}} \mathcal{V}_h^*\,. \tag{2.3}$$

In particluar, this introduces a scalar product on $\mathcal{V}$ and a linear functional $\langle\Omega, \cdot\rangle$, which will be needed in the definition of correlation functions below.

Next, we assume an operator/state correspondence, or in more mathematical terms, a *vertex operator map* $V(\cdot, z)$. That is, to every state $\Psi \in \mathcal{V}$ we associate an operator $V(\Psi, z)$ which acts on the vector space $\mathcal{V}$ and is called a (chiral) *vertex operator*[3]. As we like the grading to play a rôle here, we first define this for a homogeneous element $\psi \in \mathcal{V}_h$

$$V(\cdot, z) : \begin{cases} \mathcal{V}_h \to (\mathrm{End}\,\mathcal{V})\,[[z, z^{-1}]] \\ \psi \mapsto V(\psi, z) = \sum_{n\in -h+\mathbb{Z}} V_n(\psi) z^{-n-h} \end{cases} \tag{2.4}$$

and then extend the definition to arbitrary $\Psi \in \mathcal{V}$ by linearity of $V(\cdot, z)$ in its first argument. The space $(\mathrm{End}\,\mathcal{V})\,[[z, z^{-1}]]$ is the space of formal Laurent series. Formal means that there is no truncation condition here, i.e. we do not require $V_n(\psi) = 0$ for n sufficiently large (positive or negative). Note that the space of formal Laurent series is therefore strictly larger than the space $(\mathrm{End}\,\mathcal{V}) \otimes \mathbb{C}\,[[z, z^{-1}]]$. The formal Laurent expansion of a vertex operator $V(\psi, z)$ given

[2] Our terminology may appear unconventional at first sight. Nevertheless, we insist on our strict distinction between chiral vertex operators and chiral fields. This may become clear in section 2.4.1.

[3] We are going to drop the adjective "chiral" from now on.

in (2.4) introduces the *modes* $V_n(\psi) \in \mathrm{End}\,(\mathcal{V})$. Given a vertex operator for $\psi \in \mathcal{V}_h$, its modes can be extracted by the formal Cauchy integration theorem

$$V_n(\psi) = \oint_{(0)} \frac{\mathrm{d}z}{2\pi \mathrm{i}} z^{n+h-1} V(\psi, z)\,, \tag{2.5}$$

where $\oint_{(0)} \frac{\mathrm{d}z}{2\pi\mathrm{i}}$ is to be understood as an instruction to pick out the (-1)st (operator valued) coefficient of the formal Laurent series that follows.

The above definition of a vertex operator needs to be supplemented by some conditions. First of all the *truncation condition*, that for all $\psi \in \mathcal{V}_h$ and for all $\Phi \in \mathcal{V}$ we have

$$V_n(\psi)\Phi = 0 \quad \text{for } n \gg 0\,. \tag{2.6}$$

(The symbol $n \gg 0$ is to be read as "n sufficiently large"). Secondly, the *vacuum property*

$$V(\Omega, z) = \mathbb{1} \tag{2.7}$$

and finally the *creation property*

$$\begin{aligned} V(\psi, z)\Omega &\in \mathcal{V}[[z]] \quad \text{for } \psi \in \mathcal{V}_h\,, \\ \lim_{z\to 0} V(\Psi, z)\Omega &= \Psi \quad \forall\ \Psi \in \mathcal{V}\,. \end{aligned} \tag{2.8}$$

The first line in (2.8) just tells us that the modes $V_n(\psi)$ in the expansion $V(\psi, z) = \sum_{n\in -h+\mathbb{Z}} V_n(\psi) z^{-n-h}$ give zero when acting on the vacuum as long as n is such that the z-exponent $(-n-h)$ is negative, i.e.

$$V_n(\psi)\Omega = 0 \quad \text{for } n > -h\,. \tag{2.9}$$

Therefore, the second line in (2.8) is always well-defined. The restriction (2.9) is noted to be a special case (with a precise n) of the truncation property (2.6). Moreover, we are going to learn later, in section 2.2.2, that modes $V_n(\psi)$ raise the grading of a state by $(-n)$. From this perspective, the truncation property is needed for the third grading restriction in (2.2) to hold. Note that the third grading restriction ensures the vacuum to be the state of minimal grading in $\mathcal{V}$.[4] From the physics point of view, it is the condition that the energy spectrum be bounded from below and the vacuum be the state of lowest energy.

The structure we have defined up to now is almost a vertex algebra. The last (and actually most important) ingredient that it lacks to be a vertex algebra is the Jacobi (or Jacobi-Cauchy) identity. We are not going to assume the Jacobi identity here, but rather replace it with two other requirements (which, taken

[4]For rational CFT this is certainly true. Note that $\mathcal{V}$ is what usually is referred to as the vacuum module and the grading is the grading by conformal weight (these notions will be introduced later). In the H_3^+ model, we are going to use states of negative conformal weight as auxiliary, non-physical states. But they do not occur in the vacuum module, but in some other $\mathcal{V}$-modules; see section 2.3 for an introduction to modules. Thus, the statement remains true for the H_3^+ model and should also hold for many more CFTs.

together, are actually equivalent to the Jacobi identity, but have the benefit that they appear much more natural to a physicist). They are: Weak commutativity (that is a certain form of locality) and the existence of a generator of translations. Let us start with *weak commutativity*. In order to explain in which sense it is "weak", we need to introduce correlation functions first. This is quickly done: For any $n \in \mathbb{Z}_{>0}$ and any collection of states $\{\Psi_1, \Psi_2, \ldots, \Psi_n\}$, $\Psi_i \in \mathcal{V}$, $i = 1, \ldots, n$, let

$$\begin{aligned} G^{(n)}(\Psi_1, \Psi_2, \ldots, \Psi_n | z_1, z_2, \ldots, z_n) = \\ = \langle \Omega, V(\Psi_1, z_1) V(\Psi_2, z_2) \ldots V(\Psi_n, z_n) \Omega \rangle \end{aligned} \tag{2.10}$$

be the *correlation function* (or vacuum expectation value) of $V(\Psi_1, \cdot)$, $V(\Psi_2, \cdot)$, $\ldots$, $V(\Psi_n, \cdot)$. It is a complex-valued function of z_1, z_2, $\ldots$, z_n. For z_1, $\ldots$, $\hat{z}_i$, $\ldots$, z_n fixed ($\hat{z}$ meaning that z is removed), it is meromorphic in z_i by the truncation condition (2.6). It is linear in the states Ψ_1, Ψ_2, $\ldots$, Ψ_n by linearity of the vertex operators in their first argument. This shall be enough about correlation functions for the moment. The condition of *weak commutativity* is now easily formulated: For arbitrary $\Psi, \Phi \in \mathcal{V}$ we require that[5]

$$V(\Psi, z) V(\Phi, w) = V(\Phi, w) V(\Psi, z) \quad \text{for } z \neq w \tag{2.11}$$

in the weak sense, i.e. when inserted into any correlation function.

Next, we assume the existence of a *generator of translations* L_{-1}. It is defined to act on vertex operators as

$$e^{\alpha L_{-1}} V(\Psi, z) e^{-\alpha L_{-1}} = V(\Psi, z + \alpha)\,, \tag{2.12}$$

or, in the infinitesimal form

$$[L_{-1}, V(\Psi, z)] = \frac{\mathrm{d}}{\mathrm{d}z} V(\Psi, z)\,. \tag{2.13}$$

In order to extract its definition on states $\Psi \in \mathcal{V}$, we further require that the vacuum is invariant under translations

$$L_{-1}\Omega = 0\,. \tag{2.14}$$

Then it is immediate that

$$e^{zL_{-1}} \Psi = V(\Psi, z)\Omega\,. \tag{2.15}$$

We are now in the position to formulate and proof two important lemmata: The first one is a weak identification theorem for vertex operators and the second one establishes a property called weak associativity for vertex operators.

[5]For our purposes, the case of bosonic states suffices.

LEMMA 1 (WEAK IDENTIFICATION THEOREM): Whenever we have an operator $\mathcal{O}(z)$ that weakly commutes with all vertex operators $V(\Phi, w)$ and that satisfies $\mathcal{O}(z)\Omega = e^{zL_{-1}}\Psi$, then this operator weakly coincides with the vertex operator associated to Ψ, i.e. $\mathcal{O}(z) = V(\Psi, z)$ holds in all correlation functions.

► PROOF: Simply note that in any correlation function

$$\begin{aligned}
&\langle \Omega, V(\Phi_1, z_1)\dots V(\Phi_i, z_i)\mathcal{O}(z)V(\Phi_{i+1}, z_{i+1})\dots V(\Phi_n, z_n)\Omega\rangle = \\
&\quad = \langle \Omega, V(\Phi_1, z_1)\dots V(\Phi_i, z_i)V(\Phi_{i+1}, z_{i+1})\dots V(\Phi_n, z_n)\mathcal{O}(z)\Omega\rangle = \\
&\quad = \left\langle \Omega, V(\Phi_1, z_1)\dots V(\Phi_i, z_i)V(\Phi_{i+1}, z_{i+1})\dots V(\Phi_n, z_n)e^{zL_{-1}}\Psi\right\rangle = \\
&\quad = \langle \Omega, V(\Phi_1, z_1)\dots V(\Phi_i, z_i)V(\Phi_{i+1}, z_{i+1})\dots V(\Phi_n, z_n)V(\Psi, z)\Omega\rangle = \\
&\quad = \langle \Omega, V(\Phi_1, z_1)\dots V(\Phi_i, z_i)V(\Psi, z)V(\Phi_{i+1}, z_{i+1})\dots V(\Phi_n, z_n)\Omega\rangle\,. \blacktriangleleft
\end{aligned}$$

As an immediate corollary of this theorem we have the weak relation

$$[L_{-1}, V(\Psi, z)] = V(L_{-1}\Psi, z)\,, \tag{2.16}$$

because $e^{zL_{-1}}(L_{-1}\Psi) = e^{zL_{-1}}[L_{-1}, V(\Psi, 0)]\,\Omega = [L_{-1}, V(\Psi, z)]\,\Omega$. Let us now state the second lemma:

LEMMA 2 (WEAK ASSOCIATIVITY): For any two vertex operators

$$V(\Psi, z)V(\Phi, w) = V\left(V(\Psi, z-w)\Phi, w\right) \tag{2.17}$$

holds in the weak sense. This property is referred to as vertex operators being weakly associative.

► PROOF: The proof is simply by noting that

$$V(\Psi, z)V(\Phi, w)\Omega = V(\Psi, z)e^{wL_{-1}}\Phi = e^{wL_{-1}}V(\Psi, z-w)\Phi$$

and applying the identification theorem again. ◄

Of course, we can also expand the right hand side of (2.17) in modes. For simplicity let us take $\psi \in \mathcal{V}_h$. Then, the right hand side of (2.17) becomes[6]

$$V\left(V(\psi, z-w)\Phi, w\right) = \sum_{n\le N} V\left(V_n(\psi)\Phi, w\right)(z-w)^{-n-h}\,. \tag{2.18}$$

Note that the existence of some real number $N = N(\Phi)$ by which the summation index n in the previous formula is bounded is guaranteed by the truncation property (2.6). An immediate consequence of weak associativity is the following *skew*

[6] In this form, (2.17) is also known as operator product expansion (OPE). We return to this in section 2.6.

symmetry: Since by weak commutativity, the left hand side of (2.17) is symmetric under the interchange of (Ψ, z) and (Φ, w), so must be the right hand side. Hence,

$$V\left(V(\Psi, z-w)\Phi, w\right) = V\left(V(\Phi, w-z)\Psi, z\right). \tag{2.19}$$

Acting with both sides on the vacuum Ω and setting $w = 0$ we obtain

$$V(\Psi, z)\Phi = e^{zL_{-1}} V(\Phi, -z)\Psi. \tag{2.20}$$

This skew symmetry property will be extensively used later, when defining duals of intertwining operators. It also ensures that the commutator we are going to define in the next paragraph is really antisymmetric.

With weak commutativity and weak associativity at hand, it is quite easy to see that also the following weak identity holds. It is a variant of what in the mathematical literature is called Jacobi-Cauchy identity. We shall refer to it as the *weak Jacobi identity*:

$$\begin{aligned} \oint_{\substack{(0)\\ |w|>|z|}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} f(z,w) V(\Psi, w) V(\Phi, z) - \oint_{\substack{(0)\\ |w|<|z|}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} f(z,w) V(\Phi, z) V(\Psi, w) = \\ = \oint_{(z)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} f(z,w) V\left(V(\Psi, w-z)\Phi, z\right), \end{aligned} \tag{2.21}$$

where $f(z,w)$ can be any function that, for fixed z, is meromorphic in w, with potential poles only at $w = 0$ or $w = z$. For states $\psi \in \mathcal{V}_{h_\psi}$, $\Phi \in \mathcal{V}$, the weak Jacobi identity serves us to define the (weak) *commutator* of one mode with a vertex operator:

$$[V_m(\psi), V(\Phi, z)] = \oint_{(z)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} w^{m+h_\psi-1} V\left(V(\psi, w-z)\Phi, z\right). \tag{2.22}$$

Using the mode expansion (2.18), this can be recast into[7]

$$[V_m(\psi), V(\Phi, z)] = \sum_{k=-h_\psi+1}^{N(\Phi)} \begin{pmatrix} m + h_\psi - 1 \\ m - k \end{pmatrix} z^{m-k} V\left(V_k(\psi)\Phi, z\right). \tag{2.23}$$

The real number $N(\Phi)$ which bounds the summation index is again due to the truncation property. Note that in case of $m > -h_\psi$ all summands with $k > m$ are zero anyway (for $m \le -h_\psi$ this is however not true; yet, note that in this situation the binomial coefficient is still well-defined). From this formula, we can also derive an expression for the commutator of two modes associated to states $\psi \in \mathcal{V}_{h_\psi}$, $\phi \in \mathcal{V}_{h_\phi}$. But in order to present it in its most simplified form, we need to introduce a gradation operator first. We shall come to this in the next section.

[7] The following is again a weak identity. Obviously, when working with commutators in the sequel, equations will usually hold in the weak sense only. We are going to remember this fact tacitly from now on, so we do not have to repeat this point over and over again. As operator identities shall only be applied within correlation functions anyway, there is actually no big issue about that.

2.2.2 Vertex Operator Algebras

Up to now we have not introduced any conformal structure, but in the next step we are going to approach it. For this purpose, we are now assuming that there is a vertex operator $T(z)$ that is a symmetry current such that L_{-1} is its corresponding charge. This implies immediately that $T(z)$ has the expansion

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,. \tag{2.24}$$

As L_{-1} is the generator of translations, the vertex operator $T(z)$ must be the *energy-momentum tensor* of the theory. By the creation property it generates a state

$$\omega = L_{-2}\Omega \tag{2.25}$$

which is known as the *conformal vector*. From the expansion (2.24) we know that it is an element of $\mathcal{V}_2$. We do not know the meaning or any of the properties of the modes L_n for $n \neq -1, -2$, but this will change soon. Let us make an assumption about the operator L_0. We take it to be the *grading operator*

$$L_0\psi = h\psi \quad \text{for } \psi \in \mathcal{V}_h\,. \tag{2.26}$$

Note that the vacuum Ω is annihilated by this operator since $\Omega \in \mathcal{V}_0$. This is consistent with the creation property (2.8). We call the L_0-eigenvalue h of $\psi \in \mathcal{V}_h$ the *conformal weight* of ψ and usually denote it $h = h_\psi$.

Having introduced the energy-momentum tensor $T(z) = V(\omega, z)$, let us take a look at the consequences for the modes L_n. From (2.23), using $h_\omega = 2$, we gain the relation

$$[L_n, V(\Psi, z)] = \sum_{k=-1}^{N(\Psi)} \begin{pmatrix} n+1 \\ n-k \end{pmatrix} z^{n-k} V\left(L_k\Psi, z\right). \tag{2.27}$$

This means in particlar that

$$[L_{-1}, V(\Psi, z)] = V\left(L_{-1}\Psi, z\right), \tag{2.28}$$

which is consistent with (2.16), as well as

$$\begin{aligned} [L_0, V(\Psi, z)] &= z \cdot V\left(L_{-}1\Psi, z\right) + V\left(L_0\Psi, z\right) \\ &= z\frac{\mathrm{d}}{\mathrm{d}z} V(\Psi, z) + V\left(L_0\Psi, z\right). \end{aligned} \tag{2.29}$$

For any $\psi \in \mathcal{V}_{h_\psi}$, we can derive the commutator of L_0 with the modes of the vertex operator $V(\psi, z)$ from the last equation:

$$[L_0, V_n(\psi)] = -nV_n(\psi)\,. \tag{2.30}$$

Thus, the modes $V_n(\psi)$ raise the conformal weight of the state they act on by $(-n)$, that is

$$V_n(\psi) : \mathcal{V}_h \to \mathcal{V}_{h-n} \,. \tag{2.31}$$

This is independent of the state ψ, so we automaticaly get

$$[L_0, L_n] = -nL_n \,. \tag{2.32}$$

We can derive the analogous formulae for commutators involving L_{-1} from (2.16) and (2.13). They read

$$[L_{-1}, V_n(\psi)] = (-n - h_\psi + 1)V_{n-1}(\psi) \,, \tag{2.33}$$

from which follows that

$$[L_{-1}, L_n] = (-n - 1)L_{n-1} \,. \tag{2.34}$$

We note from (2.32) and (2.34) that commutators involving only L_{-1} and L_0 close to form an algebra. This is not surprising: We have introduced L_{-1} as the translation operator and from (2.29), we can infer that the finite transformation generated by L_0 is (for $\lambda \in \mathbb{C}$)

$$e^{\lambda L_0} V(\psi, z) e^{-\lambda L_0} = e^{\lambda h_\psi} V\left(\psi, e^\lambda z\right) , \tag{2.35}$$

that is, it generates complex dilations. These correspond to the dilations and rotations in the plane and are well-known to form a group together with the translations in two dimensions. We also note from (2.32) and (2.34) that adjoining L_1 to L_{-1} and L_0 we have again a closed algebra and that this is the largest algebra that contains L_{-1} and L_0 together with finitely many other modes from the set $\{L_n\}$. Its commutation relations can be written in one formula as

$$[L_m, L_n] = (m - n)L_{m+n} \quad \text{for } m, n \in \{-1, 0, 1\} \,. \tag{2.36}$$

This is recognized to be the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, the algebra of the *global conformal group* in two dimensions, which is $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$. Consequently, L_1 has to be identified with the generator of special conformal transformations. This is again not surprising, since it is well-known that a chiral energy-momentum tensor together with invariance under translations and complex dilations implies invariance under the full global conformal group.

Let us recall what we have achieved up to now: We have introduced a new vertex operator corresponding to a state ω of conformal weight 2 (the conformal vector). It has an interpretation as energy-momentum tensor and introduces a set of modes $\{L_n\}$. Out of this set, we could islolate L_{-1}, L_0 and L_1 as the generators of global conformal transformations. We know that this algebra cannot be enlarged by adjunction of only finitely many of the L_n to form a bigger algebra. But what we do not know yet is what happens if we take all the modes L_n. Will they close to form an algebra? In order to answer this question, we look back at equation (2.23) and use it to derive a formula for the commutator between two arbitrary modes of vertex operators $\psi \in \mathcal{V}_{h_\psi}$, $\phi \in \mathcal{V}_{h_\phi}$:

$$[V_m(\psi), V_n(\phi)] = \sum_{k=-h_\psi+1}^{N(\phi)} \begin{pmatrix} m + h_\psi - 1 \\ m - k \end{pmatrix} V_{m+n}\left(V_k(\psi)\phi\right). \tag{2.37}$$

We could not derive this formula earlier, because one needs to use $h_{(V_k(\psi)\phi)} = h_\phi - k$, what only follows because of (2.31). With this knowledge, the derivation of (2.37) is just a straightforward exercise in comparing coefficients. Let us use it to investigate the commutator between two of the L_n. We obtain

$$\begin{aligned}[L_m, L_n] = V_{m+n}\left(L_{-1}\omega\right) + (m+1)V_{m+n}\left(L_0\omega\right) + \\ + \frac{m(m+1)}{2}V_{m+n}\left(L_1\omega\right) + \frac{m(m^2-1)}{6}V_{m+n}\left(L_2\omega\right).\end{aligned} \tag{2.38}$$

There are no more terms, since $L_k\omega$ would have conformal weight $(2-k) < 0$ for $k > 2$, but by the third grading restriction $\mathcal{V}_h = 0$ for $h < 0$. We can simplify the first two terms due to (2.13) and (2.26). Moreover, since $L_2\omega$ has conformal weight zero and $\dim \mathcal{V}_0 = 1$, $L_2\omega$ must be proportional to the vacuum. The constant of proportionality is conventionally chosen to be

$$L_2\omega = \frac{c}{2}\Omega. \tag{2.39}$$

With this input, the algebra becomes

$$\begin{aligned}[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{(m+n)} + \\ + \frac{m(m+1)}{2}V_{m+n}\left(L_1\omega\right).\end{aligned} \tag{2.40}$$

But this is only consistent with antisymmetry of the commutator as well as our knowledge about the modes L_n expressed in (2.32), (2.34) and (2.36), if $V_k(L_1\omega) = 0$ for all $k \in \mathbb{Z}$. This in turn is only the case if we require

$$L_1\omega = 0. \tag{2.41}$$

If an eigenstate of L_0 is annihilated by L_1 but not by L_2, it has a property that we call *quasi-primarity*. We shall assume it for the conformal vector ω. Then, the L_n form the following algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{(m+n)}. \tag{2.42}$$

This is the celebrated *Virasoro algebra* and we have shown that it is the only consistent way to define a commutator between all modes L_n. The constant c is called *central charge* and its normalization is such that it takes the value 1 in the CFT of a massless free boson.

The appearance of the Virasoro algebra (2.42) is actually not too surprising. The conformal algebra in two dimensions consists of the elements $T_n = z^{-n+1}\frac{\mathrm{d}}{\mathrm{d}z}$, $n \in \mathbb{Z}$, which satisfy the *Witt algebra*

$$[T_m, T_n] = (m-n)T_{m+n}\,. \tag{2.43}$$

In view of (2.36) and our desire to include two dimensional conformal transformations, having this algebra for the modes L_n appears very suggestive and would be nothing but the extension of (2.36) to all n. What we really have obtained is a very natural deformation of the Witt algebra that, being not semi-simple, allows for a central extension. The central term that appears in (2.42) is indeed the most general term that can possibly occur.

2.3 Virasoro Representation Theory I: Modules for a Vertex Operator Algebra

In order to make contact with physics, we need a space of physical states on which the conformal transformations are realized. That is, the space of physical states will furnish a representation of the Virasoro algebra (2.42). We have shown in the last section, that a vertex operator algebra by itself provides such a situation. Yet, we need more states and more structure: In what is to come, we shall define the notion of the full vertex operator algebra acting on some space of states, i.e. we define the notion of a module for a vertex operator algebra.

2.3.1 Modules for a Vertex Operator Algebra

The first step in the definition of a *module* for a given vertex operator algebra $\mathcal{V}$ is again to take a vector space $\mathcal{W}$ graded with respect to a discrete index set $\mathcal{I}$ such that

$$\mathcal{W} = \coprod_{h\in\mathcal{I}} \mathcal{W}_h\,, \quad \mathcal{W}_h = 0 \ \text{ for } \ h \ll 0\,. \tag{2.44}$$

Note that we do not require the spaces $\mathcal{W}_h$ to be finite dimensional[8]. We also admit a finite number of the $\mathcal{W}_h$ with negative h to be non-zero. The space $\mathcal{W}$ is again accompanied by a *vertex operator map* $V^{(\mathcal{W})}(\cdot, z)$, but note the crucial difference to the case before: To every homogeneous state $\psi \in \mathcal{V}_h$ (!), we associate a vertex operator $V^{(\mathcal{W})}(\psi, z)$ which acts on the vector space $\mathcal{W}$ (!) by

$$V^{(\mathcal{W})}(\cdot, z) : \begin{cases} \mathcal{V}_h \to (\mathrm{End}\,\mathcal{W})\,[[z, z^{-1}]] \\ \psi \mapsto V^{(\mathcal{W})}(\psi, z) = \sum_{n\in -h+\mathbb{Z}} V_n^{(\mathcal{W})}(\psi) z^{-n-h} \end{cases}. \tag{2.45}$$

[8]In the H_3^+ model these spaces are indeed infinite dimensional.

Thus, the elements of the vertex operator algebra V have a device by which they can act on the space $\mathcal{W}$. As before, the modes of $V^{(\mathcal{W})}(\psi,z)$ can be obtained by the formal Cauchy theorem. Again, we extend the definition to arbitrary $\Psi \in \mathcal{V}$ by linearity of $V^{(\mathcal{W})}(\cdot,z)$ in its first argument.

Next, we introduce as much as possible of the structure that we have had for a vertex operator algebra. Note that the creation property cannot be formulated here. But we still need to have the *truncation condition*

$$V_n^{(\mathcal{W})}(\psi)\mathfrak{A} = 0 \quad \text{for } n \gg 0\,. \tag{2.46}$$

for all $\psi \in \mathcal{V}_h$ and all $\mathfrak{A} \in \mathcal{W}$. Also, the *vacuum property*

$$V^{(\mathcal{W})}(\Omega, z) = \mathbb{1}\,, \tag{2.47}$$

where $\mathbb{1}$ is now the identity on $\mathcal{W}$, is assumed. Also, we assume weak commutativity for the vertex operators $V^{(\mathcal{W})}(\cdot,z)$, although it is not yet clear how they can be inserted into a correlation function at all (up to now, this would not make sense, but we shall come to it in section 2.4, where we introduce intertwiners). Due to the lack of a creation property, we also do not have a weak identification theorem and cannot derive weak associativity here. Therefore, we are requiring weak associativity for this situation:

$$V^{(\mathcal{W})}(\Psi,z)V^{(\mathcal{W})}(\Phi,w) = V^{(\mathcal{W})}\left(V(\Psi,z-w)\Phi,w\right). \tag{2.48}$$

Note that we have to use $V(\cdot,z-w)$ on the right hand side. With weak commutativity and weak associativity, we have the analogues of the commutator formulae from before, in particular (2.23). The vertex operator $V^{(\mathcal{W})}(\omega,z)$ associated to the conformal vector is again taken to be the energy-momentum tensor, i.e. its modes[9] are denoted L_n, with L_{-1} being assumed to be the generator of translations and L_0 the generator of complex dilations (or, respectively, the grading operator) on $\mathcal{W}$ (!). So, we are actually extending the domain of definition of the L_n to $\mathcal{W}$. The analogue of (2.37) together with quasi-primarity of ω then implies the Virasoro algebra (2.42) on $\mathcal{W}$ by the same reasoning as before.

With the definitions we have given up to now, we are capable of associating vertex operators to states in $\mathcal{V}$ that then act on the space $\mathcal{W}$. What is still missing is of course an operator/state correspondence for the elements of $\mathcal{W}$. How can we associate vertex operators to them? The key to such a definition is the skew symmetry property (2.20). We simply define an operator

$$V^{(\mathcal{V},\mathcal{W})}(\cdot,z) : \mathcal{W} \to (\mathrm{Hom}(\mathcal{V},\mathcal{W})\left[\left[z,z^{-1}\right]\right] \tag{2.49}$$

by requiring skew symmetry of its action on $\mathcal{V}$:

$$V^{(\mathcal{V},\mathcal{W})}(\mathfrak{A},z)\Psi = e^{zL_{-1}}V^{(\mathcal{W})}(\Psi,-z)\mathfrak{A}\,. \tag{2.50}$$

[9]More precisely they should be denoted $L_n^{(\mathcal{W})}$.

It obviously satisfies the creation property

$$V^{(\mathcal{V},\mathcal{W})}(\mathfrak{A},0)\Omega = \mathfrak{A}\,. \tag{2.51}$$

For this kind of operators, we can readily proof a weak commutativity property with the $V^{(\mathcal{W})}(\cdot,z)$ operators. Namely, using the weak associativity (2.48) we get for $\Phi \in \mathcal{V}$:

$$\begin{aligned} V^{(\mathcal{W})}(\Psi,z)V^{(\mathcal{V},\mathcal{W})}(\mathfrak{A},w)\Phi &= V^{(\mathcal{W})}(\Psi,z)e^{wL_{-1}}V^{(\mathcal{W})}(\Phi,-w)\mathfrak{A} \\ &= e^{wL_{-1}}V^{(\mathcal{W})}(\Psi,z-w)V^{(\mathcal{W})}(\Phi,-w)\mathfrak{A} \\ &= e^{wL_{-1}}V^{(\mathcal{W})}\left(V(\Psi,z)\Phi,-w\right)\mathfrak{A} \\ &= V^{(\mathcal{V},\mathcal{W})}(\mathfrak{A},w)V^{(\mathcal{W})}(\Psi,z)\Phi\,. \end{aligned} \tag{2.52}$$

Due to this weak commutativity property, we can establish a weak identification theorem as before and then proof weak associativity, which now takes the form[10]

$$V^{(\mathcal{W})}(\Psi,z)V^{(\mathcal{V},\mathcal{W})}(\mathfrak{A},w) = V^{(\mathcal{V},\mathcal{W})}\left(V^{(\mathcal{W})}(\Psi,z-w)\mathfrak{A},w\right). \tag{2.53}$$

As a consequence, analogous commutator formulae as before also hold for the vertex operators $V^{(\mathcal{V},\mathcal{W})}(\cdot,z)$.

With the vertex operators V and $V^{(\mathcal{V},\mathcal{W})}$, we have established an operator/state correspondence for all states in $\mathcal{V}$ and $\mathcal{W}$. Generally when constructing a CFT, one will first start to build a vertex operator algebra $\mathcal{V}$ and then adjoin a family of modules $\left\{\mathcal{W}^{(i)}; i \in S\right\}$ labelled by an index set S to it. We have taken notice of the existence of a scalar product on $\mathcal{V}$ in section 2.2.1. Assume a scalar product on the spaces $\mathcal{W}^{(i)}$ as well[11]. The space of states $\mathcal{H}$ of the theory is then the completion β (with respect to the scalar product on the direct sum) of the direct sum of these spaces

$$\mathcal{H} = \beta\left(\bigoplus_{i\in S}\mathcal{W}^{(i)}\right). \tag{2.54}$$

In writing this formula, we include the possibility that $\mathcal{V}$ is one of the $\mathcal{W}^{(i)}$. This is however not necessary in general.

In the next section, we are going to describe the concrete construction of a vertex operator algebra and its modules.

[10] This weak associativity can be checked to be indeed consistent with weak commutativity (2.52) by acting on $\Phi \in \mathcal{V}$ with both sides of (2.53) and using weak associativity of the $V^{(\mathcal{W})}$ as well as skew symmetry of the V vertex operators on the right hand side.

[11] Again, this is fine for RCFT and remains to hold for the H_3^+ model.

2.3.2 Virasoro Representations: Construction of Vertex Operator Algebras and Modules

Every vertex operator algebra contains two states: The vacuum Ω and the conformal vector ω. The associated vertex operators are the identity $\mathbb{1}$ and the energy momentum tensor $T(z)$, respectively. Since the modes of $T(z)$ are the Virasoro generators L_n, we need to consider their action on these states. Recall that the vacuum is the state of minimal conformal weight, that is

$$L_n\Omega = 0 \quad \text{for all } n \geq -1 \tag{2.55}$$

(remember that the vacuum is annihilated by the generators of global conformal transformations anyway). Then, the space of states we have to form is (recall that $\omega = L_{-2}\Omega$)

$$\mathcal{V} = \text{span}\left\{L_{-n_k}\dots L_{-n_1}\Omega; k \geq 0 \text{ and } n_k \geq \cdots \geq n_1 \geq 2\right\}. \tag{2.56}$$

The grading is implemented by

$$\mathcal{V}_n = \text{span}\left\{L_{-n_k}\dots L_{-n_1}\Omega; k \geq 0,\ n_k \geq \cdots \geq n_1 \geq 2,\ n_1 + \cdots + n_k = n\right\} \tag{2.57}$$

and the grading restrictions are obviously obeyed. Associated vertex operators can be defined via

$$V\left(L_{-n_k}\dots L_{-n_1}\Omega, z\right) = T(z)_{-n_k+1}\dots T(z)_{-n_1+1}\mathbb{1}. \tag{2.58}$$

The meaning of the expression on the right hand side is as follows: In general,

$$T(z)_n A(z) = \oint_{(z)} \frac{\mathrm{d}w}{2\pi\mathrm{i}}(w-z)^n T(w)A(z) \tag{2.59}$$

for some product of vertex operators denoted collectively $A(z)$. An expression like $T(z)_{n_k}\dots T(z)_{n_1}\mathbb{1}$ has to be evaluated iteratively from right to left, that is first evaluating $A_1(z) = T(z)_{n_1}\mathbb{1}$, then $A_2(z) = T(z)_{n_2}A_1(z)$, and so on. If all of the n_i ($i = 1, \dots, k$) are negative, as is the case in (2.58), the result is

$$T(z)_{-n_k}\dots T(z)_{-n_1}\mathbb{1} = \prod_{i=1}^{k} \frac{1}{(n_i-1)!}\left(\partial_z^{(n_i-1)}T(z)\right). \tag{2.60}$$

It is readily checked, that $V(\Omega, z) = \mathbb{1}$ and $V(\omega, z) = T(z)$ are consistent with the definitions. There is actually a general theory behind this construction, the theory of so-called weak vertex operators and weak vertex operator algebras [27]. Within that framework, it is quite easily shown that the so defined vertex operators (2.58) do indeed satisfy all the properties required of a vertex operator. We shall not go into this here, but rather refer to [27]. For theories with Virasoro symmetry only, the vertex operator algebra we have just constructed is what one generically uses. For theories with extended (i.e. more than just Virasoro) symmetry, see section 2.7.

In the above construction, our strategy was to fix a state of lowest conformal weight. Such a state has to exist by the grading restrictions. This method, to construct a whole vector space from one such "primary" state, is also the key principle in the construction of modules for a vertex operator algebra. From now on, we call a state $\mathfrak{A}$ a *(Virasoro) primary state* if it is annihilated by all positive Virasoro modes:

$$L_n\mathfrak{A} = 0 \quad \text{for all } n > 0\,. \tag{2.61}$$

A module $\mathcal{W}^{(h)}$ is constructed by assuming a primary state $\mathfrak{A}_h$ of conformal weight h from which the whole module is generated

$$\mathcal{W}^{(h)} = \text{span}\left\{L_{-n_k}\dots L_{-n_1}\mathfrak{A}_h; k \geq 0 \text{ and } n_k \geq \cdots \geq n_1 \geq 1\right\}. \tag{2.62}$$

The obvious grading is implemented as above. Vertex operators are defined analogous to before by

$$V^{(\mathcal{W})}\left(L_{-n_k}\dots L_{-n_1}\Omega, z\right) = T(z)_{-n_k+1}\dots T(z)_{-n_1+1}\mathbb{1}\,, \tag{2.63}$$

$\mathbb{1}$ being the identity on $\mathcal{W}^{(h)}$ now. Also recall from section 2.3.1 that the energy momentum tensor and hence the Virasoro generators sensibly act on $\mathcal{W}^{(h)}$. The operator/state correspondence for states in $\mathcal{W}^{(h)}$ is now given through the operators $V^{(\mathcal{V},\mathcal{W}^h)}$, see equations (2.49) and (2.50) in the preceding section. A vertex operator $V^{(\mathcal{V},\mathcal{W})}(\mathfrak{A}, z)$, associated to a primary state $\mathfrak{A}$, will be called a *primary vertex operator* from now on.

What is still missing in our discussion is the question of reducibility of modules. We shall take it up in section 2.5. Before, we prefer to extend our notion of correlation functions, introducing chiral intertwining operators, chiral fields and descendant states along the way.

2.4 Correlation Functions

In section 2.2.1 we have defined a correlation function of states $\Psi_1, \Psi_2, \dots, \Psi_n$, $\Psi_i \in \mathcal{V}$ $(i = 1, \dots, n)$ as the vacuum expectation value of their associated vertex operators

$$\begin{aligned} G^{(n)}(\Psi_1, \Psi_2, \dots, \Psi_n | z_1, z_2, \dots, z_n) &= \\ &= \langle \Omega, V(\Psi_1, z_1)V(\Psi_2, z_2)\dots V(\Psi_n, z_n)\Omega\rangle\,. \end{aligned} \tag{2.64}$$

Now, our desire is to be able to also define correlation functions involving operators that are associated to states $\mathfrak{A}_{h_1}$, ..., $\mathfrak{A}_{h_k}$ that are elements of $\mathcal{V}$-modules

$\mathcal{W}^{(h_1)}, \ldots, \mathcal{W}^{(h_k)}$. Note that the vertex operators $V^{(\mathcal{W}^{(h_i)})}$ are completely useless for this purpose. Also, the operators $V^{(\mathcal{V},\mathcal{W}^{(h)})}$ can only serve as the rightmost insertions in a correlation function, since they map $\Omega \in \mathcal{V}$ to $\mathcal{W}^{(h)}$. But then, there is no operator at our disposal that brings us back to $\mathcal{V}$ and consequently, a correlation function can again not be defined. Hence, for the purpose of defining general correlators, we need to introduce a new kind of operator: Intertwiners. They are the subject of section 2.4.1. Afterwards, in 2.4.3, we shall define descendant states and show how correlation functions involving some of their corresponding operators can be obtained from correlators involving only primary operators. Section 2.4.4 then analyses the correlators of primary operators. In the following we shall frequently use the abbreviation $\langle \ldots \rangle$ instead of $\langle \Omega, \ldots \Omega \rangle$.

2.4.1 Intertwining Operators

Given three $\mathcal{V}$-modules $\mathcal{W}^{(i)}$, $\mathcal{W}^{(j)}$, $\mathcal{W}^{(k)}$, a *chiral intertwining operator* of type $\left({}_{k}{}^{i}{}_{j} \right)$ is a map

$$Y_k^i(\cdot, z) : \begin{cases} \mathcal{W}_h^{(j)} \to \left(\mathrm{Hom}\left(\mathcal{W}^{(i)}, \mathcal{W}^{(k)}\right)\right)[[z, z^{-1}]] \\ \mathfrak{A} \mapsto Y_k^i(\mathfrak{A}, z) = \sum_{n \in -h+\mathbb{Z}} V_n(\mathfrak{A})_k^i z^{-n-h} \end{cases}, \tag{2.65}$$

extended to $\mathcal{W}^{(j)}$ by linearity in its first component. As usual, L_{-1} is assumed to act as generator of translations and the truncation property

$$V_n(\mathfrak{A})_k^i \mathfrak{B} = 0 \quad \text{for all } \mathfrak{B} \in \mathcal{W}^{(i)} \tag{2.66}$$

is required. Note that vacuum and creation property generally do not make sense here. Furthermore, the following weak associativity with vertex operators $V^{(\mathcal{W}^{(k)})}$ is assumed ($\Psi \in \mathcal{V}$, $\mathfrak{A} \in \mathcal{W}^{(j)}$):

$$V^{(\mathcal{W}^{(k)})}(\Psi, z) Y_k^i(\mathfrak{A}, w) = Y_k^i\left(V^{(\mathcal{W}^{(j)})}(\Psi, z-w)\mathfrak{A}, w\right). \tag{2.67}$$

This establishes the analogues of all preceding commutator formulae again. Note that V, $V^{(\mathcal{W}^{(j)})}$ and $V^{(\mathcal{V},\mathcal{W}^{(j)})}$ are special intertwining operators[12] of type $\left({}_{0}{}^{0}{}_{0} \right)$, $\left({}_{j}{}^{j}{}_{0} \right)$ and $\left({}_{j}{}^{0}{}_{j} \right)$ respectively (a 0 entry indicating a mapping to $\mathcal{V}$).

Given an intertwining operator of type $\left({}_{k}{}^{i}{}_{j} \right)$, we define its *dual*, which is an intertwining operator of type $\left({}_{k}{}^{j}{}_{i} \right)$, via skew symmetry ($\mathfrak{A}_i \in \mathcal{W}^{(i)}$, $\mathfrak{A}_j \in \mathcal{W}^{(j)}$)

$$Y_k^j(\mathfrak{A}_i, z)\mathfrak{A}_j = e^{zL_{-1}} Y_k^i(\mathfrak{A}_j, -z)\mathfrak{A}_i. \tag{2.68}$$

[12] We are again dropping the “chiral” from now on.

Using this relation, one easily establishes the following commutativity property:

$$\begin{aligned} V^{(\mathcal{W}^{(k)})}(\Psi,z)Y_k^i(\mathfrak{A}_j,w)\mathfrak{A}_i &= V^{(\mathcal{W}^{(k)})}(\Psi,z)e^{wL_{-1}}Y_k^j(\mathfrak{A}_i,-w)\mathfrak{A}_j \\ &= e^{wL_{-1}}V^{(\mathcal{W}^{(k)})}(\Psi,z-w)Y_k^j(\mathfrak{A}_i,-w)\mathfrak{A}_j \\ &= e^{wL_{-1}}Y_k^j\left(V^{(\mathcal{W}^{(i)})}(\Psi,z)\mathfrak{A}_i,-w\right)\mathfrak{A}_j \\ &= Y_k^i(\mathfrak{A}_j,w)V^{(\mathcal{W}^{(i)})}(\Psi,z)\mathfrak{A}_i\,. \end{aligned} \tag{2.69}$$

With intertwining operators at our disposal, we can now finally say what we understand by a correlation function $G^{(n)}(\mathfrak{A}_1,\mathfrak{A}_2,\ldots,\mathfrak{A}_n|z_1,z_2,\ldots,z_n)$ involving arbitrary states $\mathfrak{A}_i \in \mathcal{W}^{(i)}$ (where some of the i may also be 0, with the understanding that $\mathcal{W}^{(0)} = \mathcal{V}$ as above). To this end, we define *chiral fields*

$$\mathfrak{A}(z) = \sum_{i,k} \alpha_{i,k} Y_k^i(\mathfrak{A},z) \tag{2.70}$$

($\alpha_{i,k} \in \mathbb{C}$). If the state $\mathfrak{A}$ is a primary state, the chiral field $\mathfrak{A}(z)$ is called *chiral primary field*. Now, define

$$G^{(n)}(\mathfrak{A}_1,\mathfrak{A}_2,\ldots,\mathfrak{A}_n|z_1,z_2,\ldots,z_n) = \langle \mathfrak{A}_1(z_1)\mathfrak{A}_2(z_2)\ldots\mathfrak{A}_n(z_n)\rangle \tag{2.71}$$

with the understanding that this is zero whenever there are two intertwiners that “do not match”, that is, whenever a product of intertwining operators of the type $Y_m^l(\mathfrak{A}_i,z_i)Y_k^j(\mathfrak{A}_{i+1},z_{i+1})$ with $\mathcal{W}^{(k)} \neq \mathcal{W}^{(l)}$ occurs, or, whenever $k \neq 0$ in $Y_k^i(\mathfrak{A}_1,z_1)$ or $i \neq 0$ in $Y_k^i(\mathfrak{A}_n,z_n)$. Implementing weak commutativity for arbitrary intertwiners and hence for the fields $\mathfrak{A}(z)$, one is lead to the braiding relations of Moore and Seiberg [81].

2.4.2 A Special OPE and some Transformation Formulae

The following developments are most crucial for the determination of correlation functions. Recall that for all operators Y_k^i that were introduced in the preceding sections, we have carefully paid attention to establish weak associativity with vertex operators $V^{(\mathcal{W}^{(j)})}$ for any module $\mathcal{W}^{(j)}$. This means in particular weak associativity with the energy momentum tensor acting on these modules. Therefore, by (2.17) and (2.18), the following holds for any (chiral)[13] field $\mathfrak{A}(z)$:

$$T(z)\mathfrak{A}(w) = \sum_{n \le N(\mathfrak{A})} [L_n\mathfrak{A}](w)(z-w)^{-n-2}\,. \tag{2.72}$$

[13] Once more we are dropping “chiral” from now on.

Here, the field $[L_n\mathfrak{A}](z)$ denotes the operator associated to the state $L_n\mathfrak{A}$.[14] For a primary state $\mathfrak{A}_h$, denoting the singular part of (2.72) only, this gives

$$T(z)\mathfrak{A}_h(w) \sim \frac{h\mathfrak{A}_h(w)}{(z-w)^2} + \frac{\partial_w\mathfrak{A}_h(w)}{z-w}\,. \tag{2.73}$$

The use of $\sim$ reminds us that we are only taking the singular part here. (2.72) is a special example of a type of relation that we call operator product expansion (OPE) (see section 2.6 for its general form).

Now, remember that along with weak associativity we have emphasized repeatedly the validity of commutator formulae for all kinds of operators that we have introduced. Indeed, (singular parts of) OPEs translate into commutator formulae and vice versa. As a consequence, recalling (2.27), all primary fields $\mathfrak{A}_h(z)$ obey[15]

$$[L_n, \mathfrak{A}_h(z)] = z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{A}_h(z) + h(n+1)z^n\mathfrak{A}_h(z)\,. \tag{2.74}$$

From this expression, one derives that the three basic finite global conformal transformations are realized on primary fields as

$$e^{\alpha L_{-1}}\mathfrak{A}_h(z)e^{-\alpha L_{-1}} = \mathfrak{A}_h(z+\alpha) \tag{2.75}$$
$$e^{\lambda L_0}\mathfrak{A}_h(z)e^{-\lambda L_0} = e^{\lambda h}\mathfrak{A}_h(e^\lambda z) \tag{2.76}$$
$$e^{\beta L_1}\mathfrak{A}_h(z)e^{-\beta L_1} = (1-\beta z)^{-2h}\mathfrak{A}_h\left(\frac{z}{1-\beta z}\right) \tag{2.77}$$

This can also be extended to arbitrary global conformal (i.e. $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$) transformations. Under

$$M : z \mapsto M(z) = \frac{az+b}{cz+d}\,, \quad \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})\,, \tag{2.78}$$

primary fields transform according to

$$\begin{aligned}\Delta(M)\mathfrak{A}(z)\Delta^{-1}(M) &= [M'(z)]^h\,\mathfrak{A}\,(M(z)) \\ &= (cz+d)^{-2h}\,\mathfrak{A}\,(M(z))\,.\end{aligned} \tag{2.79}$$

Here, the representation Δ is given by

$$\Delta(M) = \exp\left(\frac{b}{d}L_{-1}\right)d^{-2L_0}\exp\left(-\frac{c}{d}L_1\right)\,, \tag{2.80}$$

if $d \neq 0$, while for $d = 0$, it is

$$\Delta(M) = (ab)^{L_0}\exp(L_1)\exp(L_{-1})\left(\frac{a}{b}\right)^{L_0}\,. \tag{2.81}$$

These are basically all formulae that we will exploit in the following two sections in order to fix the general form that correlation functions must take.

[14] In terms of intertwiners, we would have $Y_k^i(L_n\mathfrak{A}, w)$. The notation (2.70) translates this into $[L_n\mathfrak{A}]$.
[15] This is also immedaite from (2.73).

