Hecke eigenforms and the arithmetic of singular K3 surfaces

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Zusammenfassung

Die Dissertation befasst sich mit der Artihmetik singulärer K3 Flächen. Ferner liefert sie eine ausgiebige Analyse der Hecke Eigenformen mit rationalen Koeffizienten und komplexer Multiplikation.

Zunächst gebe ich eine kurze Einführung in die Theorie der K3 Flächen. Dabei hebe ich singuläre K3 Flächen über $\mathbb{C}$ hervor, d.h. K3 Flächen mit maximaler Picard-Zahl $\rho = 20$. Insbesondere wiederhole ich die Ergebnisse von Shioda-Inose [S-I] und Livnés Theorem, dass eine singuläre K3 Fläche über $\mathbb{Q}$ modular ist [L]. Daraufhin formuliere ich die Frage, welche Neuformen tatsächlich zu singulären K3 Flächen über $\mathbb{Q}$ assoziiert sind.

Dieses Problem motiviert die Untersuchung der Hecke Eigenformen mit rationalen Koeffizienten und komplexer Multiplikation (CM) für beliebiges Gewicht. Diese Untersuchung findet im zweiten Kapitels der Arbeit statt. Das Hauptresultat lautet wie folgt:

Theorem
Für festes Gewicht k stehen die Neuformen mit rationalen Koeffizienten und CM, bis auf Twisten, in einer bijektiven Korrespondenz zu den imaginär-quadratischen Körpern, deren Exponent $(k-1)$ teilt. Dies wird erreicht, indem eine Neuform auf den zugehörigen CM-Körper abgebildet wird.

Der Beweis des Theorems gebraucht ein Resultat von Ribet [R], das einer Neuform mit CM einen Größencharakter des zugehörigen CM-Körpers mit entsprechenden Spuren zuordnet. Dank Ergebnissen von Stark [St] und Weinberger [W] impliziert das Theorem das folgende

Korollar
Bis auf Twisten gibt es nur endlich viele CM-Neuformen mit rationalen Koeffizienten und Gewicht $2, 3$ oder 4. Für beliebiges festes Gewicht gilt diese Aussage unter Annahme der verallgemeinerten Riemannschen Vermutung.

Das Korollar wird von expliziten Tabellen für Gewicht 3 und 4 begleitet.

Das dritte Kapitel widmet sich der Suche nach singulären K3 Flächen über $\mathbb{Q}$. Diese werden von rationalen elliptischen Flächen durch einen Basiswechsel oder durch andere Manipulationen der Weierstrass-Gleichung hergeleitet. Alle resultierenden Faserungen sind extremal. Gemeinsam mit anderen bekannten Beispielen finde ich so elliptische singuläre K3 Flächen über $\mathbb{Q}$ für 24 der 65 Gewicht 3-Formen aus der erwähnten Liste.

Im letzten Kapitel studiere ich eine spezielle singuläre K3 Fläche detailiert. Diese Fläche besitzt eine extremale elliptische Faserung mit [1,1,1,12,3*]-Konfiguration. Ich bestimme die assoziierte Neuform des Levels 27 und zeige, dass die Néron-Severi Gruppe $NS(X)$ durch Divisoren über $\mathbb{Q}$ erzeugt werden kann. Dies ergibt ein Gegenbeispiel zu einer Behauptung Shiodas [Sh2, Thm. 1]. Eine besondere

Eigenschaft der Fläche ist die gute Reduktion bei 2. Sowohl diese Reduktion als auch die entsprechende Fläche über $\mathbb{F}_3$ wird ausgiebig analysiert. Dabei verifiziere ich Vermutungen von Tate, Shioda und Artin.

Schließlich diskutiere ich die reichhaltigen Twists, welche die Fläche offeriert. Diese produzieren alle Gewicht 3-Formen mit rationalen Koeffizienten und CM mit $\mathbb{Q}(\sqrt{-3})$. Ein analoges Resultat kann für den CM-Körper $\mathbb{Q}(\sqrt{-1})$ erzielt werden, etwa über die Fermat-Quartik in $\mathbb{P}^3$. Da eine allgemeine Weierstrass-Gleichung trivialerweise quadratische Twists zulässt, komme ich zum folgenden Schluss:

Satz
Es sei f eine der 24 Neuformen vom Gewicht 3 mit rationalen Koeffizienten, für die wir eine assoziierte elliptische singuläre K3 Fläche über $\mathbb{Q}$ kennen. Dann lässt sich jeder Twist von f geometrisch in einer extremalen elliptischen K3 Faserung über $\mathbb{Q}$ realisieren.

Schlagworte: Hecke-Eigenform, komplexe Multiplikation, singuläre K3 Fläche

Abstract

My thesis investigates the arithmetic of singular K3 surfaces. It also gives an extensive analysis of Hecke eigenforms with rational coefficients and complex multiplication.

The thesis starts with a brief introduction on K3 surfaces. I emphasize the notion of singular K3 surfaces over $\mathbb{C}$, i.e. K3 surfaces with maximal Picard number $\varrho = 20$. In particular, I recall the theory of Shioda-Inose [S-I] and Livné's theorem that a singular K3 surface over $\mathbb{Q}$ is modular [L]. I then formulate the question which newforms might actually be associated to singular K3 surfaces over $\mathbb{Q}$.

This motivates the study of Hecke eigenforms with rational Fourier coefficients and complex multiplication (CM) of arbitrary weight. This investigation is the content of the second chapter of the thesis. The main result is the following

Theorem
For fixed weight k, the newforms with rational Fourier coefficients and CM are, up to twisting, in 1:1-correspondence with the imaginary quadratic fields K with exponent dividing $k-1$. This can be achieved by mapping a newform to its CM-field.

The theorem is established using the Größencharakter associated to a newform with CM by Ribet [R]. Due to work of Stark [St] and Weinberger [W], it implies

Corollary
Up to twisting, there are only finitely many CM-newforms with rational coefficients and weight $2, 3,$ or 4. For arbitrary fixed weight, this holds subject to the Generalized Riemann Hypothesis.

The corollary is supplemented by explicit tables for weights 3 and 4.

The third chapter is devoted to finding singular K3 surfaces over $\mathbb{Q}$. These are derived from rational elliptic surfaces by base change or other manipulation of the Weierstrass equation. All resulting fibrations are extremal. Together with other known examples, this approach enables me to find corresponding elliptic K3 surfaces over $\mathbb{Q}$ for 24 out of the 65 weight 3 forms from the list.

In the final chapter, I discuss one particular singular K3 surface in detail. This admits an extremal elliptic fibration with [1,1,1,12,3*] configuration. I determine the associated newform of level 27 and show that the Néron-Severi group $NS(X)$ can be generated by divisors over $\mathbb{Q}$. This gives a counterexample to a claim of Shioda [Sh2, Thm. 1]. One special property of this surface is the good reduction at 2. This reduction is studied in detail as is the corresponding surface over $\mathbb{F}_3$. I verify conjectures of Tate, Shioda and Artin for these cases.

Finally, I comment on the twists of the surface. These produce all weight 3 newforms with rational coefficients and CM by $\mathbb{Q}(\sqrt{-3})$. An analogous result can be derived

for the CM-field $\mathbb{Q}(\sqrt{-1})$, for instance using the Fermat quartic in $\mathbb{P}^3$. Since a general Weierstrass equation trivially admits quadratic twisting, I deduce the

Proposition
Let f be one of the 24 newforms of weight 3 with rational coefficients for which we know an associated elliptic singular K3 surface over $\mathbb{Q}$. Then any twist of f can be realized geometrically as an extremal elliptic K3 fibration over $\mathbb{Q}$.

Keywords: Hecke eigenform, complex multiplication, singular K3 surface.

Contents

Notation

$\mathbb{N}$	natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{Q}_\ell$	ℓ-adic rationals
$\mathbb{F}_q$	finite field of q elements
$\bar{k}$	algebraic closure of a field k
$\mathbb{P}^n$	projective n-space
ι	complex conjugation
Re	real part (of a complex number)
Im	imaginary part (of a complex number)
$\mathbb{H}$	upper-half complex plane, i.e. $\mathbb{H} = \{z \in \mathbb{C}; \operatorname{Im}(z) > 0\}$
$(\frac{\cdot}{\cdot})$	Jacobi symbol
j	j-invariant (of an elliptic curve)
Cl	class group (of a number field)
im	image (of a map)
$h^{p,q}$	Hodge numbers (for the complex cohomology of a Kähler mannifold)

Chapter I

Singular K3 surfaces

I.1 Introduction

An (algebraic) K3 surface X is a smooth projective surface with trivial canonical bundle $\omega_X = \mathcal{O}_X$ and trivial first cohomology group $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Hence, it is a Calabi-Yau variety. The Euler number $e(X)$ can be computed from the Noether formula

$$e(X) = 12\chi(\mathcal{O}_X) - K_X^2 = 24.$$

Thus the Hodge diamond of a K3 surface over $\mathbb{C}$ looks as follows

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Example I.1.1

- *Consider a quartic in $\mathbb{P}^3$ with at most isolated ADE-singularities. Then the resolution of singularities does not affect the adjunction. Hence, the minimal desingularization is a K3 surface.*
- *The Kummer surface $Km(A)$ for an abelian surface A. Here we blow-up the fixed points of the involution on A and then divide out by the involution (or vice versa).*
- *An elliptic surface over $\mathbb{P}^1$ with a section and with Euler number 24 (cf. Section III.1 for details). Or, slightly more generally, an elliptic surface S over $\mathbb{P}^1$ with $\chi(\mathcal{O}_S) = 2$, but without multiple fibres.*
- *A double sextic, i.e. the double cover of $\mathbb{P}^2$, branched along a smooth sextic curve. Here, we could also allow ADE-singularities.*

We will mainly be concerned with K3 surfaces X over number fields, i.e. in characteristic zero. Let $NS(X)$ denote the Néron-Severi group of X, i.e. the group of divisors on X up to algebraic equivalence. For the Picard number $\rho(X) = \mathrm{rk}\ NS(X) \geq 1$, we have the Lefschetz bound

$$\rho(X) \leq h^{1,1}(X) = 20.$$

This notation, without indicating the ground field, will always refer to the variety considered over the algebraic closure of the ground field.

Remark I.1.2
For the above examples, we can easily find restrictions for the Picard number. For instance, an elliptic fibration with a section has obviously $\rho \geq 2$. We will study elliptic fibrations in chapters III and IV. On the other hand, the exceptional divisors of a Kummer surface already imply

$$\rho(Km(A)) \geq \rho(A) + 16 \geq 17.$$

Similarly, the Picard number of a double sextic may be expressed in terms of the branch curve, although this might be complicated.

I.2 Lattice theory

To understand the Néron-Severi group in more detail, we consider the torsion-free cohomology group

$$\mathrm{H}^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}.$$

Equipped with the cup-product, this admits the structure of a lattice which is unimodular and even. Since its signature is $(3, 19)$, it can be uniquely identified with the so-called *K3-lattice*

$$\Lambda = U^3 \oplus E_8(-1)^2.$$

In this context, U denotes the hyperbolic plane $\mathbb{Z}^2$ with form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is the root lattice $\mathbb{Z}^8$ with form

$$\begin{pmatrix} 2 & & & -1 & & & & \\ & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ -1 & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

Then $E_8(-1)$ denotes $\mathbb{Z}^8$ endowed with the negative form.

As an even lattice of signature $(1, \rho(X) - 1)$ with respect to cup-product, $NS(X)$ embeds into Λ. We call its orthogonal complement the *transcendental lattice* and denote it by T_X:

$$T_X = NS(X)^{\perp} \subset \mathrm{H}^2(X, \mathbb{Z}).$$

This lattice is also even with signature $(2, 20 - \rho(X))$. The discriminants of $NS(X)$ and T_X only differ by the sign due to the unimodularity of Λ.

In order to relate these lattices more closely, we recall the work of Nikulin [N]. For a general non-degenerate lattice L with bilinear form $(\cdot, \cdot)$, we consider the dual lattice

$$L^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) = \{v \in L \otimes_{\mathbb{Z}} \mathbb{Q}; (v, x) \in \mathbb{Z} \ \ \forall\, x \in L\}.$$

We recall the following classical result which partly uses the natural embedding of L into $L^{\vee}$:

Lemma I.2.1 ([BHPV, I. Lemma (2.1)])
Let L be a non-degenerate lattice. If M is a sublattice of L of the same rank, then

$$[L:M]^2 = \frac{d_M}{d_L}.$$

In particular, $[L^\vee : L] = |d_L|$.

Since $|d_{T_X}| = |d_{NS(X)}|$ due to unimodularity, this lemma implies

$$[T_X^\vee : T_X] = [NS(X)^\vee : NS(X)].$$

To improve this relation, we introduce the quadratic form

$$q_L : L^\vee/L \to \mathbb{Q}/2\mathbb{Z}$$

which is induced from the $\mathbb{Q}$-valued map $a \mapsto (a,a)$ on $L^\vee$. This will be called the *discriminant form* of the lattice L. One particular property of this quadratic form is the

Proposition I.2.2 (Nikulin [N, Cor. 1.9.4])
The genus of an even lattice is determined by its signature and its discriminant form.

We will not give the definition of a genus here. Let us but mention that each genus consists of a finite number of isomorphism classes of lattices.

Proposition I.2.2 becomes very effective if the genus in consideration consists only of one isomorphism class. In fact, this will be always the case for the surfaces over $\mathbb{Q}$ which we shall consider (cf. the discussion in Section III.7). We now come to the link between T_X and $NS(X)$.

Proposition I.2.3 (Nikulin [N, Prop. 1.6.1])
Let L be a unimodular even lattice and $M \subset L$ a primitive non-degenerate sublattice. Denote the orthogonal complement of M in L by $M^\perp$. Then

$$M^\vee/M \cong (M^\perp)^\vee/(M^\perp).$$

Furthermore, $q_{M^\perp} = -q_M$.

As a consequence, since the K3-lattice is unimodular and even, the Néron-Severi group $NS(X)$ of a K3 surface X determines the discriminant form q_{T_X} of the transcendental lattice T_X. In particular, if the genus of T_X consists only of one class, then $NS(X)$ already gives the transcendental lattice itself up to isomorphism. This can be used for classification purposes, as explained in the next section.

I.3 Singular K3 surfaces

In this section, we shall introduce singular K3 surfaces. These are particularly well-understood. For instance, the question of modularity (or in a wider context the Langlands conjecture) has been solved completely for singular K3 surfaces (cf. Thm. I.3.2, I.4.2). This thesis will mostly be concerned with a further study of these surfaces.

Definition I.3.1
A K3 surface X over a field of characteristic zero is called singular *(or exceptional), if its Picard number is the maximal possible:*

$$\rho(X) = 20.$$

One particular aspect of these surfaces is that they involve no moduli. In fact, the general moduli space of complex K3 surfaces with Picard number $\rho \geq \rho_0$ has dimension $20 - \rho_0$. Nevertheless, the singular K3 surfaces are everywhere dense in the period domain of K3 surfaces (with respect to the analytic topology).

For a singular K3 surface X, the transcendental lattice T_X has rank 2 and signature $(2, 0)$, hence it is positive definite. We will refer to its discriminant $d_{T_X} > 0$ as the discriminant of X. The following classification was originally established by Shioda and Inose [S-I]:

Theorem I.3.2 (Torelli theorem for singular K3 surfaces)
The map

$$X \mapsto T_X$$

induces a (1:1)-correspondence between isomophism classes of singular K3 surfaces and isomorphism classes of positive definite even oriented lattices of rank two.

The orientation of T_X is given by the 2-form on X.

The proof of this theorem can be derived from the Torelli theorem, the surjectivity of the period map and general lattice theory as explained in [P-S, Appendix to §6]. Shioda and Inose used a more explicit approach. This relied on the fact that two positive definite even oriented lattices of rank two are isomorphic if and only if their intersection matrices are conjugated by some element of $SL_2(\mathbb{Z})$.

Consider an isomorphism class of positive definite even oriented lattices of rank two. Let d be the discriminant of these lattices. Then the isomorphism class uniquely corresponds to a *reduced* positive definite even quadratic matrix

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

of discriminant $4ac - b^2 = d$. Here $a > 0$ and $-a < b \leq a \leq c$ (and $b \geq 0$ if $a = c$).

Shioda-Inose's proof of the surjectivity of the above map uses a construction of Shioda and Mitani [S-M]; for every reduced positive definite even quadratic matrix, they produce an explicit singular abelian surface A (i.e. it has maximal Picard number $\rho(A) = 4$) with this very intersection form on the transcendental lattice T_A. This surface arises as product of two isogenous elliptic curves with complex multiplication by $\mathbb{Q}(\sqrt{-d})$. These can be given in terms of the matrix entries a, b, c as follows:

For Q as above, consider the elliptic curves E, E' which are given as complex torus

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$$

with respective values

$$\tau = \frac{-b + \sqrt{-d}}{2a} \quad \text{and} \quad \tau' = \frac{b + \sqrt{-d}}{2}.$$

By construction, E and E' have complex multiplication (CM) by $\mathbb{Q}(\sqrt{-d})$ and are isogenous. Hence, their product $A = E \times E'$ is a singular abelian surface, and its transcendental lattice T_A has rank two.

Theorem I.3.3 (Shioda-Mitani [S-M, §3])
The transcendental lattice T_A has intersection form Q.

In order to obtain a K3 surface from this abelian surface A, Shioda and Inose first apply the Kummer construction as in Example I.1.1. Here, we have

$$T_{\mathrm{Km}(A)} = T_A(2). \tag{I.1}$$

The brackets indicate that the intersection form is multiplied by 2.

To obtain a K3 surface with the original intersection form on the transcendental lattice, Shioda and Inose subsequently exhibit an elliptic fibration on $\mathrm{Km}(A)$ with a fibre of type II^* and two fibres of type I_b^*. Therefore, a quadratic base change which is ramified at the two cusps of the I_b^* fibres, gives rise to another K3 surface X.

Theorem I.3.4 (Shioda-Inose [S-I, Thm. 3])
X is a singular K3 surface with $T_X = T_A$.

In fact, $\mathrm{Km}(A)$ can be obtained from X as the minimal desingularization of the quotient of X by a Nikulin involution. This is an involution of X with exactly 8 fixed points which leaves the 2-form invariant. For the elliptic fibrations above, the resolution of the singularities at the fixed points results in the simple components of the I_b^* fibres.

We sketch the situation in the following diagram

$$\begin{array}{ccccc} X & & & & A \\ & \searrow & & \swarrow & \\ & & \mathrm{Km}(A) & & \end{array}$$

Remark I.3.5
Equation (I.1) shows that the last step is in fact necessary to obtain all singular K3 surfaces from singular abelian surfaces via the Kummer construction.

Example I.3.6
The perhaps best known singular K3 surface is the Fermat quartic in $\mathbb{P}^3$:

$$X: \quad x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0. \tag{I.2}$$

This surface contains 48 obvious lines. These generate the Néron-Severi group $NS(X)$ which has discriminant -64. The transcendental lattice is known to have the intersection form $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$. As a Kummer surface, X therefore can be realized as $Km(E_i \times E_{2i})$ with $i = \sqrt{-1}$. By construction, these elliptic curves have CM by $\mathbb{Q}(i)$.

Example I.3.7
The next natural objects to study would be Shioda modular elliptic surfaces which are K3. These are universal elliptic curves associated to congruence subgroups of $SL_2(\mathbb{Z})$. Shioda modular elliptic K3 surfaces have been analyzed in [L-Y] and [T-Y]. For instance, the surface corresponding to the congruence subgroup $\Gamma(4)$ is isomorphic to $Km(E_i \times E_i)$ with intersection form $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. Note that this surface as well as the Fermat quartic $X \subset \mathbb{P}^3$ admits an elliptic fibration with six singular fibres of type I_4. However, only the former fibration has a section (it is the Jacobian of the latter).

I.4 The modularity of singular K3 surfaces

By construction, singular K3 surfaces are very much in analogy with elliptic curves with complex multiplication. In fact, it is an immediate consequence of Shioda-Inose's construction that every singular K3 surface can be defined over a number field L. Basically, this follows from the corresponding statement for elliptic curves with CM.

This approach enables us to compute the zeta-function of a singular K3 surface X. Assume that the ground field L includes the CM-field $\mathbb{Q}(\sqrt{-d})$ and the field of definition of the isogeny $E \to E'$. Furthermore, let all generators of $NS(X)$ be defined over L.

By the work of Deuring (cf. [Si, Thm. 9.2]), there is a Größencharakter $\psi_{E/L}$ associated to the elliptic curve E over L (or equivalently to E', since the isogeny is defined over L). This can be used to describe the zeta-function $\zeta(X/L, s)$ of X over L. In the following, let $\stackrel{\circ}{=}$ denote equality up to a finite number of Euler factors.

Theorem I.4.1 (Shioda-Inose [S-I, Thm. 6])
In the above notation, the zeta-function of the singular K3 surface X over L is given by

$$\zeta(X/L, s) \stackrel{\circ}{=} \zeta_L(s)\, \zeta_L(s-1)^{20}\, L(\psi_{E/L}^2, s)\, L(\bar{\psi}_{E/L}^2, s)\, \zeta_L(s-2).$$

We shall now turn to singular K3 surfaces X which are defined over $\mathbb{Q}$. Here, we cannot assume that $NS(X)$ is generated by divisors over $\mathbb{Q}$, although this is in fact possible, contrary to a claim of Shioda (cf. Remark IV.2.1).

Let X be a singular K3 surface over $\mathbb{Q}$. Fix a prime ℓ and consider the Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$. The divisors of X give rise to a finite 20-dimensional subrepresentation of $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$. The L-series of this subrepresentation is often easy to compute (cf. Ex. I.4.4). Instead, we shall now consider the non-trivial part of the zeta-function of X. This comes from the Galois representation of the transcendental lattice T_X and will be denoted by $L(T_X, s)$.

Let d denote the discriminant of X. The above theorem suggests that $L(T_X, s)$ should be associated to a Größencharakter of $\mathbb{Q}(\sqrt{-d})$ of ∞-type 2, hence to a newform of weight 3. This has been made explicit by Livné:

Theorem I.4.2 (Livné [L, Thm. 1.3])
Let X be a singular K3 surface over $\mathbb{Q}$ with discriminant d. Then

$$L(T_X, s) = L(\phi, s) = L(f, s)$$

for a Größencharakter ϕ of $\mathbb{Q}(\sqrt{-d})$ of ∞-type 2 resp. a newform f of weight 3 with CM by $Q(\sqrt{-d})$.

For details on Größencharaktere, we refer to Chapter II, in particular Section II.2. Originally, Livné considers general motives of rank two with certain properties. For the transcendental lattice of a singular K3 surface over $\mathbb{Q}$, this is worked out in Example 1.6 in [L]. Note that this example contains a misprint, since in Livné's notation with $d < 0$, the nebentypus character should be $\epsilon_{\text{Dir}} = \left(\frac{d}{\cdot}\right)$.

Remark I.4.3
The corresponding newform heavily depends on the $\mathbb{Q}$-isomorphism class of surfaces. To circumvent this, we will mostly consider the question of the corresponding newform up to twisting. A general singular K3 surface over $\mathbb{Q}$ is not guaranteed to possess a $\overline{\mathbb{Q}}$-isomorphism which corresponds to a given twist. However, this approach

is justified by the circumstance that we will mainly consider elliptic fibrations with a section. If taken to Weierstrass form, these trivially admit quadratic twisting. For the additional twisting over the fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, the reader is referred to Section IV.7. Partial result can be found in Corollary IV.7.2 and Remark IV.7.4. They are summarized in Conclusion IV.7.5.

Example I.4.4
Let us return to the Fermat quartic $X \subset \mathbb{P}^3$. It is easy to verify that $L(T_X, s)$ corresponds to the Größencharakter of $\mathbb{Q}(\sqrt{-1})$ with conductor (2) and ∞-type 2. Respectively, this is associated to the newform f of level 16 and weight 3 from Table II.1. This can be proved using the Lefschetz fixed point formula as explained in Section IV.2. The proof is easy since X has good reduction at any odd prime. Due to the classification of Theorem II.3.4 (cf. also Rem. II.3.5), we therefore only have to count points on X over $\mathbb{F}_5$ to verify the associated newform.

We can also see the twisting on X. For instance, inserting a minus sign instead of one plus sign in the defining equation (I.2), gives an isomorphism over $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$, the field of eighth roots of unity. Since f has CM by $\mathbb{Q}(\sqrt{-1})$ (see Section II.1 for the definition), this results in the twist $f \otimes \left(\frac{2}{\cdot}\right)$.

Finally, we can determine the zeta-function of $X/\mathbb{Q}$ up to the Euler factor at 2. For this, we have to compute the Galois action on $NS(X)$. Since all 48 lines are defined over $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$, it suffices to further count points modulo 3 and 7. We find

$$\zeta(X/\mathbb{Q}, s) = \zeta(s)\,\zeta(s-1)^5\,\zeta(\chi_{-1}, s-1)^3\,\zeta(\chi_2, s-1)^6\,\zeta(\chi_{-2}, s-1)^6\,L(f,s)\,\zeta(s-2)$$

where $\chi_\bullet$ denotes the respective Legendre symbol as above. This corrects the formula from [Y, Prop. 8.12].

It is to be emphasized that we do not know a proof of Theorem I.4.2 which follows directly from Theorem I.4.1. Livné's original proof does not make use of Shioda-Inose's result at all, although he could use it at one step in order to replace a purity result of Faltings (cf. [L, Rem. 1.7]).

More importantly, it does happen that the Größencharakter ϕ of Theorem I.4.2 cannot be obtained as a square of a Größencharakter ψ as in Theorem I.4.1. (The conjugate would only appear if the field of definition contained the CM-field.) This is illustrated by the following example:

Example I.4.5
Figure III.13 shows an elliptic fibration on a singular K3 surface over $\mathbb{Q}$ with discriminant $d = 20$. Explicitly, this corresponds to the newform of weight 3 and level 20 from Table II.1, or in other terms, to a Größencharakter ϕ of $\mathbb{Q}(\sqrt{-5})$ with ∞-type 2 and rational traces. On the other hand, it is easy to verify that there cannot be a Größencharakter ψ of $\mathbb{Q}(\sqrt{-5})$ with ∞-type 1 whose square ψ^2 has rational traces. Hence $\phi \neq \psi^2$. The explicit argument follows from the general theory of Größencharaktere and eigenforms with CM as sketched in Chapter II.

I.5 Discussion

The preceding results lead to some interesting questions which motivated most of this thesis. We shall briefly sketch them in this section.

First of all, we might ask which singular K3 surfaces possess a model over $\mathbb{Q}$. This issue is completely in analogy with the classical theory of elliptic curves with complex multiplication. A general answer to this was given by Šafarevič:

Theorem I.5.1 (Šafarevič [Sf, Thm. 1])
Fix $n \in \mathbb{N}$. Then there is only a finite number of ($\mathbb{C}$-isomorphism classes of) singular K3 surfaces which have a model defined over any number field of degree at most n over $\mathbb{Q}$.

As a consequence of the above theorem, our initial problem is finite:

Question I.5.2
Which singular K3 surfaces can be defined over $\mathbb{Q}$?

A substantial step towards solving this problem was achieved by Serre, who noticed that the corresponding CM-field $\mathbb{Q}(\sqrt{-d})$ can only have 2-torsion in its class group (cf. Section II.4 and Proposition II.13.1). This criterion can often be used to deduce that a singular K3 surface cannot be defined over $\mathbb{Q}$. At the end of Section III.5, we will apply this to the extremal elliptic K3 fibrations with configuration [1,1,2,2,4,14] and [1,2,2,4,7,8].

It turns out that Question I.5.2 is beyond the scope of this thesis, since the discriminant of T_X can nevertheless grow quite large. To our knowledge, the current record is 3600 as attained by the extremal elliptic K3 fibration with configuration [3,3,4,4,5,5] (cf. [B-M]).

Instead of this problem, we shall turn to a coarser classification. This concerns the modular form (or Größencharakter) associated to $L(T_X, s)$ for a singular K3 surface X over $\mathbb{Q}$ by Theorem I.4.2.:

Question I.5.3
Up to twisting, which newforms occur in the zeta-functions of singular K3 surfaces over $\mathbb{Q}$?

As we will see in Proposition II.13.1, this approach classifies singular K3 surfaces over $\mathbb{Q}$ in terms of their discriminants up to squares.

Let X be a singular K3 surface over $\mathbb{Q}$ with discriminant d. Then the corresponding newform has CM by $\mathbb{Q}(\sqrt{-d})$. By Serre's observation, the class group of the CM-field consists only of 2-torsion. This gives a severe restriction to any answer to Question I.5.3.

As indicated in Remark I.4.3, we shall also take the twisting into account. This will enable us to produce a complete list consisting of the 65 newforms (up to one possibly, and up to twisting) which *a priori* might be associated to singular K3 surfaces over $\mathbb{Q}$. These are given in Table II.1.

This result will be a particular consequence of a general study of newforms with rational coefficients and complex multiplication which constitutes the main part of this thesis (Chapter II). For fixed weight, we will establish a bijective correspondence between newforms with rational Fourier coefficients and CM up to twisting on the one side and their respective CM-fields on the other (Thm. II.3.4).

At this point, we should emphasize the difference between singular K3 surfaces (or higher-dimensional modular varieties in general) and elliptic curves over $\mathbb{Q}$ in terms of the associated modular forms: To each newform of weight 2 with rational Fourier coefficients, a classical construction associates an elliptic curve over $\mathbb{Q}$. The other direction of this relation, the modularity of elliptic curves over $\mathbb{Q}$, was subject to the Taniyama-Shimura-Weil Conjecture, as proved by Wiles [Wi] et al.

For newforms of weight 3, however, there is no such construction available which would lead to singular K3 surfaces (or even to singular abelian surfaces). Hence, the table of 65 possible newforms leaves the following question:

Question I.5.4
Of the 65 a priori possible newforms, which do actually occur in the zeta-function of a singular K3 surface over $\mathbb{Q}$?

This problem is addressed in Chapter III where we produce a number of singular K3 surfaces over $\mathbb{Q}$ by way of base change from rational elliptic surfaces. In terms of the resulting extremal elliptic fibrations, this was recently supplemented by the announced results of Beukers-Montanus [B-M], as explained in Section III.8. In total, one finds singular K3 surfaces over $\mathbb{Q}$ for 24 out of the 65 possible newforms.

So far, we have not been very concerned with the arithmetic of explicit examples. Therefore, we will spend the fourth chapter with an extensive study of one particular singular K3 surface over $\mathbb{Q}$.

In the first instance, we will analyze its Néron-Severi group and thereby derive a counterexample to a claim of Shioda (cf. Rem. IV.2.1). Furthermore we will determine the associated newform and the zeta-function of the surface completely.

Then we want to discuss some questions and conjectures of Tate and Shioda which are related to positive characteristic. These concern the fields of definition for generators of the Néron-Severi group and supersingular reductions. We refer to Section IV.3 for the precise statements.

Finally, we comment on the rich twists which our particular K3 surface will admit in Section IV.7. We will also very briefly come back to the Fermat quartic (Rem. IV.7.4). We will conclude with the following observation (Conc. IV.7.5): Let f be any of the 24 newforms mentioned above. Then, any twist of f can be realized geometrically by some extremal elliptic K3 surface over $\mathbb{Q}$.

Chapter II

Hecke eigenforms with rational coefficients and complex multiplication

This chapter grew out of the wish to understand and eventually determine the modular forms which could possibly be associated to singular K3 surfaces over $\mathbb{Q}$. By Theorem I.4.2, any singular K3 surface over $\mathbb{Q}$ is modular, with the associated modular form of weight 3. Question I.5.3 asks which newforms occur. Our original motivation was to determine all newforms of weight 3 with rational coefficients, that is, all possibly occurring newforms. Up to twisting, we will produce a finite list in Table II.1. This result leads to the formulation of Question I.5.4 which will be addressed in the next chapter.

All the ideas involved in our analysis can be applied to newforms of general weight which have rational coefficients and CM. Hence we will treat this question in the most possible generality. A guiding and illustrating example is provided by the newforms of weight 2 with rational Fourier coefficients and CM. Since these correspond to elliptic curves over $\mathbb{Q}$ with CM, the finiteness result is guaranteed by the classical theory of elliptic curves with CM, and in particular, by class field theory. It will be interesting to note that all the phenomena occurring throughout the analysis of Hecke eigenforms with CM already appear (and can be explained) from this basic level.

The chapter is organized as follows: First we recall some of the basic theory of modular forms and Hecke Größencharaktere, including some examples (Sect. II.1, II.2). Then we state our results concerning the finiteness and shape of CM-newforms with rational coefficients for fixed weight (Sect. II.3). The subsequent sections, which constitute the main part of this chapter, are devoted to a detailed complete proof of the results. After a short application, we conclude the chapter by an explicit description of CM-newforms of weight 3 and 4 with rational coefficients (Sect. II.13, II.14). In particular, this will answer the motivating question. In an outlook, we also briefly comment on the geometric realizations of CM-newforms of weight at least 4. For weight 3, the reader will be referred to the next chapter.

II.1 Modular forms

In this section we recall some facts about modular forms. Our main aim is to sketch the connection between eigenforms with complex multiplication (CM) and Hecke Größencharakteren. The fact that a Größencharakter gives rise to a modular form goes back to Hecke. On the other hand, Ribet showed that any newform with CM comes from a Größencharakter (Thm. II.1.1). For details, the reader is referred to the paper of Ribet [R].

We are concerned with the newforms in the space of (elliptic) cusp forms $\mathcal{S}_k(\Gamma_1(N))$. Here, we fix an integral weight $k > 1$ and the level $N \in \mathbb{N}$. Mainly, we will work with the Fourier expansion at $i\infty$ for a cusp form

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n, \qquad \text{where} \quad q = e^{2\pi i \tau}.$$

We shall use the natural decomposition

$$\mathcal{S}_k(\Gamma_1(N)) = \bigoplus \mathcal{S}_k(\Gamma_0(N), \varepsilon)$$

where the sum runs over the possible *nebentypus* characters ε modulo N with $\varepsilon(-1) = (-1)^k$. For a cusp form $f \in \mathcal{S}_k(\Gamma_0(N), \varepsilon)$, the transformation formula reads:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \varepsilon(d) f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ and } \tau \in \overline{\mathbb{H}}.$$

Here, $\mathbb{H}$ denotes the complex upper half plane. The compactification $\overline{\mathbb{H}}$ is achieved by adjoining $\mathbb{P}^1(\mathbb{Q})$ (the so-called cusps).

The space of cusp forms $\mathcal{S}_k(\Gamma_1(N))$ is equipped with an algebra of endomorphisms, consisting of the Hecke operators T_m for $m \in \mathbb{N}$ with $(m, N) = 1$. The Hecke operators preserve the subspaces $\mathcal{S}_k(\Gamma_0(N), \varepsilon)$ and can be diagonalized simultaneously. Hence, $\mathcal{S}_k(\Gamma_0(N), \varepsilon)$ has a basis consisting of eigenforms of the Hecke operators T_m $(m \in \mathbb{N}, (m, N) = 1)$. If g is an eigenform for all Hecke operators, then the eigenvalues a_n appear as Fourier coefficients of a unique newform. A newform f can be characterized by the minimality of its level and the property that its Mellin transform $L(f, s)$ possesses an Euler product expansion

$$L(f, s) = \sum_{n \in \mathbb{N}} a_n n^{-s} = \prod_p (1 - a_p p^{-s} + \varepsilon(p) p^{k-1-2s})^{-1}. \tag{II.1}$$

Let $f \in \mathcal{S}_k(\Gamma_0(N), \varepsilon)$ be a newform. Consider the finite extension of $\mathbb{Q}$ which is generated by the set of Fourier coefficients $\{a_p\}$ of f. This field is well-known to be totally real if and only if the nebentypus ε is either trivial or quadratic with $\varepsilon(p) a_p = a_p$.

This relation occurs also in the definition of CM: A cusp form $f = \sum a_n q^n \in \mathcal{S}_k(\Gamma_1(N)$ is said to have *complex multiplication (CM)* by a Dirichlet character ϕ if $f = f \otimes \phi$. Here, we write the twist

$$f \otimes \phi = \sum_{n \in \mathbb{N}} \phi(n) a_n q^n.$$

Since a CM-character is necessarily quadratic, we will also refer to CM by the corresponding quadratic field K.

The main statement of this section is the following result of Ribet [R] which establishes the mentioned connection between newforms with CM and Hecke Größencharaktere. We will give details about Größencharaktere in the next section.

Theorem II.1.1 (Ribet [R, Prop. (4.4), Thm. (4.5)])
A newform has CM by K (a quadratic field) if and only if it comes from a Größencharakter of K. In particular, the field K is imaginary and unique.

We want to emphasize the following implication of Ribet's theorem:

Corollary II.1.2
Let f be a newform with totally real coefficients. Then the following relations hold:

(i) If the weight k is even, then the nebentypus ε is trivial.

(ii) If the weight k is odd, then f has CM by its nebentypus ε. In particular, ε is quadratic.

II.2 Hecke Größencharaktere

In this section we will explain Hecke's original definition of a Größencharakter (as opposed to Chevalley's adelic formulation) and sketch the connection to modular forms as well as a few examples. In this context, K shall denote an imaginary quadratic field.

Definition II.2.1
Let $\mathfrak{m}$ be an ideal of K and $l \in \mathbb{N}$. A Größencharakter *ψ of K modulo $\mathfrak{m}$ with ∞-type l is a homomorphism on the group of fractional ideals of K which are prime to $\mathfrak{m}$,*

$$\psi : \left\{ \begin{array}{c} \text{fractional ideals of } K \\ \text{which are prime to } \mathfrak{m} \end{array} \right\} \to \mathbb{C}^*,$$

such that

$$\psi((\alpha)) = \alpha^l$$

for all $\alpha \in K^*$ with $\alpha \equiv 1 \mod \mathfrak{m}$. The ideal $\mathfrak{m}$ is called the conductor *of ψ if it is minimal in the following sense: If ψ is defined modulo $\mathfrak{m}'$, then $\mathfrak{m}|\mathfrak{m}'$.*

Strictly speaking, we should refer to multiplicative congruence in this context. To simplify notation, we shall omit this in the following and also tacitly extend ψ by 0 for all fractional ideals of K which are not prime to $\mathfrak{m}$.

In order to explain the connection with modular forms, we consider the L-series of ψ where we denote the norm homomorphism of the Galois extension $K/\mathbb{Q}$ by $\mathcal{N}$:

$$L(\psi, s) = \sum_{\mathfrak{a} \text{ integral}} \psi(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{-s} = \prod_{\mathfrak{p} \text{ prime}} (1 - \psi(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-s})^{-1}.$$

Then, the inverse Mellin transform f_ψ of $L(\psi, s)$ is known to be a Hecke eigenform:

$$f_\psi = \sum_{n \in \mathbb{N}} a_n q^n = \sum_{\mathfrak{a} \text{ integral}} \psi(\mathfrak{a}) q^{\mathcal{N}(\mathfrak{a})} \qquad (q = e^{2\pi i \tau}).$$

We introduce the following notation: Let $-\Delta = -\Delta_K$ be the discriminant of K and $\chi = \chi_K$ the associated quadratic character of conductor Δ. Denote $M = \mathcal{N}(\mathfrak{m})$ and consider the Dirichlet character $\eta_\mathbb{Z}$ mod M which is given by

$$\eta_\mathbb{Z} : a \mapsto \frac{\psi((a))}{a^l} \quad (a \in \mathbb{Z}, (a, M) = 1).$$

Theorem II.2.2 (Hecke, Shimura)
f_ψ is a cusp form of weight $l+1$, level ΔM and nebentypus character $\chi\eta_{\mathbb{Z}}$:

$$f_\psi \in \mathcal{S}_{l+1}(\Gamma_0(\Delta M), \chi\eta_{\mathbb{Z}}).$$

f_ψ is a newform (of level ΔM) if and only if $\mathfrak{m}$ is the conductor of ψ.

In the following, we shall always work with the conductor.

By construction, the eigenform f_ψ has CM by K (or equivalently χ). From Corollary II.1.2, we obtain the following restrictions on the nebentypus $\varepsilon = \chi\eta_{\mathbb{Z}}$:

Corollary II.2.3
Assume that the newform f_ψ has totally real coefficients. Then the following conditions hold:

(i) If l is odd, then $\varepsilon = 1$ and hence $\eta_{\mathbb{Z}} = \chi$. In particular, $(\sqrt{-\Delta}) \mid \mathfrak{m}$.

(ii) If l is even, then $\varepsilon = \chi$ and hence $\eta_{\mathbb{Z}} = 1$.

Let ι denote the complex conjugation. Define the trace of ψ as $\psi + \psi \circ \iota$. Then the assumption that f has totally real field of coefficients is equivalent to the corresponding statement for the traces of ψ. In other words, ψ commutes with ι:

$$\psi \circ \iota = \iota \circ \psi.$$

Remark II.2.4
The first statement of Corollary II.2.3 can be strengthened in two special cases as follows: If ψ is a Größencharakter of $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$ with totally real traces and odd ∞-type, then

$$(2+2\sqrt{-1}) \mid \mathfrak{m} \quad \textit{resp.} \quad (4\sqrt{-2}) \mid \mathfrak{m}.$$

To see this, we note the following: As an ideal, the conductor of $\eta_{\mathbb{Z}}$ equals $\mathfrak{m} \cap \mathbb{Z}$. Thus the claim follows, since the conductor has to be divisible by 4 resp. 8.

The remainder of this section is devoted to the elaboration of two examples and some ideas concerning twisting. The first example gives Größencharaktere with rational traces, while the second produces traces in a real quadratic extension.

Example II.2.5
Let K be any of the seven imaginary quadratic fields with class number 1 and trivial unit group $\mathcal{O}_K^* = \{\pm 1\}$, i.e. $-\Delta_K = -7, -8, -11, -19, -43, -67, -163$. We are going to give a Größencharakter of K for any ∞-type $l \in \mathbb{N}$.

