The Kodaira dimension of Siegel modular varieties of genus 3 or higher

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Abstract

An Abelian variety is a g-dimensional complex torus which is a projective variety. In order to obtain an embedding into projective space one has to chose an ample line bundle. To each such line bundle one can associate a polarisation which only depends on the class of the line bundle in the Néron-Severi group. The type of the polarisation is given by a g-tuple of integers $e_1, \ldots, e_g$ with the property that $e_i | e_{i+1}$ for $i = 1, \ldots, g-1$. If $e_1 = \cdots = e_g = 1$, the polarisation is said to be principal. If the values $\frac{e_{i+1}}{e_i}$ are pairwise coprime, the polarisation is said to be coprime. Furthermore, we can give a symplectic basis for the group of n-torsion points, which is then called a level-n structure.

Not only Abelian varieties but also their moduli spaces have been of interest for many years. The moduli space of Abelian varieties with a fixed polarisation (and optionally a level-n structure for a fixed n) can be constructed from the Siegel upper half space by dividing out the action of the appropriate arithmetic symplectic group. These spaces are quasi-projective algebraic varieties and can made into projective algebraic varieties by the method of toroidal compactification.

Many different aspects of these varieties have been investigated, such as the type of singularities that arise in the interior and on the boundary, and the question whether a given compactification has specific desired properties. Another task is to determine their Kodaira dimension.

For principally polarised Abelian varieties, the Kodaira dimension of the moduli space is known (except for the case $g = 6$). If we approach the situation from another direction and ask for a lower bound on the level such that the moduli space is of general type, we can also give an explicit answer.

However, the case of non-principal polarisations is much less investigated. For $g = 2$ there are still several results: the Kodaira dimension is known for all but a few polarisations, and a level of 4 is known to be enough for the moduli space to be of general type for any polarisation (if one extra condition is satisfied).

This thesis considers moduli spaces of higher-dimensional, non-principally polarised Abelian varieties with a level-n structure. The main result establishes that the moduli space is of general type for any fixed coprime polarisation if the level is higher than an explicitely given bound (if one extra condition depending on the polarisation is satisfied).

To be able to prove this theorem we had to generalise a result on the minimum of integer quadratic forms that plays a central role in the construction. Furthermore, we give the orbits of one- and g-dimensional isotropic subspaces of $\mathbb{Q}^{2g}$ under the action of the arithmetic symplectic groups that correspond to polarised Abelian varieties.

Zusammenfassung

Eine Abelsche Varietät ist ein g-dimensionaler komplexer Torus, der eine projektive Varietät ist. Um eine Einbettung in einen projektiven Raum zu erhalten, muss man ein amples Geradenbündel wählen. Jedem dieser Geradenbündel kann man eine Polarisierung zuordnen, die nur von der Klasse des Geradenbündels in der Néron-Severi-Gruppe abhängt. Der Typ dieser Polarisierung wird durch ein g-Tupel $e_1, \dots, e_g$ von ganze Zahlen gegeben, die die Eigenschaft $e_i | e_{i+1}$ für $i = 1, \dots, g-1$ erfüllen. Falls $e_1 = \cdots = e_g$ gilt, nennt man die Polarisierung prinzipal. Sind die Zahlen $\frac{e_{i+1}}{e_i}$ paarweise teilerfremd, nennt man die Polarisierung teilerfremd. Außerdem kann man eine Basis für die Gruppe der n-Teilungs-Punkte angeben, was dann als Level-n Struktur bezeichnet wird.

Seit vielen Jahren sind nicht nur die Abelschen Varietäten, sondern auch ihre Modulräume von Interesse. Der Modulraum Abelscher Varietäten mit einer bestimmten Polarisierung (und eventuell einer Level-n Struktur für festes n) kann aus dem Siegelschen oberen Halbraum konstruiert werden, indem man die Operation der entsprechenden arithmetischen symplektischen Gruppe austeilt. Diese Räume sind quasi-projektive algebraische Varietäten und können durch toroidale Kompaktifizierung zu projektiven algebraischen Varietäten gemacht werden.

Viele verschiedene Aspekte dieser Varietäten wurden bereits untersucht, wie zum Beispiel welche Singularitäten im Inneren oder am Rand auftreten, und die Frage, ob eine gegebene Kompaktifizierung spezielle, gewünschte Eigenschaften hat. Eine weitere Aufgabe ist es, die Kodaira-Dimension zu bestimmen.

Für prinzipal polarisierte Abelsche Varietäten ist die Kodaira-Dimension des Modulraums bekannt (außer für den Fall $g = 6$). Andererseits können wir auch nach einer unteren Schranke für das Level fragen, so dass der Modulraum von allgemeinem Typ ist. Auch hier ist eine explizite Antwort bekannt.

Die Modulräume nicht-prinzipal polarisierter Abelscher Varietäten sind jedoch deutlich weniger untersucht. Für $g = 2$ gibt es noch einige Ergebnisse: die Kodaira-Dimension ist bis auf ein paar Ausnahmen für fast alle Polarisierungen bekannt, und man weiß, dass Level 4 bei jeder Polarisierung ausreicht, um einen Modulraum von allgemeinem Typ zu erhalten (wenn eine weitere Bedingung erfüllt ist).

Die vorliegende Doktorarbeit beschäftigt sich mit Modulräumen nicht-prinzipal polarisierter Abelscher Varietäten von höherer Dimension mit Level-n Struktur. Das zentrale Ergebnis zeigt, dass für eine fest gewählte, teilerfremde Polarisierung der Modulraum von allgemeinem Typ ist, sobald das Level über einer explizit angegebene Schranke liegt (und eine weitere Bedingung, die von der Polarisierung abhängt, erfüllt ist).

Um diesen Satz zu beweisen, mussten wir ein Ergebnis über das Minimum ganzzahliger quadratischer Formen verallgemeinern, das eine zentrale Rolle in der Konstruktion spielt. Außerdem geben wir die Orbits der ein- und g-dimensionalen isotropen Unterräume von $\mathbb{Q}^{2g}$ unter der Operation der arithmetischen symplektischen Gruppen an, die zu polarisierten Abelschen Varietäten gehören.