2.4.3 Correlation Functions of Descendant Operators

From (2.72) we see that the field associated to a state $L_n\mathfrak{A}$ appears in the expansion of the field $\mathfrak{A}(z)$ with the energy momentum tensor $T(w)$. We can easily obtain it by a contour integration:

$$[L_n\mathfrak{A}](z) = \oint_{(z)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} (w-z)^{n+1} T(w)\mathfrak{A}(z)\,. \tag{2.82}$$

If the state $\mathfrak{A}$ is a primary state, we call all states of the form $L_{-n_k}\dots L_{-n_1}\mathfrak{A}$ *descendant states* and their associated fields *descendant fields*. Equation (2.82) shows us that we can always determine the descendant fields from the primary field that they originate from. This fact is particularly useful in correlation functions. Consider a correlator of n primary fields $\mathfrak{A}_1(z_1),\dots,\mathfrak{A}_n(z_n)$ and one descendant $[L_{-k}\mathfrak{B}](z)$ ($\mathfrak{B}$ is again primary). Then,

$$\begin{aligned} &\langle [L_{-k}\mathfrak{B}](z)\mathfrak{A}_1(z_1)\dots\mathfrak{A}_n(z_n)\rangle = \\ &\quad = \oint_{(z)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} (w-z)^{-k+1} \langle T(w)\mathfrak{B}(z)\mathfrak{A}_1(z_1)\dots\mathfrak{A}_n(z_n)\rangle\,. \end{aligned} \tag{2.83}$$

Reinterpreting the counterclockwise contour around (z) as a clockwise contour around the (z_i) $(i=1,\dots,n)$ and then using the OPE (2.73) at each of the n singularities in $(w-z_i)$, yields

$$\begin{aligned} &\langle [L_{-k}\mathfrak{B}](z)\mathfrak{A}_1(z_1)\dots\mathfrak{A}_n(z_n)\rangle = \\ &\quad = -\sum_{i=1}^{n} \oint_{(z_i)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} (w-z)^{-k+1} \langle T(w)\mathfrak{B}(z)\mathfrak{A}_1(z_1)\dots\mathfrak{A}_n(z_n)\rangle \\ &\quad = -\sum_{i=1}^{n} \left[\frac{1}{(z_i-z)^{k-1}}\partial_{z_i} + \frac{(-k+1)h_i}{(z_i-z)^k} \right] \langle \mathfrak{B}(z)\mathfrak{A}_1(z_1)\dots\mathfrak{A}_n(z_n)\rangle\,. \end{aligned} \tag{2.84}$$

This result generalizes immediately to correlators involving arbitrary descendants. Accordingly, any correlator that involves descendant fields can be obtained by applying appropriate differential operators

$$\mathcal{L}_{-k}(z) = -\sum_{i=1}^{n} \left[\frac{1}{(z_i-z)^{k-1}}\partial_{z_i} + \frac{(-k+1)h_i}{(z_i-z)^k} \right] \tag{2.85}$$

to a correlation function involving only primary fields. Therefore, the problem of fixing the form of arbitrary correlators reduces to determining the form of correlation functions of primary fields only.

2.4.4 Correlation Functions of Primary Operators

In section (2.2.1), we have introduced a natural pairing on $\mathcal{V}$. This gave us a linear functional $\langle\Lambda,\cdot\rangle \in \mathcal{V}^*$ for each $\Lambda \in \mathcal{V}$. Define the *adjoint vertex operator* $[V(\Psi,z)]^\dagger$, $\Psi \in \mathcal{V}$, acting on $\mathcal{V}^*$ as

$$\langle\Lambda,\cdot\rangle \mapsto \left\langle\Lambda,[V(\Psi,z)]^\dagger\,\cdot\right\rangle \tag{2.86}$$

by setting

$$\left\langle\Lambda,[V(\Psi,z)]^\dagger\,\Phi\right\rangle = \left\langle\Lambda,V\left(e^{zL_1}\left(-z^{-2}\right)^{L_0}\Psi,z^{-1}\right)\Phi\right\rangle \tag{2.87}$$

for all $\Phi \in \mathcal{V}$. With this action, the dual space $\mathcal{V}^*$ acquires the structure of a $\mathcal{V}$-module. We shall call it the *adjoint module*. Let us not go into the proof of this fact now, but rather refer to [28].

For a quasi-primary state ψ with conformal weight h, the definition (2.87) simplifies to

$$\left\langle\Lambda,[V(\psi,z)]^\dagger\,\Phi\right\rangle = \left(-\frac{1}{z^2}\right)^h\left\langle\Lambda,V\left(\psi,\frac{1}{z}\right)\Phi\right\rangle. \tag{2.88}$$

Expanding both sides in modes and comparing coefficients in z, one finds

$$\psi_n^\dagger = (-)^h\psi_{-n}\,. \tag{2.89}$$

In particular for the Virasoro modes, this means

$$L_n^\dagger = L_{-n}\,. \tag{2.90}$$

Assuming that the vacuum Ω is the primary state of lowest conformal weight in $\mathcal{V}$ (what is always the case in our considerations), one infers without difficulty that its adjoint $\langle\Omega,\cdot\rangle$ is also a primary state under the action of the operators $L_n^\dagger$ with conformal weight zero. Futhermore, if all states in $\mathcal{V}_1$ are annihilated by L_1 (what is the case in all our considerations), the adjoint vacuum is also translation invariant, i.e. it is annihilated by $L_{-1}^\dagger$.

With the information just assembled, we are now in position to derive the *global conformal Ward identities* for a correlation function $\langle\mathfrak{A}_1(z_1)\dots\mathfrak{A}_n(z_n)\rangle$ involving only primary fields $\mathfrak{A}_k(z_k)$. Let $i \in \{-1,0,1\}$. Since the adjoint vacuum is annihilated by $L_{-i}^\dagger$ and the vacuum by the L_i, we have

$$\begin{aligned} 0 &= \left\langle\Omega,L_{-i}^\dagger\mathfrak{A}_1(z_1)\dots\mathfrak{A}_n(z_n)\Omega\right\rangle \\ &= \langle\Omega,L_i\mathfrak{A}_1(z_1)\dots\mathfrak{A}_n(z_n)\Omega\rangle \\ &= \sum_{k=1}^n\langle\Omega,\mathfrak{A}_1(z_1)\dots[L_i,\mathfrak{A}_k(z_k)]\dots\mathfrak{A}_n(z_n)\Omega\rangle \\ &= \sum_{k=1}^n z_k^i\left[z_k\partial_{z_k}+(i+1)h_k\right]\langle\mathfrak{A}_1(z_1)\dots\mathfrak{A}_n(z_n)\rangle\,. \end{aligned} \tag{2.91}$$

In the last step, (2.74) was put to use. For $i = -1$, the Ward identities express translation invariance, for $i = 0$ scale invariance and for $i = 1$ special conformal invariance of the correlation functions. In the following, let us make use of the identities (2.91) and see how they restrict the correlation functions of primary fields.

One Point Function

From translation invariance, the one point function $\langle\mathfrak{A}_h\rangle$ must be a constant and scale invariance requires it to be zero unless $h = 0$. From uniqueness of the vacuum, we therefore have

$$\begin{aligned}\langle\mathfrak{A}_h\rangle &= 0\\ \langle\mathbb{1}\rangle &= 1\,.\end{aligned} \tag{2.92}$$

Two Point Function

The two point function $\langle\mathfrak{A}_h(z)\mathfrak{A}_{h'}(w)\rangle$ can only depend on $(z-w)$ by translation invariance. Scale invariance then implies that it is proportional to $(z-w)^{-(h+h')}$. Finally, special conformal invariance requires the conformal weights to coincide $h = h'$. We therefore get

$$\langle\mathfrak{A}_h(z)\mathfrak{A}_{h'}(w)\rangle = C\delta_{h-h'}(z-w)^{-2h} \tag{2.93}$$

where C is a constant related to the normalization of the fields. Let us make a check and use the transformation formula for primary fields (2.79) in the two point correlator. It yields

$$\langle\mathfrak{A}_h\left(M(z)\right)\mathfrak{B}_h\left(M(w)\right)\rangle = (cz+d)^{2h}(cw+d)^{2h}\,\langle\mathfrak{A}_h(z)\mathfrak{B}_h(w)\rangle\,, \tag{2.94}$$

if $M(z) = \frac{az+b}{cz+d}$. Indeed, the right hand side of (2.93) shows the same behaviour:

$$(M(z)-M(w))^{-2h} = (cz+d)^{2h}(cw+d)^{2h}(z-w)^{-2h}\,. \tag{2.95}$$

Three Point Function

Translation and scale invariance request that

$$\Big\langle\mathfrak{A}_{h_1}(z_1)\mathfrak{A}_{h_2}(z_2)\mathfrak{A}_{h_3}(z_3)\Big\rangle = C_{123}(z_1-z_2)^{h_{12}}(z_1-z_3)^{h_{13}}(z_2-z_3)^{h_{23}} \tag{2.96}$$

with $h_{12}+h_{13}+h_{23} = h_1+h_2+h_3$. Special conformal invariance then determines

$$h_{12} = h_1+h_2-h_3\,,\quad h_{13} = h_1+h_3-h_2\,,\quad h_{23} = h_2+h_3-h_1\,. \tag{2.97}$$

The *structure constant* C_{123} is a non-trivial piece of information. It cannot be determined by global conformal invariance, but more elaborate methods are needed in order to compute it. The conformal bootstrap is one possibility, but it is usually not feasible in practice. While other more specialized methods have been invented, there is in general no silver bullet that will reliably (i.e. always and "automatically") determine the structure constants.

Four Point Function

The existence of crossing ratios is the reason why beginning with the four point function, global conformal invariance alone does no longer suffice to fix correlators up to constants. Instead, arbitrary functions of the crossing ratios are left undetermined. Crossing ratios are combinations of the coordinates that stay invariant under arbitrary conformal transformations. In case of the four point function $\left\langle \mathfrak{A}_{h_1}(z_1)\mathfrak{A}_{h_2}(z_2)\mathfrak{A}_{h_3}(z_3)\mathfrak{A}_{h_4}(z_4)\right\rangle$, there is only one independent crossing ratio

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\,, \tag{2.98}$$

all other imaginable combinations are related to it by inversion, translation or a combination thereof. Translation and scale invariance therefore dictate that the four point functions looks like

$$\left\langle \mathfrak{A}_{h_1}(z_1)\mathfrak{A}_{h_2}(z_2)\mathfrak{A}_{h_3}(z_3)\mathfrak{A}_{h_4}(z_4)\right\rangle = \prod_{i<j=1}^{4} (z_i - z_j)^{-\mu_{ij}} F(z)\,, \tag{2.99}$$

where $\sum_{i<j}\mu_{ij} = h_1 + h_2 + h_3 + h_4$ and $F(z)$ remains undetermined. Special conformal invariance fixes

$$\mu_{ij} = h_i + h_j - \frac{1}{3}\Delta\,, \tag{2.100}$$

where $\Delta = h_1 + h_2 + h_3 + h_4$.

2.5 Virasoro Representation Theory II: Submodules and Reducibility

We have not yet discussed the important question of reducibility of Virasoro representations, that is reducibility of $\mathcal{V}$-modules. Given a module $\mathcal{W}$, a *submodule* $\mathcal{U}$ is a subset that is itself a module. A module $\mathcal{W}$ is said to be *reducible* if it contains a proper submodule $\mathcal{U}$. In order to obtain an irreducible module, one has to quotient out the submodule, i.e. form the space $\mathcal{W}/\mathcal{U}$. Clearly, this is again a module.

Recall that the modules we have studied above were all generated from one heighest weight state. This is the generic situation we are dealing with. Typically, a submodule $\mathcal{U}$ is also generated from one element only, which is a descendant in the module $\mathcal{W}$. For a descendant state $\mathfrak{A}^{(-)}$ to generate a submodule, it is enough that it is annihilated by all positive Virasoro modes. Such a state is called *singular vector*. Let us assume that it is present in our module. Quotiening out the submodule amounts to setting $\mathfrak{A}^{(-)}$ to zero. Since it is a descendant of the highest weight state $\mathfrak{A}$, $\mathfrak{A}^{(-)}$ is of the form $L_{-n_k} \dots L_{-n_1}\mathfrak{A}$. Setting this to zero, the corresponding field must be zero as well. By (2.84) this in turn means that any correlation function of primary fields involving also the primary $\mathfrak{A}(z)$ must satisfy a differential equation

$$\mathcal{L}_{-n_k}(z) \dots \mathcal{L}_{-n_1}(z) \langle \mathfrak{A}(z) \dots \rangle = 0. \tag{2.101}$$

This puts severe restrictions on such correlators. In the case of four point functions, these differential equations are usually powerful enough to determine the function of the crossing ratio that is left unknown from global conformal invariance alone. A field $\mathfrak{A}(z)$ that gives rise to differential equations in the manner just described is called a *degenerate field*.

2.6 The General Operator Product Expansion

For a vertex operator algebra we have already seen an example of an operator product expansion (OPE) in the form of weak associativity relations (2.17), (2.18). Indeed, weak associativity provides an exact and well defined OPE. Moreover, as weak associativity could be carried over to the product of vertex operators V and $V^{(\mathcal{W})}$ with arbitrary intertwining operators, these OPEs are also exact and well defined. Especially, the OPE of the energy momentum tensor $T(z)$ with an arbitrary conformal field is if that type; see equation (2.73).

Between arbitrary chiral fields $\mathcal{A}(z)$, $\mathcal{B}(w)$, we do not have such an exact notion of OPE, because we did not define weak associativity for them. Actually, a reasonable definition of weak associativity is not possible in this situation but instead, the fusing matrix of Moore and Seiberg [81] is an appropriate tool here. We do not go into this, but have decided to take the following route, following [5]: We define a formal weak operator product, that is we require

$$\mathcal{A}_h(z)\mathcal{A}_{h'}(w) = \sum_{h''} \sum_K C^{(K)}(h, h'|h'')(z-w)^{-h-h'+h''+|K|}\mathcal{A}^{(K)}_{h''}(w) \tag{2.102}$$

to hold in all correlators. K is a set of integers $K = \{k_1, \dots, k_n\}$ with varying number of entries n. It takes care of the occurence of descendant fields

$$\mathcal{A}^{(K)}_{h''}(w) = \left[L_{-k_n} \dots L_{-k_1}\mathcal{A}_{h''}\right](w) \tag{2.103}$$

on the right hand side of (2.102). The modulus of K is simply the sum of its entries $|K| = k_1 + \cdots + k_n$. The z, w dependence in (2.102) is fixed from global conformal covariance. Only the conformal weights h'' that are summed over on the right hand side cannot be fixed from global conformal covariance alone. This is just like the situation for the three point correlator, see (2.96). The determination of the set $\mathcal{N}_{(h,h')}(h'')$ such that $h'' \in \mathcal{N}_{(h,h')}(h'')$ for the purpose of (2.102) is known as the problem of finding the *fusion rules* of a CFT. The *operator product coefficients* $C^{(K)}(h,h'|h'')$ (which we also refer to as OPE coefficients) have a connection to the structure constants $C(h,h',h'')$, namely

$$C^{(\emptyset)}(h,h'|h'') = C(h,h',h'') \,. \tag{2.104}$$

Usually, we shall only write the most singular parts in the OPE, that is only the contributions of primary fields, and denote the descendant contributions with dots:

$$\mathcal{A}_h(z)\mathcal{A}(_{h'}w) = \sum_{h''} C(h,h',h'')(z-w)^{-h-h'+h''}\mathcal{A}_{h''}(w) + \dots \,. \tag{2.105}$$

The OPE coefficients that contribute here are really just the structure constants and we sometimes use these terms interchangeably.

Finally, let us comment on the OPE with a degenerate field. We have explained in section 2.4.3 that correlators involving a degenerate field are highly restricted by a differential equation that they have to satisfy. In the same way, OPEs with a degenerate field are highly restricted. One can usually extract them from an appropriate four point function. If a degenerate module has a singular vector $\mathfrak{A}^{(K)}$ with $|K| = k$, then there are at most k channels that contribute to the OPE of any conformal field with the associated degenerate field (i.e. the set $\mathcal{N}_{(h,h')}(h'')$ has at most k entries). For our purposes, cases with two or three contributing channels are of interest; see sections 3.5.2 and 3.5.3.

2.7 Conformal Field Theory with Lie Algebra Symmetry

Up to now we have learned how to realize conformal symmetry by building vertex operator algebras and associated modules from primary states. In the following sections, we are going to explain how the same programme can be carried out for a Lie symmetry. Starting from a Lie algebra whose generators are associated to corresponding currents, we will arrive at a construction that realizes an emerging affine Lie algebra and a related Virasoro algebra on a vertex operator algebra and its (affine) modules.

2.7.1 Vertex Operator Algebra build from Lie Algebra

We start with a set of chiral currents $\{J^a(z); a = 1, \dots, \dim(\mathfrak{g})\}$ whose charges

$$J_0^a = \oint_{(0)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} J^a(z) \tag{2.106}$$

are the generators of a Lie symmetry, that is they form a *Lie algebra* $\mathfrak{g}$:

$$\left[J_0^a, J_0^b\right] = f^{abc} J_0^c \,. \tag{2.107}$$

We take $\mathfrak{g}$ to be semi-simple, so we can assume the f^{abc} to be totally antisymmetric. From (2.106), the mode expansion of the currents must be

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J_n^a \,. \tag{2.108}$$

Now we consider the trivial representation of (2.107). It is a one dimensional vector space out of which we pick a generating element and call it Ω, the vacuum. We define

$$J_n^a \Omega = 0 \quad \forall\, n > 0 \tag{2.109}$$

(note that also $J_0^a\Omega = 0$, since Ω is in the trivial representation of $\mathfrak{g}$) and form the space

$$\mathcal{V} = \text{span}\left\{J_{-n_k}^{a_k} \ldots J_{-n_1}^{a_1}\Omega; k \geq 0 \text{ and } n_k \geq \cdots \geq n_1 \geq 1\right\}. \tag{2.110}$$

On this we define a $\mathbb{Z}$-grading by

$$\mathcal{V}_n = \text{span}\left\{J_{-n_k}^{a_k} \ldots J_{-n_1}^{a_1}\Omega; k \geq 0,\; n_k \geq \cdots \geq n_1 \geq 1,\; n_1 + \cdots + n_k = n\right\}, \tag{2.111}$$

that is to say that

$$J_{-n}^a : \mathcal{V}_h \mapsto \mathcal{V}_{h+n} \quad \text{and} \quad \mathcal{V}_h = 0 \quad \forall\, h < 0 \,. \tag{2.112}$$

Vertex operators associated to states in $\mathcal{V}$ are defined by

$$V\left(J_{-n_k}^{a_k} \ldots J_{-n_1}^{a_1}\Omega, z\right) = J^{a_k}(z)_{-n_k} \ldots J^{a_1}(z)_{-n_1} \mathbb{1}_{\mathcal{V}}. \tag{2.113}$$

The meaning of the expression on the right hand side is analogous to what we have seen before in (2.59):

$$J^a(z)_n A(z) = \oint_{(z)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} (w-z)^n J^a(w) A(z) \tag{2.114}$$

for some product of vertex operators denoted collectively $A(z)$. Expressions like $J^{a_k}(z)_{-n_k} \ldots J^{a_1}(z)_{-n_1}\mathbb{1}$ are again evaluated iteratively from right to left. Clearly, $V(\Omega, z) = \mathbb{1}$, that is the vacuum property holds, and

$$V(J_{-1}^a\Omega, z) = J^a(z) \,. \tag{2.115}$$

Interestingly, from the given input, we can establish a very nice result for the modes $J_n^a = V_n(J_{-1}^a\Omega)$. Using the commutator formula (2.37) (which is at our disposal with the present input) yields

$$\left[V_m(J_{-1}^a\Omega), V_n(J_{-1}^b\Omega)\right] = V_{m+n}\left(J_0^a J_{-1}^b\Omega\right) + mV_{m+n}\left(J_1^a J_{-1}^b\Omega\right). \tag{2.116}$$

In the second summand, $J_1^a J_{-1}^b \Omega$ must be proportional to the vacuum, so

$$J_1^a J_{-1}^b \Omega = k^{ab} \Omega \,. \tag{2.117}$$

Moreover, for $J_0^a J_{-1}^b \Omega$ from the first summand to lie in $\mathcal{V}$, we need to have

$$J_0^a J_{-1}^b \Omega = \tilde{f}^{abc} J_{-1}^c \Omega \tag{2.118}$$

with some constants $\tilde{f}^{abc}$. Yet, using (2.37) once more, we can argue that

$$f^{abc} J_0^c = \left[J_0^a, J_0^b\right] = \left[V_0(J_{-1}^a), V_0(J_{-1}^b)\right] = V_0(J_0^a J_{-1}^b \Omega) = \tilde{f}^{abc} J_0^c \,, \tag{2.119}$$

hence $\tilde{f}^{abc} = f^{abc}$. Therefore, we obtain the remarkable result that the modes J_n^a form the *affine Lie algebra* associated to $\mathfrak{g}$

$$\left[J_m^a, J_n^b\right] = f^{abc} J_{m+n}^c + m k^{ab} \delta_{(m+n)} \,. \tag{2.120}$$

Therefore, we call the currents $J^a(z)$ *affine currents* in the following. A further analysis of the central term k^{ab} reveals that it has to be of the form [82]

$$k^{ab} = k \delta^{ab} \,. \tag{2.121}$$

We denote this affine Lie algebra with the symbol $\hat{\mathfrak{g}}_k$. The constant k appearing here is called *(affine Lie algebra) level*. What we now like to show is that $\mathcal{V}$, together with the constructions introduced so far, also carries the structure of a vertex operator algebra. For this, we need to identify a conformal vector $\omega \in \mathcal{V}$ and establish that the modes L_n of the associated energy momentum tensor do obey the Virasoro algebra (2.42). Having this, it follows automatically that $L_1\omega = 0$ and $L_2\omega = \frac{c}{2}\Omega$, i.e. that the conformal vector is quasi-primary. Furthermore, it is then also immediate that L_{-1} acts as a generator of translations on the energy momentum tensor and that $L_0\omega = 2\omega$. Thus, one still needs to show that L_{-1} acts as a generator of translations on all of the vertex operators (2.113) and that L_0 implements the grading. Let us start to work out what we have just outlined.

As a conformal vector, we identify

$$\omega = \frac{1}{2(k + g^\vee)} J_{-1}^a J_{-1}^a \Omega \,. \tag{2.122}$$

Here and in the following we always sum over doubly occuring Lie algebra indices. k is the level of $\hat{\mathfrak{g}}_k$ (2.121) and $g^\vee$ is the *dual Coxeter number* of $\mathfrak{g}$. It is defined as

$$-f^{abc} f^{abd} = 2g^\vee \delta^{cd} \tag{2.123}$$

(if the length of the highest root is normalized to be two). Let us now derive an expression for the modes L_n. According to (2.113), we have

$$T(z) = V(\omega, z) = \frac{1}{2(k + g^\vee)} \sum_{n\in\mathbb{Z}} z^{-n-2} \left(\sum_{m\in\mathbb{Z}} J_n^a J_{-n+m}^a \right) . \tag{2.124}$$

Note that there is no ordering ambiguity in the term

$$L_n = \frac{1}{2(k+g^\vee)} \sum_{m\in\mathbb{Z}} J^a_n J^a_{-n+m} \tag{2.125}$$

as long as $m \neq 0$. For $m = 0$, the definition of $J^a_n J^a_{-n}$ is that the negative modes are to stand on the left. With (2.125) and (2.120) one readily checks that the modes L_n satisfy the Virasoro algebra (2.42) with central charge

$$c = \frac{k \dim(\mathfrak{g})}{k+g^\vee}\,. \tag{2.126}$$

This implies $L_1\omega = 0$ and $L_2\omega = \frac{c}{2}\Omega$ byÊ(2.37). Moreover, the commutator between L_m and J^a_n comes out to be

$$[L_m, J^a_n] = -nJ^a_{m+n}\,. \tag{2.127}$$

Therefore, L_0 is really the gradation operator. (2.127) can also be used to check $L_1\omega = 0$ and $L_2\omega = \frac{c}{2}\Omega$ directly. One also straightforwardly confirms that the vacuum Ω is a primary state of weight zero which is invariant under the action of L_{-1}. Finally, to show that L_{-1} is really the generator of translations on $\mathcal{V}$, we compute

$$\left[L_{-1}, V(J^a_{-k}\Omega, z)\right] = \sum_{n\in\mathbb{Z}} z^{-n-k}\left[L_{-1}, V_n(J^a_{-k}\Omega)\right] \tag{2.128}$$

and note that by (2.113)

$$V_n(J^a_{-k}\Omega) = (-)^{k-1}\binom{n+k-1}{k-1} J^a_n\,. \tag{2.129}$$

Plugging this in,

$$\left[L_{-1}, V(J^a_{-k}\Omega, z)\right] = \frac{\mathrm{d}}{\mathrm{d}z} V(J^a_{-k}\Omega, z) \tag{2.130}$$

follows. This extends to arbitrary states $J^{a_k}_{-n_k}\ldots J^{a_1}_{-n_1}\Omega \in \mathcal{V}$.

Let us summarize: We have shown that the space $\mathcal{V}$ (2.110) generated from the current modes J^a_n is a vertex operator algebra with energy momentum tensor given by (2.124) and central charge (2.126). In the literature this goes under the name *Sugawara construction*. Additionally, we have seen that the current modes J^a_n form an affine Lie algebra (also called *current algebra*) (2.120) that is entangled with the Virasoro algebra via (2.127).

2.7.2 Affine Modules

Next, we wish to construct modules for the vertex operator algebra just defined. This is accomplished very naturally, in the same way as before in section 2.3.2. Pick a representation of the Lie algebra $\mathfrak{g}$ labelled j, say (for a finite dimensional representation of a semi-simple Lie algebra j would label a highest weight vector; since we are interested in infinite dimensional representations without any highest or lowest weights, we do not specify the exact nature of j for the moment; later, when turning to the H_3^+ model, there will be an exact and well-defined meaning, of course). The representation space be spanned by the basis elements $\mathfrak{A}_j(u)$. Here, u is an index that labelles the different basis vectors, we might call it an "isospin coordinate"; again, there will a precise meaning when treating the H_3^+ model. Then, we set

$$J_n^a \mathfrak{A}_j(u) = 0 \quad \forall\, n > 0 \tag{2.131}$$

and build the module

$$\mathcal{W}^{(j)} = \operatorname{span}\left\{ J_{-n_k}^{a_k} \dots J_{-n_1}^{a_1} \mathfrak{A}_j(u); k \geq 0 \text{ and } n_k \geq \cdots \geq n_1 \geq 1 \right\}, \tag{2.132}$$

which is graded in the by now obvious way and comes with the vertex operators

$$V\left(J_{-n_k}^{a_k} \dots J_{-n_1}^{a_1} \mathfrak{A}_j(u), z \right) = J^{a_k}(z)_{-n_k} \dots J^{a_1}(z)_{-n_1} \mathbb{1}_{\mathcal{W}^{(j)}} . \tag{2.133}$$

Note that this is not only a module for the Virasoro algebra, but even a module for the affine Lie algebra (2.120). This is why we shall also call it an *affine module.* Condition (2.131) is referred to as $\mathfrak{A}_j(u)$ being an *affine highest weight state* or an *affine primary state* (in contrast to a Virasoro highest weight state, or Virasoro primary state (2.61), respectively). From the definition of Virasoro modes L_n (2.125), every affine primary is also a Virasoro primary, i.e.

$$L_n \mathfrak{A}_j(u) = 0 \quad \forall\, n > 0 . \tag{2.134}$$

Moreover, since the $\mathfrak{A}_j(u)$ form a representation of the zero mode Lie algebra spanned by the $\{J_0^a\}$, J_0^a acts on $\mathfrak{A}_j(u)$ by the appropriate operator in the representation j

$$J_0^a \mathfrak{A}_j(u) = \mathcal{D}_j^a \mathfrak{A}_j(u) . \tag{2.135}$$

(This is sometimes referred to as the *zero mode representation*). We therefore obtain

$$L_0 \mathfrak{A}_j(u) = \frac{\mathcal{D}_j^a \mathcal{D}_j^a}{2(k + g^\vee)} \mathfrak{A}_j(u) , \tag{2.136}$$

what is to say that $\mathfrak{A}_j(u)$ has conformal weight

$$h = h(j) = \frac{\mathcal{D}_j^a \mathcal{D}_j^a}{2(k + g^\vee)} . \tag{2.137}$$

Therein, $\mathcal{D}^a_j\mathcal{D}^a_j$ is the quadratic Casimir operator for $\mathfrak{g}$ in the representation j. It is thus independent of the isospin coordinate u.

The discussion of reducibility of the modules that we have defined here is taken up in the next section.

2.7.3 Further Restrictions on Correlation Functions

Besides the usual restrictions on correlators from global conformal Ward identities (see section 2.4.4), there are three more concepts available in the case of Lie symmetry[16]: Affine Ward identities, affine singular vectors (which are the current algebra analogues of the Virasoro singular vectors) and the famous Knizhnik-Zamolodchikov equations, which basically implement the Sugawara construction (2.125) on the level of correlation functions.

Affine Ward Identities

Since $J^a_0\Omega = 0$ and $(J^a_0)^\dagger = J^a_0$, we can derive so-called *affine Ward identities* (albeit “Lie algebra Ward identities” were a more appropriate nomination, because only the zero modes of the currents, i.e. the Lie algebra generators, play a rôle here). This works in just the same manner as we have demonstrated in section 2.4.4 for global conformal transformations:

$$\begin{aligned} 0 &= \left\langle \Omega, (J^a_0)^\dagger \mathfrak{A}_{j_1}(z_1)\ldots\mathfrak{A}_{j_n}(z_n)\Omega \right\rangle \\ &= \sum_{k=1}^{n} \left\langle \Omega, \mathfrak{A}_{j_1}(z_1)\ldots\left[J^a_0, \mathfrak{A}_{j_k}(z_k)\right]\ldots\mathfrak{A}_{j_n}(z_n)\Omega \right\rangle \\ &= \sum_{k=1}^{n} \mathcal{D}^a_{j_k} \left\langle \mathfrak{A}_{j_1}(z_1)\ldots\mathfrak{A}_{j_n}(z_n) \right\rangle . \end{aligned} \tag{2.138}$$

The Lie algebra index runs through $a = 1, \ldots, \dim(\mathfrak{g})$. Hence, these are $\dim(\mathfrak{g})$ many constraints.

Affine Singular Vectors

If an affine module $\mathcal{W}^{(j)}$ contains a descendant state $\mathfrak{B}^{(-)}_j$ that is annihilated by all positive current modes J^a_n, this descendant generates a proper affine submodule. The module $\mathcal{W}^{(j)}$ is then reducible and the state $\mathfrak{B}^{(-)}_j$ is called *affine singular vector*. In order to make the module irreducible, the proper submodule is quotient

[16]Virasoro singular vectors (section 2.5) are not available here, as they exist only for the minimal models, which have central charges $c < 1$. They are not contained in theories with Lie symmetry, because for them, $c \geq 1$.

out, hence the singular vector $\mathfrak{B}_j^{(-)}$ is set to zero. The corresponding descendant field

$$\left[J^{a_k}_{-n_k}\dots J^{a_1}_{-n_1}\mathfrak{B}_j\right](z) = \oint_{(z)}\frac{\mathrm{d}w_k}{2\pi\mathrm{i}}(w_k - z)^{-n_k}J^{a_k}(w_k)\dots \\ \dots\oint_{(z)}\frac{\mathrm{d}w_1}{2\pi\mathrm{i}}(w_1 - z)^{-n_1}J^{a_1}(w_1)\mathfrak{B}_j(z) \tag{2.139}$$

must then be set to zero as well. By arguments analogous to those in sections 2.4.3 and 2.5, any correlation function involving the primary field $\mathfrak{B}_j(z)$ together with primaries $\mathfrak{A}_{j_1}(z_1), \dots \mathfrak{A}_{j_n}(z_n)$ must satisfy an equation

$$\mathcal{J}^{a_k}_{-n_k}(z)\dots\mathcal{J}^{a_1}_{-n_1}(z)\left\langle\mathfrak{B}_j(z)\mathfrak{A}_{j_1}(z_1)\dots\mathfrak{A}_{j_n}(z_n)\right\rangle = 0 \tag{2.140}$$

with operators

$$\mathcal{J}^{a_i}_{-n_i}(z) = \sum_{\ell=1}^{n}\left[\frac{\mathcal{D}^{a_i}_{j_\ell}}{(z_\ell - z)^{n_i}}\right], \tag{2.141}$$

where the operator $\mathcal{D}^{a_i}_{j_\ell}$ acts on the field ℓ-th field $\mathfrak{A}_{j_\ell}(z_\ell)$ and is therefore taken in the representation labelled j_ℓ. A field $\mathfrak{B}_j(z)$ that gives rise to differential equations in this manner is called an *(affine) degenerate field*.

Knizhnik-Zamolodchikov Equations

Due to the Sugawara construction (2.125), from any field $\mathfrak{A}_{j_\ell}(z)$ we can build the degenerate field

$$\left[\left(L_{-1} - \frac{1}{2(k+g^\vee)}\sum_{n\in\mathbb{Z}}J^a_nJ^a_{-n-1}\right)\mathfrak{A}_{j_\ell}\right](z) = 0\,. \tag{2.142}$$

Taking $\mathfrak{A}_{j_\ell}(z)$ to be an affine primary yields

$$\left[\left(L_{-1} - \frac{1}{k+g^\vee}J^a_{-1}\mathcal{D}^a_{j_\ell}\right)\mathfrak{A}_{j_\ell}\right](z) = 0\,. \tag{2.143}$$

Inserting this into a correlator with $(n-1)$ further primaries, recalling that $[L_{-1}\mathfrak{A}](z) = \partial_z\mathfrak{A}(z)$ and using a simple contour argument once again, one ends up with

$$\left[\partial_{z_\ell} + \frac{1}{k+g^\vee}\sum_{k\neq\ell}^{n}\frac{\mathcal{D}^a_{j_\ell}\otimes\mathcal{D}^a_{j_k}}{(z_\ell - z_k)}\right]\left\langle\mathfrak{A}_{j_1}(z_1)\dots\mathfrak{A}_{j_n}(z_n)\right\rangle = 0 \tag{2.144}$$

($\ell \in \{1,\dots,n\}$). These are the celebrated Knizhnik-Zamolodchikov equations. They will be of great importance for our work in chapters 5, 6 and 8, where we use them to determine various boundary two point functions.

2.8 Modular Invariance and Conformal Fields: Gluing Together Chiral Halves

Up to this point, we have only dealt with (chiral) vertex operators and (chiral) fields. In order to treat a non-chiral theory, we have to add an additional antichiral sector (which is completeley analogous to the chiral sector that we have discussed so far) and entangle it with its chiral counterpart. This is actually important for the theory to be consistent at one loop (and higher loops; we will understand what is meant by loops in a minute). For this purpose, we define a *conformal field* $\Theta(z,\bar{z})$ to the (formal) sum of (formal) products of chiral and antichiral fields:

$$\Theta(z,\bar{z}) = \sum_{k,\ell} M_{k\ell}\mathfrak{A}_k(z)\mathfrak{B}_\ell(\bar{z})\,, \tag{2.145}$$

with constants $M_{k\ell} \in \mathbb{C}$. The indices k, ℓ are representation labels for the underlying symmetry algebra (i.e. Virasoro or affine Lie). For the purpose of studying correlation functions, it is enough to consider one contribution $\mathfrak{A}_k(z)\mathfrak{B}_\ell(\bar{z})$ only, by linearity. Correlators of such products factorize into a product of a chiral and an antichiral correlator, so there is nothing new here. Note however that the OPE for fields $\Theta(z,\bar{z})$ can become more entangled[17].

Not all "gluings" (2.145) of chiral and antichiral parts lead to a consistent CFT. So far, we have always considered a CFT on the sphere. But studying it on higher genus Riemann surfaces (genus g corresponding to a g-loop closed string amplitude), a restriction on the coefficients $M_{k\ell}$ is inevitable. In order to state this, consider the *chiral torus amplitudes* (or *characters*)

$$\chi_k = \left\langle \mathfrak{A}_k, \mathrm{e}^{2\pi\mathrm{i}\tau(L_0-\frac{c}{24})}\mathfrak{A}_k \right\rangle \tag{2.146}$$

(we had argued for the existence of a scalar product on the modules $\mathcal{W}^{(k)}$). τ is the modulus of the torus. An analogous definition is given to the antichiral characters. Since τ and $\tau' = \frac{a\tau+b}{c\tau+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$ correspond to equivalent tori, the *full torus amplitude* (or *partition function*)

$$\mathcal{Z} = \sum_{k,\ell} M_{k\ell}\chi_k\bar{\chi}_\ell \tag{2.147}$$

must be invariant under $\tau \mapsto \tau'$. This restricts the coefficients $M_{k\ell}$ severly. Usually, one considers diagonal theories: $M_{k\ell} \propto \delta_{(k-\ell)}$. This is also what we restrict to in the following.

[17] In the H_3^+ model, chiral factorization for the correlation functions was shown to hold [66]. Yet, the OPE does not factorize, due to the internal isospin space; see (3.6) and (3.30).

3 Review of the Bulk H_3^+ CFT

The bulk H_3^+ model has been fairly well studied, see [83, 66, 84] and [85]. Here, we essentially fix our notation (which follows very closely [72]) and summarize those facts and formulae that will be indispensable for our work. They can basically all be found in [83, 66] and [72]. In the last subsection (section 3.5.3), we give the explicit expressions for the $b^{-2}/2$-OPE coefficients that are needed for our calculations in chapters 6 and 8 and that we have found in [67].

3.1 Action and Spectrum

The space H_3^+ is euclidean three dimensional Anti-de-Sitter space (euclidean AdS_3). Thinking of AdS_3 as the group manifold of $SL(2,\mathbb{R})$, i.e. in terms of matrices

$$\begin{pmatrix} X^0 + X^1 & X^2 - X^3 \\ X^2 + X^3 & X^0 - X^1 \end{pmatrix} \quad \text{with} \quad \left(X^0\right)^2 - \left(X^1\right)^2 - \left(X^2\right)^2 + \left(X^3\right)^2 = 1\,, \tag{3.1}$$

one sees that under euclidean rotation $X^3 \mapsto \mathrm{i}X^3$, we end up with the space H_3 of matrices that are parametrized by a three dimensional hyperboloid

$$\begin{pmatrix} X^0 + X^1 & X^2 - \mathrm{i}X^3 \\ X^2 + \mathrm{i}X^3 & X^0 - X^1 \end{pmatrix} \quad \text{with} \quad \left(X^0\right)^2 - \left(X^1\right)^2 - \left(X^2\right)^2 - \left(X^3\right)^2 = 1\,. \tag{3.2}$$

Concentrating on the upper shell of that hyperboloid only, gives the space H_3^+

$$H_3^+ = \left\{h \in SL(2,\mathbb{C}); h = h^\dagger\,, \mathrm{tr}(h) > 0\right\}. \tag{3.3}$$

Obviously, the group $SL(2,\mathbb{C})$ acts on this space via

$$g : \begin{cases} H_3^+ \to H_3^+ \\ h \mapsto ghg^\dagger \end{cases} \qquad \text{for } g \in SL(2,\mathbb{C})\,. \tag{3.4}$$

This is to say that H_3^+ is a homogeneous space with respect to $SL(2,\mathbb{C})$. It has been analysed in [86] how the square integrable functions (with respect to the existent Haar measure $\mathrm{d}h$) on this space decompose under the $SL(2,\mathbb{C})$ action with the result (see also [87])

$$L^2(H_3^+|\mathrm{d}h) = \int_{C^+}^{\oplus} \mathrm{d}j\,[\mathrm{Im}(j)]^2\,\mathcal{R}^{(0,j)}\,. \tag{3.5}$$

The direct integral is taken over $C^+ = -\frac{1}{2} + \mathrm{i}\mathbb{R}_{\geq 0}$ and the spaces $\mathcal{R}^{(0,j)}$ are representation spaces of the *unitary principal continuous series representations* of $\mathrm{SL}(2,\mathbb{C})$. (The complete list of theses spaces is parametrized by $\mathcal{R}^{(k,j)}$ for integer k and $j \in C^+$. They are certain L^2 spaces over the Riemann sphere; see appendix A for a proper description.) A basis of generalized plane waves for functions in $L^2(\mathrm{H}_3^+|\mathrm{d}h)$ according to the decomposition (3.5) is provided by [86]

$$\mathfrak{A}_j(u|h) = \frac{2j+1}{\pi}\left(\begin{pmatrix} 1 & u \end{pmatrix} \cdot h \cdot \begin{pmatrix} 1 \\ \bar{u} \end{pmatrix}\right)^{2j} \tag{3.6}$$

with $j \in C^+$ and $u \in \hat{\mathbb{C}}$, i.e. by vectors $\mathfrak{A}_j(u|\cdot)$ whose components are labelled by elements $h \in \mathrm{H}_3^+$. These results suggest that the spectrum of highest weight states in the H_3^+ model coincides with $\left\{\mathfrak{A}_j(u|\cdot); j \in C^+, u \in \hat{\mathbb{C}}\right\}$ and that the states $\mathfrak{A}_j(u|\cdot)$ should be understood in a distributional sense [66]. By this it is meant that $\mathfrak{A}_j(u|\cdot)$ becomes meaningful only if integrated against a test function, which is chosen from the Schwartz space $\mathfrak{S}(\mathrm{H}_3^+)$, the dense subspace of $L^2(\mathrm{H}_3^+|\mathrm{d}h)$ of rapidly decreasing functions.

Formulating a bosonic string theory in the H_3^+ background, i.e. writing down the corresponding nonlinear sigma model, one needs to incorporate a nontrivial B-field (NSNS two form)[1] [88]. If one chooses the following parametrization of the space H_3^+

$$h = \begin{pmatrix} e^{\phi} & e^{\phi}\bar{\gamma} \\ e^{\phi}\gamma & e^{\phi}\gamma\bar{\gamma} + e^{-\phi} \end{pmatrix} \in \mathrm{H}_3^+ \tag{3.7}$$

with real ϕ and complex γ, the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}\phi^2 + e^{2\phi}\mathrm{d}\gamma\mathrm{d}\bar{\gamma}\,. \tag{3.8}$$

In these coordinates, a possible choice of B-field[2] reads [72]

$$B = e^{2\phi}\mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma}\,. \tag{3.9}$$

The resulting action

$$S = \frac{k}{\pi}\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\left(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\gamma\bar{\partial}\bar{\gamma}\right) \tag{3.10}$$

has an $\mathfrak{sl}(2,\mathbb{C})$ symmetry and according to section 2.7, the associated currents will build an $\hat{\mathfrak{sl}}(2,\mathbb{C})_k$ affine Lie algebra (2.120) from which the spectrum of the theory is generated. The action (3.10) has also been derived in [89] from the viewpoint of gauged WZNW models (viewing H_3^+ as the left coset $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$). The author of [89] carried out a path integral quantization with that action and

[1] The dilaton may remain constant.
[2] Only the field strength $H = \mathrm{d}B$ is unambiguous.

thus derived the partition function of the model. Reading off the spectrum of the theory confirms the expectations from the previous and the present paragraph: The space of states is an affine module (as in section 2.7.2) generated from the representations $\mathcal{R}^{(0,j)}$ by $\hat{\mathfrak{sl}}(2,\mathbb{C})_k$ currents $J^a(z) = \sum_n z^{-n-1} J^a_n$, $a \in \{+,-,3\}$ (plus a corresponding antichiral sector). The partition function of [89] can be obtained from the diagonal combination of affine characters (see section 2.8). For a discussion of its modular invariance see [90].

3.2 The Vertex Operator Algebra and its Modules

Following section 2.7.1, we build a vertex operator algebra from the vacuum Ω. Besides Ω, it contains three more Virasoro primaries (that are however not affine primaries), namely the states $J^a_{-1}\Omega$ ($a \in \{+,-,3\}$) with conformal weight 1. Moreover, with the conformal vector there is also a (Virasoro) quasi-primary with conformal weight 2. The associated vertex operator are of course the currents $J^a(z)$ and the energy momentum tensor $T(z)$. The vertex operator algebra thus encodes the symmetries of the theory. The Sugawara construction described in section 2.7.1 expresses the energy momentum tensor in terms of the currents and thereby establishes the usual formula for the central charge:

$$c = \frac{3k}{k+2}\,, \tag{3.11}$$

since $g^\vee = 2$ for the $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra.

The construction of modules for that vertex operator algebra works as described in section 2.7.2. From what we have discussed in the previous section, we need to provide a module corresponding to each $\mathfrak{sl}(2,\mathbb{C})$ representation of the principal continuous series with $j \in -\frac{1}{2} + \mathrm{i}\mathbb{R}_{\geq 0}$ (plus antichiral counterparts). Denote such a module $\mathcal{W}^{(j)}$. It is build from the state $\mathfrak{A}_j(u|\cdot)$ that we have introduced above and which transforms in the $\mathfrak{sl}(2,\mathbb{C})$ representation $\mathcal{R}^{(0,j)}$. We denote its integrated (against a test function) version by $\mathfrak{A}_j(u)$ for simplicity (we should actually denote the test function as an index). The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ acts on it by the differential operators

$$\mathcal{D}^+_j(u) = -u^2\partial_u + 2ju, \qquad \mathcal{D}^-_j(u) = \partial_u, \qquad \mathcal{D}^3_j(u) = u\partial_u - j\,, \tag{3.12}$$

i.e. we have

$$J^a_0 \mathfrak{A}_j(u) = \mathcal{D}^a_j(u)\mathfrak{A}_j(u)\,. \tag{3.13}$$

Analogous formulae hold for the antichiral sector. Primary fields corresponding to states $\mathfrak{A}_j(u) \otimes \bar{\mathfrak{A}}_j(\bar{u})$ will be denoted $\Theta_j(u,\bar{u}|z,\bar{z})$. However, from now on we will always suppress the barred variables. Their OPE with a chiral current reads

$$J^a(z)\Theta_j(u|w) \sim \frac{\mathcal{D}^a_j(u)\Theta_j(u|w)}{(z-w)} \tag{3.14}$$

and the Sugawara construction induces a relation between conformal weight h and 'spin'-label j (see (2.137)):

$$h(j) = \frac{j(j+1)}{k+2}\,, \tag{3.15}$$

since the Casimir operator in the representation $\mathcal{R}^{(0,j)}$ is $\mathcal{D}_j^a(u)\mathcal{D}_j^a(u) = j(j+1)$. Note that for $j = -\frac{1}{2} + \mathrm{i}s$ ($s \in \mathbb{R}$) the Casimir is always negative: $j(j+1) = -\frac{1}{4} - s^2$. As the conformal weight $h(j)$ must be bounded from below by the grading restrictions, the model does only make sense if we take the level to be such that $k+2 < 0$ and thus $h(j) > 0$. Consequently, the level must be negative and we shall replace it by its negative $k \mapsto (-k)$ for convenience. With this altered convention, the formulae for conformal weight and central charge become

$$\begin{aligned} h(j) &= -\frac{j(j+1)}{k-2} \equiv -b^2 j(j+1) \\ c &= \frac{3k}{k-2} \end{aligned} \tag{3.16}$$

and we need to take k in the range $k \in (2, \infty)$. In the equation for $h(j)$ we have implicitely defined the parameter b for later use.

Note that there is a *reflection symmetry*, namely $h(-j-1) = h(j)$. This leads one to identify the representations with labels j and $-j-1$ and gives rise to a relation between primary fields $\Theta_j(u|z)$ and $\Theta_{-j-1}(u|z)$:

$$\Theta_j(u|z) = -R(-j-1)\frac{2j+1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2u'\,|u-u'|^{4j}\Theta_{-j-1}(u'|z)\,, \tag{3.17}$$

where the *reflection amplitude* $R(j)$ is given by[3]

$$R(j) = -\nu_b^{2j+1}\frac{\Gamma(1+b^2(2j+1))}{\Gamma(1-b^2(2j+1))}\,. \tag{3.18}$$

3.3 Correlation Functions

In the following we assemble and comment on the expressions for the two and three point functions. Crucially, the latter involve the structure constants of the theory which also determine the OPE coefficients. They were derived in [83].

[3] The following expression can be derived from knowledge of the three point function; see section 3.3.3.

3.3.1 A Remark on the Ward Identities

The structure of the correlators in isospin and worldsheet coordinates is entirely fixed from affine and global conformal Ward identities respectively. Looking back at the affine Ward identities (2.138) and the definition of the operators that realize the $\mathfrak{sl}(2,\mathbb{C})$ action (3.12), one recognizes that the affine Ward identities take just the same form (up to signs) as the global conformal Ward identities (2.91) in this case (of course with the difference that the former act on the isospin variable u, but the latter on the worldsheet coordinate z). This is not at all surprising, since both algebras that generate the Ward identities (i.e. the underlying Lie algebra and the global conformal algebra) are $\mathfrak{sl}(2,\mathbb{C})$ for the H_3^+ model. Therefore, correlation functions must look almost the same (up to some signs again) in the variables u and z.