For even l, there is a Größencharakter of K with trivial conductor and ∞-type l. This is given by

$$\psi((\alpha)) = \alpha^l \quad (\alpha \in K^*).$$

By the above arguments, ψ gives rise to a newform $f_\psi \in S_{l+1}(\Gamma_0(\Delta_K), \chi_K)$. In Theorem II.3.4 we will see that these newforms provide all newforms of weight $l+1$ with rational coefficients and CM by the respective nebentypus χ_K by means of quadratic twisting.

If the ∞-type l is odd, we can define

$$\psi((\alpha)) = \chi_K(Re(\alpha))\alpha^l \quad (\alpha \in K^*).$$

For $K \neq \mathbb{Q}(\sqrt{-2})$, this gives a unique Größencharakter with conductor $(\sqrt{-\Delta})$. This conductor is minimal by Corollary II.2.3. In case $K = \mathbb{Q}(\sqrt{-2})$, however, the minimal conductor can only be $\mathfrak{m} = (4\sqrt{-2})$ by Remark II.2.4. As a result, we obtain two Größencharaktere which differ by a twist by $\chi_{-1} \circ \mathcal{N}$. Here χ_{-1} denotes the unique quadratic character of conductor 4. We will see in Theorem II.3.4 that all newforms of weight $l+1$ with rational coefficients and CM by one of these fields can be obtained from the Größencharaktere given above after twisting.

At this point, we shall briefly explain the relation between twisting for ψ and f_ψ. For this purpose, let ϕ be a Dirichlet character. By definition, twisting f_ψ by ϕ corresponds to twisting ψ by $\phi \circ \mathcal{N}$. In other words,

$$f_\psi \otimes \phi = f_{\psi \otimes (\phi \circ \mathcal{N})}.$$

We emphasize that the latter twist is well-defined on ideals. If $\mathcal{O}_K^*$ includes extra roots of unity, then we also have to take characters of greater order into account. This happens for $K = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$. Here, a quartic resp. sextic character Ξ on the ideals of K gives rise to a twist $\psi \otimes \Xi$ with rational traces. Therefore, we shall also consider the additional twist

$$f_\psi \otimes \Xi := f_{\psi \otimes \Xi}.$$

However, its effect on the conductor resp. level is not as obvious as in the quadratic case. This will be part of the discussions in Sections II.9 and II.10. We will often refer to these twists as *admissible* when we want to emphasize the rational traces.

Throughout, we shall freely employ the convention that a quadratic character has order precisely 2 and likewise for cubic characters etc.

In the following example, we shall construct two eigenforms which come from a Größencharakter with totally real, but not rational traces. Nevertheless, this will be important for our purposes.

Example II.2.6
Consider the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-15})$ of class number $h_K = 2$. We are going to define two Größencharaktere ψ of K of conductor $(\sqrt{-15})$ and ∞-type $l = 1$ with traces in $\mathbb{Z}(\sqrt{5})$. Note that the quotient field $\mathbb{Q}(\sqrt{5})$ of $\mathbb{Z}(\sqrt{5})$ is the minimal field of definition $\mathbb{Q}(j(E))$ of the elliptic curve E with complex multiplication by $\mathcal{O}_K$.

We define ψ on all principal fractional ideals of K (prime to $(\sqrt{-15})$) by

$$\psi((a + b\sqrt{-15})) = \chi_{-15}(a)(a + b\sqrt{-15}) = \left(\frac{a}{15}\right)(a + b\sqrt{-15}).$$

For the non-principal ideals, we have to apply a normalization. For instance, this can be achieved by considering the two prime factors of (2):

$$2 = \mathfrak{p}\,\bar{\mathfrak{p}} = \left(2, \frac{1+\sqrt{-15}}{2}\right)\left(2, \frac{1-\sqrt{-15}}{2}\right).$$

Since $\psi(\mathfrak{p}^2) = \frac{1+\sqrt{-15}}{2}$, we easily derive

$$\psi(\mathfrak{p}) = \pm\frac{\sqrt{5}+\sqrt{-3}}{2}.$$

The choice of sign fixes a Größencharakter with conductor $(\sqrt{-15})$ and ∞-type 1. The opposite sign corresponds to a twist by $\chi_{-3} \circ \mathcal{N}$ or equivalently $\chi_5 \circ \mathcal{N}$. Here χ_{-3} and χ_5 represent the Legendre symbols and give the two fundamental characters dividing χ_K (see the proof of Lemma II.5.7 for details).

The relevance of the above example for our purposes lies in the fact that by construction all even powers ψ^{2l} have rational traces. Thus they give rise to newforms of respective weights $2l+1$ with rational coefficients. Moreover, the same method can also be applied to all other imaginary quadratic fields K with class group consisting of 2-torsion if Δ_K is odd or divisible by 8. However, we have already pointed out that not all newforms with rational coefficients and CM can be produced in such a way (cf. Ex. I.4.5).

II.3 Formulation of the results

In this section we formulate our main result for CM-newforms (Thm. II.3.1). To prove this, we translate the statement to Größencharaktere (Thm. II.3.3). Due to work of Weinberger, this theorem follows from a classification result for Größencharaktere (Thm. II.3.4). The proof of this result will be the content of the next eight sections.

Theorem II.3.1
Assume the Generalized Riemann Hypothesis (GRH). Then there are only finitely many CM-newforms with rational coefficients for fixed weight up to twisting.
For weights 2,3,4 this holds unconditionally.

For weight 2 this result is known from the theory of elliptic curves with CM. For the general case, we shall first prove the corresponding statement for Größencharaktere in Theorem II.3.3. Then we shall use Ribet's Theorem II.1.1 to deduce Theorem II.3.1. For this purpose, we need the following easy lemma:

Lemma II.3.2
Let ψ be a Größencharakter of an imaginary quadratic field K. Denote the corresponding newform by f. Then the following statements are equivalent:

(i) f has rational coefficients,

(ii) ψ has rational traces,

(iii) $\operatorname{im} \psi \subseteq \mathcal{O}_K$.

As a consequence, Theorem II.3.1 is equivalent to

Theorem II.3.3
Assume GRH. For fixed ∞-type there are only finitely many Größencharaktere of quadratic imaginary fields with rational traces up to twisting.
For ∞-types 1,2,3 this is unconditionally true.

Having established this equivalent formulation, we are yet again going to relate Theorem II.3.3 to another theorem. This theorem gives an unconditional classification of Größencharaktere which is slightly stronger and more precise than Theorem II.3.3. In it, we consider the *exponent* e_K of an imaginary quadratic field K, that is, the exponent of the class group $Cl(\mathcal{O}_K)$.

Theorem II.3.4
For fixed ∞-type l, there is a bijective correspondence

$$\left\{\begin{matrix}\text{Hecke Größencharaktere}\\ \text{with } \infty\text{-type } l \text{ and rational traces}\\ \text{up to twisting}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{Imaginary quadratic fields } K\\ \text{with exponent } e_K \mid l\end{matrix}\right\}.$$

Before we sketch the various steps of the proof, we briefly want to indicate why Theorem II.3.3 follows from Theorem II.3.4.

We recall a result of Weinberger: Subject to a general condition on the zeroes of the Dirichlet L-functions $L(s,\chi_K)$ for imaginary quadratic fields K, he shows that

$$e_K \to \infty \quad \text{as} \quad \Delta_K \to \infty \qquad \text{[W, Thm. 3].}$$

In particular this holds if the Generalized Riemann hypothesis for $L(s,\chi_K)$ is true (cf. [W, Thm. 4]). Hence, Theorem II.3.4 implies the conditional part of Theorem II.3.3 as claimed. Of course, we could also formulate Theorem II.3.3 with the weaker condition, but we omit this here for brevity.

Furthermore, Weinberger proves the finiteness of imaginary quadratic fields K with exponent $e_K = 2$ or 3 unconditionally [W, Thms. 1, 2] (the latter is also due to Boyd-Kisilevski [B-K]). Taking also into account the classical case of class number one as proved in [St], we obtain the extra statement for ∞-types 1,2, and 3. Subject to Theorem II.3.4, this completes the proof of Theorem II.3.3.

We will pay special attention to the unconditional cases in Sections II.13 and II.14 where we list all (or almost all) CM-newforms of weight 3 and 4 with rational coefficients up to twisting. This will answer our motivating question which originated from singular K3 surfaces over $\mathbb{Q}$ (Qu. I.5.3).

We shall conclude this section by a sketch of how the proof of Theorem II.3.4 is organized. In order to give a systematic treatment, we denote the map of Theorem II.3.4, which sends a Größencharakter to its CM-field, by $\mathfrak{j}$:

$$\mathfrak{j}: \left\{\begin{array}{c}\text{Hecke Größencharaktere}\\ \text{with } \infty\text{-type } l \text{ and rational traces}\\ \text{up to twisting}\end{array}\right\} \quad \to \quad \left\{\begin{array}{c}\text{Imaginary quadratic fields } K\\ \text{with exponent } e_K \mid l\end{array}\right\}$$
$$\psi \quad \mapsto \quad K$$

With this notation, the proof of Theorem II.3.4 amounts to checking that $\mathfrak{j}$ is well-defined (i.e. all CM-fields occurring have exponent $e_K \mid l$) and bijective. This will be established as follows:

In the next section, we use an argument of Serre to check that $\mathfrak{j}$ is well-defined (Proposition II.4.1).

Then, Section II.5 gives the proof of the surjectivity of $\mathfrak{j}$ (Corollary II.5.5). Using elementary genus theory, we will also achieve a first step towards the injectivity of $\mathfrak{j}$ (Lemma II.5.7).

Finally, the proof of the injectivity of $\mathfrak{j}$ will cover Sections II.6-II.11. The main result can be found in Theorem II.6.1 and Corollary II.6.2. Essentially, this will only require explicit twisting. A reduction step takes care of the unramified primes, as indicated in Proposition II.6.3. We start with the odd primes in Section II.7 and then come to the special prime 2 in Section II.8. Throughout, we will exclude the fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ since these involve extra roots of unity and therefore extra twists. They will be addressed in Sections II.9 and II.10. The proof will be completed by considering the ramified primes in Section II.11. Here, the reader is referred to Proposition II.11.1.

Remark II.3.5
Theorem II.3.4 can be used to determine a Größencharakter with rational traces resp. a newform with rational coefficients and CM from few explicit data. Given the bad primes, it will always suffice to check a finite number of primes to deduce

the precise conductor resp. level. This has been used in Example I.4.4. See the proof of Proposition IV.2.2 for an elaboration for another example.

II.4 j is well-defined

In this section, we will prove that j is well-defined. Clearly, j is consistent with twisting. Thus, an equivalent formulation is provided by the following proposition which goes back to an argument of J.-P. Serre in [L, Rem. 1.8]:

Proposition II.4.1
Let K be an imaginary quadratic field. Let ψ be a Größencharakter of K with rational traces and ∞-type l. Then $e_K \mid l$.

At this point, we shall introduce the character η. This will be useful for the proof of the proposition and for the injectivity of j. Its definition is based on the observation that a Größencharakter ψ acts trivially on the set $1+\mathfrak{m}$. Hence, it factors through the quotient group $(\mathcal{O}_K/\mathfrak{m})^*$. This allows us to consider the following character on the group $G=(\mathcal{O}_K/\mathfrak{m})^*$:

$$\begin{array}{rccc} \eta: & G & \to & \mathbb{C}^* \\ & \alpha & \mapsto & \dfrac{\psi((\alpha))}{\alpha^l}. \end{array}$$

We shall now study η in some detail. Formally, its definition coincides with that of $\eta_{\mathbb{Z}}$, the character describing the operation of ψ on $\mathbb{Z}/M$. Recall from Corollary II.2.3, that

$$\eta_{\mathbb{Z}} = \chi_K^l$$

if ψ has totally real traces. Here we tacitly use the convention that $\chi_K^2 = 1$. The two characters η and $\eta_{\mathbb{Z}}$ only differ by the respective module of definition. Thus we deduce that η is determined on the $\mathbb{Z}$-part of G in terms of the ∞-type l. By the $\mathbb{Z}$-part of G, we mean the residue classes of integers in G. Denote their set by

$$G_{\mathbb{Z}} = G \cap \operatorname{im}((\mathbb{Z}/M)^* \to G) = \{\alpha \in G\,;\exists\, a \in \mathbb{Z} : \alpha \equiv a \mod \mathfrak{m}\}.$$

Precisely, we obtain the following

Corollary II.4.2
Let η be the character constructed from a Größencharakter ψ of an imaginary quadratic field K as above. If ψ has totally real traces and ∞-type l, then

$$\eta|_{G_{\mathbb{Z}}} = \chi_K^l.$$

The other essential ingredient for the proof of Proposition II.4.1 is provided by the next

Lemma II.4.3
Let η as above. Assume furthermore that the traces of ψ are rational. Then

$$im\ \eta \subseteq \mathcal{O}_K^*.$$

Proof: Since G is finite, η clearly has image some roots of unity. In particular, these are algebraic integers. On the other hand, η has values in K by Lemma II.3.2. Hence, the claim follows.

As a consequence of Lemma II.4.3, we deduce the following

Corollary II.4.4
Let ψ a Größencharakter of K with rational traces and ∞-type l. Then, for any $\alpha \in \mathcal{O}_K$ coprime to the conductor $\mathfrak{m}$,

$$(\psi((\alpha))) = (\alpha^l) = (\alpha)^l$$

as (principal) ideals of $\mathcal{O}_K$.

This corollary easily enables us to prove Proposition II.4.1: We let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ with order n in $Cl(\mathcal{O}_K)$, i.e. $\mathfrak{a}^n = (\alpha)$ for some $\alpha \in \mathcal{O}_K$. If $\mathfrak{a}$ is prime to $\mathfrak{m}$, then Corollary II.4.4 gives

$$(\psi(\mathfrak{a}))^n = (\alpha)^l.$$

Since both $(\psi(\mathfrak{a}))$ and (α) are principal ideals, n has to divide l by the minimality of the order. This completes the proof of Proposition II.4.1.

For convenience, we conclude this section with the following corollary of Proposition II.4.1 which will be derived from genus theory.

Corollary II.4.5 (Odd ∞-type)
Let $l \in \mathbb{N}$ be odd. If there is a Größencharakter of K with rational traces and ∞-type l, then either Δ_K is prime (with $\Delta_K \equiv 3 \mod 4$) or $\Delta_K = 4$ or 8.

Proof: From Proposition II.4.1, we obtain that $e_K \mid l$. In particular, e_K is odd. On the other hand, it is known from genus theory (cf. the discussion succeeding Lemma II.5.7), that the 2-torsion in $Cl(\mathcal{O}_K)$ corresponds to the different genera. The genera come from the divisors of Δ_K. If e_K is odd, $CL(\mathcal{O}_K)$ consists of one genus. Hence Δ_K can only have one prime divisor, and the claim follows.

Corollary II.4.5 simplifies the study of Größencharaktere with odd ∞-type significantly. We will see this in the next section.

II.5 The surjectivity of j

In order to prove the surjectivity of $\mathfrak{j}$, we refer to Example II.2.5 as a guide and illustration. For the imaginary quadratic fields with class number $h_K = 1$ and trivial unit group $\mathcal{O}_K^* = \{\pm 1\}$, we gave a natural definition for Größencharaktere with rational traces and conductor trivial or $(\sqrt{-\Delta_K})$ (with one exception). Our main aim in this section is to extend these ideas to fields with class number (or exponent) greater than 1 by an argument analogous to Example II.2.6. Before giving details, we shall, however, give Größencharaktere of the two remaining fields with class number one, $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. The existence of extra roots of unity makes these fields special. We have to take particular care in the conductors.

Example II.5.1
Let $K = \mathbb{Q}(\sqrt{-1})$ and $l \in \mathbb{N}$. There is a unique Größencharakter ψ of K modulo $(2+2i)$ with ∞-type l. The conductor $\mathfrak{m}$ of ψ is given by

$$\mathfrak{m} = 1, \text{ if } 4 \mid l; \quad \mathfrak{m} = (2), \text{ if } 2 \mid\mid l; \quad \mathfrak{m} = (2+2i), \text{ if } 2 \nmid l.$$

Here, $||$ refers to exact divisibility in $\mathbb{N}$, i.e. $m \mid\mid n \Leftrightarrow m \mid n$ and $(m, \frac{n}{m}) = 1$.

Example II.5.2
Let $K = \mathbb{Q}(\sqrt{-3})$ and $l \in \mathbb{N}$. There is a unique Größencharakter ψ of K modulo (3) with ∞-type l. The conductor $\mathfrak{m}$ of ψ is given by

$$\mathfrak{m} = 1, \text{ if } 6 \mid l; \quad \mathfrak{m} = (\sqrt{-3}), \text{ if } 3 \mid l, \text{ but } 2 \nmid l; \quad \mathfrak{m} = (3), \text{ if } 3 \nmid l.$$

Remark II.5.3
Alternatively, we could have proceeded as follows for all fields of class number one: Define a Größencharakter of ∞-type 1 (as in Ex. II.2.5, II.5.1, II.5.2). Then consider its l-th powers. This procedure gives the same Größencharaktere as before.

So far, we have produced Größencharaktere with rational traces and arbitrary ∞-type for every imaginary quadratic field with class number 1. To prove the surjectivity of $\mathfrak{j}$, we now turn to fields with class number greater than 1. Combining the approaches of Examples II.2.5 and II.2.6, we will show the

Lemma II.5.4
Let K be an imaginary quadratic field with $h_K > 1$ and l a multiple of e_K. Then there is a Größencharakter of K with rational traces, ∞-type l and with trivial conductor if l is even, resp. with conductor $\mathfrak{m} = (\sqrt{-\Delta})$ if l is odd.

Proof: For the principal ideals of $\mathcal{O}_K$, we define a Größencharakter ψ as in Example II.2.5 by

$$(\alpha) \mapsto (\chi_K(\mathrm{Re}(\alpha))\alpha)^l.$$

This leaves only the non-principal ideals. These can be dealt with as in Example II.2.6. We consider a factorization of the abelian group $Cl(\mathcal{O}_K)$ into cyclic groups

$$Cl(\mathcal{O}_K) = C_{n_1} \times \ldots \times C_{n_r}.$$

Then, ψ is completely determined by its operation on a set of generators $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ (which are to be chosen coprime to the conductor). Letting $\mathfrak{a}_i^{n_i} = (\alpha_i)$ for some $\alpha_i \in \mathcal{O}_K$, we obtain

$$\psi(\mathfrak{a}_i)^{n_i} = \psi((\alpha_i)) = (\chi_K(\mathrm{Re}(\alpha_i))\alpha_i)^l.$$

In case n_i is odd, this determines the root

$$\psi(\mathfrak{a}_i) = (\chi_K(\mathrm{Re}(\alpha_i))\alpha_i)^{\frac{l}{n_i}} \in \mathcal{O}_K \tag{II.2}$$

uniquely due to the triviality of $\mathcal{O}_K^*$. On the other hand, if n_i is even, there is a choice of sign in the image of $\mathfrak{a}_i$ under ψ:

$$\psi(\mathfrak{a}_i) = \pm\alpha^{\frac{l}{n_i}} \in \mathcal{O}_K. \tag{II.3}$$

Any fixed choice gives rise to a Größencharakter with rational traces and conductor as claimed.

Corollary II.5.5
The map $\mathfrak{j}$ is surjective.

Proof: Lemma II.5.4 together with Examples II.2.5, II.5.1, II.5.2.

Remark II.5.6 (Minimality)
The conductors given in Lemma II.5.4 are minimal possible by Corollary II.2.3. Here, minimality refers to both, norm and divisibility of ideals. With two exceptions, this also holds for the Größencharaktere of fields of class number one which we gave in Examples II.2.5, II.5.1 and II.5.2:

In Example II.5.1, if $2||l$, then the conductor (2) is only minimal with respect to its norm. E.g., for any ∞-type $l \equiv 2 \mod 4$, there are two Größencharaktere of $\mathbb{Q}(\sqrt{-1})$ with conductor (3). (They are twisted by $\chi_{-3} \circ \mathcal{N}$.)

In Example II.5.2, if $3 \nmid l$, but $2 \mid l$, then minimality does not hold at all since $\mathfrak{m} = (2)$ is possible. An example for this is defined in (II.15). This exceptional case is a consequence of cubic twisting as discussed in Section II.10 (compare also Corollary II.11.13).

Nevertheless we shall employ the convention to refer to the named Größencharaktere and their conductors as minimal *throughout this chapter.*

We want to conclude this section with a first step towards the injectivity of $\mathfrak{j}$. Here, we will check this property for the minimal Größencharaktere which we constructed in the proof of Lemma II.5.4.

Let K be an imaginary quadratic field with $h_K > 1$ and l a multiple of e_K. On the one hand, if e_K is odd (i.e. all n_i are odd), equation (II.2) guarantees the uniqueness of the corresponding Größencharakter. On the other hand, for every even n_i we have two options due to equation (II.3). Let g be such that

$$Cl(\mathcal{O}_K)[2] = \{Q \in Cl(\mathcal{O}_K);\ Q^2 = 1\} \cong (\mathbb{Z}/2)^{g-1}.$$

Then the choices of sign in equation (II.3) give exactly 2^{g-1} Größencharaktere of K with rational traces, ∞-type l and minimal conductor. In terms of their traces at the primes of K, they only differ by signs at the non-principal primes. With a view towards proving the injectivity of $\mathfrak{j}$, we claim

Lemma II.5.7
The 2^{g-1} minimal Größencharaktere of K constructed above are quadratic twists.

To prove this lemma, we use genus theory. Consider the unique factorization of the discriminant $-\Delta_K$ into fundamental discriminants

$$-\Delta_K = \delta_1 \cdots \delta_{g'}$$

where the δ_i are pairwise co-prime and $\delta_i = -4, \pm 8$ or $(\frac{-1}{p})\, p$ for $p > 2$. Equivalently, χ_K factors into g' fundamental characters $\chi_{\delta_i} = (\frac{\delta_i}{\cdot})$ of respective conductor $|\delta_i|$. Genus theory asserts that the class group $Cl(\mathcal{O}_K)$ consists of $2^{g'-1}$ genera. Equivalently, the number of 2-torsion elements in $Cl(\mathcal{O}_K)$ is $2^{g'-1}$ (cf. [Co, 3.B]). Hence $g = g'$.

We now consider a Größencharakter ψ as above and twist it by a character χ which is a product of some of the χ_{δ_i}. Since ψ has complex multiplication by $\chi_K \circ \mathcal{N}$, twisting it by the complementary character $\frac{\chi_K}{\chi} \circ \mathcal{N}$ gives rise to the same Größencharakter ψ'. As χ and $\frac{\chi_k}{\chi}$ are co-prime (in the sense that they have co-prime conductors), the twist does not affect the conductor. Hence, ψ' has the same minimal conductor as ψ.

Consider the 2^g products of the χ_{δ_i}. They are divided into pairs of complementary characters. The 2^{g-1} pairs give rise to as many Größencharaktere with minimal conductor by way of twisting ψ. These twists differ due to the uniqueness of the CM-field (Thm. II.1.1). Hence, the twists produce exactly the minimal Größencharaktere considered in Lemma II.5.7. In other words, each pair of fundamental characters uniquely corresponds to a genus. This finishes the proof of Lemma II.5.7.

II.6 The injectivity of $\mathfrak{j}$

In this section, we will show how the injectivity of $\mathfrak{j}$ can be proven. The general idea consists in exhibiting an admissible twist (in the sense of p. 15) which gives rise to a minimal Größencharakter in the notion of Remark II.5.6. This will imply the injectivity of $\mathfrak{j}$ because of Lemma II.5.7.

Theorem II.6.1
Let ψ be a Größencharakter with rational traces. Then there is an admissible twist $\psi \otimes \Xi$ with minimal conductor.

Before we sketch the proof of this theorem, which will actually cover the next five sections, let us explain how it implies the injectivity of $\mathfrak{j}$.

Fix the imaginary quadratic field K and the ∞-type l. By Lemma II.5.7, all minimal Größencharaktere of K with ∞-type l are equivalent under twisting. After twisting according to Theorem II.6.1, the same holds for any two Größencharaktere of K with rational traces and ∞-type l. We obtain the

Corollary II.6.2
$\mathfrak{j}$ is injective.

Theorem II.6.1 concerns the conductor of a Größencharakter ψ and its twists. The main advantage of this approach is that the conductor is completely controlled by the character η which was defined in Section II.4. Moreover it is immediate that twisting ψ corresponds to twisting η:

$$(\psi, \eta) \leftrightarrow (\psi \otimes \Xi, \eta \otimes \Xi). \tag{II.4}$$

In other words, up to twisting with fundamental characters, all the information about ψ is encoded in η. Therefore, we will work with η rather than with ψ in what follows.

In the proof of Theorem II.6.1, the twisting will be exhibited primewise. For every prime $\mathfrak{p}$ dividing the conductor $\mathfrak{m}$, we perform an admissible twist such that the $\mathfrak{p}$-parts of the conductor of the twist and of the minimal conductor agree. Generally the twisting character will have p-power conductor. Here, we write $\mathfrak{p}$ (and $\bar{\mathfrak{p}}$) for the primes of $\mathcal{O}_K$ and p for the corresponding primes of $\mathbb{Z}$.

If a prime $\mathfrak{p}$ divides $\mathfrak{m}$, we factor $\mathfrak{m} = \mathfrak{m}'\mathfrak{p}^{e_\mathfrak{p}}$ or $\mathfrak{m} = \mathfrak{m}'\mathfrak{p}^{e_\mathfrak{p}}\bar{\mathfrak{p}}^{e_{\bar{\mathfrak{p}}}}$, such that $\mathfrak{m}'$ is prime to p. Then we restrict η to the multiplicative subgroup

$$\Omega = \{\alpha \in (\mathcal{O}_K/\mathfrak{m})^* : \alpha \equiv 1 \mod \mathfrak{m}'\} \subset G.$$

We denote the restrictions

$$\tilde{\eta} = \eta|_\Omega \quad \text{and} \quad \Omega_\mathbb{Z} = \Omega \cap G_\mathbb{Z}.$$

By the Chinese remainder theorem,

$$\Omega = (\mathcal{O}_K/\mathfrak{p}^{e_\mathfrak{p}})^* \quad \text{or} \quad \Omega = (\mathcal{O}_K/\mathfrak{p}^{e_\mathfrak{p}})^* \oplus (\mathcal{O}_K/\bar{\mathfrak{p}}^{e_{\bar{\mathfrak{p}}}})^*.$$

The further factorization of this abelian group will allow us to determine $\tilde{\eta}$ explicitly. Then we will exhibit the corresponding twist which produces the right conductor. This will also give us the possible values for the exponents $e_\mathfrak{p}$. We collect them in Proposition II.12.1.

We shall first exhibit the described reduction step for the unramified primes. This will be the major step towards the proof of Theorem II.6.1.

Proposition II.6.3 (Reduction step)
Let ψ be a Größencharakter with rational traces. Then there is an admissible twist $\psi \otimes \Xi$ with conductor composed of the ramifying primes.

The proof of this proposition is divided into the next four sections, according to the occurring twists.

In the first two sections, we restrict to imaginary quadratic fields K with trivial unit group $\mathcal{O}_K^* = \{\pm 1\}$. Hence, we only have to consider quadratic twists due to Lemma II.4.3. We first take care of the odd primes p (Proposition II.7.1). Each of these comes with a unique quadratic character χ_p of conductor p. Then we treat the primes above 2, where the quadratic characters χ_{-1} and $\chi_{\pm 2}$ of respective conductors 4 and 8 occur (Lemma II.8.1).

Finally, Sections II.9 and II.10 are devoted to the fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ with larger unit groups. For the quartic and sextic characters in question, we apply methods in analogy with the corresponding elliptic curves with CM. The relevant results can be found in Proposition II.9.8 and II.10.6.

After the proof of the reduction step, we analyze the ramified primes in Section II.11. Proposition II.11.1 will finish the proof of Theorem II.6.1.

The following lemma will be useful for the study of the unramified primes:

Lemma II.6.4
Let ψ be a Größencharakter of K with rational traces and conductor $\mathfrak{m}$. Let $\mathfrak{p}$ be an unramified prime of $\mathcal{O}_K$ which divides $\mathfrak{m}$. Consider $\tilde{\eta}$ as constructed above. Then

$$\tilde{\eta}|_{\Omega_{\mathbb{Z}}} = 1.$$

Proof: If the ∞-type l of ψ is even, the statement follows from Corollary II.4.2. If l is odd, then we know that

$$\eta|_{G_{\mathbb{Z}}} = \chi_K \quad \text{(Cor. II.4.2).} \tag{II.5}$$

Recall that in this setting Δ_K is either an odd prime or 4 or 8 (Cor. II.4.5). In the first case, we can use the property $(\sqrt{-\Delta_K}) \mid \mathfrak{m}$ from Corollary II.2.3. Since $\mathfrak{p}$ is unramified, this gives $(\sqrt{-\Delta_K}) \mid \mathfrak{m}'$. Hence we have the implication

$$a \in \Omega_{\mathbb{Z}} \Rightarrow a \equiv 1 \mod \Delta_K.$$

Since $\tilde{\eta}|_{\Omega_{\mathbb{Z}}} = (\eta|_{G_{\mathbb{Z}}})|_{\Omega_{\mathbb{Z}}}$, equation (II.5) now gives the claim. For $\Delta_K = 4$ or 8, the same argument can be applied using the relation from Remark II.2.4. This finishes the proof of Lemma II.6.4.

II.7 Reduction step for odd primes and trivial unit group $\mathcal{O}_K^*$

In this section, we will restrict to fields with trivial unit group $\mathcal{O}_K^* = \{\pm 1\}$ and prove the reduction step (Prop. II.6.3) for all (unramified) odd primes.

Proposition II.7.1
Let $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ and ψ be a Größencharakter of K with rational traces and conductor $\mathfrak{m}$. Consider an odd prime p of $\mathbb{Z}$ which does not ramify in K. If $(p, \mathfrak{m}) \neq 1$, then let $\mathfrak{m}'$ be the prime to p part of $\mathfrak{m}$. Then, the quadratic twist $\psi \otimes (\chi_p \circ \mathcal{N})$ has conductor $\mathfrak{m}'$.

Applying this proposition primewise, we can twist away all unramified odd primes. We obtain a Größencharakter whose conductor is composed of the ramified primes and the primes at 2.

The remainder of this section is devoted to the proof of Proposition II.7.1. Due to the correspondence (II.4), this will be achieved by establishing the claim

$$\tilde{\eta} = \chi_p \circ \mathcal{N}. \tag{II.6}$$

We shall now distinguish between two cases according to the splitting behaviour of p in K.

II.7.1 p stays inert in K

In this case, $\tilde{\eta}$ operates on the quotient $\Omega \cong (\mathcal{O}_K/p^e)^*$. As an abelian group, it has a factorization

$$\Omega = (\mathcal{O}_K/p^e)^* = C_{p^2-1} \times \tilde{\Omega}$$

where $\tilde{\Omega}$ is a group of cardinality $p^{2(e-1)}$. In particular, this cardinality is odd. Hence $\tilde{\eta}$ as a quadratic character operates trivially on $\tilde{\Omega}$. This gives $e \leq 1$.

We deduce that $\Omega = (\mathcal{O}_K/p)^* = C_{p^2-1}$ with a non-trivial action of the quadratic character $\tilde{\eta}$. Then we realize that $\chi_p \circ \mathcal{N}$ defines another non-trivial quadratic character on Ω. For this, we first have to check the well-definedness. This can be read off from the implication

$$p \mid (\alpha - \beta) \;\Rightarrow\; p \mid (\mathcal{N}(\alpha) - \mathcal{N}(\beta)).$$

For the non-triviality, we recall that p splits in $\mathcal{O}_K$ as a principal ideal if and only if $\chi_\delta(p) = 1$ for every fundamental discriminant δ dividing Δ_K. Hence $\chi_p \circ \mathcal{N} \not\equiv 1$ on Ω for an unramified prime p. Since Ω is cyclic, any two non-trivial quadratic characters on Ω have to coincide. Thus the claim (II.6) follows.

II.7.2 p splits in K

We write $p = \mathfrak{p}\bar{\mathfrak{p}}$ such that $\mathfrak{m} = \mathfrak{m}'\mathfrak{p}^{e_\mathfrak{p}}\bar{\mathfrak{p}}^{e_{\bar{\mathfrak{p}}}}$. Hence the canonical factorization becomes

$$\Omega = (\mathcal{O}_K/\mathfrak{p}^{e_\mathfrak{p}})^* \oplus (\mathcal{O}_K/\bar{\mathfrak{p}}^{e_{\bar{\mathfrak{p}}}})^* = \Omega_1 \oplus \Omega_2.$$

The respective quotient maps $\Omega \to \Omega_i$ shall be denoted by $[\cdot]_i$. Since $\mathfrak{p} \neq \bar{\mathfrak{p}}$, the summands are isomorphic to $(\mathbb{Z}/p^{e_\mathfrak{p}})^*$ and $(\mathbb{Z}/p^{e_{\bar{\mathfrak{p}}}})^*$, respectively.

We factor $\tilde{\eta} = \tilde{\eta}_1\tilde{\eta}_2$ with the characters $\tilde{\eta}_i$ operating on Ω_i. We can view them as acting on $(\mathbb{Z}/p^e)^*$ for the respective exponent e. With this notion, our next result is that the characters are conjugate:

$$\tilde{\eta}_1 = \overline{\tilde{\eta}_2}. \tag{II.7}$$

This is a direct consequence of the equality from Lemma II.6.4

$$1 = \tilde{\eta}(a) = \tilde{\eta}_1([a]_1)\tilde{\eta}_2([a]_2) \;\; \forall\, a \in \Omega_\mathbb{Z}.$$

To deduce the result, we only need that $[a]_1 \equiv [a]_2 \mod p^{\min(e_\mathfrak{p}, e_{\bar{\mathfrak{p}}})}$ for $a \in \Omega_\mathbb{Z}$. In particular, the conjugacy (II.7) implies that

$$e = e_\mathfrak{p} = e_{\bar{\mathfrak{p}}}. \tag{II.8}$$

Our next claim is that the characters $\tilde{\eta}_i$ are in fact quadratic; then, by conjugation

$$\tilde{\eta}_1 = \tilde{\eta}_2.$$

To prove this, let us assume on the contrary that the character $\tilde{\eta}_1$ is not quadratic, and establish a contradiction. By assumption, there is an element $\alpha \in \Omega$ with $\tilde{\eta}_1([\alpha]_1) \notin \{\pm 1\}$. Let $n_\mathfrak{p}, n_{\mathfrak{m}'}$ be the respective orders of $\mathfrak{p}$ and $\mathfrak{m}'$ in the class group $Cl(K)$, so that $\mathfrak{p}^{n_\mathfrak{p}} = (\pi)$ and $\mathfrak{m}'^{n_{\mathfrak{m}'}} = (\mu)$ for some $\pi, \mu \in \mathcal{O}_K$. We then consider the elements

$$\alpha + k\pi\mu \in \Omega \quad \text{for } \; k \in \mathbb{Z}.$$

Taking k in the range of $0, \ldots, p^e - 1$, the residue classes $[\alpha + k\pi\mu]_2$ run through the whole of $\mathbb{Z}/p^e$ by definition. Hence there is a k_0 such that $[\alpha + k_0\pi\mu]_2 = [1]_2$. This gives the required contradiction, since $\tilde{\eta}$ is quadratic by Lemma II.4.3:

$$\begin{aligned}\tilde{\eta}(\alpha + k_0\pi\mu) &= \tilde{\eta}_1([\alpha + k_0\pi\mu]_1)\, \tilde{\eta}_2([\alpha + k_0\pi\mu]_2) \\ &= \tilde{\eta}_1([\alpha]_1)\, \tilde{\eta}_2([1]_2) = \tilde{\eta}_1([\alpha]_1) \notin \{\pm 1\}.\end{aligned}$$

Since the $\tilde{\eta}_i$ are quadratic, we obtain the bound $e \leq 1$. This comes from the factorization

$$\Omega_i \cong (\mathbb{Z}/p^e)^* = C_{p-1} \times C_{p^{e-1}}$$

due to the odd cardinality of the second factor.

As a non-trivial quadratic character on $\mathbb{F}_p^*$, $\tilde{\eta}_i$ coincides with $\chi_p (i = 1, 2)$. We will now show that this gives $\tilde{\eta} = \chi_p \circ \mathcal{N}$ as in claim (II.6).

To see this, we note that $[\alpha]_2 = [\bar{\alpha}]_1$ in $\mathbb{F}_p^*$; in particular, for $\alpha \in \Omega_\mathbb{Z}$, we can delete the subscript, regardless of the factor. For general $\alpha \in \Omega$ we thereby obtain

$$\begin{aligned}\tilde{\eta}(\alpha) &= \tilde{\eta}_1([\alpha]_1)\tilde{\eta}_2([\alpha]_2) \\ &= \chi_p([\alpha]_1)\chi_p([\bar{\alpha}]_1) \\ &= \chi_p([\mathcal{N}(\alpha)]) \\ &= \chi_p(\mathcal{N}(\alpha)) = \chi_p \circ \mathcal{N}(\alpha).\end{aligned}$$

This proves claim (II.6) and subsequently Proposition II.7.1.

We end this section with the following observation:

Corollary II.7.2
Let $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ and ψ be a Größencharakter of K with rational traces. Consider an unramified prime $\mathfrak{p}$ of $\mathcal{O}_K$ with odd norm. Then

$$e_\mathfrak{p} = e_{\bar{\mathfrak{p}}} \quad \text{and} \quad e_\mathfrak{p} \leq 1.$$

II.8 Reduction step for $p = 2$ and trivial unit group $\mathcal{O}_K^*$

This section complements Proposition II.7.1 by treating the (unramified) primes at 2. This will finish the proof of the reduction step (Prop. II.6.3) for all fields with trivial unit group $\mathcal{O}_K^* = \{\pm 1\}$. In detail, we are going to prove the following

Lemma II.8.1
Let $K \neq \mathbb{Q}(\sqrt{-3})$ and ψ be a Größencharakter of K with rational traces and conductor $\mathfrak{m}$. Assume that 2 does not ramify in K and $(2, \mathfrak{m}) \neq 1$. Let $\mathfrak{m}'$ be the part of $\mathfrak{m}$ which is prime to 2. Then, there is a quadratic twist $\psi \otimes (\chi \circ \mathcal{N})$ with conductor $\mathfrak{m}'$. Here $\chi = \chi_{-1}, \chi_2$ or χ_{-2}.

As in the previous section, the lemma will be proved by establishing the equality

$$\tilde{\eta} = \chi \circ \mathcal{N}. \tag{II.9}$$

The analysis will be slightly more involved since we have to consider three quadratic characters instead of one. Moreover, all groups but possibly one which are involved in the factorization of Ω are 2-groups. Hence we cannot use the cardinality arguments of the previous section. Instead, we will perform some explicit computations. We shall first come to the split case where we use the known shape of $(\mathbb{Z}/2^e)^*$.

II.8.1 2 splits in K

We apply the argumentation developed for the odd primes in Section II.7.2. A careful analysis shows that only the factorization of $(\mathbb{Z}/2^e)^*$ makes a substantial difference. Hence, in the first instance, we derive that $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are conjugate and quadratic, thus coincide. In particular, we thereby obtain $e = e_{\mathfrak{p}} = e_{\bar{\mathfrak{p}}}$.

Then, $(\mathbb{Z}/2)^*$ is trivial, so $e \neq 1$. For $e > 1$, the factorization of Ω_i gives two factors:

$$\Omega_i \cong (\mathbb{Z}/2^e)^* = C_2 \times C_{2^{e-2}}.$$

We read off the three possible quadratic characters χ_{-1} and $\chi_{\pm 2}$ for $\tilde{\eta}_i$. Furthermore, we see that $e \leq 3$.

We conclude as in Section II.7.2 that $\tilde{\eta} = \tilde{\eta}_i \circ \mathcal{N}$. This proves equation (II.9) and thereby Lemma II.8.1 for the split case.

II.8.2 2 stays inert in K

2 stays inert in $\mathcal{O}_K$ if and only if $\Delta_K \equiv 3 \mod 8$. Write $\Delta_K = 4\delta - 1$ for some odd squarefree $\delta \in \mathbb{N}$. As in the split case, we can easily exclude the exponent $e = 1$, since $\mathcal{O}_K/2 \cong \mathbb{F}_4$ with group of units C_3. This does not admit a non-trivial quadratic character. For greater exponents, we will compute generators of the factors of $(\mathcal{O}_K/2^e)^*$. For this purpose, let α denote a root of $x^2 - x + \delta$, so $\alpha^2 = \alpha - \delta$ and $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

The proof of equality (II.9) is based on the constant number of factors of Ω. More precisely, we find generators of the 2-factors of Ω which are independent of the actual exponent. For $e > 1$, a straight inductive argument gives

$$\begin{array}{rcccccccc} (\mathcal{O}_K/2^e)^* & = & C_2 & \times & C_{2^{e-1}} & \times & C_{2^{e-2}} & \times & C_3 \\ & = & <-1> & \times & <1-2\alpha> & \times & <1+4\alpha> & \times & <\alpha^{2^{e-1}}>. \end{array}$$

The first two factors include $\Omega_{\mathbb{Z}} \cong (\mathbb{Z}/2^e)^* =< -1 > \times < 5 >$ as a subgroup of index 2. By cardinality reasons and Lemma II.6.4, the quadratic character $\tilde{\eta}$ can only act non-trivially on the factors generated by $1-2\alpha$ and $1+4\alpha$. This gives the bound

$$e \leq 3.$$

For both generators we have two options in choosing signs. Together these correspond to the four quadratic characters in question (including the trivial one). Let us make this explicit for $e = 2$.