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Keywords

- Abelian varieties
- non-principal polarisations
- Kodaira dimension

Schlagworte

- Abelsche Varietäten
- Nicht-prinzipale Polarisierungen
- Kodaira-Dimension

Contents

Chapter 1

Introduction

1.1 Overview: The Kodaira dimension of Siegel modular varieties

In one (complex) dimension, an elliptic curve E can be given as $E = \mathbb{C}/\mathfrak{L}$, where $\mathfrak{L}$ is a non-degenerate lattice. Without loss of generality, we may assume $\mathfrak{L}$ to be given in the form $\mathfrak{L} = \mathbb{Z}\tau + \mathbb{Z}$ with $\operatorname{Im}(\tau) > 0$. Two elliptic curves E_τ and $E_{\tau'}$ are isomorphic if and only if there are integers a,b,c,d with $ad-bc=1$ such that $\tau' = \frac{a\tau+b}{c\tau+d}$. This means that their moduli space, i. e. the space parametrising elliptic curves, is

$$\mathscr{A}_1 := \mathfrak{S}_1/\Gamma_1,$$

where $\mathfrak{S}_1 := \left\{\tau \in \mathbb{C} \,\middle|\, \operatorname{Im}(\tau) > 0\right\}$ is the Siegel upper half plane and $\Gamma_1 := \operatorname{SL}(2,\mathbb{Z})$. The action of $M \in \Gamma_1$ on $\mathfrak{S}_1$ is defined by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau+b}{c\tau+d}.$$

The concept corresponding to an elliptic curve in higher dimension is that of an abelian variety; this is defined to be a complex torus which admits an embedding into projective space. Such an embedding exists if and only if there exists a positive definite Riemann form[1]. It can be given by $f(x,y) = x\Lambda\,{}^t\!y$ with a matrix of the form

$$\Lambda = \begin{pmatrix} & \Delta \\ -\Delta & \end{pmatrix} \quad \text{where } \Delta = \operatorname{diag}(e_1,\dots,e_g).$$

We call it a polarisation of type $(e_1,\dots,e_g)$. The e_i are positive integers which, without changing the group, may be chosen such that $e_i | e_{i+1}$ for $i = 1,\dots,g-1$. The special case of type $(1,\dots,1)$ is called a principal polarisation.

Again, we ask for the moduli space. Let us first consider g-dimensional abelian varieties with a principal polarisation. The moduli space of these varieties is

$$\mathscr{A}_g := \mathfrak{S}_g/\Gamma_g,$$

where $\mathfrak{S}_g := \left\{\tau \in \operatorname{Sym}(g,\mathbb{C}) \,\middle|\, \operatorname{Im}(\tau) > 0\right\}$ is the Siegel upper half space. As it turns out, we cannot use $\operatorname{SL}(2g,\mathbb{Z})$ as one might expect, but

$$\Gamma_g := \operatorname{Sp}(2g,\mathbb{Z}) := \left\{M \in \operatorname{GL}(2g,\mathbb{Z}) \,\middle|\, MJ\,{}^t\!M = J\right\},$$

[1] This is a non-degenerate alternating bilinear form satisfying certain properties.

where $J := \begin{pmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{pmatrix}$. Since for a principal polarisation Δ is the unit matrix, we could also write Λ instead of J. The action of $M \in \Gamma_g$ on $\mathfrak{S}_g$ is defined by

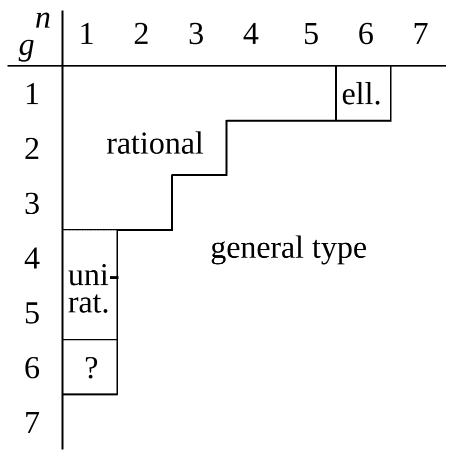
$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}.$$

$\mathscr{A}_g$ is a quasi-projective variety. We want to ask for the Kodaira dimension of these varieties or, more precisely, that of a projective model. So we have to compactify, and the best way to do this is via toroidal compactifications; the details will be discussed in chapter 2, but for now assume that there exists a compactification with $\mathscr{A}_g$ as a Zariski-open subset. Since the Kodaira dimension is birationally invariant and these compactifications are birational to each other we may choose any. We define the Kodaira dimension of $\mathscr{A}_g$ to be that of a smooth projective model.

It is known that $\mathscr{A}_g$ is rational for $g \leq 3$, unirational for $g = 4, 5$ and of general type for $g \geq 7$. The results about (uni-)rationality are due to S. Mori, S. Mukai, J.-I. Igusa and a number of other people, and those proving general type to Y.-S. Tai, E. Freitag and D. Mumford.

In many cases one has an additional structure on the abelian varieties, namely level structures. The subgroups $\Gamma_g(n) \subset \Gamma_g$, which we will define in section 1.2.2, allow to define $\mathscr{A}_g(n) := \mathfrak{S}_g/\Gamma_g(n)$, the moduli space of g-dimensional principally polarised abelian varieties with a full level-n structure. And again, the question is for the Kodaira dimension.

We can summarise the known results in the following table:



This shows the combinations for which $\mathscr{A}_g(n)$ is known to be rational or at least unirational, to have Kodaira dimension 0, or to be of general type. Note that still nothing is known about the Kodaira dimension of $\mathscr{A}_6$.

Let us now consider the case of non-principal polarisations. We can construct the moduli space for these abelian varieties with given type of polarisation using a group $\tilde{\Gamma}_{\text{pol}}$ defined analogously to Γ_g, only that Δ is not the unit matrix. The action of $M \in \tilde{\Gamma}_{\text{pol}}$ on $\mathfrak{S}_g$ is defined by

$$M : \tau \mapsto (A\tau + B\Delta)(C\tau + D\Delta)^{-1}\Delta.$$

The fact that this action depends on Δ and hence on the polarization is denoted by the tilde on $\tilde{\Gamma}_{\text{pol}}$.