3.3.2 Two Point Function

Due to the reflection symmetry $h(j) = h(-j-1)$, the two point function is a sum of two terms. Therefore, we have to include one a priori unknown coefficient that cannot be fixed by choosing the normalization of the fields:

$$\begin{aligned}\left\langle \Theta_{j_2}(u_2|z_2)\Theta_{j_1}(u_1|z_1)\right\rangle = |z_2-z_1|^{-4h(j_1)}\cdot \\ \cdot\left\{-\left(\frac{\pi}{2j_1+1}\right)^2\delta^{(2)}(u_2-u_1)\delta(j_1+j_2+1)+\right. \\ \left. +B^{-1}(j_1)|u_2-u_1|^{4j_1}\delta(j_1-j_2)\right\}.\end{aligned} \tag{3.19}$$

The normalization that we chose in the first term may seem a bit awkward at first sight. Yet, it is convenient for working in the boundary H_3^+ CFT which is our ultimate goal. See also the remarks in [66] and [72]. Moreover, as the standard reference on the boundary theory [72] works with the same normalization as we do here, it is ensured that we can later conveniently compare our expressions to theirs.

The z- and u-dependencies in the two point function (3.19) are fixed from the global conformal and affine Ward identities as usual. They do indeed allow for both, the $\propto \delta^{(2)}(u_2-u_1)$ as well as the $\propto |u_2-u_1|^{4j_1}$ term. Note however, that the form given above is the only one consistent with the reflection relation (3.17) of the fields. Using it in the two point function, one can determine the coefficient $B(j)$ in terms of the reflection amplitude. It must be

$$B(j) = \frac{2j+1}{\pi}R(j)\,. \tag{3.20}$$

3.3.3 Three Point Function

Global conformal invariance and the affine Ward identities fix the z- and u-dependence as usual and determine the three point function up to the structure constants:

$$\begin{aligned}\left\langle \Theta_{j_3}(u_3|z_3)\Theta_{j_2}(u_2|z_2)\Theta_{j_1}(u_1|z_1)\right\rangle &= C(j_3,j_2,j_1)\cdot \\ &\cdot|z_3-z_2|^{-2h_{32}}|z_3-z_1|^{-2h_{31}}|z_2-z_1|^{-2h_{21}}\cdot \\ &\cdot|u_3-u_2|^{2j_{32}}|u_3-u_1|^{2j_{31}}|u_2-u_1|^{2j_{21}}\,.\end{aligned} \tag{3.21}$$

Here, $h_{12} = h_1 + h_2 - h_3$, etc. and $j_{12} = j_1 + j_2 - j_3$, etc. The crucial part of information are the structure constants $C(j_3, j_2, j_1)$. They have been derived in [83] and are given by

$$C(j_3,j_2,j_1) = \frac{G(j_1+j_2+j_3+1)G(j_1+j_2-j_3)G(j_1+j_3-j_2)G(j_2+j_3-j_1)}{\nu_b^{j_1+j_2+j_3+1}G_0G(2j_1)G(2j_2)G(2j_3)}\,. \tag{3.22}$$

The parameters ν_b and G_0 are left arbitrary from the original derivation. Still, they can be fixed by taking one field in the three point correlator to be the identity and matching the resulting expression with the two point function. (Taking one field to be the identity is to be understood as a limit here that one has to take carefully in order to recover the $\propto \delta^{(2)}(u_2 - u_1)$ term; see [66]). With our desired normalization of the two point function (3.19), the resulting expressions read

$$\nu_b = \pi\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\,, \tag{3.23}$$

$$G_0 = 2b^{-4}G(-1)\,. \tag{3.24}$$

Γ is just the ordinary Euler gamma function. The function G is more involved. It is related to the Υ function (that also occurs in the Liouville three point function; see [91, 92] and section 7.1) via $G(j) = b^{-b^2j(j+1+b^{-2})}\Upsilon^{-1}(-bj)$, where Υ is constructed from Barnes' double gamma function Γ_2:

$$\Upsilon^{-1}(s) = \Gamma_2(s|b,b^{-1})\Gamma_2(b+b^{-1}-s|b,b^{-1})\,, \tag{3.25}$$

$$\log\Gamma_2(s|\omega_1,\omega_2) = \lim_{t\to 0}\frac{\partial}{\partial t}\sum_{n_1,n_2=0}^{\infty}\frac{1}{(s+n_1\omega_1+n_2\omega_2)^t}\,. \tag{3.26}$$

The G-function has the following properties [66]:

$$\begin{gathered}G(j) = G(-j-1-b^{-2})\,, \\ G(j-1) = \frac{\Gamma(1+b^2j)}{\Gamma(-b^2j)}G(j)\,, \qquad G(j-b^{-2}) = b^{2(2j+1)}\frac{\Gamma(1+j)}{\Gamma(-j)}G(j)\,.\end{gathered} \tag{3.27}$$

Using these functional relations, it can be analytically continued to the complex plane and then has poles at

$$j = n + mb^{-2} \quad \text{and} \quad j = -(n+1) - (m+1)b^{-2}\,, \tag{3.28}$$

$n, m \in \mathbb{Z}_{\geq 0}$. The last point we like to mention about the structure constants is that we can determine the reflection coefficient (3.18) with their help. This works simply by using the reflection relation (3.17) between field Θ_j and Θ_{-j-1} in the three point function and employing the explicit form of the structure constants (3.22). In particular the relation

$$\begin{aligned} C(j, j_2, j_1) = (2j+1)\nu_b^{-(2j+1)} \frac{\Gamma(1 - b^2(2j+1))}{\Gamma(1 + b^2(2j+1))} \cdot \\ \cdot \frac{\gamma(j_1 - j_2 - j)\gamma(j_2 - j_1 - j)}{\gamma(-2j)} C(-j-1, j_2, j_1)\,, \end{aligned} \tag{3.29}$$

where $\gamma(z) = \frac{\Gamma(z)}{\Gamma(1-z)}$, is crucial here. One derives it with the help of (3.27). The result for $R(j)$ is written in equation (3.18).

3.4 Operator Product Expansion

Due to the continuous spectrum of heighest weight states, the OPE involves an integral here (rather than a finite sum as in ordinary RCFT). This works fine for the operators to be fused corresponding to states in the physical spectrum $j \in -\frac{1}{2} + \mathrm{i}\mathbb{R}_{\geq 0}$ and even for j lying in a strip around that line (we are making this precise in 3.4.1). This is the generic case and we will describe it first. However, as soon as the initial strip is left, one has to take care of singularities of the integrand that happen to cross the contour of integration. In order to reach these regions of the complex j-plane, one needs to make an analytic continuation of the OPE. That process is explained in the second subsection. The complete OPE was first constructed by Teschner in [66].

3.4.1 Generic Case

The OPE takes the form

$$\begin{aligned} \Theta_{j_2}(u_2|z_2)\Theta_{j_1}(u_1|z_1) \sim \int_{C^+} \mathrm{d}\mu(j) C(j, j_2, j_1) |z_2 - z_1|^{-2h_{12}} |u_2 - u_1|^{2j_{12}} \\ \cdot \left[\mathcal{J}_{21}(j)\Theta\right]_{-j-1}(u|z_1)\,, \end{aligned} \tag{3.30}$$

where, as usual, by $\sim$ we mean weak equality up to descendant contributions. The contour of integration is $C^+ = -\frac{1}{2} + \mathrm{i}\mathbb{R}_{\geq 0}$ and the normalization of the measure

$d\mu(j) = -\frac{(2j+1)^2}{\pi^2}dj$ is due to the normalization of fields chosen earlier in the two point function. Furthermore, the operator $\mathcal{J}_{21}(j)$ is defined by

$$[\mathcal{J}_{21}(j)\Theta]_{-j-1}(z) = \int_{\mathbb{C}} d^2u|u-u_2|^{2(j+j_2-j_1)}|u-u_1|^{2(j+j_1-j_2)}\Theta_{-j-1}(u|z)\,. \tag{3.31}$$

Using this OPE in the three point function, one readily confirms its consistency. It is instructive to rewrite the integral a little. Using (3.29) and the reflection relation (3.17), one can show that the integrand in (3.30) is invariant under $j \mapsto (-j-1)$. Consequently, the integral can be extended to the full contour $C = -\frac{1}{2} + i\mathbb{R}$:

$$\begin{aligned}\Theta_{j_2}(u_2|z_2)\Theta_{j_1}(u_1|z_1) \sim \frac{1}{2}\int_C & d\mu(j)C(j,j_2,j_1)|z_2-z_1|^{-2h_{12}}|u_2-u_1|^{2j_{12}} \\ & \cdot [\mathcal{J}_{21}(j)\Theta]_{-j-1}(u|z_1)\,.\end{aligned} \tag{3.32}$$

The integrand has poles that stem from the G-functions in the structure constants and from the operator $[\mathcal{J}_{21}(j)\Theta]_{-j-1}(u|z_1)$. These latter poles come from the terms $\propto |u-u_2|^{2(j+j_2-j_1)}$ and so on, since they have poles in j whenever $j = j_1 - j_2 - 1 - n$ for $n \in \mathbb{Z}_{\geq 0}$ (and analogously for the other factors). The list of poles has been studied in [66] and is given by

$$\begin{aligned} j &= j_{21}^{\pm} - 1 - n - mb^{-2}\,, \qquad & j &= j_{21}^{\pm} + n + mb^{-2}\,, \\ j &= -j_{21}^{\pm} - 1 - n - mb^{-2}\,, \qquad & j &= -j_{21}^{\pm} + n + mb^{-2}\,, \end{aligned} \tag{3.33}$$

where $j_{21}^{+} = j_2 + j_1 + 1$, $j_{21}^{-} = j_2 - j_1$ and $n, m \in \mathbb{Z}_{\geq 0}$. Therefore, as long as

$$|\mathrm{Re}(j_{21}^{\pm})| < \frac{1}{2}\,, \tag{3.34}$$

no poles lie on the contour C. Even more, the poles in the left column of (3.33) lie entirely to the left of the contour in this case and the poles in the right column of (3.33) all lie to the right of C. Particularly, for $j_1, j_2 \in -\frac{1}{2} + i\mathbb{R}_{\geq 0}$ (i.e. in the physical spectrum) the bound (3.34) is obeyed.

3.4.2 Analytic Continuation

As soon as one of the $j_{21}^{\pm}$ leaves the region (3.34), some of the poles (3.33) cross the contour C. The analytic continuation of the OPE (3.32) to these regions in the j-plane is defined in [66] by deforming the contour in such a way that again all the poles lie either entirely to its left or to its right, as it is the case in the initial region (3.34). The integral over this deformed contour can then be rewritten as one over C plus a sum of residue terms from the poles that have crossed the original contour. In this way, additional contributions to the OPE are picked up. Note that the procedure just described is very reminiscent of what one does when analytically continuing Gauss' hypergeometric function in the integral representation.

3.5 OPEs Involving Degenerate Fields

3.5.1 Reducible $\hat{\mathfrak{sl}}(2,\mathbb{C})_k$ Modules

By the results of [83], the affine modules $\mathcal{W}^{(j)}$ are irreducible for $j \in C = -\frac{1}{2} + \mathrm{i}\mathbb{R}$, but become reducible, if

$$j = j_{r,s} = -\frac{1}{2} + \frac{1}{2}r + \frac{b^{-2}}{2}s, \tag{3.35}$$

where either $r \geq 1$, $s \geq 0$ or $r < -1$, $s < 0$. The primary fields $\Theta_{j_{r,s}}$ are consequently degenerate fields. As the structure constants (3.22) are analytic functions of the "spin"-labels and the OPE has an analytic continuation to spins $j_{r,s}$ as in (3.35), correlation functions with $\Theta_{j_{r,s}}$ field insertions are well-defined [83]. This is known in the literature as "Teschner's trick" (see for example [77]). In the course of our work, we shall make use of the degenerate fields associated to $j_{2,0} = 1/2$ and $j_{1,1} = b^{-2}/2$.

Taking the OPE with a degenerate field $\Theta_{j_{r,s}}$, only a finite set of operators is produced. One might wonder how this is incorporated by the above OPE (3.32). The answer is the following: For $j_{r,s}$ as in (3.35), the OPE coefficients are generically zero, due to the factor of $G(2j_{r,s})$ in the denominator (recall (3.28)). The continuous part of the OPE (3.32) therefore vanishes. But still, finitely many contributions from poles that cross the contour C are picked up. The analysis is slightly tedious here, as one needs to look out for *double* poles in the numerator of the integrand that cancel the present simple pole of the denominator in order to yield an overall simple pole. Also, several poles of the numerator coincide and one needs to separate them carefully by adding small imaginary parts, before the contour can be deformed in a reasonable way [93]. Through the deformation, these poles may however become truly separated, that is they will not coincide again when the imaginary parts are removed. This is why one needs to be extremely careful when looking out for poles of the numerator that really become double poles. In the following, we do not go into this tedious business, but rather refer to the literature [66, 93] and state the results for the OPE coefficients that are needed later, in chapters 5, 6 and 8.

3.5.2 OPE with Degenerate Field $\Theta_{1/2}$

The OPE coefficients with the degenerate field $\Theta_{1/2}$ are written in [72], using the same normalization as we do. Since $\Theta_{1/2}$ is degenerate, the OPE is highly restricted. Only the field operators with $j_+ = j + 1/2$ and $j_- = j - 1/2$ do occur. This can also be seen directly from the OPE (3.32). The corresponding coefficients are

$$C_+(j) = 1\,, \qquad C_-(j) = \frac{1}{\nu_b}\frac{\Gamma(-b^2(2j+1))\Gamma(1+2b^2j)}{\Gamma(1+b^2(2j+1))\Gamma(-2b^2j)}\,. \tag{3.36}$$

3.5.3 OPE with Degenerate Field $\Theta_{b^2/2}$

The singular vector labelled by $b^{-2}/2$ restricts the possibly occuring field operators in the operator product to those with labels $j_+ := j + b^{-2}/2$, $j_- := j - b^{-2}/2$ and $j_\times := -j-1-b^{-2}/2$. The corresponding OPE coefficients have been calculated by us in [67]. We obtain

$$
\begin{gathered}
C_+(j) = 1, \qquad C_-(j) = -\nu_b^{-b^{-2}} \left[b^2(2j+1)\right]^{-2}, \\
C_\times(j) = -\frac{\nu_b^{-2j-1-b^{-2}}}{b^4} \frac{\Gamma(1+b^{-2})}{\Gamma(1-b^{-2})} \frac{\Gamma(1+2j)\Gamma(-1-2j-b^{-2})\Gamma(-b^2(2j+1))}{\Gamma(-2j)\Gamma(2+2j+b^{-2})\Gamma(1+b^2(2j+1))}.
\end{gathered}
\tag{3.37}
$$

Part II

Branes and Boundaries

> [. . .] the scientist must premise current theory as the rules of his game. His object is to solve a puzzle, preferably one at which others have failed, [. . .]
>
> *Thomas S. Kuhn, Logic of Discovery or Psychology of Research?*

4 Boundary Conformal Field Theory

In this chapter we review the basic techniques and results of conformal field theory with boundary that are relevant for our work. We shall always take the real axis $\mathrm{Im}(z) = 0$ to be the boundary and consider a conformal field theory on the (closure of the) upper half plane $\bar{\mathbb{H}} = \{z \in \mathbb{C}; \mathrm{Im}(z) \geq 0\}$. Global conformal transformations $z \mapsto \frac{az+b}{cz+d}$ have to leave the boundary invariant and hence, half of the global conformal $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ symmetry is broken, leaving only

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 \,. \tag{4.1}$$

In addition, local conformal tranformations need to have real coefficients. This will entangle chiral and antichiral currents when passing from a given bulk CFT to its associated boundary CFT and henceforth we can no longer focus on one chirality only. We will discuss this fact and its consequences in the next section. Section 4.2 gives an overview of the sewing constraints in boundary CFT which were found by Cardy-Lewellen [70] and Lewellen [71].

4.1 Basic Techniques of Boundary Conformal Field Theory

4.1.1 Gluing Conditions and Transformation Formulae

In order to discuss a CFT on the upper half plane only, we need to decouple it from the lower half plane. This is achieved by imposing the condition that no energy and momentum flow across the boundary, which is the real axis. This amounts to requiring that

$$T(z) = \bar{T}(z) \quad \text{at} \quad \mathrm{Im}(z) = 0 \tag{4.2}$$

This is called the *conformal gluing condition*. Comparing coefficients, it readily implies that chiral and antichiral Virasoro generators are no longer independent, but must be related as

$$L_n = \bar{L}_n \,. \tag{4.3}$$

Consequently, there is only one set of independent Virasoro generators and the distinction between chiral and antichiral fields is lost. Let us pick all the chiral generators to form one set of independent symmetry generators. Then, what previously (in the bulk CFT) were antichiral parts of conformal fields, now transform under the action of chiral generators: $\left[L_n, \bar{\mathfrak{A}}_{\bar{h}}(\bar{z})\right] = \left[\bar{L}_n, \bar{\mathfrak{A}}_{\bar{h}}(\bar{z})\right]$. Essentially, previously antichiral fields are reinterpreted as chiral ones and also their coordinates $\bar{z}$ behave like chiral coordinates. A primary field $\Theta_{h,\bar{h}}(z,\bar{z}) = \mathfrak{A}_h(z)\bar{\mathfrak{A}}_{\bar{h}}(\bar{z})$ hence transforms as

$$\begin{aligned}\left[L_n, \Theta_{h,\bar{h}}(z,\bar{z})\right] = &\left\{z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z} + h(n+1)z^n\right\}\Theta_{h,\bar{h}}(z,\bar{z}) + \\ &+\left\{\bar{z}^{n+1}\frac{\mathrm{d}}{\mathrm{d}\bar{z}} + \bar{h}(n+1)\bar{z}^n\right\}\Theta_{h,\bar{h}}(z,\bar{z})\,.\end{aligned} \tag{4.4}$$

For the purpose of subjecting correlators of these fields to differential equations (like Ward identities or singular vector equations) the coordinate $\bar{z}$ remains independent of z. Only after the differential equations have been solved are we taking the physical cut, which is still $\bar{z} = z^*$ (as in bulk CFT). In boundary CFT, the interpretation of this is that one thinks of the antichiral degrees of freedom as being mapped to chiral ones living in the lower half plane. But it is very important to keep in mind that this interpretation is only appropriate after having subjected the correlation functions to the constraining differential equations.

For affine currents $J^a(z)$, we usually also impose gluing conditions, so-called *affine gluing conditions*. The related boundary conditions are then referred to as *maximal symmetry preserving boundary conditions*. We write them as

$$J^a(z) = \rho^a{}_b \bar{J}^b(z) \quad \text{at} \quad \mathrm{Im}(z) = 0\,. \tag{4.5}$$

The map ρ is the *gluing map*. It must be such that the conformal gluing condition (4.2) is preserved (recall that the Sugawara construction (2.125) expresses the energy momentum tensor in terms of the affine currents). Usually, there is more than one possible choice for ρ. This groups the boundary conditions into a number of different classes. By the same reasoning as before, an affine gluing condition leaves only one half of the original Lie symmetry generators independent. Again, we take these to be the chiral ones, with the consequence that previously antichiral fields are reinterpreted as chiral ones: $\left[J^a_n, \bar{\mathfrak{A}}_{\bar{j}}(\bar{z})\right] = \left[\left(\rho^a{}_b \bar{J}^b_n\right), \bar{\mathfrak{A}}_{\bar{j}}(\bar{z})\right]$. Affine primary fields $\Theta_{j,\bar{j}}(z,\bar{z}) = \mathfrak{A}_j(z)\bar{\mathfrak{A}}_{\bar{j}}(\bar{z})$ transform as

$$\left[J^a_n, \Theta_{j,\bar{j}}(z,\bar{z})\right] = z^n \mathcal{D}^a_j \Theta_{j,\bar{j}}(z,\bar{z}) + \bar{z}^n\left(\rho^a{}_b \bar{\mathcal{D}}^b_{\bar{j}}\right)\Theta_{j,\bar{j}}(z,\bar{z})\,. \tag{4.6}$$

Again, it is very important to note that the physical cut $\bar{z} = z^*$ is only taken after having imposed differential equations (like Ward identities, singular vector or Knizhnik-Zamolodchikov equations) to correlation functions of the fields. In the process of solving differential equations, the coordinate $\bar{z}$ remains independent of z.

4.1.2 Boundary Ward Identities

Proceeding as in sections 2.4.4 and 2.7.3, but taking care of (4.4) and (4.6), one derives the *global conformal boundary Ward identities*

$$\begin{aligned}\sum_{k=1}^{n}\Big\{ & z_k^{i+1}\partial_{z_k}+w_k^{i+1}\partial_{w_k}+\\ & +(i+1)\left[z_k^i h_k+w_k^i\bar{h}_k\right]\Big\}\left\langle\Theta_{h_1,\bar{h}_1}(z_1,w_1)\ldots\Theta_{h_n,\bar{h}_n}(z_n,w_n)\right\rangle=0,\end{aligned}\tag{4.7}$$

where $i\in\{-1,0,1\}$, and the *affine boundary Ward identities*

$$\sum_{k=1}^{n}\left[\mathcal{D}^a_{j_k}+\left(\rho^a{}_b\bar{\mathcal{D}}^b_{\bar{j}_k}\right)\right]\left\langle\Theta_{j_1,\bar{j}_1}(z_1,w_1)\ldots\Theta_{j_n,\bar{j}_n}(z_n,w_n)\right\rangle=0,\tag{4.8}$$

with $a=1,\ldots,\dim(\mathfrak{g})$. We have named the second coordinate w_k to indicate that it is just another (at this instant independent) chiral coordinate. After having solved theses equations, one imposes $w_k=z_k^*$ ($k=1,\ldots,n$).

4.1.3 Boundary Knizhnik-Zamolodchikov Equations

The same reasoning as in section 2.7.3 that lead to (2.144) applies. Implementing the Sugawara construction (2.125) at the level of correlators and accounting for euqations (4.4) and (4.6), one obtains the *boundary Knizhnik-Zamolodchikov equations*

$$\begin{aligned}&\Bigg[(k+g^\vee)\partial_{z_\ell}+\sum_{k\neq\ell}^{n}\frac{\mathcal{D}^a_{j_\ell}\otimes\mathcal{D}^a_{j_k}}{(z_\ell-z_k)}+\\ &\qquad+\sum_{k=1}^{n}\frac{\mathcal{D}^a_{j_\ell}\otimes\left(\rho^a{}_b\bar{\mathcal{D}}^b_{\bar{j}_k}\right)}{(z_\ell-w_k)}\Bigg]\left\langle\Theta_{j_1,\bar{j}_1}(z_1,w_1)\ldots\Theta_{j_n,\bar{j}_n}(z_n,w_n)\right\rangle=0\\ &\Bigg[(k+g^\vee)\partial_{w_\ell}+\sum_{k\neq\ell}^{n}\frac{\left(\rho^a{}_b\bar{\mathcal{D}}^b_{\bar{j}_\ell}\right)\otimes\left(\rho^a{}_b\bar{\mathcal{D}}^b_{\bar{j}_k}\right)}{(w_\ell-w_k)}+\\ &\qquad+\sum_{k=1}^{n}\frac{\left(\rho^a{}_b\bar{\mathcal{D}}^b_{\bar{j}_\ell}\right)\otimes\mathcal{D}^a_{j_k}}{(w_\ell-z_k)}\Bigg]\left\langle\Theta_{j_1,\bar{j}_1}(z_1,w_1)\ldots\Theta_{j_n,\bar{j}_n}(z_n,w_n)\right\rangle=0\,,\end{aligned}\tag{4.9}$$

$\ell\in\{1,\ldots,n\}$. The same comments about the coordinates z_k and w_k as before apply.

4.1.4 Additional Operators and OPEs

Boundary Field Operators

In addition to bulk field operators $\Theta_h(z)$ (from now on suppressing the antichiral part), we can also have operators that are inserted on the boundary. Denote them $\Psi(x)$ ($x \in \mathbb{R}$). On primary fields $\Psi(x)$, the Virasoro generators L_n act as

$$[L_n, \Psi(x)] = x^{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x) + h_\Psi(n+1)x^n\Psi(x)\,. \qquad (4.10)$$

Global conformal covariance is reflected in the $\mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$ transformation

$$\Delta(g)\Psi(x)\Delta^{-1}(g) = (cx+d)^{-2h_\Psi}\Psi(g\cdot x)\,, \quad g\cdot x = \frac{ax+b}{cx+d}\,. \qquad (4.11)$$

Boundary operators can also carry boundary condition changing labels. Consider a set of boundary conditions labelled by a parameter α. A boundary field $\Psi^{\beta\alpha}(x)$ changes boundary condition α to the one labelled β. A boundary field without such labels (as we have written down above) does actually not exist. But most of the time (at least for our purposes), it will just be of the type $\Psi^{\alpha\alpha}(x)$, i.e. boundary condition preserving. We shall sometimes take the freedom to leave out such boundary preserving indices.

Bulk-Boundary OPE

Besides the usual OPE between two bulk fields (section 2.6), there are now additional operator expansions: The boundary OPE between two boundary fields and the *bulk-boundary OPE* in which a bulk field is expanded in terms of boundary fields. We only discuss the latter here. It reads[1] (assuming a diagonal CFT, i.e. $h = \bar{h}$)

$$\Theta_h(z) = \sum_\Psi (z-z^*)^{-2h+h_\Psi} C(h, h_\Psi|\alpha)\left\{\Psi^{\alpha\alpha}(z)\Big|_{(z=z^*)} + \mathcal{O}(z-z^*)\right\} \qquad (4.12)$$

and can be thought of as the bulk OPE of the chiral parts of $\Theta_h(z) = \mathfrak{A}_h(z)\bar{\mathfrak{A}}_h(\bar{z})$ after having taken the physical cut. The z-dependence is fixed from conformal covariance and the $\mathcal{O}(z-z^*)$ corrections to the leading term indicate descendant contributions. The sum over boundary fields Ψ is taken over "all Ψ that contribute" and the determination of that set is the problem of determining the values of h, h_Ψ and α for which the *bulk-boundary OPE coefficients* $C(h, h_\Psi|\alpha)$ are non-zero. The bulk-boundary OPE coefficients are related to certain two point functions in the same manner as the bulk OPE coeffcients (for the leading terms)

[1] In the nonrational H_3^+ model, there is a subtlety here which requires the distinction of two different forms of the bulk-boundary OPE. See section 5.2.4.

are related to the three point functions (structure constants). This is easily seen when one writes (4.12) in the following way

$$\Theta_h(z) = \sum_{\Psi} |x-z|^{2h_\Psi} \langle \Theta_h(z)\Psi(x)\rangle_\alpha \left\{ \Psi^{\alpha\alpha}(z)\Big|_{(z=z^*)} + \mathcal{O}(z-z^*) \right\}. \qquad (4.13)$$

The factor of $|x-z|^{-2h_\Psi}$ is included to make the expression on the right hand side independent of the auxiliary variable x. The correlation function $\langle\Theta_h(z)\Psi(x)\rangle_\alpha$ measures the overlap of the boundary field[2] $\Psi(x)$ with the bulk field $\Theta_h(z)$ in the presence of boundary condition α. Demanding conformal covariance, it is given by

$$\langle\Theta_h(z)\Psi(x)\rangle_\alpha = |x-z|^{-2h_\Psi}(z-z^*)^{-2h+h_\Psi} C(h, h_\Psi|\alpha) \qquad (4.14)$$

wherein we recover the bulk-boundary OPE coefficient.

4.1.5 Correlation Functions

Correlation Functions in the presence of a boundary condition are again partly determined from Ward identities (see (Conf-Bdry-Ward-Ids) and (Affine-Bdry-Ward-Ids)). Besides the bulk-boundary OPE coefficient and its related correlator, there are more additional basic correlators in boundary CFT. We do not discuss the correlators between boundary fields here (since we do not need them later), but comment on the important one point functions and the two point functions. From now on, we usually talk about correlation functions of bulk fields in the presence of some boundary condition. For brevity, we shall refer to these as correlation functions. Whenever we mean a different situation this will be indicated.

One Point Function

Global conformal covariance determines the one point function up to a constant, the *one point amplitude* $A(h|\alpha)$:

$$\langle\Theta_h(z)\rangle_\alpha = |z-z^*|^{-2h} A(h|\alpha)\,. \qquad (4.15)$$

Seen as a special case of the bulk-boundary OPE coefficient, the one point amplitude measures the coupling of a bulk field with conformal weight h to the identity field on the boundary:

$$A(h|\alpha) = C(h,0|\alpha)\,. \qquad (4.16)$$

These couplings encode much information about the boundary conditions. Cardy [75] and Ishibashi [94] have invented the concept of a boundary state in order to describe boundary conditions in rational CFT. Here, the set of one amplitudes contains all information to determine the boundary state. Demanding S-duality

[2] As remarked above, $\Psi(x)$ should more correctly be denoted $\Psi^{\alpha\alpha}(x)$.

of the boundary partition function, one can then use the boundary state formalism to derive the spectrum of boundary operators associated to given boundary conditions [75]. Thus, the determination of the set of one point amplitudes is a central problem in boundary CFT and it is also at the heart of our study in chapters 5, 6 and 8. In RCFT, this problem can generally be addressed by studying the so-called sewing relations (or Cardy-Lewellen constraints, see section 4.2) that correlation functions need to obey in order to yield a consistent boundary CFT. We will make use of this approach as well, but we shall see that in the nonrational H_3^+ model one needs to assume a certain continuation prescription in order to make this method work. As we can show that two different assumptions are possible, the status of the sewing relations in nonrational CFT is not completely settled. We discuss this issue in chapters 6, 8 and in the conclusion, chapter 9.

Remark on the Two Point Function

In the presence of a boundary, it is already in the two point function that global conformal covariance ceases to be sufficient for the determination of the full coordinate dependence of correlation functions. From the point of view of differential equations, determining the two point function is the same as calculating a chiral four point function in bulk CFT. That is, it involves conformal blocks that depend on a crossing ratio. The sewing relation we are going to examine in detail in chapters 5, 6 and 8 (and that we review among others in the next section) is a constraint on the two point function. The explicit construction of two point functions is therefore a reoccuring theme in our analysis.

4.2 Cardy-Lewellen Constraints

It is a well-known fact that any bulk CFT (i.e. closed string) amplitude on any desired Riemann surface can be obtained by "sewing together" three point functions on the sphere (also known under the figurative term "pair of pants"), provided that the four point function is crossing symmetric and the partition function modular invariant. Lewellen [71] has shown that in case of boundary CFT four additional sewing constraints arise. Furthermore, Cardy and Lewellen [70] and Lewellen [71] have shown that in the case of rational CFT, these sewing relations can be solved in terms of bulk quantities alone. The bulk structure data which are needed in this enterprise are the S-matrix that acts on the characters under modular S-transformations, the bulk OPE coefficients and the conformal blocks. The method is basically a generalization of the conformal bootstrap to boundary CFT and provides a tool for the determination of the additional boundary CFT structure data like the one point amplitude, bulk-boundary and boundary OPE coefficients out of the given bulk data.

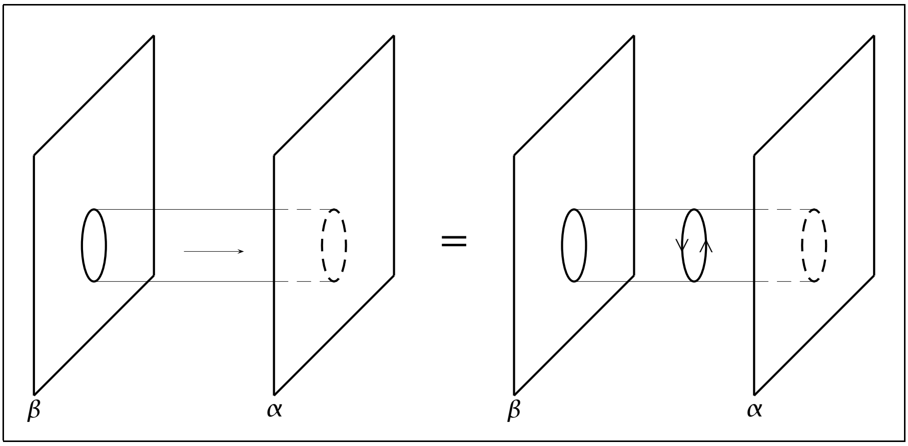

Figure 4.1: S-Duality in BCFT. The amplitude for closed strings to be created from the brane-vacuum β, propagate and be absorbed by the brane α equals the one loop vacuum amplitude for open strings with one end attached to the brane β and the other end to brane α.

So what are the additional four constraints? The first one is concerned with the consistency of the CFT of boundary fields. It demands crossing symmetry of the four point functions involving boundary operators (i.e. open strings) only. The second constraint is the S-duality of the boundary partition function that we have already mentioned above and that is drawn in figure 4.1. Condition number three concerns the boundary-boundary-bulk (or open-open-closed) amplitude (see figure 4.2). Finally, the constraint that is of major interest to us, is a relation for the closed-closed-open amplitude, see figure 4.3. In this original form it is rather tedious to implement, but if we take the open string to be the identity, it clearly becomes the crossing symmetry relation of a chiral four point function. It is in this form that we shall implement the constraint in our work. From the graphical representation in figure 4.3, we can see that the left hand side (or rather upper side), when the open string is taken to be the identity, involves terms $\propto \sum_q C(j_1, q|\alpha)C(j_2, q|\alpha)$, where we sum over propagating open strings q. The expression on the right hand side (lower side) contains $\propto \sum_j C(j_1, j_2, j)A(j|\alpha)$ with a sum over closed strings j. Projecting both sides onto the contribution of the identity open string channel only, the expression that we have written for the left hand side becomes a product of two one point amplitudes[3] $\propto A(j_1|\alpha)A(j_2|\alpha)$. On the right hand side, taking this projection

[3]Due to a subtlety in the bulk-boundary OPE, one actually needs to distinguish two different possibilities in the case of the nonrational H_3^+ model, see section 5.2.4. We had already mentioned that

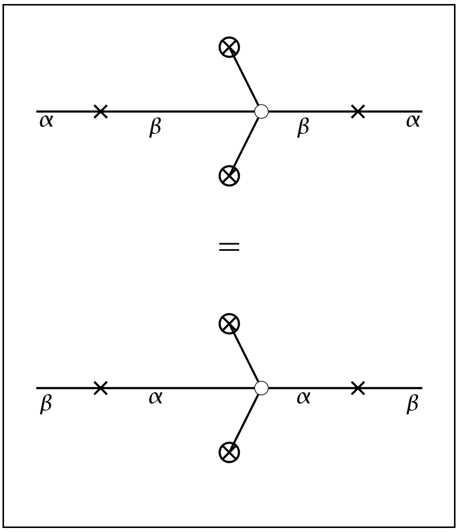

Figure 4.2: Sewing relation for the open-open-closed amplitude. The open insertions are marked with a cross and change the boundry condition.

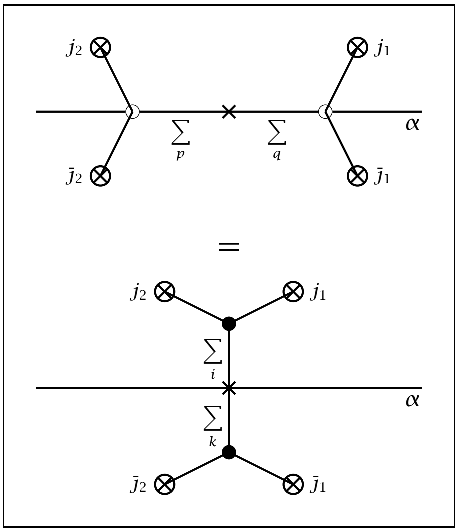

Figure 4.3: Sewing relation for the closed-closed-open amplitude. We make use of this constraint with the open string insertion being the identity; see text.

requires some work. It amounts to taking one bulk field close to the boundary, such that it can be expanded using its bulk-boundary OPE. This is the *factorization limit* and here, the continuation prescription alluded to before becomes necessary, since in order to be able to take the factorization limit, one needs to continue the two point function to a suitable patch. All this will become clear when we carry out the explicit constructions for the H_3^+ model in chapters 6 and 8.

The projected form of the constraint involves the one point amplitudes together with four point conformal blocks and bulk OPE coefficients. Thus, given the latter two structure data (which are bulk quantities), one gains an equation that one should be able to solve for the one point amplitudes. Generically, a solution for the one point amplitude will not exist for arbitrary boundary conditions, but restrictions will apply. By the same token, the labels j of closed strings that do couple consistently are expected to be constrained.

In case of RCFT, the procedure described above has been shown to work out in

earlier.

quite some generality [70, 71]. What happens in the case of generic nonrational CFT is not at all clear. The H_3^+ model has the benefit that one can analytically continue the symmetry representation labels to degenerate field labels ("Teschner's trick"; see section 3.4.2). Using degenerate fields, the model looks very much like a rational CFT. In particular the conformal bootstrap as described in this section becomes feasible. Luckily, from correlators involving a degenerate field one can still infer expressions that are of general validity. We shall see this in the next two chapters. In chapter 5, the constraint of figure 4.3 resulting from H_3^+ degenerate field $\Theta_{1/2}$ (recall section 3.5.1) is discussed following [72] and our work [67, 69]. This is however not sufficient to fix the general one point amplitude uniquely; a further constraint is needed. Our work carried out in [69] is devoted to the study of an additional constraint from degenerate field $\Theta_{b^{-2}/2}$ (section 3.5.1), choosing an analytic continuation prescription. It is the content of chapter 6. The same kind of constraint is also examined for a continuous continuation prescription in chapter 8. See [68] for our original work on this issue.

5 Boundary H_3^+ CFT

In this chapter, we review the basic features of the boundary H_3^+ model and add some of our own observations and results. An overview of the gluing conditions is followed by a discussion of additional patterns that organize the H_3^+ model branes. Our contributions here are the systematic distinction between regular and irregular branes [67] explained in section 5.2.1 and, in section 5.2.2, a clear discussion of the isospin dependencies and potential isomorphies [69]. Then, we summarize the relevant bulk-boundary OPEs in section 5.3 and go on with the introduction of the first constraint on the one point amplitudes in section 5.4. This constraint is not a Cardy-Lewellen sewing relation, but rather H_3^+ model specific: The reflection symmetry (3.17) implies a symmetry of the one point amplitude. We are careful to distinguish between regular and irregular case in the formulae we give. In sections 5.5 and 5.6, we systematically derive the Cardy-Lewellen constraints (as in figure 4.3 with the open string insertion being the identity) associated to degenerate field $\Theta_{1/2}$ for all cases of AdS_2 branes. The constraint takes the form of a 1/2-shift equation. Some of these equations had been given before, but many new ones (irregular discrete ρ_1 (section 5.5.2), regular discrete ρ_2 (section 5.5.3), regular discrete ρ_1 (section 5.5.4), regular continuous ρ_2 (section 5.6.2) and regular continuous ρ_1 (section 5.6.3)) were given by us for the first time in [67]. See table 5.1 in section 5.2.1 for an overview. As we shall see in sections 5.5 and 5.6, the derivation of 1/2-shift equations does not need a continuation of the relevant two point function. Thus, the problems that we encountered and solved in [68, 69] do not arise at this level.

5.1 Gluing Conditions

We choose maximal symmetry preserving boundary conditions. This is done by imposing a gluing condition along the boundary (which is taken to be the real axis)

$$J^a(z) = \rho^a{}_b \bar{J}^b(\bar{z}) \qquad \text{at } z = \bar{z}, \tag{5.1}$$

where ρ is the gluing map. By the Sugawara construction, we also have

$$T(z) = \bar{T}(\bar{z}) \qquad \text{at } z = \bar{z}, \tag{5.2}$$

and hence not only is a subgroup of the current algebra symmetry preserved, but also half of the conformal symmetry. In our case there are four possible gluing maps $\rho_1, \ldots, \rho_4$:

$$\begin{aligned} \rho_1 \bar{J}^3 &= \bar{J}^3 & \rho_1 \bar{J}^\pm &= \bar{J}^\pm , \\ \rho_2 \bar{J}^3 &= \bar{J}^3 & \rho_2 \bar{J}^\pm &= -\bar{J}^\pm , \\ \rho_3 \bar{J}^3 &= -\bar{J}^3 & \rho_3 \bar{J}^\pm &= \bar{J}^\mp , \\ \rho_4 \bar{J}^3 &= -\bar{J}^3 & \rho_4 \bar{J}^\pm &= -\bar{J}^\mp . \end{aligned} \tag{5.3}$$

We will only be concerned with the first and second case, ρ_1 and ρ_2. The associated branes are conventionally called AdS_2 branes [72] and we follow this nomination (although they have euclidean AdS_2 worldvolume and should more accurately be named H_2^+ branes).

5.2 Various Types of Branes

From the gluing conditions, branes fall into two great classes: AdS_2 and S^2 branes. In each class, there are more distinctions to make. In our study of the boundary H_3^+ model, we will distinguish between discrete and continuous as well as regular and irregular branes. The adjectives discrete and continuous allude to the open string spectra an the branes, whereas regular and irregular refer to the (isospin) u-dependence of the one point functions. We elaborate on these notions in the following subsections.

5.2.1 Regular and Irregular Branes

In this section we argue that the possible one point amplitudes must be distinguished by their regularity behaviour when approaching the boundary in internal u-space. Let us explain in detail why this is the case for the example that the gluing map is $\rho = \rho_2$ (the other cases can clearly be treated in just the same way). It is the affine boundary Ward identites (4.8) that fix the u-dependence of the one point function $G^{(1)}_{j,\alpha}(u|z) := \left\langle \Theta_j(u|z) \right\rangle_\alpha$ in the presence of boundary condition α entirely. The equation for J^- tells us that it is a function of $u + \bar{u}$ only, which we can just see as one complex variable since we have not taken the physical cut yet. The equations for J^3 and J^+ determine the one point function to be

$$G^{(1)}_{j,\alpha}(u|z) = (u + \bar{u})^{2j} A_{j,\alpha}(z) . \tag{5.4}$$

We call this the regular dependence. One might think that $2j \in \mathbb{Z}$ should be required in order to have trivial monodromy. But note that the physical cut has still not been taken and monodromy invariance is only required for the physical amplitude. Now, setting $\bar{u} = u^*$, $u + \bar{u}$ becomes a real variable. Therefore, a monodromy does not exist for the physical amplitude and there is no restriction on j here[1]. The only problem that arises in (5.4) for general j is when $(u + u^*) < 0$, because one needs to define $(-)^{2j}$. This problem does actually not arise for the 1/2-shift equations (sections 5.5 and 5.6) and we argue in chapter 6 that a preferred definition of $(-)^{2j}$ is suggested when analytically continuing the two point function in order to obtain $b^{-2}/2$-shift equations. Thus, the regular dependence can be treated in a meaningful way.

But one can also choose a different route. In order to circumvent the need for a definition of $(-)^{2j}$, the physical cut can be taken in two steps. Setting $\bar{u} = u^*$, first assume that $(u + u^*) = 2u_1 > 0$ and write

$$G^{(1)}_{j,\alpha}(u; u_1 > 0|z) = (u + \bar{u})^{2j} A^{+}_{j,\alpha}(z)\,. \tag{5.5}$$

Then, assume that $(u + u^*) = 2u_1 < 0$ and define

$$G^{(1)}_{j,\alpha}(u; u_1 < 0|z) = (-u - \bar{u})^{2j} A^{-}_{j,\alpha}(z)\,. \tag{5.6}$$

In this way, the lack of an a priori definition of $(-)^{2j}$ is shifted into a new a priori unknown integration "constant" $A^{-}_{j,\alpha}(z)$, that is thought of as being unrelated (!) to the first constant of integration $A^{+}_{j,\alpha}(z)$. The ansatz for the one point amplitude is summarized in the following formula

$$G^{(1)}_{j,\alpha}(u|z) = |u + \bar{u}|^{2j} A^{\sigma}_{j,\alpha}(z)\,, \tag{5.7}$$

with $\sigma = \mathrm{sgn}(u + u^*)$. This form is called the irregular dependence. In this and the next chapter, we will compute the one point amplitudes resulting from both these ansätze and find that they are indeed very different in nature. The corresponding branes will be called *regular* or *irregular*, respectively. Let us mention that in the literature, both kinds of solutions, regular and irregular ones, have been studied. For example, [72] and [79] look at irregular $AdS_2^{(c)}$ and [76] treats irregular $AdS_2^{(d)}$ branes, whereas [78] studies regular solutions. But up to now nobody has pointed out that for every case of boundary condition $\rho_1, \ldots \rho_4$, we should actually look for both kinds of solutions. Table 5.1 shows how little of the 'landscape' has actually been explored so far (before our work). It also shows that, except for one case in [78], it has always been only one consistency condition on which the proposed solutions were based, namely the shift equation for the degenerate field $\Theta_{1/2}$. The solutions to this equation are not unique and at least a second consistency condition should be derived that can fix the solution uniquely. The shift equations for the degenerate field $\Theta_{b^{-2}/2}$ that we derived for all cases of AdS_2 branes [69] (and also [68]) can do this job.

[1] Interestingly, the $b^{-2}/2$-shift equation will require $j \in \frac{1}{2}\mathbb{Z}$; see chapter 6.

	u-dependence	shift equation (continuous) for $\Theta_{1/2}$?	for $\Theta_{b^{-2}/2}$?	shift equation (discrete) for $\Theta_{1/2}$?	for $\Theta_{b^{-2}/2}$?
ρ_1	$\|u-\bar{u}\|^{2j}$	[79]	—	—	—
	$(u-\bar{u})^{2j}$	[78]	—	[78]	[78]
ρ_2	$\|u+\bar{u}\|^{2j}$	[72]	—	[76]	—
	$(u+\bar{u})^{2j}$	—	—	—	—
ρ_3	$\|1-u\bar{u}\|^{2j}$	—	—	—	—
	$(1-u\bar{u})^{2j}$	—	—	[78]	[78]
ρ_4	$(1+u\bar{u})^{2j}$	—	—	[72]	—

Table 5.1: Classes of D-brane solutions and status of their exploration before our work [67, 68, 69]. Compare also table 9.1, chapter 9. [78] did not pay attention to the occurence of possible signums σ, which is however inevitable, even in the regular cases (see chapter 6). We are therefore reconsidering their results. Note that only one version of u-dependence appears for ρ_4, as the expression is always strictly positive.

5.2.2 AdS_2 and S^2 Branes

From the u dependence of the one point functions (see table 5.1), we can determine what subgroup of the $SL(2,\mathbb{C})$ isospin symmetry is preserved by the varying gluing conditions. Since a primary field $\Theta_j(u|z)$ transforms under an $SL(2,\mathbb{C})$ isospin transformation $u \mapsto u' := \frac{au+b}{cu+d}$ as

$$\Theta_j(u|z) \mapsto \Theta'_j(u'|z) = |cu+d|^{-4j}\Theta_j(u|z)\,, \tag{5.8}$$

one needs to check for every u dependence which $SL(2,\mathbb{C})$ subgroup it preserves up to a factor of $|cu+d|^{-4j}$. The result is that the dependencies $|u\pm\bar{u}|^{2j}$ and $(u\pm\bar{u})^{2j}$ preserve an $SL(2,\mathbb{R})$ subgroup and are therefore AdS_2 branes, whereas $|1-u\bar{u}|^{2j}$ and $(1\pm u\bar{u})^{2j}$ preserve an $SU(2)$ subgroup and are thus S^2 branes[2]. The cases of gluing maps ρ_1 and ρ_2 should therefore be isomorphic, as should be those of ρ_3 and ρ_4. However, such a conclusion, which would suggest to leave half of the gluing maps unstudied, might be drawn too quickly here. Indeed, at least one issue is unclear: How can ρ_3 and ρ_4 belong to isomorphic branes if the irregular ρ_3 dependence allows the inclusion of a signum σ and ρ_4 does not (see table 5.1)? The answer must be that only further consistency checks will forbid the inclusion of a signum for irregular ρ_3. We take this as a hint that consistency checks will add to the analysis described in this subsection and therefore consider both gluing maps, ρ_1 as well as ρ_2, separately. Interestingly, we shall find that in case of irregular branes the shift equations for both gluing

[2] The names are also justified from a classical analysis of the worldvolumes of the branes [95, 72], if one remembers that AdS_2 should mean euclidean AdS_2.

maps are isomorphic, whereas in the regular case there are crucial differences (see sections 6.1.3 and 6.1.4).