If $e = 2$, then $< 1+4\alpha >$ is trivial, and $\tilde{\eta}$ operates non-trivially on $< 1-2\alpha >$. We claim that

$$\tilde{\eta} = \chi_{-1} \circ \mathcal{N}.$$

This character is well-defined on Ω, since for $a, b \in \mathcal{O}_K$

$$2^e \mid (a-b) \Rightarrow 2^{e+1} \mid (\mathcal{N}(a) - \mathcal{N}(b)). \tag{II.10}$$

Clearly, the characters coincide on the first and last factor of

$$\begin{array}{rcccccc} \Omega \;=\; (\mathcal{O}_K/2^2)^* & = & C_2 & \times & C_2 & \times & C_3 \\ & = & < -1 > & \times & < 1-2\alpha > & \times & < \alpha^2 > . \end{array}$$

For the middle factor, we use that $\mathcal{N}(a+b\alpha) = a^2 + ab + \delta b^2$ for $a, b \in \mathbb{Z}$. Hence $\chi_{-1} \circ \mathcal{N}(1-2\alpha) = -1$, and this proves the claim.

For $e = 3$, i.e. $\tilde{\eta}(1+4\alpha) = -1$, the same computations give $\tilde{\eta} = \chi_2 \circ \mathcal{N}$ or $\tilde{\eta} = \chi_{-2} \circ \mathcal{N}$. This concludes the proof of equation (II.9) and subsequently Lemma II.8.1. Together with Proposition II.7.1, this implies the reduction step (Prop. II.6.3) for all imaginary quadratic fields except for $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. These fields will be considered in the next two sections. Before this, we note the following

Corollary II.8.2
Let $K \neq \mathbb{Q}(\sqrt{-3})$ and ψ be a Größencharakter of K with rational traces. If $\mathfrak{p}$ is an unramified prime of $\mathcal{O}_K$ at 2, then

$$e_{\mathfrak{p}} = e_{\bar{\mathfrak{p}}} \quad \textit{and} \quad e_{\mathfrak{p}} \in \{0, 2, 3\}.$$

II.9 Reduction step for $\mathbb{Q}(\sqrt{-1})$

In this section we will prove the reduction step (Prop. II.6.3) for the field $K = \mathbb{Q}(\sqrt{-1})$. Since K contains the primitive fourth root of unity i, we have to take quartic characters into account. As a motivation, we consider the elliptic curve E with CM by $\mathcal{O}_K$:

$$E: \quad y^2 = x^3 - x.$$

The associated Größencharakter Ψ was given in Example II.5.1 with conductor $\mathfrak{m} = (2+2i)$ and ∞-type $l = 1$.

Quartic twists can be derived from the change of variables

$$x \mapsto d^{-\frac{1}{2}}x, \quad y \mapsto d^{-\frac{3}{4}}y.$$

This gives rise to the twisted elliptic curve

$$E_d: \quad y^2 = x^3 - dx.$$

The associated Größencharakter Ψ_d can be related to Ψ by biquadratic reciprocity. Biquadratic reciprocity is expressed by the fourth-power residue symbol

$$\left(\frac{d}{\cdot}\right)_4 .$$

This character is well-defined on ideals of $\mathcal{O}_K$. It sends a prime $\mathfrak{p}$ of K, which is coprime to $2d$, to the unique fourth root of unity ζ, such that

$$d^{\frac{\mathcal{N}(\mathfrak{p})-1}{4}} \equiv \zeta \mod \mathfrak{p}.$$

Using Jacobi sums, one can show that

Theorem II.9.1 (Ireland, Rosen [I-R, 18, § 4, 6])

$$\Psi_d = \Psi \otimes \left(\frac{d}{\cdot}\right)_4^3.$$

By the classical theory of elliptic curves with CM, this implies the injectivity of $\mathfrak{j}$ for this special case:

Corollary II.9.2
Any two Größencharaktere of K with rational traces and ∞-type 1 are equivalent under twisting.

For general ∞-type, the injectivity of $\mathfrak{j}$ requires to prove the analogous statement. In this section, we prove the reduction step (Prop. II.6.3) for the unramified primes of K. Throughout we employ the usual notation.

Fix $l \in \mathbb{N}$. Let ψ be a Größencharakter of K with rational traces and ∞-type l. Denote the conductor of ψ by $\mathfrak{m}$. Assume that $\mathfrak{p}$ is an unramified prime of K with $(\mathfrak{p}, \mathfrak{m}) \neq 1$. Let $\mathfrak{m}'$ the part of $\mathfrak{m}$ which is prime to p.

We are concerned with the character $\tilde{\eta}$ on Ω as constructed in Section II.6. By Lemma II.4.3, $\tilde{\eta}$ is at most quartic. If $\tilde{\eta}$ is quadratic, we can apply the methods of Section II.7 to deduce the following analogue of Proposition II.7.1:

Lemma II.9.3
In the above notation, assume that $\tilde{\eta}$ is quadratic. Then $\tilde{\eta} = \chi_p \circ \mathcal{N}$, and $\psi \otimes (\chi_p \circ \mathcal{N})$ has conductor $\mathfrak{m}'$.

In general, exactly the same arguments can be used for quartic $\tilde{\eta}$ instead of quadratic. We only have to replace the target group $\{\pm 1\}$ by $< i >$, the actual $\mathcal{O}_K^*$. In the split case, this gives quarticity and conjugacy

$$\tilde{\eta}_1 = \overline{\tilde{\eta}_2} \quad \text{as quartic characters on } (\mathbb{Z}/p^e)^*. \tag{II.11}$$

As a consequence, we obtain the same bounds as in the quadratic case:

Corollary II.9.4
For a Größencharakter of K with rational traces and an unramified prime $\mathfrak{p}$ of K, we have

$$e_{\mathfrak{p}} = e_{\bar{\mathfrak{p}}} \quad \textit{and} \quad e_{\mathfrak{p}} \leq 1.$$

In the inert case, the bound follows from the usual cardinality argument.

Since quadratic twisting has been settled completely in Lemma II.9.3, we shall now assume that $\tilde{\eta}$ has order 4. We want to compare $\tilde{\eta}$ to the quartic character

$$\left(\frac{p^*}{\cdot}\right)_4.$$

Here $p^* = \chi_{-1}(p)p$ is the fundamental discriminant associated to p. We emphasize that this character is well-defined on the ideals of $\mathcal{O}_K$. However, it does not have conductor p in general, contrary to the quartic character $\left(\frac{\cdot}{p}\right)_4$.

To determine the precise conductor, we recall the notion of primary elements of K: $\alpha \in K$ is called *primary* if

$$\alpha \equiv 1 \mod (2+2i).$$

By definition, p^* is primary. We will deduce the conductor from the following relation between $\left(\frac{p^*}{\cdot}\right)_4$ and $\left(\frac{\cdot}{p}\right)_4$:

Theorem II.9.5 (Law of biquadratic reciprocity [I-R, 9, Thm. 2])
Let p be an odd prime. Restricted to primary elements, we have

$$\left(\frac{p^*}{\cdot}\right)_4 = \left(\frac{\cdot}{p}\right)_4.$$

Remark II.9.6
The law of biquadratic reciprocity also holds if we only restrict to elements

$$\alpha \equiv 1 \mod 2.$$

This is because $\left(\frac{-1}{p}\right)_4 = 1$. We shall call these elements semi-primary.

Corollary II.9.7
Let p be an odd prime. Then $\left(\frac{p^*}{\cdot}\right)_4$ has conductor $\begin{cases} p, & \text{if } p \equiv \pm 1 \mod 8,\\ 2p, & \text{if } p \equiv \pm 3 \mod 8.\end{cases}$

Proof: Since $\left(\frac{-1}{p}\right)_4^2 = \chi_p \circ \mathcal{N}$, the conductor is at least p. On the other hand, it is at most $2p$ by Remark II.9.6.

To determine the precise conductor, we shall now consider $\alpha, \beta \in K$ which are coprime to $2p$, such that

$$\alpha \equiv \beta \mod (1+i)p, \quad \text{but} \quad \alpha \not\equiv \beta \mod 2.$$

Let ζ be the unique fourth root of unity, such that $\zeta\alpha$ is primary. Then $i\zeta\beta$ is semi-primary. Theorem II.9.5, combined with Remark II.9.6, gives

$$\left(\frac{p^*}{\alpha}\right)_4 = \left(\frac{p^*}{\beta}\right)_4 \Leftrightarrow \left(\frac{\zeta\alpha}{p}\right)_4 = \left(\frac{i\zeta\beta}{p}\right)_4 \Leftrightarrow \left(\frac{i}{p}\right)_4 = 1 \Leftrightarrow p \equiv \pm 1 \mod 8.$$

Here, the last equivalence is directly derived from the definition of the fourth power residue symbol. If $p \equiv \pm 3 \mod 8$, it follows that the conductor is $2p$. In the other case, we note that

$$\left(\frac{i}{p}\right)_4 = 1 \Leftrightarrow \left(\frac{\cdot}{p}\right)_4 \text{ is defined on ideals} \Leftrightarrow \left(\frac{\cdot}{p}\right)_4 = \left(\frac{p^*}{\cdot}\right)_4.$$

Using this identification, we conclude that $\left(\frac{p^*}{\cdot}\right)_4$ has conductor p if $p \equiv \pm 1 \mod 8$. This completes the proof of Corollary II.9.7.

As a consequence, $\left(\frac{p^*}{\cdot}\right)_4$ is not well-defined on the initial group $G = (\mathcal{O}_K/\mathfrak{m})^*$ if $p \equiv \pm 3 \mod 8$ and $2 \nmid \mathfrak{m}$. We will therefore consider the given Größencharakter ψ only modulo the least common multiple

$$\tilde{\mathfrak{m}} = [2, \mathfrak{m}]. \text{ Define } \tilde{\mathfrak{m}}' = [2, \mathfrak{m}'].$$

Then we start with $\tilde{G} = (\mathcal{O}_K/\tilde{\mathfrak{m}})^*$ and consider the restriction

$$\tilde{\Omega} = \{\alpha \in (\mathcal{O}_K/\tilde{\mathfrak{m}})^*; \alpha \equiv 1 \mod \tilde{\mathfrak{m}}'\}.$$

By Corollary II.9.4, we have $\mathfrak{m} = p\,\mathfrak{m}'$ and $\tilde{\mathfrak{m}} = p\,\tilde{\mathfrak{m}}'$, so $\Omega \cong \tilde{\Omega} \cong (\mathcal{O}_K/p)^*$. Hence we may abuse notation and consider $\tilde{\eta}$ as a character on the modified group $\tilde{\Omega}$. By definition, $\tilde{\Omega}$ only consists of semi-primary elements. Hence, the quartic character $\left(\frac{p^*}{\cdot}\right)_4$ is defined on $\tilde{\Omega}$. Thus we can finally compare the two characters $\tilde{\eta}$ and $\left(\frac{p^*}{\cdot}\right)_4$:

Proposition II.9.8
Let ψ be a Größencharakter of K with rational traces. Let p is be an odd prime with $(p,\mathfrak{m}) \neq 1$. Then there is a twist $\psi \otimes \left(\frac{p^*}{\cdot}\right)_4^j$ modulo $\tilde{\mathfrak{m}}'$ for some $j \in \{1,2,3\}$.

Proof: We have seen that Lemma II.9.3 corresponds to $j = 2$. Hence we can assume $\tilde{\eta}$ to have order 4. We distinguish two cases.

If p stays inert in K, i.e. $p \equiv -1 \mod 4$, $\tilde{\Omega} \cong \mathbb{F}_{p^2}^*$ is cyclic. Here we need that $e_p = 1$ by Corollary II.9.4. Hence, the two quartic characters $\tilde{\eta}$ and $\left(\frac{-p}{\cdot}\right)_4$ on $\tilde{\Omega}$ are either equal or conjugate. This is equivalent to the claim.

If p splits as $p = \mathfrak{p}\bar{\mathfrak{p}}$, we factor $\tilde{\eta} = \tilde{\eta}_1\tilde{\eta}_2$ with respective quartic characters on $\mathcal{O}_K/\mathfrak{p}$ and $\mathcal{O}_K/\bar{\mathfrak{p}}$ in the notation of Section II.7.2. But then, the only quartic characters on $\mathcal{O}_K/\mathfrak{p}$ are

$$\left(\frac{\cdot}{\mathfrak{p}}\right)_4 \quad \text{and} \quad \left(\frac{\cdot}{\mathfrak{p}}\right)_4^3$$

and likewise for $\mathcal{O}_K/\bar{\mathfrak{p}}$ due to cyclicity. We now use that the characters $\tilde{\eta}_1$, $\tilde{\eta}_2$ are conjugate when considered on $\mathbb{Z}/p$ (cf. eq. (II.11)). Since $[\alpha]_2 = [\bar{\alpha}]_1$, this gives

$$\tilde{\eta}_2(\alpha) = \tilde{\eta}_2([\alpha]_2) = \tilde{\eta}_2([\bar{\alpha}]_1) = \overline{\tilde{\eta}_1([\bar{\alpha}]_1)} = \overline{\left(\frac{[\bar{\alpha}]_1}{\mathfrak{p}}\right)_4^j} = \overline{\left(\frac{\bar{\alpha}}{\mathfrak{p}}\right)_4^j} = \left(\frac{\alpha}{\bar{\mathfrak{p}}}\right)_4^j$$

for some $j \in \{1,3\}$. With Remark II.9.6, the law of biquadratic reciprocity (Thm. II.9.5) leads to

$$\tilde{\eta} = \tilde{\eta}_1\tilde{\eta}_2 = \left(\frac{\cdot}{\mathfrak{p}}\right)_4^j \left(\frac{\cdot}{\bar{\mathfrak{p}}}\right)_4^j = \left(\frac{\cdot}{p}\right)_4^j = \left(\frac{p^*}{\cdot}\right)_4^j.$$

This finishes the proof of Proposition II.9.8.

We conclude this section by recalling that Proposition II.9.8 implies the reduction step (Prop. II.6.3) for $K = \mathbb{Q}(\sqrt{-1})$. This is because a biquadratic twist for an unramified prime p does only affect the factors of the conductor at the ramified prime $(1+i)$ and at p. Since the latter disappears, all unramified primes can be successively twisted away.

II.10 Reduction step for $\mathbb{Q}(\sqrt{-3})$

To complete the proof of the reduction step (Prop. II.6.3), there is one more imaginary quadratic field to examine: $K = \mathbb{Q}(\sqrt{-3})$. In this case, $\mathcal{O}_K^*$ consists of the sixth roots of unity. Let us fix one primitive third root of unity and denote it by ϱ.

Working with $\tilde{\eta}$ as usual, the techniques of Sections II.7 and II.8 can directly be applied to the unramified primes. In doing so, we only have to replace "quadratic" by "sextic" and insert the adequate $\mathcal{O}_K^*$. In this way, we obtain completely analogous properties for $\tilde{\eta}$, e.g.:

Corollary II.10.1
For a Größencharakter of K with rational traces and an unramified prime $\mathfrak{p}$ of K, we have

$$e_{\mathfrak{p}} = e_{\bar{\mathfrak{p}}} \quad \textit{and} \quad e_{\mathfrak{p}} \leq \begin{cases} 1, & \textit{if } \mathcal{N}(\mathfrak{p}) \textit{ is odd,} \\ 3, & \textit{if } \mathfrak{p} = (2). \end{cases}$$

Remark II.10.2
We can deal with quadratic and cubic twisting completely separately. The quadratic twisting behaves exactly as in Sections II.7 and II.8. Hence, we can assume $\tilde{\eta}$ to be cubic (or trivial) in the following. Then, cardinality reasoning gives the following sharpening of Corollary II.10.1:

Let $\mathfrak{p}$ be an unramified prime of K. If $\tilde{\eta}$ is cubic, then $e_{\mathfrak{p}} = 1$.

Following this remark, let us assume $\tilde{\eta}$ to be cubic. Recall that $\tilde{\eta}$ operates trivially on $\Omega_{\mathbb{Z}}$ by Lemma II.6.4. We will compare $\tilde{\eta}$ to the cubic residue symbol

$$\left(\frac{p}{\cdot}\right)_3 .$$

This sends a prime $\mathfrak{q}$ of $\mathcal{O}_K$ which is co-prime to $3p$, to the unique third root of unity ζ, such that

$$p^{\frac{\mathcal{N}(\mathfrak{q})-1}{3}} \equiv \zeta \mod \mathfrak{q}.$$

As in the previous section, we have to take special care of the conductor of this cubic character. For this purpose, we introduce the notion of primary elements in K:

Definition II.10.3
We call an element $\alpha \in K$ primary, *if it satisfies the congruence $\alpha \equiv 1 \mod 3$.*

We note that this definition differs from the classical one which requires a congruence to -1. However, for cubic reciprocity, this does not make a difference, since -1 is a third power. Hence we obtain the

Theorem II.10.4 (Law of cubic reciprocity [I-R, 9, Thm. 1])
Let $p \neq 3$ be a prime. Restricted to primary elements, we have the equality

$$\left(\frac{p}{\cdot}\right)_3 = \left(\frac{\cdot}{p}\right)_3 .$$

The law of cubic reciprocity enables us to find the conductor of $\left(\frac{p}{\cdot}\right)_3$:

Corollary II.10.5
Let $p \neq 3$ be a prime. Then $\left(\frac{p}{\cdot}\right)_3$ has conductor $\begin{cases} p, & \text{if } p \equiv \pm 1 \mod 18, \\ 3p, & \text{if } p \equiv \pm 5, \pm 7 \mod 18. \end{cases}$

The proof of this corollary follows exactly the lines of the proof of Corollary II.9.7. In particular, this means that

$$\left(\frac{p}{\cdot}\right)_3 = \left(\frac{\cdot}{p}\right)_3 \Leftrightarrow p \equiv \pm 1 \mod 18.$$

As a consequence, we have to make the same modification as in the previous section. That is, we consider the Größencharakter ψ only modulo the least common multiple

$$\tilde{\mathfrak{m}} = [3, \mathfrak{m}] \quad \text{and let} \quad \tilde{\mathfrak{m}}' = [3, \mathfrak{m}'].$$

Then, the previous argument can be copied word by word (with the third-power residue symbol replacing the fourth-power). This leads to the

Proposition II.10.6
Let ψ be a Größencharakter of K with rational traces. Let $p \neq 3$ be a prime with $(p, \mathfrak{m}) \neq 1$. Then there is a twist $\psi \otimes \Xi$ modulo $\tilde{\mathfrak{m}}'$. Ξ is either a product of a quadratic twist $\chi \circ \mathcal{N}$ and $\left(\frac{p}{\cdot}\right)_3$ or $\left(\frac{p}{\cdot}\right)_3^2$ or it is a single character of these. Here, $\chi = \chi_p$ (or χ_{-1}, χ_{-2} if $p = 2$).

This proposition concludes the proof of the reduction step (Prop. II.6.3). The proof of Theorem II.6.1 now only requires to deal with the ramified primes. Before this will be done in the next section, we briefly note:

Remark II.10.7
Consider the elliptic curve with CM by $\mathcal{O}_K$,

$$E: \ y^2 + y = x^3.$$

This is a very instructive example, since all phenomena discussed above already occur here. The associated Größencharakter Φ has conductor (3) and ∞-type 1 as given in Example II.5.2. The corresponding statement concerning the twisting may be found in [I-R, 18, § 3, 7].

II.11 Ramification step

In this section we will finish the proof of the injectivity of $\mathfrak{j}$ (Cor. II.6.2). For this purpose, we have to show Theorem II.6.1: Every Größencharakter with rational traces can be admissibly twisted to obtain the minimal conductor of Remark II.5.6. In the reduction step (Prop. II.6.3), we eliminated the unramified primes from the conductor. Hence, Theorem II.6.1 will now follow from

Proposition II.11.1 (Ramification step)
Let ψ be a Größencharakter with rational traces and conductor composed of the ramified primes. Then there is an admissible twist of ψ with the minimal conductor of Remark II.5.6.

We will exhibit the proof of the ramification step case by case-wise, as in the proof of the reduction step (Prop. II.6.3). Throughout, we write p for the prime of $\mathbb{Z}$ which ramifies as $p = \mathfrak{p}^2$ in K.

For later use, let us note the following analogue statement to Lemma II.6.4:

Lemma II.11.2
Let ψ be a Größencharakter of K with rational traces and ∞-type l. If $\mathfrak{p}$ is a ramified prime dividing the conductor of ψ, then

$$\tilde{\eta}|_{\Omega_{\mathbb{Z}}} = \chi_K^l.$$

This lemma follows from Corollary II.2.3 in connection with Corollary II.4.5.

II.11.1 Odd p ramifying in $K \neq \mathbb{Q}(\sqrt{-3})$

In this case, $\tilde{\eta}$ is a quadratic character on

$$\Omega = (\mathcal{O}_K/\mathfrak{p}^e)^* = C_{p-1} \times \tilde{\Omega}$$

where $\tilde{\Omega}$ denotes a group of cardinality p^{e-1}. Hence $e \leq 1$. But then,

$$\Omega = (\mathcal{O}_K/\mathfrak{p})^* \cong (\mathbb{Z}/p)^* = \Omega_{\mathbb{Z}}.$$

Thus, η is uniquely determined by Lemma II.11.2. This uniqueness implies the ramification step (Prop. II.11.1) for the case under consideration. As an application, we obtain the

Corollary II.11.3
Let ψ be a Größencharakter of $K \neq \mathbb{Q}(\sqrt{-3})$ with rational traces and ∞-type l. If p is an odd ramifying prime, then

$$e_{\mathfrak{p}} = \begin{cases} 0, & \text{if } l \text{ is even,} \\ 1, & \text{if } l \text{ is odd.} \end{cases}$$

II.11.2 $p = 2$ ramifies in $K \neq \mathbb{Q}(\sqrt{-1})$

Let $K \neq \mathbb{Q}(\sqrt{-1})$. Then 2 ramifies in K if and only if $2 \mid \Delta_K$. We distinguish between two settings: $4 \,||\, \Delta_K$ and $8 \mid \Delta_K$.

$8 \mid \Delta_K$

Write $\Delta_K = 8\delta$ for some odd squarefree $\delta \in \mathbb{N}$. Then $\mathfrak{p} = (2, \sqrt{-2\delta})$ is not principal unless $\delta = 1$. We shall now compute the factorization of $\Omega = (\mathcal{O}_K/\mathfrak{p}^e)^*$ in terms of the exponent $e = e_{\mathfrak{p}}$.

Since $\mathcal{O}_K/\mathfrak{p}$ is trivial, $e = 1$ is ruled out. If $e = 2$ or 3, we calculate

$$(\mathcal{O}_K/\mathfrak{p}^e)^* = C_{2^{e-1}} =< 1 + \sqrt{-2\delta} > .$$

In general, a straight inductive argument gives

$$\Omega = (\mathcal{O}_K/\mathfrak{p}^e)^* = \Omega_{\mathbb{Z}} \times C_{2^\epsilon} =< -1 > \times < 5 > \times < 1 + \sqrt{-2\delta} > .$$

Here, for $e > 3$, $1 + \sqrt{-2\delta}$ has order 2^ϵ with $\epsilon = \lfloor \frac{e}{2} \rfloor$. As a consequence, $\tilde{\eta}$ is determined completely by its operation on $1 + \sqrt{-2\delta}$ due to Lemma II.11.2. We shall now treat the cases of odd and even ∞-type l of ψ separately.

If l is even, then $\tilde{\eta}|_{\Omega_{\mathbb{Z}}} = 1$. Hence, $e = 2$ is the only non-trivial possibility. Then, $\tilde{\eta} = \chi_{-1} \circ \mathcal{N}$ by inspection. Here, $\chi_{-1} \circ \mathcal{N}$ is well-defined on Ω by implication (II.10).

For odd l, Corollary II.4.5 implies $\delta = 1$. Thus, $e \geq 5$ by Remark II.2.4. But then, this is the only possible exponent due to the factorization of $\Omega_{\mathbb{Z}}$. We obtain two Größencharaktere with the same conductor, corresponding to the choice of sign in $\tilde{\eta}(1 + \sqrt{-2\delta})$. This result agrees with Example II.2.5 and proves the ramification step (Prop. II.11.1) for this case.

Corollary II.11.4
Let $8 \mid \Delta_K$ and ψ be a Größencharakter of K with rational traces and ∞-type l. If $\mathfrak{p}$ denotes the prime above 2 in K, then

$$e_{\mathfrak{p}} = \begin{cases} 0 \text{ or } 2, & \text{if } l \text{ is even,} \\ 5, & \text{if } l \text{ is odd.} \end{cases}$$

$4 \,||\, \Delta_K$

Let $\Delta_K = 4\delta$ for some squarefree natural number $\delta \equiv 1 \bmod 4$. The field $\mathbb{Q}(\sqrt{-1})$, i.e. $\delta = 1$, will be treated separately in Section II.11.3. For $\delta > 1$, the ∞-type l of ψ has to be even by Corollary II.4.5. Hence, Corollary II.11.3 implies $\mathfrak{m} = \mathfrak{p}^e$. Therefore, we will work directly with G and η instead of restricting to Ω. Note that $\mathfrak{p} = (2, 1+\sqrt{-\delta})$ is non-principal for $\delta > 1$..

The factorization of G rules out exponent $e = 1$ as before. For $e > 1$, an inductive argument shows that

$$G = G_{\mathbb{Z}}[\sqrt{-\delta}] \times < 1 + 2\sqrt{-\delta} > . \tag{II.12}$$

Here, $G_{\mathbb{Z}}$ has index 2 in $G_{\mathbb{Z}}[\sqrt{-\delta}]$ and the order of $1 + 2\sqrt{-\delta}$ in G is 2^ϵ with $\epsilon = \lfloor \frac{e}{2} \rfloor - 1$.

Because of Lemma II.11.2, η operates trivially on $G_{\mathbb{Z}}$. Hence, it is determined by its operation on $\sqrt{-\delta}$ and $1 + 2\sqrt{-\delta}$. Our next claim is that the second element already suffices. In other words,

$$\eta(1 + 2\sqrt{-\delta}) = 1 \Rightarrow e = 0. \tag{II.13}$$

To see this, let $\eta(1 + 2\sqrt{-\delta}) = 1$. We note that

$$G_{\mathbb{Z}} \times < 1 + 2\sqrt{-\delta} >= \{\alpha \in G; \alpha \equiv 1 \mod 2\}$$

Therefore, our assumption $\eta(1 + 2\sqrt{-\delta}) = 1$ implies $\mathfrak{m} \mid (2)$, i.e. $e \leq 2$ by the definition of a Größencharakter and its conductor. We shall now assume $e = 2$ and establish a contradiction.

Since $(\mathcal{O}_K/\mathfrak{p}^2)^* = C_2 =< \sqrt{-\delta} >$, the assumption $e = 2$ implies $\eta(\sqrt{-1}) = -1$. As a consequence, the operation of ψ on an principal ideal $(a + b\sqrt{-\delta})$ of $\mathcal{O}_K$ with $a, b \in \mathbb{Z}$ can be described as

$$(a + b\sqrt{-\delta}) \mapsto \sigma(a,b)\,(a + b\sqrt{-\delta})^l \quad \text{with} \begin{cases} \sigma(a,b) = 0, & \text{if } 2 \mid (a,b), \\ \sigma(a,b) = 1, & \text{if } 2 \mid b, 2 \nmid a, \\ \sigma(a,b) = -1, & \text{if } 2 \mid a, 2 \nmid b. \end{cases}$$

Here, we only need that $\mathfrak{m} = 2$ and $\eta(\sqrt{-\delta}) = -1$. We shall now derive a contradiction. Consider the following non-principal ideal $\mathfrak{a}$ of K:

$$\mathfrak{a} = \left(\frac{\delta+1}{2}, \frac{\delta-1}{2} + \sqrt{-\delta} \right).$$

This is prime to $\mathfrak{p}$, since $2 \nmid \frac{\delta+1}{2}$. The order of $\mathfrak{a}$ in $Cl(\mathcal{O}_K)$ is two:

$$\mathfrak{a}^2 = \left(\frac{\delta+1}{2}, \frac{\delta-1}{2} + \sqrt{-\delta} \right)^2 = \left(\frac{\delta-1}{2} + \sqrt{-\delta} \right).$$

Since $\frac{\delta-1}{2}$ is divisible by 2, we have $\sigma(\mathfrak{a}^2) = -1$. For the non-principal ideal $\mathfrak{a}$, this leads to

$$\psi(\mathfrak{a}) = \pm\sqrt{-1} \left(\frac{\delta-1}{2} + \sqrt{-\delta} \right)^{\frac{l}{2}}.$$

As $\delta > 1$, $\psi(\mathfrak{a})$ does not have rational trace, giving the required contradiction. This rules out $e = 2$ if $\eta(1 + 2\sqrt{-\delta}) = 1$. Hence, it proves the claim (II.13).

Then, the only non-trivial case consists of $\eta(1+2\sqrt{-\delta}) = -1$. We claim that this corresponds to $e = 4$. This can be seen as follows: On the one hand, this is the first exponent where $1+2\sqrt{-\delta}$ contributes non-trivially to G (cf. eq. (II.12)), so $e \geq 4$. On the other hand, η operates trivially on

$$G_{\mathbb{Z}} \times \{\alpha^2; \alpha \in < 1+2\sqrt{-\delta} >\} \supseteq \{\alpha \in G; \alpha \equiv 1 \mod 4\}.$$

Thus $\mathfrak{m} \mid (4)$, i.e. $e \leq 4$. This proves the claim. We deduce the

Corollary II.11.5
Let $K \neq \mathbb{Q}(\sqrt{-1})$ have discriminant $-\Delta_K \equiv 4 \mod 8$ and ψ be a Größencharakter of K with rational traces. If $\mathfrak{p}$ denotes the prime above 2 in K, then

$$e_{\mathfrak{p}} \in \{0, 4\}.$$

We now finish the proof of the ramification step (Prop. II.11.1) for this case. This will be achieved by proving that (if $e = 4$)

$$\eta = \chi_2 \circ \mathcal{N}. \tag{II.14}$$

From implication (II.10) we derive that $\chi_2 \circ \mathcal{N}$ is well-defined on G. Then we consider the twist by this character. Since $\eta \otimes (\chi_2 \circ \mathcal{N})(1+2\sqrt{-\delta}) = 1$ by construction, this twist has zero exponent due to relation (II.13). In other words, the conductor becomes trivial, and equation (II.14) as well as the ramification step (Prop. II.11.1) follow.

II.11.3 $(1+i)$ as the ramified prime in $\mathbb{Q}(\sqrt{-1})$

Let $K = \mathbb{Q}(\sqrt{-1})$. In this case, the only ramified prime is $\mathfrak{p} = (1+i)$, such that $\mathfrak{m} = \mathfrak{p}^e$ by assumption. We consider $G = (\mathcal{O}_K/\mathfrak{m})^*$ and the quartic character η on G. As usual, $e \neq 1$, since otherwise G would be trivial. For $e > 1$, we obtain the factorization of G from equation (II.12) with $\epsilon = \lfloor \frac{e}{2} \rfloor - 1$:

$$G = (\mathcal{O}_K/\mathfrak{p}^e)^* = G_{\mathbb{Z}}[i] \times C_{2^\epsilon} = < 5 > \times < i > \times < 1+2i > .$$

The action of the character η on $G_{\mathbb{Z}}[i]$ is known a priori. For $G_{\mathbb{Z}}$, this is immediate from Lemma II.11.2. For the units $\mathcal{O}_K^* = < i >$, it follows from the fact, that ψ operates on ideals:

$$\eta|_{G_{\mathbb{Z}}} = \chi_{-1}^l \quad \text{and} \quad \eta|_{\mathcal{O}_K^*} = (\alpha \mapsto \alpha^{-l}).$$

As a consequence, η is determined by its operation on $1+2i$. In particular, it is a priori unique if $e \leq 3$. This exponent corresponds to the minimal conductor of Example II.5.1.

Remark II.11.6
In Section II.9, we had to consider the Größencharakter modulo a subideal in order to compare $\tilde{\eta}$ to the fourth power residue symbol. We want to show that this modification does not cause any ambiguity. We replaced the conductor $\mathfrak{m}$ by the least common multiple $\tilde{\mathfrak{m}} = [2, \mathfrak{m}]$. Then, the given twist was defined modulo $\tilde{\mathfrak{m}}' = [2, \mathfrak{m}']$. Hence, if $\tilde{\mathfrak{m}}' \neq \mathfrak{m}'$, then $e_{(1+i)} \leq 2$ for the twist. In other words, if the twisting of the unramified primes requires this modification at any stage, then $e \leq 2$ in the end. Now, the above uniqueness applies. (In particular, we deduce from Remark II.2.4 that no modifications are necessary if l is odd.)

We now investigate the exponents $e > 3$. If $\eta(1+2i) = -1$, i.e. $e = 4$, we twist by $\chi_2 \circ \mathcal{N}$ exactly as in the previous section (cf. eq. (II.14)).

The only other possibilities are $\eta(1+2i) = \pm i$. These correspond to $e = 6$ since then the order of $1+2i$ in G is four. We want to compare η to the character $\left(\frac{2}{\cdot}\right)_4$ which maps $1+2i$ to i. For this purpose, we need that $\left(\frac{2}{\cdot}\right)_4$ is well-defined on G:

Lemma II.11.7
$\left(\frac{2}{\cdot}\right)_4$ has conductor 8.

Proof: Because of the factorization of G, the conductor is at least 8. The fact that this is also an upper bound, relies on the following

Theorem II.11.8 (Dirichlet [Co, Thm. 4.23])
Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ which is co-prime to 2. Write $\mathfrak{a} = (a + 2bi)$ with $a, b \in \mathbb{Z}$. Then

$$\left(\frac{2}{\mathfrak{a}}\right)_4 = i^{ab}.$$

From this theorem, it is elementary to deduce the following: Let $\alpha, \beta \in \mathcal{O}_K$ with odd norm. Then $\left(\frac{2}{\alpha}\right)_4 = \left(\frac{2}{\beta}\right)_4$ if $\alpha \equiv \beta \mod 8$. Hence, $\left(\frac{2}{\cdot}\right)_4$ is defined modulo 8. This finishes the proof of Lemma II.11.7.

As a result, if $e = 6$, we can twist ψ by $\left(\frac{2}{\cdot}\right)_4$ (which operates trivially on $G_{\mathbb{Z}[i]}$). One of the twists $\eta \otimes \left(\frac{2}{\cdot}\right)_4$ or $\eta \otimes \left(\frac{2}{\cdot}\right)_4^3$ acts trivially on $1+2i$. Hence, it has minimal conductor by our above considerations. This concludes the proof of the ramification step (Prop. II.11.1) for $K = \mathbb{Q}(\sqrt{-1})$.

We shall now collect all possible exponents e. For $e > 3$, these are immediate from the above analysis. For $e \leq 3$, we first of all use the description of the minimal conductors in Example II.5.1. However, we also have to take the extra effect of biquadratic twisting into account. By Remark II.2.4, this can only occur for even ∞-type:

Corollary II.11.9
Let ψ be a Größencharakter of $\mathbb{Q}(\sqrt{-1})$ with rational traces and ∞-type l. Then

$$e_{(1+i)} \in \begin{cases} \{0,2,4,6\}, & \text{if } 2 \mid l, \\ \{3,4,6\}, & \text{if } 2 \nmid l. \end{cases}$$

II.11.4 $(\sqrt{-3})$ as the ramified prime in $\mathbb{Q}(\sqrt{-3})$

In this subsection we let $K = \mathbb{Q}(\sqrt{-3})$ with ramified prime $\mathfrak{p} = (\sqrt{-3})$. Hence, the conductor is $\mathfrak{m} = \mathfrak{p}^e$ and η is a sextic character on $G = (\mathcal{O}_K/\mathfrak{p}^e)^*$. Again this is pre-determined on $G_{\mathbb{Z}[\varrho]}$ by Lemma II.11.2 and the fact that ψ is defined on ideals. Here, ϱ denotes a primitive third root of unity. The first quotients are

$$(\mathcal{O}_K/\mathfrak{p})^* \cong \mathbb{F}_2^* = <2> \text{ and } (\mathcal{O}_K/\mathfrak{p}^2)^* \cong \mathcal{O}_K^* = <2> \times <\varrho>.$$

Since on these factors η is pre-determined, a Größencharakter of K as in Proposition II.11.1 with exponent $e \leq 2$ is unique (for fixed ∞-type). Hence it has to be the minimal Größencharakter of Example II.5.2.

Remark II.11.10
Recall the modification of the conductor, which was exhibited during the cubic twisting of the unramified primes in Section II.10. As in Remark II.11.6, we conclude that this modification does not cause any ambiguity. This follows from the uniqueness of the resulting Größencharakter for $e \leq 2$.

For general exponent $e > 2$, a straight inductive argument leads to the factorization

$$\begin{array}{rcccc} (\mathcal{O}_K/\mathfrak{p}^e)^* & = & G_{\mathbb{Z}}[\varrho] & \times & C_{3^\epsilon} \\ & = & <2> \times <\varrho> & \times & <1+3\sqrt{-3}>. \end{array}$$

Here, $\epsilon = \lfloor \frac{e}{2} \rfloor - 1$. Since η has order at most six, we have to consider a non-trivial contribution of the last factor. So let us assume that $\eta(1+3\sqrt{-3}) \neq 1$. This is equivalent to $e = 4$. We shall now twist η with the cubic character $\left(\frac{3}{\cdot}\right)_3$ or its conjugate. For this purpose, we need the

Lemma II.11.11
$\left(\frac{3}{\cdot}\right)_3$ *has conductor 9.*

This is a direct consequence of:

Theorem II.11.12 (Eisenstein [I-R, 9, Ex. 5])
Let $\alpha = a + 3b\sqrt{-3}$ be primary in $\mathcal{O}_K$. Then

$$\left(\frac{3}{\alpha}\right)_3 = \varrho^{-b}.$$

Because of Lemma II.11.11, we can twist ψ by $\left(\frac{3}{\cdot}\right)_3$ or $\left(\frac{3}{\cdot}\right)_3^2$. Since

$$\left(\frac{3}{1+3\sqrt{-3}}\right)_3 = \varrho^{-1},$$

one of the twists operates trivially on $1+3\sqrt{-3}$. This leads to exponent $e \leq 2$ and therefore minimal conductor. This completes the proof of the ramification step (Prop. II.11.1).

Finally, we collect the generally possible exponents $e_{(\sqrt{-3})}$. Here, we also have to consider the special effect of cubic twisting on the conductor.

Corollary II.11.13
Let ψ be a Größencharakter of $\mathbb{Q}(\sqrt{-3})$ with rational traces and ∞-type l. Then

$$e_{(\sqrt{-3})} \in \begin{cases} \{0,2,4\}, & \text{if } 2 \mid l, \\ \{1,2,4\}, & \text{if } 2 \nmid l. \end{cases}$$

II.12 Conclusion

In the preceding section, a detailed analysis has shown that every Größencharakter with rational traces whose conductor is composed of the ramified primes, can be twisted to obtain the minimal conductor. This proves the ramification step (Prop. II.11.1). Together with the reduction step (Prop. II.6.3), this enables us to deduce the corresponding statement for general Größencharaktere with rational traces, independent of the conductor (Theorem II.6.1).

On the other hand, for a fixed field and ∞-type, all minimal Größencharaktere are equivalent under twisting by Lemma II.5.7. Hence, Theorem II.6.1 suffices to prove the injectivity of $\mathfrak{j}$ (Cor. II.6.2). This settles the remaining part of the proof of Theorem II.3.4: Größencharaktere with rational traces and fixed ∞-type l are, up to twisting, in 1:1-correspondence to imaginary quadratic fields K with exponent $e_K \mid l$.

In particular, this implies, subject to GRH, that there are only finitely many CM-newforms with rational coefficients and fixed weight up to twisting (Thm. II.3.1). We will spend the next two sections with an investigation of the cases of weight 3 and 4 where this holds unconditionally. For weight 2, the corresponding result can be derived from [Si, App. A, §3]. The associated Größencharaktere can be found in Examples II.2.5, II.5.1 and II.5.2.

The main content of this section is the following table. For a Größencharakter with rational traces, the table lists the possible exponents $e_\mathfrak{p}$ of its conductor at a prime $\mathfrak{p}$. The values are collected from Corollaries II.7.2, II.8.2, II.9.4, II.10.1, II.11.3, II.11.4, II.11.5, II.11.9, II.11.13.

Proposition II.12.1
Let K be an imaginary quadratic field. Let ψ be a Größencharakter of K with rational traces and ∞-type l. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$. Denote the exponent of the conductor of ψ at $\mathfrak{p}$ by $e_\mathfrak{p}$. Then $e_\mathfrak{p} = e_{\bar{\mathfrak{p}}}$ with the following possible values:

prime	*$\mathfrak{p}$ prime to 2*			*$\mathfrak{p}$ above 2*				
ramification	*ramified*		*unram.*	$4 \Vert \Delta_K$		$8 \mid \Delta_K$		*unram.*
∞-type	*odd*	*even*	*arb.*	*odd*	*even*	*odd*	*even*	*arb.*
$\Delta_K \neq 3, 4$	*1*	*0*	*0,1*	-	*0,4*	*5*	*0,2*	*0,2,3*
$\Delta_K = 4$	-	-	*0,1*	*3,4,6*	*0,2,4,6*	-	-	-
$\Delta_K = 3$	*1,2,4*	*0,2,4*	*0,1*	-	-	-	-	*0,1,2,3*

In terms of these exponents, we can also give the corresponding statement for the level of the associated newforms. These possible values can be compared to results of Serre [Se, (4.8.8) & Application à (4.8.8) b)]. He gives the maximal exponents for the conductor $N = \prod p^{e_p}$ of a modular integral two-dimensional Galois representation of the middle cohomology group of an odd-dimensional projective variety over $\mathbb{Q}$:

$$\begin{aligned} e_p &\leq 2, \text{ if } p > 3, \\ e_p &\leq 5, \text{ if } p = 3, \\ e_p &\leq 8, \text{ if } p = 2. \end{aligned}$$

Note that these bounds only apply to even weight. Meanwhile, our results only hold for CM-forms, but of arbitrary weight.

Corollary II.12.2
In arbitrary weight, Serre's bounds on the exponents of the level hold for CM-newforms with rational coefficients.