Again, we ask for the Kodaira dimension of these spaces $\mathscr{A}_{\text{pol}}$. There are various results, due to Y. Tai, V. Gritsenko, G. K. Sankaran, K. Hulek and others.

For example, it is known that

$\mathscr{A}_{e_1,\dots,e_g}$ is of general type, if
$g \geq 16$ or
$g \geq 8$ and the e_i are odd and sums of two squares
$\forall t \in \mathbb{N}\ \exists g(t) \in \mathbb{N}\ \forall g \geq g(t)\ :\ \mathscr{A}_{1,\dots,1,t}$ is of general type
(V. Gritsenko also showed that $g(2) = 13$)

And for two dimensions:
$\mathscr{A}_{1,1} (= \mathscr{A}_2), \mathscr{A}_{1,2}, \mathscr{A}_{1,3}$ are rational
$\mathscr{A}_{1,4}$ is unirational
$\mathscr{A}_{1,p}$ for $p \geq 73$ prime is of general type (G. K. Sankaran and C. Erdenberger)
$\kappa(\mathscr{A}_{1,t}) \geq 0$ for $t \geq 13$ (there are 14 exceptions for which this is not yet known)

As before, let us now consider full level structures by using the subgroups $\tilde{\Gamma}_{\text{pol}}(n) \subset \tilde{\Gamma}_{\text{pol}}$ (defined in section 1.2.2) to construct $\mathscr{A}_{\text{pol}}(n)$. We can ask the following question: Which level do we need so that $\mathscr{A}_{\text{pol}}(n)$ is of general type for all polarisations?

For $g = 2$, we already know that $\mathscr{A}_{1,1}(n)$ is rational for $n \leq 3$, so that we need at least level 4. K. Hulek showed in [H] that indeed

$$\mathscr{A}_{1,t}(n) \text{ is of general type for } n \geq 4, \gcd(n,t) = 1.$$

The condition that t and n are to be coprime is technical and can probably be weakened or even dropped.

For $g = 3$, we know that $\mathscr{A}_{1,1,1}(n)$ is rational for $n \leq 2$, so that we need at least level 3. The claim is the following:

Theorem: Let the type of the polarisation be $(1, e_1, e_2)$ with $e_1, e_2 \in \mathbb{N}$ where $e_1 | e_2$, $e_2 \neq 2$ and $\gcd(e_1, \frac{e_2}{e_1}) = 1$. Then $\mathscr{A}_{1,e_1,e_2}(n)$ is of general type provided $\gcd(n, e_2) = 1, n \geq 3$ and

$$n > \frac{9e_2}{4} \min\left\{ \frac{e_1^2}{\min\{e_1e_2, e_2\sqrt{3e_1}, e_1\sqrt{3}\sqrt[3]{e_1e_2}\}}, \frac{e_2}{\min\{e_1e_2, e_1\sqrt{3e_1e_2}, \sqrt{3}\sqrt[3]{e_1^2e_2^2}\}} \right\}.$$

Furthermore, we can even generalise to higher genus. There is, however, one drawback in form of Proposition 3.3.8 that we have only been able to prove up to genus 9. So, the result is the following:

Theorem: Let $3 \leq g \leq 9$. Let the type of the polarisation be $(1, e_1, \dots, e_{g-1})$ with $e_i | e_{i+1}$, $e_{g-1} \neq 2$ and $\gcd(\frac{e_i}{e_{i-1}}, \frac{e_j}{e_{j-1}}) = 1$ for all $i \neq j$. Then $\mathscr{A}_{1,e_1,\dots,e_{g-1}}(n)$ is of general type provided $\gcd(n, e_{g-1}) = 1$, $n \geq 3$ and

$$n > \frac{2^g + 1}{(g+1)2^{g-3}} \min\left\{ \frac{e_{g-2}}{C(\mathbb{L}_1)}, \frac{e_{g-1}}{e_1 C(\mathbb{L}_2)} \right\}$$

where $\mathbb{L}_1 = \mathbb{L}(e_1, \frac{e_2}{e_1}, \dots, \frac{e_{g-1}}{e_{g-2}})$ and $\mathbb{L}_2 = \mathbb{L}(\frac{e_{g-1}}{e_{g-2}}, \dots, \frac{e_2}{e_1}, e_1)$ and

$$C(\mathbb{L}(x_1, \dots, x_{g-1})) = \min\left\{ \min_{2 \leq r \leq g} \left\{ \frac{\sqrt{3}}{\sqrt[r]{\prod_{i=1}^{r-1} x_i^i}} \right\}, 1 \right\}.$$

1.2 Basic definitions and theorems

Now we shall state in detail what we have only sketched so far.

1.2.1 Siegel modular forms

Definition 1.2.1: Siegel space.
We define the *Siegel space of dimension g* to be

$$\mathfrak{S}_g := \left\{ \tau \in \mathrm{Sym}(g,\mathbb{C}) \,\middle|\, \mathrm{Im}(\tau) > 0 \right\},$$

the space of symmetric complex $g \times g$ matrices with positive definite imaginary part.

Definition 1.2.2: Symplectic Group.
Let $\mathbb{E}$ be an euclidian domain. The *symplectic group* $\mathrm{Sp}(2g,\mathbb{E})$ is the subgroup of $\mathrm{GL}(2g,\mathbb{E})$ defined by

$$\mathrm{Sp}(2g,\mathbb{E}) := \left\{ M \in \mathrm{GL}(2g,\mathbb{E}) \,\middle|\, MJ{}^tM = J \right\}, \quad \text{where} \quad J := \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

In the case of $\mathbb{E} = \mathbb{Z}$ we also use the notation $\Gamma_g := \mathrm{Sp}(2g,\mathbb{Z})$.