5.2.3 AdS_2 Boundary Fields

As we are focussing on AdS_2 branes, let us give a brief account of their associated primary boundary fields. Denote such a field $\Psi_\ell^{\beta\alpha}(t|x)$, $x \in \mathbb{R}$. Living in euclidean AdS_2, these fields fall into representations of $\mathrm{SL}(2,\mathbb{R})$. ℓ is the corresponding "spin" label and t the isospin coordinate. See appendix A for details about the different $\mathrm{SL}(2,\mathbb{R})$ representations. From a semiclassical analysis, the relevant representations are again the principal continuous series. There are actually two such series, but only one (given in (A.40)) occurs here [72, 73]. Therefore, $\ell \in -\frac{1}{2} + \mathrm{i}\mathbb{R}_{\geq 0}$ and $t \in \bar{\mathbb{R}}$. Under isospin transformations, the fields transform as

$$\Delta(g)\Psi_\ell(t|x)\Delta^{-1}(g) = |-cx+a|^{2\ell}\Psi_\ell(g^{-1}\cdot t|x) \tag{5.9}$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\,, \quad g^{-1}\cdot t = \frac{dt-b}{-ct+a}\,. \tag{5.10}$$

Boundary changing operators are more complicated and were shown in [73] to transform under $\tilde{\mathrm{SL}}(2,\mathbb{R})$, the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. We do not need them in the sequel.

5.2.4 Discrete and Continuous Branes

Another subtlety one needs to take care of is the possibility of having discrete and continuous branes. The characterising adjectives "continuous" and "discrete" relate to the parameter spaces of these solutions or, equivalently, to the open string spectra on the branes. For example, in [72], a solution for the *continuous* AdS_2 branes was proposed, whereas [76] proposed a solution for the *discrete* AdS_2 branes. From now on, we will carefully distinguish these different kinds of solutions, by adding a superscript (c) in case of a continuous brane and (d) for a discrete one, as it has already been done in [76]. Let us now explain where the difference between continuous and discrete branes originates and how it leads to different factorization constraints. For convenience, let us choose the irregular u-dependence and fix the gluing map to be $\rho = \rho_2$. The discussion for irregular ρ_1 is completely analogous.

Assuming a discrete open string spectrum on the brane, the bulk-boundary OPE for $\Theta_{j_{r,s}}$ is

$$\begin{aligned} \Theta_{j_{r,s}}(u_2|z_2) = \sum_{\{\ell_0\}} |z_2-\bar{z}_2|^{-2h(j_{r,s})+h(\ell_0)}\,|u_2+\bar{u}_2|^{2j_{r,s}+\ell_0+1}\,\cdot \\ \cdot\, C_\sigma(j_{r,s},\ell_0|\alpha)\,(\mathcal{J}\Psi)^{\alpha\alpha}_{\ell_0}(u_2\,|\mathrm{Re}(z_2))\,\{1+\mathcal{O}(z_2-\bar{z}_2)\}\,, \end{aligned} \tag{5.11}$$

where $\{\ell_0\}$ is a discrete set of SL(2)-'spin' labels, $\Psi_\ell^{\alpha\alpha}(t|x)$ are primary boundary fields ($t, x \in \mathbb{R}$) and we have defined

$$(\mathcal{J}\Psi)_\ell^{\alpha\alpha}(u|x) = \int_{\mathbb{R}} \frac{\mathrm{d}t}{2\pi} \, |u + \mathrm{i}t|^{-2\ell-2} \, \Psi_\ell^{\alpha\alpha}(t|x) \,. \tag{5.12}$$

Note that under a scaling $u \mapsto \lambda u$, this transforms as

$$(\mathcal{J}\Psi)_{\ell_0}^{\alpha\alpha}(\lambda u|x) = \lambda^{-\ell_0-1}(\mathcal{J}\Psi)_{\ell_0}^{\alpha\alpha}(u|x) \,, \tag{5.13}$$

so that the scaling properties of $\Theta_{j_{r,s}}(u_2|z_2)$ on the L.H.S are matched correctly. Now, the kind of factorization constraint we are seeking for arises when looking at the identity contribution of the bulk-boundary OPE. The corresponding bulk-boundary OPE coefficient $C_\sigma(j_{r,s}, 0|\alpha)$ can be identified with a one-point amplitude:

$$C_\sigma(j_{r,s}, 0|\alpha) = A_\sigma(j_{r,s}|\alpha) \,. \tag{5.14}$$

Therefore, starting with a two point function and taking the factorization limit leads, in the discrete case, to a product $A_{\sigma_2}(j_{r,s}|\alpha)A_{\sigma_1}(j|\alpha)$.

On the other hand, assuming a continuous open string spectrum on the brane, the bulk-bundary OPE of $\Theta_{b^{-2}/2}$ contains

$$\tilde{c}_\sigma(j_{r,s}, l_0|\alpha) = \mathrm{Res}_{\ell=\ell_0} C_\sigma(j_{r,s}, \ell|\alpha) \tag{5.15}$$

rather than $C(j_{r,s}, \ell_0|\alpha)$. The reason for this is given in [77]. Let us summarize it here briefly: Since we are using Teschner's Trick, i.e. we are analytically continuing the field label j_2 to the label of a degenerate representation $j_2 = j_{r,s}$, we should look at the generic bulk-boundary OPE

$$\begin{aligned}\Theta_{j_2}(u_2|z_2) = \int_{C^+} \mathrm{d}\ell \, |z_2 - \bar{z}_2|^{-2h(j_2)+h(\ell)} \, |u_2 + \bar{u}_2|^{2j_2+\ell+1} \, \cdot \\ \cdot \, C_\sigma(j_2, \ell|\alpha) \, (\mathcal{J}\Psi)_\ell^{\alpha\alpha} \, (u_2 \, |\mathrm{Re}(z_2)) \, \{1 + \mathcal{O}(z_2 - \bar{z}_2)\} \,,\end{aligned} \tag{5.16}$$

where the contour of integration is $C^+ = -\frac{1}{2} + \mathrm{i}\mathbb{R}_{\geq 0}$. Since $j_2 = j_{r,s}$ is a degenerate representation, only a discrete set of open string modes is excited in the bulk-boundary OPE of its corresponding field operator. Accordingly, when deforming the contour in the process of analytic continuation, only finitely many contributions $\{\ell_0\}$ are picked up. They come from poles in the $C_\sigma(j_{r,s}, \ell|\alpha)$ that cross the contour of integration. Therefore, not the bulk-boundary coefficients themselves, but only their residua occur. Henceforth, we obtain

$$\begin{aligned}\Theta_{j_{r,s}}(u_2|z_2) = \sum_{\{\ell_0\}} |z_2 - \bar{z}_2|^{-2h(j_{r,s})+h(\ell_0)} \, |u_2 + \bar{u}_2|^{2j_{r,s}+\ell_0+1} \, \cdot \\ \cdot \, \tilde{c}_\sigma(j_{r,s}, \ell_0|\alpha) \, (\mathcal{J}\Psi)_{\ell_0}^{\alpha\alpha} \, (u_2 \, |\mathrm{Re}(z_2)) \, \{1 + \mathcal{O}(z_2 - \bar{z}_2)\} \,.\end{aligned} \tag{5.17}$$

In the factorization limit, we are looking at the identity contribution again, but this time, the residuum of the appropriate bulk-boundary coefficient does not have an obvious relation to a one-point-amplitude. Thus, in the continuous case, we are left with a product $\tilde{c}_{\sigma_2}(j_{r,s}, 0|\alpha)A_{\sigma_1}(j|\alpha)$.

5.3 Bulk-Boundary OPEs

In this section we give the explicit form of the specific bulk-boundary OPEs needed for our calculations in chapters 6 and 8. For convenience, let us write the cases of gluing maps ρ_1 and ρ_2 in one formula. Also, we just write down the case of discrete open string spectrum, as the continuous case is easily obtained by changing C_σ to $\tilde{c}_\sigma$. See also section 5.2.4, where we have introduced the generic bulk-boundary OPE and discussed the difference between discrete and continuous branes. Also note that further difference has to be made between the cases of regular and irregular branes. The formulae given below work for the irregular case, whereas for the discrete case, we need to replace the modulus $|\dots|$ by ordinary brackets $(\dots)$ (in both, z and u variables, since we are working with a regular z dependence as well; see e.g. section 5.5.3). This is necessary to ensure that the identification $C_\sigma = A_\sigma$, equation (5.14), still holds true.

5.3.1 Bulk-Boundary OPE for $\Theta_{1/2}$

For the irregular dependence, the bulk-boundary OPE reads

$$\begin{aligned}\Theta_{1/2}(u_2|z_2) = {} & |z_2 - \bar{z}_2|^{\frac{3}{2}b^2}\,|u_2 \pm \bar{u}_2|\,C_\sigma(1/2,0|\alpha)\mathbb{1}\,\{1 + \mathcal{O}\,(z_2 - \bar{z}_2)\} + \\ & + |z_2 - \bar{z}_2|^{-\frac{1}{2}b^2}\,|u_2 \pm \bar{u}_2|^2\,C_\sigma(1/2,1|\alpha)\cdot \\ & \cdot\,(\mathcal{J}\Psi)_1^{\alpha\,\alpha}\,(u_2\,|\mathrm{Re}(z)\,)\,\{1 + \mathcal{O}\,(z_2 - \bar{z}_2)\}\ .\end{aligned} \tag{5.18}$$

The upper sign corresponds to gluing map ρ_2, the lower sign to ρ_1. In order to obtain the regular dependence, one simply replaces $|\dots|$ by $(\dots)$ and leaves out the subscript σ. This ensures (5.14) to remain true.

5.3.2 Bulk-Boundary OPE for $\Theta_{b^{-2}/2}$

In the irregular case, one has

$$\begin{aligned}\Theta_{b^{-2}/2}(u_2|z_2) = {} & |z_2 - \bar{z}_2|^{1+b^{-2}/2}\,|u_2 \pm \bar{u}_2|^{b^{-2}}\,C_\sigma(b^{-2}/2,0|\alpha)\mathbb{1}\,\{1 + \mathcal{O}\,(z_2 - \bar{z}_2)\} + \\ & + |z_2 - \bar{z}_2|^{-b^{-2}/2}\,|u_2 \pm \bar{u}_2|^{2b^{-2}+1}\,C_\sigma(b^{-2}/2,b^{-2}|\alpha)\cdot \\ & \cdot\,(\mathcal{J}\Psi)_{b^{-2}}^{\alpha\,\alpha}\,(u_2\,|\mathrm{Re}(z)\,)\,\{1 + \mathcal{O}\,(z_2 - \bar{z}_2)\} + \\ & + |z_2 - \bar{z}_2|^{-b^{-2}/2}\,C_\sigma(b^{-2}/2,-b^{-2}-1|\alpha)\cdot \\ & \cdot\,(\mathcal{J}\Psi)_{-b^{-2}-1}^{\alpha\,\alpha}\,(u_2\,|\mathrm{Re}(z)\,)\,\{1 + \mathcal{O}\,(z_2 - \bar{z}_2)\}\ .\end{aligned} \tag{5.19}$$

Again, upper sign corresponds to gluing map ρ_2, lower sign to ρ_1 and the regular dependence is obtained by replacing $|\dots|$ by $(\dots)$ and dropping the σ.

5.4 A First Constraint on the One Point Amplitude from Reflection Symmetry

The main constraints on the one point amplitude we want to study are of the Cardy-Lewellen type (section 4.2). Yet, in the H_3^+ model the first nontrivial constraint arises from the reflection symmetry (3.17). We give the details of its derivation for irregular as well as regular branes in this section.

5.4.1 Irregular One Point Amplitudes

Due to the reflection symmetry (3.17), the one point amplitude has to obey

$$\begin{aligned}\frac{\pi}{2j+1}\,|u \mp \bar{u}|^{2j}\,A_\sigma(j|\alpha) &= \\ &= -R(-j-1)\int_{\mathbb{C}} \mathrm{d}^2u'\,\left|u-u'\right|^{4j}\left|u'\mp\bar{u}'\right|^{-2j-2}A_{\sigma'}(-j-1|\alpha).\end{aligned} \tag{5.20}$$

The upper sign corresponds to gluing map ρ_1, the lower sign to ρ_2. Note that $\sigma' \equiv \sigma(u')$. Since we can always expand $A_{\sigma'}(-j-1|\alpha) = A^0(-j-1|\alpha) + \sigma' A^1(-j-1|\alpha)$, we need to compute the integrals ($\epsilon \in \{0,1\}$):

$$I_\epsilon^\mp := \int_{\mathbb{C}} \mathrm{d}^2u'\,\left|u-u'\right|^{4j}\left|u'\mp\bar{u}'\right|^{-2j-2}(\sigma')^\epsilon. \tag{5.21}$$

Gluing Map ρ_1 - Calculation of I_ϵ^-

Assume $u_2 > 0$. We split the integral into

$$\begin{aligned}I_\epsilon^- &= (-)^\epsilon\int_{-\infty}^{+\infty}\mathrm{d}u_1'\int_{-\infty}^{0}\mathrm{d}u_2'\left[(u_1-u_1')^2+(u_2-u_2')^2\right]^{2j}(-2u_2')^{-2j-2}+\\ &+\int_{-\infty}^{+\infty}\mathrm{d}u_1'\int_{0}^{u_2}\mathrm{d}u_2'\left[(u_1-u_1')^2+(u_2-u_2')^2\right]^{2j}(2u_2')^{-2j-2}+\\ &+\int_{-\infty}^{+\infty}\mathrm{d}u_1'\int_{u_2}^{+\infty}\mathrm{d}u_2'\left[(u_1-u_1')^2+(u_2-u_2')^2\right]^{2j}(2u_2')^{-2j-2}\\ &\equiv (-)^\epsilon I_1^> + I_2^> + I_3^>.\end{aligned} \tag{5.22}$$

Being careful about signs and using some Gamma function identities (see appendix C), we obtain

$$I_1^> = -\frac{\pi}{2j+1}\,|u-\bar{u}|^{2j}, \qquad I_2^> = -I_3^>. \tag{5.23}$$

Now, assume $u_2 < 0$. In this case, we choose the following splitting

$$\begin{aligned} I_\epsilon^- &= (-)^\epsilon \int_{-\infty}^{+\infty} \mathrm{d}u_1' \int_{-\infty}^{u_2} \mathrm{d}u_2' \left[(u_1 - u_1')^2 + (u_2 - u_2')^2\right]^{2j} (-2u_2')^{-2j-2} + \\ &+ (-)^\epsilon \int_{-\infty}^{+\infty} \mathrm{d}u_1' \int_{u_2}^{0} \mathrm{d}u_2' \left[(u_1 - u_1')^2 + (u_2 - u_2')^2\right]^{2j} (-2u_2')^{-2j-2} + \\ &+ \int_{-\infty}^{+\infty} \mathrm{d}u_1' \int_{0}^{+\infty} \mathrm{d}u_2' \left[(u_1 - u_1')^2 + (u_2 - u_2')^2\right]^{2j} (2u_2')^{-2j-2} \\ &\equiv (-)^\epsilon I_1^< + (-)^\epsilon I_2^< + I_3^<. \end{aligned} \tag{5.24}$$

This time we get

$$I_1^< = -I_2^<, \qquad I_3^< = -\frac{\pi}{2j+1} \left|u - \bar{u}\right|^{2j}. \tag{5.25}$$

Assembling, we obtain

$$I_\epsilon^- = -\frac{\pi}{2j+1} \left|u - \bar{u}\right|^{2j} (-\sigma)^\epsilon. \tag{5.26}$$

Gluing Map ρ_2 - Calculation of I_ϵ^+

Splitting the integral as before and renaming the integration variables, it is easy to see that

$$I_\epsilon^+ = I_\epsilon^-(u_1 \leftrightarrow u_2) = -\frac{\pi}{2j+1} \left|u + \bar{u}\right|^{2j} (-\sigma)^\epsilon. \tag{5.27}$$

The Constraint for Irregular One Point Amplitudes

Putting things together, we arrive at the constraint

$$A_\sigma(j|\alpha) = R(-j-1) A_{-\sigma}(-j-1|\alpha). \tag{5.28}$$

Using the definition of the reflection amplitude (3.18), we are led to redefine the one point amplitude

$$f_\sigma(j) := \nu_b^j \Gamma(1 + b^2(2j+1)) A_\sigma(j|\alpha) \tag{5.29}$$

(note that we have dropped the α-dependence of f_σ). For this redefined one point amplitude, the constraint simply reads

$$f_\sigma(j) = -f_{-\sigma}(-j-1). \tag{5.30}$$

5.4.2 Regular One Point Amplitudes

This time, there is no signum, so we only need to compute the integrals:

$$I^{\mp} = \int_{\mathbb{C}} d^2u' \, |u - u'|^{4j} (u' \mp \bar{u}')^{-2j-2} . \tag{5.31}$$

Up to a sign and the missing signum, the result is very much the same as before:

$$I^{\mp} = \frac{\pi}{2j+1} (u \mp \bar{u})^{2j} . \tag{5.32}$$

Therefore, in the regular case, the constraint for the redefined one point amplitude is

$$f(j) = +f(-j-1). \tag{5.33}$$

5.5 1/2-Shift Equations for the Discrete Branes

In this section we give details concerning the derivation of 1/2-shift equations for discrete AdS_2 branes. The equation for irregular $AdS_2^{(d)}$ with gluing map ρ_2 had been given in [76]. This is why we do not go into any detail there. The other 1/2-shift equations in this section are new and have originally been derived by us in [67].

5.5.1 Irregular $AdS_2^{(d)}$ Branes - Gluing Map ρ_2

Ansatz for the One Point Functions

The gluing map is ρ_2. Choosing the irregular u-dependence, it restricts the one point function in the presence of boundary condition α to be of the form

$$\left\langle \Theta_j(u|z) \right\rangle_\alpha = |z - \bar{z}|^{-2h(j)} \, |u + \bar{u}|^{2j} \, A_\sigma(j|\alpha). \tag{5.34}$$

Note that the one point amplitude depends on $\sigma = \mathrm{sgn}(u + \bar{u})$.

1/2-Shift Equation

Making use of a relation to Liouville theory (that we shall review for other reasons in chapter 7), the following shift equation has been derived in [76] for the redefined one point amplitude (5.29). It is

$$-\frac{1}{\pi}\Gamma(-b^2) \sin[2\pi b^2] \sin[\pi b^2(2j+1)] f_\sigma \left(\frac{1}{2}\right) f_\sigma(j) =$$
$$= \sin[\pi b^2(2j+2)] f_\sigma \left(j + \frac{1}{2}\right) - \sin[\pi b^2 2j] f_\sigma \left(j - \frac{1}{2}\right). \tag{5.35}$$

We can rederive this by studying the two point function with a degenerate field $\Theta_{1/2}(u|z)$, but we do not go into this now. We detail a similar calculation in the next subsection.

5.5.2 Irregular $AdS_2^{(d)}$ Branes - Gluing Map ρ_1

Ansatz for the One Point and Two Point Functions

Choosing the irregular u-dependence, the gluing map ρ_1 restricts the one point function in the presence of boundary condition α to be of the form

$$\left\langle \Theta_j(u|z) \right\rangle_\alpha = |z-\bar{z}|^{-2h(j)}\,|u-\bar{u}|^{2j}\,A_\sigma(j|\alpha). \tag{5.36}$$

Our ansatz for the two point function $G^{(2)}_{j,\alpha}(u_i|z_i) = \left\langle \Theta_{1/2}(u_2|z_2)\Theta_j(u_1|z_1) \right\rangle_\alpha$ with degenerate field $1/2$ (fixing the u_i and z_i dependence up to a dependence on the crossing ratios) is

$$\begin{aligned} G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) = |z_1-\bar{z}_1|^{2[h(1/2)-h(j)]}\,|z_1-\bar{z}_2|^{-4h(1/2)}\,\cdot \\ \cdot\,|u_1-\bar{u}_1|^{2j-1}\,|u_1-\bar{u}_2|^2\,H^{(2)}_{j,\alpha}(u|z), \end{aligned} \tag{5.37}$$

with crossing ratios

$$z = \frac{|z_2-z_1|^2}{|z_2-\bar{z}_1|^2} \qquad \text{and} \qquad u = \frac{|u_2-u_1|^2}{|u_2-\bar{u}_1|^2}. \tag{5.38}$$

Knizhnik-Zamolodchikov Equation

Mapping $z_1 \to 0$, $\bar{z}_2 \to 1$ and $\bar{z}_1 \to \infty$ (i.e. $z_2 \to z$), and analogously in the u's, the Knizhnik-Zamolodchikov equation (4.9) for z_2 is brought to standard form

$$\begin{aligned} -b^{-2}z(z-1)\partial_z H^{(2)}_{j,\alpha}(u|z) &= u(u-1)(u-z)\partial_u^2 H^{(2)}_{j,\alpha}(u|z)+ \\ &+\left[u^2-(2j+1)uz+(2j+1)u+z\right]\partial_u H^{(2)}_{j,\alpha}(u|z)+ \\ &+\left[u+\frac{1}{2}(2j-1)z-j\right]H^{(2)}_{j,\alpha}(u|z)\,. \end{aligned} \tag{5.39}$$

Since $\Theta_{1/2}$ is a degenerate field, $H^{(2)}_{j,\alpha}(u|z)$ also satisfies the singular vector equation $\partial_u^2 H^{(2)}_{j,\alpha}(u|z) = 0$ and hence, it is a simple polynomial in u:

$$H^{(2)}_{j,\alpha}(u|z) = H_0(z) + uH_1(z)\,. \tag{5.40}$$

Plugging this into (5.39) and comparing coefficients in powers of u, we obtain three equations (from $\mathcal{O}(u^0)$, $\mathcal{O}(u^1)$ and $\mathcal{O}(u^2)$). The $\mathcal{O}(u^2)$ equation is trivially satisfied, the other two read

$$\begin{aligned} -b^{-2}z(z-1)\partial_z H_0 &= \left[\frac{1}{2}(2j-1)z - j\right] H_0 + zH_1 \,, \\ -b^{-2}z(z-1)\partial_z H_1 &= H_0 + \left[-\frac{3}{2}z - jz + j + 1\right] H_1 \,. \end{aligned} \tag{5.41}$$

Solving the first of these two equations for H_1 gives

$$H_1 = -b^{-2}(z-1)\partial_z H_0 - \left[\frac{1}{2}(2j-1) - \frac{j}{z}\right] H_0 \tag{5.42}$$

and thus, knowing H_0, we can immediately compute H_1 from (5.42). Plugging (5.42) into the second equation results in a second order differential equation for H_0. It reads

$$\begin{aligned} z(1-z)^2\partial_z^2 H_0 = b^2 &\left[(1-z) - z(1-z)(2-b^{-2})\right]\partial_z H_0 - \\ - b^4 &\left[(2j-1+b^{-2}j) - z^{-1}j(j+1+b^{-2}) - z(j^2+j-\frac{3}{4})\right] H_0 \,. \end{aligned} \tag{5.43}$$

This equation can be transformed to a differential equation of hypergeometric type. Setting $H_0 = z^p(1-z)^q h_0$ yields

$$\left\{z(1-z)\partial_z^2 + \left[\gamma - (\alpha+\beta+1)z\right]\partial_z - \alpha\beta\right\} h_0 = 0 \,, \tag{5.44}$$

provided that $p = -b^2 j$ and $q = -\frac{1}{2}b^2$. The parameters are

$$\begin{aligned} &\alpha = -b^2 \,, \quad \beta = -b^2(2j+2) \,, \\ &\gamma = -b^2(2j+1) \,. \end{aligned} \tag{5.45}$$

Two linearly independent basis solutions are chosen from requiring their asymptotics to be those of an s-channel conformal block. Namely, we expand

$$H^{(2)}_{j,\alpha}(u|z) = \sum_{\epsilon=\pm} a^j_\epsilon \mathcal{F}^s_{j,\epsilon}(u|z) \tag{5.46}$$

and require that for $z \to 0$ followed by $u \to 0$:[3]

$$\begin{aligned} \mathcal{F}^s_{j,+}(u|z) &\simeq z^{h(j+1/2)-h(j)-h(1/2)} u^0 \left[1 + \mathcal{O}(u) + \mathcal{O}(z)\right] \\ &= z^{-b^2 j}\left[1 + \mathcal{O}(u) + \mathcal{O}(z)\right] \,, \\ \mathcal{F}^s_{j,-}(u|z) &\simeq z^{h(j-1/2)-h(j)-h(1/2)} u^1 \left[1 + \mathcal{O}(u) + \mathcal{O}(z)\right] \\ &= z^{b^2(j+1)} u \left[1 + \mathcal{O}(u) + \mathcal{O}(z)\right] \,. \end{aligned} \tag{5.47}$$

[3] The order of the limits is important and determines the normalization of our conformal blocks; see [72, 83].

This selects the basis solutions

$$\begin{aligned}
\mathcal{F}^s_{j,+}(u|z) &= z^{-b^2 j}(1-z)^{-b^2/2}\Big\{F(\alpha,\beta;\gamma|z) - \\
&\qquad - u\left(\frac{\alpha}{\gamma}\right)F(\alpha+1,\beta;\gamma+1|z)\Big\}, \\
\mathcal{F}^s_{j,-}(u|z) &= z^{b^2(j+1)}(1-z)^{-b^2/2}\Big\{uF(\beta-\gamma,\alpha-\gamma+1;1-\gamma|z) - \\
&\qquad - z\left(\frac{\beta-\gamma}{1-\gamma}\right)F(\beta-\gamma+1,\alpha-\gamma+1;2-\gamma|z)\Big\},
\end{aligned} \tag{5.48}$$

where $F(\alpha,\beta;\gamma|z)$ is the ordinary Gauss hypergeometric function.

Expansion Coefficients

The only thing left to determine in (5.46) are the expansion coefficients a^j_ϵ. This is done by using the OPE between $\Theta_j(u_1|z_1)$ and $\Theta_{1/2}(u_2|z_2)$ on the left hand side of (5.37). Since $|z_2-z_1|\to 0$ implies $z\to 0$ and we have chosen the $z\to 0$ asymptotics of the conformal blocks to be s-channel asymptotics, the comparison of left and right hand side becomes easy: The z-asymptotics that accompanies the coefficient a^j_ϵ on the right hand side is adjusted to match the asymptotics coming with the emergence of the field Θ_{j_ϵ} from the OPE on the left hand side ($j_\pm = j \pm \frac{1}{2}$). To see this in detail, the $z\to 0$ asymptotics coming from the $\mathcal{F}^s_{j,+}$ block on the right hand side in (5.37) is

$$\begin{aligned}
\mathrm{RHS}^+ &\simeq z^{-b^2 j}|z_1-\bar{z}_1|^{-2[h(j)+h(1/2)]}|u_1-\bar{u}_1|^{2j+1}a^j_+ \\
&= |z_2-z_1|^{-2b^2 j}|z_1-\bar{z}_1|^{-2[h(j)+h(1/2)]+2b^2 j}|u_1-\bar{u}_1|^{2(j+1/2)}a^j_+ \\
&= |z_2-z_1|^{-2b^2 j}|z_1-\bar{z}_1|^{-2h(j+1/2)}|u_1-\bar{u}_1|^{2(j+1/2)}a^j_+ ,
\end{aligned} \tag{5.49}$$

what matches the following asymptotic contribution from the OPE on the left hand side (see the OPE given in section 3.5.2)

$$\begin{aligned}
\mathrm{LHS}^+ &\simeq |z_2-z_1|^{-2[h(j)+h(1/2)-h(j+1/2)]}|z_1-\bar{z}_1|^{-2h(j+1/2)}\cdot \\
&\qquad \cdot |u_1-\bar{u}_1|^{2(j+1/2)}C_+(j)A_\sigma(j_+|\alpha) \\
&= |z_2-z_1|^{-2b^2 j}|z_1-\bar{z}_1|^{-2h(j+1/2)}|u_1-\bar{u}_1|^{2(j+1/2)}\cdot \\
&\qquad \cdot C_+(j)A_\sigma(j_+|\alpha)\,.
\end{aligned} \tag{5.50}$$

Here, $C_+(j)$ is the OPE coefficient $C(j+1/2,1/2,j)$ and $A_\sigma(j_+|\alpha)$ the one point amplitude of the field $\Theta_{j+1/2}$ appearing in the OPE. In just the same way, the

$z \to 0$ asymptotics coming from the $\mathcal{F}^s_{j,-}$ conformal block on the right hand side of (5.37)

$$\begin{aligned}\mathrm{RHS}^+ &\simeq z^{b^2(j+1)} u |z_1 - \bar{z}_1|^{-2[h(j)+h(1/2)]} |u_1 - \bar{u}_1|^{2j+1} a^j_- \\ &= |z_2 - z_1|^{2b^2(j+1)} |u_2 - u_1|^2 |z_1 - \bar{z}_1|^{-2[h(j)+h(1/2)]-2b^2(j+1)} \cdot \\ &\qquad\qquad \cdot |u_1 - \bar{u}_1|^{2j-1} a^j_- \\ &= |z_2 - z_1|^{2b^2(j+1)} |u_2 - u_1|^2 |z_1 - \bar{z}_1|^{-2h(j-1/2)} \cdot \\ &\qquad\qquad \cdot |u_1 - \bar{u}_1|^{2(j-1/2)} a^j_-\end{aligned} \tag{5.51}$$

is seen to match the asymptotic contribution resulting from the appearance of the field $\Theta_{j-1/2}$ in the OPE on the left hand side:

$$\begin{aligned}\mathrm{LHS}^- &\simeq |z_2 - z_1|^{-2[h(j)+h(1/2)-h(j-1/2)]} |u_2 - u_1|^2 |z_1 - \bar{z}_1|^{-2h(j-1/2)} \cdot \\ &\qquad\qquad \cdot |u_1 - \bar{u}_1|^{2(j-1/2)} C_-(j) A_\sigma(j_-|\alpha) \\ &= |z_2 - z_1|^{2b^2(j+1)} |u_2 - u_1|^2 |z_1 - \bar{z}_1|^{-2h(j-1/2)} \cdot \\ &\qquad\qquad \cdot |u_1 - \bar{u}_1|^{2(j-1/2)} C_-(j) A_\sigma(j_-|\alpha) \,.\end{aligned} \tag{5.52}$$

Consequently, we have the following simple expressions for the expansion coefficients

$$a^j_\epsilon(\alpha) = C_\epsilon(j) A_\sigma(j_\epsilon|\alpha) \tag{5.53}$$

(but as mentioned before, this just reflects the normalization of the conformal blocks). Thus, as usual in boundary CFT, the correlator's dependence on the boundary condition is captured by the expansion coefficients.

Factorization Limit and 1/2-Shift Equation

In the factorization limit, the degenerate field $\Theta_{1/2}(u_2|z_2)$ is taken to approach the boundary, i.e. one takes $\mathrm{Im}(z_2) \to 0$, what implies that $z \to 1$ from below. This limit can be taken without difficulty, since the behaviour of the conformal blocks (5.48) for $z \to 1-$ is determined from the limiting behaviour of the occuring hypergeometric functions. Indeed, the limit $z \to 1-$ for $F(\alpha, \beta; \gamma|z)$ is very well known. The $\mathcal{F}^s_{j,+}$ conformal block goes like

$$\mathcal{P}_{\mathbb{1}} \mathcal{F}^s_{j,+} \simeq (1-z)^{\frac{3}{2}b^2} (1-u) \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} a^j_+(\alpha) \,, \tag{5.54}$$

where we have projected onto the identity contribution, since the constraint we are aiming at makes use of this channel only (see section 4.2). We can do the same for the $\mathcal{F}^s_{j,-}$ conformal block. It behaves like

$$\mathcal{P}_{\mathbb{1}}\mathcal{F}^s_{j,-} \simeq (1-z)^{\frac{3}{2}b^2}(1-u)\frac{\Gamma(1-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\beta-\gamma)\Gamma(\alpha-\gamma+1)}a^j_-(\alpha)\,. \tag{5.55}$$

The leading u, z dependence $(1-z)^{3b^2/2}(1-u)$ becomes, together with the prefactor in (5.37):

$$\begin{aligned}(1-z)^{\frac{3}{2}b^2}(1-u)|z_1-\bar{z}_1|^{2[h(1/2)-h(j)]}|z_1-\bar{z}_2|^{4h(1/2)}|u_1-\bar{u}_1|^{2j-1}|u_1-\bar{u}_2|^2 = \\ = |z_1-\bar{z}_1|^{-2h(j)}|u_1-\bar{u}_1|^{2j}|z_2-\bar{z}_2|^{\frac{3}{2}b^2}|u_2-\bar{u}_2|\,,\end{aligned} \tag{5.56}$$

where we have used that

$$1-z = \frac{|z_1-\bar{z}_1||z_2-\bar{z}_2|}{|z_2-\bar{z}_1|^2}\,, \quad 1-u = \frac{\sigma_1\sigma_2|u_1-\bar{u}_1||u_2-\bar{u}_2|}{|u_2-\bar{u}_1|^2} \tag{5.57}$$

together with $\mathrm{Im}(z_1) > 0$, $\mathrm{Im}(z_2) > 0$ and $\sigma_1 = \sigma_2$. As a result, the right hand side of (5.37), projected onto the identity contribution yields

$$\begin{aligned}\mathcal{P}_{\mathbb{1}}G^{(2)}_{j,\alpha}(u_i|z_i) \simeq |z_1-\bar{z}_1|^{-2h(j)}|u_1-\bar{u}_1|^{2j}|z_2-\bar{z}_2|^{\frac{3}{2}b^2}|u_2-\bar{u}_2|\cdot \\ \cdot\left\{\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}C_+(j)A_\sigma(j+1/2|\alpha)+\right. \\ \left. + \frac{\Gamma(1-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\beta-\gamma)\Gamma(\alpha-\gamma+1)}C_-(j)A_\sigma(j-1/2|\alpha)\right\}.\end{aligned} \tag{5.58}$$

Making use of the bulk-boundary OPE (5.18) for $\Theta_{1/2}$ in the two point function (5.37) and projecting onto the identity, we also obtain

$$\mathcal{P}_{\mathbb{1}}G^{(2)}_{j,\alpha}(u_i|z_i) \simeq |z_2-\bar{z}_2|^{\frac{3}{2}b^2}|u_2-\bar{u}_2|C_\sigma(1/2,0|\alpha)|z_1-\bar{z}_1|^{-2h(j)}|u_1-\bar{u}_1|^{2j}A_\sigma(j|\alpha)\,. \tag{5.59}$$

With the identification $C_\sigma(1/2,0|\alpha) = A_\sigma(1/2|\alpha)$ and redefining the one point amplitude as in (5.29), we finally obtain our desired 1/2-shift equation:

$$\begin{aligned}-\frac{1}{\pi}\Gamma(-b^2)\sin[2\pi b^2]\sin[\pi b^2(2j+1)]f_\sigma\left(\frac{1}{2}\right)f_\sigma(j) = \\ = \sin[\pi b^2(2j+2)]f_\sigma\left(j+\frac{1}{2}\right) - \sin[\pi b^2 2j]f_\sigma\left(j-\frac{1}{2}\right)\end{aligned} \tag{5.60}$$

Remarkably enough, it is the same as for gluing map ρ_2 in section 5.5.1!

5.5.3 Regular $AdS_2^{(d)}$ Branes - Gluing Map ρ_2

Ansatz for the One Point and Two Point Functions

This time choosing the regular u-dependence, the gluing map ρ_2 fixes the one point function as (note that we are not including a signum as discussed in section 5.2.1)

$$\left\langle \Theta_j(u|z) \right\rangle_\alpha = (z-\bar{z})^{-2h(j)}\,(u+\bar{u})^{2j}\,A(j|\alpha). \tag{5.61}$$

The boundary two point function with degenerate field $1/2$ is

$$\begin{aligned} G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) = (z_1-\bar{z}_1)^{-2h(j)}\,(z_2-\bar{z}_2)^{-2h(1/2)}\,\cdot \\ \cdot\,(u_1+\bar{u}_1)^{2j}\,(u_2+\bar{u}_2)\,H^{(2)}_{j,\alpha}(u|z), \end{aligned} \tag{5.62}$$

with crossing ratios

$$z = \frac{|z_1-z_2|^2}{(z_1-\bar{z}_1)\,(z_2-\bar{z}_2)} \qquad \text{and} \qquad u = -\frac{|u_1-u_2|^2}{(u_1+\bar{u}_1)\,(u_2+\bar{u}_2)}. \tag{5.63}$$

They take values in $z \in (-\infty, 0)$ (because z_1, z_2 are in the upper half plane) and $u \in (-\infty, 0)$ (if we take $\sigma_1 = \sigma_2$).

Knizhnik-Zamolodchikov Equation

Mapping $z_1 \to 0$, $\bar{z}_1 \to 1$ and $\bar{z}_2 \to \infty$ (i.e. $z_2 \to z$), and doing analogously in the u's, the standard form of the Knizhnik-Zamolodchikov equation (4.9) for z_2 is

$$\begin{aligned} -b^{-2}z(z-1)\partial_z H^{(2)}_{j,\alpha}(u|z) &= u(u-1)(u-z)\partial_u^2 H^{(2)}_{j,\alpha}(u|z) + \\ &+ \left[-2ju^2 - 2uz + (2j+1)u + z\right]\partial_u H^{(2)}_{j,\alpha}(u|z) + \\ &+ \left[2ju - j\right] H^{(2)}_{j,\alpha}(u|z)\,. \end{aligned} \tag{5.64}$$

The procedure of solving this together with the singular vector equation

$$\partial_u^2 H^{(2)}_{j,\alpha}(u|z) = 0 \tag{5.65}$$

is precisely as described in the previous subsection. We end up with the following s-channel conformal blocks:

$$\begin{aligned} \mathcal{F}^s_{j,+}(u|z) &= z^{-b^2 j}(1-z)^{-b^2 j}\left\{F(\alpha,\beta;\gamma|z) - \right. \\ &\qquad \left. -u\left(\frac{\alpha}{\gamma}\right)F(\alpha+1,\beta;\gamma+1|z)\right\}, \\ \mathcal{F}^s_{j,-}(u|z) &= z^{b^2(j+1)}(1-z)^{-b^2 j}\left\{uF(\beta-\gamma,\alpha-\gamma+1;1-\gamma|z) - \right. \\ &\qquad \left. -\,z\left(\frac{\beta-\gamma}{1-\gamma}\right)F(\beta-\gamma+1,\alpha-\gamma+1;2-\gamma|z)\right\}, \end{aligned} \tag{5.66}$$

with parameters

$$\alpha = -b^2(2j)\,,\quad \beta = -b^2(2j+2)\,,\quad \gamma = -b^2(2j+1)\,. \tag{5.67}$$

Expansion Coefficients

The expansion coefficients have to be modified slightly in this case. We have

$$a^j_\epsilon(\alpha) = \epsilon C^{1/2}_\epsilon(j) A(j_\epsilon|\alpha)\,. \tag{5.68}$$

The sign in a^j_- is due to the minus sign in the definition of u in (5.63).

Factorization Limit and 1/2-Shift Equation

Taking the limit $\mathrm{Im}(z_2) \to 0$, we have this time that $z \to -\infty$. This limit of the hypergeometric function is also well known, so there is again no problem here. The $z \to 1$ asymptotics of the $\mathcal{F}^s_{j,+}$ conformal block projected onto the identity contribution reads

$$\mathcal{P}_{\mathbb{1}}\mathcal{F}^s_{j,+} \simeq z^0 e^{i\pi b^2 j}\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}\,, \tag{5.69}$$

while for $\mathcal{F}^s_{j,-}$ it is

$$\mathcal{P}_{\mathbb{1}}\mathcal{F}^s_{j,-} \simeq -z^0 e^{-i\pi b^2(j+1)}\frac{\Gamma(1-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma)\Gamma(1-\alpha)}\,. \tag{5.70}$$

Therefore, we obtain

$$\begin{aligned}\mathcal{P}_{\mathbb{1}}G^{(2)}_{j,\alpha}(u_i|z_i) \simeq{}& (z_1-\bar{z}_1)^{-2h(j)}(z_2-\bar{z}_2)^{-2h(1/2)}(u_1+\bar{u}_1)^{2j}(u_2+\bar{u}_2)\,\cdot\\ &\cdot\left\{e^{i\pi b^2 j}\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}a^j_+ - e^{-i\pi b^2(j+1)}\frac{\Gamma(1-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma)\Gamma(1-\alpha)}a^j_-\right\}.\end{aligned} \tag{5.71}$$

On the other hand, when making use of the bulk-boundary OPE of $\Theta_{1/2}$ inside the correlator (5.62), we get

$$\begin{aligned}\mathcal{P}_{\mathbb{1}}G^{(2)}_{j,\alpha}(u_i|z_i) \simeq{}& (z_1-\bar{z}_1)^{-2h(j)}(z_2-\bar{z}_2)^{-2h(1/2)}(u_1+\bar{u}_1)^{2j}(u_2+\bar{u}_2)\,\cdot\\ &\cdot C(1/2,0|\alpha)A(j|\alpha)\,.\end{aligned} \tag{5.72}$$

Identifying $C(1/2,0|\alpha) = A(1/2|\alpha)$ and using the explicit expressions for the OPE coefficients (section 3.5.2), we arrive at the following 1/2-shift equation for the redefined one point amplitude (5.29):

$$\begin{aligned}&e^{i\pi b^2 j}\sin[\pi b^2(2j+2)]f\left(j+\frac{1}{2}\right) - e^{-i\pi b^2(j+1)}\sin[\pi b^2 2j]f\left(j-\frac{1}{2}\right) =\\ &\quad = -\frac{1}{\pi}\Gamma(-b^2)\sin[2\pi b^2]\sin[\pi b^2(2j+1)]f\left(\frac{1}{2}\right)f(j)\,.\end{aligned} \tag{5.73}$$

5.5.4 Regular $AdS_2^{(d)}$ Branes - Gluing Map ρ_1

Ansatz for the One Point and Two Point Functions

Again we choose the regular u-dependence, so that the gluing map ρ_1 fixes the one point function to be

$$\left\langle \Theta_j(u|z) \right\rangle_\alpha = (z-\bar{z})^{-2h(j)} (u-\bar{u})^{2j} A(j|\alpha). \tag{5.74}$$

The boundary two point function with degenerate field $1/2$ is

$$\begin{aligned} G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) = (z_1-\bar{z}_1)^{-2h(j)} (z_2-\bar{z}_2)^{-2h(1/2)} \cdot \\ \cdot (u_1-\bar{u}_1)^{2j} (u_2-\bar{u}_2) H^{(2)}_{j,t,\alpha}(u|z), \end{aligned} \tag{5.75}$$

with crossing ratios

$$z = \frac{|z_1-z_2|^2}{(z_1-\bar{z}_1)(z_2-\bar{z}_2)} \qquad \text{and} \qquad u = \frac{|u_1-u_2|^2}{(u_1-\bar{u}_1)(u_2-\bar{u}_2)}. \tag{5.76}$$

Note that again $z \in (-\infty, 0)$, $z \in (-\infty, 0)$, as in the preceding subsection.

Knizhnik-Zamolodchikov Equation

Mapping $z_1 \to 0$, $\bar{z}_1 \to 1$ and $\bar{z}_2 \to \infty$ (i.e. $z_2 \to z$), the Knizhnik-Zamolodchikov equation (4.9) for z_2 is brought to standard form

$$\begin{aligned} -b^{-2}z(z-1)\partial_z H^{(2)}_{j,\alpha}(u|z) = u(u-1)(u-z)\partial_u^2 H^{(2)}_{j,\alpha}(u|z) + \\ + \left[-2ju^2 - 2uz + (2j+1)u + z\right] \partial_u H^{(2)}_{j,\alpha}(u|z) + \\ + [2ju - j]\, H^{(2)}_{j,\alpha}(u|z)\,. \end{aligned} \tag{5.77}$$

This equals exactly the Knizhnik-Zamolodchikov equation in the previous section. Thus, we obtain the same conformal blocks as above (5.66).

Expansion Coefficients

The definition of u does not contain a minus sign. Therefore, the expansion coefficients are this time given by

$$a^j_\epsilon(\alpha) = C_\epsilon(j) A(j_\epsilon|\alpha). \tag{5.78}$$

Factorization Limit and 1/2-Shift Equation

In the limit $\mathrm{Im}(z_2) \to 0$, the same comments as in section 5.5.3 apply. The 1/2-shift equation that we produce reads

$$\begin{aligned} &e^{i\pi b^2 j} \sin[\pi b^2(2j+2)] f_\sigma\left(j+\frac{1}{2}\right) + e^{-i\pi b^2(j+1)} \sin[\pi b^2 2j] f_\sigma\left(j-\frac{1}{2}\right) = \\ &\quad = -\frac{1}{\pi}\Gamma(-b^2)\sin[2\pi b^2]\sin[\pi b^2(2j+1)] f_\sigma\left(\frac{1}{2}\right) f_\sigma(j)\,. \end{aligned} \tag{5.79}$$

Note that this 1/2-shift equation differs from the one for ρ_2 (5.73) only in the sign between the two terms on the right hand side. It is this little detail that will later (in section 6.1.4) allow for a more general solution than in the case of ρ_2.

5.6 1/2-Shift Equations for the Continuous Branes

In this section we assemble the 1/2-shift equations for the continuous AdS_2 branes. The ansätze for the one point functions are always as in the preceding section and the two point functions are computed in exactly the same way. Therefore we just state our results here. Actually, the 1/2-shift equations can readily be obtained from their counterparts for the discrete branes. One only needs to replace $C_\sigma(1/2,0|\alpha)$ by $\tilde{c}_\sigma(1/2,0|\alpha)$ (recall section 5.2.4) and notice that an additional factor of $\sqrt{\nu_b}\,\Gamma(1+2b^2)$ appears along with $\tilde{c}_\sigma(1/2,0|\alpha)$ from redefining the one point amplitudes according to (5.29). Using the relation

$$\sin[2\pi b^2] = -\frac{\pi}{\Gamma(1+2b^2)\Gamma(-2b^2)}\,, \tag{5.80}$$

one easily sees that the standard factor appearing along with $\tilde{c}_\sigma(1/2,0|\alpha)$ is $\sqrt{\nu_b}\frac{\Gamma(-b^2)}{\Gamma(-2b^2)}$. The 1/2-shift equations for the irregular continuous branes with gluing maps ρ_2/ρ_1 had already been stated in [72]/[79]. The regular cases are new [69].

5.6.1 Irregular $AdS_2^{(c)}$ Branes - Gluing Maps ρ_1, ρ_2

As before in the discrete case, we find identical 1/2-shift equations for gluing maps ρ_1, ρ_2. The equation reads

$$\begin{aligned} &\sqrt{\nu_b}\frac{\Gamma(-b^2)}{\Gamma(-2b^2)}\tilde{c}_\sigma(1/2,0|\alpha)\sin[\pi b^2(2j+1)] f_\sigma(j) = \\ &\qquad = \sin[\pi b^2(2j+2)] f_\sigma\left(j+\frac{1}{2}\right) - \sin[\pi b^2 2j] f_\sigma\left(j-\frac{1}{2}\right). \end{aligned} \tag{5.81}$$

5.6.2 Regular $AdS_2^{(c)}$ Branes - Gluing Map ρ_2

Using the above recipe, we read off the following 1/2-shift equation from (5.73):

$$\begin{aligned} &e^{i\pi b^2 j}\sin[\pi b^2(2j+2)]f\left(j+\frac{1}{2}\right)-e^{-i\pi b^2(j+1)}\sin[\pi b^2 2j]f\left(j-\frac{1}{2}\right)= \\ &\quad=\sqrt{\nu_b}\frac{\Gamma(-b^2)}{\Gamma(-2b^2)}\tilde{c}(1/2,0|\alpha)\sin[\pi b^2(2j+1)]f(j)\,. \end{aligned} \tag{5.82}$$

5.6.3 Regular $AdS_2^{(c)}$ Branes - Gluing Map ρ_1

This 1/2-shift equation differs from the previous one only in a sign on the left hand side:

$$\begin{aligned} &e^{i\pi b^2 j}\sin[\pi b^2(2j+2)]f\left(j+\frac{1}{2}\right)+e^{-i\pi b^2(j+1)}\sin[\pi b^2 2j]f\left(j-\frac{1}{2}\right)= \\ &\quad=\sqrt{\nu_b}\frac{\Gamma(-b^2)}{\Gamma(-2b^2)}\tilde{c}(1/2,0|\alpha)\sin[\pi b^2(2j+1)]f(j)\,. \end{aligned} \tag{5.83}$$

6 More Shift Equations via Analytic Continuation

Having discussed $1/2$-shift equations, there is an immediate call for solutions. Indeed, solutions to the $1/2$-shift equations given in the previous chapter together with the reflection symmetry constraint (5.30), (5.33) can be given and have been given (partly by us and partly by others; see the next two sections for precise citations) [72, 79, 78, 76, 67, 68, 69]. We shall discuss these solutions in the present chapter. One problem is, however, that the $1/2$-shift equations do not fix the solutions uniquely. Thus, the goal is to derive a second independent shift equation, in order to make the solutions for the one point amplitude unique and hence back up their consistency. The natural candidate from which to derive a second factorization constraint is the two point function involving the next simple degenerate field which has $\mathfrak{sl}(2,\mathbb{C})$ label $j = b^{-2}/2$. The aim of the present chapter is to study this two point function, analyse how it provides us with the desired $b^{-2}/2$-shift equation and study the implications of the new constraints. In this, we follow closely our article [69].