II.13 The CM-newforms of weight 3

In this section we will give all CM-newforms of weight 3 with rational Fourier coefficients up to twisting (except for possibly one). These constitute exactly the modular forms which can a priori appear as non-trivial factor $L(T_X, s)$ in the L-series of singular K3 surfaces over $\mathbb{Q}$ (cf. Thm. I.4.2). Hence this section answers the question which was our original motivation (cf. Qu. I.5.3). In particular, we deduce the

Proposition II.13.1
Consider the following classifications of singular K3 surfaces over $\mathbb{Q}$:

(i) by the discriminant d of the surface up to squares,

(ii) by the associated newform up to twisting,

(iii) by the level of the associated newform up to squares,

(iv) by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.

Then, all these classifications are equivalent. In particular, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

Proof: The classifications by (ii) and (iv) coincide due to Theorem II.3.4 which also gives the extra claim. With the help of this theorem, the equivalence of (ii) and (iv) with (i) follows from Theorem I.4.2. Finally, (iii)⇔(iv) can be read off from Proposition II.12.1. This completes the proof of Proposition II.13.1.

We shall now come explicitly to the newforms of weight 3. They correspond to Größencharaktere of ∞-type 2. By Theorem II.3.4 (or the above lemma), we are therefore concerned with those imaginary quadratic fields whose class group consists only of 2-torsion. In other words, every genus consists of a single class. These fields are related to Euler's *convenient numbers* (or *idoneal*) and were also studied by Gauss.

In 1934, Chowla [Ch] proved that the number of imaginary quadratic fields with exponent 1 or 2 is finite. In 1973, Weinberger [W, Thm. 2] showed that the list of known fields was almost complete. More precisely, he stated that there was at most one imaginary quadratic field with exponent 2 and negative discriminant $\Delta_K > 5460$. He showed that this would have $\Delta_K > 2 \cdot 10^{11}$. Subject to some condition on Dirichlet L-functions, which in particular would be a consequence of GRH, he proved that the known fields with $\Delta_K \leq 5460$ give the complete list.

In this section, we will give a newform $f = \sum_n a_n q^n$ of weight 3 with rational Fourier coefficients and CM by K for every such field K. As explained, the list is complete except for possibly one more newform (of level at least $2 \cdot 10^{11}$). The Fourier coefficients were calculated by a straight forward computer program. We used the following normalization to choose one particular newform out of the 2^{g-1} twisted minimal ones of Lemma II.5.7: If $g > 1$, order the primes ramifying in K as $0 < p_1 < \ldots < p_g$. Then the table lists the newform with $a_{p_g} = -p$ while $a_{p_j} = p$ for all $j < g$.

A further comment concerns the question of minimal conductor resp. level as discussed in Remark II.5.6. The table lists the CM-form of weight 3 with minimal 3-power level, which is 27. As explained in the remark, there is also a weight 3 newform with CM by $\mathbb{Q}(\sqrt{-3})$ and level 12. The corresponding Größencharakter of conductor (2) is defined by

$$(\alpha) \mapsto \left(\frac{\alpha}{2}\right)_3 \alpha^2. \tag{II.15}$$

Here, we see the effect of cubic twisting on the conductor.

In the following table, the first column lists the level N of the respective CM-newforms. By Proposition II.13.1, such a newform has CM by $\mathbb{Q}(\sqrt{-N})$. This also gives its nebentypus. We then list the Fourier coefficients at the first 30 primes.

N	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53
7	-3	0	0	-7	-6	0	0	0	18	-54	0	-38	0	58	0	-6
8	-2	-2	0	0	14	0	2	-34	0	0	0	0	-46	14	0	0
11	0	-5	-1	0	-11	0	0	0	35	0	-37	-25	0	0	50	-70
15	-1	3	-5	0	0	0	14	-22	-34	0	2	0	0	0	14	86
16	0	0	-6	0	0	10	-30	0	0	42	0	-70	18	0	0	90
19	0	0	-9	-5	3	0	15	-19	-30	0	0	0	0	-85	75	0
20	2	-4	-5	4	0	0	0	0	-44	-22	0	0	62	76	4	0
24	2	-3	-2	-10	10	0	0	0	0	-50	38	0	0	0	0	94
27	0	0	0	-13	0	-1	0	11	0	0	-46	47	0	-22	0	0
35	0	-1	5	-7	-13	19	-29	0	0	23	0	0	0	0	31	0
40	2	0	-5	-6	-18	6	0	-2	26	0	0	54	-78	0	-86	-74
43	0	0	0	0	-21	-17	-9	0	3	0	19	0	39	-43	-78	63
51	0	3	-7	0	5	-25	-17	-13	29	-10	0	0	65	35	0	0
52	2	0	0	-12	-4	-13	-18	12	0	6	36	0	0	0	68	-102
67	0	0	0	0	0	0	-33	-29	-21	-9	0	7	0	0	27	0
84	2	3	-4	-7	-20	0	20	10	4	0	-50	-10	68	0	0	0
88	2	0	0	0	-11	-18	0	-6	-42	14	-26	0	0	42	6	0
91	0	0	3	-7	0	13	0	-25	-45	-33	55	0	-30	-5	-81	15
115	0	0	5	-9	0	0	11	0	-23	-57	-53	51	-33	-6	0	-101
120	2	3	-5	0	2	-14	-26	0	-14	38	-58	34	0	-74	34	0
123	0	3	0	0	-19	0	-7	0	0	17	-61	-49	-41	-37	53	-58
132	2	3	0	-8	-11	0	-32	16	2	-8	0	-58	16	64	-82	0
148	2	0	0	0	0	0	0	-36	-28	0	-12	-37	-66	12	0	-42
163	0	0	0	0	0	0	0	0	0	0	0	0	-81	-77	-69	-57
168	2	3	0	-7	0	-2	-22	0	-38	-26	34	0	26	-82	0	22
187	0	0	0	-3	11	0	-17	0	0	41	0	0	-71	0	-93	-81
195	0	3	5	1	-17	-13	-31	0	-19	0	0	61	43	0	0	41
228	2	3	0	0	-16	0	0	-19	8	-56	-14	0	-32	0	56	-8
232	2	0	0	0	0	0	0	0	0	-29	-54	-42	0	0	-22	0
235	0	0	5	0	0	-21	0	0	-1	0	0	0	0	39	-47	0
267	0	3	0	0	0	0	0	0	-43	-31	0	0	-7	0	0	0
280	2	0	5	-7	0	0	6	-18	0	0	0	-66	0	-54	66	-34
312	2	3	0	0	0	-13	0	-14	0	-46	0	22	-74	0	-62	2
340	2	0	5	0	-12	0	-17	0	0	0	28	6	0	-84	-76	0
372	2	3	0	0	0	0	-28	0	0	-4	-31	0	0	-38	-92	44
403	0	0	0	0	-9	13	0	0	0	0	-31	43	0	0	0	0
408	2	3	0	0	0	0	-17	0	-22	0	0	-62	14	0	0	-98
420	2	3	5	-7	8	-16	0	-32	0	0	-8	0	-2	-26	-46	-104
427	0	0	0	7	0	0	-27	0	0	0	1	0	0	0	0	0
435	0	3	5	0	-7	0	0	0	-41	-29	0	-71	53	-59	0	19
483	0	3	0	7	-1	0	0	-31	-23	0	0	0	-79	0	-67	83
520	2	0	5	0	0	-13	0	0	-6	0	-42	0	0	0	0	54
532	2	0	0	7	0	-12	0	-19	0	0	0	0	44	0	18	0
555	0	3	5	0	0	-11	0	0	0	-53	0	-37	0	49	-91	-79
595	0	0	5	7	0	9	-17	0	-39	0	-57	-11	-37	0	-59	0
627	0	3	0	0	11	7	-23	-19	0	0	0	0	0	0	0	-103
660	2	3	5	0	-11	4	0	-28	0	-52	0	0	-28	0	0	-26
708	2	3	0	0	0	0	0	0	0	0	-56	0	0	-32	0	0
715	0	0	5	0	11	-13	21	-27	0	0	0	-69	17	-31	-49	0
760	2	0	5	0	0	0	0	-19	0	-18	0	0	0	-66	0	0
795	0	3	5	0	0	0	-19	0	0	0	0	0	-77	0	41	-53
840	2	3	5	-7	0	0	0	0	0	2	-22	46	-58	0	-74	0
1012	2	0	0	0	11	0	-12	0	-23	0	0	0	0	0	0	0
1092	2	3	0	7	0	-13	8	0	-32	0	0	0	0	0	-88	0
1155	0	3	5	7	-11	0	1	-17	-31	47	0	0	0	-79	0	29
1320	2	3	5	0	-11	0	0	0	0	0	0	-14	38	-46	0	0
1380	2	3	5	0	0	0	0	-8	-23	0	0	-64	0	0	0	0
1428	2	3	0	7	0	0	-17	4	0	-44	0	0	0	0	0	0
1435	0	0	5	7	0	0	0	-3	0	0	0	0	-41	0	0	-99
1540	2	0	5	7	-11	0	0	0	24	0	-48	0	-72	0	0	0
1848	2	3	0	7	-11	0	0	0	0	0	0	0	0	-2	-38	62
1995	0	3	5	7	0	0	0	-19	-11	-37	43	-59	0	0	0	0
3003	0	3	0	7	11	-13	0	0	0	19	-29	0	-61	0	0	0
3315	0	3	5	0	0	13	-17	0	0	7	-23	0	0	0	0	89
5460	2	3	5	7	0	-13	0	0	0	0	0	-4	0	-44	0	-76

Table II.1: The CM-newforms of weight 3 with rational Fourier coefficients

59	61	67	71	73	79	83	89	97	101	103	107	109	113
0	0	-118	114	0	-94	0	0	0	0	0	186	106	-222
-82	0	62	0	-142	0	158	146	-94	0	0	-178	0	98
107	0	35	-133	0	0	0	-97	95	0	-190	0	0	215
0	-118	0	0	0	98	-154	0	0	0	0	-106	-22	206
0	-22	0	0	-110	0	0	-78	130	-198	0	0	-182	-30
0	103	0	0	-25	0	90	0	0	-102	0	0	0	0
0	-58	-116	0	0	0	76	-142	0	122	-44	124	38	0
10	0	0	0	50	-58	-134	0	-190	190	-10	-86	0	0
0	-121	-109	0	-97	131	0	0	167	0	-37	0	-214	0
0	0	0	2	34	-157	-86	0	-149	0	199	0	-97	0
78	0	0	0	0	0	0	18	0	0	186	0	0	0
-54	0	91	0	0	-14	123	0	-193	159	-181	42	-169	0
0	0	-70	-130	0	0	0	0	0	0	155	-211	0	-199
-116	-86	108	-92	0	0	-68	0	0	-6	0	0	0	174
51	0	-67	-126	79	0	-102	111	0	0	-62	147	0	0
0	0	0	100	0	0	0	-172	0	-148	94	-164	-118	0
0	78	0	54	0	0	122	-174	-158	-194	118	-182	174	-126
90	0	0	0	-29	67	159	-165	131	0	0	-150	0	135
3	0	111	27	0	0	-41	0	-174	87	114	191	0	19
98	0	-26	0	0	38	0	0	0	182	0	0	0	166
0	-1	0	101	23	0	0	14	0	161	83	0	0	0
74	0	0	-34	0	136	0	0	62	136	0	0	0	0
44	0	0	0	-2	84	0	0	0	54	132	0	0	0
0	-41	0	-21	0	0	3	0	31	0	0	0	0	63
-106	94	-34	58	0	0	-58	122	0	0	178	0	0	0
-69	54	-53	0	129	-114	0	-9	0	0	19	61	-207	0
-38	-73	-74	103	94	-37	0	-173	181	0	0	149	0	-34
0	-106	58	0	-82	-146	128	64	0	0	-98	0	0	112
-114	6	-98	0	0	42	-66	0	0	86	0	-18	0	0
-117	-113	-54	-93	99	-77	0	-57	0	-33	0	-209	0	179
29	0	-133	0	-121	-109	77	-89	-73	-154	0	0	-49	137
62	-102	-6	-138	-106	-122	0	0	166	-22	-46	74	0	0
0	0	82	-14	0	-154	0	22	0	98	-106	-202	166	0
0	0	-36	108	-126	-148	-4	-162	126	-138	36	0	0	-46
-68	0	0	-44	0	34	0	116	-178	0	0	28	-154	0
0	0	0	0	-133	0	42	147	0	-201	-197	-189	0	-177
-86	-14	0	74	0	0	-38	0	0	-2	-202	0	82	158
0	0	106	128	104	0	26	-158	-184	118	0	0	-202	16
57	-61	0	0	0	0	0	-66	0	141	0	-213	-209	-201
0	0	0	0	1	0	79	62	49	173	0	-134	-217	0
-43	53	0	0	0	0	0	0	-82	41	137	122	0	-142
0	0	-126	38	-114	0	-94	0	-66	0	154	0	-198	0
0	0	-132	-124	0	-108	-138	-164	156	0	0	-52	0	0
7	0	0	0	0	0	-19	67	46	0	169	29	0	0
0	3	0	0	0	0	149	0	0	0	-66	129	0	-114
-91	0	0	-67	0	139	109	-31	0	-26	0	0	-86	17
-58	0	-86	98	124	92	-164	0	0	92	-14	-116	0	94
-59	0	16	-94	0	0	0	-176	0	-152	88	-22	0	-128
0	0	-9	0	0	0	0	0	51	0	0	6	-42	0
42	0	0	0	0	0	14	0	-186	0	-174	0	142	-154
0	0	-131	-17	-119	0	0	0	0	43	-59	161	0	14
-22	38	0	-82	-134	0	0	38	-86	0	0	-206	0	-194
0	76	42	0	0	0	0	0	0	0	-162	0	-196	0
-64	0	-22	0	-62	0	-16	0	142	176	-158	136	0	0
-113	67	0	0	0	0	-131	-53	-191	0	-179	0	0	149
0	-98	0	-122	14	-62	0	0	0	0	-146	0	-2	-214
-112	0	0	-88	0	112	74	-98	56	0	0	-154	0	0
0	-116	0	0	-92	-46	0	-94	-44	134	172	0	0	124
0	0	-71	0	-141	0	-121	137	0	-167	-81	0	0	0
8	-32	-64	0	0	-18	-54	0	-114	48	0	0	0	0
0	0	0	0	-118	0	-142	46	0	-106	0	0	-134	0
0	0	1	47	-139	0	0	0	0	-197	0	0	0	0
0	109	0	-131	0	0	23	0	103	0	0	-137	-211	0
-103	0	-121	0	0	0	0	-43	0	0	0	0	133	0
92	0	0	0	0	0	0	0	0	-188	0	0	0	44

Table II.1: The CM-newforms of weight 3 with rational Fourier coefficients

II.14 The CM-newforms of weight 4

This section is concerned with CM-newforms of weight 4 with rational Fourier coefficients. By Theorem II.1.1, these forms correspond to Größencharaktere of quadratic imaginary fields with rational traces and ∞-type 3. Hence, Theorem II.3.4 shows that we can restrict our attention to those imaginary quadratic fields whose class group consists only of 3-torsion. The number of these fields is known to be finite. This goes back to Boyd and Kisilevski [B-K], but can be also be found in Weinberger's article [W].

Unfortunately, we have not found an explicit list of imaginary quadratic fields with exponent 3. Hence, we searched for them with a straight forward computer program. Among the imaginary quadratic fields with discriminant down to -100.000, we found 17 such fields. The lowest discriminant of them is -4027. (This is also the only field among them with class group different from $\mathbb{Z}/3$: $Cl(\mathbb{Q}(\sqrt{-4027})) \cong \mathbb{Z}/3\times\mathbb{Z}/3$, as already computed by A. Scholz and O. Taussky in 1934 [S-T].) We expect this list to be complete, but do not know a proof of this statement.

For each of these imaginary quadratic fields K with exponent 3 and those of class number 1, the following table lists the unique newform with CM by K, rational Fourier coefficients and minimal level N. To be precise, the uniqueness only holds with the exception of $\mathbb{Q}(\sqrt{-2})$ where the twist by χ_{-1} has the same level; this has been discussed in Example II.2.5 and Section II.11.2. In the table, the level constitutes the first column and is expressed as $N = M\Delta_K^2$ with Δ_K the negative discriminant of K as usual. Then the Fourier coefficients at the first 25 primes follow.

II.15 Outlook

Finally we want to return to Question I.5.3 which it all started with. Or in fact, we turn it around.

Originally, we wanted to understand the modularity of singular K3 surfaces in terms of the associated newforms of weight 3. This has been addressed by the explicit table in Section II.13. The natural question now is which of these newforms actually occur in the L-series of some K3 surface over $\mathbb{Q}$ (Qu. I.5.4). This question will be addressed in the next chapter.

Meanwhile, we shall now formulate this problem in general form:

Question II.15.1
Which CM-newforms with rational coefficients (and weight k) have geometric realizations, say in a projective variety X over $\mathbb{Q}$ with $h^{k-1,0}(X) = 1$?

In other words we ask whether the Galois representation associated to a newform occurs as 2-dimensional motive of some projective variety which is defined over $\mathbb{Q}$. Here we tacitly only consider varieties different from the classical $(k-2)$-fold fibre product of the universal elliptic curve with level N-structure which was fundamental to Deligne's construction of the associated Galois representation for a newform of weight k and level N. (In general, these fibre products do not meet the given criterion.)

What makes our situation special, is that we have very precise associated geometric objects, different from Deligne's fibre products. Namely, these objects are the

N	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97
3^2	0	0	0	20	0	-70	0	56	0	0	308	110	0	-520	0	0	0	182	-880	0	1190	884	0	0	-1330
$2\,4^2$	0	0	22	0	0	-18	-94	0	0	-130	0	214	-230	0	0	518	0	830	0	0	1098	0	0	-1670	594
7^2	-5	0	0	0	-68	0	0	0	-40	-166	0	450	0	-180	0	590	0	0	-740	688	0	-1384	0	0	0
$4\,8^2$	0	10	0	0	-18	0	90	106	0	0	0	0	-522	-290	0	0	846	0	-70	0	430	0	-1350	-1026	-1910
11^2	0	8	18	0	0	0	0	0	-108	0	340	-434	0	0	-36	-738	-720	0	-416	612	0	0	0	1674	-34
19^2	0	0	14	-36	40	0	14	0	212	0	0	0	0	128	364	0	0	630	0	0	1078	0	-112	0	0
23^2	3	4	0	0	0	-74	0	0	0	282	-344	0	426	0	48	0	708	0	0	1176	-1226	0	0	0	0
31^2	1	0	2	16	0	0	0	156	0	0	0	0	-278	0	-616	0	-740	0	-804	1000	0	0	0	0	-1906
43^2	0	0	0	0	32	-90	-130	0	-140	0	108	0	22	0	500	-130	-904	0	-360	0	0	-1116	-680	0	290
59^2	0	-7	-21	-29	0	0	126	-119	0	-159	0	0	-525	0	0	327	0	0	0	-1065	0	-835	0	0	0
67^2	0	0	0	0	0	0	50	-144	-220	-266	0	270	0	0	-220	0	-104	0	0	788	-90	0	-1480	-374	0
83^2	0	5	0	-25	57	0	-75	0	180	309	-317	-335	378	0	0	0	471	-943	0	0	0	0	0	0	0
107^2	0	-1	0	0	-9	-11	0	-137	189	234	0	353	279	0	540	-711	0	911	0	0	0	-1127	1245	-1467	0
139^2	0	0	19	-11	43	93	0	0	0	-155	-345	-306	490	0	644	0	0	0	-29	575	0	-1091	1433	-1549	0
163^2	0	0	0	0	0	0	0	0	0	0	0	0	122	-360	-580	-770	0	918	0	-1012	0	0	-1040	0	990
211^2	0	0	17	0	7	-83	0	165	0	0	0	-331	0	507	461	158	760	0	0	425	1242	-1285	1408	0	0
283^2	0	0	0	33	-71	79	0	0	-29	223	0	0	-61	0	0	0	-163	691	0	212	-630	0	1400	1423	537
307^2	0	0	0	12	-64	0	2	128	0	0	0	398	-506	0	0	34	0	0	0	-1196	0	-684	-1120	1526	1890
331^2	0	0	13	0	0	0	-139	-25	0	0	333	0	0	277	0	-707	0	0	-639	-1013	0	1141	248	1210	0
379^2	0	0	11	0	0	0	0	155	53	0	0	429	-197	0	0	0	0	-907	889	0	0	609	-1403	0	-846
499^2	0	0	-1	0	0	0	0	0	0	115	83	0	0	313	389	0	0	0	-1029	0	1173	0	0	0	0
547^2	0	0	0	0	-56	-6	0	88	0	298	0	0	0	0	-388	766	0	0	-488	0	1114	0	0	0	-1758
643^2	0	0	0	27	0	0	0	0	-131	-257	259	0	0	0	0	751	0	0	0	0	0	0	931	-1183	-1897
883^2	0	0	0	0	0	-29	-137	0	0	-233	-111	0	0	0	0	0	0	0	1047	-1151	0	147	809	0	0
907^2	0	0	0	0	0	-25	0	69	65	0	0	0	-503	0	0	-665	0	0	0	1187	0	0	0	1501	0
4027^2	0	0	0	0	0	-69	125	-153	0	-182	0	0	134	416	0	-329	0	-43	-1095	1111	1109	443	-1511	-1649	1842

Table II.2: The CM-newforms of weight 4 with rational Fourier coefficients

elliptic curves with CM by the respective field K. By the classical theory of CM, this gives a positive answer to our question for weight $k = 2$. For higher weight, it is probably most natural to ask for geometric realizations in Calabi-Yau varieties over $\mathbb{Q}$. For weight 3, i.e. dimension 2, this leads exactly to the singular K3 surfaces we started with. In this respect, they are classified in terms of their discriminants up to squares (cf. Prop. II.13.1). We will find a number of such surfaces in the next chapter.

Let us now consider newforms with rational coefficients and CM of weight 4. Here we ask for a corresponding (preferably rigid) Calabi-Yau threefold over $\mathbb{Q}$. The occurrence of the level 9-form from Table II.2 was established by Werner with assistance of van Geemen [WvG]. The other confirmed forms of level 32, 49, and 256 have only recently appeared in a preprint of Cynk and Meyer [C-M]. To find more examples, we could consider triple products or Weil restrictions of elliptic curves with CM. In order to produce a Calabi-Yau threefold, we would then have to divide out by some group action. Apart from finding such a group, the major problem consists in preserving the field of definition and resolving singularities appropriately. This is still unsolved.

We conclude this chapter with the remark that for higher dimensions even less is known. Only recently, Cynk and Hulek exhibited a general approach [C-H]. One particular construction involves the n-fold product of the elliptic curve E with CM by $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ as given in Remark II.10.7. Recall the Größencharakter Φ associated to E. After dividing out E^n by a certain finite group action, Cynk and Hulek derive a resolution of singularities X, which is Calabi-Yau and defined over $\mathbb{Q}$. For n odd, this has middle cohomology group

$$\mathrm{H}^n(X) = \mathrm{H}^{n0}(X) \oplus \mathrm{H}^{0n}(X)$$

which is two-dimensional. Its L-series comes from the Größencharakter Φ^n. If n is even, the corresponding statement can be established for the transcendental lattice $T(X) \subset \mathrm{H}^n(X, \mathbb{Z})$.

Chapter III

Elliptic fibrations of some extremal K3 surfaces

The main goal of this chapter is to find singular K3 surfaces over $\mathbb{Q}$. In particular, we will emphasize for which newforms of weight 3 (with rational coefficients) we can find an associated surface. This concerns the classification of singular K3 surfaces (over $\mathbb{Q}$) in terms of the discriminant up to squares (cf. Prop. II.13.1).

Our approach consists in exhibiting extremal elliptic K3 fibrations which can be derived from rational elliptic surfaces by direct, relatively simple manipulations of the Weierstrass equations. Our aim is to exhaust this approach completely. The main technique will be pull-back by a base change. We only exclude the general construction involving the induced J-map of the fibration (considered as a base change generally of degree 24, conf. [MP2, section 2]). The base changes we construct will have degree at most 8. Additionally there is another effective method if we allow the extremal K3 surface to have non-reduced fibres. Then we can also manipulate the Weierstrass equations by adding or transferring common factors, thus changing the shape of singular fibres rather than introducing new cusps. In total, this approach will enable us to realize 201 out of the 325 configurations of singular fibres which exist for extremal elliptic K3 fibrations by the classification of [S-Z]. Note, however, that the configuration does in general not determine the isomorphism class of the complex surface (cf. Section III.7 for a particular example).

For most of this chapter, we will concentrate on extremal elliptic K3 fibrations with only semi-stable fibres. The determination of the 112 possible configurations of singular fibres originally goes back to Miranda and Persson [MP2]. For 20 of them, Weierstrass equations over $\mathbb{Q}$ (or in one case $\mathbb{Q}(\sqrt{5})$) have been obtained in [Ir] or [Sh3], [L-Y] and [T-Y]. These give rise to the elliptic K3 surface with the maximal singular fibre (the first one from the list in [MP2]), the Shioda modular ones and those coming from semi-stable extremal (hence Shioda modular) rational elliptic surfaces after a quadratic base change. Because of the construction, it is easy to determine the unique $\mathbb{C}$-isomorphism classes of these surfaces over $\mathbb{Q}$ using [S-Z]. Up to squares, the surfaces have discriminants 3, 4, 7, 8 and 19.

The main idea of this chapter consists in applying a base change of higher degree to other extremal rational elliptic surfaces (namely those with three cusps). In the pull-back surface, we replace the non-reduced singular fibres in such a way by semi-stable fibres that it is an extremal K3. Indeed, five of the Shioda modular rational elliptic surfaces can also be obtained in such a way according to [MP1, section 7]. Here, we investigate those base changes which do not factor through the Shioda modular surfaces. We find Weierstrass equations for 38 further extremal semi-stable elliptic K3 fibrations, only 6 of which are not defined over $\mathbb{Q}$. Again, the isomorphism classes of these surfaces can be determined in advance since we know the Mordell-Weil groups. The surfaces over $\mathbb{Q}$ realize the following 32 configurations of singular fibres in the notation of [MP2]:

[1,1,1,2,3,16]	[1,1,1,2,5,14]	[1,1,1,3,3,15]	[1,1,1,3,6,12]
[1,1,1,5,6,10]	[1,1,2,2,3,15]	[1,1,2,3,3,14]	[1,1,2,4,4,12]
[1,1,2,4,6,10]	[1,1,3,3,8,8]	[1,1,3,4,6,9]	[1,2,2,2,3,14]
[1,2,2,2,5,12]	[1,2,2,2,7,10]	[1,2,2,3,4,12]	[1,2,2,3,6,10]
[1,2,2,5,6,8]	[1,2,2,6,6,7]	[1,2,3,3,3,12]	[1,2,3,4,4,10]
[1,2,4,4,6,7]	[1,2,4,5,6,6]	[1,3,3,3,5,9]	[1,3,3,5,6,6]
[1,3,4,4,4,8]	[2,2,2,4,6,8]	[2,2,2,3,5,10]	[2,2,3,3,4,10]
[2,2,3,4,5,8]	[2,2,4,4,6,6]	[2,3,3,3,4,9]	[2,3,4,4,5,6]

The corresponding discriminants are 15, 20, 24, 35, 40, 84, and 120 up to squares.

The additional fibrations which result from this approach, but can only be defined over some quadratic or cubic extension of $\mathbb{Q}$, are

[1,1,2,2,4,14], [1,1,2,6,6,8], [1,2,2,4,5,10]
[1,2,2,4,7,8], [1,2,3,3,6,9], [1,2,3,4,6,8].

For the non-semi-stable extremal K3 fibrations which can be derived from rational elliptic surfaces, we produce two long tables in Section III.6. Together with another fibration which is described in Section III.8, these further provide the discriminants 11 and 168. These results are collected in Table III.3.

This chapter is organized as follows: After shortly recalling some basic facts about elliptic surfaces in the next section (III.1), we will spend the major part of it with constructing the base changes and giving the resulting equations for the semi-stable extremal elliptic K3 surfaces (Sections III.2, III.3, III.4, III.5). Eventually we will also consider the non-semi-stable fibrations in Section III.6 although we will keep their treatment quite concise. Section III.7 concerns the question whether a configuration determines the isomorphism class of its surface. Two examples are studied in detail. We conclude with some comments in Section III.8. These refer to our original question concerning the associated newforms. We will also mention a recent announcement of Beukers and Montanus which supplements our results (cf. Table III.4).

One final remark seems to be in order: There are, of course, many other ways to produce extremal (or singular) elliptic K3 surfaces. The perhaps best known is the concept of double sextics as introduced in [P]. We will not pursue this approach here, so the interested reader is also referred to [MP3] and [ATZ] for instructive applications.

III.1 Elliptic surfaces over $\mathbb{P}^1$ with a section

An elliptic surface over $\mathbb{P}^1$, say $Y \xrightarrow{r} \mathbb{P}^1$, with a section is given by a minimal Weierstrass equation

$$y^2 = x^3 + Ax + B$$

where A and B are homogeneous polynomials in the two variables of $\mathbb{P}^1$ of degree $4M$ and $6M$, respectively, for some $M \in \mathbb{N}$. Then the section is the point at ∞. The term minimal refers to the common factors of A and B: They are not allowed to have a common irreducible factor with multiplicity greater than 3 in A and greater than 5 in B. Otherwise we could cancel these factors by a change of variables. This convention restricts the singularities of the Weierstrass equation to rational double points. Then, the surface Y is the minimal desingularization. In this chapter, we are mainly interested in examples where both A and B have rational coefficients (while the results of [MP2] will only imply the existence of A and B over some number field). Of course, we can also assume A and B to have (minimal) integer coefficients, but we will not go into detail here.

As announced in the introduction, we are going to pay special attention to the singular fibres of Y. For a general choice of A and B there will be $12M$ of them (each a rational curve with a node). The types of the singular fibres, which were first classified by Kodaira in [Ko], can be read off directly from the j-function of Y (cf. [Si, IV.9, table 4.1]). Up to a factor, this is the quotient of A^3 by the discriminant Δ of Y which is defined as $\Delta = -16\,(4A^3 + 27B^2)$:

$$j = -\frac{1728\,(4A)^3}{\Delta}.$$

Then Y has singular fibres above the zeroes of Δ which we call the *cusps* of Y. Let x_0 be a cusp and n the order of vanishing of Δ at x_0. The fibre above x_0 is called *semi-stable* if it is a rational curve with a node (for $n = 1$) or a cycle of n lines, $n > 1$. In Kodaira's notation, this is type I_n. The fibre above x_0 is semi-stable if and only if A does not vanish at x_0 (i.e. iff x_0 is not a common zero of A and B). On the other hand, we get either a non-reduced fibre (distinguished by a *) over x_0 if A and B both vanish at x_0 to order at least 2 and 3, respectively, or an additive fibre of type II, III or IV otherwise.

One common property of the singular fibres is that in every case the vanishing order of Δ at the cusp x_0 equals the Euler number of the fibre above x_0. Recall that $\mathrm{H}^1(Y, \mathcal{O}_Y) = 0$ and $p_g = \dim \mathrm{H}^2(Y, \mathcal{O}_Y) = M - 1$, while the canonical divisor $K_Y = (M-2)\,F$ for a general fibre F (cf. e.g. [M1, Lecture III]). Hence, Y is K3 (resp. rational) if and only if $M = 2$ (resp. $M = 1$). The Euler number of Y equals the sum of the Euler numbers of its (singular) fibres. By the above considerations, this coincides with the degree $12M$ of Δ. Hence, we obtain that Y is a K3 surface if and only if its Euler number equals 24. (On the contrary, Y is rational if and only if $e(Y) = 12$.)

Before discussing the effect of a base change on the elliptic surface Y, let us at first introduce the following notation: We say that a map

$$\pi : \mathbb{P}^1 \to \mathbb{P}^1$$

has *ramification index* $(n_1, ..., n_k)$ at $x_0 \in \mathbb{P}^1$ if x_0 has k pre-images under π with respective orders $n_i \;\; (i = 1, \dots, k)$. The base change of Y by π is simply defined as the pull-back surface

$$X \xrightarrow{\pi \circ r} \mathbb{P}^1.$$

Here, we substitute π into the Weierstrass equation and j-function of Y and subsequently normalize to obtain X. Let π have ramification index $(n_1, \dots, n_k)$ at the cusp x_0 of Y. Then a semi-stable singular fibre of type I_n above x_0 is replaced by k fibres of types $I_{n_1 n}, \dots, I_{n_k n}$ in the pull-back X.

For a non-semi-stable singular fibre the substitution process is non-trivial for two reasons: On the one hand the Weierstrass equation might simply lose its minimality through the substitution. On the other hand, the minimalized Weierstrass equation can still become *inflated*. This means that the pull-back surface has more than one non-reduced fibre. Then there is a quadratic twist of the surface, sending $x \mapsto \alpha^2 x$ and $y \mapsto \alpha^3 y$, which replaces an even number of non-reduced fibres by their reduced relatives (i.e. I_n^* by I_n and II^*, III^*, IV^* by IV, III, II, respectively). Here, α is the vanishing polynomial of the cusps of the non-reduced fibres. Following [M2] this process will be called *deflation*. Since our main interest lies in elliptic (K3) surfaces with only semi-stable fibres, deflation provides a useful tool to construct such surfaces via base change.

It is exactly these two methods (minimalization and deflation after a suitable base change) which we will use to resolve the non-semi-stable fibres of the base surface Y. The explicit behaviour of the singular fibres under a base change can be derived from [Si, IV.9, table 4.1] or found in [MP1, section 7]. We sketch it in the next figure where the number next to an arrow denotes the order of ramification under π (of one particular pre-image). The fibres of type I_n^* are exceptional in that they allow two possibilities of substitution by semi-stable fibres: Either by ramification of even index or by pairwise deflation.

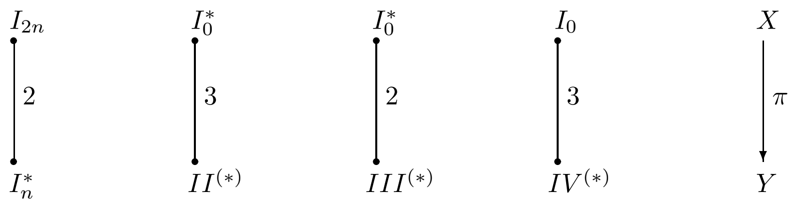

Figure III.1: The resolution of the non-semi-stable fibres

III.2 The deflated base changes

Our main interest lies in finding equations over $\mathbb{Q}$ for *extremal* semi-stable elliptic K3 fibrations. By definition, these are singular elliptic K3 surfaces with finite Mordell-Weil group and only semi-stable fibres. These assumptions are quite restrictive. It is an immediate consequence that the number of cusps has to be 6, and one finds the 112 possible configurations of singular fibres in the classification of [MP2, Thm (3.1)]. The main idea of this chapter is to produce some of these K3 fibrations by the methods described in the previous section via the pull-back of a rational elliptic surface by a base change. This approach is greatly helped by the good explicit knowledge one has of the rational elliptic surfaces (cf. [H], [MP1], [S-H]). One only has to construct suitable base changes.

The starting point for our considerations is a rational elliptic surface with a section

$$Y \xrightarrow{r} \mathbb{P}^1.$$

For a base change $\pi : \mathbb{P}^1 \to \mathbb{P}^1$, let

$$X \xrightarrow{\pi \circ r} \mathbb{P}^1$$

denote the pullback via π. Since the Mordell-Weil group $MW(Y)$ injects into $MW(X)$, we will assume Y to be extremal. Note, however, that the process of deflation can a priori change the Mordell-Weil group. Therefore it could also seem worth considering non-extremal rational elliptic surfaces, especially those with a small number of cusps as presented in [H] and [S-H]. A close observation nevertheless shows that these would not produce any configurations different from those known or obtained in this chapter, unless one turns to the general case where π has degree 24. This can also be derived from [Kl, Thm. 1.2]. We will comment on the general case in Section III.8.

Extremal rational elliptic surfaces have been completely classified by Miranda-Persson in [MP1]. There are six semi-stable surfaces with four cusps. These had previously been identified as Shioda modular by Beauville [B]. As explained, these surfaces have been treated exhaustively in [T-Y], giving rise to the semi-stable extremal elementary fibrations of [P].

Furthermore there are four surfaces with only two cusps (one of them appearing in a continuous family). We will not use these surfaces since they have no fibre of type I_n or I_n^* with $n > 0$ at all.

The remaining six extremal rational elliptic surfaces have three cusps. Each has exactly one non-reduced fibre while the other two singular fibres are semi-stable. These are the surfaces we are going to investigate for a pull-back via a base change. For the remaining part of this section we will concentrate on those ("deflated") base changes π which give rise to a non-inflated pull-back K3 surface $X \stackrel{\pi\circ r}{\longrightarrow} \mathbb{P}^1$ after minimalizing. The pull-back surfaces coming from inflating base changes will be derived in Section III.5.

Let Y be an extremal rational elliptic surface Y with three cusps. There are a number of conditions on the base change π if one wants an extremal semi-stable K3 pull-back. The Euler number $e(X) = 24$ of the pull-back surface X predicts the degree of π, only depending on the type of the non-reduced fibre W^* of Y.

On the one hand, if W is of additive type (i.e. $W \in \{II, III, IV\}$), we will eventually replace it by smooth fibres after minimalizing and deflating, if necessary. Let the other two singular fibres be of type I_m and I_n with $m, n \in \mathbb{N}$. Then $m + n \leq 4$, and $e(X) = (\deg\, \pi)(m + n)$. On the other hand, let the non-reduced fibre have type I_k^*. If the other singular fibres have again m and n components, then we have $k + m + n = 6$. Therefore we require $\deg\, \pi = 4$.

Our assumption, that the semi-stable pull-back X is extremal, implies the minimal number of six singular fibres. This gives another stringent restriction. In some cases, this leads to a contradiction to the Hurwitz formula

$$-2 \geq -2\deg\, \pi + \sum_{x\in\mathbb{P}^1} (\deg\, \pi - \#\pi^{-1}(x)).$$

Finally, as we decided to concentrate on resolving the non-reduced fibre W^* by a deflated base change, the ramification index at the corresponding cusp has to be divisible by 2, 3, 4, or 6 if $W = I_n^*, IV, III$, or II, respectively (cf. Fig. III.1).

It will turn out that some of the base changes can only be defined over an extension of $\mathbb{Q}$ of low degree. Nevertheless, for any base change, the pull-back surface X will have at least two rational cusps. For simplicity and without loss of generality, we will choose these by a Möbius transformation to be 0 and ∞ (and a further third rational cusp, if it exists, to be 1). For every rational surface, we will derive a Weierstrass equation over $\mathbb{Q}$ such that the cusps are 0,1 and ∞. This gives us the opportunity to construct the base changes before considering the surfaces.

We require that the pull-back of a base change π does not factor through a Shioda modular rational elliptic surface. Equivalently, π does not factor into a composition $\pi'' \circ \pi'$ of a degree 2 map π'' and a further map π', such that the non-reduced fibre is already resolved by π'. Otherwise, the intermediate pull-back $X' \stackrel{\pi'\circ r}{\longrightarrow} \mathbb{P}^1$ would be semi-stable and have at most 4 cusps, hence it would be Shioda modular by [B].

We shall now investigate the deflated base changes with the above listed properties. For an extremal rational elliptic surface Y with three cusps, our analysis depends only on the type of the non-reduced fibre W^*. Throughout we employ the notation of [MP1]. For the computations we wrote a straight forward Maple program. In all but two cases, this sufficed to determine the minimal base change. For the remaining two base changes, we used Macaulay to compute a solution mod p for some small primes p and then lift to characteristic 0.

At first assume $\mathbf{W}$ to be $\mathbf{II}$. According to [MP1, section 5] there is, up to isomorphism, a unique rational elliptic surface with this fibre whose other two singular fibres are both of type I_1. Denote this by $\mathbf{X_{211}}$. For the pull-back X to have Euler number $e(X) = 24$, we would need π to have degree 12 and ramification index 12 or (6,6) at the cusp of the non-reduced fibre. Then, the restriction of the other two cusps to have exactly six pre-images under π leads to a contradiction to the Hurwitz formula.

The situation is similar if $\mathbf{W} = \mathbf{III}$. This implies $Y = \mathbf{X_{321}}$ to have further singular fibres I_2, I_1. The III^* fibre requires ramification of index a multiple of 4, so π must have degree 8 with ramification index 8 or (4,4) at the cusp of this fibre. Again, the Hurwitz formula rules out the pull-back surface X to have only six (semi-stable) singular fibres. In Section III.5, we will construct inflating base changes for this surface which resolve the non-reduced fibre and are compatible with the Hurwitz formula.

$\mathbf{W} = \mathbf{IV}$ gives a priori two possibilities for the elliptic surface Y. One of them, $\mathbf{X_{431}}$ with singular fibres of type IV^*, I_3, I_1, does actually exist by [MP1]. Since a IV^* fibre requires the ramification index to be divisible by 3, we need π to have degree 6 and ramification of order (3,3) at the cusp of this fibre (since ramification index 6 would contradict the other two fibres having six pre-images again by the Hurwitz formula). The suitable maps are presented in the next paragraphs. Throughout we assume the IV^* fibre to sit above 1. If such a base change was totally ramified above one of the two remaining cusps, then it would necessarily be composite. Since this was excluded, we only have to deal with those maps such that 0 has two or three pre-images (and then exchange 0 and ∞).

Consider a base change π of degree 6 with ramification index (3,3) at 1. At first, let 0 and ∞ have three pre-images. Our restriction that π does not factor, implies that at least one of the cusps has ramification index (3,2,1). We will assume ∞ to have this ramification index. Then we search for maps such that 0 has ramification index (4,1,1), (3,2,1) or (2,2,2). However, a map with the last ramification cannot exist since the resulting pull-back of $\mathbf{X_{431}}$ does not appear in the list of [MP2]. In the following, we will frequently use this argument. Here, let us once give the details.

Assume that such a base change π exists. Then we can realize the configuration [2,2,2,3,6,9] as pull-back X from $\mathbf{X_{431}}$. We claim that this fibration has a 2-section. Otherwise, the fibre types would imply that

$$(\mathbb{Z}/2)^4 \subseteq NS(X)^\vee / NS(X).$$

This will follow from Lemma III.7.1. So the 2-length of the quotient would be at least 4. But by Proposition I.2.3,

$$NS(X)^\vee / NS(X) \cong T_X^\vee / T_X.$$

Since T_X has rank 2, this can maximally have 2-length 2. This gives the required contradiction.