Lemma 1.2.3.
For an euclidian domain $\mathbb{E}$ the symplectic group $\mathrm{Sp}(2g,\mathbb{E})$ is generated by $J = \left(\begin{smallmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{smallmatrix}\right)$ and the matrices of the form $\left(\begin{smallmatrix} \mathbb{1} & S \\ 0 & \mathbb{1} \end{smallmatrix}\right)$, where $S \in \mathrm{Sym}(g,\mathbb{E})$.

Proof.
See [Fre83, Satz A 5.4 in the appendix]. □

Definition 1.2.4: Action of the symplectic group.
The symplectic groups $\mathrm{Sp}(2g,\mathbb{E})$ for $\mathbb{E} = \mathbb{Z},\mathbb{Q},\mathbb{R}$ act on $\mathfrak{S}_g$ the following way: For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\tau \in \mathfrak{S}_g$ let $M\tau := (A\tau + B)(C\tau + D)^{-1}$.

Definition 1.2.5: Siegel modular form.
Assume $g \geq 2$. A function $f : \mathfrak{S}_g \to \mathbb{C}$ is called *Siegel modular form with respect to an arithmetic subgroup* $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Q})$ *of degree g and weight w* if the following properties apply:

(i) f is holomorphic

(ii) For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ we have the equality $f(M\tau) = \det(C\tau + D)^w f(\tau)$

Remark 1.2.6.
For $g = 1$ we need the additional condition that f is bounded around any cusp: for any $M \in \mathrm{Sp}(2,\mathbb{Q})$ and any $y \in \mathbb{R}, y > 0$ we have that $\{ f(M\tau) \,|\, \mathrm{Im}(\tau) > y \}$ is bounded.

Property (ii) needs only to be shown for the generators of the modular group; in the case of $\Gamma = \Gamma_g = \mathrm{Sp}(2g,\mathbb{Z})$ this means:

- For all $S \in \mathrm{Sym}(g,\mathbb{Z}) : f(\tau + S) = f(\tau)$
- $f(-\tau^{-1}) = (\det \tau)^w f(\tau)$.

1.2.2 Abelian varieties

Definition 1.2.7: Abelian variety.
An *Abelian variety* is a complex torus which admits an embedding into projective space, i. e. we have $A = \mathbb{C}^g/\mathfrak{L}$ where $\mathfrak{L} = \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_{2g}$ for an $\mathbb{R}$-basis $\{b_1, \ldots, b_{2g}\}$ of $\mathbb{C}^g$ and a map $\phi : A \hookrightarrow \mathbb{P}^N$.

Remark 1.2.8.
For $g = 1$ such an embedding always exists. If $\mathfrak{L} = \mathbb{Z}\tau + \mathbb{Z}$ it can be found e. g. by using the Weierstraß $\wp$-function associated to the lattice $\mathfrak{L}$:

$$\phi : \begin{cases} \mathbb{C}/\mathfrak{L} \hookrightarrow \mathbb{P}^2 \\ \quad z \mapsto (1 : \wp(z) : \wp'(z)) \text{ for } z \neq \infty \\ \quad \infty \mapsto (0 : 0 : 1) \end{cases}.$$

This means that every elliptic curve is an abelian variety. In general:

Lemma 1.2.9.
A complex torus $A = \mathbb{C}^g/\mathfrak{L}$ admits an embedding into projective space if and only if there exists a non-degenerate alternating bilinear form $H' : \mathfrak{L} \otimes \mathfrak{L} \to \mathbb{Z}$ whose $\mathbb{R}$-linear extension to $\mathbb{C}^g$ satisfies $H'(ix, iy) = H'(x, y)$.

Proof.
See [Mu1, p. 35]. □

Remark 1.2.10.
This is equivalent to saying that there exists a Riemann form on $\mathbb{C}^g$ with respect to $\mathfrak{L}$, which is a positive-definite Hermitian form H with the property that its imaginary part is integer-valued on $\mathfrak{L}$. This form can be constructed from H' by $H(x,y) = H'(ix,y) + iH'(x,y)$.

Lemma 1.2.11.
With respect to a suitable basis of $\mathfrak{L}$ the aforementioned form H' can be given by

$$H'(x,y) = x\Lambda^t y \text{ where } \Lambda = \begin{pmatrix} & \Delta \\ -\Delta & \end{pmatrix} \text{ and } \Delta = \operatorname{diag}(e_1, \ldots, e_g).$$

The values e_i can be chosen to be positive integers satisfying $e_i | e_{i+1}$ for all $i = 1, \ldots, g-1$. This choice is unique.

Proof.
See [Igu72, p. 65]. □

Definition 1.2.12: Polarization.
The form H' corresponding to $\Delta = \operatorname{diag}(e_1, \ldots, e_g)$ is called a *polarization of type* $(e_1, \ldots, e_g)$. A polarization of type $(1, \ldots, 1)$ is called a *principal polarization*.

Lemma 1.2.13.
The moduli space of principally polarized g-dimensional abelian varieties can be given by the normal complex analytic space

$$\mathscr{A}_g := \mathfrak{S}_g / \Gamma_g.$$

The action of Γ_g on $\mathfrak{S}_g$ has already been given in Definition 1.2.4. $\mathscr{A}_g$ is a quasi-projective variety.

Proof.
See [LB, Chapter 8, Corollary 2.7] and [LB, Remark 10.4]. □

Definition 1.2.14: Symplectic groups for polarizations.
For a given polarization $(e_1,\ldots,e_g)$ define

$$\tilde{\Gamma}_{\text{pol}} := \tilde{\Gamma}_{e_1,\ldots,e_g} := \left\{ M \in \mathrm{GL}(2g,\mathbb{Z}) \middle| M\Lambda {}^t\!M = \Lambda \right\}.$$

The action of $\tilde{\Gamma}_{\text{pol}}$ on $\mathfrak{S}_g$ is defined to be

$$M\tau := (A\tau + B\Delta)(C\tau + D\Delta)^{-1}\Delta \quad \text{where } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{\Gamma}_{\text{pol}}.$$

Remark 1.2.15.
Note that this is in fact a generalization of the principally polarized case, where now Δ is no longer the unit matrix. The tilde on $\tilde{\Gamma}_{\text{pol}}$ is used to point out that the group action involves $\Delta \neq \mathbb{1}$.