Our derivation of $b^{-2}/2$-shift equations starts with solving the associated Knizhnik-Zamolodchikov equation for the two point function involving the degenerate field $\Theta_{b^{-2}/2}$. Contrary to the situation that we encountered for the $1/2$-shift equations in the last chapter, the solution for the correlator is not everywhere defined, but only in a certain (u,z)-patch. Unfortunatley, in order to take the factorization limit, one has to move out of this initial region. Consequently, a continuation prescription for the two point correlator is needed. Since, in its initial domain, it is in fact an analytic function of both variables (u,z), we shall assume here that it can be extended to other regions by analytic continuation in (u,z). Then, a definition of the two point correlator can be given in the patch that is relevant to the factorization limit and we can derive the desired $b^{-2}/2$-shift equations.

With these new equations at our disposal, we then go on and study solutions for the one point amplitudes. The strategy is to write down the most general solution to the $1/2$-shift equation together with the reflection symmetry constraint in a first step and then insert this solution into the new $b^{-2}/2$-shift equation. This will usually restrict the solution further or exclude it as beeing inconsistent [69].

Having described the programme to be carried out in this chapter, let us mention briefly what is to follow afterwards. In fact, the continuation prescription alluded to above is not the only possibility. From a correspondence between the H_3^+ model and Liouville theory [74, 73], a certain continuity prescription has been proposed [73]. We are going to review the correspondence in chapter 7 and also explain this continuity proposal. Then, in chapter 8, we describe our work [68], where we have shown how the continuity proposal can be realized explicitly in the H_3^+ model and have analysed the consequences for $b^{-2}/2$-shift equations and their solutions. But before we come to that, let us now turn to the approach that uses analytic continuation.

6.1 Discrete Branes

In subsection 6.1.1, we give the details of our derivation [69] of the $b^{-2}/2$-shift equation for the irregular discrete AdS_2 branes with gluing map ρ_2. With the help of our new shift equation, we then discuss consistency of the solution proposed in [76]. Afterwards, we proceed with the $b^{-2}/2$-shift equations and their solutions for the cases of irregular discrete AdS_2 branes with gluing map ρ_1, regular discrete ρ_2 and regular discrete ρ_1. All the $b^{-2}/2$-shift equations and results about the one point amplitude solutions that we discuss for these cases have been given by us in [69].

6.1.1 Irregular $AdS_2^{(d)}$ Branes - Gluing Map ρ_2

Ansatz for the One Point and Two Point Functions

Recall the form of the one point function from section 5.5.1

$$\left\langle \Theta_j(u|z) \right\rangle_\alpha = |z - \bar{z}|^{-2h(j)} \, |u + \bar{u}|^{2j} \, A_\sigma(j|\alpha). \tag{6.1}$$

Using the Ward identities, the form of the two point function

$$G^{(2)}_{j,\alpha}(u_i|z_i) = \left\langle \Theta_{b^{-2}/2}(u_2|z_2)\Theta_j(u_1|z_1) \right\rangle_\alpha \tag{6.2}$$

can be partially fixed as

$$\begin{aligned} G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) = {} & |z_1 - \bar{z}_1|^{2[h(b^{-2}/2)-h(j)]} \, |z_1 - \bar{z}_2|^{-4h(b^{-2}/2)} \cdot \\ & \cdot |u_1 + \bar{u}_1|^{2j-b^{-2}} \, |u_1 + \bar{u}_2|^{2b^{-2}} \, H^{(2)}_{j,\alpha}(u|z), \end{aligned} \tag{6.3}$$

where $H^{(2)}_{j,\alpha}(u|z)$ is an unknown function of the crossing ratios

$$z = \frac{|z_2 - z_1|^2}{|z_2 - \bar{z}_1|^2} \quad \text{and} \quad u = \frac{|u_2 - u_1|^2}{|u_2 + \bar{u}_1|^2}. \tag{6.4}$$

Taking the physical cut, the crossing ratios are real and take values $0 \le z \le 1$ and $0 \le u \le 1$.

Knizhnik-Zamolodchikov Equation

We use the Knizhnik-Zamolodchikov equations (4.9) for z_2. Mapping $z_1 \to 0$, $\bar{z}_2 \to 1$ and $\bar{z}_1 \to \infty$ (i.e. $z_2 \to z$) and analogously in the u's, brings this equation to the following standard form

$$\begin{aligned}
-b^{-2}z(z-1)\partial_z H^{(2)}_{j,\alpha}(u|z) &= u(u-1)(u-z)\partial_u^2 H^{(2)}_{j,\alpha}+ \\
&+\left\{\left[1-2b^{-2}\right]u^2+\left[b^{-2}-2j-2\right]uz+\left[2j+b^{-2}\right]u+z\right\}\partial_u H^{(2)}_{j,\alpha}+ \\
&+\left\{b^{-4}u+\left[b^{-2}j-b^{-4}/2\right]z-b^{-2}j\right\}H^{(2)}_{j,\alpha}.
\end{aligned} \tag{6.5}$$

Since one field operator is the degenerate field $\Theta_{b^{-2}/2}$, the space of solutions is finite dimensional. In fact it consists of three conformal blocks only, namely those for $j_\pm := j \pm b^{-2}/2$ and $j_\times := -j-1-b^{-2}/2$. The general solution therefore reads

$$H^{(2)}_{j,\alpha} = \sum_{\epsilon=+,-,\times} a^j_\epsilon(\alpha)\mathcal{F}^s_{j,\epsilon} \tag{6.6}$$

with (see [83] and [96])

$$\begin{aligned}
\mathcal{F}^s_{j,+}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}F_1(\alpha,\beta,\beta';\gamma|u;z)\,, \\
\tilde{\mathcal{F}}^s_{j,-}(u|z) &= z^{\beta-\gamma+1-j}(1-z)^{\gamma-\alpha-1-b^{-2}/2}(u-z)^{-\beta}\cdot \\
&\quad\cdot F_1\left(1-\beta',\beta,\alpha+1-\gamma;2+\beta-\gamma\,\middle|\,\frac{z}{z-u};\frac{z}{z-1}\right)\,, \\
\tilde{\mathcal{F}}^s_{j,\times}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}e^{i\pi(\alpha+1-\gamma)}\frac{\Gamma(\alpha)\Gamma(\gamma-\beta)}{\Gamma(\alpha+1-\beta)\Gamma(\gamma-1)}\cdot \\
&\quad\cdot\left\{u^{-\alpha}F_1\left(\alpha,\alpha+1-\gamma,\beta';\alpha+1-\beta\,\middle|\,\frac{1}{u};\frac{z}{u}\right)-\right. \\
&\quad\left.-e^{-i\pi\alpha}\frac{\Gamma(\alpha+1-\beta)\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)\Gamma(1-\beta)}F_1(\alpha,\beta,\beta';\gamma|u;z)\right\}.
\end{aligned} \tag{6.7}$$

The function $F_1(\alpha,\beta,\beta',\gamma|u;z)$ is a generalized hypergeometric functions, the first one of Appell's double hypergeometric functions. We give an overview and summarize its most important properties in appendix C. See the books [96, 97] for more information. Splitting the common factor $z^{-j}(1-z)^{-b^{-2}/2}$, these functions are found in [96] as (respectively) $\mathcal{Z}_1$, $\mathcal{Z}_5$ and the last one is a combination of $\mathcal{Z}_8$ and $\mathcal{Z}_1$. For the occuring parameters we find

$$\alpha=\beta=-b^{-2}, \quad \beta'=-2j-1-b^{-2}, \quad \gamma=-2j-b^{-2}\,. \tag{6.8}$$

For our purposes, we like to replace the $\tilde{\mathcal{F}}^s_{j,-}$ block by

$$\begin{aligned}
\mathcal{F}^s_{j,-}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}u^{-\beta}z^{1+\beta-\gamma}\cdot \\
&\quad\cdot F_1\left(1+\beta+\beta'-\gamma,\beta,1+\alpha-\gamma;2+\beta-\gamma\,\middle|\,\frac{z}{u};z\right)\,.
\end{aligned} \tag{6.9}$$

This is found in [96] as $\mathcal{Z}_{15}$ and coincides with the conformal block given in (6.7) in the overlap of their domains of convergence [96] and thus, (6.9) is a continuation of the former $\tilde{\mathcal{F}}^s_{j,-}$ block. Also, we continue the first summand of $\tilde{\mathcal{F}}^s_{j,\times}$ to $(\frac{1}{u}, \frac{z}{u}) \equiv (\eta, \xi) \approx (\infty, 0)$. Then, one of the resulting two terms precisely cancels the second summand of $\tilde{\mathcal{F}}^s_{j,\times}$ and we are only left with

$$\begin{aligned} \mathcal{F}^s_{j,\times}(u|z) = z^{-j}(1-z)^{-b^{-2}/2}u^{1-\gamma}\cdot \\ \cdot\, G_2\left(\beta', 1+\alpha-\gamma; 1+\beta-\gamma, \gamma-1 \,\middle|\, -\frac{z}{u}; u\right). \end{aligned} \tag{6.10}$$

$G_2(\beta, \beta'; \alpha, \alpha'|u; z)$ is again of the generalized hypergeometric type. It is the second function appearing on Horn's list. We also give an introduction to this function in appendix C and refer to [98, 97] for more information. With our modest improvements, the boundary two point function is now defined in the region $z < u < 1$. For convenience, let us once and for all assemble the conformal blocks we are using:

$$\begin{aligned} \mathcal{F}^s_{j,+}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}F_1(\alpha, \beta, \beta'; \gamma|u; z)\,, \\ \mathcal{F}^s_{j,-}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}u^{-\beta}z^{1+\beta-\gamma}\cdot \\ &\quad \cdot\, F_1\left(1+\beta+\beta'-\gamma, \beta, 1+\alpha-\gamma; 2+\beta-\gamma \,\middle|\, \frac{z}{u}; z\right), \\ \mathcal{F}^s_{j,\times}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}u^{1-\gamma}\cdot \\ &\quad \cdot\, G_2\left(\beta', 1+\alpha-\gamma; 1+\beta-\gamma, \gamma-1 \,\middle|\, -\frac{z}{u}; u\right) \end{aligned} \tag{6.11}$$

with parameters

$$\alpha = \beta = -b^{-2}, \quad \beta' = -2j-1-b^{-2}, \quad \gamma = -2j-b^{-2}\,. \tag{6.12}$$

(6.11) constitutes a linearly independent set of three solutions. By general theory, any other solution can be expressed as a linear combination of them [97]. This reflects nicely the fact that the degenerate field $\Theta_{b^{-2}/2}$ restricts the propagating fields to only three possibilities, namely those belonging to representations $j_\pm$ and $j_\times$, as we have mentioned above.

Expansion Coefficients

The constants $a^j_\epsilon(\alpha)$ in (6.6) are some still undetermined coefficients. They are fixed by using the bulk OPE of the two field operators on the left hand side and taking the appropriate limit $|z_2 - z_1| \to 0$ on the right hand side of (6.3). The $a^j_\epsilon(\alpha)$ will then generically turn out to be some product of bulk OPE coefficient times one point amplitude, which is why the α-dependence occurs in the a^j_ϵ-coefficients. Using a bulk OPE on the left hand side of (6.3), we obtain

$$\begin{aligned}
&G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) \simeq \\
&\simeq |z_2-z_1|^{-2j}\,|z_1-\bar z_1|^{-2h(j_+)}\,|u_1+\bar u_1|^{2j+b^{-2}}\,C_+(j)A_\sigma(j_+|\alpha)+ \\
&+|z_2-z_1|^{2j+2}\,|u_2-u_1|^{2b^{-2}}\,|z_1-\bar z_1|^{-2h(j_-)}\cdot \\
&\qquad\qquad\qquad\cdot|u_1+\bar u_1|^{2j-b^{-2}}\,C_-(j)A_\sigma(j_-|\alpha)+ \\
&+|z_2-z_1|^{-2j}\,|u_2-u_1|^{2(2j+1+b^{-2})}\,|z_1-\bar z_1|^{-2h(j_\times)}\cdot \\
&\qquad\qquad\qquad\cdot|u_1+\bar u_1|^{-2j-2-b^{-2}}\,C_\times(j)A_\sigma(j_\times|\alpha)\,.
\end{aligned} \tag{6.13}$$

We have used here that

$$\begin{aligned}
h(j_+) &= h(j_\times) = h(j)+h(b^{-2}/2)-j \\
h(j_-) &= h(j)+h(b^{-2}/2)+j+1.
\end{aligned} \tag{6.14}$$

For the explicit expressions of the bulk OPE coefficients $C_\epsilon(j)$ see section 3.5.3. On the right hand side of (6.3) we can also take the limit $|z_2-z_1|\to 0$ ($\Rightarrow z\to 0+$) followed by $|u_2-u_1|\to 0$ ($\Rightarrow u\to 0+$). The conformal blocks (6.11) behave as follows:

$$\begin{aligned}
\mathcal{F}^s_{j,+}(u|z) &\simeq z^{-j}\,, \\
\mathcal{F}^s_{j,-}(u|z) &\simeq z^{j+1}u^{b^{-2}}\,, \\
\mathcal{F}^s_{j,\times}(u|z) &\simeq z^{-j}u^{2j+1+b^{-2}}\,.
\end{aligned} \tag{6.15}$$

Together with the prefactor of (6.3)

$$\begin{aligned}
&|z_1-\bar z_1|^{2[h(b^{-2}/2)-h(j)]}\,|z_1-\bar z_2|^{-4h(b^{-2}/2)}\,|u_1+\bar u_1|^{2j-b^{-2}}\,|u_1+\bar u_2|^{2b^{-2}}\simeq \\
&\qquad\simeq |z_1-\bar z_1|^{-2[h(b^{-2}/2)+h(j)]}\,|u_1+\bar u_1|^{2j+b^{-2}}
\end{aligned} \tag{6.16}$$

and recalling that

$$z=\frac{|z_2-z_1|^2}{|z_2-\bar z_1|^2}\qquad\text{and}\qquad u=\frac{|u_2-u_1|^2}{|u_2+\bar u_1|^2}, \tag{6.17}$$

we find precisely

$$a^j_\epsilon(\alpha)=C_\epsilon(j)A_\sigma(j_\epsilon|\alpha)\,. \tag{6.18}$$

Factorization Limit and $b^{-2}/2$-Shift Equation

The idea is just as before in the case of $1/2$-shift equations (chapter 5): In order to obtain a shift equation, take the limit $\mathrm{Im}(z_2) \to 0$ (implying $z \to 1-$). The problem that occurs here is that the original conformal blocks (6.11) are only defined for $z < u < 1$ and therefore, the factorization limit cannot be taken straight away. Hence, we have to make an analytic continuation to a suitable region. This is a little involved. Let us take the $\mathcal{F}^s_{j,-}$ block as prototype example to illustrate the problems that one encounters. It shows all features that can become important. Recall that

$$\begin{aligned}\mathcal{F}^s_{j,-}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}u^{-\beta}z^{1+\beta-\gamma}\cdot \\ &\cdot F_1\left(1+\beta+\beta'-\gamma, \beta, 1+\alpha-\gamma; 2+\beta-\gamma \,\middle|\, \frac{z}{u}; z\right).\end{aligned} \tag{6.19}$$

The conformal blocks we are using (6.11) are well defined in the region $z < u < 1$. Remember that the crossing ratios u and z, as given in equation (6.4), are both real with $0 \leq u, z \leq 1$ in the physical cut. For now, we can work in that region. Later (see below), we shall need to relax the reality condition on u, z slightly, i.e. we will move away from the physical cut in a controlled way. But for the moment, we like to maintain $u < 1$, which corresponds to equal signs $\sigma_1 = \sigma_2$. Since the factorization limit requires $z \to 1-$, we necessarily need to continue to a patch where $z > u$, $u < 1$, $z \approx 1$. We cannot use the standard analytic continuation of Appell's function F_1 as given in [97], because some coefficients turn out to become infinite in these formulae. This is due to the following relation between the parameters (6.12):

$$1 + \beta' - \gamma = 0. \tag{6.20}$$

The invalidation of the continuation formulae in [97] can be traced back to a special (logarithmic) case in the continuation of Gauss' hypergeometric function, when expanding F_1 appropriately (see appendix C). We will see this in detail shortly. In order to continue $\mathcal{F}^s_{j,-}$ sensibly, the first step is to expand the occuring $F_1(\ldots|\frac{z}{u}; z)$ in powers of the first variable $\frac{z}{u}$ (see appendix C)

$$\begin{aligned}&F_1\left(\beta, \beta, 1+\beta-\gamma; 2+\beta-\gamma \,\middle|\, \frac{z}{u}; z\right) = \\ &= \sum_{n=0}^{\infty} \frac{(\beta)_n(\beta_n)}{(2+\beta-\gamma)_n} F(\beta+n, 1+\beta-\gamma; 2+\beta-\gamma+n|z)\frac{(z/u)^n}{n!}\end{aligned} \tag{6.21}$$

and then use a standard analytic continuation (as found e.g. in [98]; we also cite it in appendix C) of Gauss' hypergeometric function $F(\ldots|z)$ in order to expand it in terms of $(1-z)$. As can be seen from the parameters, this is a generic case. Furthermore, since $0 \leq z < 1$, also $0 < (1-z) \leq 1$, meaning that no branch cuts are met and convergence in the domain needed is ensured. Two different terms arise from this continuation:

$$F_1\left(\beta,\beta,1+\beta-\gamma;2+\beta-\gamma\,\middle|\,\frac{z}{u};z\right) =: \frac{\Gamma(2+\beta-\gamma)\Gamma(1-\beta)}{\Gamma(2-\gamma)} I\left(\frac{z}{u};1-z\right) + \\ + \frac{\Gamma(2+\beta-\gamma)\Gamma(\beta-1)}{\Gamma(\beta)\Gamma(1+\beta-\gamma)}(1-z)^{1-\beta} II\left(\frac{z}{u};1-z\right). \tag{6.22}$$

Let us focus on the first one. After some minor manipulations it reads:

$$I\left(\frac{z}{u};1-z\right) = \sum_{n=0}^{\infty}\frac{(\beta_n)(\beta)_n}{(1)_n}F(\beta+n,1+\beta-\gamma;\beta|1-z)\frac{(z/u)^n}{n!}. \tag{6.23}$$

Now, we expand the hypergeometric function in powers of $(1-z)$ to yield a double expansion. Afterwards, the whole expression can be resummed and written as a single expansion again, but this time in powers of $(1-z)$ only:

$$I\left(\frac{z}{u};1-z\right) = \sum_{m=0}^{\infty}(1+\beta-\gamma)_m F\left(\beta,\beta+m;1\,\middle|\,\frac{z}{u}\right)\frac{(1-z)^m}{m!}. \tag{6.24}$$

In order to reach the desired patch, the "inner" hypergeometric function must now be continued to yield an expansion in the variable $\frac{u}{z}$. This, however, is no longer a generic case, but a logarithmic one. It is precisely where the formula for the full Appell function F_1 in [97] breaks down. Nevertheless, we can do it right here. The appropriate continuation formula for the Gauss function is found in [98], for example. For convenience, we have also included it in appendix C. After its use, the resulting series it not easily resummend again to yield some familiar functions. But since we are taking the limit $z \to 1$ anyway, we can isolate the leading term in $(1-z)$, which is just the term with $m = 0$ in the above expansion. Thus, for $z \to 1$, the result is

$$I \simeq \frac{e^{i\pi\beta}u^{\beta}}{\Gamma(1-\beta)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(\beta)_n(\beta)_n}{(1)_n}\frac{u^n}{n!}\left[-\log(u)+h_n(\beta)-i\pi\right]\{1+\mathcal{O}(1-z)\}. \tag{6.25}$$

Note that the u dependence $\propto \log(u)$ looks rather unfamiliar, but is actually nothing to worry about: The correct expansion variable for the OPE needed here is actually $(1-u)$ and $-\log(u) = -\log(1-(1-u)) = (1-u)\{1+\mathcal{O}(1-u)\}$. Together with the prefactor $z^{-j}(1-z)^{-b^{-2}/2}u^{-\beta}z^{1+\beta-\gamma}$, which belongs to the definition of $\mathcal{F}^{s}_{j,-}$, this term has the correct asymptotics corresponding to propagation of the modes b^{-2} and $-b^{-2}-1$ (see 5.3.2). It does however not contribute to the propagation of the identity and consequently does not enter the factorization constraint.

There is still one more comment to make about the above continuation of Gauss' hypergeometric function from $\eta \equiv \frac{z}{u}$ to $\frac{u}{z} = \frac{1}{\eta}$. The Gauss function $F(a,b;c|\eta)$ has a branch cut along the line $\eta \in \mathbb{R}_{>1}$. Continuation formulae are invalidated if η or its transformed counterpart take values in this line. This is, however, precisely the situation we need to handle, if we stay inside the physical cut region. We overcome the problem by relaxing the reality condition on u, z slightly: Given the Gauss function $F(a,b;c|\eta)$, with $\eta \in \mathbb{R}$, $\eta > 1$, let $\eta \mapsto \eta e^{-i\epsilon}$ ($\epsilon > 0$). Gauss' hypergeometric function is continuous from below in η (but not from above for $\eta > 1$, so there is no choice involved here), i.e.

$$F(a,b;c|\eta) = \lim_{\epsilon \to 0+} F(a,b;c|\eta e^{-i\epsilon}). \tag{6.26}$$

We therefore take occuring phases to be in $(-2\pi, 0]$. In particular, this means $(-) = e^{-i\pi}$. On the right hand side of (6.26), an analytic continuation formula (see [98] or appendix C) can now be used. In the end, the epsilon is removed by taking it to be zero and we are back inside the physical cut. This procedure automatically selects the correct phases. In practice, all we need to do is keep the phase prescription in mind and write down everything without the epsilon. In the logarithmic case $b = a + m$, $m \in \mathbb{Z}_{\geq 0}$, the continuation is

$$\begin{aligned} F\left(a, a+m; c\,|\eta\right) &= \frac{\Gamma(c)\,(-\eta)^{-a-m}}{\Gamma(a+m)\Gamma(c-a)} \cdot \\ \cdot \sum_{n=0}^{\infty} \frac{(a)_{n+m}(1-c+a)_{n+m}}{n!(n+m)!} &\left(\frac{1}{\eta}\right)^n \left[\log(-\eta) + h_n(a,c,m)\right] + \\ &+ \frac{\Gamma(c)\,(-\eta)^{-a}}{\Gamma(a+m)} \cdot \sum_{n=0}^{m-1} \frac{\Gamma(m-n)(a)_n}{\Gamma(c-a-n)n!} \left(\frac{1}{\eta}\right)^n . \end{aligned} \tag{6.27}$$

With our phase prescription, the logarithm becomes $\log(-\eta) = -\log(\frac{1}{\eta}) - i\pi$ and $(-\eta)^{-a} = e^{i\pi a}(\frac{1}{\eta})^a$. This is how the phase $e^{i\pi\beta}$ and the $-i\pi$ in (6.25) arise.

Let us now turn to the second term in the continuation of $F_1(\ldots|\frac{z}{u}; z)$:

$$II\left(\frac{z}{u}; 1-z\right) = \sum_{n=0}^{\infty} (\beta)_n F(2-\gamma, 1+n; 2-\beta|1-z)\frac{(z/u)^n}{n!}. \tag{6.28}$$

Expanding and resumming as above, this can equally be written as

$$II\left(\frac{z}{u}; 1-z\right) = \sum_{m=0}^{\infty} \frac{(2-\gamma)_m(1)_m}{(2-\beta)_m} F\left(\beta, 1+m; 1 \,\middle|\, \frac{z}{u}\right) \frac{(1-z)^m}{m!} . \tag{6.29}$$

This time, the continuation from $\frac{z}{u}$ to $\frac{u}{z}$ follows a generic case. The phase prescription is exactly as above. In the end, as $z \to 1$, we obtain

$$II \simeq e^{i\pi\beta} u^{\beta} F(\beta, \beta; \beta|u)\,\{1 + \mathcal{O}(1-z)\}\ . \tag{6.30}$$

Together with the overall prefactor $z^{-j}(1-z)^{1+b^{-2}/2}u^{-\beta}z^{1+\beta-\gamma}$ (coming from the definition of $\mathcal{F}^s_{j,-}$ together with the $(1-z)^{1-\beta}$ from the first continuation) and using that

$$F(\beta,\beta;\beta|u) = (1-u)^{-\beta}\,, \tag{6.31}$$

this shows precisely the asymptotic behaviour of the propagating identity. This term therefore enters the factorization constraint.

The $\mathcal{F}^s_{j,\times}$ block

$$\begin{aligned}\mathcal{F}^s_{j,\times}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}u^{1-\gamma}\cdot\\ &\cdot G_2\left(\beta', 1+\alpha-\gamma; 1+\beta-\gamma, \gamma-1 \,\middle|\, -\frac{z}{u}; u\right).\end{aligned} \tag{6.32}$$

can be treated along similar lines. Here, we first continue the second variable u to $(1-u)$. As this turns out to be a generic case, the resumming works out again and we can then continue in the firrst variable from $\frac{z}{u}$ to $\frac{u}{z}$. This is again generic. The overall result does not contain a term corresponding to the identity propagating and hence no contribution to the shift equation is generated here.

Finally, the $\mathcal{F}^s_{j,+}$ block

$$\mathcal{F}^s_{j,+}(u|z) = z^{-j}(1-z)^{-b^{-2}/2}F_1(\alpha,\beta,\beta';\gamma|u;z) \tag{6.33}$$

is easily continued using standard formulae of e.g. [97]. We obtain

$$\begin{aligned}\mathcal{F}^s_{j,+} &\simeq (1-z)^{1+b^{-2}/2}(1-u)^{b^{-2}}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta'-\gamma)}{\Gamma(\alpha)\Gamma(\beta')}\cdot[1+\mathcal{O}(1-z)]+\\ &+(1-z)^{b^{-2}/2}\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta')}F(\alpha,\beta;\gamma-\beta'|u)\cdot[1+\mathcal{O}(1-z)]\,.\end{aligned} \tag{6.34}$$

The first summand gives the identity contribution and enters the shift equation.

While the original two point function, using the conformal blocks (6.11), was defined in the patch $0 \le z < u < 1$, the analytically continued expressions are valid for $0 \le u < z \le 1$ (we always have $z \le 1$ by definition) and therefore allow for the derivation of the factorization constraint. Using expansions in $1-\frac{z}{u}$ in (6.11), this two point function can be shown to possess a finite limit at $u = z$. This has been anticipated in [73]. Moreover, since we are using analytic continuations, it must also be continuous at $u = z$. This feature has been postulated as an axiom in [73]. Our two point function shows all their requirements except the anticipated weakening of the Cardy-Lewellen factorization constraint, that is, the two point function in the patch $0 \le u < z \le 1$ is completely determined from its expression in $0 \le z < u < 1$, by analytic continuation.

Finally, let us state here the asymptotic behaviour of the analytically continued conformal blocks in full. As we have already noted earlier, the parameters α, β, β', γ are not all independent of each other, but obey the relations $\alpha = \beta$ and

$1 + \beta' - \gamma = 0$. We therefore eliminate α and β' and only work with β and γ. Up to terms which are of order $\{1 + \mathcal{O}(1-z, 1-u)\}$ we find that

$$
\begin{aligned}
\mathcal{F}^s_{j,+}(u|z) \simeq{}& \frac{\Gamma(\gamma)\Gamma(\beta-1)}{\Gamma(\beta)\Gamma(\gamma-1)}(1-z)^{1+b^{-2}/2}(1-u)^{b^{-2}} + \\
&+ \frac{\Gamma(\gamma)\Gamma(1-2\beta)}{\Gamma(\gamma-\beta)\Gamma(1-\beta)}(1-z)^{-b^{-2}/2} + \\
&+ \frac{\Gamma(\gamma)\Gamma(1-\beta)\Gamma(2\beta-1)}{\Gamma(\gamma-\beta)\Gamma(\beta)\Gamma(\beta)}(1-z)^{-b^{-2}/2}(1-u)^{2b^{-2}+1}, \\
\mathcal{F}^s_{j,-}(u|z) \simeq{}& \frac{\Gamma(2+\beta-\gamma)\Gamma(\beta-1)}{\Gamma(\beta)\Gamma(1+\beta-\gamma)}\mathrm{e}^{\mathrm{i}\pi\beta}(1-z)^{1+b^{-2}/2}(1-u)^{b^{-2}} + \\
&+ \frac{\Gamma(2+\beta-\gamma)\Gamma(1-2\beta)}{\Gamma(2-\gamma)\Gamma(1-\beta)\Gamma(1-\beta)\Gamma(\beta)}\mathrm{e}^{\mathrm{i}\pi\beta}(1-z)^{-b^{-2}/2} + \\
&+ \frac{\Gamma(2+\beta-\gamma)\Gamma(2\beta-1)}{\Gamma(2-\gamma)\Gamma(\beta)\Gamma(\beta)\Gamma(\beta)}\mathrm{e}^{\mathrm{i}\pi\beta}(1-z)^{-b^{-2}/2}(1-u)^{2b^{-2}+1} + \\
&+ \frac{\Gamma(2+\beta-\gamma)}{\Gamma(2-\gamma)\Gamma(\beta)}\mathrm{e}^{\mathrm{i}\pi\beta}(1-z)^{-b^{-2}/2}\sum_{n=0}^{\infty} h_n(\beta)\frac{(\beta)_n(\beta)_n}{(1)_n}\frac{u^n}{n!}, \\
\mathcal{F}^s_{j,\times}(u|z) \simeq{}& \frac{\Gamma(2-\gamma)\Gamma(1-2\beta)\Gamma(\gamma-\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)\Gamma(1-\beta)}(1-z)^{-b^{-2}/2} + \\
&+ \Bigg[\frac{\Gamma(2-\gamma)\Gamma(2\beta-\gamma)}{\Gamma(1+\beta-\gamma)\Gamma(1+\beta-\gamma)}\mathrm{e}^{\mathrm{i}\pi(\gamma-1)} - \\
&\quad - \frac{\Gamma(2-\gamma)\Gamma(\gamma-2\beta)\Gamma(2\beta-1)\Gamma(\gamma-\beta)}{\Gamma(\beta)\Gamma(1-\beta)\Gamma(1-\beta)\Gamma(\gamma-1)}\mathrm{e}^{\mathrm{i}\pi 2\beta}\Bigg] \cdot \\
&\cdot (1-z)^{-b^{-2}/2}(1-u)^{2b^{-2}+1}.
\end{aligned}
\tag{6.35}
$$

This must be compard to the asymptotics of (6.3) when using the bulk-boundary OPE of $\Theta_{b^{-2}/2}$ (see the discussion in section 5.2.4 together with section 5.3.2)[1]:

$$
\begin{aligned}
G^{(2)}_{j,\alpha}(u_1, u_2|z_1, z_2) \simeq{}& |z_1 - \bar{z}_1|^{-2h(j)}\, |z_2 - \bar{z}_2|^{-2h(b^{-2}/2)} \cdot \\
&\cdot |u_1 + \bar{u}_1|^{2j}\, |u_2 + \bar{u}_2|^{b^{-2}} A_\sigma(j|\alpha)A_\sigma(b^{-2}/2|\alpha).
\end{aligned}
\tag{6.36}
$$

We see that the terms $\propto (1-z)^{1+b^{-2}/2}(1-u)^{b^{-2}}$ correspond to propagation of the identity. These are the terms that enter the shift equation. Note that the $\mathcal{F}^s_{j,\times}$ block does not contribute to these. The other terms which have a leading

[1] We take the signs $\sigma_1 = \mathrm{sgn}(u_1 + \bar{u}_1)$ and $\sigma_2 = \mathrm{sgn}(u_2 + \bar{u}_2)$ to be equal here: $\sigma_1 = \sigma_2 = \sigma$. One could, of course, also take $\sigma_1 = -\sigma_2 = \sigma$. This would lead to further constraints. Note, however, that for $\sigma_1 = \sigma_2$, the crossing ratio u lies in the interval $0 < u < 1$, whereas for $\sigma_1 = -\sigma_2$ we have $u > 1$. In the latter case one has to use different analytic continuations when taking $z \to 1-$. We have not worked that out.

z-dependence $\propto (1-z)^{-b^{-2}/2}$ can be identified with propagation of the two other possible boundary fields $\Psi_{b^{-2}}$ (which has leading u-dependence $\propto (1-u)^0$) and $\Psi_{-b^{-2}-1}$ (u-dependence $\propto (1-u)^{2b^{-2}+1}$). (Recall that, because $\Theta_{b^{-2}/2}$ is a degenerate field, its bulk boundary OPE is highly restricted). Conveniently, all terms that appear fit in nicely with this interpretation. Only in the fourth summand in the block $\mathcal{F}^s_{j,-}$ we cannot extract the explicit $(1-u)$-dependence, because of the additional coefficients $h_n(\beta)$. They stem from the analytic continuation of a Gauss hypergeometric function in an exceptional (logarithmic) case (see appendix C). Yet, from its $(1-z)$-dependence it is clear that this term does not come from a propagation of the identity and therefore does not affect the shift equation. Collecting the terms that stem from the identity propagation on both sides yields the desired $b^{-2}/2$-shift equation for the redefined one point amplitude (5.29)

$$\left[\Gamma(1+b^2)\right]^{-1} f_\sigma\left(\frac{b^{-2}}{2}\right) f_\sigma(j) = f_\sigma\left(j+\frac{b^{-2}}{2}\right) + \mathrm{e}^{-\mathrm{i}\pi b^{-2}} f_\sigma\left(j-\frac{b^{-2}}{2}\right). \quad (6.37)$$

Solving the Shift Equations

The $1/2$-shift equation (5.35) is solved by [76]

$$f_\sigma(j|m,n) = \frac{\mathrm{i}\pi\sigma\mathrm{e}^{\mathrm{i}\pi m}}{\Gamma(-b^2)\sin[\pi n b^2]}\mathrm{e}^{-\mathrm{i}\pi\sigma(m-\frac{1}{2})(2j+1)}\frac{\sin[\pi n b^2(2j+1)]}{\sin[\pi b^2(2j+1)]}, \quad (6.38)$$

with $n, m \in \mathbb{Z}$.[2] Note that this also satisfies the reflection symmetry constraint (5.30). One checks however quite easily that it does not satisfy our new shift equation (6.37). Interestingly, the obstruction is precisely the term that stems from the $\mathcal{F}^s_{j,-}$ conformal block. Without it, the equation would be obeyed. Nevertheless, we need to conclude that the irregular $AdS_2^{(d)}$ brane with gluing map ρ_2 is not consistent.

6.1.2 Irregular $AdS_2^{(d)}$ Branes - Gluing Map ρ_1

Ansatz for the One Point and Two Point Functions

The form of the one point function was (recall section 5.5.2)

$$\left\langle\Theta_j(u|z)\right\rangle_\alpha = |z-\bar{z}|^{-2h(j)}\,|u-\bar{u}|^{2j}\,A_\sigma(j|\alpha). \quad (6.39)$$

The ansatz for the boundary two point function with degenerate field $b^{-2}/2$ (fixing the u_i and z_i dependence up to a dependence on the crossing ratios) is

$$\begin{aligned} G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) = {} & |z_1-\bar{z}_1|^{2[h(b^{-2}/2)-h(j)]}\,|z_1-\bar{z}_2|^{-4h(b^{-2}/2)}\,\cdot \\ & \cdot\,|u_1-\bar{u}_1|^{2j-b^{-2}}\,|u_1-\bar{u}_2|^{2b^{-2}}\,H^{(2)}_{j,\alpha}(u|z), \end{aligned} \quad (6.40)$$

[2] This is how the solution has been given in [76]. In fact we only need $m \in \mathbb{Z}$ to satisfy equation (5.35).

with crossing ratios

$$z = \frac{|z_2 - z_1|^2}{|z_2 - \bar{z}_1|^2} \quad \text{and} \quad u = \frac{|u_2 - u_1|^2}{|u_2 - \bar{u}_1|^2}. \tag{6.41}$$

Knizhnik-Zamolodchikov Equation

The conformal blocks that solve the Knizhnik-Zamolodchikov equation for z_2 turn out to be just the same ones as for gluing map ρ_2, so they are given by (6.11) with parameters

$$\alpha = \beta = -b^{-2}, \quad \beta' = -2j - 1 - b^{-2}, \quad \gamma = -2j - b^{-2}. \tag{6.42}$$

Expansion Coefficients

The expansion coefficients also stay as before:

$$a^j_\epsilon(\alpha) = C_\epsilon(j) A_\sigma(j_\epsilon|\alpha). \tag{6.43}$$

Factorization Limit and $b^{-2}/2$-Shift Equation

Taking the limit $\mathrm{Im}(z_2) \to 0$, we obtain the same $b^{-2}/2$-shift equation as for gluing map ρ_2 (6.37). As the $1/2$-shift equations (sections 5.5.1, 5.5.2) also coincide, this means that the irregular discrete branes that arise from gluing maps ρ_1 and ρ_2 respectively, are indeed isomorphic. Compare to our remarks in section 5.2.2, where we speculated that this may happen. However, we do not find a solution (other than the trivial one) that satsifies both factorization constraints, as we have already explained in section 6.1.1.

6.1.3 Regular $AdS_2^{(d)}$ Branes - Gluing Map ρ_2

Ansatz for the One Point and Two Point Functions

The form of the one point function was (section 5.5.3)

$$\left\langle \Theta_j(u|z) \right\rangle_\alpha = (z - \bar{z})^{-2h(j)} (u + \bar{u})^{2j} A(j|\alpha). \tag{6.44}$$

The boundary two point function with degenerate field $\Theta_{b^{-2}/2}$ is

$$\begin{aligned} G^{(2)}_{j,\alpha}(u_1, u_2|z_1, z_2) = (z_1 - \bar{z}_1)^{-2h(j)} (z_2 - \bar{z}_2)^{-2h(b^{-2}/2)} \cdot \\ \cdot (u_1 + \bar{u}_1)^{2j} (u_2 + \bar{u}_2)^{b^{-2}} H^{(2)}_{j,\alpha}(u|z), \end{aligned} \tag{6.45}$$

with crossing ratios

$$z = \frac{|z_1 - z_2|^2}{(z_1 - \bar{z}_1)\,(z_2 - \bar{z}_2)} \qquad \text{and} \qquad u = -\frac{|u_1 - u_2|^2}{(u_1 + \bar{u}_1)\,(u_2 + \bar{u}_2)}. \tag{6.46}$$

Again, they take values in $z \in (-\infty, 0)$ (because z_1, z_2 are in the upper half plane), $u \in (-\infty, 0)$ (if we take $\sigma_1 = \sigma_2$).

Knizhnik-Zamolodchikov Equations

Solving the Knizhnik-Zamolodchikov equations for z_2 results in the following conformal blocks:

$$\begin{aligned}
\mathcal{F}^s_{j,+}(u|z) &= z^{-j}(1-z)^{-j}F_1(\alpha,\beta,\beta';\gamma|u;z),\\
\mathcal{F}^s_{j,-}(u|z) &= z^{-j}(1-z)^{-j}u^{-\beta}z^{1+\beta-\gamma}\cdot\\
&\qquad \cdot F_1\left(1+\beta+\beta'-\gamma,\beta,1+\alpha-\gamma;2+\beta-\gamma\,\middle|\,\frac{z}{u};z\right),\\
\mathcal{F}^s_{j,\times}(u|z) &= z^{-j}(1-z)^{-j}u^{1-\gamma}\cdot\\
&\qquad \cdot G_2\left(\beta',1+\alpha-\gamma;1+\beta-\gamma,\gamma-1\,\middle|\,-\frac{z}{u};u\right),
\end{aligned} \tag{6.47}$$

this time with parameters

$$\alpha = -2j, \quad \beta = -b^{-2}, \quad \beta' = -2j-1-b^{-2}, \quad \gamma = -2j-b^{-2}. \tag{6.48}$$

Note that the common $(1-z)$-dependence is changed here to $(1-z)^{-j}$.