So X has a 2-section σ. Consider the quotient of X by (translation by) σ. The minimal desingularization Z of this is again an elliptic K3 surface. The resulting singular fibres can be computed from the components which σ meets. In particular, the fibres of types I_9 and I_3 result in I_{18} and I_6 in Z. But this is impossible, since then $e(Z) > 24$. Hence, X cannot admit a 2-section. Since the existence of the 2-section followed from the construction via π, this base change cannot exist.

The computations show that the base change with the second ramification, $\tilde{\pi}$, can only be realized over a cubic extension of $\mathbb{Q}$ (cf. Section III.8 and Figure III.38 for details). Here, we only construct the first base change:

$$\begin{array}{rccl} \pi_{\mathbf{3,4}}: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \\ & (s:t) & \mapsto & (27s^4(125t^2-90st-27s^2) : -3125t^3(t-s)^2(5t+4s)). \end{array}$$

Then $27s^4(125t^2-90st-27s^2)+3125t^3(t-s)^2(5t+4s) = (25t^2-10st-9s^2)^3$, so $\pi_{3,4}$ has the required properties.

Now, let 0 have only two pre-images and ∞ four. The respective ramification indices are (5,1), (4,2) or (3,3) and (2,2,1,1) or (3,1,1,1). Only the first and the last of these do not allow a composition as a degree 3 and a degree 2 map. Hence, at least one of these two ramification indices must occur for the base change to meet our criteria.

Let us first construct the maps with ramification index (5,1) at 0. They are:

$$\begin{array}{rccl} \pi_{\mathbf{5,3}}: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \\ & (s:t) & \mapsto & (729s^5(s-t) : -t^3(135s^3+9st^2+t^3)) \end{array}$$

with $729s^5(s-t)+t^3(135s^3+9st^2+t^3) = (9s^2-3st-t^2)^3$ and

$$\begin{array}{rccl} \pi_{\mathbf{5,2}}: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \\ & (s:t) & \mapsto & (2^6 3^3 s^5 t : -(s^2-4st-t^2)^2(125s^2+22st+t^2)) \end{array}$$

with $2^6 3^3 s^5 t + (s^2-4st-t^2)^2(125s^2+22st+t^2) = (5s^2+10st+t^2)^3$.

The other base changes have ramification index (3,1,1,1) at ∞. It is immediate that there is no such map π with ramification index (3,3) at 0: After exchanging 1 and ∞, the map π would have to look like $(f_0^3 g_0^3 : f_1^3 g_1^3)$ with distinct linear homogeneous factors f_i, g_i. Then, with ϱ a primitive third root of unity,

$$f_0^3 g_0^3 - f_1^3 g_1^3 = (f_0 g_0 - f_1 g_1)(f_0 g_0 - \varrho f_1 g_1)(f_0 g_0 - \varrho^2 f_1 g_1).$$

This polynomial cannot have a cubic factor. Therefore, the next map completes the list of suitable base changes for $\mathbf{X_{431}}$. Remember that we still have to take the permutation of 0 and ∞ via exchanging s and t into account.

$$\begin{array}{rccl} \pi_{\mathbf{4,3}}: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \\ & (s:t) & \mapsto & (729s^4t^2 : -(s-t)^3(8s^3+120s^2t-21st^2+t^3)) \end{array}$$

with $729s^4t^2 + (s-t)^3(8s^3+120s^2t-21st^2+t^3) = (2s^2-8st-t^2)^3$.

We conclude this section by considering the non-reduced fibre $\mathbf{W}^*$ to equal $\mathbf{I_n^*}$ for some $n > 0$. By [MP1] there are three extremal rational elliptic surfaces with such a singular fibre. All of them have two further singular fibres, both semi-stable. The surfaces will be introduced in the next section. Independent of the surface, we have already seen that an adequate deflated base change π must have degree 4 and ramification of index (2,2) or 4 at the cusp of the I_n^* fibre. Then X is extremal if and only if the two other cusps have 4 or 5 pre-images, respectively. Assume that the non-reduced fibre sits above the cusp $\infty = (1:0)$. We now construct the base changes π which do not factor into two maps of degree 2. Equivalently, one of the cusps has ramification index (3,1).

At first, let the base change π be totally ramified at ∞. By the above considerations, the other two cusps have ramification indices (3,1) and (2,1,1). Up to exchanging them, for example by

$$\phi : (s:t) \mapsto (t-s:t),$$

the map π can be realized as

$$\begin{array}{rccc} \pi_{\mathbf{4}}: & \mathbb{P}^1 & \to & \mathbb{P}^1 \\ & (s:t) & \mapsto & (256s^3(s-t) : -27t^4), \end{array}$$

since $(256s^3(s-t)+27t^4) = (4s-3t)^2(16s^2+8st+3t^2)$.

In the other case, π has ramification index (2,2) at ∞. Our restrictions imply ramification of index (3,1) at one of the other two cusps. Without loss of generality, let this cusp be 1. Then the last cusp also has two pre-images and thus ramification index (3,1) or (2,2). The second cannot exist: Given such a map, it could be expressed as $(f_0^2g_0^2 : f_1^2g_1^2)$ with distinct homogeneous linear forms f_i, g_i in s,t. The factorization $f_0^2g_0^2 - f_1^2g_1^2 = (f_0g_0+f_1g_1)(f_0g_0-f_1g_1)$ cannot have a cubic factor. Hence 1 cannot have ramification index (3,1).

We conclude this section with a base change $\pi_{\mathbf{2}}$ of degree 4 with ramification indices (3,1), (3,1) and (2,2):

$$\begin{array}{rccc} \pi_2: & \mathbb{P}^1 & \to & \mathbb{P}^1 \\ & (s:t) & \mapsto & (64s^3(s-t) : (8s^2-4st-t^2)^2). \end{array}$$

In the next two sections (III.3, III.4), we will substitute the base changes $\pi_{3,4}, \pi_{5,3}$, $\pi_{5,2}$, $\pi_{4,3}, \pi_2$ and π_4 into the normalized Weierstrass equations of the extremal rational elliptic surfaces with three singular fibres. We will derive equations over $\mathbb{Q}$ for extremal K3 surfaces with six singular fibres, all of which semi-stable.

III.3 The fibrations coming from degree 4 base changes

In this and the next section (III.4), we proceed as follows to obtain equations for extremal K3 surface with six semi-stable fibres: Consider the Weierstrass equations given for the extremal rational elliptic surfaces in [MP1, Table 5.2]. We apply the normalizing Möbius transformation which maps the cusps to 0, 1 and ∞. Then we exhibit the deflated base changes π_i from the previous section. After minimalizing by an admissible change of variables, this gives the Weierstrass equations for 16 extremal elliptic K3 surfaces from the list of [MP2]. Throughout we choose the coefficients of the polynomials A and B involved in the Weierstrass equation to be minimal by rescaling, if necessary. By construction, the pull-back surface X always inherits the sections of the rational elliptic surface Y. As a consequence, we are able to derive the isomorphism class of X (in terms of the intersection form on its transcendental lattice) from the classification in [S-Z].

In this section we consider only the extremal rational elliptic surfaces with an $\mathbf{I}_\mathbf{n}^*$ fibre ($n>0$). As explained, they require a base change of degree 4. Before substituting by the base changes π_4 or π_2, we choose the normalizing Möbius transformation in such a way that the I_n^* fibre sits above ∞.

Let us start with $\mathbf{X_{411}}$ which has Weierstrass equation

$$y^2 = x^3 - 3\,t^2(s^2 - 3t^2)\,x + st^3(2s^2 - 9t^2)$$

The singular fibres are I_4^* over ∞ and two I_1 over ± 2. Substituting $(s : t) \mapsto (4s - 2t : t)$ maps the two I_1 fibres to 0 and 1. This gives

$$y^2 = x^3 - 3\,t^2(16s^2 - 16st + t^2)\,x + 2\,t^3(2s - t)(32s^2 - 32st - t^2).$$

Then, we substitute by $\pi_{\mathbf{4}}$ and get the Weierstrass equation of an extremal K3 surface:

$$\begin{aligned} y^2 \quad &= \quad x^3 - 3\,(9s^8 + 48s^7t + 48s^4t^4 + 64s^6t^2 + 128s^3t^5 + 16t^8)\,x \\ &\qquad -2\,(3s^4 + 8s^3t + 8t^4)(9s^8 + 48s^7t + 48s^4t^4 + 64s^6t^2 + 128s^3t^5 - 8t^8). \end{aligned}$$

This provides a realization of the configuration **[1,1,1,2,3,16]** in the notation of [MP2] (i.e. 3 fibres of type I_1, one of types I_2, I_3 and I_{16} each).

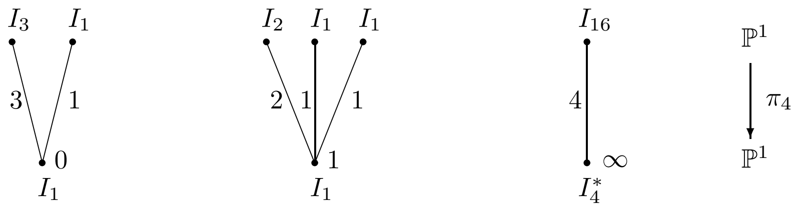

Figure III.2: A realization of **[1,1,1,2,3,16]**

On the other hand, we can also substitute by $\pi_{\mathbf{2}}$ in the normalized Weierstrass equation. We obtain:

$$\begin{aligned} y^2 \quad &= \quad x^3 - 3\,(16s^8 - 64s^7t - 224s^5t^3 + 392t^4s^4 + 64s^6t^2 + 112t^5s^3 + 16t^6s^2 + 8t^7s + t^8)\,x \\ &\qquad -2\,(2s^2 - 4st - t^2)(2s^2 + t^2) \\ &\qquad\quad (16s^8 - 64s^7t + 544s^5t^3 - 952t^4s^4 + 64s^6t^2 - 272t^5s^3 + 16t^6s^2 + 8t^7s + t^8). \end{aligned}$$

This realizes **[1,1,3,3,8,8]**.

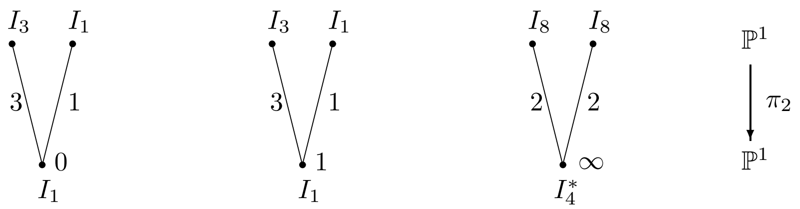

Figure III.3: A realization of **[1,1,3,3,8,8]**

We shall apply the same procedure to the surface $\mathbf{X_{141}}$ with Weierstrass equation

$$y^2 = x^3 - 3\,(s - 2t)^2(s^2 - 3t^2)\,x + s(s - 2t)^3(2s^2 - 9t^2).$$

This surface will be studied in some detail in Section IV.1. The normalization of the cusps leads to the Weierstrass equation

$$y^2 = x^3 - 3\,t^2(16t^2 - 16st + s^2)\,x + 2\,t^3(s - 2t)(s^2 + 32st - 32t^2).$$

This has an I_4 fibre above 0, I_1 above 1 and I_1^* above ∞. From this, we will derive three examples, two of them connected by the Möbius transformation ϕ. Substitution of $\pi_{\mathbf{4}}$ produces a realization of **[1,1,2,4,4,12]**:

$$\begin{aligned} y^2 \quad &= \quad x^3 - 3\,(16t^8 + 48s^4t^4 - 64s^3t^5 + 9s^8 - 24s^7t + 16s^6t^2)\,x \\ &\qquad -2\,(2t^4 + 3s^4 - 4s^3t)(-32t^8 - 96s^4t^4 + 128s^3t^5 + 9s^8 - 24s^7t + 16s^6t^2) \end{aligned}$$

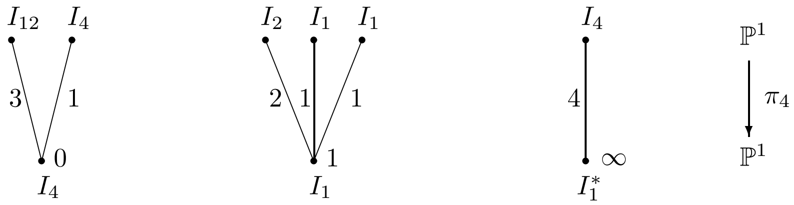

Figure III.4: A realization of **[1,1,2,4,4,12]**

Conjugation by ϕ gives the Weierstrass equation

$$y^2 = x^3 - 3\,t^2(t^2 + 14st + s^2)\,x - 2\,(t + s)t^3(t^2 - 34st + s^2).$$

This leads to the following realization of **[1,3,4,4,4,8]**

$$\begin{aligned} y^2 \quad &= \quad x^3 + 3\,(-24s^7t - 16s^6t^2 - 9s^8 - t^8 + 42s^4t^4 + 56s^3t^5)\,x \\ &\qquad -2\,(9s^8 + 24s^7t + 16s^6t^2 + 102s^4t^4 + 136s^3t^5 + t^8)(-3s^4 - 4s^3t + t^4) \end{aligned}$$

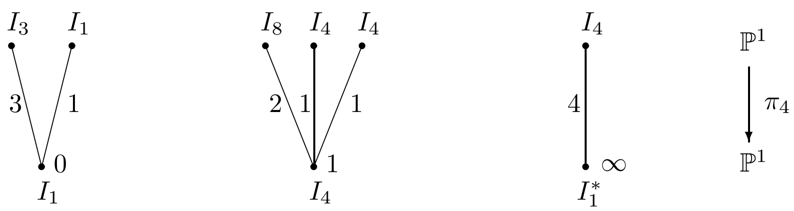

Figure III.5: A realization of **[1,3,4,4,4,8]**

On the other hand, substitution of $\pi_{\mathbf{2}}$ in the normalized Weierstrass equation gives:

$$\begin{aligned} y^2 \quad &= \quad x^3 - 3\,(s^8 - 4s^7t + 16s^5t^3 - 28t^4s^4 + 4s^6t^2 - 8t^5s^3 + 16s^2t^6 + 8st^7 + t^8)\,x \\ &\qquad -(2s^4 - 4s^3t + 4st^3 + t^4) \\ &\qquad\quad (s^8 - 4s^7t - 32s^5t^3 + 56t^4s^4 + 4s^6t^2 + 16t^5s^3 - 32s^2t^6 - 16st^7 - 2t^8). \end{aligned}$$

This realization of **[1,2,2,3,4,12]** is invariant under conjugation by ϕ.

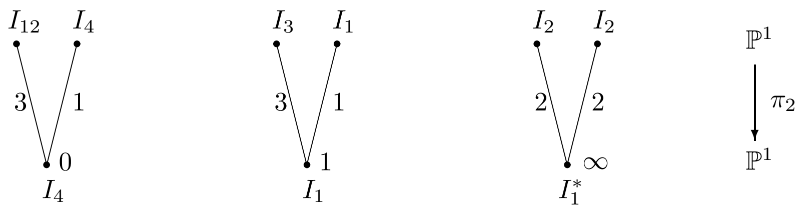

Figure III.6: A realization of **[1,2,2,3,4,12]**

Finally, we turn to the surface $\mathbf{X_{222}}$ in the notation of [MP1]. Miranda-Persson give the Weierstrass equation

$$y^2 = x^3 - 3\,st(s-t)^2\,x + (s-t)^3(s^3+t^3).$$

This has cusps the third roots of unity with an I_2^* fibre above 1 and I_2 fibres above the two primitive roots ω, ω^2. Take ω in the upper half plane. Consider the Möbius transformation which maps ω to ∞ and ω^2 to 0 while fixing 1. This gives a Weierstrass equation which is not defined over $\mathbb{Q}$. However, the change of variables $x \mapsto \xi^2 x, y \mapsto \xi^3 y$ with $\xi = 3\sqrt{-3}$ leads to a Weierstrass equation over $\mathbb{Q}$ with the same cusps:

$$y^2 = x^3 - 3\,(s^2 - st + t^2)(s-t)^2 x + (s-2t)(2s-t)(t+s)(s-t)^3.$$

We exchange the cusps 1 and ∞ and subsequently substitute by $\pi_{\mathbf{4}}$ or $\pi_{\mathbf{2}}$. The first substitution gives a realization of **[2,2,2,4,6,8]**:

$$\begin{aligned} y^2 \;&=\; x^3 - 3\,(9s^8 - 24s^7t + 16s^6t^2 + 3s^4t^4 - 4s^3t^5 + t^8)\,x \\ &\quad -(2t^4 + 3s^4 - 4s^3t)(3s^4 - 4s^3t - t^4)(6s^4 - 8s^3t + t^4) \end{aligned}$$

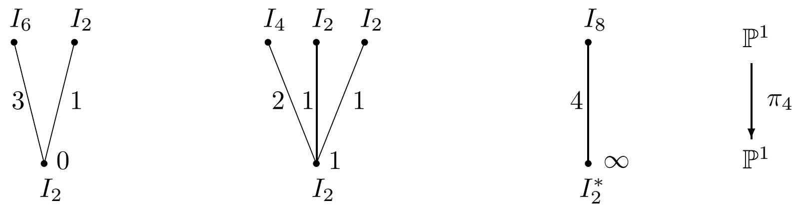

Figure III.7: A realization of **[2,2,2,4,6,8]**

The second substitution realizes **[2,2,4,4,6,6]**:

$$\begin{aligned} y^2 \;&=\; x^3 - 3\,(16s^8 - 64ts^7 + 64t^2s^6 + 16t^3s^5 - 28t^4s^4 - 8t^5s^3 + 16t^6s^2 + 8t^7s + t^8)\,x \\ &\quad +2\,(2s^2 - 4st - t^2)(8s^4 - 16s^3t + 4st^3 + t^4)(t^2 + 2s^2)(2s^4 - 4s^3t + 4st^3 + t^4). \end{aligned}$$

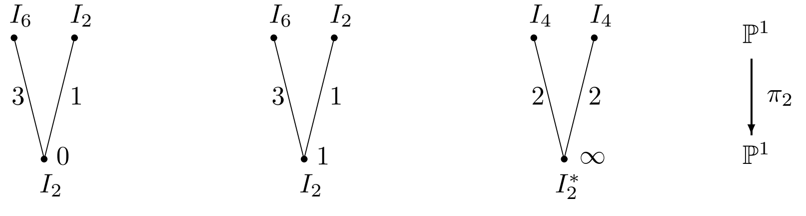

Figure III.8: A realization of **[2,2,4,4,6,6]**

III.4 The fibrations coming from degree 6 base changes

We consider the extremal rational elliptic surface $\mathbf{X_{431}}$. By base change, it will give rise to 9 extremal elliptic K3 surfaces, 8 of which can be realized over $\mathbb{Q}$.

We modify the Weierstrass equation of $\mathbf{X_{431}}$ given in [MP1] by exchanging 1 and ∞. It becomes

$$y^2 = x^3 - 3\,(s-t)^3(s-9t)\,x - 2(s-t)^4(s^2+18st-27t^2).$$

One finds the fibre of type I_3 above 0 and I_1 above ∞ while the IV^* fibre sits above 1. We can resolve the non-reduced fibre by the degree 6 base changes from Section III.2. Before the substitution, we can also permute 0 and ∞.

At first, we shall use $\pi_{\mathbf{3,4}}$. This leads to the Weierstrass equation

$$\begin{aligned} y^2 \;&=\; x^3 - 3\,(-15s^4t^2 + 54s^5t + 81s^6 + 15s^2t^4 - 100s^3t^3 + 6st^5 - t^6)(9s^2+2st-t^2)\,x \\ &\quad -2\,(19683s^{12} + 26244s^{11}t + 1458s^{10}t^2 + 43740s^9t^3 + 25785s^8t^4 - 16776s^7t^5 \\ &\quad -10108s^6t^6 + 3864s^5t^7 + 885s^4t^8 - 380s^3t^9 - 6s^2t^{10} + 12st^{11} - t^{12}). \end{aligned}$$

This realizes **[1,2,3,3,3,12]**.

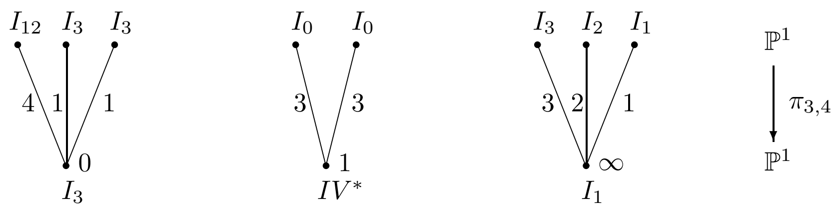

Figure III.9: A realization of **[1,2,3,3,3,12]**

Permuting 0 and ∞ before the substitution gives a realization of **[1,1,3,4,6,9]**:

$$\begin{aligned} y^2 \;&=\; x^3 - 3\,(-t^6 + 6st^5 + 15s^2t^4 - 100s^3t^3 - 1215s^4t^2 + 4374s^5t + 6561s^6) \\ &\quad (9s^2+2st-t^2)\,x \\ &\quad +2\,(14348907s^{12} + 19131876s^{11}t + 1062882s^{10}t^2 - 4855140s^9t^3 \\ &\quad -185895s^8t^4 + 452952s^7t^5 - 7084s^6t^6 - 20328s^5t^7 \\ &\quad +3405s^4t^8 - 380s^3t^9 - 6s^2t^{10} + 12st^{11} - t^{12}). \end{aligned}$$

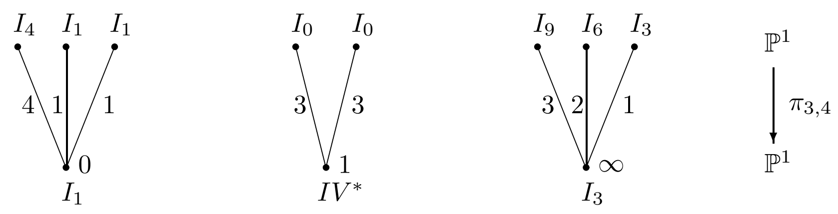

Figure III.10: A realization of **[1,1,3,4,6,9]**

We now turn to the substitutions by $\pi_{\mathbf{5,3}}$. These provide the following realizations of **[1,1,1,3,3,15]** as

$$\begin{aligned} y^2 \;&=\; x^3 - 3\,(s^2-ts-t^2)(s^6 - 3s^5t + 45t^3s^3 - 27t^5s - 9t^6)\,x \\ &\quad +2\,(-s^{12} + 6s^{11}t - 9s^{10}t^2 + 90s^9t^3 - 270t^4s^8 - 54t^5s^7 + 819t^6s^6 \\ &\quad +54t^7s^5 - 810t^8s^4 - 270t^9s^3 + 243t^{10}s^2 + 162st^{11} + 27t^{12}) \end{aligned}$$

and of **[1,3,3,3,5,9]** as

$$\begin{aligned} y^2 \;&=\; x^3 - 3,(s^2-st-t^2)(5s^3t^3 - 3st^5 - t^6 + 9s^6 - 27s^5t)\,x \\ &\quad +2\,(27s^{12} - 162s^{11}t + 243s^{10}t^2 + 90s^9t^3 - 270s^8t^4 - 54s^7t^5 \\ &\quad +119s^6t^6 + 54s^5t^7 + 30s^4t^8 + 10s^3t^9 - 9s^2t^{10} - 6st^{11} - t^{12}). \end{aligned}$$

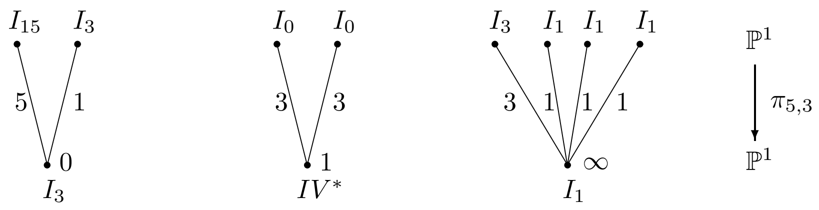

Figure III.11: A realization of [**1,1,1,3,3,15**]

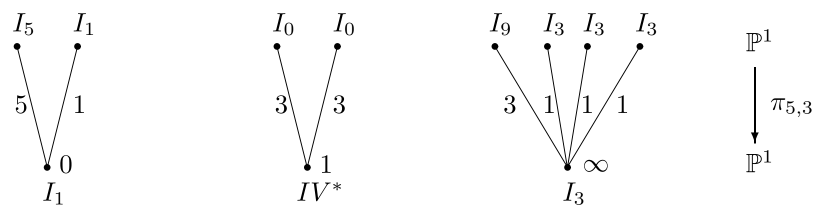

Figure III.12: A realization of [**1,3,3,3,5,9**]

The substitution by $\pi_{\mathbf{5},\mathbf{2}}$ allows us to realize [**1,1,2,2,3,15**] by virtue of the Weierstrass equation

$$\begin{aligned} y^2 \; = \; & x^3 - 3\,(125s^6 - 786s^5t + 1575s^4t^2 + 1300s^3t^3 + 315s^2t^4 + 30st^5 + t^6) \\ & \quad (5s^2 + 10st + t^2)\,x \\ & +2\,(15625s^{12} + 112986s^{10}t^2 - 100500s^{11}t - 941300s^9t^3 \\ & \quad +1514175s^8t^4 + 3849240s^7t^5 + 2658380s^6t^6 + 912696s^5t^7 \\ & \quad +t^{12} + 180375s^4t^8 + 21500s^3t^9 + 1530s^2t^{10} + 60st^{11}). \end{aligned}$$

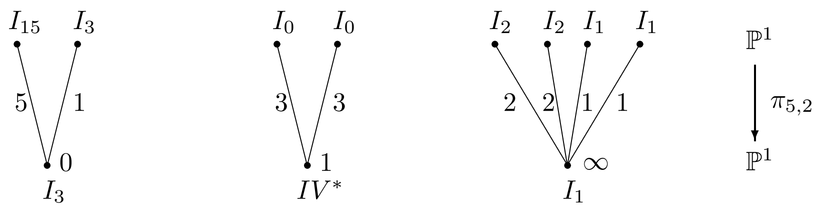

Figure III.13: A realization of [**1,1,2,2,3,15**]

Then, the configuration [**1,3,3,5,6,6**] comes from

$$\begin{aligned} y^2 \; = \; & x^3 - 3\,(125s^6 + 14574s^5t + 1575s^4t^2 + 1300s^3t^3 + 315s^2t^4 + 30st^5 + t^6) \\ & \quad (5s^2 + 10st + t^2)\,x \\ & -2\,(15625s^{12} - 4132500s^{11}t - 48851622s^{10}t^2 - 51744500s^9t^3 \\ & \quad -40418625s^8t^4 - 6311400s^7t^5 + 1690700s^6t^6 + 880440s^5t^7 \\ & \quad +180375s^4t^8 + 21500s^3t^9 + 1530s^2t^{10} + 60st^{11} + t^{12}). \end{aligned}$$

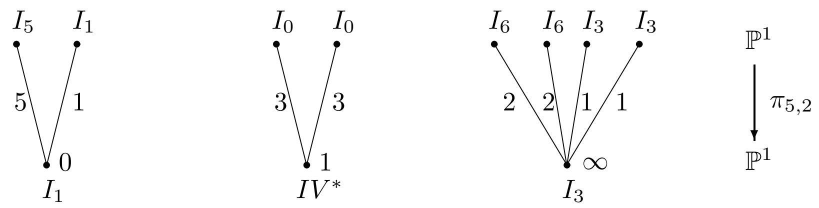

Figure III.14: A realization of **[1,3,3,5,6,6]**

Finally, we can also substitute by $\pi_{\mathbf{4,3}}$ and produce Weierstrass equations for **[1,1,1,3,6,12]** as

$$\begin{aligned} y^2 &= x^3 - 3\,(-276s^4t^2 + 8s^6 + 96s^5t + 416s^3t^3 - 186s^2t^4 + 24st^5 - t^6) \\ &\qquad (2s^2 + 8st - t^2)\,x \\ &\quad +2\,(11160s^8t^4 + 7392s^{10}t^2 - 15232s^9t^3 - 130176s^7t^5 \\ &\qquad +220056s^6t^6 - 160416s^5t^7 + 54792s^4t^8 + 64s^{12} \\ &\qquad +1536s^{11}t - 9760s^3t^9 + 948s^2t^{10} - 48st^{11} + t^{12}) \end{aligned}$$

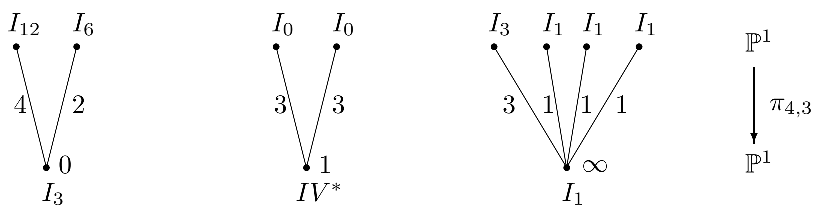

Figure III.15: A realization of **[1,1,1,3,6,12]**

and for **[2,3,3,3,4,9]** as

$$\begin{aligned} y^2 &= x^3 - 3\,(8s^6 + 96s^5t + 6204s^4t^2 + 416s^3t^3 - 186s^2t^4 + 24st^5 - t^6) \\ &\qquad (2s^2 + 8st - t^2)\,x \\ &\quad -2\,(68400s^4t^8 + 64s^{12} - 101472s^{10}t^2 + 1536s^{11}t \\ &\qquad +2751144s^6t^6 - 1321600s^9t^3 - 9460008s^8t^4 - 5791104s^7t^5 \\ &\qquad -487008s^5t^7 - 9760s^3t^9 + 948s^2t^{10} - 48st^{11} + t^{12}). \end{aligned}$$

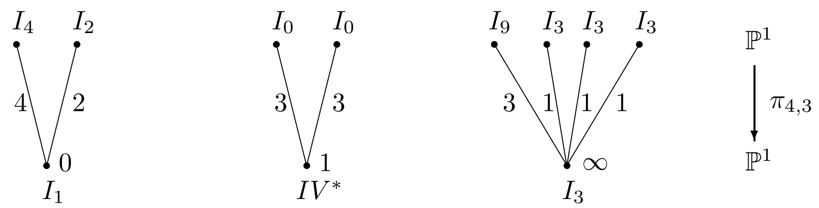

Figure III.16: A realization of **[2,3,3,3,4,9]**

We remark that substitution by $\tilde{\pi}$ allows us to realize the configuration **[1,2,3,3,6,9]** over the number field $\mathbb{Q}(x^3 + 12x - 12)$, but we will not give the equations here (cf. Section III.8).

III.5 The inflating base changes and the resulting fibrations

We now turn to the other possibility to resolve non-reduced fibres by a base change π. In this case we allow π to be inflating, i.e. the pull-back X may contain non-reduced fibres of type I_n^* for $n \geq 0$ apart from its semi-stable fibres. The only additional assumption is that their number is even. Via deflation, we can substitute these fibres by their reduced relatives. The resulting semi-stable surface is a quadratic twist of X.

One might hope to produce a number of new configurations by this method. A close inspection, however, shows that none arise from the extremal rational elliptic surfaces considered in the last two sections. This approach is nevertheless useful, since it enables us to work with the extremal rational elliptic surface $\mathbf{X_{321}}$ as well. The reason is that the Hurwitz formula is not violated if we choose the degree 8 base change π of $\mathbb{P}^1$ to have ramification index (2,2,2,2) at the cusp of the III^* fibre (instead of (4,4) or 8 before). The fibre is thus replaced by four fibres of type I_0^* which can easily be twisted away.

We have to assume the other two cusps to have six pre-images in total. Up to exchanging the cusps 0 and ∞, there are 13 such base changes which cannot be factored through an extremal rational elliptic surface. In the following we concentrate on the 9 of these which can be defined over $\mathbb{Q}$. They result in 17 further extremal semi-stable elliptic K3 surfaces which arise by pull-back from $\mathbf{X_{321}}$. The four base changes which are not defined over $\mathbb{Q}$ will briefly be sketched at the end of this section.

Remember that $\mathbf{X_{321}}$ has singular fibres of type III^*, I_2 and I_1. We normalize the Weierstrass equation given in [MP1] such that the III^* fibre sits above 1 and the I_2 and I_1 fibres above 0 and ∞, respectively:

$$y^2 = x^3 - 3\,(s-t)^3(s-4t)\,x - 2\,(s-t)^5(s+8t).$$

In the following, we investigate the degree 8 base changes of $\mathbb{P}^1$ with ramification index (2,2,2,2) at 1 such that the further cusps 0 and ∞ have six pre-images in total. We will concentrate on those base changes which give rise to new configurations of extremal elliptic K3 fibrations over $\mathbb{Q}$. Without loss of generality, we assume that 0 does not have more pre-images than ∞ (since we can exchange the cusps).

The three base changes which are totally ramified at 0 realize configurations known from [L-Y] or [T-Y]. We now turn to the base changes π such that 0 has two pre-images. Throughout we can exclude ramification of index (2,2,2,2) at both other cusps since this would not allow ramification of high order at 0. We list the base changes according to the ramification index at 0.

Let π have ramification index **(7,1)** at 0. There are four base changes to consider. The computations show that the base change with ramification index (4,2,1,1) at ∞ can only be defined over the Galois extensions $\mathbb{Q}(\sqrt{-7})$. It is given at the end of this section. Here we give the remaining three base changes over $\mathbb{Q}$:

The first base change has ramification index (5,1,1,1) at ∞. It can be given as

$$\begin{array}{rccl} \pi: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \\ & (s:t) & \mapsto & (s^7(s-4t) : -4t^5(14s^3+14ts^2+20t^2s+25t^3)) \end{array}$$

since $s^7(s-4t)+4t^5(14s^3+14ts^2+20t^2s+25t^3) = (10t^4+4st^3+2s^2t^2-2s^3t-s^4)^2$.

Substituting π into the normalized Weierstrass equation of $\mathbf{X_{321}}$ gives, after deflation, a realization of **[1,1,1,2,5,14]**:

$$\begin{aligned} y^2 &= x^3 - 3\,(16s^8 + 32s^7t - 112t^5s^3 + 56t^6s^2 - 40t^7s + 25t^8)\,x \\ &\quad -2\,(-5t^4 + 4t^3s - 4s^2t^2 + 8s^3t + 8s^4) \\ &\qquad (8s^8 + 16s^7t + 112t^5s^3 - 56t^6s^2 + 40t^7s - 25t^8). \end{aligned}$$

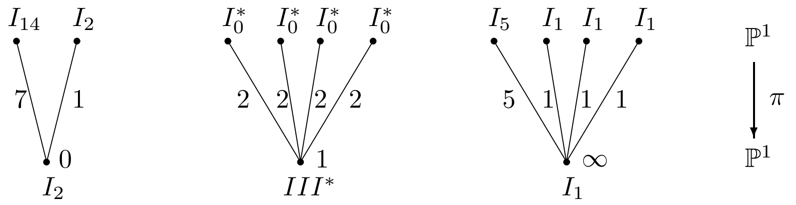

Figure III.17: A realization of **[1,1,1,2,5,14]**

Permuting the cusps 0 and ∞ before the substitution leads to the following realization of **[1,2,2,2,7,10]**:

$$\begin{aligned} y^2 &= x^3 - 3\,(-14t^5s^3 + 14t^6s^2 - 20t^7s + 25t^8 + s^8 + 4s^7t)\,x \\ &\quad +(-10t^4 + 4t^3s - 2s^2t^2 + 2s^3t + s^4) \\ &\qquad (14t^5s^3 - 14t^6s^2 + 20t^7s - 25t^8 + 2s^8 + 8s^7t). \end{aligned}$$

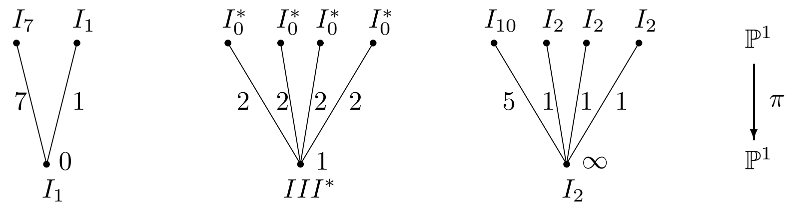

Figure III.18: A realization of **[1,2,2,2,7,10]**

The second base change has ramification index (3,3,1,1) at ∞. It can be chosen as

$$\begin{aligned} \pi: \quad \mathbb{P}^1 &\rightarrow \mathbb{P}^1 \\ (s:t) &\mapsto (1728s^7t : -(s^2-5st+t^2)^3(7s^2-13st+t^2)) \end{aligned}$$

since $1728s^7t + (s^2-5st+t^2)^3(7s^2-13st+t^2) = (t^4-14st^3+63s^2t^2-70s^3t-7s^4)^2$.

As pull-back of $\mathbf{X_{321}}$ via π we realize the constellations **[1,1,2,3,3,14]**:

$$\begin{aligned} y^2 &= x^3 - 3\,(49s^8 - 316s^7t + 4018s^6t^2 - 8624s^5t^3 + 5915s^4t^4 - 1904s^3t^5 + 322s^2t^6 - 28st^7 + t^8)\,x \\ &\quad +2(49s^8 - 964s^7t + 4018s^6t^2 - 8624s^5t^3 + 5915s^4t^4 - 1904s^3t^5 + 322s^2t^6 - 28st^7 + t^8) \\ &\qquad (t^4 - 14st^3 + 63s^2t^2 - 70s^3t - 7s^4) \end{aligned}$$

and **[1,2,2,6,6,7]**:

$$\begin{aligned} y^2 &= x^3 - 3\,(49s^8 + 6164s^7t + 4018s^6t^2 - 8624s^5t^3 + 5915s^4t^4 - 1904s^3t^5 + 322s^2t^6 - 28st^7 + t^8)\,x \\ &\quad -2\,(49s^8 - 14572s^7t + 4018s^6t^2 - 8624s^5t^3 + 5915s^4t^4 - 1904s^3t^5 + 322s^2t^6 - 28st^7 + t^8). \\ &\qquad (t^4 - 14st^3 + 63s^2t^2 - 70s^3t - 7s^4). \end{aligned}$$

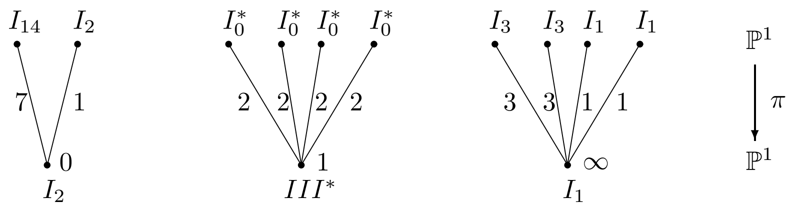

Figure III.19: A realization of **[1,1,2,3,3,14]**

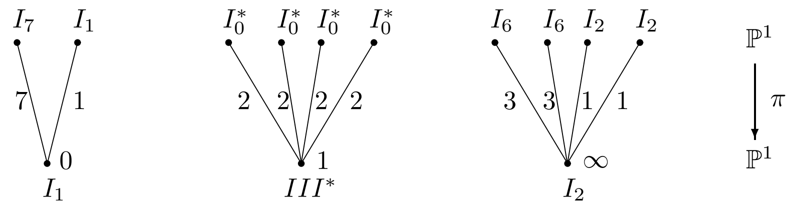

Figure III.20: A realization of **[1,2,2,6,6,7]**

Recall that the pull-back surfaces inherit the group of sections of the rational elliptic surfaces. As a consequence the above fibration with configuration [1,1,2,3,3,14] necessarily has Mordell-Weil group $\mathbb{Z}/(2)$. It differs from the surface with the same configuration and $MW = (0)$, as obtained as a double sextic over $\mathbb{Q}$ in [ATZ, p.55]. The underlying surfaces are not isomorphic, since their discriminants differ.

The third base change has ramification index (3,2,2,1) at ∞. We consider the map

$$\begin{array}{rccc}\pi: & \mathbb{P}^1 & \to & \mathbb{P}^1\\ & (s:t) & \mapsto & (s^7(s+24t):16t^3(7s^2-14st+6t^2)^2(2s-t))\end{array}$$

with $s^7(s+48t)-16t^3(7s^2-14st+6t^2)^2(2s-t) = (24t^4-80t^3s+72s^2t^2-12s^3t-s^4)^2$.