Note also that we may chose $e_1 = 1$ without changing the group $\tilde{\Gamma}_{\text{pol}}$.

Lemma 1.2.16.
The moduli space of abelian varieties with a polarization of type $(e_1,\ldots,e_g)$ can be given by the normal complex analytic space

$$\mathscr{A}_{\text{pol}} := \mathfrak{S}_g / \tilde{\Gamma}_{\text{pol}}.$$

Proof.
See [LB, Chapter 8, Corollary 2.7]. □

Definition 1.2.17: Full level structures.
Let $A = \mathbb{C}^g/\mathfrak{L}$ be a polarized abelian variety. A symplectic basis $\lambda_1,\ldots,\lambda_g,\mu_1,\ldots,\mu_g$ of $\mathfrak{L}$ for the polarization determines a basis for the group of n-torsion points $A^{[n]}$ in A, namely $\overline{\frac{1}{n}\lambda_1},\ldots,\overline{\frac{1}{n}\lambda_g},\overline{\frac{1}{n}\mu_1},\ldots,\overline{\frac{1}{n}\mu_g}$. A *full level-n structure* on A is a basis of $A^{[n]}$ coming from a symplectic basis in this way.

Definition 1.2.18: Principal congruence subgroups.
Define the *principal congruence subgroups* $\Gamma_{\text{pol}}(n)$ and $\tilde{\Gamma}_{\text{pol}}(n)$ by

$$\Gamma_{\text{pol}}(n) := \left\{ M \in \Gamma_{\text{pol}} \middle| M \equiv \mathbb{1} \mod n \right\} \quad \text{and}$$
$$\tilde{\Gamma}_{\text{pol}}(n) := \left\{ M \in \tilde{\Gamma}_{\text{pol}} \middle| M \equiv \mathbb{1} \mod n \right\}.$$

The action of $\Gamma_{\mathrm{pol}}(n)$ and $\tilde{\Gamma}_{\mathrm{pol}}(n)$ on $\mathfrak{S}_g$ are the actions induced from Γ_{pol} and $\tilde{\Gamma}_{\mathrm{pol}}$, respectively.

Lemma 1.2.19.
The moduli space of $(e_1,\dots,e_g)$-polarised abelian varieties (whether principally or non-principally polarised) with a (full) level-n structure can be given by the normal complex analytic spaces

$$\mathscr{A}_g(n) := \mathfrak{S}_g/\Gamma_g(n) \quad \text{or} \quad \mathscr{A}_{\mathrm{pol}}(n) := \mathfrak{S}_g/\tilde{\Gamma}_{\mathrm{pol}}(n),$$

respectively.

Proof.
See [LB, Chapter 8, Theorem 3.2]. □

Definition 1.2.20: Dual lattice.
Given a non-degenerate form $\Lambda : \mathfrak{L}\times\mathfrak{L} \to \mathbb{Z}$ (actually, we extend this map to a map $\mathfrak{L}\times(\mathfrak{L}\otimes\mathbb{Q}) \to \mathbb{Q}$ which we also denote by Λ), the *dual lattice* $\mathfrak{L}^\vee$ is defined by

$$\mathfrak{L}^\vee := \left\{ y \in \mathfrak{L}\otimes\mathbb{Q} \middle| \forall x \in \mathfrak{L} \ : \ \Lambda(x,y) \in \mathbb{Z} \right\}.$$

Remark 1.2.21.
The group $\mathfrak{L}^\vee/\mathfrak{L}$ is non-canonically isomorphic to $(\mathbb{Z}_{e_1}\times\cdots\times\mathbb{Z}_{e_g})^2$ where the e_i are the elementary divisors of the polarisation. It carries a skew form induced by Λ, and the group $(\mathbb{Z}_{e_1}\times\cdots\times\mathbb{Z}_{e_g})^2$ has a $\mathbb{Q}/\mathbb{Z}$-valued form which with respect to the canonical generators is given by

$$\begin{pmatrix} & \Delta^{-1} \\ -\Delta^{-1} & \end{pmatrix}.$$

The details can be found in [HKW, p. 9], so we only mention that we assume the polarisation of A to be given by the class of an ample line bundle (or its divisor $D \in \operatorname{Pic}(A)$) and define

$$\lambda : \begin{cases} A & \to & \operatorname{Pic}^0(A) \\ x & \mapsto & [(D+x)-D] \end{cases}.$$

Now, $\mathfrak{L}^\vee/\mathfrak{L} \simeq \ker\lambda$ and we can define

Definition 1.2.22: A canonical level structure.
A *canonical level structure* is a symplectic isomorphism

$$\alpha : \ker\lambda \to (\mathbb{Z}_{e_1}\times\cdots\times\mathbb{Z}_{e_g})^2$$

where the groups are equipped with the forms described above. Define the group $\tilde{\Gamma}^{\mathrm{lev}}_{\mathrm{pol}}$ by

$$\tilde{\Gamma}^{\mathrm{lev}}_{\mathrm{pol}} := \left\{ M \in \tilde{\Gamma}_{\mathrm{pol}} \middle| M|_{\mathfrak{L}^\vee/\mathfrak{L}} = \mathrm{id}_{\mathfrak{L}^\vee/\mathfrak{L}} \right\} \subset \tilde{\Gamma}_{\mathrm{pol}}$$

with the induced action on $\mathfrak{S}_g$.

Remark 1.2.23.
The concept of canonical level structures does only make sense for non-principally polarised abelian varieties, since otherwise $\mathfrak{L}^\vee = \mathfrak{L}$. Note, however, that a full level-n structure in the principally polarised case is the same as a canonical level structure for a polarisation of type $(n,\dots,n)$.