Expansion Coefficients

Using the OPE on the left hand side of (6.45), we find

$$\begin{aligned}
&G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) \simeq\\
&\simeq |z_2 - z_1|^{-2j}\,(z_1-\bar{z}_1)^{-2h(j_+)}\,(u_1+\bar{u}_1)^{2j+b^{-2}}\,C_+(j)A(j_+|\alpha)+\\
&+ |z_2 - z_1|^{2j+2}\,|u_2-u_1|^{2b^{-2}}\,(z_1-\bar{z}_1)^{-2h(j_-)}\,(u_1+\bar{u}_1)^{2j-b^{-2}}\\
&\qquad\qquad\qquad\qquad \cdot C_-(j)A(j_-|\alpha)+\\
&+ |z_2 - z_1|^{-2j}\,|u_2-u_1|^{2(2j+1+b^{-2})}\,(z_1-\bar{z}_1)^{-2h(j_\times)}\,(u_1+\bar{u}_1)^{-2j-2-b^{-2}}\\
&\qquad\qquad\qquad\qquad \cdot C_\times(j)A(j_\times|\alpha).
\end{aligned} \tag{6.49}$$

On the right hand side, taking $|z_2 - z_1| \to 0$ ($\Rightarrow z \to 0-$) followed by $|u_2 - u_1| \to 0$ ($\Rightarrow u \to 0-$), the conformal blocks (6.47) show the behaviour

$$\begin{aligned}
\mathcal{F}^s_{j,+}(u|z) &\simeq z^{-j}\,,\\
\mathcal{F}^s_{j,-}(u|z) &\simeq z^{j+1}u^{b^{-2}}\,,\\
\mathcal{F}^s_{j,\times}(u|z) &\simeq z^{-j}u^{2j+1+b^{-2}}\,.
\end{aligned} \tag{6.50}$$

Remember that they are accompanied by the prefactor

$$\begin{aligned}(z_1-\bar z_1)^{-2h(j)}\,(z_2-\bar z_2)^{-2h(b^{-2}/2)}\,(u_1+\bar u_1)^{2j}\,(u_2+\bar u_2)^{b^{-2}} &\simeq \\ \simeq (z_1-\bar z_1)^{-2[h(j)+h(b^{-2}/2)]}\,(u_1+\bar u_1)^{2j+b^{-2}}&\end{aligned} \tag{6.51}$$

from (6.45) and that

$$z=\frac{|z_1-z_2|^2}{(z_1-\bar z_1)\,(z_2-\bar z_2)} \qquad \text{and} \qquad u=-\frac{|u_1-u_2|^2}{(u_1+\bar u_1)\,(u_2+\bar u_2)}\,. \tag{6.52}$$

Now, we need to be careful about phase factors that arise from z^{-j}, $u^{b^{-2}}$, and so on. In order to be consistent with the choice of phase we have to make because of the branch cut of the hypergeometric functions (that is, we take phases to be in $(-2\pi,0]$ — see section 6.1.1), we have to use the relations ($\nu\in\mathbb{C}$)

$$\begin{aligned} z^\nu &= \mathrm{e}^{-4\pi\mathrm{i}\nu}\,|z_1-z_2|^{2\nu}\,(z_1-\bar z_1)^{-\nu}\,(z_2-\bar z_2)^{-\nu}\,,\\ u^\nu &= \mathrm{e}^{\mathrm{i}\pi(\sigma-2)\nu}\,|u_1-u_2|^{2\nu}\,(u_1+\bar u_1)^{-\nu}\,(u_2+\bar u_2)^{-\nu}\,,\end{aligned} \tag{6.53}$$

where $\sigma=\mathrm{sgn}(u_1+\bar u_1)=\mathrm{sgn}(u_2+\bar u_2)$. One can quickly check that this is correct by comparing the complex phases on both sides of the equations. With the help of this, one sees that the coefficients $a_\epsilon^{j,b^{-2}/2}$ need to be defined with phases as follows

$$\begin{aligned} a_+^{j,b^{-2}/2}(\alpha) &= \mathrm{e}^{-4\pi\mathrm{i}j}C_+^{b^{-2}/2}(j)A(j_+|\alpha),\\ a_-^{j,b^{-2}/2}(\alpha) &= \mathrm{e}^{4\pi\mathrm{i}j-\mathrm{i}\pi(\sigma-2)b^{-2}}C_-^{b^{-2}/2}(j)A(j_-|\alpha),\\ a_\times^{j,b^{-2}/2}(\alpha) &= \mathrm{e}^{-4\pi\mathrm{i}j-\mathrm{i}\pi(\sigma-2)(2j+1+b^{-2})}C_\times^{b^{-2}/2}(j)A(j_\times|\alpha)\,.\end{aligned} \tag{6.54}$$

Factorization Limit and $b^{-2}/2$-Shift Equation

Taking the limit $\mathrm{Im}(z_2)\to 0$, we have this time that $z\to-\infty$ and $u\to-\infty$ ($\sigma_1=\sigma_2$). Therefore, as the conformal blocks must be expanded in the variables $\left(\frac{1}{z},\frac{1}{u}\right)$, we need to take different analytic continuations of the occuring Appell and Horn functions than before. Yet, the procedure and the issues one needs to take care of, and that we have illustrated in great detail in section 6.1.1, stay the same: We obtain the expansions of Appell's function F_1 and Horn's function G_2 by making repeated use of their one variable expansions with ordinary hypergeometric functions as coefficients (see appendix C). To the coefficient functions we can apply standard analytic continuation formulae (collected, again, in appendix C), resum and repeat everything if necessary — just as in section 6.1.1. For the regular branes, note that the parameters α, β, β', γ are once again not all independent of each other, but obey the relations $\alpha+\beta-\gamma=0$ and $1+\beta'-\gamma=0$

(6.48), which we use to eliminate α and β' and only work with β and γ. Up to terms which are of order $\left\{1+\mathcal{O}(\frac{1}{z},\frac{1}{u})\right\}$ we find that

$$
\begin{aligned}
\mathcal{F}^s_{j,+}(u|z) \simeq{}& \left[\frac{\Gamma(\gamma)\Gamma(2\beta-\gamma)\Gamma(1-2\beta+\gamma)\Gamma(\beta-1)}{\Gamma(\beta)\Gamma(\beta)\Gamma(\gamma-1)\Gamma(1-\beta)}+\right.\\
&+\left.\frac{\Gamma(\gamma)\Gamma(\gamma-2\beta)\Gamma(1+2\beta-\gamma)\Gamma(\beta-1)}{\Gamma(\gamma-\beta)\Gamma(\gamma-1)\Gamma(\beta)\Gamma(1+\beta-\gamma)}\mathrm{e}^{\mathrm{i}\pi(2\beta-\gamma)}\right]\mathrm{e}^{-\mathrm{i}\pi j}z^0u^0+\\
&+\left[\frac{\Gamma(\gamma)\Gamma(1-2\beta)}{\Gamma(\gamma-\beta)\Gamma(1-\beta)}\mathrm{e}^{\mathrm{i}\pi(2j+1+2b^{-2})}\right]\mathrm{e}^{-\mathrm{i}\pi j}z^{1+b^{-2}}u^{b^{-2}}+\\
&+\left[\frac{\Gamma(\gamma)\Gamma(2\beta-\gamma)\Gamma(1-2\beta+\gamma)\Gamma(1-\beta)}{\Gamma(\beta)\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(2-2\beta)}\mathrm{e}^{\mathrm{i}\pi(\beta-1)}+\right.\\
&+\left.\frac{\Gamma(\gamma)\Gamma(\gamma-2\beta)\Gamma(2\beta-1)\Gamma(1+2\beta-\gamma)\Gamma(1-\beta)}{\Gamma(\gamma-\beta)\Gamma(\gamma-1)\Gamma(\beta)\Gamma(\beta)\Gamma(2-\gamma)}\mathrm{e}^{-\mathrm{i}\pi(1+\gamma-3\beta)}\right]\cdot\\
&\cdot\mathrm{e}^{-\mathrm{i}\pi j}z^{1+b^{-2}}u^{-1-b^{-2}},\\
\mathcal{F}^s_{j,-}(u|z) \simeq{}& \left[\frac{\Gamma(2+\beta-\gamma)\Gamma(\beta-1)}{\Gamma(\beta)\Gamma(1+\beta-\gamma)}\mathrm{e}^{-\mathrm{i}\pi(1-2\beta)}\right]\mathrm{e}^{-\mathrm{i}\pi j}z^0u^0+\\
&+\left[\frac{\Gamma(2+\beta-\gamma)\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(2-\gamma)}\mathrm{e}^{-\mathrm{i}\pi\beta}\right]\mathrm{e}^{-\mathrm{i}\pi j}z^{1+b^{-2}}u^{b^{-2}}+\\
&+\left[\frac{\Gamma(2+\beta-\gamma)\Gamma(2\beta-1)\Gamma(1-\beta)}{\Gamma(\beta)\Gamma(\beta)\Gamma(2-\gamma)}\mathrm{e}^{-\mathrm{i}\pi(2-3\beta)}\right]\mathrm{e}^{-\mathrm{i}\pi j}z^{1+b^{-2}}u^{-1-b^{-2}},\\
\mathcal{F}^s_{j,\times}(u|z) \simeq{}& \left[\frac{\Gamma(2-\gamma)\Gamma(\gamma-2\beta)\Gamma(\gamma-\beta)\Gamma(1+2\beta-\gamma)\Gamma(\beta-1)}{\Gamma(1-\beta)\Gamma(1-\beta)\Gamma(\gamma-1)\Gamma(\beta)\Gamma(1+\beta-\gamma)}\mathrm{e}^{-\mathrm{i}\pi(1-2\beta)}+\right.\\
&+\left.\frac{\Gamma(2-\gamma)\Gamma(2\beta-\gamma)\Gamma(1-2\beta+\gamma)\Gamma(\beta-1)}{\Gamma(1+\beta-\gamma)\Gamma(1+\beta-\gamma)\Gamma(\gamma-1)\Gamma(1-\beta)}\mathrm{e}^{-\mathrm{i}\pi(1-\gamma)}\right]\mathrm{e}^{-\mathrm{i}\pi j}z^0u^0+\\
&+\left[\frac{\Gamma(2-\gamma)\Gamma(\gamma-\beta)\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)\Gamma(1-\beta)}\mathrm{e}^{-\mathrm{i}\pi\beta}\right]\mathrm{e}^{-\mathrm{i}\pi j}z^{1+b^{-2}}u^{b^{-2}}+\\
&+\left[\frac{\Gamma(\gamma-2\beta)\Gamma(\gamma-\beta)\Gamma(2\beta-1)\Gamma(1+2\beta-\gamma)\Gamma(1-\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)\Gamma(\gamma-1)\Gamma(\beta)\Gamma(\beta)}\mathrm{e}^{-\mathrm{i}\pi(2-3\beta)}+\right.\\
&+\left.\frac{\Gamma(2-\gamma)\Gamma(2\beta-\gamma)\Gamma(1-2\beta+\gamma)\Gamma(1-\beta)}{\Gamma(1+\beta-\gamma)\Gamma(1+\beta-\gamma)\Gamma(\gamma-\beta)\Gamma(2-2\beta)}\mathrm{e}^{-\mathrm{i}\pi(2-\beta-\gamma)}\right]\cdot\\
&\cdot\mathrm{e}^{-\mathrm{i}\pi j}z^{1+b^{-2}}u^{-1-b^{-2}}.
\end{aligned}
\tag{6.55}
$$

We have written out these long and tedious terms for the reader to appreciate that all terms that arise at leading order are again grouped into three different asymptotics: z^0u^0 (corresponding to the propagating identity), $z^{1+b^{-2}}u^{b^{-2}}$

(corresponding to the field $\Psi_{b^{-2}}$) and $z^{1+b^{-2}}u^{-1-b^{-2}}$ (corresponding to $\Psi_{-b^{-2}-1}$). Presumably, all sums can be simplified. We have only done so for the identity contributions, because this is all we need. Writing down the identity contributions only and simplifying the occuring terms, the result looks much more convenient:

$$\begin{aligned} \mathcal{F}^s_{j,+}(u|z) &\simeq \frac{2j+1+b^{-2}}{1+b^{-2}}\mathrm{e}^{\mathrm{i}\pi j} + \dots , \\ \mathcal{F}^s_{j,-}(u|z) &\simeq -\frac{2j+1}{1+b^{-2}}\mathrm{e}^{-\mathrm{i}\pi(j+1-2b^{-2})} + \dots , \\ \mathcal{F}^s_{j,\times}(u|z) &\simeq \frac{\Gamma(2j+2+b^{-2})\Gamma(-1-b^{-2})\Gamma(-2j)}{\Gamma(-2j-1-b^{-2})\Gamma(1+b^{-2})\Gamma(2j+1)}\mathrm{e}^{-\mathrm{i}\pi(3j+2b^{-2})} + \dots . \end{aligned} \tag{6.56}$$

The dots now represent the contributions of the two other fields that are different from the identity. Using the bulk boundary OPE for $\Theta_{b^{-2}/2}$ (5.19) in the point function (6.45), the leading contribution of the identity is

$$\begin{aligned} G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) \simeq (z_2-\bar{z}_2)^{1+b^{-2}/2}\,(u_2\pm\bar{u}_2)^{b^{-2}}\,(z_1-\bar{z}_1)^{-2h(j)}\,\cdot \\ \cdot\,(u_1\pm\bar{u}_1)^{2j}\,C(b^{-2}/2,0|\alpha)A(j|\alpha) \end{aligned} \tag{6.57}$$

Comparing this to the expressions resulting from the conformal blocks, we obtain the following $b^{-2}/2$-shift equation

$$\begin{aligned} \left[\Gamma(1+b^2)\right]^{-1} f\left(\frac{b^{-2}}{2}\right) f(j) = \mathrm{e}^{-\mathrm{i}\pi 3j} f\left(j+\frac{b^{-2}}{2}\right) - \\ -\mathrm{e}^{\mathrm{i}\pi 3j}\mathrm{e}^{-\mathrm{i}\pi\sigma b^{-2}} f\left(j-\frac{b^{-2}}{2}\right) + \mathrm{e}^{-\mathrm{i}\pi 3j}\mathrm{e}^{-\mathrm{i}\pi\sigma(2j+b^{-2})} f\left(-j-1-\frac{b^{-2}}{2}\right). \end{aligned} \tag{6.58}$$

Solving the Shift Equations

An immediate drawback of (6.58) is the occurence of σ's on its right hand side. Lacking a σ-dependence of the (redefined) one point amplitude f, it is not clear how these σ's could reasonably be incorporated. One might be tempted to require, for example, $b^{-2} \in \mathbb{Z}$, $2j \in \mathbb{Z}$ in order to make the σ-dependence vanish. However, we should not hasten and require such conditions already at this point, since it might still be possible that a solution f exists such that the σ-dependent terms cancel out without any additional assumptions. (We will actually encounter such a case in the next section, when solving the shift equations for regular AdS_2 branes with gluing map ρ_1). Yet, in the present situation we can rule out such a possibility. Without making any further assumptions, we can proof that there is no solution that satisfies both shift equations (5.73), (6.58) together with the reflection symmetry constraint (5.33).

A No Solution Theorem

In order to give the proof that there is no solution to both factorization constraints together with the reflection symmetry constraint in the case of regular discrete branes with gluing map ρ_2, let us make the redefinition

$$f(j) \equiv -\frac{\pi e^{i\frac{\pi}{4}b^2}}{\Gamma(-b^2)} \frac{e^{-i\pi\frac{b^2}{4}(2j+1)^2}}{\sin[\pi b^2(2j+1)]} g(j)$$

and work with $g(j)$ here. Note that it has opposite parity of $f(j)$. The shift equations (5.73) and (6.58) in terms of $g(j)$ are given as (**2**) and (**3**) in the following

THEOREM: The system of equations

$$(\mathbf{1})\quad g(j) = -g(-j-1)$$

$$(\mathbf{2})\quad g(1/2)g(j) = g(j+1/2) - g(j-1/2)$$

$$\begin{aligned}(\mathbf{3})\quad g(b^{-2}/2)g(j) = {} & e^{-i\pi 4j} g(j+b^{-2}/2) + e^{i\pi 4j} e^{-i\pi\sigma b^{-2}} g(j-b^{-2}/2) - \\ & - e^{-i\pi 4j} e^{-i\pi\sigma(2j+b^{-2})} g(-j-1-b^{-2}/2)\end{aligned}$$

does not admit a non-trivial solution $g(j)$.

► PROOF: In order to proof this result, we proceed in two steps. The first one is to show that any functions satisfying (**1**) and (**2**) must be 1-periodic. The second step establishes that any 1-periodic function cannot satisfy (**3**).

1st Step. Any solution to (**1**) *and* (**2**) *must be periodic with period* 1*:* (**1**) tells us that $g(-1/2) = -g(-1/2)$, i.e. $g(-1/2) = 0$. Thus, using (**2**) at $j = -1/2$ together with (**1**) $g(-1) = -g(0)$, we obtain $g(0) = 0$. (**2**) taken at $j = 0$ then reveals immediately that

$$g(1/2) = g(-1/2) = 0\,.$$

Hence, (**2**) implies 1-periodicity of $g(j)$.

2nd Step. The 1*-periodic function* $g(j)$ *cannot satisfy* (**3**)*:* Using (**3**) at $j = 0$ and the 1-periodicity of $g(j)$ yields (recall $g(0) = 0$)

$$g(b^{-2}/2) = 0\,.$$

Thus, equation (**3**) becomes

$$\begin{aligned}0 = {} & e^{-i\pi 4j} g(j+b^{-2}/2) + e^{i\pi 4j} e^{-i\pi\sigma b^{-2}} g(j-b^{-2}/2) - \\ & - e^{-i\pi 4j} e^{-i\pi\sigma(2j+b^{-2})} g(-j-1-b^{-2}/2).\end{aligned} \tag{6.59}$$

Taking this for $j \mapsto -j$, using 1-periodicity together with (**1**) and multiplying by $\exp[-\mathrm{i}\pi\sigma 2j]$ produces

$$\begin{aligned} 0 = &-\mathrm{e}^{\mathrm{i}\pi 4j}\mathrm{e}^{-\mathrm{i}\pi\sigma 2j}g(j-b^{-2}/2) - \mathrm{e}^{\mathrm{i}\pi 4j}\mathrm{e}^{-\mathrm{i}\pi\sigma b^{-2}}g(j-b^{-2}/2)+ \\ &+\mathrm{e}^{-\mathrm{i}\pi 4j}\mathrm{e}^{-\mathrm{i}\pi\sigma(2j+b^{-2})}g(-j-1-b^{-2}/2). \end{aligned} \tag{6.60}$$

Adding (6.59) and (6.60), we obtain

$$\mathrm{e}^{-\mathrm{i}\pi 4j}g(j+b^{-2}/2) = \mathrm{e}^{\mathrm{i}\pi 4j}\mathrm{e}^{-\mathrm{i}\pi\sigma 2j}g(j-b^{-2}/2). \tag{6.61}$$

Plugging this back into (6.59), we can derive the relation

$$g(j-b^{-2}/2) = -2\mathrm{e}^{-\mathrm{i}\pi\sigma b^{-2}}\cos[\pi\sigma 2j]g(j-b^{-2}/2) \tag{6.62}$$

Consequently, if $g(j-b^{-2}/2) \neq 0$, we must have

$$1 = -2e^{-i\pi\sigma b^{-2}}\cos[\pi\sigma 2j]. \tag{6.63}$$

For the right hand side to be independent of σ, we would need b^{-2} to be an integer. But this does actually not prevent $g(j)$ from vanishing everywhere: Assume that $b^{-2} \in \mathbb{Z}$. Then, because of 1-periodicity, we have that

$$g(j-b^{-2}/2) = g(j+b^{-2}/2). \tag{6.64}$$

Therefore, (6.61) implies $2j \in \mathbb{Z}$ as long as $g(j+b^{-2}/2)$ does not vanish. But for such values of j, $\cos[\pi\sigma 2j] = \pm 1$ and hence, (6.62) requires (remember that we are still assuming $b^{-2} \in \mathbb{Z}$)

$$g(j-b^{-2}/2) = \pm 2g(j-b^{-2}/2), \tag{6.65}$$

which is only consistent with an everywhere vanishing function g: $g(j) \equiv 0$. This concludes the proof of our no solution theorem. ◂

6.1.4 Regular $AdS_2^{(d)}$ Branes - Gluing Map ρ_1

Ansatz for the One Point and Two Point Functions

The one point function was (section 5.5.4)

$$\left\langle \Theta_j(u|z) \right\rangle_\alpha = (z-\bar{z})^{-2h(j)}(u-\bar{u})^{2j}A(j|\alpha). \tag{6.66}$$

The boundary two point function with degenerate field $b^{-2}/2$ is

$$\begin{aligned} G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) = &(z_1-\bar{z}_1)^{-2h(j)}(z_2-\bar{z}_2)^{-2h(b^{-2}/2)}\cdot \\ &\cdot (u_1-\bar{u}_1)^{2j}(u_2-\bar{u}_2)^{b^{-2}}H^{(2)}_{j,\alpha}(u|z), \end{aligned} \tag{6.67}$$

with crossing ratios

$$z = \frac{|z_1-z_2|^2}{(z_1-\bar{z}_1)(z_2-\bar{z}_2)} \quad \text{and} \quad u = \frac{|u_1-u_2|^2}{(u_1-\bar{u}_1)(u_2-\bar{u}_2)}. \tag{6.68}$$

Note that again $z \in (-\infty, 0)$, $z \in (-\infty, 0)$.

Knizhnik-Zamolodchikov Equations

The Knizhnik-Zamolodchikov equations for z_2 are identical to those in the former case. Hence, they yield the conformal blocks (6.47) with parameters (6.48) again.

Expansion Coefficients

The expansion coefficients again acquire complex phases. Using the OPE on the left hand side of (6.67), we find

$$\begin{aligned}
&G^{(2)}_{j,\alpha}(u_1, u_2|z_1, z_2) \simeq \\
&\simeq |z_2 - z_1|^{-2j}\,(z_1 - \bar{z}_1)^{-2h(j_+)}\,(u_1 - \bar{u}_1)^{2j+b^{-2}}\,C_+(j)A(j_+|\alpha)+ \\
&+ |z_2 - z_1|^{2j+2}\,|u_2 - u_1|^{2b^{-2}}\,(z_1 - \bar{z}_1)^{-2h(j_-)}\,(u_1 - \bar{u}_1)^{2j-b^{-2}} \\
&\qquad\qquad\cdot C_-(j)A(j_-|\alpha)+ \\
&+ |z_2 - z_1|^{-2j}\,|u_2 - u_1|^{2(2j+1+b^{-2})}\,(z_1 - \bar{z}_1)^{-2h(j_\times)}\,(u_1 - \bar{u}_1)^{-2j-2-b^{-2}} \\
&\qquad\qquad\cdot C_\times(j)A(j_\times|\alpha).
\end{aligned} \tag{6.69}$$

Taking $|z_2 - z_1| \to 0$ ($\Rightarrow z \to 0-$) followed by $|u_2 - u_1| \to 0$ ($\Rightarrow u \to 0-$) on the right hand side, the conformal blocks (6.47) again show the behaviour

$$\begin{aligned}
\mathcal{F}^s_{j,+}(u|z) &\simeq z^{-j}\,, \\
\mathcal{F}^s_{j,-}(u|z) &\simeq z^{j+1}u^{b^{-2}}\,, \\
\mathcal{F}^s_{j,\times}(u|z) &\simeq z^{-j}u^{2j+1+b^{-2}}\,.
\end{aligned} \tag{6.70}$$

This time they are accompanied by the prefactor

$$\begin{aligned}
(z_1 - \bar{z}_1)^{-2h(j)}\,(z_2 - \bar{z}_2)^{-2h(b^{-2}/2)}\,(u_1 - \bar{u}_1)^{2j}\,(u_2 - \bar{u}_2)^{b^{-2}} \simeq \\
\simeq (z_1 - \bar{z}_1)^{-2[h(j)+h(b^{-2}/2)]}\,(u_1 - \bar{u}_1)^{2j+b^{-2}}
\end{aligned} \tag{6.71}$$

from (6.67) with

$$z = \frac{|z_1 - z_2|^2}{(z_1 - \bar{z}_1)\,(z_2 - \bar{z}_2)} \qquad \text{and} \qquad u = \frac{|u_1 - u_2|^2}{(u_1 - \bar{u}_1)\,(u_2 - \bar{u}_2)}. \tag{6.72}$$

To be careful about phase factors, we have to use ($\nu \in \mathbb{C}$)

$$\begin{aligned}
z^\nu &= \mathrm{e}^{-4\pi\mathrm{i}\nu}\,|z_1 - z_2|^{2\nu}\,(z_1 - \bar{z}_1)^{-\nu}\,(z_2 - \bar{z}_2)^{-\nu}\,, \\
u^\nu &= \mathrm{e}^{-\mathrm{i}\pi(\sigma+3)\nu}\,|u_1 - u_2|^{2\nu}\,(u_1 - \bar{u}_1)^{-\nu}\,(u_2 - \bar{u}_2)^{-\nu}\,,
\end{aligned} \tag{6.73}$$

where $\sigma = \mathrm{sgn}(u_1 - \bar{u}_1) = \mathrm{sgn}(u_2 - \bar{u}_2)$. With the help of this, it is easy to see that the coefficients $a_\epsilon^{j,b^{-2}/2}$ need to be defined with the following phases

$$\begin{aligned} a_+^{j,b^{-2}/2}(\alpha) &= \mathrm{e}^{-4\pi \mathrm{i} j} C_+^{b^{-2}/2}(j) A(j_+|\alpha), \\ a_-^{j,b^{-2}/2}(\alpha) &= \mathrm{e}^{4\pi \mathrm{i} j + \mathrm{i}\pi(\sigma+3)b^{-2}} C_-^{b^{-2}/2}(j) A(j_-|\alpha), \\ a_\times^{j,b^{-2}/2}(\alpha) &= \mathrm{e}^{-4\pi \mathrm{i} j + \mathrm{i}\pi(\sigma+3)(2j+b^{-2})} C_\times^{b^{-2}/2}(j) A(j_\times|\alpha). \end{aligned} \tag{6.74}$$

Factorization Limit and $b^{-2}/2$-Shift Equation

About the limit $\mathrm{Im}(z_2) \to 0$ the same comments as in the last section 6.1.3 for ρ_2 apply. Our $b^{-2}/2$-shift equation reads

$$\begin{aligned} \left[\Gamma(1+b^2)\right]^{-1} f\left(\frac{b^{-2}}{2}\right) f(j) = \mathrm{e}^{-\mathrm{i}\pi 3j} f\left(j + \frac{b^{-2}}{2}\right) - \\ -\mathrm{e}^{\mathrm{i}\pi(3j+b^{-2})} \mathrm{e}^{\mathrm{i}\pi\sigma b^{-2}} f\left(j - \frac{b^{-2}}{2}\right) - \mathrm{e}^{-\mathrm{i}\pi(j-b^{-2})} \mathrm{e}^{\mathrm{i}\pi\sigma(2j+b^{-2})} f\left(-j - 1 - \frac{b^{-2}}{2}\right). \end{aligned} \tag{6.75}$$

Solving the Shift Equations

We take a first step by solving the $1/2$-shift equation (5.79) together with the reflection symmetry constraint (5.33). We find the following one parameter solution

$$f(j|n) = -\frac{\pi \mathrm{e}^{\mathrm{i}\pi \frac{b^2}{4}}}{\Gamma(-b^2)\sin[\pi n b^2]} \mathrm{e}^{-\mathrm{i}\pi \frac{b^2}{4}(2j+1)^2} \frac{\sin[\pi n b^2(2j+1)]}{\sin[\pi b^2(2j+1)]}. \tag{6.76}$$

Up to now there are no restrictions on the parameter n. Note, that due to the missing σ-dependence of $f(j)$, we cannot include a factor $\propto \mathrm{e}^{-\sigma(2j+1)r}$. Such a term would actually be typical of an AdS_2 brane [72] (also compare to all the previous AdS_2 brane solutions), so what we have here is rather a degenerate AdS_2 brane (degenerate is a good adjective here; see below for an explanation). Comparing to the solution for the irregular branes (6.38), the $\propto \mathrm{e}^{-\mathrm{i}\pi\sigma(2j+1)m}$ behaviour is replaced by the term $\propto \exp\left[-\mathrm{i}\pi\frac{b^2}{4}(2j+1)^2\right]$. Note that this is an additional quantum deformation, since for $b^2 \to 0$ (corresponding to $k \to \infty$) this term goes to one.

Inserting (6.76) into our second shift equation (6.75), we find, very remarkably, that it is obeyed provided that $n \in \mathbb{Z}$ and $j \in \frac{1}{2}\mathbb{Z}$. Thus, in (6.76) with $n \in \mathbb{Z}$ and $j \in \frac{1}{2}\mathbb{Z}$, we have given a fully consistent solution to both factorization constraints. Seeing the restriction on j, one might ask for the meaning of a $b^{-2}/2$-shift equation with arbitrary b^{-2}. We like to view (6.76) as an interpolating solution, that is a solution which is mathematically well-defined for arbitrary complex values of j and as such an object can be inserted into a $b^{-2}/2$-shift equation. However, physically sensible information is only provided for values $j \in \frac{1}{2}\mathbb{Z}$. Moreover, one should note that the (missing) factor $\propto e^{-i\pi\sigma(2j+1)m}$ with $m \in \mathbb{Z}$ does not depend on σ when $j \in \frac{1}{2}\mathbb{Z}$. Hence, one can really view the absence of this typical (for AdS_2 branes) term as a degeneracy which is related to the restriction $j \in \frac{1}{2}\mathbb{Z}$.

6.2 Continuous Branes

In this section we assemble our results (shift equations and solutions) concerning the continuous branes. The two point functions are always determined as shown in the corresponding sections of chapter 6.1 and thus, we do not write them down here again, but merely state our results. Recall from section 5.2.4 that, instead of $C_\sigma = A_\sigma$, we now encounter the residua $\tilde{c}_\sigma$.

6.2.1 Irregular $AdS_2^{(c)}$ Branes - Gluing Maps ρ_1, ρ_2

For both gluing maps the $b^{-2}/2$ shift equation is

$$(1+b^2)\nu_b^{\frac{b^{-2}}{2}}\,\tilde{c}_\sigma(b^{-2}/2,0|\alpha)f_\sigma(j) = f_\sigma\left(j+\frac{b^{-2}}{2}\right) + e^{-i\pi b^{-2}}f_\sigma\left(j-\frac{b^{-2}}{2}\right). \tag{6.77}$$

Therefore, we discover that the irregular continuous branes are isomorphic for gluing maps ρ_1, ρ_2, just as before in the discrete case. In [72] and [79], the following solution to the $1/2$-shift equation (5.81) and the reflection symmetry constraint (5.30) has been proposed

$$f_\sigma(j|\alpha) = -\frac{\pi A_b}{\sqrt{\nu_b}}\frac{e^{-\alpha(2j+1)\sigma}}{\sin[\pi b^2(2j+1)]}. \tag{6.78}$$

To obtain this solution, it was used that

$$\tilde{c}_\sigma(1/2,0|\alpha) = -\frac{\sigma}{\sqrt{\nu_b}}\frac{\Gamma(-2b^2)}{\Gamma(-b^2)}2\sinh(\alpha). \tag{6.79}$$

Plugging the solution (6.78) into the $b^{-2}/2$-shift equation (6.77), we can infer an expression for the unknown $\tilde{c}_\sigma(b^{-2}/2,0|\alpha)$:

$$\tilde{c}_\sigma(b^{-2}/2,0|\alpha) = -\frac{e^{-i\pi b^{-2}/2}}{\nu_b^{b^{-2}/2}(1+b^2)}2\cosh\left[\left(\alpha\sigma - i\frac{\pi}{2}\right)b^{-2}\right]. \tag{6.80}$$

Hence, the known irregular continuous AdS_2 branes are fully consistent with both factorization constraints. Note that for $b^{-2} = b^2 = 1$, the bulk-boundary OPE coefficients (6.79), (6.80) coincide, as do the two shift equations (5.81) and (6.77).

6.2.2 How to Approach the Regular $AdS_2^{(c)}$ Branes

We have just seen that knowledge of the occuring coefficients $\tilde{c}_\sigma(1/2,0|\alpha)$ and $\tilde{c}_\sigma(b^{-2}/2,0|\alpha)$ is needed to decide whether the continuous branes are consistent or not. In [72], $\tilde{c}_\sigma(1/2,0|\alpha)$ has been given for irregular branes. We can however not expect the corresponding coefficients in the regular case $\tilde{c}(1/2,0|\alpha)$, $\tilde{c}(b^{-2}/2,0|\alpha)$ to coincide with the irregular ones. Indeed, the latter are independent of σ, while the former ones show an explicit σ-dependence (see equations (6.79) and (6.80)).

Therefore, as no explicit expressions for $\tilde{c}(1/2,0|\alpha)$, $\tilde{c}(b^{-2}/2,0|\alpha)$ are known, our approach to the regular continuous AdS_2 branes with gluing map ρ_1 will be to make a certain ansatz for the form of the (redefined) one point amplitude. By inserting this ansatz into the shift equations, we will infer expressions for $\tilde{c}(1/2,0|\alpha)$ and $\tilde{c}(b^{-2}/2,0|\alpha)$ that we then discuss. For gluing map ρ_2, we even fail to write down an ansatz and we shall argue that the shift equations do not admit a solution in that case.

6.2.3 Regular $AdS_2^{(c)}$ Branes - Gluing Map ρ_1

The $b^{-2}/2$-shift equation that we derive is

$$
\begin{aligned}
(1+b^2)\nu_b^{\frac{b^{-2}}{2}}\,\tilde{c}(b^{-2}/2,0|\alpha)f(j) &= \mathrm{e}^{-\mathrm{i}\pi 3j} f\left(j+\frac{b^{-2}}{2}\right) - \\
-\mathrm{e}^{\mathrm{i}\pi(3j+b^{-2})}\mathrm{e}^{\mathrm{i}\pi\sigma b^{-2}} f\left(j-\frac{b^{-2}}{2}\right) &- \mathrm{e}^{-\mathrm{i}\pi(j-b^{-2})}\mathrm{e}^{\mathrm{i}\pi\sigma(2j+b^{-2})} f\left(-j-1-\frac{b^{-2}}{2}\right).
\end{aligned}
\tag{6.81}
$$

In order to study a solution to both shift equations (5.83) and (6.81) we shall make the following ansatz: From our experience in the discrete case (section 6.1.4), we expect the additional quantum deformation $\exp[-\mathrm{i}\pi\frac{b^2}{4}(2j+1)]$ to occur. (You might remember from the discrete branes that, technically, this is the term that cancels the $\exp[\mathrm{i}\pi b^2 j]$ and $\exp[-\mathrm{i}\pi b^2(j+1)]$ factors on the right hand side of the $1/2$-shift equation; these factors are of course also present here, in the continuous case). Secondly, just like in the irregular continuous solution, we also expect the deformation $\sin^{-1}[\pi b^2(2j+1)]$ to be present. Along with it, an additional factor with odd parity under $j \mapsto (-j-1)$ must be included in order to get the overall parity of the solution right (recall equation (5.33)). A choice for this factor is suggested from the following observation: We can again not include the term $\propto \exp[-\sigma(2j+1)r]$ (which would be typical for an AdS_2 brane, see [72] and compare our remarks in section 6.1.4), due to the missing σ-dependence of $f(j)$. But in order to get the parity right, we can just use its odd part, so we include $\sin[\pi b^2(2j+1)\alpha]$ into our ansatz. Putting all this together, we believe that the most natural ansatz for a regular continuous brane is

$$f(j|\alpha) = A_b^{(reg)} \mathrm{e}^{-\mathrm{i}\pi \frac{b^2}{4}(2j+1)^2} \frac{\sin[\pi b^2(2j+1)\alpha]}{\sin[\pi b^2(2j+1)]}, \tag{6.82}$$

with an arbitrary, but only b-dependent constant $A_b^{(reg)}$. Just as above, in the irregular continuous case, it cannot be fixed, because the continuous shift equations are always linear in the one point amplitude.

Using our ansatz (6.82) in the $1/2$-shift equation (5.83) results in

$$\tilde{c}(1/2,0|\alpha) = \frac{\mathrm{e}^{-\mathrm{i}\pi\frac{3}{4}b^2}}{\sqrt{\nu_b}} \frac{\Gamma(-2b^2)}{\Gamma(-b^2)} 2\cos(\pi b^2 \alpha). \tag{6.83}$$

The $b^{-2}/2$-shift equation (6.81) gives more restrictions: It is only sensible for $j \in \frac{1}{2}\mathbb{Z}$. Furthermore, $\alpha = m \in \mathbb{Z}$ is needed in order to handle it. With these two restrictions, we obtain

$$\tilde{c}(b^{-2}/2, 0|\alpha = m) = \mathrm{i}\frac{\nu_b^{b^{-2}/2}}{(1+b^2)}(-)^m \mathrm{e}^{-\mathrm{i}\pi\frac{b^{-2}}{4}}. \tag{6.84}$$

Interestingly, this coincides almost (up to a factor of two) with $\tilde{c}(1/2,0|\alpha)$ (6.83) when taking $b^2 = b^{-2} = 1$ (provided that $\alpha = m \in \mathbb{Z}$). We take this as a strong hint that this brane is consistent. The restrictions on α and j that we find, together with its form (6.82) actually make it look like a discrete brane rather than a continuous one. We do not have an explanation for this strange behaviour. But we like to notice that together with the continuous brane of the previous section and the other one-parameter discrete brane from section 6.1.4, we now have a brane spectrum that is labelled by one continuous and two discrete parameters. This can indeed be expected from a generalization of Cardy's condition [75, 76]. We take up and elaborate more on this point in the conclusion, chapter 9.

6.2.4 Regular $AdS_2^{(c)}$ Branes - Gluing Map ρ_2

Our $b^{-2}/2$-shift equation reads

$$\begin{aligned}(1+b^2)\nu_b^{\frac{b^{-2}}{2}}\tilde{c}(b^{-2}/2,0|\alpha)f(j) &= \mathrm{e}^{-\mathrm{i}\pi 3j} f\left(j+\frac{b^{-2}}{2}\right) - \\ -\mathrm{e}^{\mathrm{i}\pi 3j}\mathrm{e}^{-\mathrm{i}\pi\sigma b^{-2}} f\left(j-\frac{b^{-2}}{2}\right) &+ \mathrm{e}^{-\mathrm{i}\pi 3j}\mathrm{e}^{-\mathrm{i}\pi\sigma(2j+b^{-2})} f\left(-j-1-\frac{b^{-2}}{2}\right).\end{aligned} \tag{6.85}$$

Trying to solve both shift equations (5.82) and (6.85), we start to think about a solution for (5.82). Yet, the ansatz that proved succesful in the preceding section, when solving (5.83), does not work out this time. The reason is a relative minus sign in (5.82) which is not present in (5.83). It prevents the shifted $\propto \sin[\pi b^2(2j+1)\alpha]$ terms from adding up correctly. Without using this sine term, however, we cannot write an ansatz that has the correct parity. Thus, we try to proceed as in section 6.1.3 and see how far we can get proving that no solution exists in this case. Redefining

$$f(j) = \frac{\mathrm{e}^{-\mathrm{i}\pi\frac{b^2}{4}(2j+1)^2}}{\sin\left[\pi b^2(2j+1)\right]} g(j)\,, \tag{6.86}$$

we write the reflection symmetry constraint (5.33) and the two shift equations (5.82), (6.85) in terms of $g(j)$ as

$$\begin{aligned}
&(\mathbf{1})\quad g(j) = -g(-j-1)\\
&(\mathbf{2})\quad c' g(j) = g(j+1/2) - g(j-1/2)\\
&(\mathbf{3})\quad c'' g(j) = \mathrm{e}^{-\mathrm{i}\pi 4j} g(j+b^{-2}/2) + \mathrm{e}^{\mathrm{i}\pi 4j}\mathrm{e}^{-\mathrm{i}\pi\sigma b^{-2}} g(j-b^{-2}/2) -\\
&\qquad\qquad - \mathrm{e}^{-\mathrm{i}\pi 4j}\mathrm{e}^{-\mathrm{i}\pi\sigma(2j+b^{-2})} g(-j-1-b^{-2}/2)\,.
\end{aligned}$$

The coefficients c' and c'' are not of interest here. They just contain all the b-dependent factors:

$$\begin{aligned}
c' &= \sqrt{\nu_b}\frac{\Gamma(-b^2)}{\Gamma(-2b^2)}\mathrm{e}^{\mathrm{i}\pi\frac{3}{4}b^2}\tilde{c}(1/2,0|\alpha)\\
c'' &= -\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\frac{b^{-2}}{4}}(1+b^2)\nu_b^{b^{-2}/2}\tilde{c}(b^{-2}/2,0|\alpha)\,.
\end{aligned} \tag{6.87}$$

Playing around with the above set of equations, we first note that from (**1**) $g(-1/2) = 0$ and $g(0) = -g(-1)$. Then, from (**2**) at $j = -1/2$, we infer $g(0) = 0 = g(-1)$. Afterwards, (**2**) for $j = 0$ yields $g(1/2) = 0$. Actually, we can now prove that

$$g\left(\frac{k}{2}\right) = 0 \quad \forall\, k \in \mathbb{Z}\,. \tag{6.88}$$

This is a simple induction argument: Noting that (**2**) just states

$$c' g\left(\frac{k+1}{2}\right) = g\left(\frac{k+2}{2}\right) - g\left(\frac{k}{2}\right) \tag{6.89}$$

as well as

$$c' g\left(\frac{k}{2}\right) = g\left(\frac{k+1}{2}\right) - g\left(\frac{k-1}{2}\right)\,, \tag{6.90}$$

it is clear that starting with $g\left(\frac{k}{2}\right) = 0$, $g\left(\frac{k+1}{2}\right) = 0$, we can deduce that also $g\left(\frac{k+2}{2}\right) = 0$, $g\left(\frac{k-1}{2}\right) = 0$. Now we use this knowledge and take (**3**) for $j = \frac{k}{2}$, $k \in \mathbb{Z}$. Employing also (**1**), we end up with

$$g\left(\frac{k - b^{-2}}{2}\right) = -\left[\mathrm{e}^{\mathrm{i}\pi\sigma b^{-2}} + (-)^k\right] g\left(\frac{k + b^{-2}}{2}\right). \tag{6.91}$$

This equation must also hold for $\sigma \mapsto (-\sigma)$. Hence, subtracting it from its counterpart which contains $(-\sigma)$ rather than σ, we get

$$-2\mathrm{i}\sin[\pi\sigma b^{-2}]g\left(\frac{k + b^{-2}}{2}\right) = 0\,. \tag{6.92}$$

Thus, we either need $b^{-2} \in \mathbb{Z}$ or $g\left(\frac{k+b^{-2}}{2}\right) = 0$. The former is quite a restriction, saying that these branes do not exist for generic, but only for integer (affine Lie algebra) level. We would rather not assume this. Then, all we are able to show is that $g(j)$ vanishes on a very specific set of points. We do not see a possibility to develop our argumentation any further at this point, but we would rather expect these branes to be inconsistent.

7 H_3^+/Liouville Correspondence

In this chapter, we just assemble some basic formulae about Liouville theory (both, in the bulk and with a boundary) that are needed in order to state its correspondence to the H_3^+ model given in [74, 73].

7.1 Liouville Theory in the Bulk

Classical Liouville theory is defined by the action

$$S = \frac{1}{\pi} \int \mathrm{d}^2 z \left[(\partial\phi)(\bar{\partial}\phi) + \mu e^{2b\phi} \right]. \tag{7.1}$$

The energy momentum tensor derived from this action is

$$T(z) = -(\partial\phi)^2 + Q\partial^2\phi\,, \tag{7.2}$$

where Q must be $Q = b + b^{-1}$ in order to make the interaction term exactly marginal. Then, Liouville theory can be quantized as a two dimensional CFT (without extended symmetry but Virasoro symmetry only). Note that there is a (quantum) symmetry $b \mapsto b^{-1}$. The central charge of the quantum theory is $c = 1 + 6Q^2$ and primary fields are denoted $V_\alpha(z)$. Their conformal weights are $h(\alpha) = \alpha(Q - \alpha)$ and the physical spectrum consists of fields with labels $\alpha \in \frac{Q}{2} + \mathrm{i}\mathbb{R}_{\geq 0}$ (compare to the H_3^+model). Also note the reflection symmetry $h(\alpha) = h(Q - \alpha)$. The Liouville two point function reads

$$\left\langle V_{\alpha_2}(z_2) V_{\alpha_1}(z_1) \right\rangle = 2\pi |z_2 - z_1|^{-4h(\alpha_1)} \left[\delta(Q - \alpha_2 - \alpha_1) + R^{(L)}(\alpha_1)\delta(\alpha_2 - \alpha_1) \right] \tag{7.3}$$

with the Liouville reflection amplitude $R^{(L)}(\alpha)$. It intertwines the representations with labels α and $Q - \alpha$

$$V_\alpha(z) = R^{(L)}(\alpha) V_{Q-\alpha}(z) \tag{7.4}$$

and is given by

$$R^{(L)}(\alpha) = \left[\pi\mu\gamma(b^2) \right]^{b^{-1}(Q-2\alpha)} \cdot \frac{\Gamma(1 + b(2\alpha - Q))\Gamma(1 + b^{-1}(2\alpha - Q))}{\Gamma(1 - b(2\alpha - Q))\Gamma(1 - b^{-1}(2\alpha - Q))}\,. \tag{7.5}$$

The three point function of Liouville field theory is given by

$$\left\langle V_{\alpha_3}(z_3) V_{\alpha_2}(z_2) V_{\alpha_1}(z_1) \right\rangle = |z_1 - z_2|^{-2h_{12}} |z_1 - z_3|^{-2h_{13}} |z_2 - z_3|^{-2h_{23}} C^{(L)}(\alpha_3, \alpha_2, \alpha_1)\,. \tag{7.6}$$

The structure constants have been derived in [91] and independently in [92] and rederived using a different technique in [99]. They are given by

$$C^{(L)}(\alpha_3,\alpha_2,\alpha_1) = \frac{\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{b^{-1}(Q-\alpha_1-\alpha_2-\alpha_3)}}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)} \cdot \\ \cdot \frac{\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)} . \tag{7.7}$$

The Υ function has already occured in the H_3^+ structure constants, see section 3.3.3. The Liouville structure constants and therefore the OPEs have an analytic continuation to field labels α outside the physical range. This is completely analogous to the situation in the H_3^+ model. In particular, one can make a continuation to degenerate representations. The degenerate representation with label $\alpha = -\frac{1}{2b}$ is of particular interest in the H_3^+/Liouville correspondence.

7.2 H_3^+/Liouville Correspondence for the Bulk Theories

The correspondence has been proven in [74]. It uses H_3^+ bulk fields in the μ-basis, see appendix B. The statement is that an H_3^+ n-point correlator ($n \geq 2$) can be computed from a $(2n-2)$-point Liouville correlator with the insertion of $(n-2)$ degenerate fields $V_{-\frac{1}{2b}}$. The precise relation is [74]

$$\left\langle \Phi_{j_n}(\mu_n|z_n)\dots\Phi_{j_1}(\mu_1|z_1)\right\rangle = \frac{\pi b}{2}(-\pi)^{-n}\delta^{(2)}(\mu_1+\dots+\mu_n)\cdot \\ \cdot|\theta_n|^2\left\langle V_{\alpha_n}(z_n)\dots V_{\alpha_1}(z_1)V_{-\frac{1}{2b}}(y_{n-2})\dots V_{-\frac{1}{2b}}(y_1)\right\rangle . \tag{7.8}$$

The function θ_n is

$$\theta_n(z_1,\dots,z_n|y_1,\dots,y_{n-2},u) = u\prod_{r<s\leq n} z_{rs}^{\frac{b^{-2}}{2}} \prod_{k<l\leq n-2} y_{kl}^{\frac{b^{-2}}{2}} \prod_{r=1}^{n}\prod_{k=1}^{n-2}(z_r-y_k)^{-\frac{b^{-2}}{2}} \tag{7.9}$$

and thus relates the coordinate dependencies of both sides. Note the left hand side depends on isospin variables $\mu_1,\dots,\mu_n$, while the right hand side involves new coordinates $u, y_1,\dots y_{n-2}$. Since $\sum_{i=1}^n \mu_i = 0$, the numbers of variables match. They are related in the following way:

$$\sum_{i=1}^{n}\frac{\mu_i}{t-z_i} = u\frac{\prod_{j=1}^{n-2}(t-y_j)}{\prod_{i=1}^{n}(t-z_i)} , \tag{7.10}$$

where the equality is to be read as an equality of rational functions of t. Note that because of the relation $\mu_1+\dots+\mu_n = 0$, this equality determines precisely

$(n-1)$ (rather than n) coefficients, which is just the right number to gain sufficiently many relations for the computation of the variables $u, y_1, \ldots, y_{n-2}$. So in principle, the positions y_j of the degenerate fields $V_{-\frac{1}{2b}}$ can be determined from the isospin labels μ_i. The purpose of the additional degenerate fields that appear on the Liouville side is to mimick the effect of the additional isospin symmetry present on the H_3^+ side[1]. The Liouville parameter b is related to the H_3^+ affine level k as $b^{-2} = (k-2)$, the Liouville coupling is fixed to be $\mu = \frac{b^2}{\pi^2}$ and the labels α_i of the Liouville fields are determined from the H_3^+ affine symmetry representations labels j_i:

$$\alpha_i = b(j_i + 1) + \frac{1}{2b}\,. \tag{7.11}$$

7.3 Liouville Theory with Boundary

As there is only Virasoro symmetry, one can only study maximal symmetry preserving branes with the conformal gluing condition $T = \bar{T}$ at $\mathrm{Im}(z) = 0$. As Liouville theory, like the H_3^+ model, has a continuous spectrum (see also section 7.1), one also has to distinguish continuous and discrete branes here (see section 5.2.4). Their one point functions have been derived following the same strategy that we have carried out for the H_3^+ model in sections 5 and 6: Via shift equations. However, the analysis is easier in Liouville theory, since the two degenerate fields that one uses, $V_{-\frac{b}{2}}$ and $V_{-\frac{1}{2b}}$, are related by the symmetry $b \mapsto b^{-1}$. In particular, one does not need any additional continuation prescription for the two point function, as it is the case in the H_3^+ model.

The continuous Liouville branes (commonly called FZZT branes) have been derived in [100]. The one point functions $\langle V_\alpha(z)\rangle = |z - z^*|^{-2h(\alpha)} U(\alpha|s)$ come with a one point amplitude

$$\begin{aligned} U(\alpha|s) = &\frac{2}{b}\left[\pi\mu\gamma(b^2)\right]^{\frac{1}{2b}(Q-2\alpha)} \Gamma(2b\alpha - b^2)\Gamma(2b^{-1}\alpha - b^{-2} - 1)\cdot \\ &\cdot \cosh\left[(2\alpha - Q)\pi s\right], \end{aligned} \tag{7.12}$$

where $s \in \mathbb{C}$ is a parameter that labels the boundary condition.

The solution for the discrete Liouville branes (ZZ branes) was given in [101]. It is labelled by two positive integers $m, n \in \mathbb{Z}_{>0}$ and has one point amplitude

$$\begin{aligned} U(\alpha|m,n) = &\frac{\sin\left[\pi b^{-1}Q\right]\sin\left[\pi m b^{-1}(2\alpha - Q)\right]\sin\left[\pi b Q\right]\sin\left[\pi n b(2\alpha - Q)\right]}{\sin\left[\pi m b^{-1}Q\right]\sin\left[\pi b^{-1}(2\alpha - Q)\right]\sin\left[\pi n b Q\right]\sin\left[\pi b(2\alpha - Q)\right]}\cdot \\ &\cdot \frac{Q\left[\pi\mu\gamma(b^2)\right]^{-b^{-1}\alpha}\Gamma(bQ)\Gamma(b^{-1}Q)}{(Q-2\alpha)\Gamma(b(Q-2\alpha))\Gamma(b^{-1}(Q-2\alpha))}\,. \end{aligned} \tag{7.13}$$

[1] Indeed, the proof given in [74] is based on the observation that Knizhnik-Zamolodchikov equations on the H_3^+ side can be read as BPZ singular vector equations on the Liouville side.

7.4 H_3^+/Liouville Correspondence with AdS_2 Boundary

In [73], the correspondence between H_3^+ model and Liouville theory that we reviewed briefly in section 7.2 was generalized to correlators in the presence of continuous AdS_2 boundary conditions on the H_3^+ side and FZZT boundary conditions on the Liouville side. For the purpose of stating the correspondence, we label the AdS_2 boundary conditions by a parameter $r \in \mathbb{R}$ for the moment (and not by α as in chapters 5, 6 and 8, since α is reserved for the Liouville momenta now). For a correlator involving n bulk and m boundary H_3^+ fields (in the μ- and τ-basis, respectively; see appendix B), the relation is [73]

$$\begin{aligned}
&\left\langle \prod_{a=1}^{n} \Phi_{j_a}(\mu_a|z_a) \prod_{b=1}^{m} \Psi_{\ell_b}^{r_b, r_{b-1}}(\tau_b|x_b) \right\rangle_{r',r} = \\
&\qquad \pi^2 \sqrt{\frac{b}{2}} (-\pi)^{-n} \delta\left(2\mathrm{Re}(\mu_1 + \cdots + \mu_n) + \tau_1 + \cdots + \tau_m\right) |u| \, |\theta_{m,n}|^{\frac{b^{-2}}{2}} \cdot \\
&\qquad \cdot \left\langle \prod_{a=1}^{n} V_{\alpha_a}(z_a) \prod_{b=1}^{m} B_{\beta_b}^{s_b, s_{b-1}}(x_b) \prod_{a'=1}^{n'} V_{-\frac{1}{2b}}(w_{a'}) \prod_{b'=1}^{m'} B_{-\frac{1}{2b}}(y_{b'}) \right\rangle_{s',s} ,
\end{aligned} \tag{7.14}$$

where the fields $B_\beta(x)$ are boundary Liouville fields. On the H_3^+ side, the boundary condition changing operators $\Psi_{\ell_b}^{r_b, r_{b-1}}(\tau_b|x_b)$ take AdS_2 boundary conditon r_{b-1} and map it to r_b. We have $r_0 = r$ and $r_n = r'$. The corresponding FZZT boundary conditions s_b on the Liouville side are related to the AdS_2 labels r_b via

$$s_b = \frac{r_b}{2\pi b} - \frac{\mathrm{i}}{4b} \mathrm{sgn}\varphi(t) , \tag{7.15}$$

where the rational function $\varphi(t)$ controls the change of variables from $\{\mu_a, \tau_b\}$ to $\{u, w_{a'}, y_{b'}\}$. It is

$$\varphi(t) = \sum_{a=1}^{n} \frac{\mu_a}{t - z_a} + \sum_{a=1}^{n} \frac{\bar{\mu}_a}{t - z_a^*} + \sum_{b=1}^{m} \frac{\tau_b}{t - x_b} , \tag{7.16}$$

and the variables $w_{a'}$, $y_{b'}$ are the zeros of $\varphi(t)$. Note that due to the condition $2\mathrm{Re}(\mu_1 + \cdots + \mu_n) + \tau_1 + \cdots + \tau_m = 0$, $\varphi(t)$ has $2n + m - 2$ many zeros. The individual numbers n', m' of additional Liouville fields to be inserted in the Liouville correlator on the right hand side of (7.14) are not fixed, but only their sum is. It has to be $2n' + m' = 2n + m - 2$, in order to accomodate all the zeros of $\varphi(t)$. Note that we only have n' (rather than $2n'$) additional bulk field insertions. The reason is the following: Whenever $\varphi(t)$ has a complex zero, an additional bulk field $V_{-\frac{1}{2b}}$ is inserted. Since complex zeros come in complex conjugate pairs, only the zero that lies in the upper half plane can be taken and its complex conjugate must be discarded. Thus, the number of additional bulk field insertions is controlled by the number of pairs of complex conjugate zeros. Then, for every real zero of $\varphi(t)$ we insert an additional boundary Liouville field $B_{-\frac{1}{2b}}$. This is important for the Hosomichi-Ribault proposal that we explain in the next section and that motivates our construction described in chapter 8. Let us quickly fill in the remaining ingredients of the correspondence. The field labels are related as before

$$\alpha_a = b(j_a + 1) + \frac{1}{2b}, \quad \beta_b = b(\ell_b + 1) + \frac{1}{2b}, \tag{7.17}$$

the variable u is given by

$$u = 2\mathrm{Re}(\mu_1 z_1 + \cdots + \mu_n z_n) + \tau_1 x_1 + \cdots + \tau_m x_m \tag{7.18}$$

and we omit the exact expression for $\theta_{m,n}$ here, since it is not relevant for our purposes. It is given in [73].