The pull-back via π gives rise to the constellations **[1,2,2,2,3,14]**:

$$\begin{aligned} y^2 \;=\;& x^3-3\,(s^8+12s^7t-784t^3s^5+1764t^4s^4-1512t^5s^3+616t^6s^2-120t^7s+9t^8)\,x\\ &+(-s^8-12s^7t-1568t^3s^5+3528t^4s^4-3024t^5s^3+1232t^6s^2-240t^7s+18t^8)\\ &\quad(-2s^4-12s^3t+36s^2t^2-20st^3+3t^4).\end{aligned}$$

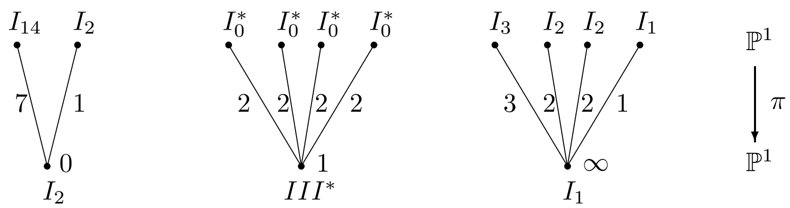

Figure III.21: A realization of **[1,2,2,2,3,14]**

and **[1,2,4,4,6,7]**:

$$\begin{aligned} y^2 \;=\;& x^3-3\,(s^8-392s^5t^3+1764s^4t^4-3024s^3t^5+2464s^2t^6-960st^7+144t^8+24s^7t)\,x\\ &+2\,(-24t^4+80st^3-72s^2t^2+12s^3t+s^4)\\ &\quad(196s^5t^3-882s^4t^4+1512s^3t^5-1232s^2t^6+480st^7-72t^8+s^8+24s^7t).\end{aligned}$$

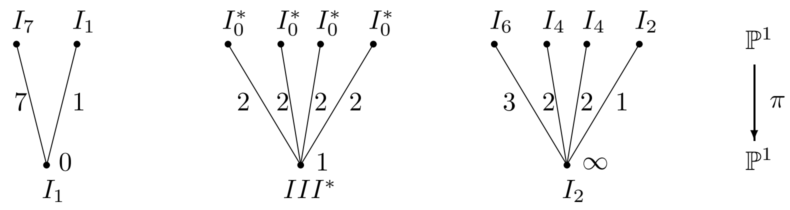

Figure III.22: A realization of **[1,2,4,4,6,7]**

We turn to base changes with ramification index **(6,2)** at 0. Excluding all those ramification indices such that the resulting configuration of singular fibres does not meet the criteria of [MP2], the remaining base changes realize configurations known from [T-Y] or the previous sections. The situation is exactly the same for ramification index **(4,4)**. Hence, the only remaining ramification index at 0 is **(5,3)**. By the above considerations, we can exclude ramification of index (3,3,1,1) at ∞. Then there are three possibilities left:

For ramification index (5,1,1,1) at ∞, consider the map

$$\begin{array}{rccc} \pi: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \\ & (s:t) & \mapsto & (s^5(s-2t)^3 : 4t^5(6s^3-22s^2t-12st^2-9t^3)) \end{array}$$

with $s^5(s-2t)^3 - 4t^5(6s^3-22s^2t-12st^2-9t^3) = (6t^4+4st^3+6s^2t^2-6s^3t+s^4)^2$.

This base change realizes **[1,1,1,5,6,10]**:

$$\begin{aligned} y^2 \;=\; & x^3 - 3\,(16s^8 - 96s^7t + 192s^6t^2 - 128s^5t^3 - 48s^3t^5 + 88s^2t^6 + 24st^7 + 9t^8)\,x \\ & -2\,(3t^4 + 4st^3 + 12s^2t^2 - 24s^3t + 8s^4) \\ & \quad (8s^8 - 48s^7t + 96s^6t^2 - 64s^5t^3 + 48s^3t^5 - 88s^2t^6 - 24st^7 - 9t^8) \end{aligned}$$

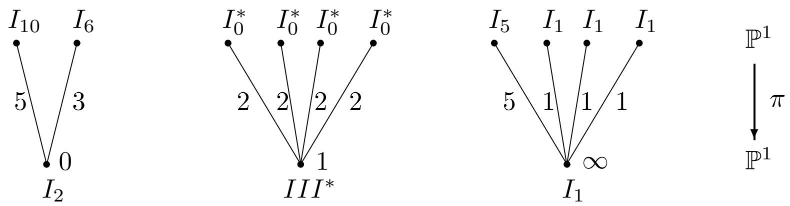

Figure III.23: A realization of **[1,1,1,5,6,10]**

and after exchanging 0 and ∞ also **[2,2,2,3,5,10]**:

$$\begin{aligned} y^2 \;=\; & x^3 - 3\,(s^8 - 6s^3t^5 + 22s^2t^6 + 12st^7 + 9t^8 - 12s^7t + 48s^6t^2 - 64s^5t^3)\,x \\ & +(6t^4 + 4st^3 + 6s^2t^2 - 6s^3t + s^4) \\ & \quad (6s^3t^5 - 22s^2t^6 - 12st^7 - 9t^8 + 2s^8 - 24s^7t + 96s^6t^2 - 128s^5t^3). \end{aligned}$$

The ramification index (4,2,1,1) at ∞ can be obtained by the following base change:

$$\begin{array}{rccc} \pi: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \\ & (s:t) & \mapsto & (s^5(s-8t)^3 : 4t^4(3s+t)^2(5s^2-42st-9t^2)) \end{array}$$

with $s^5(s-8t)^3 - 4t^4(3s+t)^2(5s^2-42st-9t^2) = (s^4-12s^3t+24s^2t^2+32st^3+3t^4)^2$.

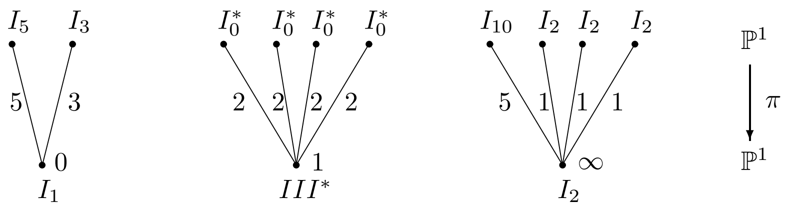

Figure III.24: A realization of **[2,2,2,3,5,10]**

This realizes **[1,1,2,4,6,10]**:

$$\begin{aligned} y^2 &= x^3 - 3\,(16s^8 - 192s^7t + 768s^6t^2 - 1024s^5t^3 - 720s^4t^4 \\ &\qquad +2784s^3t^5 + 1312s^2t^6 + 192st^7 + 9t^8)\,x \\ &\quad -2\,(8s^4 - 48s^3t + 48s^2t^2 + 32st^3 + 3t^4) \\ &\qquad (8s^8 - 96s^7t + 384s^6t^2 - 512s^5t^3 + 720s^4t^4 \\ &\qquad\quad -2784s^3t^5 - 1312s^2t^6 - 192st^7 - 9t^8). \end{aligned}$$

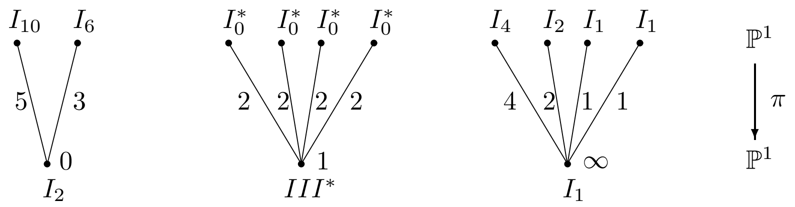

Figure III.25: A realization of **[1,1,2,4,6,10]**

The permutation of 0 and ∞ leads to the constellation **[2,2,3,4,5,8]**:

$$\begin{aligned} y^2 &= x^3 - 3\,(-45s^4t^4 + 348s^3t^5 + 328s^2t^6 + 96st^7 + 9t^8 + s^8 - 24s^7t + 192s^6t^2 - 512s^5t^3)\,x \\ &\quad +(s^4 - 12s^3t + 24s^2t^2 + 32st^3 + 6t^4) \\ &\qquad (45s^4t^4 - 348s^3t^5 - 328s^2t^6 - 96st^7 - 9t^8 + 2s^8 - 48s^7t + 384s^6t^2 - 1024s^5t^3) \end{aligned}$$

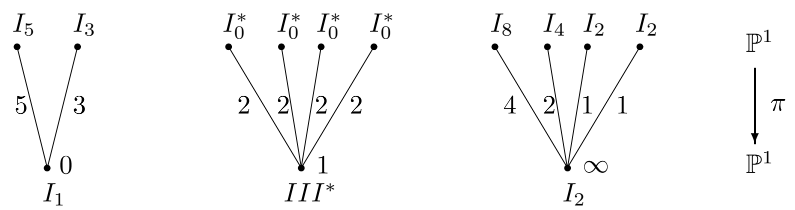

Figure III.26: A realization of **[2,2,3,4,5,8]**

The final possible ramification index at ∞ is (3,2,2,1). This is encoded in the following base change:

$$\begin{aligned} \pi : \quad \mathbb{P}^1 &\quad\rightarrow\quad \mathbb{P}^1 \\ (s:t) &\quad\mapsto\quad (4s^5(9s-4t)^3 : -t^3(4s+t)(10s^2-6st+t^2)^2) \end{aligned}$$

with
$4s^5(9s-4t)^3 + t^3(4s+t)(10s^2-6st+t^2)^2 = (54s^4 - 36s^3t + 4s^2t^2 + 4st^3 - t^4)^2.$

This base change enables us to realize [**1,2,2,3,6,10**]:

$$\begin{aligned} y^2 \;=\;& x^3 - 3\,(9s^8 - 36s^7t + 48s^6t^2 + 112s^5t^3 - 380s^4t^4 + 312s^3t^5 + 72s^2t^6 - 216st^7 + 81t^8)\,x \\ &+(-6s^4 + 12s^3t - 4s^2t^2 - 12st^3 + 9t^4) \\ &\quad(-9s^8 + 36s^7t - 48s^6t^2 + 288s^5t^3 - 760s^4t^4 + 624s^3t^5 + 144s^2t^6 - 432st^7 + 162t^8) \end{aligned}$$

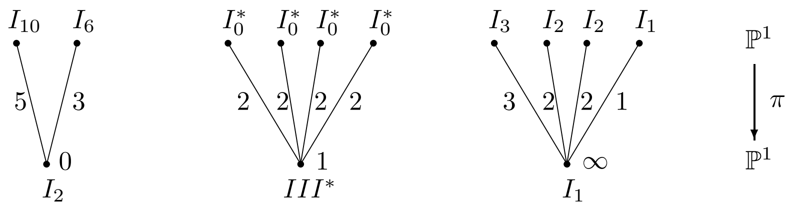

Figure III.27: A realization of [**1,2,2,3,6,10**]

and furthermore [**2,3,4,4,5,6**]:

$$\begin{aligned} y^2 \;=\;& x^3 - 3\,(-208s^5t^3 - 380s^4t^4 + 312s^3t^5 + 72s^2t^6 - 216st^7 \\ &\qquad +81t^8 + 144s^8 - 576s^7t + 768s^6t^2)\,x \\ &-2\,(-6s^4 + 12s^3t - 4s^2t^2 - 12st^3 + 9t^4) \\ &\quad(816s^5t^3 - 380s^4t^4 + 312s^3t^5 + 72s^2t^6 - 216st^7 \\ &\qquad +81t^8 - 288s^8 + 1152s^7t - 1536s^6t^2). \end{aligned}$$

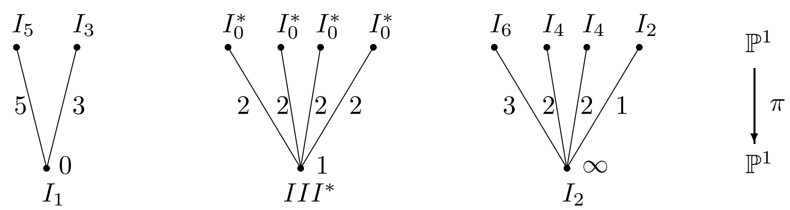

Figure III.28: A realization of [**2,3,4,4,5,6**]

We shall now consider those base changes such that both cusps, 0 and ∞, have three pre-images. As a starting point, consider ramification of index **(6,1,1)** at 0. There are three respective ramification indices at ∞ such that π does not admit a factorization. However, (4,2,2) cannot occur, since the resulting fibration does not exist by [MP2]. The base change with index (4,3,1) can only be defined over the number field $\mathbb{Q}(\sqrt{-3})$. It is given at the end of this section. The remaining base change with ramification index (5,2,1) at ∞ can be defined by

$$\begin{aligned} \pi:\quad \mathbb{P}^1 \;&\rightarrow\; \mathbb{P}^1 \\ (s:t) \;&\mapsto\; (4s^6(9s^2+24st+70t^2) : t^5(14s-5t)^2(4s-t)) \end{aligned}$$

with $4s^6(9s^2+24st+70t^2) - t^5(14s-5t)^2(4s-t) = (5t^4 - 24t^3s + 18s^2t^2 + 8s^3t + 6s^4)^2$.

The pull-back surface by π has the singular fibres [**1,2,2,2,5,12**]:

$$\begin{aligned} y^2 \;=\;& x^3 - 3\,(9s^8 + 24s^7t + 70s^6t^2 - 784t^5s^3 + 756t^6s^2 - 240t^7s + 25t^8)\,x \\ &+(50t^82 - 9s^8 - 24s^7t - 70s^6t^2 - 1568t^5s^3 + 1512t^6s^2 - 480t^7s) \\ &\quad(5t^4 - 24st^3 + 18s^2t^2 + 8s^3t + 6s^4) \end{aligned}$$

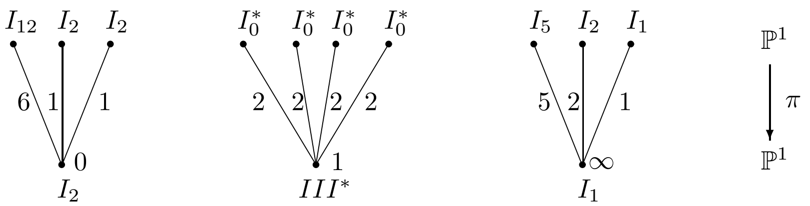

Figure III.29: A realization of **[1,2,2,2,5,12]**

Permuting 0 and ∞ gives rise to the constellation **[1,1,2,4,6,10]**. This was already realized in the preceding paragraphs where 0 was assumed to have only two preimages (Fig. III.25). Although the fibrations differ, the underlying complex surfaces are isomorphic. This will be discussed in Section III.7. There we will also include a figure of the last fibration (Fig. III.35).

We now come to base changes with ramification index **(5,2,1)** at 0. Again, the other cusp ∞ cannot have ramification index (4,2,2). Furthermore, the base change with ramification index (5,2,1) at ∞ can only be defined over the number field $\mathbb{Q}(7x^3 + 19x^2 + 16x + 8)$. It is given at the end of this section. Here, we only construct the two remaining base changes with ramification index (4,3,1) or (3,3,2) at ∞. For the first, consider the map

$$\begin{array}{rccc} \pi: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \\ & (s:t) & \mapsto & (2^8 s^5 (24s-7t)^2(4s+3t) : -t^4(7s-t)^3(15s-t)) \end{array}$$

with
$2^8s^5(24s-7t)^2(4s+3t)+t^4(7s-t)^3(15s-t) = (t^4-18t^3s+69s^2t^2-32s^3t-384s^4)^2$.

This enables us to realize the configurations **[1,2,3,4,4,10]**:

$$\begin{aligned} y^2 \; = \; & x^3 - 3\,(36864s^8 + 6144s^7t - 12992s^6t^2 + 2352t^3s^5 \\ & \qquad +5145t^4s^4 - 2548t^5s^3 + 462t^6s^2 - 36t^7s + t^8)\,x \\ & -2\,(-t^4 + 18st^3 - 69s^2t^2 + 32s^3t + 384s^4) \\ & \quad (18432s^8 + 3072s^7t - 6496s^6t^2 + 1176t^3s^5 \\ & \qquad -5145t^4s^4 + 2548t^5s^3 - 462t^6s^2 + 36t^7s - t^8) \end{aligned}$$

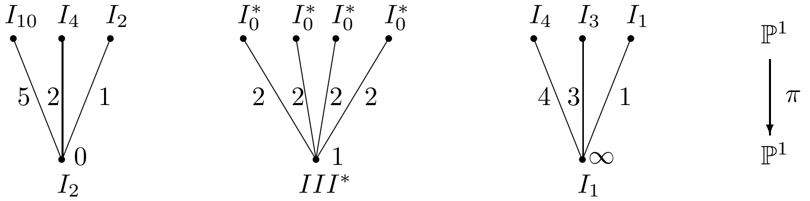

Figure III.30: A realization of **[1,2,3,4,4,10]**

and **[1,2,2,5,6,8]**:

$$\begin{aligned} y^2 \; = \; & x^3 - 3\,(5145t^4s^4 - 2548t^5s^3 + 462t^6s^2 - 36t^7s + t^8 \\ & \qquad +589824s^8 + 98304s^7t - 207872s^6t^2 + 37632t^3s^5)\,x \\ & +2\,(-t^4 + 18st^3 - 69s^2t^2 + 32s^3t + 384s^4) \\ & \quad (-5145t^4s^4 + 2548t^5s^3 - 462t^6s^2 + 36t^7s - t^8 \\ & \qquad +1179648s^8 + 196608s^7t - 415744s^6t^2 + 75264t^3s^5). \end{aligned}$$

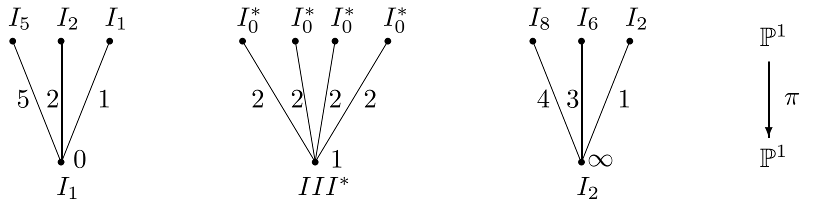

Figure III.31: A realization of **[1,2,2,5,6,8]**

The second base change with ramification index (3,3,2) at ∞ can be constructed in the following way:

$$\begin{array}{rccc} \pi: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \\ & (s:t) & \mapsto & (9s^5(s+6t)^2(9s+4t) : -4t^2(10s^2+24st+9t^2)^3) \end{array}$$

with
$9s^5(s+6t)^2(9s+4t)+4t^2(10s^2+24st+9t^2)^3 = (9s^4+56s^3t+234s^2t^2+216st^3+54t^4)^2.$

It enables us to realize the configurations **[2,2,3,3,4,10]**:

$$\begin{aligned} y^2 &= x^3 - 3\,(1296s^8 + 8064s^7t + 77392s^6t^2 + 232992s^5t^3 + 319680s^4t^4 \\ &\qquad +214272s^3t^5 + 71928s^2t^6 + 11664st^7 + 729t^8)\,x \\ &\quad -2\,(27t^4 + 216st^3 + 468s^2t^2 + 224s^3t + 72s^4) \\ &\qquad (648s^8 + 4032s^7t - 57304s^6t^2 - 229104s^5t^3 - 319680s^4t^4 \\ &\qquad -214272s^3t^5 - 71928s^2t^6 - 11664st^7 - 729t^8) \end{aligned}$$

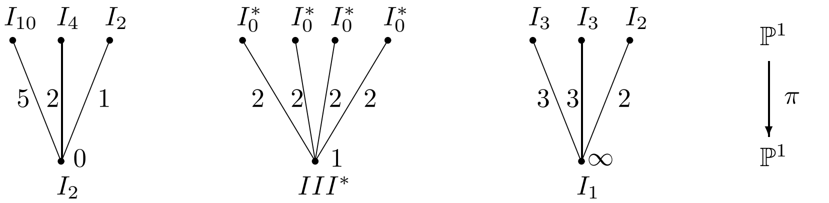

Figure III.32: A realization of **[2,2,3,3,4,10]**

and **[1,2,4,5,6,6]**:

$$\begin{aligned} y^2 &= x^3 - 3\,(4348s^6t^2 + 8496s^5t^3 + 19980t^4s^4 + 26784t^5s^3 \\ &\qquad +17982t^6s^2 + 5832t^7s + 729t^8 + 81s^8 + 1008s^7t)\,x \\ &\quad +(54t^4 + 216st^3 + 234s^2t^2 + 56s^3t + 9s^4) \\ &\qquad (5696s^6t^2 - 4608s^5t^3 - 19980t^4s^4 - 26784t^5s^3 \\ &\qquad -17982t^6s^2 - 5832t^7s - 729t^8 + 162s^8 + 2016s^7t). \end{aligned}$$

Eventually, we come to the remaining base changes with ramification index **(4,3,1)**, **(4,2,2)** or **(3,3,2)** at 0. All but one of them either cannot exist or give rise to known configurations of singular fibres. The remaining base change has ramification index (4,3,1) at both cusps, 0 and ∞. It can only be defined over the quadratic extension $\mathbb{Q}(\sqrt{2})$ of $\mathbb{Q}$. We give it below.

We conclude this section by collecting the four base changes which appeared above, but are only defined over an extension of $\mathbb{Q}$. They are presented in the order of appearance in the course of this section:

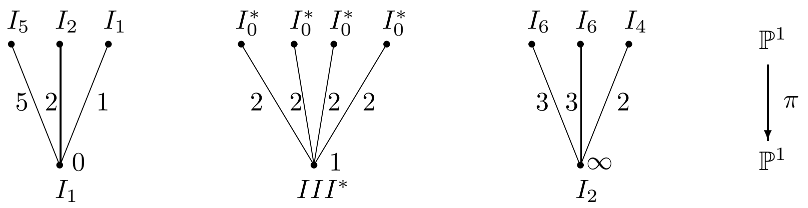

Figure III.33: A realization of **[1,2,4,5,6,6]**

- A base change with ramification indices (7,1) and (4,2,1,1) at 0 and ∞: Let v be a solution of $2x^2 - 7x + 28$. Then the base change can be given by $\pi((s:t)) = (s^7(s{+}2vt) : (10633/4v{-}2401)/40\, t^4(s{-}t)^2(s^2{+}(6{-}2v)st/5{+}(3v{-}14)t^2/10))$. It realizes the configurations **[1,1,2,2,4,14]** and **[1,2,2,4,7,8]** over $\mathbb{Q}(\sqrt{-7})$. This is the minimal field of definition not only for the fibrations, but for the surfaces. This follows from Proposition II.13.1: The surfaces have discriminant 56 and 224, so $\mathbb{Q}(\sqrt{-d}) = \mathbb{Q}(\sqrt{-14})$. This field has exponent 4.

- The base change with ramification indices (6,1,1) and (4,3,1) can be defined over $\mathbb{Q}(\sqrt{-3})$. If v is a solution to $3x^2 - 3x + 7$, then we have $\pi((s:t)) = (s^6(s^2 + 4vst - (19v+14)/5\, t^2) : (1763v - 259)/20\, t^4(s-t)^3(s - (4v-7)/5\, t))$. On the one hand we can thereby produce the configuration **[1,1,2,6,6,8]**. The field of definition of this fibration is the same as for the double sextic in [P, p. 313]. The other pull-back has the configuration **[1,2,2,3,4,12]**, which has already been obtained over $\mathbb{Q}$ in Figure III.6.

  We will now show that these two fibrations have different Mordell-Weil groups: For each fibration, consider the pull-back of a primitive section of the basic rational elliptic surface $\mathbf{X_{321}}$ resp. $\mathbf{X_{141}}$. This section can be directly computed in terms of the components of the singular fibres which it meets. Comparing this shape to [ATZ, Remark 0.4 (5)], we conclude that in both cases the induced section generates the Mordell-Weil group of the pull-back. In particular, MW of basic surface and pull-back coincide. Since $MW(\mathbf{X_{321}}) = \mathbb{Z}/2$ and $MW(\mathbf{X_{141}}) = \mathbb{Z}/4$, we obtain the claim.

  The above fibrations over $\mathbb{Q}(\sqrt{-3})$ provide the first examples where a fibration cannot be defined over $\mathbb{Q}$ (cf. III.8.2), although the underlying surface can. To see this, we use the intersection form on the transcendental lattice and Theorem I.3.2.

  By [S-Z], the configuration **[1,1,2,6,6,8]** implies the intersection form $diag(6,24)$. The complex K3 surface with this form admits another elliptic fibration, No. 144 in the notation of [S-Z]. Table III.2 derives this fibration over $\mathbb{Q}$. Similarly, the configuration **[1,2,2,3,4,12]** with $MW = \mathbb{Z}/2$ as above implies intersection form $12I$. This coincides with the intersection form of the **[2,2,4,4,6,6]** configuration. This fibration was realized over $\mathbb{Q}$ in Figure III.8.

- A base change with ramification index (5,2,1) at both cusps, 0 and ∞: Let v be a zero of $7x^3{+}19x^2{+}16x{+}8$. Then consider $\pi((s:t)) = (167s^5(s{-}2t)^2(s{+}4(v{+}1)\,t) : -(15v^2{+}55v{+}52)\,t^5(4s{+}(3v^2{+}3v{-}4)\,t)^2(8s{-}(7v^2{+}15v{+}4)\,t))$. The pull-back gives rise to the configuration **[1,2,2,4,5,10]** over the extension $\mathbb{Q}(x^3 - 75x + 5150)$.

- The final base change of this section has ramification index (4,3,1) at 0 and ∞. It is defined over $\mathbb{Q}(\sqrt{2})$. Let v be a root of $7x^2{+}8x{+}2$. Then consider $\pi((s:t)) = (2^4 7^3 s^4(s{-}t)^3(s{+}(8v{+}3)\,t) : (8v{+}3)\,t^4(14s{+}(9v{+}4)\,t)^3(2s{-}(5v{+}4)\,t))$. This map leads to the extremal K3 surface with singular fibres **[1,2,3,4,6,8]**.

Together with [T-Y], the previous three sections exhaust the extremal semi-stable elliptic K3 surfaces which can be realized as non-general pull-back of rational elliptic surfaces. Here the term "general" refers to the general pull-back construction involving the induced J-map and the rational elliptic surface with singular fibres III^*, II, I_1 (cf. Rem. III.8.2). Its construction as a base change of degree 24 with very restricted ramification is beyond the scope of this chapter, as it essentially makes no use of the elliptic fibration of the basic rational elliptic surface. A general solution to this problem has recently been announced by Beukers and Montanus [B-M]. We comment on this in Section III.8.

III.6 The non-semi-stable fibrations

This section is devoted to a brief analysis of the non-semi-stable extremal elliptic K3 fibrations. We will determine all fibrations which can be derived from rational elliptic surfaces. The treatment is significantly simplified by the fact that every such surface has a non-reduced fibre. This follows from [Kl, Thm. 1.2]. Moreover, every K3 fibration with more than one non-reduced fibre is easily transformed into a rational elliptic surface by the deflation process described in Section III.1. Such an extremal K3 surface necessarily has three or four cusps. In the case of three cusps, we find the K3 surfaces directly in [Ng]. If the K3 has four cusps, the corresponding rational elliptic surfaces is given in [H]. All but one of the corresponding rational surfaces can be uniquely defined over $\mathbb{Q}$ up to $\mathbb{C}$-isomorphism. Except for three cases which are specified below, the deflation is also defined over $\mathbb{Q}$.

Table III.1 collects the extremal K3 fibrations with three or four cusps. The numbering refers to the classification in [S-Z] and will be employed throughout this section. Each fibration can be derived from a rational elliptic surface by manipulating the Weierstrass equation. In other words, we reobtain the rational elliptic surface by deflation.

No.	Config.	No.	Config.	No.	Config.	No.	Config.
113	5,5,1*,1*	206	1,1,2*,8*	255	1,1,8*,IV*	293	2,5,III*,IV*
121	2,8,1*,1*	209	1,1,1*,9*	256	3,3,III*,III*	294	1,6,III*,IV*
124	1,9,1*,1*	219	IV*,IV*,IV*	257	2,4,III*,III*	295	1*,III*,IV*
136	2*,2*,2*	220	4,4,IV*,IV*	258	1,5,III*,III*	296	2,2 II*,II*
137	4,4,2*,2*	222	2,4,IV*,IV*	279	0*,III*,III*	297	IV,II*,II*
153	3,6,1*,2*	226	1,7,IV*,IV*	280	3,5,1*,III*	313	1*,1*,II*
154	1,8,1*,2*	243	4,5,1*,IV*	281	2,6,1*,III*	314	2,5,1*,II*
155	3,3,3*,3*	244	2,7,1*,IV*	282	1,7,1*,III*	315	1,6,1*,II*
167	2,6,1*,3*	245	1,8,1*,IV*	283	3,4,2*,III*	316	3,3,2*,II*
168	1,6,2*,3*	246	2*,IV*,IV*	284	1,6,2*,III*	317	1,5,2*,II*
169	2,2,4*,4*	247	3,5,2*,IV*	285	1*,2*,III*	318	2,3,3*,II*
177	1*,1*,4*	248	1,7,2*,IV*	286	2,4,3*,III*	319	1,2,5*,II*
178	2,4,2*,4*	249	2,5,3*.IV*	287	1,5,3*,III*	320	1,1,6*,II*
179	1,1,5*,5*	250	1*,3*,IV*	288	2,3,4*,III*	321	2,4,II*,IV*
187	1,5,1*,5*	251	1,5,4*,IV*	289	1,3,5*,III*	322	1,5,II*,IV*
195	2,3,1*,6*	252	2,3,5*,IV*	290	1,2,6*,III*	323	0*,II*,IV*
196	1,3,2*,6*	253	1,4,5*,IV*	291	1,1,7*,III*	324	2,3,II*,III*
197	1,2,3*,6*	254	1,2,7,IV*	292	3,4,III*,IV*	325	1,4,II*,III*
205	1,2,1*,8*						

Table III.1: The extremal K3 fibrations with three or four cusps

There are four K3 fibrations in the above table which cannot be defined over $\mathbb{Q}$ by this approach. For three of them, the basic rational elliptic surfaces nevertheless is defined over $\mathbb{Q}$. However, since it has cusps which are conjugate in some quadratic field, certain manipulations are not defined over $\mathbb{Q}$. As a result, for the fibrations with No. 187, No. 245 and No. 282, fields of definition can only be given as $\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{-7})$, respectively. The rational elliptic surface which corresponds to No. 294 can itself only be defined over $\mathbb{Q}(\sqrt{-3})$.

For these four fibrations, we shall briefly discuss whether the underlying K3 surface can be defined over $\mathbb{Q}$. This is possible for No. 187, 245, and 294: Again we use the intersection form on the transcendental lattice, as determined in [S-Z]. This gives respective isomorphisms with the following surfaces: No. 259 from Table III.2, No. 33 from Figure III.7, and No. 84 from Figure III.6 (which has $MW = \mathbb{Z}/4$). On the other hand, the surface admitting configuration No. 282 cannot be defined over $\mathbb{Q}$ by Proposition II.13.1. Here the field of definition $\mathbb{Q}(\sqrt{-7})$ is minimal.

We now come to the extremal K3 fibrations which are still missing with respect to the classification in [S-Z]. These have exactly one non-reduced fibre. We will derive half of them from rational elliptic surfaces. We will apply base change and further make use of the "transfer of *" as explained in [M2]. Essentially this just moves the * from one fibre to another (a priori not necessarily singular). This can be achieved by translating the common factor of the polynomials A and B in the Weierstrass equation. We will take base changes of degree 2 to 6 into account, depending on the singular fibres of the basic rational elliptic surface. For each degree, we are going to exploit one example in more detail. The remaining fibrations will only be sketched very roughly.

Degree 2: These base changes involve the rational elliptic surfaces with four singular fibres, which have one fibre of type III. For example, take the surface Y with singular fibres I_1, I_3, I_5, III. According to [H], this surface and all its cusps can be defined over $\mathbb{Q}$. Consider a quadratic base change π of $\mathbb{P}^1$ which is ramified at the cusp of the III-fibre and at one further cusp. The pull-back of Y via π is a K3 surface over $\mathbb{Q}$ with five semi-stable singular fibres and one of type I_0^*. Transferring the * gives rise to three extremal K3 surfaces with one non-reduced and four semi-stable fibres. For instance, the configuration [2,3,3,5,5,0*] can be transformed to **[3,3,5,5,2*]** (No. 138), **[2,3,5,5,3*]** (No. 157) or **[2,3,3,5,5*]** (No. 180).

Degree 3: Let $Y = \mathbf{X_{141}}, \mathbf{X_{222}}$ or $\mathbf{X_{411}}$ as defined in Section III.3. After deflation, any base change of degree 3 gives a K3 surface. This is extremal if and only if the base change is only ramified at the three cusps (i.e. the cusps have 5 pre-images in total). We will refer to the possible base changes as triple covers without specifying the particular one. They are all defined over $\mathbb{Q}$.

Degree 4: Consider $\mathbf{X_{431}}$ as introduced in Section III.4. We want to apply a base change of degree 4. This gives an extremal K3 fibration, if we select the ramification index (3,1) at the cusp of the IV^*-fibre and minimize the number of pre-images of the other two cusps at 4. In fact, we can adequately choose both base changes π_2 and π_4 from the third section after exchanging cusps. The third useful base change has ramification index (3,1) at every cusp. It can be given by

$$\pi_{\mathbf{3}}((s:t)) = (s^3(s-2t) : t^3(t-2s)).$$

This base change, for instance, gives rise to the constellation **[1,3,3,9,IV*]** (No. 233).

Degree 5: For the base changes of the next two paragraphs, the basic rational elliptic surface will be $\mathbf{X_{321}}$ (defined in Section III.5). A base change of degree 5 with ramification index (2,2,1) at 1 (the cusp of III^*) leads to a K3 surface. After deflation, only one III^* remains in the pull-back. The resulting K3 fibrations is extremal if and only if the other two cusps have the minimal number of 4 pre-images. There are five such base changes, all but one defined over $\mathbb{Q}$. We will only go into detail for one of them and then list the others:

- The first base change can be given by

$$\pi_{\mathbf{E}}((s:t)) = (s(s^2-5st+5t^2)^2 : 4t^5),$$

since $s(s^2-5st+5t^2)^2-4t^5 = (s-4t)(s^2-3ts+t^2)^2$. As pull-back we realize **[2,4,4,5,III*]** (No. 259) and **[1,2,2,10,III*]** (No. 275).

- $\pi_{\mathbf{F}}((s:t)) = (4s^3(3s-5t)^2 : t^4(15s+2t))$.
- $\pi_{\mathbf{G}}((s:t)) = (s^3(4s-5t)^2 : t^3(4t-5s)^2)$.
- $\pi_{\mathbf{H}}((s:t)) = (64t^5 : (t-s)^3(9s^2-33st+64t^2))$.
- $\pi_{\mathbf{I}}((s:t)) = (s^4(s-5t) : t^4(2i-11)(5s+(3+4i)t)$ with $i^2=-1$.

Degree 6: Consider a base change of degree 6 with ramification index (2,2,2) at 1. We apply this base change to $\mathbf{X_{321}}$. After deflation and transfer of *, we obtain an elliptic K3 surface whose singular fibres sit above the pre-images of the other two cusps 0 and ∞. Restricting their number to the minimum 5, we achieve an extremal K3 fibration. The transfer of * turns a distinct semi-stable fibre I_n into its non-reduced relative of type I_n^*. Choosing different I_n, one base change gives rise to at most 9 different fibrations. There are six base changes with the above properties. We give the three maps without factorization:

- $\pi_{\mathbf{A}}((s:t)) = (4(s^2-4st+t^2)^3 : 27t^4s(s-4t))$ or alternatively $\pi'_{\mathbf{A}}((s:t)) = (4s^3(s-2t)^3 : t^4(3s^2-6st-t^2))$.
- $\pi_{\mathbf{B}}((s:t)) = (-4t^5(6s+t) : s^3(2s-5t)^2(s-4t))$.
- $\pi_{\mathbf{C}}((s:t)) = (-s^4(s^2+2st+5t^2) : 4t^5(t-2s))$.

Let us mention one particular example in more detail: We want to realize the configuration **[1,2,3,10,2*]** (No. 148). This can be achieved as pull-back from $\mathbf{X_{321}}$ via the base change π_B. Here we need ramification index (3,2,1) at the I_1-fibre and (5,1) at the I_2. The remarkable point about this construction is that we still have a choice of where to move the * after the pull-back: We can transfer it either to the I_2 at 5/2 which sits above the original I_1, or to the I_2 over $-1/6$ which comes from the I_2-fibre of $\mathbf{X_{321}}$. We will prove in the next section that the resulting two surfaces are not isomorphic. This is the only example where there is such an ambiguity concerning the transfer of *.

We are now in the position to compute all the remaining extremal elliptic K3 fibrations which can be derived from rational elliptic surfaces by our simple methods. The following table collects their configurations together with the number at which they appear in [S-Z]. We further add the Mordell-Weil group MW and the reduced coefficients of the intersection form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ on the transcendental lattice which determine the isomorphism class of the surface (up to orientation if there is ambiguity in the sign of b). The right-hand part of the table gives a very brief description of the construction and the field of definition for the fibration. For shortness we will indicate the occurrence of a transfer of * only by a * in the description of the construction. We will not mention deflation.

No.	Config.	MW	a	b	c	Construction	def.
114	1,4,6,6,1*	$\mathbb{Z}/2$	12	0	12	pull-back from X_{321} via π_A *	$\mathbb{Q}$
115	1,5,5,6,1*	0	20	0	30	double cover of [1,3,5,III] *	$\mathbb{Q}$
116	2,4,5,6,1*	$\mathbb{Z}/2$	12	0	20	pull-back from X_{321} via π_B *	$\mathbb{Q}$
117	1,2,7,7,1*	0	14	0	28	double cover of [1,1,7,III] *	$\mathbb{Q}(\sqrt{-7})$
122	2,3,4,8,1*	$\mathbb{Z}/4$	6	0	8	triple cover of X_{141}	$\mathbb{Q}$
123	2,2,5,8,1*	$\mathbb{Z}/2$	8	0	20	pull-back from X_{321} via π_C *	$\mathbb{Q}$
127	1,3,3,10,1*	0	6	0	60	double cover of [1,3,5,III] *	$\mathbb{Q}$
128	2,2,3,10,1*	$\mathbb{Z}/2$	2	0	60	pull-back of X_{321} via π_B *	$\mathbb{Q}$
129	1,2,4,10,1*	$\mathbb{Z}/2$	8	4	12	pull-back from X_{321} via π_C *	$\mathbb{Q}(\sqrt{-1})$
132	1,2,2,12,1*	$\mathbb{Z}/4$	2	0	6	triple cover of X_{141}	$\mathbb{Q}$
133	1,1,3,12,1*	$\mathbb{Z}/2$	6	0	6	triple cover of X_{141}	$\mathbb{Q}$
135	1,1,1,14,1*	0	6	2	10	double cover of [1,1,7,III] *	$\mathbb{Q}(\sqrt{-7})$
138	3,3,5,5,2*	0	30	0	30	double cover of [1,3,5,III] *	$\mathbb{Q}$
139	2,2,6,6,2*	$\mathbb{Z}/2\times\mathbb{Z}/2$	6	0	6	triple cover of X_{222}	$\mathbb{Q}$
140	2,4,4,6,2*	$\mathbb{Z}/2\times\mathbb{Z}/2$	4	0	12	triple cover of X_{222}	$\mathbb{Q}$
141	1,4,5,6,2*	$\mathbb{Z}/2$	4	0	30	pull-back from X_{321} via π_B *	$\mathbb{Q}$
142	1,1,7,7,2*	0	14	0	14	double cover of [1,1,7,III] *	$\mathbb{Q}(\sqrt{-7})$
144	2,3,3,8,2*	$\mathbb{Z}/2$	6	0	24	pull-back from X_{321} via π_A *	$\mathbb{Q}$
145	1,3,4,8,2*	$\mathbb{Z}/2$	4	0	24	triple cover of X_{141} *	$\mathbb{Q}$
146	1,2,5,8,2*	$\mathbb{Z}/2$	6	2	14	pull-back from X_{321} via π_C *	$\mathbb{Q}(\sqrt{-1})$
148	1,2,3,10,2*	$\mathbb{Z}/2$	6 4	0 2	10 16	pull-back from X_{321} via π_B *	$\mathbb{Q}$
149	1,1,4,10,2*	$\mathbb{Z}/2$	4	0	10	pull-back from X_{321} via π_C *	$\mathbb{Q}$
151	1,1,2,12,2*	$\mathbb{Z}/2$	4	0	6	triple cover of X_{141} *	$\mathbb{Q}$
156	3,4,4,4,3*	$\mathbb{Z}/4$	8	4	8	triple cover of X_{141}	$\mathbb{Q}$
157	2,3,5,5,3*	0	10	0	60	double cover of [1,3,5,III] *	$\mathbb{Q}$
161	2,2,3,8,3*	$\mathbb{Z}/2$	4	0	24	pull-back from X_{321} via π'_A *	$\mathbb{Q}$
162	1,2,4,8,3*	$\mathbb{Z}/4$	2	0	8	triple cover of X_{141}	$\mathbb{Q}$
163	1,2,2,10,3*	$\mathbb{Z}/2$	4	0	10	pull-back from X_{321} via π_B *	$\mathbb{Q}$
164	1,1,3,10,3*	0	2	0	60	double cover of [1,3,5,III] *	$\mathbb{Q}$
166	1,1,1,12,3*	$\mathbb{Z}/4$	2	1	2	triple cover of X_{141}	$\mathbb{Q}$
170	3,3,4,4,4*	$\mathbb{Z}/2$	12	0	12	double cover of [2,3,4,III] *	$\mathbb{Q}$
171	1,1,6,6,4*	$\mathbb{Z}/2$	6	0	6	pull-back from X_{321} via π_A *	$\mathbb{Q}$
172	2,2,4,6,4*	$\mathbb{Z}/2\times\mathbb{Z}/2$	2	0	12	triple cover of X_{222} *	$\mathbb{Q}$
173	1,2,5,6,4*	$\mathbb{Z}/2$	2	0	30	pull-back from X_{321} via π_B *	$\mathbb{Q}$
175	1,2,3,8,4*	$\mathbb{Z}/2$	2	0	24	triple cover of X_{411}	$\mathbb{Q}$
176	1,1,2,10,4*	$\mathbb{Z}/2$	2	0	10	pull-back from X_{321} via π_C *	$\mathbb{Q}$
180	2,3,3,5,5*	0	12	0	30	double cover of [1,3,5,III] *	$\mathbb{Q}$
181	1,2,4,6,5*	$\mathbb{Z}/2$	4	0	12	pull-back from X_{321} via π_B *	$\mathbb{Q}$
182	1,1,5,6,5*	0	4	0	30	double cover of [1,3,5,III] *	$\mathbb{Q}$
184	1,2,2,8,5*	$\mathbb{Z}/2$	4	0	8	pull-back from X_{321} via π_C *	$\mathbb{Q}$
188	2,2,4,4,6*	$\mathbb{Z}/2\times\mathbb{Z}/2$	4	0	4	triple cover of X_{222}	$\mathbb{Q}$
189	1,1,5,5,6*	0	10	0	10	double cover of [1,3,5,III] *	$\mathbb{Q}$
190	1,2,4,5,6*	$\mathbb{Z}/2$	2	0	20	pull-back from X_{321} via π_B *	$\mathbb{Q}$
191	2,2,2,6,6*	$\mathbb{Z}/2\times\mathbb{Z}/2$	4	2	4	triple cover of X_{222}	$\mathbb{Q}$
192	1,1,4,6,6*	$\mathbb{Z}/2$	2	0	12	pull-back from X_{321} via π'_A *	$\mathbb{Q}$
201	1,1,2,7,7*	0	6	2	10	double cover of [1,1,7,III] *	$\mathbb{Q}(\sqrt{-7})$
202	2,2,3,3,8*	$\mathbb{Z}/2$	6	0	6	pull-back from X_{321} via π_A *	$\mathbb{Q}$
203	1,2,3,4,8*	$\mathbb{Z}/2$	4	0	6	triple cover of X_{411} *	$\mathbb{Q}$
204	1,2,2,5,8*	$\mathbb{Z}/2$	2	0	4	pull-back from X_{321} via π_C *	$\mathbb{Q}$
210	1,1,3,3,10*	0	6	0	6	double cover of [1,3,5,III] *	$\mathbb{Q}$
211	1,2,2,3,10*	$\mathbb{Z}/2$	2	0	6	pull-back from X_{321} via π_B *	$\mathbb{Q}$
212	1,1,2,4,10*	$\mathbb{Z}/2$	2	0	4	pull-back from X_{321} via π_C *	$\mathbb{Q}$
215	1,1,2,2,12*	$\mathbb{Z}/2$	2	0	2	triple cover of X_{411}	$\mathbb{Q}$
216	1,1,1,3,12*	$\mathbb{Z}/2$	2	1	2	triple cover of X_{411}	$\mathbb{Q}$
218	1,1,1,1,14*	0	2	0	2	double cover of [1,1,7,III] *	$\mathbb{Q}$
223	1,3,6,6,IV*	$\mathbb{Z}/3$	6	0	6	pull-back from X_{431} via π_2	$\mathbb{Q}$
224	3,3,4,6,IV*	$\mathbb{Z}/3$	6	0	12	pull-back from X_{431} via π_4	$\mathbb{Q}$
233	1,3,3,9,IV*	$\mathbb{Z}/3$	6	3	6	pull-back from X_{431} via π_3	$\mathbb{Q}$

Table III.2: The extremal K3 fibrations with five cusps

No.	Config.	MW	a	b	c	Construction	def.
234	2,2,3,9,IV*	$\mathbb{Z}/3$	2	0	18	pull-back from X_{431} via π_2	$\mathbb{Q}$
241	1,1,2,12,IV*	$\mathbb{Z}/3$	2	0	4	pull-back from X_{431} via π_4	$\mathbb{Q}$
259	2,4,4,5,III*	$\mathbb{Z}/2$	4	0	20	pull-back from X_{321} via π_E	$\mathbb{Q}$
261	1,4,4,6,III*	$\mathbb{Z}/2$	4	0	12	pull-back from X_{321} via π_F	$\mathbb{Q}$
262	2,3,4,6,III*	$\mathbb{Z}/2$	6	0	12	pull-back from X_{321} via π_G	$\mathbb{Q}$
263	2,2,5,6,III*	$\mathbb{Z}/2$	8	2	8	pull-back from X_{321} via π_H	$\mathbb{Q}$
270	2,2,3,8,III*	$\mathbb{Z}/2$	2	0	24	pull-back from X_{321} via π_F	$\mathbb{Q}$
271	1,2,4,8,III*	$\mathbb{Z}/2$	4	0	8	pull-back from X_{321} via π_I	$\mathbb{Q}(\sqrt{-1})$
275	1,2,2,10,III*	$\mathbb{Z}/2$	2	0	10	pull-back from X_{321} via π_E	$\mathbb{Q}$
276	1,1,3,10,III*	$\mathbb{Z}/2$	4	1	4	pull-back from X_{321} via π_H	$\mathbb{Q}$
298	3,3,4,4,II*	0	12	0	12	double cover of [3,4,II,III] *	$\mathbb{Q}$
299	2,2,5,5,II*	0	10	0	10	double cover of [2,5,II,III] *	$\mathbb{Q}$
301	1,1,6,6,II*	0	6	0	6	double cover of [1,6,II,III] *	$\mathbb{Q}$

Table III.2: The extremal K3 fibrations with five cusps

Note that No. 135 and 201 are fibrations on the same surface as No. 282 and the semi-stable fibration with configuration [1,1,2,2,4,14]. Since this surface cannot be defined over $\mathbb{Q}$ by Proposition II.13.1, the field of definition $\mathbb{Q}(\sqrt{-7})$ is minimal.