Lemma 1.2.24.
The moduli space of non-principally polarised abelian varieties with a canonical level structure can be given by the normal complex analytic space

$$\mathscr{A}_{\mathrm{pol}}^{\mathrm{lev}} := \mathfrak{S}_g/\tilde{\Gamma}_{\mathrm{pol}}^{\mathrm{lev}}.$$

Proof.
Again, see [LB, Chapter 8, Theorem 3.1]. □

1.2.3 Theta functions

Notation 1.2.25.
For $x \in \mathbb{C}$, let $\mathbf{e}[x] := e^{2\pi i x}$.

Definition 1.2.26: Theta function.
Let $m = (m'm'') \in \mathbb{R}^{2g}, \tau \in \mathfrak{S}_g, z \in \mathbb{C}^g$. We define the *theta function of characteristic m and modulus* τ to be

$$\theta_m(\tau,z) := \sum_{\zeta\in\mathbb{Z}^g} \mathbf{e}[\tfrac{1}{2}(\zeta+m')\tau\,{}^t(\zeta+m') + (\zeta+m')\,{}^t(z+m'')].$$

Lemma 1.2.27.
$\theta_m(\tau,z)$ is a holomorphic function on $\mathfrak{S}_g \times \mathbb{C}^g$.

Proof.
See [Igu72, p. 49]. □

Notation 1.2.28.
For any square matrix $A \in \mathbb{C}^{g\times g}$ let $(A)_0 := (a_{11},\dots,a_{gg}) \in \mathbb{C}^g$ be the vector of the diagonal elements of A.

Theorem 1.2.29: Theta transformation formula.
Let $m = (m'm'') \in \mathbb{R}^{2g}$ be a character, $\tau \in \mathfrak{S}_g, z \in \mathbb{C}^g$ and choose a transformation $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}(2g,\mathbb{Z})$. Then the following equation holds:

$$\theta_{m^\#}(\tau^\#,z^\#) = \mathbf{e}[\tfrac{1}{2}z(C\tau+D)^{-1}C\,{}^tz]\det(C\tau+D)^{\frac{1}{2}}u\theta_m(\tau,z),$$

in which the new values are given by

$$\tau^\# = (A\tau+B)(C\tau+D)^{-1}, \quad z^\# = z(C\tau+D)^{-1} \quad \text{and}$$

$$m^\# = mM^{-1} + \frac{1}{2}((C\,{}^t\!D)_0 (A\,{}^t\!B)_0).$$

The value u is an eighth root of unity, independent of τ and z.

Proof.
See [Igu72, p. 85]. □

Remark 1.2.30.
For the expression $\det(C\tau+D)^{\frac{1}{2}}$ we may choose either of the roots, since $\mathfrak{S}_g$ is connected and simply connected.[2] This choice may influence u but this is not a problem; it is only important that once a root is chosen we do not change the selection.

When we want to use theta functions to construct Siegel modular forms, there are some points in this transformation formula that may cause problems:

- the value u; but since u is a root of unity, we can get rid of it by taking the appropriate power
- the factor $\mathbf{e}[\frac{1}{2}z(C\tau+D)^{-1}C\,{}^t\!z]$; by letting $z=0$, this factor becomes 1
- the fact that the characteristic changes, so that one theta function is transformed into another one — the classical way to solve this problem by taking the product of theta functions, so that both θ_m and $\theta_{m^\#}$ are factors of the modular form we want to construct. We will come back to this idea shortly.

These observations motivate the following definitions:

Definition 1.2.31: Thetanullwert or theta constant.
A function $\theta_m(\tau) := \theta_m(\tau,0) : \mathfrak{S}_g \to \mathbb{C}$ is called *thetanullwert* or *theta constant*.

Definition 1.2.32: Even and odd characteristics.
Let $m = (m'm'') \in \{0, \frac{1}{2}\}^{2g}$. Then the characteristic m, the theta function $\theta_m(\tau,z)$ and the theta constant $\theta_m(\tau)$ are called *even*, if and only if $4m'\,{}^t\!m'' \equiv 0 \mod 2$; otherwise they are called *odd*.

Lemma 1.2.33.
$\theta_m(\tau)$ vanishes for all $\tau \in \mathfrak{S}_g$ if and only if m is an odd characteristic.

Proof.
See [Fre83, p. 42, Folgerung 3.2_1]. □

Lemma 1.2.34.
The set of even (resp. odd) theta functions is closed under the operation of $\mathrm{Sp}(2g,\mathbb{Z})$.

Proof.
We show this by applying the theta transformation formula to the character $m = (m'm'')$ for the generators of $\mathrm{Sp}(2g,\mathbb{Z})$:

[2] cf. [Igu72, p. 83]

For $M_1 = \left(\begin{smallmatrix} \mathbb{1} & S \\ 0 & \mathbb{1} \end{smallmatrix}\right)$ with $S = {}^tS$, we get $M_1^{-1} = \left(\begin{smallmatrix} \mathbb{1} & -S \\ 0 & \mathbb{1} \end{smallmatrix}\right)$ and therefore

$$\begin{aligned}
m^\# = (m'^\# m''^\#) &= (m'm'') \begin{pmatrix} \mathbb{1} & -S \\ 0 & \mathbb{1} \end{pmatrix} + \tfrac{1}{2}\big((0)_0 (S)_0\big) \\
&= (m', m'' - m'S + \tfrac{1}{2}(S)_0), \quad \text{which gives} \\
4m'^\# \, {}^tm''^\# &= 4m' \, {}^tm'' - 4m' \, {}^t(m'S) + 2m' \, {}^t(S)_0 \\
&= 4m' \, {}^tm'' - 4m'S \, {}^tm' + 2m' \, {}^t(S)_0 \\
&= 4m' \, {}^tm'' - 4 \sum_{1 \le i \le j \le g} m_i m_j s_{i,j} (2 - \delta_{ij}) + 2 \sum_{1 \le i \le g} m_i s_{i,i} \\
&= 4m' \, {}^tm'' - \sum_{1 \le i < j \le g} 2(2m_i)(2m_j) s_{i,j} - \sum_{1 \le i \le g} [-(2m_i)^2 + (2m_i)] s_{i,i} \\
&\equiv 4m' \, {}^tm'' \mod 2,
\end{aligned}$$

where the last equivalence follows from

$$\begin{aligned}
2m_i \in \mathbb{Z} &\implies 2(2m_i)(2m_j) \equiv 0 \mod 2 \quad \text{and} \\
2m_i \in \{0,1\} &\implies (2m_i)^2 = 2m_i \implies -(2m_i)^2 + (2m_i) = 0.
\end{aligned}$$