7.5 The Hosomichi-Ribault Proposal

Consider an H_3^+ correlator $\left\langle \Phi_{j_2}(\mu_2|z_2)\Phi_{j_1}(\mu_1|z_1)\right\rangle)_r$ of two bulk field insertions under a continuous AdS_2 boundary condition r, just what we need in order to derive shift equations (see chapters 5 and 6 where we work, however, in the u-basis). Following the previous section, it is mapped to a Liouville correlator with additional degenerate field insertions with positions given by the zeros of

$$\varphi(t) = \sum_{a=1}^{2} \left(\frac{\mu_a}{t - z_a} + \frac{\bar{\mu}_a}{t - z_a^*} \right). \tag{7.19}$$

Because of the condition $\mathrm{Re}(\mu_1 + \mu_2) = 0$, this has two zeros y_1, y_2. They can either be both real, in which case the corresponding Liouville correlator is $\left\langle V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)B_{-\frac{1}{2b}}(y_2)B_{-\frac{1}{2b}}(y_1)\right\rangle_s$, i.e. it has two additional boundary field insertions. The situation is called boundary regime in [73]. The other possible case is that of complex conjugate y_1, y_2. This is called the bulk regime in [73] and the occuring Liouville correlator is $\left\langle V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)V_{-\frac{1}{2b}}(y_1)\right\rangle_s$ (where we have taken y_1 to be the zero with positive imaginary part $\mathrm{Im}(y_1) > 0$). But note that there is also the possibility of vanishing discriminant, i.e. one real zero of order two. In this case, that we like to call the transition point, the H_3^+/Liouville correspondence breaks down! It has been shown in [73] that in the variable $z = \frac{|z_1 - z_2|}{|z_1 - z_2^*|}$ (i.e. the boundary CFT crossing ratio, see chapter 5), the transition point occurs at $z = \mu \equiv \frac{|\mu_1 + \mu_2|}{|\mu_1| + |\mu_2|}$ and that the bulk regime corresponds to $0 < z < \mu$ and the boundary regime to $\mu < z < 1$. It is also argued there, that the value of μ becomes simply the coordinate u in the u-basis and the correspondence therefore breaks down along the line $u = z$. Such a behaviour is also known from the relation between Knizhnik-Zamolodchikov and BPZ singular vector equations in the $\widehat{\mathfrak{su}}(2)_k$ WZNW model [102]. In [73], the transition point is seen as a novel kind of "singularity" of H_3^+ model correlators. The Hosomichi-Ribault proposal (in the following also called continuity proposal) put forward in [73] now states, that H_3^+ correlators should be continuous at all transition point singularities. It is furthermore argued, that all correlators have finite limits (from both sides) at the transition point and that, assuming only continuity at the transition point, the correlator in the boundary regime would not be entirely determined from its expression in the bulk regime. This is however a big problem for the implementation of the Cardy-Lewellen constraints (see section 4.2): If an arbitrariness remains in the correlator, it is highly questionable that a unique shift equation can be derived in this way. The conclusion of [73] is that the Cardy-Lewellen constraints will be weakened.

Yet, the continuity proposal could not be made explicit in [73]. In the next chapter, we demonstrate along the lines of our article [68], how the proposal can be implemented in the H_3^+ model and construct a two point function that has exactly the properties required by the continuity proposal: It has finite limits for $z \to u$ from both sides ($z < u$ and $z > u$), it is continuous along $u = z$ and it has a one-parameter arbitrariness in the boundary regime $z > u$. However, contrary to the expectations of [73], we can show [68] that the factorization limit remains unambiguous and can hence derive a sensible $b^{-2}/2$-shift equation. We discuss its consequences for the H_3^+ model brane spectrum.

8 More Shift Equations from the Hosomichi-Ribault Proposal

We have already motivated the need for additional shift equations in chapter 6: The $1/2$-shift equations do not fix the one point amplitudes for AdS_2 branes uniquely and therefore, the derivation of $b^{-2}/2$-shift equations is an important problem. We have also already seen that their derivation is not as straightforward as that of $1/2$-shift equations, because the two point function that one considers does initially not cover the domain in which the factorization limit is to be taken. Therefore, a continuation prescription is needed. We have shown in chapter 6 that an analytic continuation of the two point function involving $\Theta_{b^{-2}/2}$ is feasible and leads to the desired $b^{-2}/2$-shift equations. We have also mentioned that a different proposal for the continuation procedure has been made in the literature, the Hosomichi-Ribault proposal reviewed and explained in chapter 7. In this chapter, we set out to implement the Hosomichi-Ribault proposal within the H_3^+ model, following closely our original article [68]. The treatment is given for irregular AdS_2 branes only.

The programme is as follows: First we give a solution to the Knizhnik-Zamolodchikov equation in the region $z < u$. It is essentially the same as in section 6.1.1. We show that this solution has a finite $u = z$ limit. Then, a solution to the Knizhnik-Zamolodchikov equation in the region $u < z$ is found. It is partially fixed from the requirement that its $u = z$ limit matches that of the previous solution. However, an ambiguity in the conformal blocks $\mathcal{F}^s_{j,-}$ and $\mathcal{F}^s_{j,\times}$ persists. Yet, the two point function is then defined everywhere in the (u, z) unit square and continuous along $u = z$. This construction is the content of section 8.1 and realizes explicitly the Hosomichi-Ribault proposal. Afterwards, in section 8.2, we take the factorization limit and derive the desired $b^{-2}/2$-shift equations for discrete as well as continuous AdS_2 branes. The key point is that the aforementioned ambiguity does not enter here, because the conformal blocks $\mathcal{F}^s_{j,-}$ and $\mathcal{F}^s_{j,\times}$ are shown not to contribute in the factorization limit. In section 8.3, we finally check that discrete as well as continuous irregular AdS_2 branes are consistent with our new $b^{-2}/2$-shift equations. In chapter 9 we discuss our results in the light of the Hosomichi-Ribault proposal [73], Cardy's work [75] and the analytic approach of chapter 6 [67, 69].

8.1 Construction of the Two Point Function

From the Ward identities of the model, the two point function

$$G^{(2)}_{j,\alpha}(u_i|z_i) = \left\langle \Theta_{b^{-2}/2}(u_2|z_2)\Theta_j(u_1|z_1)\right\rangle_\alpha \tag{8.1}$$

is restricted to be of the form

$$\begin{aligned} G^{(2)}_{j,\alpha}(u_1,u_2|z_1,z_2) = |z_1-\bar z_1|^{2[h(b^{-2}/2)-h(j)]}\,|z_1-\bar z_2|^{-4h(b^{-2}/2)}\,\cdot \\ \cdot\,|u_1+\bar u_1|^{2j-b^{-2}}\,|u_1+\bar u_2|^{2b^{-2}}\,H^{(2)}_{j,\alpha}(u|z)\,. \end{aligned} \tag{8.2}$$

The parameter α again labels the AdS_2 boundary conditions. The reduced two point function $H^{(2)}_{j,\alpha}(u|z)$ is a still unknown function of the crossing ratios

$$z = \frac{|z_2-z_1|^2}{|z_2-\bar z_1|^2} \qquad \text{and} \qquad u = \frac{|u_2-u_1|^2}{|u_2+\bar u_1|^2}\,. \tag{8.3}$$

The two point function (8.2) has to satisfy a Knizhnik-Zamolodchikov equation (4.9) for z_2. Mapping $z_1\to 0$, $\bar z_2\to 1$ and $\bar z_1\to\infty$ (i.e. $z_2\to z$), it is brought to standard form

$$\begin{aligned} -b^{-2}z(z-1)\partial_z H^{(2)}_{j,\alpha}(u|z) &= u(u-1)(u-z)\partial_u^2 H^{(2)}_{j,\alpha}+ \\ &+\left\{\left[1-2b^{-2}\right]u^2+\left[b^{-2}-2j-2\right]uz+\left[2j+b^{-2}\right]u+z\right\}\partial_u H^{(2)}_{j,\alpha}+ \\ &+\left\{b^{-4}u+\left[b^{-2}j-b^{-4}/2\right]z-b^{-2}j\right\}H^{(2)}_{j,\alpha}\,. \end{aligned} \tag{8.4}$$

This is solved by (see [96, 83] and compare to section 6.1.1)

$$H^{(2)}_{j,\alpha} = \sum_{\epsilon=+,-,\times} a^j_\epsilon(\alpha)\mathcal{F}^s_{j,\epsilon} \tag{8.5}$$

with

$$\mathcal{F}^s_{j,+}(u|z) = z^{-j}(1-z)^{-b^{-2}/2}F_1(\alpha,\beta,\beta';\gamma|u;z)\,, \tag{8.6}$$

$$\begin{aligned}\mathcal{F}^s_{j,-}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}u^{-\beta}z^{1+\beta-\gamma}\times \\ &\times F_1\left(1+\beta+\beta'-\gamma,\beta,1+\alpha-\gamma;2+\beta-\gamma\,\middle|\,\frac{z}{u};z\right)\,,\end{aligned} \tag{8.7}$$

$$\begin{aligned}\mathcal{F}^s_{j,\times}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}u^{1-\gamma}\times \\ &\times G_2\left(\beta',1+\alpha-\gamma;1+\beta-\gamma,\gamma-1\,\middle|\,-\frac{z}{u};-u\right)\,.\end{aligned} \tag{8.8}$$

The appearance of only three conformal blocks is due to the presence of degenerate field $\Theta_{b^{-2}/2}$. The propagating modes are denoted $j_\pm := j\pm b^{-2}/2$ and $j_\times := -j-1-b^{-2}/2$. We identify the parameters to be

$$\alpha=\beta=-b^{-2},\quad \beta'=-2j-1-b^{-2},\quad \gamma=-2j-b^{-2}\,. \tag{8.9}$$

So far, everything is as in chapter 6 (section 6.1.1). The conformal blocks (8.6), (8.7), (8.8) are obviously well defined in the patch $z < u$ (when talking about the patches, it is always tacitly understood that $0 \le u < 1$ and $0 \le z < 1$). Their linear combinations, i.e. the coefficients $a^j_\epsilon(\alpha)$, are determined from comparison with the OPE in the limit $z \to 0$ followed by $u \to 0$. The result is simply

$$a^j_\epsilon(\alpha) = C_\epsilon(j) A_\sigma(j_\epsilon|\alpha)\,, \tag{8.10}$$

$C_\epsilon(j)$ being the coefficients occuring in the OPE of $\Theta_{b^{-2}/2}(u_2|z_2)$ with $\Theta_j(u_1|z_1)$, see 3.5.3.

Let us now see how this solution can be extended to the region $u < z$. Clearly, $\mathcal{F}^s_{j,+}$ is already everywhere defined, so we do not have to worry about it in the following. But let us analyse how $\mathcal{F}^s_{j,-}$ and $\mathcal{F}^s_{j,\times}$ behave when we move to $u = z$ from the region $z < u$. Using the generalized series representations of F_1 and G_2 (see appendix C), we find

$$\begin{aligned}
\mathcal{F}^s_{j,-}(u=z) &= z^{1-\gamma-j}(1-z)^{-b^{-2}/2}\frac{\Gamma(1-\beta-\beta')\Gamma(2+\beta-\gamma)}{\Gamma(1-\beta')\Gamma(2-\gamma)}\cdot \\
&\quad \cdot F(1+\beta+\beta'-\gamma, 1+\alpha-\gamma; 2-\gamma|z)\,, \\
\mathcal{F}^s_{j,\times}(u=z) &= z^{1-\gamma-j}(1-z)^{-b^{-2}/2}\frac{\Gamma(1-\beta-\beta')\Gamma(\gamma-\beta)}{\Gamma(1-\beta)\Gamma(\gamma-\beta-\beta')}\cdot \\
&\quad \cdot F(1+\beta+\beta'-\gamma, 1+\alpha-\gamma; 2-\gamma|z)\,.
\end{aligned} \tag{8.11}$$

Here, F denotes the standard hypergeometric function. Interestingly, the linearly independent solutions (8.7), (8.8) degenerate at $u = z$ and become essentially the same function (up to factors). We will see shortly that it is this fact that prevents us from fixing a solution for $u < z$ uniquely.

The task is now to find a solution to the Knizhnik-Zamolodchikov equation in the region $u < z$ that matches the above for $u = z$. One building block is, of course, $\mathcal{F}^s_{j,+}$. The other two are

$$\begin{aligned}
\tilde{\mathcal{F}}^s_{j,-}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}u^{1+\beta'-\gamma}z^{-\beta'}\cdot \\
&\quad \cdot F_1\left(1+\beta+\beta'-\gamma, 1+\alpha-\gamma, \beta'; 2+\beta'-\gamma \,\middle|\, u; \frac{u}{z}\right),
\end{aligned} \tag{8.12}$$

$$\begin{aligned}
\tilde{\mathcal{F}}^s_{j,\times}(u|z) &= z^{-j}(1-z)^{-b^{-2}/2}z^{1-\gamma}\cdot \\
&\quad \cdot G_2\left(\beta, 1+\alpha-\gamma; 1+\beta'-\gamma, \gamma-1 \,\middle|\, -\frac{u}{z}; -z\right).
\end{aligned} \tag{8.13}$$

The tilde indicates that this is the solution in region $u < z$. Again, splitting the common factor $z^{-j}(1-z)^{b^{-2}/2}$, the first function is found in [96] as $\mathcal{Z}_{14}$ and the second one is related to $\mathcal{Z}_9$. Note that the third argument of G_2 is $1+\beta'-\gamma = 0$ for our specific parameter values (8.9) which are dictated by the Knizhnik-Zamolodchikov equation. Nevertheless, the function G_2 stays well-defined and a generalized series representation can be derived (see appendix C). By making use of the general series representations of F_1 and G_2, one can show that the conformal blocks (8.12), (8.13) agree along $u = z$ with those from patch $z < u$ up to factors:

$$\begin{aligned}\tilde{\mathcal{F}}^s_{j,-}(u=z) &= z^{1-\gamma-j}(1-z)^{-b^{-2}/2}\frac{\Gamma(1-\beta-\beta')\Gamma(2+\beta'-\gamma)}{\Gamma(1-\beta)\Gamma(2-\gamma)}\cdot \\ &\quad \cdot F(1+\beta+\beta'-\gamma, 1+\alpha-\gamma; 2-\gamma|z)\,, \\ \tilde{\mathcal{F}}^s_{j,\times}(u=z) &= z^{1-\gamma-j}(1-z)^{-b^{-2}/2}\frac{\Gamma(2-\beta-\gamma)}{\Gamma(1-\beta)\Gamma(2-\gamma)}\cdot \\ &\quad \cdot F(1+\beta+\beta'-\gamma, 1+\alpha-\gamma; 2-\gamma|z)\,.\end{aligned} \tag{8.14}$$

These factors are absorbed through a suitable definition of the expansion coefficients $\tilde{a}^j_\epsilon(\alpha)$ in the patch $u < z$. They must therefore be related to the former ones $a^j_\epsilon(\alpha)$ as

$$\tilde{a}^j_+(\alpha) = a^j_+(\alpha)\,, \tag{8.15}$$

$$\begin{aligned}\tilde{a}^j_-(\alpha)\frac{\Gamma(1-\beta-\beta')\Gamma(2+\beta'-\gamma)}{\Gamma(1-\beta)\Gamma(2-\gamma)} + \tilde{a}^j_\times(\alpha)\frac{\Gamma(2-\beta-\gamma)}{\Gamma(1-\beta)\Gamma(2-\gamma)} &= \\ = a^j_-(\alpha)\frac{\Gamma(1-\beta-\beta')\Gamma(2+\beta-\gamma)}{\Gamma(1-\beta')\Gamma(2-\gamma)} + a^j_\times(\alpha)\frac{\Gamma(1-\beta-\beta')\Gamma(\gamma-\beta)}{\Gamma(1-\beta)\Gamma(\gamma-\beta-\beta')}\,.\end{aligned} \tag{8.16}$$

Thus, we cannot uniquely fix the coefficients $\tilde{a}^j_-(\alpha)$ and $\tilde{a}^j_\times(\alpha)$. An ambiguity remains in the two dimensional subspace spanned by $\tilde{\mathcal{F}}^s_{j,-}$ and $\tilde{\mathcal{F}}^s_{j,\times}$. It is good to realize, that for the values of the parameters α, β, β', γ which are given in (8.9) and SL(2)-label j in the physical range $j \in -\frac{1}{2} + \mathrm{i}\mathbb{R}_{\geq 0}$, we never catch any poles of the gamma functions. The reduced two point function $H^{(2)}_{j,\alpha} = \sum_{\epsilon=+,-,\times} a^j_\epsilon(\alpha)\mathcal{F}^s_{j,\epsilon}$ is now defined in the (semi-open) unit square $0 \leq u < 1$, $0 \leq z < 1$. The lines $u = 1$, $z = 1$ have to be understood as limiting cases.

8.2 Factorization Limit and Shift Equations

Using our solution (8.6), (8.12), (8.13) in the patch $u < z$, we can now take the limit $z \to 1$ from below while $u < 1$. Performing it on the conformal blocks, we find

$$\begin{aligned}
\tilde{\mathcal{F}}^s_{j,+} &\simeq (1-z)^{1+b^{-2}/2}(1-u)^{b^{-2}}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta'-\gamma)}{\Gamma(\alpha)\Gamma(\beta')}\cdot[1+\mathcal{O}(1-z)]+\\
&\quad+(1-z)^{-b^{-2}/2}\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta')}F(\alpha,\beta;\gamma-\beta'|u)\cdot[1+\mathcal{O}(1-z)]\ ,\\
\tilde{\mathcal{F}}^s_{j,-} &\simeq (1-z)^{-b^{-2}/2}u^{1+\beta'-\gamma}\cdot\\
&\quad\cdot F(1+\beta+\beta'-\gamma,1+\alpha+\beta'-\gamma;2+\beta'-\gamma|u)\cdot[1+\mathcal{O}(1-z)]\ ,\\
\tilde{\mathcal{F}}^s_{j,\times} &\simeq (1-z)^{-b^{-2}/2}F(\alpha,\beta;1|u)\,[1+\mathcal{O}(1-z)]\ .
\end{aligned} \tag{8.17}$$

The limit $z \to 1$ from below corresponds to using a bulk-boundary OPE in the correlator. Now, we have to distinguish between discrete and continuous case again (recall section 5.2.4). Assuming a discrete open string spectrum on the brane, the bulk-boundary OPE for $\Theta_{b^{-2}/2}$ is

$$\begin{aligned}
\Theta_{b^{-2}/2}(u_2|z_2) &= |z_2-\bar{z}_2|^{1+b^{-2}/2}\,|u_2+\bar{u}_2|^{b^{-2}}\,C_\sigma(b^{-2}/2,0|\alpha)\mathbb{1}\,\{1+\mathcal{O}\,|z_2-\bar{z}_2|\}+\\
&\quad+|z_2-\bar{z}_2|^{-b^{-2}/2}\,|u_2+\bar{u}_2|^{2b^{-2}+1}\,C_\sigma(b^{-2}/2,b^{-2}|\alpha)\cdot\\
&\qquad\cdot(\mathcal{J}\Psi)^{\alpha\,\alpha}_{b^{-2}}\,(u_2\,|\mathrm{Re}(z)\,)\,\{1+\mathcal{O}\,|z_2-\bar{z}_2|\}+\\
&\quad+|z_2-\bar{z}_2|^{-b^{-2}/2}\,C_\sigma(b^{-2}/2,-b^{-2}-1|\alpha)\cdot\\
&\qquad\cdot(\mathcal{J}\Psi)^{\alpha\,\alpha}_{-b^{-2}-1}\,(u_2\,|\mathrm{Re}(z)\,)\,\{1+\mathcal{O}\,|z_2-\bar{z}_2|\}\ ,
\end{aligned} \tag{8.18}$$

For the purpose of deriving the factorization constraint, we concentrate on the contribution of the identity field $\mathbb{1}$ only. Identifying $C_\sigma(b^{-2}/2,0|\alpha) = A_\sigma(b^{-2}/2|\alpha)$, we deduce the following $b^{-2}/2$-shift equation

$$f_\sigma(b^{-2}/2)f_\sigma(j) = \Gamma(1+b^2)f_\sigma(j+b^{-2}/2)\,, \tag{8.19}$$

where we have suppressed the α-dependence and used the redefined one point amplitude (5.29). Note that on the left hand side, the one point amplitudes again carry identical σ's. As usual, this is because we are in a region where $u < 1$. In a domain with $1 < u$ they would indeed carry opposite signs.

On the other hand, assuming a continuous open string spectrum on the brane, the bulk-bundary OPE of $\Theta_{b^{-2}/2}$ contains

$$\tilde{c}_\sigma(b^{-2}/2,j_\epsilon|\alpha) = \mathrm{Res}_{j_2=b^{-2}/2}C_\sigma(j_2,j_\epsilon|\alpha) \tag{8.20}$$

instead of $C(b^{-2}/2,j_\epsilon|\alpha)$ (as usual, $\epsilon = +,-,\times$). The $b^{-2}/2$-shift equation we obtain for the redefined one point amplitude (5.29) then reads

$$\nu_b^{b^{-2}/2}(1+b^2)\tilde{c}(b^{-2}/2,0|\alpha)f_\sigma(j) = f_\sigma(j+b^{-2}/2)\,. \tag{8.21}$$

8.3 Consistency of Discrete and Continuous AdS_2 Branes

The discrete $AdS_2^{(d)}$ branes of [76] have one point amplitudes

$$f_\sigma(j|m,n) = \frac{i\pi\sigma e^{i\pi m}}{\Gamma(-b^2)\sin[\pi n b^2]} e^{-i\pi\sigma(m-\frac{1}{2})(2j+1)} \frac{\sin[\pi n b^2(2j+1)]}{\sin[\pi b^2(2j+1)]}, \quad (8.22)$$

with $n, m \in \mathbb{Z}$. It is absolutely straightforward to check that they satisfy the $b^{-2}/2$-shift equation (8.19). Note that checking the $1/2$-shift equation, we actually only need $m \in \mathbb{Z}$. The additional restriction $n \in \mathbb{Z}$ is required when checking our novel $b^{-2}/2$-shift equation (8.19). The above amplitudes also satisfy the reflection symmetry constraint (5.30), a fact that has of course already been checked in [76].

Let us now turn our attention to the continuous $AdS_2^{(c)}$ branes of [72]. Their one point amplitudes read

$$f_\sigma(j|\alpha) = -\frac{\pi A_b}{\sqrt{\nu_b}} \frac{e^{-\alpha(2j+1)\sigma}}{\sin[\pi b^2(2j+1)]}, \quad (8.23)$$

with $\alpha \in \mathbb{R}$. Plugging that into the appropriate $b^{-2}/2$-shift equation (8.21), we can infer an expression for the residuum of the bulk-boundary OPE coefficient

$$\tilde{c}(b^{-2}/2, 0|\alpha) = -\frac{\mathrm{e}^{-\alpha\sigma b^{-2}}}{\nu_b^{b^{-2}/2}(1+b^2)}. \quad (8.24)$$

9 Summary and Discussion

The central topic of this thesis is the consistency of branes (boundary states) in the noncompact nonrational H_3^+ CFT. It is addressed by studying the famous Cardy-Lewellen sewing relations for the two point function. They become feasible in this context through the use of Teschner's trick, which makes RCFT techniques available for this noncompact CFT. Essentially, we derive so-called shift equations from the appropriate sewing relation and analyse their solutions. This results in explicit expressions for the one point functions of the corresponding branes.

Our main contributions can be divided into two great themes. The first one is a solution to the problem of deriving $b^{-2}/2$-shift equations. The issue here is that a continuation prescription for the two point function is needed (see chapter 6, in particular 6.1.1, section 7.5 and chapter 8). We show that an analytic continuation, although technically quite involved, can be carried out explicitly. The resulting shift equations lead to a brane spectrum which consists of one continuous set (6.78) and two discrete sets, (6.76) and (6.82), of AdS_2 solutions. In the following we refer to this as the analytic approach. Motivated by the Hosomichi-Ribault proposal [73] that we review in 7.5, we also explore a different continuation prescription that we refer to as the continuous approach for the rest of this chapter. It is the content of chapter 8 and results in one continuous (8.23) and one two-parameter discrete set (8.22) of AdS_2 solutions. The crucial point about the continuous approach is that it leaves an ambiguity in the continued two point function that could potentially be passed on to the shift equation and hence invalidate the whole procedure. Our merit here is, that we can demonstrate this not to happen in the H_3^+ model and succeed in deriving a meaningful and unambiguous $b^{-2}/2$-shift equation for this case as well.

The second theme is the initiation of a systematic exploration of branes in the H_3^+ model. Before our work, these branes were grouped into two great classes, AdS_2 and S^2 branes, and these two classes were each again divided into discrete and continuous branes. To this pattern, we propose to add an additional subdivision into regular and irregular solutions (section 5.2). Moreover, we point out that an analysis of potential equivalences between the H_3^+ model branes on the level of their isospin dependence is not sufficient and must be supplemented by further input from the study of consistency conditions like the Cardy-Lewellen constraints. Therefore, isospin dependencies that seem equivalent at first sight must nevertheless be studied separately. The importance of these observations is demonstrated by our subsequent analysis in chapters 5 and 6: Without the regular branes, we would miss some important parts of the brane spectrum in the

	u-dependence	shift equation (continuous) for $\Theta_{1/2}$?	shift equation (continuous) for $\Theta_{b^{-2}/2}$?	shift equation (discrete) for $\Theta_{1/2}$?	shift equation (discrete) for $\Theta_{b^{-2}/2}$?
ρ_1	$\lvert u-\bar{u}\rvert^{2j}$	[79]/✓	$\star/\circledast$	$\circledast$	$\star/\circledast$
	$(u-\bar{u})^{2j}$	[78]/$\circledast$	$\circledast$	[78]/$\circledast$	[78]/$\circledast$
ρ_2	$\lvert u+\bar{u}\rvert^{2j}$	[72]/✓	$\star/\circledast$	[76]/✓	$\star/\circledast$
	$(u+\bar{u})^{2j}$	$\circledast$	$\circledast$	$\circledast$	$\circledast$
ρ_3	$\lvert -1+u\bar{u}\rvert^{2j}$	—	—	—	—
	$(-1+u\bar{u})^{2j}$	—	—	[78]	[78]
ρ_4	$(1+u\bar{u})^{2j}$	—	—	[72]	—

Table 9.1: Classes of brane solutions: Our contributions are marked with a $\circledast$ for the analytic approach and a $\star$ for the continuous approach (see text). Confirmed results are ticked ✓. Recall that there is no distinction between analytic and continuous approach for the 1/2-shift equations. Also remember that we have reconsidered the results of [78] for reasons explained in section 5.2.

analytic approach (sections 6.1.4 and 6.2.3). Furthermore, concerning the isospin dependence we learn that gluing maps ρ_1 and ρ_2 are not equivalent for regular branes (compare 6.1.3 and 6.1.4 as well as 6.2.4 and 6.2.3). In view of these results, we are the first ones to give an exhaustive discussion of AdS_2 branes within the analytic approach, offering a complete treatment of the different isospin dependencies, following systematically the patterns discrete/continuous as well as regular/irregular and deriving two independent shift equations for each case. See table 9.1 for an overview. From the point of view of the systematics, our merit is the computation of many new 1/2-shift equations (namely (5.60), (5.73), (5.79), (5.82) and (5.83)) and much more importantly the derivation of the full set of $b^{-2}/2$-shift equations in case of the analytic approach (see (6.37), (6.58), (6.75), (6.77), (6.81) and (6.85)) as well as some equations of this type, (8.19) and (8.21), within the continuous approach. In addition, we also succeed in giving either a solution to the constraints or a proof of their non-existence (except for one case, section 6.2.4, where we could only make a conjecture).

Our results on the spectrum of AdS_2 branes may be interpreted as follows: In RCFT, according to an analysis carried out by Cardy [75] and, independently, Ishibashi [94], it is known that branes are in one to one correspondence with the physical spectrum of the bulk theory. In this light, we should only have one continuous set of branes. So how do the additional branes, labelled by two discrete parameters, fit in here? To give an interpretation, it has been proposed in [76] that the discrete AdS_2 branes should be thought of as beeing associated to the degenerate $\hat{\mathfrak{sl}}(2,\mathbb{C})_k$ representations with $j_{m,n} = -\frac{1}{2} + \frac{m}{2} + \frac{n}{2}b^{-2}$, $m,n \in \mathbb{Z}$. The brane spectrum that we find from the Hosmichi-Ribault proposal fits in nicely with this interpretation. This might indicate that the Cardy-Ishibashi analysis needs to be modified at some point to make it applicable to noncompact nonrational CFT. On the other hand, the brane spectrum that we obtain from the analytic approach gives (besides one continuous series) two one-parameter sets of discrete branes (see 6.1.4 and 6.2.3). In view of the above proposal, one would naturally associate them to the spins $j_{m,0}$ and $j_{0,n}$ and think that some branes are missing. However, the discrete branes in the analytic approach couple only to bulk fields that transform in finite dimensional $\mathfrak{sl}(2,\mathbb{C})$ representations $j \in \frac{1}{2}\mathbb{Z}$. Thus, they do not couple to the physical states of the model, which fall into the representations $j \in -\frac{1}{2}+\mathrm{i}\mathbb{R}_{\geq 0}$, and one could argue that these branes can actually be discarded. Then, we are only left with one continuous set of branes and the Cardy-Ishibashi analysis (which was originally devised for RCFT) remains correct. Consequently, if one adapts the Cardy-Ishibashi analysis to noncompact nonrational CFT, one should get a device that may be able to select one of the two brane spectra and hence one of the two approaches.

An adaption of the Cardy-Ishibashi analysis to noncompact nonrational CFT would probably als suggest a classification strategy for the H_3^+ branes. Recall that for our systematic treatment we have used the patterns AdS_2/S^2, discrete/continuous, regular/irregular and analytic/continuous (note that continuous appears with two different meanings). We have however not achieved a classification, as there are no results that guarantee this list of patterns to be complete. Hence, the question still remains what classification really means here, or phrased differently, how a complete list of patterns will look like. A noncompact nonrational version of the Cardy-Ishibashi analysis would certainly help here again. Interestingly, there were speculations about additional branes in the H_3^+ model only quite recently [103], but no additional patterns could be given.

Let us also look at the question of what approach, analytic or continuous, might be preferable from a different perspective. While we have shown that the continuity proposal works and even yields quite reasonable results, there are some issues about it that we want to criticize here: First of all, the H_3^+/Liouville map is singular at the transition point, that is, the correspondence really breaks down here. The "singularity" that is discussed in [73] is not seen on the H_3^+ side of the correspondence, but on the Liouville side. It occurs when two Liouville boundary fields collide. But this is in the first place a singularity of the map and not of the H_3^+ model. By the same reasoning, the continuity proposal is merely to be viewn as an assumption about the map, and not about the H_3^+ model. For these reasons, we want to advocate a model-intrinsic approach to the H_3^+ model here. From this perspective, the analytic approach is very natural and preferred for three simple reasons: Firstly, the two point function is an analytic object in its initial domain of definition. Secondly, no intractable problems are encountered when carrying out the analytic continuations. Thirdly, there is no danger here that the shift equation might become invalidated, since the two point function is clearly completely determined from analytic continuation. Moreover, the transition point is seen in the analytic approach as well (as the demarcation of the initial patch), but it is very well behaved and by no means singular. In fact, the requirement (of the Hosomichi-Ribault proposal) of finiteness and continuity at the $u = z$ "singularity" is also met here; the regularity behaviour is even better. Therefore, one might actually be tempted to reverse the argumentation of [73] and rather ask if anything new may be learned about Liouville theory, or at least about the correspondence, from the behaviour of the H_3^+ model (within the analytic approach) at the transition point. Another advantage of a model-intrinsic treatment is also its generalizability. Ultimately, one will be interested in general noncompact nonrational CFTs that may not have a correspondence to some Liouville-like theory. For such models, one needs results that are independent of possible mappings or correspondences. The lesson for the H_3^+ model is that one would make an analytic continuation to cross the $u = z$ singularity in a model-intrinsic study.

Concerning the H_3^+/Liouville correspondence, we have just indicated that one may ask if anything new can be learned in the opposite direction, transferring knowledge about the well-behaved transition point on the H_3^+ side to the Liouville side. This is a new perspective, since originally the correspondence was designed to work into the other direction only (transferring knowledge from Liouville theory to H_3^+). Another interesting direction in view of the correspondence is the incorporation of other branes: In its present form, the H_3^+/Liouville map works only for AdS_2/FZZT branes. Can it be generalized to include also S^2 branes (and on the Liouville side ZZ branes)?

Dwelling a litter longer in the vicinity of the H_3^+ model, we also want to point out that the impact of our work on string theory on AdS_3 [104] as well as the cigar CFT [105, 106] (which describes a bosonic string in an euclidean black hole background), which are both related to the H_3^+ model, is also to be worked out. For example, we have remarked above that the regular branes require $2j \in \mathbb{Z}$ and henceforth decouple from the physical spectrum of the H_3^+ model. Yet, these branes could still turn out to be important in view of string theory on AdS_3 or the cigar CFT, because the physical spectrum of these theories is richer (see [90] and [107], respectively).

Admittedly, the study we have carried out appears to be quite specifically tailored for a treatment of the H_3^+ model. But is this really true? Let us speculate which of the features that we have encountered may carry over to more general models. We would expect that the occurence of $u = z$ "singularities" is a general feature of nonrational models with an underlying noncompact symmetry. The reason is that a nonpolynomial dependence on the isospin variable is typical of the representations of noncompact groups. One might suspect that the Appell and Horn functions (which are generalized hypergeometric functions in two complex variables) become Lauricella functions in more general models with, say, $\mathfrak{sl}(n, \mathbb{C})$ symmetry. This statement may be understood as a call for a revival of interest in these special functions and, in particular, their analytic continuations. These functions have very generic singularities whenever two (or more) of their variables coincide (but analytic continuation over the singularities should in general be possible). These will be the generalizations of the $u = z$ singularity encountered here. It is then expected, that the analytic approach which has been pursued here is also applicable in such situations, only that one has more singularities to take care of. Moreover, possible mappings to Liouville-like theories (for $\mathfrak{sl}(n, \mathbb{C})$ WZNW models these could be the $\mathfrak{sl}(n)$ Toda theories; see below) will probably break down again, just like the H_3^+/Liouville correspondence.

Seeing the progress made towards an understanding of the H_3^+ model and Liouville theory, one can hope that more general nonrational CFTs will be studied in the near future. Noncompact WZNW models with an $\mathfrak{sl}(n, \mathbb{C})$ symmetry together with the $\mathfrak{sl}(n)$ conformal Toda field theories [108] are natural next candidates. They are expected to generalize the H_3^+/Liouville correspondence, a speculation that becomes understandable if one realizes that Liouville theory can be viewn as the $\mathfrak{sl}(2)$ Toda theory. Other interesting and promising developments are emerging just these days. In [109], a whole family of solvable nonrational CFT models was introduced and one can hope for new lessons to be learned from them. Moreover, the first paper [110] that is devoted to a study of the general structures and foundations of nonrational CFT has appeared by now. Thus, the prospects for the study of nonrational CFT are very good. It is a rather young field that just starts to develop from the study of explicit examples towards a more general understanding. It is expected that one will get a handle on a large variety of nonrational CFT models soon. This is highly desirable, as such models provide the framework for a treatment of noncompact string backgrounds.

Part III

Appendices

Wortarm stolpert ihr,
selbstvergessen,
getrieben vom Engel der Abstraktion,
über Galois-Felder und Riemann-Flächen,
knietief im Cantor-Staub,
durch Hausdorffsche Räume.

Hans Magnus Enzensberger, Die Mathematiker

A Representation Theory of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$

In this appendix, we shall describe elements of the representation theory for locally compact topological groups. The literature we follow is [111, 112, 113]. Although we give explicit formulae only for $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$, we shall in fact arrive at a complete description of the unitary irreducible representations of $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{R})$ in this appendix. Our aim is to give a flavour of the general theory, but without introducing most of its vast and rich terminology and technology.

A.1 Locally Compact Groups

Let us start with some definitions and assumptions that allow us to fix our notation. In this we follow [114, 115].

A *topological group* is a group G that is at the same time also a topological space, such that group multiplication $G \times G \to G$ and inversion $G \to G$ are continuous (the former with respect to the product topology on $G \times G$). A topological group G is *locally compact*, if every point in G has a compact neighbourhood (in Hausdorff spaces this is equivalent to having a local base of compact neighbourhoods). We also want to assume that G has the *Hausdorff property*, that is to any two distinct points $a, b \in G$, $a \neq b$, there are neighbourhoods U of a and V of b such that $U \cap V = \emptyset$. The assumptions of local compactness and of the Hausdorff property are of great importance here, as they guarantee the existence of a left-invariant Radon measure on G: The famous *Haar measure* which is unique up to a positive factor. We typically denote it by μ. When talking about representations, this measure will be indispensable because it gives a notion of scalar product, norm and therefore also unitarity. In the following we denote by G a locally compact topological group with the Hausdorff property and for brevity just call it a locally compact Hausdorff group. The rôle of the Hausdorff property is that it ensures the compact sets to be Borel sets. This is important, since the theory makes great use of inner regular Borel measures and the notion of an inner regular measure is based on the approximation of measurable sets by compact subsets, which therefore need to be measurable as well. For more details and the importance of local compactness in the proof of Haar's theorem and the Riesz representation theorem (which enters the proof of Haar's result), see [114].

The next ingredient we are going to use is a subgroup $H \subset G$. It comes with the canonical subspace topology, of course, and is itself a topological group. With

G being locally compact, the left coset G/H is also locally compact. In order to establish the Hausdorff property for G/H, we need to assume that H is closed in G. Note that we have not assumed H to be a normal subgroup of G. Therefore, the homogeneous space G/H is not necessarily a group and we cannot make use of Haar's theorem again to ensure the existence of an invariant measure on it (if H was a normal subgroup, this conclusion would however be correct). Yet, what is guaranteed in this situation is the existence of a *quasi-invariant* measure ν on the homogeneous space G/H [112]. Quasi invariance of the measure ν means that every ν-null set is also a null set with respect to any G-translate of ν, i.e. with respect to the measures $\nu \circ g$ for all $g \in G$. This property is usually also referred to by saying that all G-translates of ν are *absolutely continuous* with respect to ν. The quasi-invariant measure ν is needed in the definition of induced representations that we give below. From now on, we will always denote by H a closed subgroup of G. G acts naturally on G/H by left translation: $x \mapsto gx$ and this action is continuous. Moreover, π denotes the canonical epimorphism $\pi : G \to G/H$. Finally, a *section* for G/H is a Borel map $s : G/H \to G$ such that $\pi \circ s = \mathbb{1}|_{G/H}$, i.e. s maps each equivalence class to an element in the same class: $s(x)H = x$ for $x \in G/H$. Such sections always exist under the given assumptions [112].

These are the basics of topological groups that we need in what is to follow. Note that for Lie groups the above assumptions are incorporated automatically, since a Lie group carries the structure of a smooth manifold, which is typically assumed to be Hausdorff, and it inherits local compactness from being homeomorphic to $\mathbb{R}^n$.

A.2 Elements of the General Representation Theory for Locally Compact Groups

A representation of a topological group G is a homomorphism Δ from G into the group $\mathcal{B}(V)$ of bounded linear operators on a Hilbert space V, such that the map

$$\begin{cases} G \times V \to V \\ (g, v) \mapsto \Delta(g)v \end{cases} \tag{A.1}$$

is continuous. Provided that $||\Delta(g)||$ (operator norm on $\mathcal{B}(V)$) is uniformly bounded in a neighbourhood of the identity, one can easily see that the continuity property follows from strong continuity of Δ, i.e. continuity of the maps

$$\begin{cases} G \to V \\ g \mapsto \Delta(g)v \end{cases} \tag{A.2}$$

for all $v \in V$ at $g = 1$. A representation Δ is said to be *unitary*, if

$$[\Delta(g)]^\dagger = [\Delta(g)]^{-1} \quad \forall \ g \in G\,. \tag{A.3}$$

It is called *irreducible*, if there are no closed invariant subspaces other than 0 and V, i.e. no closed vector subspaces $U \subset V$ such that

$$\Delta(g)U \subset U \quad \forall \ g \in G\,, \tag{A.4}$$

with $U \neq 0, V$.

On any locally compact Hausdorff group G, there is one representation we can write down immediately: The so-called *(left) regular representation* λ

$$\begin{cases} G \to \mathcal{B}(L^2(G|\mu)) \\ g \mapsto \lambda(g) \end{cases} \tag{A.5}$$

which acts on $L^2(G|\mu)$ as

$$[\lambda(g)f](g') := f(g^{-1}g')\,. \tag{A.6}$$

Thanks to the translation invariance of the Haar measure μ, this representation is clearly unitary. It is however usually reducible. What the irreducible unitary representations are and how the regular representation decomposes into them, belong to the central questions of representation theory. In the theory of compact groups, it is a well-established fact that all unitary irreducible representations do occur in the decomposition of the regular representation. We shall see, however, that this statement does not remain true for noncompact groups.

A.2.1 Induced Representations

One important method to construct irreducible unitary representations is the method of *induced representations*. This is of course known to a physicist from Wigner's construction of induced representations for the Poincaré group which lies at the heart of relativistic quantum theory. The basic idea is that from a given representation χ (with representation space V) of some subgroup H and a section $s : G/H \to G$, a representation of the whole group G is induced on the Hilbert space $L^2(G/H, V|\nu)$ via

$$[\Delta(g)f](x) := \left(\frac{\mathrm{d}\nu(g^{-1}x)}{\mathrm{d}\nu(x)}\right)^{\frac{1}{2}} \chi\left([s(x)]^{-1}gs\left(g^{-1}x\right)\right) f\left(g^{-1}x\right). \tag{A.7}$$

We need to explain the normalization $\left(\frac{\mathrm{d}\nu(g^{-1}x)}{\mathrm{d}\nu(x)}\right)^{1/2}$. Since we are working with a quasi-invariant measure ν (i.e. any set that is a ν-null set is also a $(\nu \circ g^{-1})$-null set) we have, by the Radon-Nikodym theorem [114], that $\frac{\mathrm{d}\nu(g^{-1}x)}{\mathrm{d}\nu(x)}$ exists as a quasi-integrable function (with respect to ν) which is non-negative, since $(\nu \circ g^{-1})$ is a measure. (A function f is quasi-integrable, if it is measurable and if at least one of $f^+ = \frac{1}{2}(f + |f|), f^- = -\frac{1}{2}(f - |f|)$ has a finite integral. If both parts are finite, f is integrable.) Thus, the expression written down in (A.7) is sensible. We will understand the rôle of the normalization factor in a little while.