Table III.2 completes the treatment of extremal elliptic K3 fibrations which can be derived from rational elliptic surfaces by direct manipulation of the Weierstrass equation or as pull-back via a non-general base change. We would like to finish this section with the following remark. It concerns K3 surfaces which possess an extremal elliptic fibration with non-trivial Mordell-Weil group. For every such surface, this chapter (combined with [T-Y]) gives at least one explicit extremal fibration which is obtained as pull-back from a rational elliptic surface. This result might be compared to the idea of elementary fibrations proposed in [MP3, section 6]. We should, however, point out that our pull-backs can in general not be called elementary in the strict sense of [P],[MP3].

III.7 Isomorphism classes

This section gives a brief discussion of the problem which isomorphism class of complex surfaces a fibration corresponds to. We will apply the lattice theory as sketched in Section I.2 to the transcendental lattices (cf. Thm. I.3.2).

We will be particularly concerned with two configurations: For [1,2,3,10,2*], we already pointed out in Section III.6 that there were two such fibrations (both with $MW \cong \mathbb{Z}/2$) which correspond to different isomorphism classes of complex surfaces. On the other hand, we will check for the two fibrations with configuration [1,1,2,4,6,10] which appeared in Section III.5 that these do indeed give the same surface. Both statements will be proved using of the discriminant form (cf. Sect. I.2).

We recall some elementary properties of the dual lattices for the root lattices A_{n-1} and D_m. These correspond to the fibre types I_n and I_{m-4}^*. Throughout, we will number the components of A_{n-1} as $\Theta_1, \ldots, \Theta_{n-1}$. They form a non-closed chain. D_{m+4} can be identified with the components of a fibre of type I_m^* after omitting one simple component. If the fibration has a section O, this will be the component meeting O. We shall denote this component by C_0. The other components can be found in the following figure.

We will now describe the dual lattices $A_{n-1}^\vee$ and $D_m^\vee$ in terms of elements of $A_{n-1} \otimes_\mathbb{Z} \mathbb{Q}$ resp. $D_m \otimes_\mathbb{Z} \mathbb{Q}$.

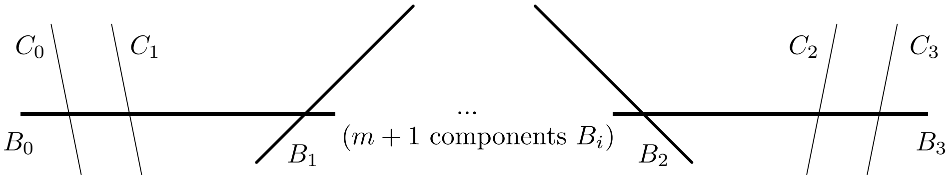

Figure III.34: A fibre of type I_m^*

Lemma III.7.1
Let $n \in \mathbb{N}$ and $Q_n = \frac{1}{n}\sum_{i=1}^{n-1} i\Theta_i \in A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $Q_n^2 = \frac{1-n}{n}$ and

$$A_{n-1}^\vee / A_{n-1} \cong \mathbb{Z}/n = < Q_n > .$$

Lemma III.7.2
Let $m \in \mathbb{N}_0$. Define $R_m = \frac{1}{2}(C_1 + \sum_{i=0}^{m}(i+2)B_i) + \frac{1}{4}((m+2)C_2 + (m+4)C_3) \in D_{m+4} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $R_m^2 = -\frac{m+4}{4}$. If m is odd, then

$$D_{m+4}^\vee / D_{m+4} \cong \mathbb{Z}/4 = < R_m > .$$

If m is even, then let $S_m = C_1 + \sum_{i=1}^{m} B_i + \frac{1}{2}(C_2 + C_3) \in D_{m+4} \otimes_{\mathbb{Z}} \mathbb{Q}$. Here $S_m^2 = -1$. In this case

$$D_{m+4}^\vee / D_{m+4} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 = < R_m, S_m > .$$

Let us now use Lemma III.7.1 to study the fibrations with configuration [1,1,2,4,6,10]. We start with the fibration X which is sketched in Figure III.25. In order to investigate $NS(X)^\vee$, we have to determine the Mordell-Weil group $MW(X)$ of the fibration. Let V denote the trivial lattice of X. This is generated by the 0-section O, a general fibre and the components of fibres which do not meet O. Since X is extremal, V has rank 20 by the formula of Shioda-Tate. We will use that

$$MW(X) \cong NS(X)/V.$$

Since X arises as pull-back from $\mathbf{X_{321}}$, the fibration has $MW(X) \cong \mathbb{Z}/2$. The non-trivial section P meets the two fibres of type I_{10}, I_6 above the I_2 of $\mathbf{X_{321}}$ at their middle components Θ_5 resp. Θ_3. This gives the identification

$$P \equiv 5Q_{10} + 3Q_6 \mod V$$

in the notation of Lemma III.7.1. Note that the lattice $< O, F >$ is isomorphic to the hyperbolic plane U. We obtain

$$NS(X) = < A_9, A_5, P > \oplus A_3 \oplus A_1 \oplus U$$

An easy calculation comparing indices shows that

$$NS(X)^\vee / NS(X) = < Q_{10} + Q_6, Q_4, Q_2 > .$$

The self-intersection numbers of these generators (mod $2\mathbb{Z}$) give

$$q_{NS(X)} = \mathbb{Z}/15(\frac{4}{15}) \oplus \mathbb{Z}/4(-\frac{3}{4}) \oplus \mathbb{Z}/2(-\frac{1}{2})$$

Lemma III.7.3
T_X has intersection form equivalent to

$$\begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}.$$

Proof: We have used in Section II.13 that the quadratic field $\mathbb{Q}(\sqrt{-30})$ has exponent 2. Equivalently, every genus of even positive definite binary matrices with discriminant 120 consists of a single class. Let L be $\mathbb{Z}^2$, endowed with the above intersection form. Then, the lemma's claim is equivalent to

$$q_L = -q_{NS(X)}$$

by Propositions I.2.2 and I.2.3. This equality can be achieved as follows: Consider the generators $(\frac{1}{5}, \frac{1}{3}), (0, \frac{1}{4}), (\frac{1}{2}, 0)$ of $L^\vee/L$. Modulo $2\mathbb{Z}$, these have exactly the above self-intersection numbers after exchanging the sign. Hence the equality follows.

We shall now compare this result to the other fibration X' with configuration [1,1,2,4,6,10]. This can be derived from Figure III.29 after exchanging the cusps 0 and ∞ downstairs (or dividing out by the 2-section and resolving singularities). The fibration can be sketched as follows:

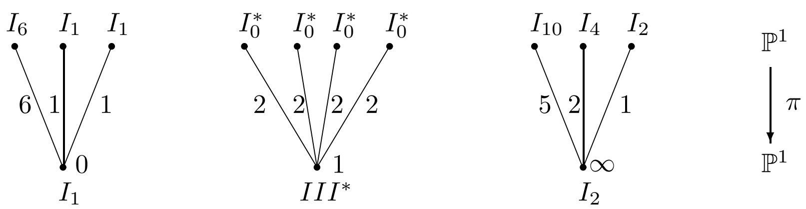

Figure III.35: The fibration X' with configuration **[1,1,2,4,6,10]**

Lemma III.7.4
As a complex surface, X' is isomorphic to X.

In practice, we would be able to deduce this lemma from the classification in [S-Z], since this lists the above configuration uniquely as No. 61. However, we decided to include an independent proof:

By Theorem I.3.2, it is sufficient to show that $T_{X'}$ comes endowed with the same intersection form as T_X (up to $SL_2(\mathbb{Z})$). Again, this will be achieved using the discriminant form. We start by computing $NS(X')$.

We have to find a generator P' for $MW(X') \cong NS(X')/V'$. This is read off from Figure III.35 as

$$P' = 5Q_{10} + 2Q_4 + Q_2$$

in the notation of Lemma III.7.1. Hence, we obtain $NS(X') = < A_9, A_3, P' > \oplus A_5 \oplus U$. In particular, this implies

$$NS(X')^\vee/NS(X') = < 2Q_{10}, Q_4 + Q_2, Q_6 > .$$

Thus

$$q_{NS(X')} = \mathbb{Z}/5(-\frac{18}{5}) \oplus \mathbb{Z}/4(-\frac{5}{4}) \oplus \mathbb{Z}/6(-\frac{5}{6}).$$

Consider T_X. Choose generators $(\frac{3}{5}, 0), (\frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{6})$ for $T_X^\vee/T_X$. Computing their self-intersection numbers, we find

$$q_{T_X} = -q_{NS(X')}.$$

In particular, $q_{T_{X'}} = q_{T_X}$. Since every genus consists of a single class, this implies $T_{X'} \cong T_X$. Therefore $X' \cong X$ by Theorem I.3.2.

Remark III.7.5
Consider the quotients of the two fibrations X and X' by the respective 2-section. Then, the minimal desingularizations do not only differ in terms of their configurations as given in Figures III.26 and III.29, but also as complex surfaces, since their discriminants do not match.

The remainder of this section is devoted to a configuration where the two associated fibrations do indeed belong to different isomorphism classes of complex surfaces. This will be proved by the same method as above.

Consider the configuration [**1,2,3,10,2***] (No. 148). It can be constructed as pull-back from $\mathbf{X_{321}}$ via the base change π_B. In fact, we obtain two fibrations, since after deflation, we can transfer the remaining * of the three fibres of type I_0^* above the original III to two different places: either to the fibre of type I_2 above the original I_1 or to the I_2 which comes from the I_2-fibre of $\mathbf{X_{321}}$. Although both fibration have $MW \cong \mathbb{Z}/2$, the Néron-Severi groups differ. We will deduce that their discriminant forms do not coincide.

Let Z denote the first fibration and Z' the second. We sketch them in the following two figures.

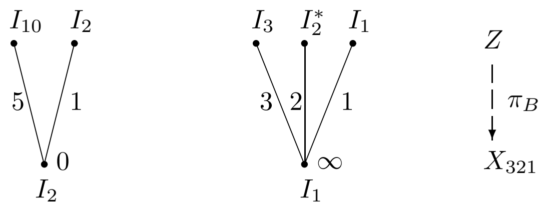

Figure III.36: The fibration Z with configuration [**1,2,3,10,2***]

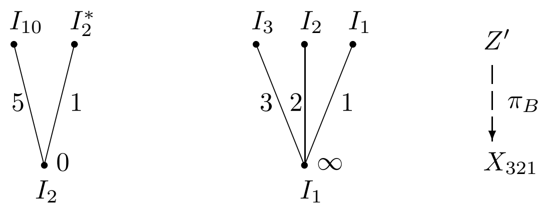

Figure III.37: The fibration Z' with configuration [**1,2,3,10,2***]

Our first aim is to determine the 2-sections of Z and Z'. For this purpose, we investigate the behaviour of a fibre of type I_{2m}^* when divided by a torsion section. The resolution of singularities results in

$$\widetilde{I_{2m}^*/O} = \widetilde{I_{2m}^*/S_{2m}} = I_{4m}^* \quad \text{and} \quad \widetilde{I_{2m}^*/R_{2m}} = I_m^*. \tag{III.1}$$

Here, the second case includes the quotient by $R_{2m} + S_{2m}$ as well. This is because, modulo the trivial lattice V, $R_{2m} + S_{2m}$ is obtained from R_{2m} by exchanging C_2 and C_3.

As a consequence, the 2-section P' of Z' can be identified with

$$P' = 5Q_{10} + R_2 \in NS(Z')/V'$$

after renumbering if necessary. Since $P'^2 \equiv 0 \mod 2$, this identification agrees with the evenness of the cup-product on the Néron-Severi group. On the other hand, this condition implies for the 2-section P of Z

$$P = 5Q_{10} + Q_2 + S_2 \in NS(Z)/V.$$

Lemma III.7.6
T_Z has intersection form $\begin{pmatrix} 4 & 2 \\ 2 & 16 \end{pmatrix}$ *while $T_{Z'}$ is endowed with* $\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$.
In particular, $Z \not\cong Z'$.

Proof: Both claims of the theorem will be established using the discriminant form on NS and T. This is possible because of Proposition I.2.2, since for discriminant 60, each genus of positive definite even binary forms consists of one class.

Let us start with Z'. Here, we easily calculate

$$NS(Z')^\vee/NS(Z') =< 2Q_{10}, R_2 + S_2, Q_3, Q_2 >$$

such that

$$q_{NS(Z')} = \mathbb{Z}/5(-\frac{18}{5}) \oplus \mathbb{Z}/2(-\frac{7}{2}) \oplus \mathbb{Z}/3(-\frac{2}{3}) \oplus \mathbb{Z}/2(-\frac{1}{2}) = \mathbb{Z}/10(-\frac{1}{10}) \oplus \mathbb{Z}/6(-\frac{1}{6}).$$

This leads to the intersection matrix for $T_{Z'}$ as in the lemma, if we choose generators $(0, \frac{1}{10}), (\frac{1}{6}, 0)$ for $T_Z^\vee/T_Z$.

For Z, we see that

$$NS(Z)^\vee/NS(Z) =< 2Q_{10}, Q_2 + R_2, Q_3, S_2 > .$$

This gives

$$q_{NS(Z)} = \mathbb{Z}/5(-\frac{18}{5}) \oplus \mathbb{Z}/2(-2) \oplus \mathbb{Z}/3(-\frac{3}{2}) \oplus \mathbb{Z}/2(-1)$$

We realize that $q_{T_Z} = -q_{NS(Z)}$ for the lemma's intersection matrix by selecting generators $(\frac{2}{5}, \frac{1}{5}), (0, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, 0)$ for $T_Z^\vee/T_Z$. This completes the proof of Lemma III.7.6.

III.8 Final comments

This chapter's investigation produced all extremal elliptic K3 fibrations which can be derived from rational elliptic surfaces by direct, relatively simple manipulations of their Weierstrass equations. Hence, we have to comment on two problems: One of them concerns the general extremal elliptic K3 fibration with a section. In other words, we ask for explicit equations for any such fibration. This problem refers to the classification of [S-Z] and in particular to the Miranda-Persson list [MP2]. We will address it later.

Meanwhile, we shall now return to the initial problem for this investigation (Qu. I.5.4): For which newforms from Table II.1 can one find an associated singular K3 surface over $\mathbb{Q}$? By Proposition II.13.1, this can be rephrased as follows:

Question III.8.1
Up to squares, which discriminants belong to singular K3 surfaces over $\mathbb{Q}$?

The fibrations of [Ir] or [Sh3], [L-Y] and [T-Y] give rise to the discriminants 3, 4, 7, 8, and 19. In the following table, we collect the additional discriminants which we have realized in this chapter. The first entry will be explained below. We list the minimal discriminant, the number of a corresponding configuration in the notation of [S-Z] and a figure or other reference where the configuration can be found. (This is not unique, of course.)

d	11	15	20	24	35	40	84	120	168
No.	312	102	103	107	100	56	97	61	244
Fig.	eq. (III.2)	III.11	III.13	III.2	III.17	III.32	III.19	III.25	Table III.1

Table III.3: The discriminants and corresponding fibrations found in this chapter

Let us now explain the first entry from the above table. This corresponds to the configuration **[1,1,1,11,II*]**. In order to find such a fibration, we applied a direct elimination procedure, starting from the general Weierstrass equation resp. J-map. After a change of variables, this resulted in the following affine Weierstrass equation:

$$y^2 = x^3 + t^2(t^2+3t+1)\,x^2 + t^4(2t+4)\,x + t^5(t+1). \tag{III.2}$$

This has discriminant $\Delta = 16\,(4t^3+17t^2+14t-27)\,t^{10}$, such that the II^* fibre sits above 0 and a fibre of type I_{11} above ∞. This is the only extremal elliptic K3 fibration with discriminant $d = 11$.

Due to lack of computer memory, we did not try to find more extremal elliptic K3 fibrations. Any such fibration which is not included in this chapter, involves a (general) base change of $\mathbb{P}^1$ of degree at least 14. However, there has recently been a major achievement towards the general case which was announced by Beukers and Montanus [B-M]. They derive Weierstrass equations for all semi-stable extremal elliptic K3 fibrations, i.e. for the entire Miranda-Persson list.

To prove that the fields of definition which they found are optimal, and that they found all possible fibrations for each configuration, Beukers and Montanus use Grothendieck's theory of *dessins d'enfant*. Consider the J-map which is associated to an elliptic fibration with given configuration of singular fibres. In the semi-stable case, this is a polynomial map (also referred to as Belji map)

$$J : \mathbb{P}^1 \to \mathbb{P}^1$$

of degree 24. The associated dessin d'enfant is the pre-image of the real interval $[0,1]$ under J. For instance, this has to be symmetric if J is defined over $\mathbb{Q}$ (or a real field).

We illustrate this by returning to the base changes of $\mathbb{P}^1$ with degree 6 and ramification indices $(3,2,1), (3,2,1)$ and $(3,3)$. In Section III.2, we claimed that these maps were only defined over some cubic extension of $\mathbb{Q}$. In detail, this is given as $\mathbb{Q}(x^3+12x-12)/\mathbb{Q}$. Let v be a root of $5x^3+12x^2+12x+4$. Then the base changes are defined by

$$\begin{array}{rccc} \tilde{\pi}: & \mathbb{P}^1 & \to & \mathbb{P}^1 \\ & (s:t) & \mapsto & (-s^3(s-t)^2(s+(2+3v)t) : (2+3v)t^3(s+(1+v)^2t)^2(s+\dfrac{t}{5v+2})). \end{array}$$

This is indicated by the three corresponding dessins d'enfant which we draw in the following figure:

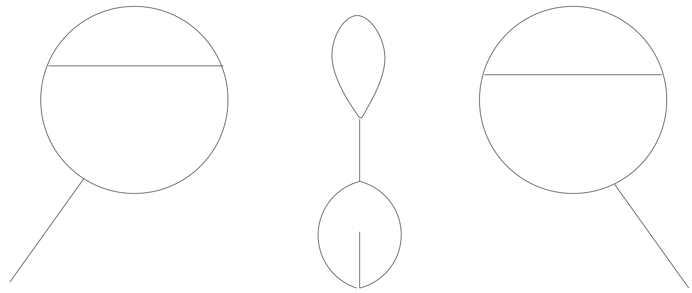

Figure III.38: The three dessins d'enfant of the base change $\tilde{\pi}$

For the pull-back, we can ask again whether the underlying surface has a model over $\mathbb{Q}$. By [S-Z], the only other extremal elliptic K3 fibration which this surface admits, has No. 273. For this fibration, one easily derives three corresponding dessins d'enfant as well. Hence, it is also not defined over $\mathbb{Q}$.

Remark III.8.2
Many configurations of [B-M] had already been achieved by our methods. The respective fields of definition agree. This can be explained as follows:

Let X be an extremal semi-stable elliptic K3 surface. The associated J-map J_X realizes X as pull-back from a rational elliptic surface Z. This has singular fibres of type III^*, II, and I_1. One way to interpret our approach is that we only investigated those base changes which factor through an extremal rational elliptic surface Y with three cusps. Since these surfaces are defined over $\mathbb{Q}$, this also holds for their J-maps. Here, J_Y realizes Y as pull-back of Z. Hence, a composition $J_X = J_Y \circ \pi$ is defined over some K if and only if π is.

For our purposes, the main effect of the results of [B-M] is that they supplement our investigations with another 10 discriminants as given below:

d	51	52	88	91	132	195	280	312	420	660
No.	110	90	69	95	68	93	65	91	19	72

Table III.4: The discriminants corresponding to fibrations over $\mathbb{Q}$ from [B-M]

We now collect the results. Up to squares, there are 24 discriminants which are known to correspond to an extremal elliptic K3 fibration over $\mathbb{Q}$. Recall the bound of 65 discriminants (or 66, if GRH does not hold) which is given by Proposition II.13.1 and Table II.1. In particular, this summation exhausts the semi-stable extremal elliptic K3 fibrations with section. As for the non-semi-stable ones, the classification of [S-Z] has not yet been further investigated with respect to fields of definition or explicit defining equations, as far as we know. However, a close inspection easily shows that the remaining fibrations can possibly only give rise to one more CM-field with exponent 2 (such that the K3 surface might have a model over $\mathbb{Q}$ by Proposition II.13.1). In the terminology of [S-Z], this comes from

$$\text{No. } 119 \quad \text{with configuration } [2,3,5,7,1^*] \quad \text{and intersection form } \begin{pmatrix} 20 & 0 \\ 0 & 42 \end{pmatrix}.$$

However, this particular fibration cannot be defined over $\mathbb{Q}$. This result is based on the observation that the configuration (or, more precisely, the J-map) has two different associated dessins d'enfant as sketched in the following figure.

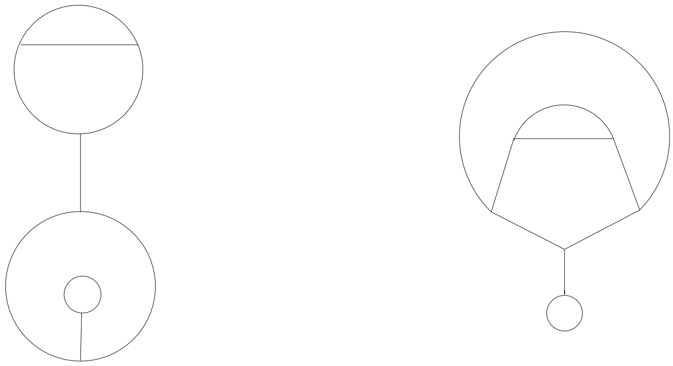

Figure III.39: The two dessins d'enfant of the configuration [2,3,5,7,1*]

On the other hand, there is only one isomorphism class of surfaces, corresponding to this configuration. This is specified by the above intersection form. Hence the minimal field of definition for this fibration has to be a quadratic extension of $\mathbb{Q}$. Since the dessins d'enfant are symmetric, this extension is real. It should be possible to verify this by an explicit calculation of the corresponding system of equations, but due to lack of computer memory, we did not do so.

This concludes the investigation of extremal elliptic K3 fibrations over $\mathbb{Q}$ in terms of the associated weight 3-newforms, terminating at 24 associated forms up to twisting. We shall end this chapter with some comments on other ways to find singular K3 surfaces (over $\mathbb{Q}$).

The first idea consists in investigating a certain family of K3 surfaces over $\mathbb{Q}$ with two II* fibres, such that the generic member has Picard number at least 18. This approach is motivated by Shioda-Inose's explicit construction of an elliptic fibration for any singular K3 surface, which contains these two singular fibres (cf. Sect. I.3). This idea is momentarily pursued by the author. First tests look promising, although it is uncertain whether this approach will eventually be successful.

At this point we should emphasize that the non-existence of a certain elliptic fibration over $\mathbb{Q}$ need not mean that the basic surface cannot be defined over $\mathbb{Q}$. In fact, we have seen this phenomenon in Sections III.5 and III.6: For some fibrations which were not defined over $\mathbb{Q}$, we pointed out that the underlying surface admitted another fibration over $\mathbb{Q}$. Therefore, one should also take other constructions into consideration. Let us name just one of them.

We can easily construct singular K3 surfaces X as Kummer surfaces $\mathrm{Km}(A)$ of singular abelian surfaces A. Indeed, taking A as a product of isogenous elliptic curves E, E' with CM leads exactly to Shioda-Inose's main approach as sketched in I.3. Still, there is the question of the field of definition. Obviously, if we choose $E = E'$ as one of the nine elliptic curves with CM which are defined over $\mathbb{Q}$, then X will also be defined over $\mathbb{Q}$. In general, however, there only seems to be the concept of Weil restriction available. This can be applied to those CM-curves which are defined over a (real) quadratic extension of $\mathbb{Q}$, i.e. for the class number 2 CM-fields. Apart from this, the question remains wide open.

Chapter IV

An explicit example

In this chapter, we are going to investigate one particular singular K3 surface X over $\mathbb{Q}$ in detail. First of all, we will determine the corresponding weight 3 form (cf. Thm. I.4.2) explicitly. For this, we will calculate the action of Frobenius on the transcendental lattice by counting points and applying the Lefschetz fixed point formula. Then we will offer two different proofs: One uses general (2-adic) two-dimensional Galois representations and relies on work of Faltings and Serre; the other is based on our previous classification of CM-forms with rational coefficients in Chapter II. In both cases, we will only have to check a few traces to deduce the weight 3 form.

This will enable us to compute the zeta-function of the surface. Then we will use the zeta-function to study the reductions of X modulo some primes p. We emphasize that we are able to find a model with good reduction at 2. We will subsequently verify conjectures of Tate, Shioda and Artin. The conjectures will be introduced in Section IV.3 and verified in Sections IV.4 - IV.6.

The final section is devoted to the twists of X. We will show that these produce all newforms of weight 3 with rational coefficients and CM by $\mathbb{Q}(\sqrt{-3})$. We also briefly return to the Fermat quartic.

Throughout we will consider the extremal elliptic K3 fibration with the [1,1,1,12,3*] configuration (No. 166 in [S-Z] and table III.2). Since this arises as cubic base change of the extremal rational elliptic surface X_{141} with singular fibres I_1^*, I_4 and I_1, we shall start by studying this surface.

IV.1 The rational elliptic surface X_{141}

In this section, we shall look for a model Y of X_{141} over $\mathbb{Q}$ which has good reduction everywhere. For this purpose, we shall manipulate the equation

$$y^2 = x^3 - 3\,(s-2t)^2(s^2-3t^2)\,x + s(s-2t)^3(2s^2-9t^2)$$

given in Section III.3. Recall that this has the I_4 fibre at $s=\infty=(1:0)$ and the other two singular fibres at ± 2. In view of the reduction at 2, we will normalize in a different way compared to Section III.3. That is, we only move the I_1^* fibre to 0, such that the equation becomes

$$y^2 = x^3 - \,s^2(s^2+4st+t^2)x + s^3(s+2t)(2s^2+8st-t^2)$$

with the I_1 fibre above -4. The resulting discriminant is

$$\Delta = 16 \cdot 27\, s^7 t^4 (4t + s).$$

We deduce good reduction of X_{141} outside 2 and 3. For these two remaining primes, we shall now study the divisors of X_{141}. This will enable us to derive the ramification locus of the Galois representation of $\mathrm{H}^2_{\text{ét}}(X_{141}, \mathbb{Q}_\ell)$. Note that X_{141}, as any rational elliptic surface with a section, can be obtained by blowing up $\mathbb{P}^2$ in the nine base points of a cubic pencil. Hence, the ramification locus of $\mathrm{H}^2_{\text{ét}}(X_{141}, \mathbb{Q}_\ell)$ coincides with the set of bad primes of an appropriate model of X_{141}.

In the following, let O denote the 0-section of a Weierstrass equation, i.e. the point at ∞. Consider the trivial lattice V of X_{141}. Recall that V is generated by O, a general fibre and those components of fibres which do not meet O. In the extremal case, $MW(X) \cong NS(X)/V$ is only torsion. Hence $NS(X_{141}) \otimes \mathbb{Q} = V \otimes \mathbb{Q}$. Our aim is to find a $\overline{\mathbb{Q}}$-isomorphic model for X_{141} over $\mathbb{Q}$ such that all generators of V are defined over $\mathbb{Q}$.

At ∞, the elliptic curve over $\mathbb{Q}(t)$ which corresponds to X_{141} has non-split multiplicative reduction. That is, the two components of the I_4 fibre of X_{141} which meet the O-component, are conjugate over $\mathbb{Q}(\sqrt{-3})$. Hence we shall twist X_{141} over this quadratic field by sending $y \mapsto 3\sqrt{-3}\, y$ (and $x \mapsto -9\, x$).

After the change of variables $x \mapsto x + \frac{s^2 - st}{3}$, the resulting Weierstrass equation reads

$$Y': \quad y^2 = x^3 + s(s-t)\, x^2 - 2s^3 t\, x + s^4 t^2. \tag{IV.1}$$

One easily deduces the good reduction at 3. This agrees with the discriminant

$$\Delta = 16\, s^7 t^4 (4t + s).$$

For the I_1^* fibre of the new surface Y' at 0, we apply Tate's algorithm [Si, IV, §9] to find that all components are defined over $\mathbb{Q}$.

As a consequence, $H^2_{\text{ét}}(Y', \mathbb{Q}_\ell)$ is completely unramified. We shall look for a model of Y' over $\mathbb{Q}$ which also has good reduction at 2. This can be achieved by mapping $t \mapsto 4t$ and dividing the whole equation by $\frac{1}{64}$ as well as sending $x \mapsto 4\, x, y \mapsto 8\, y$. Then, the change of variables $y \mapsto y + \frac{sx - s^2 t}{2}$ leads to the equation

$$y^2 + s\, xy - s^2 t\, y = x^3 - st\, x^2. \tag{IV.2}$$

with discriminant

$$\Delta = s^7 t^4 (s + 16t).$$

This eventually has good reduction at 2 and 3. However, in view of the reduction at 2, we will rather work with another equation. Modulo 2, this will reduce to the equation which was given in the classification of extremal rational elliptic surfaces in characteristic 2 in [Ito]. We map $x \mapsto x + st$ and obtain

$$X_{141}: \quad y^2 + s\, x\, y = x^3 + 2st\, x^2 + s^2 t^2\, x. \tag{IV.3}$$

Upon reducing mod 2, this surface only inherits the two singular fibres of types I_4 and I_1^*. There is wild ramification at the latter fibre.

IV.2 The extremal K3 fibration

Let Y be the model of X_{141} given by equation (IV.3). Consider the cubic base change

$$\pi : (s : t) \mapsto (s^3 : t^3).$$

Via pull-back, this gives rise to an extremal elliptic K3 surface X. The resulting Weierstrass equation reads

$$y^2 + s^2\, x\, y = x^3 + 2st^3\, x^2 + s^2t^6\, x. \tag{IV.4}$$

This has an D_7 resp. A_{11} singularity in the fibre above 0 resp. ∞. Let X denote the minimal desingularization. This fibration has configuration [1,1,1,12,3*]. The reducible singular fibres I_3^* and I_{12} sit at 0 and ∞. The three fibres of type I_1 can be found at the cube roots of -16. We will describe the explicit resolution of the singularities in Section IV.5.

By construction, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on the trivial lattice V of X. Since X is extremal, tensoring V with $\mathbb{Q}$ gives the corresponding statement for $NS(X)$.

Remark IV.2.1
Shioda in [Sh2, Thm. 1] claimed that $\rho(X/\mathbb{Q}) = 20$ were impossible for any K3 surface X over $\mathbb{Q}$. However, his proof contained a serious gap (cf. [Sh4]), and an explicit counterexample (the elliptic fibration with configuration [1,1,1,1,1,19]) was analyzed by J. Top and the author in [ST]. Here, we have another counterexample.

The basic surface Y has everywhere good reduction. Furthermore the base change π is nowhere degenerate upon reducing. Hence, the pull-back X can only have bad reduction at the prime divisors of the degree of π, i.e. at 3. Modulo 3, equation (IV.4) obtains an additional A_2 singularity, so the reduction is in fact bad. For details, confer Section IV.4. In terms of $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$, the ramification is reflected in the contribution of T_X (considered as a two-dimensional ℓ-adic Galois representation ρ). The reduction properties of X imply that ρ is only ramified at 3 and the respective prime ℓ.

This agrees with the discriminant of X which is 3. To see this, recall that

$$d_{T_X} = -d_{NS(X)} \quad \text{(cf. Section I.2)}.$$

Consider the trivial lattice V of X. Since

$$V = A_{11} \oplus D_7 \oplus U,$$

V has discriminant $d_V = -12 \cdot 4 = -48$. The Néron-Severi group $NS(X)$ is obtained from V by adding the sections. Here, X is extremal, so $MW(X)$ is finite. Then Lemma I.2.1 gives

$$d_{NS(X)} = \frac{d_V}{|MW(X)|^2}.$$

Explicitly, we have $MW(X) \cong MW(Y)$, consisting of four elements. For details, the reader might consult Section IV.5. We obtain $d_{NS(X)} = -3$ and $d_{T_X} = 3$.

Then the (reduced) intersection form on T_X can only be

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

since this is the only even positive definite quadratic form of discriminant 3 up to equivalence under $SL_2(\mathbb{Z})$.

We have seen that the reduction X_p of X at any (unramified) prime $p \neq 3$ is smooth and that X has discriminant 3. Theorem I.4.2 implies that

$$\det\, \rho(\mathrm{Frob}_p) = \left(\frac{p}{3}\right) p^2. \tag{IV.5}$$

To find the trace of ρ, we use the Lefschetz fixed point formula. We have already seen that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates trivially on $NS(X)$. Hence, the Lefschetz fixed point formula returns

$$\#X(\mathbb{F}_p) = 1 + 20\,p + \mathrm{tr}\ \rho(\mathrm{Frob}_p) + p^2.$$

Using a straight computer program, we calculated the following traces and determinants at the first good primes.

p	tr ρ	det ρ
2	0	-4
5	0	-25
7	-13	49
11	0	-121
13	-1	169
17	0	-289
19	11	361
23	0	-529
29	0	-841
31	-46	961
37	47	1369

By inspection, these traces coincide with the Fourier coefficients of the newform $f = \sum_n a_n q^n$ of level 27 and weight 3 from table II.1. We shall now prove that this holds at every prime:

Proposition IV.2.2

$$L(T_X, s) = L(f, s).$$

This proposition can be established by two different methods which we shall explain in the following:

The first approach makes explicit use of Livné's theorem I.4.2 which states that ρ is modular. Let g denote the associated newform. By Theorem I.4.2, g has CM by $\mathbb{Q}(\sqrt{-3})$, so it is a twist of f because of Theorem II.3.4. In our special situation, g is unramified outside 3 (i.e. it has level 3^r), since X has good reduction elsewhere. Hence we only have to compare the few possible newforms with such a level. For instance, such a procedure has also been described in [D]. There it was necessary to search tables of newform of weight 4 and level up to 6400. In our case, however, the odd weight (or the CM-property) allows us to make use of the techniques involved in the proof of Theorem II.3.4 (cf. Rem. II.3.5).

Let ψ_f denote the Größencharakter of conductor 3 which corresponds to f, and accordingly ψ_g. Then Proposition II.11.1 and explicitly the reasoning of Section II.11.4 show that there are three possibilities in total:

$$\psi_g = \psi_f \quad \text{or} \quad \psi_g = \psi_f \otimes \left(\frac{3}{\cdot}\right)_3 \quad \text{or} \quad \psi_g = \psi_f \otimes \left(\frac{3}{\cdot}\right)_3^2. \tag{IV.6}$$

These Größencharaktere differ at any split prime p with nontrivial third residue symbol at its factors. We have seen in connection with Theorem II.11.12 that $p = 7$ is such a prime. In order to determine the right associated newform, it is therefore sufficient to compare the Fourier coefficients (or traces) at 7. In particular, the trace of ρ at Frob_7 shows that $f = g$.

A priori, this only guarantees that all but finitely many traces coincide. But here, the associated Galois representations ρ and ρ_f (see below) are simple, since the traces are not even. Hence Proposition IV.2.2 follows.

We shall now sketch the second possible method to prove the proposition. Essentially, this consists of the claim that the 2-adic Galois representations ρ and ρ_f are isomorphic. Here we make use of a classical construction of Deligne. To a (normalized) newform, Deligne associates a compatible system of ℓ-adic Galois representations with the same traces and determinants at the Frobenius elements Frob_p. This proof does not use Livné's theorem on the modularity of X. We only need his description of the determinant of ρ in equation (IV.5). In general, this depends only on the discriminant of X.

By Deligne's construction, the determinants of ρ and ρ_f coincide. Therefore, we can apply Proposition 1 of [S]. For this, we furthermore need that the traces at 11 are even. Given this, the proposition tells us that ρ and ρ_f are isomorphic if and only if the traces at the above listed primes (with the exception of 2 and 29) coincide. Since this is exactly how we found f, we deduce the claim.

Corollary IV.2.3
The zeta-function of X is

$$\zeta(X, s) = \zeta(s)\,\zeta(s-1)^{20}\,L(f, s)\,\zeta(s-2).$$

Proof: We verify the corollary at every local Euler factor. On the one hand, at all primes $p \neq 3$, this follows from Proposition IV.2.2 and $\rho(X/\mathbb{Q}) = 20$.

On the other hand, the reduction X_3 at 3 is singular. Hence, the local ζ-factor is only defined via the number of points of this singular variety over the fields $\mathbb{F}_{3^r}$, since the Lefschetz fixed point formula is not available.

An explicit resolution of the singularity is studied in Section IV.4. With respect to the number of points, X_3 looks like $\mathbb{P}^2$ blown up in 19 rational points. Let us explain what we mean by this: We compare X_3 to Y_3, the reduction of Y modulo 3. The base change π is purely inseparable mod 3. Hence, over any finite field $\mathbb{F}_{3^r}$, the smooth fibres of X_3 have the same number of points as the corresponding fibres of Y_3. On the other hand, X_3 has ten additional $\mathbb{P}^1$s in the singular fibres at 0 and ∞. These are all defined over $\mathbb{F}_3$. Thus

$$\# X_3(\mathbb{F}_{3^r}) = \# Y_3(\mathbb{F}_{3^r}) + 10 \cdot 3^r.$$

Recall that Y_3 is $\mathbb{P}^2$, blown up in nine points. These points are all rational over $\mathbb{F}_3$, since the absolute Galois group operates trivially. We deduce that

$$\# X_3(\mathbb{F}_{3^r}) = \#(\mathbb{P}^2(19))(\mathbb{F}_{3^r}).$$

This gives the local Euler factor

$$\zeta_3(X, T) = \frac{1}{(1-T)(1-3\,T)^{20}(1-3^2\,T)}.$$

The level 27 implies $L_3(f, s) = 1$ by the classical theory of modular forms. By equation (II.1), this also follows from Table II.1 and the nebentypus $\varepsilon = \chi_{-3}$ of f. This completes the proof of Corollary IV.2.3.