For $M_2 = \left(\begin{smallmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{smallmatrix}\right)$, we get $M_2^{-1} = \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$ and

$$(m'^\# m''^\#) = (m'm'') \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (m'', -m'),$$

which immediately leads to

$$4m'^\# \, {}^tm''^\# \equiv 4m'm'' \mod 2.$$

This shows that the transformed character is even if and only if the original one is, which proves the lemma. □

Now we can construct Siegel modular forms by taking the product of some even theta constants, such that for each θ_m contained in $\chi := \theta_{m_1} \dots \theta_{m_k}$ the theta constant $\theta_{m^\#}$ transformed according to Theorem 1.2.29 is also a factor of χ. As it turns out, in order to fulfil this property we have to include all even theta constants in the product. Taking into account that we still need the correct exponent to get rid of the factor u in the transformation formula, we obtain the following theorem:

Theorem 1.2.35.
The functions

$$\chi^{(g)}(\tau) := \prod_{m \text{ even}} \theta_m(\tau)^{k_g}$$

with

$$k_g := \begin{cases} 8 & \text{for } g = 1 \\ 2 & \text{for } g = 2 \\ 1 & \text{for } g \ge 3 \end{cases}$$

are non-zero modular forms of degree g and weight

$$r = \begin{cases} 12 & \text{for } g = 1 \\ 10 & \text{for } g = 2 \\ (2^g + 1)2^{g-2} & \text{for } g \ge 3 \end{cases}.$$

Proof.
See [Fre83, p. 42, Satz 3.3]. □

Theorem 1.2.36.
Let $g \geq 3$. Then the order of vanishing of $\chi^{(g)}$ on the boundary of $\mathscr{A}_g$ is 2^{2g-5}.
Proof.
This is part of [Mu3, Theorem 2.10]. □

1.2.4 Number theoretic functions

We recall and define some functions that will help us in counting later on.

Definition 1.2.37: Generalised phi function.
Let $n, k \in \mathbb{N}$. A set of integers $x_1, \dots, x_k$ is said to be *relatively prime to n* if $\gcd(x_1, \dots, x_k, n) = 1$. Define

$$\varphi_k(n) := \left|\{(x_1, \dots, x_k) \in \mathbb{Z}_n^k \middle| \gcd(x_1, \dots, x_k, n) = 1\}\right|.$$

For $k = 1$ this function is known as the *Euler phi function* which we also denote by φ.

Lemma 1.2.38.
The functions φ_k are multiplicative[3].
Proof.
This proof is a generalisation of [Nath, Theorem 2.7].

Fix $k \in \mathbb{N}$ and assume $\gcd(m, n) = 1$. There are $\varphi_k(mn)$ congruence classes in $(\mathbb{Z}/mn\mathbb{Z})^k$ that are relatively prime to mn. By [Nath, Theorem 2.6], every congruence class modulo mn can be written uniquely in the form

$$(ma_1 + nb_1 + mn\mathbb{Z}, \dots, ma_k + nb_k + mn\mathbb{Z}),$$

where a_i and b_i are integers such that $0 \leq a_i \leq n-1$ and $0 \leq b_i \leq m-1$. (This is essentially the Chinese Remainder Theorem).

Let $c := \gcd(x_1, \dots, x_k)$. Then $\gcd(m, n) = 1$ implies that

$$\begin{aligned}\gcd(c, m) = \gcd(x_1, \dots, x_k, m) &= \gcd(ma_1 + nb_1, \dots, ma_k + nb_k, m)\\ &= \gcd(nb_1, \dots, nb_k, m) = \gcd(b_1, \dots, b_k, m)\end{aligned}$$

and, analogously, $\gcd(c, n) = \gcd(a_1, \dots, a_k, n)$. It follows that $\gcd(c, mn) = 1$ if and only if $\gcd(c, m) = \gcd(c, n) = 1$ if and only if

$$\gcd(a_1, \dots, a_k, n) = \gcd(b_1, \dots, b_k, m) = 1.$$

Since the numbers of $a \in [0, \dots, n-1]^k$ and $b \in [0, \dots, m-1]^k$ satisfying these conditions are given by $\varphi_k(n)$ and $\varphi_k(m)$, respectively, we have $\varphi_k(mn) = \varphi_k(m)\varphi_k(n)$, and so the function φ_k is multiplicative. □

[3] A number theoretic function $f : \mathbb{N} \to \mathbb{N}$ is called *multiplicative* if for all coprime $n, m \in \mathbb{N}$ we have $f(nm) = f(n)f(m)$.

Lemma 1.2.39.
For any integer n the following equation holds:

$$\sum_{d|n} \varphi(d) = n.$$

Proof.
See [Nath, Theorem 2.8]. □

Lemma 1.2.40.
For any square-free integer n we have $\varphi(n^2) = n\varphi(n)$.

Proof.
With [Nath, Theorem 2.7] we know $\varphi(m) = m\prod_{p|m}(1-\frac{1}{p})$ for all $m \in \mathbb{N}$ and p prime. If n is square-free, i. e. a product of distinct primes, we obtain

$$\varphi(n^2) = n^2 \prod_{p|n^2}(1-\tfrac{1}{p}) = n\prod_{p|n}(p-1) = n\prod_{p|n}\varphi(p) = n\varphi\big(\prod_{p|n} p\big) = n\varphi(n).$$

□

Definition 1.2.41: Sigma functions.
For $n \in \mathbb{N}, \alpha \in \mathbb{C}$ let

$$\sigma_\alpha(n) := \sum_{d|n} d^\alpha.$$

For $\alpha = 0$ this function is known as the function τ that gives the number of divisors.

Lemma 1.2.42.
The functions σ_α are multiplicative.