It is easy to check that (A.7) does indeed define a homomorphism and that the operator $\Delta(g)$ is linear. Note that the group element $[s(x)]^{-1} gs\,(g^{-1}x)$ is really in H, as it has to be since χ is a representation of H only. Also, if χ is a character (or quasi-character) of H, i.e. a homomorphism from H to S^1 (or to $\mathbb{C}^\times := \mathbb{C} \setminus 0$), the definition is independent of the choice of section $s : G/H \to G$. This is, because for any section $s(x) \in xH \subset G$ (since $\pi \circ s = \mathbb{1}|_{G/H}$, $\pi : G \to G/H$ is the canonical epimorphism), which implies that if $t : G/H \to G$ is another section, we have

$$s(x) = t(x) \cdot h(x) \tag{A.8}$$

for some $h(x) \in H$. Therefore,

$$[s(x)]^{-1} gs\left(g^{-1}x\right) = [h(x)]^{-1} \cdot [t(x)]^{-1} gt\left(g^{-1}x\right) \cdot h(x) \tag{A.9}$$

and thus, if χ is a (quasi-) character, independence of the choice of section follows. The representation space $L^2(G/H, V|\nu)$ is the space of square integrable functions (with respect to the measure ν on G/H) from G/H to V. The norm is

$$||f||^2_{L^2} := \int_{G/H} ||f(x)||^2_V \mathrm{d}\nu(x)\,, \tag{A.10}$$

where $||\cdot||_V$ is the norm on the Hilbert space V. It is quite obvious that $\Delta(g)f$ is again in $L^2(G/H, V|\nu)$: As mentioned above, $\frac{d\nu(g^{-1}x)}{d\nu(x)}$ is a non-negative, quasi-integrable function. Moreover, the left translation $x \mapsto g^{-1}x$ is continuous, hence measurable, and the composed map

$$x \mapsto \left[||\cdot||^2_V \circ f \circ g^{-1}\right](x) \equiv ||f\left(g^{-1}x\right)||^2_V \tag{A.11}$$

is therefore measurable and, since it is non-negative, also quasi-integrable. The product of non-negative quasi-integrable functions is again quasi-integrable, so we can have a look at

$$\begin{aligned}
&\int_{G/H} \left|\frac{\mathrm{d}\nu\,(g^{-1}x)}{\mathrm{d}\nu(x)}\right| \cdot ||f\left(g^{-1}x\right)||^2_V \mathrm{d}\nu(x) = \\
&= \int_{G/H} ||f\left(g^{-1}x\right)||^2_V \mathrm{d}\nu\left(g^{-1}x\right) && \text{since } \frac{\mathrm{d}\nu(g^{-1}x)}{\mathrm{d}\nu(x)} \geq 0 \\
&= \int_{G/H} ||f(x)||^2_V \mathrm{d}\nu(x) && \text{since } g^{-1} : G/H \cong G/H \\
&= ||f||^2_{L^2} < \infty\,.
\end{aligned}$$

Thus, $\left|\frac{\mathrm{d}\nu(g^{-1}x)}{\mathrm{d}\nu(x)}\right| \cdot ||f\,(g^{-1}x)\,||^2_V$ is not only quasi-integrable, but even integrable. And as χ, being a representation, is a bounded operator, we can conclude that

$$[\Delta(g)f] \in L^2(G/H, V|\nu)\,. \tag{A.12}$$

From this argument, we see that the Radon-Nikodym derivative $\frac{\mathrm{d}\nu(g^{-1}x)}{\mathrm{d}\nu(x)}$ repairs the missing translation invariance of the measure and is an important ingredient in making the representation unitary. Using this once again, we have immediately that

$$||\Delta(g)f||^2 = \int_{G/H} \left\|\sigma\left([s(x)]^{-1} gs\left(g^{-1}x\right)\right) f\left(g^{-1}x\right)\right\|_V^2 \mathrm{d}\nu\left(g^{-1}x\right) \tag{A.13}$$

and therefore, if χ is unitary, i.e.

$$||\chi(h)f\left(g^{-1}x\right)||_V = ||f\left(g^{-1}x\right)||_V\,, \tag{A.14}$$

we gain that also Δ is unitary. Even more, equation (A.13) tells us that (even for nonunitary χ) $\Delta(g)$ is a bounded operator with norm

$$||\Delta(g)||_{\mathcal{B}(L^2)} \leq ||\sigma||_{\mathcal{B}(V)}\,. \tag{A.15}$$

From the description of induced representations that we have given, it is clear that these are always infinite dimensional, since their representation spaces are certain $L^2(\dots)$ spaces. Therefore, although induced representations as described above are also well-defined for compact groups, they are only of marginal importance there, since all unitary irreducible representations of compact groups are finite dimensional. However, in the representation theory of noncompact groups, it is the infinite dimensional representations that are the unitary and irreducible ones (except for the trivial representation, of course) and thus, the induced representations are indispensable here. Now, as we have seen, an induced representation depends on two data: A subgroup H and a character χ of that subgroup. One should therefore ask, which subgroups and characters one can use in order to generate the maximum of inequivalent irreducible unitary representations. It turns out that we do not need to consider all subgroups, but rather the so-called *parabolic subgroups* and certain characters for them. A parabolic sugroup $H \subset G$ is such that G/H is a projective variety. For our purposes, the cases of $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{R})$ are sufficient, and for these groups the parabolic subgroups are just the *block-upper-triangular subgroups*. They consist of $n \times n$ matrices of the form

$$P = \begin{pmatrix} g_{11} & & * \\ & \ddots & \\ 0 & & g_{rr} \end{pmatrix}, \tag{A.16}$$

where the g_{ii} are complex (in the case of $\mathrm{SL}(n,\mathbb{C})$) or real (for $\mathrm{SL}(n,\mathbb{R})$) matrices of size $m_i \times m_i$ with non-zero determinant. Of course, $m_1 + \cdots + m_r = n$. The matrices P have a standard decomposition, the *Levi-Langlands decomposition*:

$$P \equiv M \cdot N := \begin{pmatrix} g_{11} & & 0 \\ & \ddots & \\ 0 & & g_{rr} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1}_{11} & & * \\ & \ddots & \\ 0 & & \mathbb{1}_{rr} \end{pmatrix}, \tag{A.17}$$

where the $\mathbb{1}_{ii}$ are identity matrices of dimension $m_i \times m_i$. Now, there are two cases to distinguish. In the general framework, there is a minimal parabolic subgroup, the so-called *Borel subgroup*. It is singled out, since it already suffices for the decomposition of the left regular representation. In case of $\mathrm{SL}(\mathrm{n},\mathbb{C})$ or $\mathrm{SL}(\mathrm{n},\mathbb{R})$, minimality means $r = n$, $m_1 = \cdots = m_n = 1$ and it is therefore simply the group of triangular matrices (with appropriate entries). We shall call such a matrix B. Induction from the Borel subgroup gives the famous *principal series representations* of Gel'fand and Naimark. The characters χ to be used are defined to be trivial on the second factor N in the Levi-Langlands decomposition, while on the first factor, M (which is now just a diagonal matrix) they run through the characters of the group of diagonal matrices. We will denote the principal series representations by Δ_χ. Gel'fand and Naimark showed that they are all irreducible (for $\mathrm{SL}(\mathrm{n},\mathbb{C})$), but not all inequivalent. We will discuss the concrete examples of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$ soon. They also showed that the regular representation decomposes into a direct integral of principal series representations. This is surprising, because the trivial representation (which is unitary and irreducible) is not contained in the principal series representations. Therefore, in contrast to the representation theory for compact groups, it is in the case of noncompact groups no longer true that all unitary irreducible representations occur in the decomposition of the regular representation.

And indeed, there are in general many more unitary irreducible representations than just the identity and the principal series ones. Induction from general parabolic subgroups P gives rise to the so-called *degenerate series representations*. The characters χ one uses here are defined as follows: Take the map

$$P = \begin{pmatrix} g_{11} & & * \\ & \ddots & \\ 0 & & g_{rr} \end{pmatrix} \mapsto (\det(g_{11}),\ldots,\det(g_{rr})) \in (\mathbb{C}^\times)^r \tag{A.18}$$

and decompose it with a character of the group $(\mathbb{C}^\times)^r$ in order to yield an appropriate character of P. The resulting representations, which we will denote $\Delta_{P,\chi}$, are again all irreducible (for $\mathrm{SL}(\mathrm{n},\mathbb{C})$) and unitary. The maximal case $r = 1$, $m_1 = n$, which is obviously the case where P=G, produces the trivial representation.

This is not yet the end of the story. There are still more unitary irreducible representations to reveal. Taking a quasi-character instead of a character in the definition of Δ_χ or $\Delta_{P,\chi}$, the representations are no longer unitary, but for some suitable quasi-characters the scalar product (and hence the canonical representation space) can be changed to yield more unitary irreducible representations. See (A.34) for an example. These additional representations (they are inequivalent to the former ones) are called *supplementary series representations*.

And even now, we are not quite at the end. In fact, for the complex semisimple Lie groups we have reached the complete list of irreducible unitary representations. But for their real forms, not all of the described induced representations are irreducible and, more importantly, there can be further unitary irreducible representations that are not induced representations. In order to construct them in generality, the theory has to go much deeper. We will not attempt to go into the many new features of these new representations here, but since they do show up in $\mathrm{SL}(2,\mathbb{R})$, we devote a separate subsection to describe and comment on them briefly.

A.2.2 A Few Remarks on Discrete Series Representations

As we have just said, the general construction of the *discrete series representations* is very different from and requires a lot more technology than the induced representations. Harish-Chandra has constructed them under the assumption that the group G has a compact Cartan subgroup. The focus on Lie groups is necessary here (while subsection A.2.1 on induced representations has more general validity) and hence, the definition of a *Cartan subgroup* is clear: It is simply a maximal connected abelian subgroup of G generated by the elements of a Cartan subalgebra, which is standard in Lie theory. In the study of noncompact groups, it is instructive and common to decompose the Lie algebra into *compact* and *noncompact generators* (in [111], this is called the *Cartan decomposition*). One arrives at it as follows: There always exist anti-linear involutions θ (i.e. $\theta^2 = 1$) of a Lie algebra. For the classical Lie groups, which are matrix groups, a common choice is the *Cartan involution*. On the group elements, it is conjugate transpose followed by inversion, i.e. on the Lie algebra it becomes conjugate transpose together with multiplication by minus one. Now, it becomes crucial that we regard the Lie algebra as an algebra over the real (!) numbers. (This does have consequences when talking about the *dimension* and the *rank* of the Lie group, which we here always mean to be the *real dimension* and the *real rank*. With this convention, $\dim \mathrm{SL}(2,\mathbb{C}) = 6$ and $\operatorname{rank} \mathrm{SL}(2,\mathbb{C}) = 2$, whereas $\dim \mathrm{SL}(2,\mathbb{R}) = 3$ and $\operatorname{rank} \mathrm{SL}(2,\mathbb{R}) = 1$.) The eigenvalues of θ are then real and the Cartan decomposition is the decomposition of the Lie algebra into θ-even (i.e. eigenvalue $+1$) and θ-odd (eigenvalue -1) elements and these are called compact and noncompact generators, respectively. For the classical Lie groups, the compact (θ-even) ones are anti-hermitian, while the noncompact (θ-odd) generators are hermitian (we follow the mathematician's convention and use the exponential map without including the imaginary unit). If the Cartan subalgebra can be chosen among the compact generators only, the resulting Cartan subgroup will be compact. As the maximal compact subgroup $K \subset G$ of G surely also has a compact Cartan subgroup, compactness of the Cartan subgroup of G is obviously equivalent to the condition $\operatorname{rank} K = \operatorname{rank} G$. This is the situation of Harish-Chandra's theorem. If it is met, so-called discrete series representations can be constructed on general grounds. One easily sees that SL(2, ℝ) is such a case: Its Lie algebra is

$$\mathrm{sl}(2,\mathbb{R}) = \mathrm{span}_{\mathbb{R}} \left\langle t_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, t_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, t_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle . \tag{A.19}$$

t_1 is clearly a compact generator, while the other two, t_2 and t_3, are noncompact. As the Cartan subalgebra is one-dimensional, it can simply be chosen to be spanned by t_1 and is hence compact. This is of course also in accord with the rank of the maximal compact subgroup, which is SO(2) in this example, being equal to the rank of the whole group. However, for the groups $\mathrm{SL}(n,\mathbb{R})$, which have maximal compact subgroup $\mathrm{SO}(n)$, we have rank $\mathrm{SL}(n,\mathbb{R}) = n-1$, rank $\mathrm{SO}(n) = \left\lfloor \frac{n}{2} \right\rfloor$ (with $\lfloor \cdot \rfloor$ the Gauss bracket) and therefore

$$\text{rank } \mathrm{SL}(n,\mathbb{R}) \neq \text{rank } \mathrm{SO}(n) \text{ for } n > 2 . \tag{A.20}$$

Hence, the Harish-Chandra construction does not work in these cases. Moreover, as the existence of a compact Cartan subgroup, i.e. rank K = rank G, is also a necessary condition for the existence of discrete series representations, the groups $\mathrm{SL}(n,\mathbb{R})$ do not admit discrete series representations whenever $n > 2$. This is the reason why we contend ourselves with just citing the discrete series representations for SL(2, ℝ) and will not delve into general constructions that in view of $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{R})$ would not bring any additional benefit.

A.3 Irreducible Unitary Representations of SL(2, ℂ)

In this section we want to construct the irreducible unitary representations of SL(2, ℂ) using the technique of induced representations introduced in section A.2.1.

First, we need to determine the parabolic subgroups. This is fairly easy here, since the only ones are either the full group itself or the upper triangular subgroup (which is precisely the Borel subgroup). Since the group itself always induces the trivial representation, we are left with one case only, namely induction from the Borel subgroup, which results in the principal series representations. As there are no generic parabolic subgroups, we see immediately that $G := \mathrm{SL}(2,\mathbb{C})$ does not possess any degenerate series representations.

So let us take H to be the upper triangular subgroup. It is easy to see that two matrices $g, g' \in G$,

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad g' = \begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix}, \tag{A.21}$$

are equivalent modulo H, $g \sim_H g'$, if and only if

$$\frac{g'_{11}}{g'_{21}} = \frac{g_{11}}{g_{21}} = z \in \hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\} . \tag{A.22}$$

Hence, we define the projection map $\pi : G \to G/H$ to be

$$\pi : \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \mapsto \frac{g_{11}}{g_{21}}. \tag{A.23}$$

This shows that $G/H = \hat{\mathbb{C}}$ is really a projective variety, and H a parabolic subgroup.

Now, we want to determine the action of G on the homogeneous space $G/H = \hat{\mathbb{C}}$. Left-translation by an element $g \in G$

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} as + bu & \dots \\ cs + du & \dots \end{pmatrix} \tag{A.24}$$

induces precisley

$$z = \pi \begin{pmatrix} s & t \\ u & v \end{pmatrix} \mapsto \pi \begin{pmatrix} as + bu & \dots \\ cs + du & \dots \end{pmatrix} = \frac{az + b}{cz + d} \tag{A.25}$$

on $G/H = \hat{\mathbb{C}}$. The space of sections $s : G/H \to G$ can therefore be parametrized by the elements (recall $\pi \circ s = \mathbb{1}|_{G/H}$)

$$s(z) = \begin{pmatrix} \alpha z & \beta - \frac{1}{\alpha} \\ \alpha & \beta z^{-1} \end{pmatrix}. \tag{A.26}$$

Note that these sections are not defined for $z = 0$ and $z = \infty$. This is not a problem, since the theory only requires Borel sections, which may be undefined on a set of measure zero. With this explicit parametrisation, the particular combination $[s(z)]^{-1} gs\left(g^{-1}z\right)$ can be checked to really lie in H. In fact, it is given by

$$[s(z)]^{-1} gs\left(g^{-1}z\right) = \begin{pmatrix} (-cz + a)^{-1} & \dots \\ 0 & (-cz + a) \end{pmatrix}. \tag{A.27}$$

What we have denoted with dots here does not play a rôle, since the quasi-characters that are used for the principal continuous representations are trivial on the off-diagonal part in the Levi-Langlands decomposition. The quasi-characters $\chi_{k,\rho}$ to be used are parametrized by $k \in \mathbb{Z}$ and $\rho \in \mathbb{C}$ and they map

$$\chi_{k,\rho} : \begin{pmatrix} (-cz + a)^{-1} & \dots \\ 0 & (-cz + a) \end{pmatrix} \mapsto \left(\frac{-cz + a}{|-cz + a|}\right)^{-k} |-cz + a|^{-\rho}. \tag{A.28}$$

The last thing we need to determine is the Radon-Nikodym derivative of the measure $\mathrm{d}^2 z = \mathrm{d}x\mathrm{d}y = \frac{\mathrm{i}}{2}\mathrm{d}z\mathrm{d}\bar{z}$. Since we have

$$\mathrm{d}\left(g^{-1}z\right) = \mathrm{d}z \cdot (-cz + a)^{-2}, \tag{A.29}$$

the Radon-Nikodym factor (already taking the square-root) is just an additional contribution of $|-cz+a|^{-2}$.

With all these ingredients in our hands, we can now finally write down the *full nonunitary principal series representations* of $\mathrm{SL}(2,\mathbb{C})$:

$$\begin{gathered}\left[\Delta_{k,\rho} f\right](z) = |-cz+a|^{-2-\rho}\left(\frac{-cz+a}{|-cz+a|}\right)^{-k} f\left(\frac{dz-b}{-cz+a}\right), \\ f \in L^2(\hat{\mathbb{C}}, \mathbb{C}^\times | \mathrm{d}^2 z), \qquad k \in \mathbb{Z}, \quad \rho \in \mathbb{C}\,.\end{gathered} \tag{A.30}$$

This is nonunitary, because the $\chi_{k,\rho}$ are generically quasi-characters. They obviously become characters for $\rho \in \mathrm{i}\mathbb{R}$. The corresponding representations are then unitary. It can also be shown that they are irreducible. These are the *irreducible unitary principle series representations* of $\mathrm{SL}(2,\mathbb{C})$:

$$\begin{gathered}\left[\Delta_{k,v} f\right](z) = |-cz+a|^{-2-\mathrm{i}v}\left(\frac{-cz+a}{|-cz+a|}\right)^{-k} f\left(\frac{dz-b}{-cz+a}\right), \\ f \in L^2(\hat{\mathbb{C}} | \mathrm{d}^2 z), \qquad k \in \mathbb{Z}, \quad v \in \mathbb{R}\,.\end{gathered} \tag{A.31}$$

However, there are equivalences among the $\Delta_{k,v}$, namely

$$\Delta_{k,v} \simeq \Delta_{-k,-v}\,. \tag{A.32}$$

Therefore, we usually restrict to a set of inequivalent representations out of this class that we parametrize as

$$\begin{gathered}\left[\Delta_{k,v} f\right](z) = |-cz+a|^{4v}\left(\frac{-cz+a}{|-cz+a|}\right)^{-k} f\left(\frac{dz-b}{-cz+a}\right), \\ f \in L^2(\hat{\mathbb{C}} | \mathrm{d}^2 z), \qquad k \in \mathbb{Z}, \quad v \in -\frac{1}{2} + \mathrm{i}\mathbb{R}_{\geq 0}\,.\end{gathered} \tag{A.33}$$

With the principal series representations at our disposal, we are already very close to the complete list. There are no degenerate series representations, as we have remarked at the beginning of this section and the complex groups do not admit any discrete series representations. The only missing ones are the supplementary series representations that are obtained from the nonunitary principal series with $k=0$ and $0<\rho<2$. They become unitary if the scalar product is changed from the standard L^2 product to

$$(f|g) := \int \mathrm{d}^2 z \int \mathrm{d}^2 w \, \frac{[f(z)]^* \, g(w)}{|z-w|^{2-\rho}} \tag{A.34}$$

(this is obviously positive-definite). Gel'fand and Naimark proofed that these are unitary and irreducible. Thus we have in addition the following *(irreducible unitary) supplementary series representations* of $\mathrm{SL}(2,\mathbb{C})$:

$$\begin{gathered}\left[\Delta_w f\right](z) = |-cz+a|^{-2-w} f\left(\frac{dz-b}{-cz+a}\right), \\ f \in L^2(\hat{\mathbb{C}} | (\cdot|\cdot)), \qquad 0 < w < 2\,.\end{gathered} \tag{A.35}$$

A.4 Irreducible Unitary Representations of $\mathrm{SL}(2,\mathbb{R})$

Let us start with the induced representations. This is completely analogous to the complex case treated in the previous section. Again, there are no degenerate series representations, since the only two parabolic subgroups are the group $G := \mathrm{SL}(2,\mathbb{R})$ itself (that induces the identity representation) and the upper triangular subgroup H. Note that this time all entries of the matrices are real, of course. Therefore, the homogeneous space is

$$G/H \simeq \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}\,, \tag{A.36}$$

again a projective variety. The action of G on this space is just as in the previous section (with all entries real — we will not bother to say this any more from now on), the canonical epimorphism π is just the same, the space of sections can be parametrized likewise, the particular combination $[s(x)]^{-1}\,gs\,(g^{-1}x)$ $(x \in \mathbb{R})$ looks completely as before. What changes here are the characters $\chi_{\epsilon,\rho}$. They are parametrized by $\epsilon \in \{0,1\}$ and $\rho \in \mathbb{C}$ and act as

$$\chi_{\epsilon,\rho} : \begin{pmatrix} (-cx+a)^{-1} & \cdots \\ 0 & (-cx+a) \end{pmatrix} \mapsto \mathrm{sgn}^{\epsilon}(-cx+a)\,|-cx+a|^{-\rho}\ . \tag{A.37}$$

Finally, the Radon-Nikodym factor (again already taking the square-root), only produces $|-cx+a|^{-1}$ this time. Assembling all that, we write down the *full nonunitary principal series representations* of $\mathrm{SL}(2,\mathbb{R})$:

$$\begin{gathered}[\Delta_{\epsilon,\rho}f]\,(x) = |-cx+a|^{-1-\rho}\,\mathrm{sgn}^{\epsilon}(-cx+a) f\left(\frac{dx-b}{-cx+a}\right), \\ f \in L^2(\bar{\mathbb{R}}, \mathbb{R}^{\times}|\mathrm{d}x), \qquad \epsilon \in \{0,1\}\,, \quad \rho \in \mathbb{C}\,. \end{gathered} \tag{A.38}$$

Being based on quasi-characters, these are again generically nonunitary, but do become unitary for $\rho \in \mathrm{i}\mathbb{R}$, i.e. precisely when the $\chi_{\epsilon,\rho}$ become characters. The resulting unitary representations are all irreducible, except for the case $(\epsilon,\rho) = (1,0)$. There are also the following equivalences between these representations:

$$\Delta_{0,\mathrm{i}v} \simeq \Delta_{0,-\mathrm{i}v}\,, \qquad \Delta_{1,\mathrm{i}v} \simeq \Delta_{1,-\mathrm{i}v}\ , \tag{A.39}$$

that we readily use to reparametrize the representations and only list inequivalent ones. Consequently, we have the following *irreducible unitary principle series representations* of $\mathrm{SL}(2,\mathbb{R})$:

$$\begin{gathered}[\Delta_{0,v}f]\,(x) = |-cx+a|^{2v}\,f\left(\frac{dx-b}{-cx+a}\right), \\ f \in L^2(\bar{\mathbb{R}}|\mathrm{d}x), \qquad v \in -\frac{1}{2} + \mathrm{i}\mathbb{R}_{\geq 0}\,, \end{gathered} \tag{A.40}$$

as well as

$$\begin{gathered}\left[\Delta_{1,v} f\right](x) = |-cx+a|^{2v} \operatorname{sgn}(-cx+a) f\left(\frac{dx-b}{-cx+a}\right), \\ f \in L^2(\bar{\mathbb{R}}|\mathrm{d}x), \qquad v \in -\frac{1}{2} + \mathrm{i}\mathbb{R}_{>0}\,.\end{gathered} \tag{A.41}$$

Note that we have to take the imaginary part of v strictly greater than zero in the last case ($\epsilon = 1$), while it is greater or equal to zero for $\epsilon = 0$. This is due to the representation $\Delta_{1,0}$ being reducible, as stated above.

Moreover, there are again supplementary series representations. Just like in the previous section, they stem from the nonunitary principal series with $\epsilon = 0$, $0 < \rho < 1$ if one uses the following scalar product for functions $f, g : \mathbb{R} \to \mathbb{C}$

$$(f|g) := \int \mathrm{d}x \int \mathrm{d}y \frac{[f(x)]^* g(y)}{|x-y|^{1-\rho}}\,. \tag{A.42}$$

The *(irreducible unitary) supplementary series representations* of SL(2, ℝ) are therefore

$$\begin{gathered}\left[\Delta_{w} f\right](x) = |-cx+a|^{-1-w} f\left(\frac{dx-b}{-cx+a}\right), \\ f \in L^2(\bar{\mathbb{R}}, \mathbb{C}|(\cdot|\cdot)), \qquad 0 < w < 1\,.\end{gathered} \tag{A.43}$$

So far, everything was just like in the complex case. Now, however, our list of irreducible unitary representations is not yet complete: We need to add the *(irreducible unitary) discrete series representations* of SL(2, ℝ). Let us cite them here, following [111]. We have

$$\begin{gathered}\left[\Delta_n^+ f\right](z) = (-cz+a)^{-n} f\left(\frac{dz-b}{-cz+a}\right), \qquad n \in \mathbb{Z}_{\geq 2}\,, \\ f : \mathbb{H}^+ \to \mathbb{C} \text{ holomorphic}, \quad ||f||_n^2 := \int_{y>0} \mathrm{d}x\mathrm{d}y |f(x+\mathrm{i}y)|^2 y^{n-2} < \infty\,,\end{gathered} \tag{A.44}$$

and an antiholomorphic version

$$\begin{gathered}\left[\Delta_n^- f\right](\bar{z}) = (-c\bar{z}+a)^{-n} f\left(\frac{d\bar{z}-b}{-c\bar{z}+a}\right), \qquad n \in \mathbb{Z}_{\geq 2}\,, \\ f : \mathbb{H}^- \to \mathbb{C} \text{ antiholomorphic}, \quad ||f||_n^2 := \int_{y>0} \mathrm{d}x\mathrm{d}y |f(x-\mathrm{i}y)|^2 y^{n-2} < \infty\,,\end{gathered} \tag{A.45}$$

where $\mathbb{H}^\pm$ are the upper and lower half-plane respectively: $\mathbb{H}^\pm := \{\mathrm{Im}(z) \gtrless 0\}$. For the value $n = 1$, the (irreducible unitary) representations $\Delta_1^\pm$ can be defined in the same way, but the scalar product has to be changed to

$$||f||_1^2 := \sup_{y>0} \int \mathrm{d}x |f(x \pm \mathrm{i}y)|^2\,. \tag{A.46}$$

$\Delta_1^{\pm}$ are sometimes called *limits of discrete series representations*. We had mentioned above, that the representation $\Delta_{1,0}$, which lies in the principal series, is reducible. In fact, it can be shown that

$$\Delta_{1,0} \simeq \Delta_1^+ \oplus \Delta_1^- \,. \tag{A.47}$$

B H_3^+ Primary Fields in Different Bases

According to the representation theory of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$ that we summarize in appendix A, we are commonly working with bulk fields $\Theta_j(u|z)$ that carry a complex isospin variable $u \in \hat{\mathbb{C}}$ and boundary fields $\Psi_\ell(t|x)$ that depend on a real isospin coordinate $t \in \bar{\mathbb{R}}$. This is the convention which is most convenient for our purposes. But of course, one is free to choose different bases for the representation space. For example, the H_3^+/Liouville correspondence reviewed in chapter 7 is more conveniently formulated in terms of μ- and τ-basis fields. For other purposes, the $(m,\bar{m})$- and (m,η)-basis is more practical. In the following, we are going to summarize the relations between these different bases. We are going to drop the worldsheet coordinates z and x.

B.1 Bulk Fields in Different Bases

B.1.1 The Transformation $\Theta_j(u) \leftrightarrow \Theta_j(\mu)$

For $\mu \in \mathbb{C}$, the transformation is

$$\Theta_j(\mu) = \frac{1}{\pi}|\mu|^{2j+2}\int_{\mathbb{C}} \mathrm{d}^2u\, \mathrm{e}^{(\mu u-\bar{\mu}\bar{u})}\Theta_j(u)\,. \tag{B.1}$$

This is inverted by

$$\Theta_j(u) = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\mu|\mu|^{-2j-2}\mathrm{e}^{-(\mu u-\bar{\mu}\bar{u})}\Theta_j(\mu)\,, \tag{B.2}$$

which one can check by making use of

$$\delta^{(2)}(u) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2k\, \mathrm{e}^{\mathrm{i}[k_1\mathrm{Re}(u)+k_2\mathrm{Im}(u)]} = \frac{1}{\pi^2}\int \mathrm{d}^2\mu\, \mathrm{e}^{-(\mu u-\bar{\mu}\bar{u})}\,, \tag{B.3}$$

where the integration variables were changed from (k_1,k_2) to $\mu = -2(k_2+\mathrm{i}k_1)$.

B.1.2 The Transformation $\Theta_j(u) \leftrightarrow \Theta_j^{m,\bar{m}}$

For $(m-\bar{m}) \in \mathbb{Z}$ and $(m+\bar{m}) \in \mathrm{i}\mathbb{R}$, i.e. $m = \frac{n+\mathrm{i}p}{2}$, $\bar{m} = \frac{-n+\mathrm{i}p}{2}$ with $n \in \mathbb{Z}$ and $p \in \mathbb{R}$, set

$$\Theta_j^{m,\bar{m}} = \int_{\mathbb{C}} \frac{\mathrm{d}^2u}{|u|^2}u^{-j+m}\bar{u}^{-j+\bar{m}}\Theta_j(u)\,. \tag{B.4}$$

The backtransformation reads

$$\Theta_j(u) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} \mathrm{d}p \sum_{n\in\mathbb{Z}} u^{j-m}\bar{u}^{j-\bar{m}}\Theta_j^{m,\bar{m}}\,, \tag{B.5}$$

with the relation between (n,p) and $(m,\bar{m})$ as stated above. To check this, one has to use that

$$\begin{aligned}(2\pi)^2|u'|^2\delta^{(2)}(u-u') &= 2\pi r'\delta(r-r')\cdot 2\pi\delta(\varphi-\varphi') \\ &= \left(\int_{\mathbb{R}}\mathrm{d}p\left(\frac{r}{r'}\right)^{-\mathrm{i}p}\right)\cdot\left(\sum_{n\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n(\varphi-\varphi')}\right) \\ &= \int_{\mathbb{R}}\mathrm{d}p\sum_{n\in\mathbb{Z}}\left(\frac{u}{u'}\right)^{-m}\left(\frac{\bar{u}}{\bar{u}'}\right)^{-\bar{m}}\,,\end{aligned} \tag{B.6}$$

if one introduces $u = r\mathrm{e}^{\mathrm{i}\varphi}$, $u' = r'\mathrm{e}^{\mathrm{i}\varphi'}$ in the intermediate step.

B.2 Boundary Fields in Different Bases

B.2.1 The Transformation $\Psi_\ell(t) \leftrightarrow \Psi_\ell(\tau)$

This transformation is simply ($\tau\in\mathbb{R}$)

$$\Psi_\ell(\tau) = |\tau|^{\ell+1}\int_{\mathbb{R}}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\tau t}\Psi_\ell(t)\,, \tag{B.7}$$

which isobviousyl inverted by

$$\Psi_\ell(t) = \int_{\mathbb{R}}\frac{\mathrm{d}\tau}{2\pi}|\tau|^{-\ell-1}\mathrm{e}^{-\mathrm{i}\tau t}\Psi_\ell(\tau)\,. \tag{B.8}$$

B.2.2 The Transformation $\Psi_\ell(t) \leftrightarrow \Psi_\ell^{m,\eta}$

This basis transformation is defined for $m\in\mathrm{i}\mathbb{R}$, $\eta\in\{0,1\}$ to be

$$\Psi_\ell^{m,\eta} = \int_{\mathbb{R}}\mathrm{d}t|t|^{-\ell-1+m}\left[\mathrm{sgn}(t)\right]^\eta\Psi_\ell(t)\,. \tag{B.9}$$

The inverse transformation is

$$\Psi_\ell(t) = -\frac{\mathrm{i}}{4\pi}\sum_{\eta=0,1}\left[\mathrm{sgn}(t)\right]^\eta\int_{\mathrm{i}\mathbb{R}}\mathrm{d}m\,|t|^{\ell-m}\Psi_\ell^{m,\eta}\,. \tag{B.10}$$

In order to check this, one makes use of

$$2\pi\mathrm{i}|t'|\left(\delta(t-t')+\delta(t+t')\right) = \int_{\mathrm{i}\mathbb{R}}\mathrm{d}m\,\left|\frac{t}{t'}\right|^{-m}\,. \tag{B.11}$$

C Hypergeometric and Generalized Hypergeometric Functions

In this appendix, we assemble some frequently needed formulae for Gauss' hypergeometric function and Appell's first as well as Horn's second generalized hypergeometric function. Most of the formula stated here are given for convenience and can also be found in standard references about the topic like [98, 97, 96].

C.1 Some Preliminary Identities

Before we come to the functions of interest, let us summarize some identities that are frequently used when manipulating (generalized) hypergeometric functions. They concern the Euler Γ function and the Pochhammer symbols.

C.1.1 Γ Function Identities

$$\int_0^1 \mathrm{d}t \; t^{a-1}(1-t)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{C.1}$$

$$\int_0^\infty \mathrm{d}t \; (1+t^2)^\alpha = \frac{\sqrt{\pi}}{2}\frac{\Gamma(-\alpha-\frac{1}{2})}{\Gamma(-\alpha)} \tag{C.2}$$

$$\Gamma(2j) = \frac{1}{\sqrt{\pi}}(2)^{2j-1}\Gamma(j)\Gamma(j+\frac{1}{2}) \tag{C.3}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{C.4}$$

C.1.2 Pochhammer Symbol Identities

The Pochhammer symbol is defined to be

$$(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}\,. \tag{C.5}$$

From this definition and the functional equation of Euler's gamma function, $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$, one easily derives the following identites:

$$\begin{aligned}
(\alpha)_{-m} &= \frac{(-)^m}{(1-\alpha)_m}\,, \\
(\alpha)_{m+n} &= \left\{\begin{array}{l} (\alpha+m)_n(\alpha)_m \\ (\alpha+n)_m(\alpha)_n \end{array}\right.\,, \\
(\alpha)_{m-n} &= \left\{\begin{array}{l} (\alpha+m)_{-n}(\alpha)_m \\ (\alpha-n)_m(\alpha)_{-n} \end{array}\right.\,.
\end{aligned} \tag{C.6}$$

C.2 Gauss' Hypergeometric Function

C.2.1 Definition

Convergent Series

The ordinary Gauss' hypergeometric function F (also denoted ${}_2F_1$, but since we do not consider the generalizations ${}_pF_q$, we do not need to make this distinction here) is defined as a convergent series by

$$F(a,b;c|z) = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{C.7}$$

for $|z| < 1$ and $c \notin \mathbb{Z}_{\leq 0}$. $(a)_n$ and so on are Pochhammer symbols as in C.1.2. It is obviously symmetric in a and b. By making use of the definition of the Pochhammer symbol, one shows

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}F(a,b;c|z) = \frac{(a)_k(b)_k}{(c)_k}F(a+k,b+k;c+k|z)\,. \tag{C.8}$$

The hypergeometric function can be continued to regions other than $|z| < 1$ from certain integral representations given by Euler and Barnes [98]. We shall not need these formulae, but just state the continuation formulae that one obtains below.

Differential Equation

Any homogeneous linear differential equation of second order which has at most three regular singularities can be reduced to the *hypergeometric differential equation*

$$\left\{z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + [c-(a+b+1)z]\,\frac{\mathrm{d}}{\mathrm{d}z} - ab\right\} y = 0 \tag{C.9}$$

A set of two linearly independent solutions is

$$\begin{aligned} y_1 &= F(a,b;c|z) \\ y_2 &= z^{1-c}F(1+a-c,1+b-c;2-c|z) \end{aligned} \tag{C.10}$$

if $c \notin \mathbb{Z}$. Note that the two solutions coincide if $c = 1$. The cases $c \in \mathbb{Z}$ are certain logarithmic cases that we do not need at this stage, but only later when coming to analytic continuations.

C.2.2 Analytic Continuations of Gauss' Hypergeometric Function

The formulae stated here are taken from [98]. Note that for the hypergeometric function to exist, we always need $c \notin \mathbb{Z}_{\leq 0}$.

Generic Case ($b - a \notin \mathbb{Z}$)

$$\begin{aligned} F\left(a,b;c\,\middle|\,\frac{1}{z}\right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\left(-\frac{1}{z}\right)^{-a} F(a,1-c+a;1-b+a|z)+ \\ &+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\left(-\frac{1}{z}\right)^{-b} F(b,1-c+b;1-a+b|z). \end{aligned} \tag{C.11}$$

Generic Case ($c - a - b \notin \mathbb{Z}$)

$$\begin{aligned} F(a,b;c\,|z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1|1-z)+ \\ &+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a,c-b;c-a-b+1|1-z). \end{aligned} \tag{C.12}$$

Logarithmic Case ($b - a \equiv m \in \mathbb{Z}_{\geq 0}$)

$$\begin{aligned} F\left(a,b;c\,\middle|\,\frac{1}{z}\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-a)}\left(-\frac{1}{z}\right)^{-b}\sum_{n=0}^{\infty}\frac{(a)_{n+m}(1-c+a)_{n+m}}{n!(n+m)!}\cdot \\ &\cdot\left(\frac{1}{z}\right)^{-n}\left[\log\left(-\frac{1}{z}\right)+h_n\right]+ \\ &+\frac{\Gamma(c)}{\Gamma(b)}\left(-\frac{1}{z}\right)^{-a}\sum_{n}^{m-1}\frac{\Gamma(m-n)(a)_n}{\Gamma(c-a-n)n!}\left(\frac{1}{z}\right)^{-n}. \end{aligned} \tag{C.13}$$

Note that if $b - a \in \mathbb{Z}$, it is no restriction to take $b - a = m \in \mathbb{Z}_{\geq 0}$, as this can always be achieved by exchanging the rôles of a and b if necessary. The occuring $h_n \equiv h_n(a,c,m)$ is defined as

$$h_n(a,c,m) = \psi(1+m+n)+\psi(1+n)-\psi(a+m+n)-\psi(c-a-m-n), \tag{C.14}$$

with $\psi(z)$ being the logarithmic derivative of the gamma function:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{C.15}$$

C.3 The Appell Function F_1

C.3.1 Definition

Convergent Series

The definition of Appell's function F_1 is as a convergent series is

$$F_1(\alpha,\beta,\beta';\gamma|u;z) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{u^m}{m!}\frac{z^n}{n!}. \tag{C.16}$$

It is convergent for complex u and z in the domain $|u| < 1$, $|z| < 1$. Clearly, it is symmetric under simultaneous exchange $\beta \leftrightarrow \beta'$ and $u \leftrightarrow z$. For the third parameter γ we need $\gamma \neq 0, -1, -2, \ldots$. There are again several integral representations of Barnes and Euler type [97] that we do not need here.

Differential Equation

Appell's function F_1 can also be defined as being the solution of the following system of two partial differential equations

$$u(1-u)\partial_u^2 y + z(1-u)\partial_u\partial_z y + [\gamma - (\alpha+\beta+1)u]\partial_u y - \beta z\partial_z y - \alpha\beta y = 0 \tag{C.17}$$

together with the same equation, but with the exchange of $u \leftrightarrow z$ and $\beta \to \beta'$. This is of course nothing but the observed symmetry of F_1. This system of equations can be brought to a different form [96], which is

$$\begin{aligned}\beta z(1-z)\partial_z y = u(1-u)(u-z)\partial_u^2 y &+ \\ + \Big[\gamma(u-z) - (\alpha+\beta+1)u^2 + (\alpha+\beta-\beta'+1)uz + \beta' z\Big]\partial_u y &- \\ - \alpha\beta(u-z)y&\end{aligned} \tag{C.18}$$

and again its counterpart with $u \leftrightarrow z$ and $\beta \leftrightarrow \beta'$, together with the third equation

$$(u-z)\partial_u\partial_z y - \beta'\partial_u y + \beta\partial_z y = 0. \tag{C.19}$$

Note that (C.18) is precisely of the form of the Knizhnik-Zamolodchikov equations like (6.5) that we need to solve in chapter 6.

C.3.2 Analytic Continuations

The following formulae are taken from [97].

$$\begin{aligned}
F_1(\alpha,\beta,\beta';\gamma|u;z) = {} & \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta')}(1-u)^{-\beta}z^{-\beta'}\cdot \\
& \cdot G_2\left(\beta,\beta';1+\beta'-\gamma,\gamma-\alpha-\beta' \,\middle|\, \frac{u}{1-u};\frac{1-z}{z}\right)+ \\
& + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta'-\gamma)}{\Gamma(\alpha)\Gamma(\beta')}(1-u)^{-\beta}(1-z)^{\gamma-\alpha-\beta'}\cdot \\
& \cdot F_1\left(\gamma-\alpha,\beta,\gamma-\beta-\beta';\gamma-\alpha-\beta'+1 \,\middle|\, \frac{1-z}{1-u};1-z\right),
\end{aligned} \tag{C.20}$$

in a neighbourhood of $(u,z)=(0,1)$. The function G_2 is the second Horn function, see next section. The parameters α, β, β', γ must be such that the Euler Γ functions do not blow up. In the cases where we need to derive different formula from the generalized series representations (next subsection), this is precisely what happens and invalidates the formulae given in this subsection.

$$\begin{aligned}
F_1(\alpha,\beta,\beta';\gamma|u;z) = {} & \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-u)^{\gamma-\alpha-\beta}(1-z)^{-\beta'}\cdot \\
& \cdot F_1\left(\gamma-\alpha,\gamma-\beta-\beta',\beta';\gamma-\alpha-\beta+1 \,\middle|\, 1-u;\frac{1-u}{1-z}\right)+ \\
& + \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)\Gamma(\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta-\beta')\Gamma(\alpha)}u^{-\beta}(1-z)^{-\beta'}\cdot \\
& \cdot D^{1,2}_{(2)}\left(\gamma-\alpha-\beta,\beta,\beta';\gamma-\beta-\beta',\beta'-\alpha+1 \,\middle|\, \frac{u-1}{u};\frac{1}{1-z}\right)+ \\
& + \frac{\Gamma(\gamma)\Gamma(\beta'-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta')}(1-z)^{-\alpha}\cdot \\
& \cdot F_1\left(\alpha,\beta,\gamma-\beta-\beta';\alpha-\beta'+1 \,\middle|\, \frac{1-u}{1-z};\frac{1}{1-z}\right),
\end{aligned} \tag{C.21}$$

in a neighbourhood of $(u,z)=(1,\infty)$. The function $D^{1,2}_{(2)}$ is defined by

$$D^{1,2}_{(2)}(\alpha,\beta,\beta';\gamma,\gamma'|u;z) := \sum_{m,n}\frac{(\alpha)_{n-m}(\beta)_m(\beta')_n}{(\gamma)_{-m}(\gamma')_n}\frac{u^m}{m!}\frac{z^n}{n!}, \tag{C.22}$$

convergent in the domain $u < 2$, $z < \frac{1}{2}$. The same comments on the parameters as above apply. Finally, there is also

$$\begin{aligned} F_1(\alpha,\beta,\beta';\gamma|u;z) &= \frac{\Gamma(\gamma)\Gamma(\alpha-\beta-\beta')}{\Gamma(\gamma-\beta-\beta')\Gamma(\alpha)}(1-u)^{-\beta}(1-z)^{\gamma-\alpha-\beta'}\cdot \\ &\cdot(-z)^{\alpha-\gamma}F_1\left(\gamma-\alpha,\beta,1-\alpha;1-\alpha+\beta+\beta'\,\middle|\,\frac{u-z}{z(u-1)};\frac{1}{z}\right)+ \\ &+\frac{\Gamma(\gamma)\Gamma(\beta+\beta'-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta+\beta')}(1-u)^{-\beta}(1-z)^{\gamma-\alpha-\beta'}\cdot \\ &\cdot(-z)^{\beta+\beta'-\gamma}G_2\left(\beta,\gamma-\beta-\beta';1-\beta-\beta',\beta+\beta'-\alpha\,\middle|\,\frac{z-u}{u-1};-\frac{1}{z}\right), \end{aligned} \tag{C.23}$$

in a neighbourhood of $(u,z)=(\infty,\infty)$.

C.3.3 Generalized Series Representations

Employing the Pochhammer symbol identites stated in C.1.2, one deduces easily that

$$F_1(\alpha,\beta,\beta';\gamma|u;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta')_n}{(\gamma)_n}F(\alpha+n,\beta;\gamma+n|u)\frac{z^n}{n!}, \tag{C.24}$$

F being the standard hypergeometric function. Of course, there is an analogous statment about the expansion in the variable u. It is simply obtained by exchanging β and β' on the right hand side. In expansions of this type, one can use an analytic continuation of the "inner" hypergeometric function C.2 and afterwards (usually, not always) resum the resulting series. In this way, one can actually derive the continuation formulae stated in the preceding subsection, if one uses the generic continuations of the Gauss function. In the logarithmic cases, other formulae arise, see for example section 6.1.1. Manipulations of this kind are very crucial and occur frequently in our work.

C.4 The Horn Function G_2

C.4.1 Definition as a Convergent Series and Differential Equation

Horn's function G_2 is defined by

$$G_2(\beta,\beta';\alpha,\alpha'|u;z)=\sum_{m,n=0}^{\infty}(\beta)_m(\beta')_n(\alpha)_{n-m}(\alpha')_{m-n}\frac{u^m}{m!}\frac{z^n}{n!}. \tag{C.25}$$

This series converges for complex u and z with $|u|<1$, $|z|<1$. Its parameters α and α' must be such that $\alpha \neq 1,2,3,\ldots$ and $\alpha' \neq 1,2,3,\ldots$. Note the symmetry under $\alpha \leftrightarrow \alpha'$ together with $\beta \leftrightarrow \beta'$ and $\gamma \leftrightarrow \gamma'$.

C.4.2 Generalized Series Representations

Employing the Pochhammer symbol identites stated in C.1.2, one deduces that

$$G_2(\beta,\beta';\alpha,\alpha'|u;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta')_n}{(1-\alpha')_n}F(\alpha'-n,\beta;1-\alpha-n|-u)\frac{(-z)^n}{n!}\,. \quad \text{(C.26)}$$

The analogous expansion in the variable u is of course obtained by exchanging α and α' as well as β and β' on the right hand side. Analytic continuation formulae for G_2 can now be derived by the same procedure that we have outlined above for Appell's function.

One should notice that for $\alpha\in\mathbb{Z}_{\leq 0}$, the above expansion breaks down, because some of the occuring hypergeometric functions cease to be well defined (for $\alpha\in\mathbb{Z}_{>0}$ the function G_2 is not defined anyway). For our purposes, the case $\alpha=0$ becomes important when taking $u=z$ in (8.13). In this case, it is however not difficult to derive a similar expansion [68]:

$$G_2(\beta,\beta';0,\alpha'|u;z)=\sum_{n=0}^{\infty}\frac{(\beta)_n(\beta')_n}{(1)_n}F(\beta+n,\alpha';1+n|-u)\frac{(u\cdot z)^n}{n!}\,. \quad \text{(C.27)}$$

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Schluss

[. . .] woher um alles in der Welt stünde es fest, dass gerade wahre Urteile mehr Vergnügen machten als falsche und, gemäß einer prästabilierten Harmonie, angenehme Gefühle mit Notwendigkeit hinter sich drein zögen? – Die Erfahrung aller strengen, tief gearteten Geister lehrt das Umgekehrte. Man hat fast jeden Schritt breit Wahrheit sich abringen müssen, man hat fast alles dagegen preisgeben müssen, woran sonst das Herz, woran unsere Liebe, unser Vertrauen zum Leben hängt. Es bedarf Größe der Seele dazu: Der Dienst der Wahrheit ist der härteste Dienst. – Was heißt denn rechtschaffen sein in geistigen Dingen? Dass man streng gegen sein Herz ist, dass man die "schönen Gefühle" verachtet, dass man sich aus jedem Ja und Nein ein Gewissen macht! [. . .]

Friedrich Nietzsche, Der Antichrist

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List of Related Publications

Hendrik Adorf and Michael Flohr:
On Factorization Constraints for Branes in the H_3^+ Model
to appear in Int. J. Mod. Phys. A, arXiv:0801.2711 [hep-th]

Abstract: We comment on the brane solutions for the boundary H_3^+ model that have been proposed so far and point out that they should be distinguished according to the patterns regular/irregular and discrete/continuous. In the literature, mostly irregular branes have been studied, while results on the regular ones are rare. For all types of branes, there are questions about how a second factorization constraint in the form of a $b^{-2}/2$-shift equation can be derived. Here, we assume analyticity of the boundary two point function, which means that the Cardy-Lewellen constraints remain unweakened. This enables us to derive unambiguously the desired $b^{-2}/2$-shift equations. They serve as important additional consistency conditions. For some regular branes, we also derive $1/2$-shift equations that were not known previously. Case by case, we discuss possible solutions to the enlarged system of constraints. We find that the well-known irregular continuous AdS_2 branes are consistent with our new factorization constraint. Furthermore, we establish the existence of a new type of brane: The shift equations in a certain regular discrete case possess a non-trivial solution that we write down explicitly. All other types are found to be inconsistent when using our second constraint. We discuss these results in view of the Hosomichi-Ribault proposal and some of our earlier results on the derivation of $b^{-2}/2$-shift equations.

Hendrik Adorf and Michael Flohr:
Continuously Crossing $u = z$ in the H_3^+ Boundary CFT
JHEP 11 (2007) 024, arXiv:0707.1463 [hep-th]

Abstract: For AdS boundary conditions, we give a solution of the H_3^+ two point function involving degenerate field with SL(2)-label $b^{-2}/2$, which is defined on the full (u, z) unit square. It consists of two patches, one for $z < u$ and one for $u < z$. Along the $u = z$ “singularity”, the solutions from both patches are shown to have finite limits and are merged continuously as suggested by the work of Hosomichi and Ribault. From this two point function, we can derive $b^{-2}/2$-shift

equations for AdS_2 D-branes. We show that discrete as well as continuous AdS_2 branes are consistent with our novel shift equations without any new restrictions.

Hendrik Adorf and Michael Flohr:
On the Various Types of D-Branes in the Boundary H_3^+ Model
submitted to Nucl. Phys. B, arXiv:hep-th/0702158

Abstract: We comment on the D-brane solutions for the boundary H_3^+ model that have been proposed so far and point out that many more types of D-branes should be considered possible. Assuming analyticity of the boundary two point function, we can derive $b^{-2}/2$-shift equations that constitute important, but so far unknown, consistency conditions. We discuss their possible solutions and show that the one point functions that have been proposed up to now do not [or do only under very restrictive conditions] satisfy these new shift equations. This shows that assuming the boundary two point function to be analytic constrains the model very strongly. We then discuss the analytically continued two point function carefully and find that it shows a singularity along the line $u = z$. The implications of these findings are discussed.

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04/2004:	Peter Rowe Memorial Prize for “outstanding performance in the M.Sc. examinations”, awarded from the Department of Mathematical Sciences, University of Durham, U.K.
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10/2000-10/2004:	scholar of the Studienstiftung des deutschen Volkes (German National Academic Foundation)