IV.3 The conjectures for the reductions

In this section, we shall discuss conjectures of Tate, Shioda and Artin for smooth projective surfaces over finite fields. In particular, these conjectures apply to (supersingular) reductions of varieties originally defined over number fields.

Corollary IV.2.3 will be very useful: If X has good reduction at p, then the corollary gives the local ζ-function of the smooth variety $X/\mathbb{F}_p$. Explicitly, let $p \neq 3$. We obtain

$$P_2(X/\mathbb{F}_p, T) = \det(1 - \mathrm{Frob}_p\, T; \mathrm{H}^2_{\text{ét}}(X/\overline{\mathbb{F}}_p, \mathbb{Q}_\ell)) = (1 - p\,T)^{20}(1 - a_p\,T + \chi_{-3}(p)\, p^2\, T^2).$$

This is exactly where the Tate conjecture enters. To formulate it, consider a finite field k and a smooth projective variety Z/k. Define the Picard number of Z over k

$$\rho(Z/k) = \mathrm{rk}\ NS(Z)^{\mathrm{Gal}(\overline{k}/k)}.$$

Recall that we employed the convention $\rho(Z) = \rho(Z/\overline{k})$ in Section I.1.

Conjecture IV.3.1 (Tate [T1, (C)], [T2])
Let $q = p^r$ and $Z/\mathbb{F}_q$ a smooth projective variety. Denote the order of the zero of $P_2(Z/\mathbb{F}_q, T)$ at $T = \frac{1}{q}$ by u. Then $u = \rho(Z/\mathbb{F}_q)$.

The Tate conjecture is known for elliptic K3 surfaces with a section in characteristic $p > 3$ [T2, Thm. (5.6)]. This implies most of the following

Proposition IV.3.2
Let p be a prime. Consider X as a surface over $\mathbb{F}_p$. Then

$$\rho(X/\mathbb{F}_p) = \begin{cases} 20, & \text{if } p \equiv 1 \mod 3, \\ 21, & \text{if } p \equiv 0, 2 \mod 3. \end{cases}$$

Here we refer to X as the minimal resolution of the A_n and D_m singularities of the surface over $\mathbb{F}_p$ which is given by equation (IV.4). We shall also write $X/\mathbb{F}_p$. This surface coincides with the reduction X_p of X if and only if p is a good prime (i.e. $p \neq 3$). On the contrary, X_3 still contains an A_2 singularity, so it is not smooth. The desingularization $X/\mathbb{F}_3$ requires two further blow-ups. This will be sketched in the next section.

If $p > 3$, Proposition IV.3.2 follows from the (known) Tate conjecture. The proofs for $p = 2$ and 3 will be given in the following three sections. We will also verify the following conjecture of Shioda.

Let L be a number field and Z a singular K3 surface over L. If $\mathfrak{p}$ is a prime of L, denote the residue field of L at $\mathfrak{p}$ by $L_\mathfrak{p}$. We call $\mathfrak{p}$ supersingular if and only if $Z/L_\mathfrak{p}$ is supersingular (i.e. $\rho(Z/\overline{L}_\mathfrak{p}) = 22$).

Shioda's conjecture concerns the surface $Z/L_\mathfrak{p}$ at a supersingular prime $\mathfrak{p}$. Then we can compare two lattices of rank two: On the one hand, we have the transcendental lattice T_Z of Z/L. On the other hand we can use the natural embedding

$$NS(Z/\overline{L}) \subseteq NS(Z/\overline{L}_\mathfrak{p})$$

to define the orthogonal complement

$$T_\mathfrak{p} = NS(Z/\overline{L})^\perp \subset NS(Z/\overline{L}_\mathfrak{p}).$$

Conjecture IV.3.3 (Shioda [Sh3, Conj. 4.1])
Let Z be a singular K3 surface over a number field L and $\mathfrak{p}$ a supersingular prime. Then, the two lattices T_Z and $T_{\mathfrak{p}}$ are similar.

In other words, the claim is that $T_{\mathfrak{p}}$ is isomorphic to $T_Z(-m)$ for some $m \in \mathbb{Q}_{>0}$. We will verify this conjecture for the extremal elliptic K3 fibration X at the primes 2 and 3 in the next three sections. This will be achieved by finding explicit generators of the respective $T_{\mathfrak{p}}$. $X/\mathbb{F}_3$ will give an example for the following

Theorem IV.3.4 (Artin [A, (6.8)])
Let p be an odd prime. Let Z be a supersingular K3 surface over a finite field k with p^r elements. If r is odd, then $Gal(\overline{k}/k)$ operates non-trivially on $NS(Z)$.

Note that this agrees perfectly with Proposition IV.3.2 for our surface X. In fact, the known part of Tate's Conjecture IV.3.1 shows that at a supersingular prime $p > 3$,

$$\rho(X/\mathbb{F}_{p^2}) = 22.$$

We will also verify Theorem IV.3.4 for $X/\mathbb{F}_{2^r}$ and $X/\mathbb{F}_{3^r}$. In the first case, this will give evidence of the following

Conjecture IV.3.5
Theorem IV.3.4 also holds for $p = 2$.

IV.4 $X/\mathbb{F}_3$

In this section, we shall consider $X/\mathbb{F}_3$. This will be special because 3 divides the degree of the base change π. More precisely, π is purely inseparable modulo 3. As a consequence, the base change $X/\mathbb{F}_3$ from $Y/\mathbb{F}_3$ via π has only three singular fibres. They have types I_3^*, I_{12} and I_3. In particular, the elliptic fibration $X/\mathbb{F}_3$ is extremal (cf. the classification of [Ito]) and supersingular.

We emphasize that we refer to the reduction X_3 of $X/\mathbb{Q}$ at 3 as bad. It has an additional A_2 singularity whose resolution results in the fibre of type I_3.

We shall now verify Tate's and Shioda's conjectures for $X/\mathbb{F}_3$. Consider the 4-torsion sections of $X/\mathbb{F}_3$ which come from $X/\mathbb{Q}$ upon reducing (see Section IV.5 for a detailed study). These sections meet the O-component of the I_3 fibre since such a fibre only admits 3-torsion. Denote the other components of the I_3 fibre (not meeting O) by Θ_1, Θ_2. By construction, they also do not meet the generators of the trivial lattice V of $X/\mathbb{Q}$ (embedded into $NS(X/\mathbb{F}_3)$). Hence, they are perpendicular to $NS(X/\mathbb{Q}) \subset NS(X/\overline{\mathbb{F}}_3)$. We claim that this is already all of $NS(X/\overline{\mathbb{F}}_3)$:

Lemma IV.4.1
Let A_2 denote the root lattice generated by Θ_1 and Θ_2. Then

$$NS(X/\overline{\mathbb{F}}_3) \cong NS(X/\mathbb{Q}) \oplus A_2.$$

The proof of this lemma will be directly derived from the following classical

Theorem IV.4.2 (Artin, Rudakov-Šafarevič)
Let X be a supersingular K3 surface over a field of characteristic p. Then

$$discr\ NS(X) = -p^{2\sigma_0} \quad \text{for some } \sigma_0 \in \{1, \dots, 10\}.$$

Here, σ_0 is called the Artin invariant.

As a consequence, in our situation, we have discr $NS(X/\overline{\mathbb{F}}_3) = -3^{2\sigma_0}$ for some $\sigma_0 \in \{1, \dots, 10\}$. But then, the above sublattice $NS(X/\mathbb{Q}) \oplus A_2$ of $NS(X/\overline{\mathbb{F}}_3)$ has rank 22 and discriminant -9. Since this is the maximal possible discriminant, we deduce the equality of the two lattices. This proves Lemma IV.4.1.

Corollary IV.4.3
The supersingular K3 surface $X/\mathbb{F}_3$ has Artin invariant $\sigma_0 = 1$.

We shall now prove Proposition IV.3.2 for $p = 3$. Since $\rho(X/\mathbb{Q}) = 20$, the reduction of $NS(X/\mathbb{Q})$ is clearly generated by divisors over $\mathbb{F}_3$. Using Lemma IV.4.1, we only have to study the field of definition of the two further generators Θ_1, Θ_2 of $NS(X/\mathbb{F}_3)$.

For this, consider the elliptic curve over $\mathbb{F}_3(s)$ given by equation (IV.4). It has non-split multiplicative reduction at $(1 : -1)$. More precisely, the components Θ_1 and Θ_2 are conjugate in $\mathbb{F}_3(\sqrt{-1})$. In particular, their sum is defined over $\mathbb{F}_3$, while the difference has eigenvalue -1 with respect to the conjugation. Hence, we deduce the claim of Proposition IV.3.2 that $\rho(X/\mathbb{F}_3) = 21$. This agrees with the Tate conjecture and Theorem IV.3.4. We also see that $\rho(X/\mathbb{F}_9) = 22$.

Finally, we come to Shioda's Conjecture. By Lemma IV.4.1, we have

$$T_3 = NS(X/\mathbb{Q})^\perp = A_2 \subset NS(X/\overline{\mathbb{F}}_3).$$

Since A_2 has intersection matrix $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, we deduce the validity of Conjecture IV.3.3 with $m = 1$.

IV.5 $NS(X)$

To verify the conjectures for the reduction X_2, we will need a better knowledge of $NS(X) = NS(X/\overline{\mathbb{Q}})$. To be precise, we want to express the sections of the original fibration X over $\mathbb{Q}$ in terms of $V \otimes_\mathbb{Z} \mathbb{Q}$ (or $V^\vee$). This is possible since the fibration is extremal, such that

$$MW(X) \cong NS(X)/V$$

is only torsion. In terms of equation (IV.4), the sections of the elliptic fibration X are given by:

$$MW(X) \cong \mathbb{Z}/4 =< P >=< (-st^3, 0) >= \{(-st^3, 0), (0, 0), (-st^3, s^3t^3), O\}.$$

The sections can be derived from the sections of X_{141}, as given in [MP1], by following them through the base and variable changes. We will later see that $MW(X)$ gives already all torsion-sections of $X/\overline{\mathbb{F}}_2$.

We want to express the section P as a $\mathbb{Q}$-divisor in $V \otimes_\mathbb{Z} \mathbb{Q}$. This can be achieved by determining the precise components of the singular fibres which P intersects.

For this purpose, we shall now exhibit the explicit desingularization of the Weierstrass equation (IV.4) in some detail. The resolutions of the A_{11} and D_7 singularities at 0 and ∞ will not differ in any positive characteristic. Since we are mainly interested in the simple components of the resulting fibres, it will always suffice to consider one particular chart in the blow-up in detail.

For the fibre of type I_3^* at $s=0$, we will not need the complete resolution in detail. The original rational double point sits at $(s,x,y)=(0,0,0)$. We blow this point up such that in the chart $s'=1$

$$x=x's, y=y's.$$

Then the transform is given by

$$y'^2+s^2x'y'=sx'(x'+1)^2 \tag{IV.7}$$

with a component $D_0=\{s=y'^2=0\}$ of multiplicity two in the fibre at 0. This contains two singularities at $x'=0$ and $x'=-1$. The resolution of the former consists of a simple smooth $\mathbb{P}^1$ which shall be denoted by

$$C_1=\{s=y''^2-x''=0\}.$$

Meanwhile the latter produces another component D_2 of multiplicity 2. After translating $x'\mapsto x'-1$, this component can be read off from the resulting equation:

$$y''^2+s^2x''y''-sy''=sx''^2(sx''-1).$$

Again, this transform contains two singularities. One of these sits outside the chart in consideration and is subsequently replaced by another component D_1 of multiplicity 2. The other singularity can be found in our chart at $(0,0,0)$. The blow-up gives

$$y'''^2+s^2x'''y'''-y'''=sx'''^2(s^2x'''-1).$$

This provides the two simple components

$$C_2=\{s=y'''-1=0\} \quad \text{and} \quad C_3=\{s=y'''=0\}.$$

The further resolution of the double point at the common intersection of D_2, C_2 and C_3 (which is not situated in this chart) produces a final components D_3 of multiplicity two. We sketch the resulting fibre of type I_3^* in the following figure. We denote the original component which meets O by C_0:

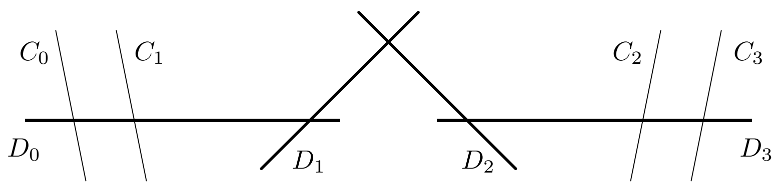

Figure IV.1: The fibre of type I_3^* at 0

We shall now come to the I_{12} fibre at ∞ and give the resolution of the rational double point at $(t,x,y)=(0,0,0)$. This will require six successive blow-ups, which can always be analyzed in the chart $t'=1$. Here $x=x't, y=y't$, such that the first blow-up gives rise to

$$y'^2+x'y'=tx'^3+2t^3x'^2+t^5x'.$$

In the notation of Section III.7, we read off the fibre components

$$\Theta_1=\{t=y'+x'=0\} \quad \text{and} \quad \Theta_{11}=\{t=y'=0\}$$

as well as the singularity at $(0,0,0)$. We continue with the resolution procedure exactly along these lines and number the components respectively. Then, the sixth blow-up produces (in the respective coordinates)

$$y''^2+x''y''=t^6x''^3+2t^3x''^2+x''.$$

This is smooth and concludes the resolution. We shall denote the resulting final component by Θ_6. The Θ_i for $i = 1, \ldots, 11$ form an A_{11} configuration, a chain of eleven smooth rational curves. This can be found in Figure IV.2 where we also add Θ_0. We emphasize that all components of both singular fibres are defined over $\mathbb{Q}$ as predicted by Tate's algorithm. Our next aim is to compute the precise components which the section P meets. This can be achieved by keeping track of the behaviour of P under the resolution:

Lemma IV.5.1
The section P meets the components C_3 and Θ_9.

Proof: At $s = 0$, $P = (-s, 0)$ meets the original node. After the first blow-up it reads $P = (-1, 0)$, meeting the node with multiplicity 2. After translating $x' \mapsto x' - 1$, P is fixed as $P = (0, 0)$ throughout the remaining resolution process. Hence, it eventually intersects the component C_3.

Similarly, the fibre at ∞ (i.e. $t = 0$) sees $P = (-t^3, 0)$ go through three blow-ups, until it reads $(-1, 0)$. In the above notation, it therefore meets Θ_9. This concludes the proof of Lemma IV.5.1.

Lemma IV.5.1 enables us to identify P in $NS(X)$. More precisely, we describe it in terms of the generators of the trivial lattice V. Since P is given by polynomials, it does not meet O. We will use the $\mathbb{Q}$-divisors

$$\begin{aligned} A &= \frac{1}{4}(\Theta_1 + 2\Theta_2 + \ldots + 9\Theta_9 + 6\Theta_{10} + 3\Theta_{11}) \quad \text{and} \\ B &= \frac{1}{4}(2C_1 + 4D_0 + 6D_1 + 8D_2 + 10D_3 + 5C_2 + 7C_3). \end{aligned}$$

In the notation of Lemma III.7.2, B equals R_3 after exchanging D_i by B_i. From Lemma III.7.1, we obtain $A \equiv 3Q_{12} \mod V$.

Corollary IV.5.2
In $V \otimes_{\mathbb{Z}} \mathbb{Q}$, the section P is given as

$$P = O + 2F - A - B.$$

Proof: Since $4\,P \equiv 0 \mod V$, it suffices to check the following intersection numbers:

$$(P.O) = 0, (P.F) = 1, (P.\Theta_i) = \delta_{i,9}, (P.D_j) = 0, (P.C_j) = \delta_{j,3} \text{ and } (P^2) = -2.$$

Remark IV.5.3
Let $[\mathcal{P} + \mathcal{Q}]$ denote the addition in $MW(X)$ with respect to the group law of the generic fibre. This can be computed in the quotient $NS(X)/V$. As a $\mathbb{Q}$-divisor in $V \otimes_{\mathbb{Z}} \mathbb{Q}$, $[3P]$ can be obtained from P by exchanging C_2 and C_3 as well as Θ_i and Θ_{12-i} $(i = 1, \ldots, 5)$. Similarly, an easy computation gives

$$\begin{aligned} [2P] &= O + 2F - \frac{1}{2}(\Theta_1 + 2\Theta_2 + \ldots + 6\Theta_6 + 5\Theta_7 + \ldots + \Theta_{11}) \\ &\quad - \frac{1}{2}(2C_1 + 2D_0 + \ldots + 2D_3 + C_2 + C_3). \end{aligned}$$

IV.6 $X/\mathbb{F}_2$

In this section, we shall consider the elliptic K3 surface $X/\mathbb{F}_2$. This coincides with the reduction X_2 of X at 2. For simplicity, we shall work with the affine coordinate

s. Reducing equation (IV.4) modulo 2, we obtain

$$X/\mathbb{F}_2: \quad y^2 + s^2\,x\,y = x^3 + s^2\,x. \tag{IV.8}$$

Recall that this has only two singular fibres. They sit at 0 and ∞ and have types I_3^* and I_{12}. Since 2 stays inert in $\mathbb{Q}(\sqrt{-3})$, the local L-factor is given by

$$P_2(X/\mathbb{F}_2, T) = (1 - 2\,T)^{21}\,(1 + 2\,T)$$

because of Corollary IV.2.3. In accordance with the Tate conjecture, we shall prove that X is a supersingular K3 surface. To be precise, we claim

Proposition IV.6.1
$X/\mathbb{F}_2$ is a supersingular K3 surface with Picard numbers

$$\rho(X/\mathbb{F}_2) = 21 \quad \textit{and} \quad \rho(X/\mathbb{F}_4) = 22.$$

This proposition will complete the proof of Proposition IV.3.2. It will be established by finding explicit generators for $NS(X/\overline{\mathbb{F}}_2)$. Additionally to the reduction of $NS(X/\mathbb{Q})$, we need two generators. By the formula of Shioda-Tate, these can be given as sections of the elliptic fibration $X/\overline{\mathbb{F}}_2$.

In detail, we compute some additional sections of $X/\overline{\mathbb{F}}_2$ which are not derived from $MW(X/\mathbb{Q})$ by way of reduction. Then we determine two among them which supplement $NS(X/\mathbb{Q})$ to generate $NS(X/\overline{\mathbb{F}}_2)$. Let α be a generator of $\mathbb{F}_4$ over $\mathbb{F}_2$, i.e. $\alpha^2+\alpha+1 = 0$. Among others, we found the following sections. Here, the inverse refers to the group law on the generic fibre.

section	inverse
$Q = (1, 1)$	$(1, 1 + s^2)$
(s^2, s^2)	$(s^2, s^2 + s^4)$
$(s + s^3, s^3 + s^4)$	$(s + s^3, s^4 + s^5)$
$R = (s + \alpha s^3, \alpha^2 s^4 + \alpha s^5)$	$(s + \alpha s^3, s^3 + \alpha^2 s^4)$
$(1 + s^4, 1 + \alpha s^2 + \alpha^2 s^6)$	$(1 + s^4, 1 + \alpha^2 s^2 + \alpha s^6)$

We shall now prove that $MW(X/\overline{\mathbb{F}}_2)$ has rank 2 and can be generated by the sections P, Q and R. This will be achieved with the help of the *height pairing* on the Mordell-Weil group, as introduced by Shioda in [Sh1]. Let V denote the trivial lattice of $X/\mathbb{F}_2$. Equivalently, reduce the trivial lattice of $X/\mathbb{Q}$. The height pairing is defined via the orthogonal projection

$$\varphi: MW(X/\overline{\mathbb{F}}_2) \to V^\perp \otimes_{\mathbb{Z}} \mathbb{Q} \subset NS(X/\overline{\mathbb{F}}_2) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This projection only uses the information which (simple) component of a reducible fibre a section meets.

Shioda's height pairing is defined by

$$\begin{array}{rccc} <\cdot,\cdot>: & MW(X/\overline{\mathbb{F}}_2) \times MW(X/\overline{\mathbb{F}}_2) & \to & \mathbb{Q} \\ & \mathcal{P} \qquad\qquad\qquad \mathcal{Q} & \mapsto & -(\varphi(\mathcal{P}).\varphi(\mathcal{Q})). \end{array}$$

Here $(\cdot.\cdot)$ denotes the intersection form on $NS(X/\overline{\mathbb{F}}_2) \otimes_{\mathbb{Z}} \mathbb{Q}$. The height pairing is symmetric and bilinear. It induces the structure of a positive definite lattice on $MW(X/\overline{\mathbb{F}}_2)/MW(X/\overline{\mathbb{F}}_2)_{\text{tor}}$ (the Mordell-Weil lattice).

Let $\mathcal{P}, \mathcal{Q} \in MW(X/\overline{\mathbb{F}}_2)$. Shioda shows that

$$<\mathcal{P}, \mathcal{Q}> = \chi(\mathcal{O}_X) - (\mathcal{P}.\mathcal{Q}) + (\mathcal{P}.O) + (\mathcal{Q}.O) - \sum_v \text{contr}_v(\mathcal{P}, \mathcal{Q})$$

where the sums runs over the cusps and the contr_v can be computed strictly in terms of the components of the singular fibres which $\mathcal{P}$ and $\mathcal{Q}$ meet [Sh1, Thm. 8.6] (cf. also [Sh1, (8.16)]).

We shall now return to the explicit resolution of singularities which was achieved for $X/\mathbb{Q}$ in the previous section. This stays valid in any positive characteristic $p \neq 3$, since we can reduce mod p at any step without difficulties. On $X/\mathbb{F}_2$, we want to determine the precise components of the singular fibres which the sections Q and R meet. For this, we have to follow the sections closely through the resolution process.

Lemma IV.6.2
Q meets C_0 and Θ_8 while R intersects C_3 and Θ_1.

Proof: In the I_3^* fibre above 0, Q meets C_0. On the other hand, R meets the node $(0,0,0)$ of equation (IV.8). After the first blow-up, the section reads $R = (1+\alpha s^2, \alpha^2 s^3 + \alpha s^4)$ such that it meets the multiplicity 2 node $(s, x', y') = (0,1,0)$. After translating $x' \mapsto x' - 1$, the section is taken to $R = (\alpha, \alpha^2 s + \alpha s^2)$ by the next two blow-ups. This intersects $C_3 = \{s = y''' = 0\}$.

We now come to the I_{12} fibre at ∞. Both sections meet the node of equation (IV.8). The first blow-up has $R = (\alpha + t^2, \alpha + \alpha^2 t)$; this only intersects Θ_1. On the other hand, Q reads $Q = (1, t^2)$ after the fourth blow-up. Thus it meets Θ_8. This completes the proof of Lemma IV.6.2.

We are now in the position to compute the projections $\varphi(Q)$ and $\varphi(R)$. However, we will postpone their explicit calculation, since we will only need this for the explicit verification of Shioda's Conjecture IV.3.3. Here, we can use Shioda's results from [Sh1] to compute the discriminant of the lattice N generated by the trivial lattice V and the sections P, Q and R. Using Theorem IV.4.2, this will suffice to prove that N equals $NS(X/\overline{\mathbb{F}}_2)$.

Lemma IV.6.3
Let Q, R be the sections of $X/\mathbb{F}_4$ as specified above. Then

$$\begin{pmatrix} < Q,Q > & < Q,R > \\ < Q,R > & < R,R > \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

Proof: Since they are polynomial, Q and R do not meet O. For the self intersection numbers, we thus only miss the contributions from the singular fibres. These are derived from [Sh1, (8.16)] with the help of Lemma IV.6.2:

$$\begin{aligned} < Q,Q > &= 4 - \frac{4 \cdot 8}{12} = \frac{4}{3} \\ < R,R > &= 4 - \frac{7}{4} - \frac{11}{12} = \frac{4}{3}. \end{aligned}$$

For the remaining entry, we furthermore have to analyze the intersection of Q and R. We have to find the common zeroes of

$$\begin{aligned} \alpha s^3 + s + 1 &= \alpha(s+\alpha^2)(s^2+\alpha^2 s+1) \quad \text{and} \\ \alpha s^5 + \alpha^2 s^4 + 1 &= \alpha(s+1)(s+\alpha^2)(s^3+\alpha^2 s+1). \end{aligned}$$

The last respective factors are irreducible over $\mathbb{F}_4$. Hence, the only common zero is $s = \alpha^2$. Since this occurs with multiplicity one, the intersection is transversal. Hence, we deduce

$$< Q,R > = 2 - 1 - \frac{1 \cdot 4}{12} = \frac{2}{3}.$$

This finishes the proof of Lemma IV.6.3.

Proposition IV.6.4
$NS(X/\overline{\mathbb{F}}_2)$ has discriminant -4. It is generated by the trivial lattice V and the sections P, Q and R, so $MW(X/\overline{\mathbb{F}}_2) = MW(X/\mathbb{F}_4) =< P, Q, R >$.

Proof: Consider the lattice N which is generated by the trivial lattice V and the sections P, Q and R. Clearly, this is a sublattice of $NS(X/\overline{\mathbb{F}}_2)$. We can identify $N' =< V, P >\cong NS(X/\mathbb{Q}) \subset N$. Recall that this sublattice has discriminant -3.

Then we use the orthogonal projection φ from $NS(X/\overline{\mathbb{F}}_2)$ to $V^\vee \otimes_\mathbb{Z} \mathbb{Q}$. Since $P \in V \otimes_\mathbb{Z} \mathbb{Q}$, it follows that

$$\operatorname{discr} N = (\operatorname{discr} N') \ \det \begin{pmatrix} (\varphi(Q).\varphi(Q)) & (\varphi(Q).\varphi(R)) \\ (\varphi(Q).\varphi(R)) & (\varphi(R).\varphi(R)) \end{pmatrix}.$$

Since $(\varphi(\mathcal{P}).\varphi(\mathcal{Q})) = - < \mathcal{P}, \mathcal{Q} >$, Lemma IV.6.3 gives that the matrix has determinant $\frac{4}{3}$. Hence, N has discriminant -4 and in particular rank 22. Since both values are maximal possible (cf. Thm. IV.4.2), we obtain $N = NS(X/\overline{\mathbb{F}}_2)$. The claim $MW(X/\overline{\mathbb{F}}_2) =< P, Q, R >$ then follows from the formula of Shioda-Tate. This proves Proposition IV.6.4.

Proposition IV.6.4 implies Proposition IV.6.1. It verifies the Tate conjecture for $X/\mathbb{F}_{2^r}$ with any $r \in \mathbb{N}$. We also deduce the validity of Conjecture IV.3.5. Furthermore, we have seen that X_2 has Artin invariant $\sigma_0 = 1$.

We shall conclude this section by verifying Conjecture IV.3.3 for $X/\mathbb{Q}$ and its reduction at 2. Note that it suffices to consider $NS(X/\mathbb{Q})$ and $NS(X/\mathbb{F}_4)$. The dual of $NS(X/\mathbb{Q})$ will always refer to the embedding into $NS(X/\mathbb{F}_4)$.

We shall now give the explicit shape of $\varphi(Q)$ and $\varphi(R)$. The projections are easily computed as

$$\begin{aligned} \varphi(Q) &= Q - O - 2F + \frac{1}{3}(\Theta_1 + 2\Theta_2 + \ldots + 8\Theta_8 + 6\Theta_9 + 4\Theta_{10} + 2\Theta_{11}) \\ \varphi(R) &= R - O - 2F + B + \frac{1}{12}(11\Theta_1 + 10\Theta_2 + \ldots + \Theta_{11}). \end{aligned}$$

Since the denominators are divisible by 3, $\varphi(Q), \varphi(R) \notin NS(X/\mathbb{F}_4)$. In other words,

$$< \varphi(Q), \varphi(R) > \supsetneq NS(X/\mathbb{Q})^\perp \supseteq < 3\varphi(Q), 3\varphi(R) > . \tag{IV.9}$$

We claim that

$$NS(X/\mathbb{Q})^\perp =< 3\varphi(Q), \varphi(Q) + \varphi(R) >=< 3\varphi(R), \varphi(Q) + \varphi(R) > . \tag{IV.10}$$

Since the ranks are two, the inequality of (IV.9) implies that the claim is equivalent to

$$\varphi(Q) + \varphi(R) \in NS(X/\mathbb{F}_4).$$

Since $P \equiv -A - B \mod V$, it suffices to show that

$$\varphi(Q) + \varphi(R) \equiv A + B \mod < V, Q, R > .$$

Explicitly, we have

$$\begin{aligned}\varphi(Q)+\varphi(R) &= Q+R-2O-4F+\frac{1}{12}(11\Theta_1+10\Theta_2+\ldots+\Theta_{11})\\ &\quad +B+\frac{1}{3}(\Theta_1+2\Theta_2+\ldots+8\Theta_8+6\Theta_9+4\Theta_{10}+2\Theta_{11})\\ &\equiv B+\frac{1}{12}\sum_j(12-j)\Theta_j+\frac{1}{3}\sum_j j\Theta_j\\ &= B+\frac{1}{12}\sum_j(12-j+4j)\Theta_j\\ &\equiv B+\frac{1}{4}\sum_j j\Theta_j\equiv A+B \mod <V,Q,R>.\end{aligned}$$

This proves the claim (IV.10). We shall now verify Conjecture IV.3.3 for $X/\mathbb{Q}$ and its reduction at 2. Then $T_2 =< 3\varphi(Q), \varphi(Q)+\varphi(R) >$ with intersection form

$$\begin{pmatrix}(3\varphi(Q).3\varphi(Q)) & (3\varphi(Q).\varphi(Q)+\varphi(R))\\ (3\varphi(Q).\varphi(Q)+\varphi(R)) & (\varphi(Q)+\varphi(R).\varphi(Q)+\varphi(R))\end{pmatrix}$$
$$= -\begin{pmatrix}12 & 6\\ 6 & 4\end{pmatrix} = -2\begin{pmatrix}6 & 3\\ 3 & 2\end{pmatrix} \sim -2\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}.$$

Hence, Conjecture IV.3.3 holds with $m=2$.

IV.7 Twisting

This section will conclude the investigation of the extremal elliptic K3 surface X by commenting on twisting. Our aim is to show that all admissible twists of the associated newform f correspond to twists of X. This is non-trivial, since f admits cubic twists.

For the quadratic twisting, the statement can be deduced from the Weierstrass equation (IV.1) after base change. For instance, it is immediate from point counting that the twist of X over $\mathbb{Q}(\sqrt{d})$ corresponds to the quadratic twist $f\otimes\left(\frac{d}{\cdot}\right)$.

By construction the twist has bad reduction at the primes dividing the discriminant of $\mathbb{Q}(\sqrt{d})$. This can be read off from the I_{12} fibre at ∞ (resp. I_3 for the rational elliptic surface Y) whose components Θ_j and Θ_{12-j} become pairwise conjugate over $\mathbb{Q}(\sqrt{d})$.

If $d\equiv 1 \mod 4$, then the twist has a model with good reduction at 2. This can be derived by exactly the same change of variables which lead from equation (IV.1) to (IV.2). This remark concludes the study of quadratic twists.

We shall now come to the cubic twists of X. They can be achieved by considering the base change

$$\pi_d : (s:t)\mapsto(s^3:dt^3)$$

instead of the original π. Here, we can restrict to positive cube-free d. Let $X^{(d)}$ denote the pull-back from the rational elliptic surface Y by π_d. An affine model is given by

$$X^{(d)}:\quad y^2+ds^2\,x\,y=x^3+2st^3\,x^2+s^2t^6\,x. \tag{IV.11}$$

Of course, $X^{(d)}$ has the same configuration of singular fibres as X and also inherits the trivial action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $NS(X^{(d)})$. By construction, it has bad reduction exactly at the prime divisors of $3d$. At prime divisors of d, this reduction is highly degenerate, since then the singularity in the fibre at ∞ is no longer an ordinary rational double point.

Proposition IV.7.1
The twists $X^{(d)}$ for positive cube-free d are in 1:1-correspondence with the cubic twists of f. The correspondence is obtained by considering the Mellin transforms of the respective L-series $L(T_{X^{(d)}}, s)$.

Recall that $f = \sum_n a_n q^n$ is the newform of weight 3 and level 27 from Table II.1. Since f has CM by $\mathbb{Q}(\sqrt{-3})$, the next corollary follows directly from the classification of CM-forms in Theorem II.3.4. We only need the quadratic twisting as explained above and the cubic twisting from Proposition IV.7.1.

Corollary IV.7.2
Let g be a newform of weight 3 with CM by $\mathbb{Q}(\sqrt{-3})$. Then there is a twist X' of X with $L(T_{X'}, s) = L(g, s)$ (up to the bad Euler factors).

We shall now give a simple proof of Proposition IV.7.1. It suffices to investigate $X^{(p)}$ (and $X^{(p^2)}$) for any prime p. Later, we will give a more detailed statement in Theorem IV.7.3.

By construction, X and $X^{(p)}$ are isomorphic over $\mathbb{Q}(\sqrt[3]{p})$. Since furthermore their L-series have CM by $\mathbb{Q}(\sqrt{-3})$, these are either equal or cubic twists by $\left(\frac{p}{\cdot}\right)_3$ or $\left(\frac{p}{\cdot}\right)_3^2$. We now have to distinguish whether $p = 3$ or not.

If $p \neq 3$, then $X^{(p)}$ has bad reduction at p as explained above. Hence, the associated newform g is ramified at p whereas f is not. Thus g is one of the two twists. By construction, $X^{(p^2)}$ gives the other twist (the squared twist). This achieves the complete twisting except for the 3-part.

Let us now assume $p = 3$. Here, we can perform an explicit calculation by point counting on $X^{(3)}$ and $X^{(9)}$. Let $\rho_{X^{(3)}}$ resp. $\rho_{X^{(9)}}$ denote the compatible system of Galois representations on $T_{X^{(3)}}$ resp. $T_{X^{(9)}}$. We collect the results in the following table:

p	tr $\rho_{X^{(3)}}$	tr $\rho_{X^{(9)}}$
2	0	0
5	0	0
7	2	11
11	0	0
13	23	−22
17	0	0
19	−37	26
23	0	0
29	0	0
31	59	−13
37	26	−73

Note that the varieties $X^{(3)}$ and $X^{(9)}$ only have bad reduction at 3. Hence, the corresponding newforms belong to those listed in (IV.6). Since the traces at Frob_7 differ, $X^{(3)}$ and $X^{(9)}$ give the two cubic twists of f from (IV.6). For details, the reader is referred to the proof of Proposition IV.2.2. Alternatively, we could argue with [S, Prop. 1]. This completes the proof of Proposition IV.7.1.

We shall now determine the corresponding newforms explicitly. Using the calculation for $\left(\frac{3}{\cdot}\right)_3$ from Section II.11.4, we compute that

$$\psi_f \otimes \left(\frac{3}{\cdot}\right)_3 \left(\left(\frac{1+3\sqrt{-3}}{2}\right)\right) = \frac{-13+3\sqrt{-3}}{2} \frac{-1-\sqrt{-3}}{2} = \frac{11+5\sqrt{-3}}{2}.$$

Hence, this twist corresponds to $X^{(9)}$. Our final goal for this chapter is to prove the analogous general statement:

Theorem IV.7.3
$X^{(d)}$ corresponds to the twist $\psi_f \otimes \left(\frac{d}{\cdot}\right)_3^2$.

The proof of this theorem is organized as follows: We exhibit another extremal elliptic fibration on the surfaces X and $X^{(d)}$. This will have fibres with CM by $\mathbb{Q}(\sqrt{-3})$ such that we can perform the twisting fibrewise. Then the claim will follow.

We want to give an elliptic fibration on $X^{(d)}$ with the component D_0 as a section. Therefore, we work with the affine equation (IV.7) and its $X^{(d)}$ analogue. Here, $D_0 = \{s = y' = 0\}$, so the fibration is the affine projection on the x'-coordinate. We employ the usual notation, i.e. replace x' by t and likewise for s and y'. Then equation (IV.4) becomes

$$y^2 + dtx^2y = t(t+1)^2x. \tag{IV.12}$$

In order to obtain a projective model for this, we first homogenize

$$s^3y^2z + ds^2tx^2y = t(t+s)^2xz^2.$$

Then the change of variable $y \mapsto \frac{t+s}{s}y$ gives

$$X^{(d)}: \;\; s(t+s)y^2z + dstx^2y = t(t+s)xz^2. \tag{IV.13}$$

This has six singularities, two in each fibre above $0, -1$ and ∞. Their resolution produces three fibres of type IV^*. These can be read off directly from the original fibration. We sketch this in Figure IV.2.

The sections of the new fibration are D_0, Θ_3 and Θ_9, as printed in blue, and the remaining components form the three singular fibres of type IV^*. In particular, they are all defined over $\mathbb{Q}$, as we have seen before.

The smooth fibres of the above fibration are elliptic curves with CM by $\mathbb{Q}(\sqrt{-3})$. However, the impact of twisting on the associated newforms is not visible so far. Therefore, we transform eq. (IV.13) to Weierstrass form. The procedure from [Ca, §8] returns

$$X^{(d)}: \;\; y^2 + ds^2t^2(s+t)^2y = x^3.$$

By [I-R, 18, Thm. 4], the Größencharaktere associated to the smooth fibres are twisted by $\left(\frac{d}{\cdot}\right)_3^2$ upon moving from $X^{(1)}$ to $X^{(d)}$. Using the Lefschetz fixed point formula, this implies Theorem IV.7.3.

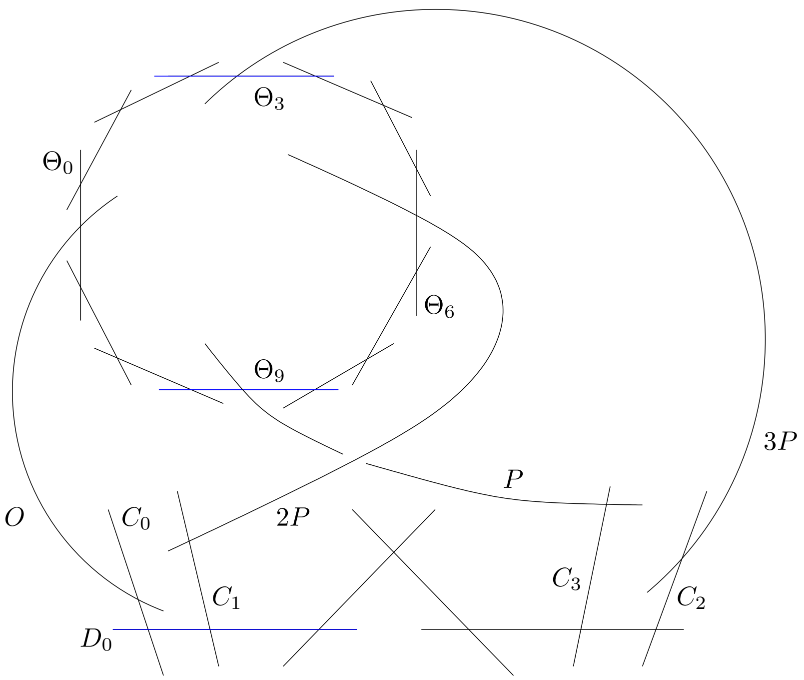

Figure IV.2: The fibration on $X^{(d)}$ with three fibres of type IV^*

In Corollary IV.7.2, we have seen that all newforms of weight 3 with rational coefficients and CM by $\mathbb{Q}(\sqrt{-3})$ can be realized geometrically by some twist $X^{(d)}$. For newforms which only admit quadratic twisting, the corresponding statement is often trivial. By Section III.8, there are, up to twisting, 22 such forms where we know an associated elliptic singular K3 surface over $\mathbb{Q}$. In fact, all these surfaces admit an elliptic fibration with a section over $\mathbb{Q}$. Then, we can twist the fibration quadratically to realize the twists of the associated newform.

The remaining newforms are those with CM by $\mathbb{Q}(\sqrt{-1})$, admitting quartic twisting. We address their geometric realizations in our final remark.

Remark IV.7.4
Consider the newforms of weight 3 with rational coefficients and CM by $\mathbb{Q}(\sqrt{-1})$. Geometric realizations for all of them are easily derived from the Fermat quartic (cf. Ex. I.3.6 and I.4.4). However, this highly affects the defining equations of the 48 lines, so the action of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Néron-Severi group becomes quite complicated. Similarly, we could argue with the extremal elliptic K3 fibration

$$Z: \quad y^2 = x^3 + s^3t^3(s+t)^2x.$$

This has singular fibres III^*, III^*, I_0^* (No. 279 in Table III.1). Here, the twisting can be interpreted fibrewise. This provides an analogous result to Theorem IV.7.3 (resp. Cor. IV.7.2) for the Größencharaktere of $\mathbb{Q}(\sqrt{-1})$ with rational traces and ∞-type 2. In particular, any newform of weight 3 with rational coefficients and CM by $\mathbb{Q}(\sqrt{-1})$ has a geometric realization by some twist $Z^{(d)}$ of Z. In terms of the Néron-Severi group, a quartic twist

$$Z^{(d)}: \quad y^2 = x^3 + d\, s^3t^3(s+t)^2x$$

only affects two simple components of the I_0^* fibre. They become conjugate over $\mathbb{Q}(\sqrt{d})$.

With respect to the associated newforms up to twisting, our analysis in Chapter III together with [B-M] exhausted the extremal elliptic K3 fibrations over $\mathbb{Q}$. This was discussed in Section III.8. This gives the first part of the following conclusion. The second statement is based on the above discussion of twisting.

Conclusion IV.7.5

Let X be an extremal elliptic K3 fibration over $\mathbb{Q}$. Up to twisting, the associated newform is one of the 24 newforms from Table II.1 which we specified in Section III.8 by giving the discriminant of X (cf. Prop. II.13.1).

Conversely, let f be any of the 24 newforms from Table II.1 for which we know, up to twisting, an associated elliptic singular K3 surface over $\mathbb{Q}$. Then any admissible twist of f has a geometric realization as an extremal elliptic K3 fibration over $\mathbb{Q}$.

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