Proof.
Assume $\gcd(m,n) = 1$. Then

$$\sigma_\alpha(nm) = \sum_{d|nm} d^\alpha = \sum_{d_1|n, d_2|m} (d_1 d_2)^\alpha = \sum_{d_1|n} d_1^\alpha \sum_{d_2|m} d_2^\alpha = \sigma_\alpha(n)\sigma_\alpha(m).$$

□

Chapter 2

Toroidal compactification

In this thesis, we will make extensive use of toric varieties and toroidal compactifications introduced by D. Mumford et al. in [AMRT]. For more detail on the background of these constructions, we refer the reader to the books by T. Oda [Oda] or K. Hulek, C. Kahn and S. Weintraub [HKW]. These methods have been used by several authors to compactify the moduli spaces of abelian varieties, both with and without polarisation. One of the interesting questions was to describe the boundary of such a compactification, with particular interest in the compactification corresponding to the second Voronoi decomposition, which turns out to represent a good functor ([AN], [Ale]). Those descriptions can be found in the publications by H.-J. Brasch [Bra94], M. Friedland [F] or M. Friedland and G. K. Sankaran [FS].

In this thesis, however, we do not need to go into as much detail. Therefore, we will only give the general construction of toroidal compactifications.

The overall idea of toroidal compactifications is that we use several toric varieties to ”add parts of a boundary” to a given open variety and then ”glue” these partial compactifications (which are compactifications in only one direction each) to obtain a global compactification of the variety. This glueing process requires that the partial compactifications are compatible in a sense that we will specify later. Let us begin by describing toric varieties, closely following the aforementioned books, namely [Oda, Chapter 1], [HKW, Section 3] and [Bra94, Kapitel 3, Paragraph 1].

2.1 Toric varieties

2.1.1 Summary of the construction

Toric varieties are special algebraic varieties that contain an algebraic torus $T = (\mathbb{C}^*)^r$ as an open and dense subset and admit an algebraic action of T extending the group multiplication of T on T to the whole of the toric variety.

The construction of toric varieties does not consist of very many steps. However, since in some way it parallels the significantly more complex construction of toroidal varieties nicely, we shall give a summary nonetheless.

(i) Decompose a vector space containing a lattice using rays through the lattice points (in other words, construct a fan).

For each of the open cones σ that make up the fan

(ii) construct an affine algebraic variety T_σ.

Combine these varieties to obtain the toric variety as follows:

(iii) Define the disjoint union of all T_σ.

(iv) Define an equivalence relation on it.

(v) Finally, define the toric variety to be the quotient of the union by the equivalence.

This method shall now be established in more detail.

2.1.2 Construction

Notation 2.1.1.
Let N be a free abelian group of rank $r \geq 1$ and $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ the vector space of dimension r with integer structure given by N. Denote the dual $\mathbb{Z}$-module by $M := \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$. We have the canonical $\mathbb{R}$-bilinear pairing $\langle , \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ which we may use to define an algebraic torus

$$T_N := \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \simeq N \otimes_\mathbb{Z} \mathbb{C}^* \simeq (\mathbb{C}^*)^r.$$

Remark 2.1.2.
This torus will be the aforementioned torus T contained in the toric variety.

Furthermore, we can regain N and M from T_N: Each $n \in N$ gives rise to a one-parameter subgroup $\gamma_n : \mathbb{C}^* \to T_N$, which is a homomorphism defined for $\lambda \in \mathbb{C}^*$ and $m \in M$ by $\gamma_n(\lambda)(m) := \lambda^{\langle m,n \rangle}$. Therefore, N is isomorphic to the groups of one-parameter subgroups of T_N which is $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}^*, T_N)$.

On the other hand, define for $m \in M$ the homomorphism $e(m) : T_N \to \mathbb{C}^*$ by $t \mapsto t(m)$. Because of the 'exponential' law $e(m+m') = e(m)e(m')$ and $e(0) = 1$ we can identify M with the character group of T_N, namely $\mathrm{Hom}_{\mathrm{alg}}(T_N, \mathbb{C}^*)$.

The construction of the toric variety depends on the choice of a fan in $N_\mathbb{R}$, by which we mean the following object:

Definition 2.1.3: Cones and fans.

(i) A *strongly convex rational polyhedral cone*[1] in $N_\mathbb{R}$ is a subset $\sigma \subset N_\mathbb{R}$ of the form

$$\sigma = \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_s \quad n_1, \ldots, n_s \in N$$

that does not contain a line.

(ii) The *dual cone* to σ is defined to be

$$\sigma^\vee := \left\{ x \in M_\mathbb{R} \middle| \forall y \in \sigma \ : \ \langle x, y \rangle \geq 0 \right\}.$$

[1] in short: scrp-cone

(iii) A *face* of an scrp-cone is a subset of σ which can be given using an element $m_0 \in \sigma^\vee$ in the way

$$\tau = \sigma \cap \{m_0\}^\perp := \{y \in \sigma \,|\, \langle m_0, y\rangle = 0\}.$$

This relation is denoted by $\tau \prec \sigma$.

(iv) A *fan* Σ is a non-empty collection of scrp-cones satisfying

(a) $\forall \sigma \in \Sigma\ \forall \tau \prec \sigma\ :\ \tau \in \Sigma$

(b) $\forall \sigma_1, \sigma_2 \in \Sigma\ :\ \sigma_1 \cap \sigma_2 \prec \sigma_i$ for $i = 1, 2$

Proposition 2.1.4.
Every scrp-cone contains $\{0\}$ as a face.

Proof.
Follows directly from strong convexity. □

Definition 2.1.5: Simplicial and regular cones.
An scrp-cone in $N_\mathbb{R}$ is called *simplicial* if the set of generators $\{n_i\}$ is linearly independent.

An scrp-cone in $N_\mathbb{R}$ is called *regular* (also *basic* or *non-singular*) if the set of generators $\{n_i\}$ can be enlarged to a $\mathbb{Z}$-basis of N.

Next we need affine varieties T_σ for each $\sigma \in \Sigma$. These will then be glued to form the toric variety. They are constructed as follows.

Notation 2.1.6.
Let $\mathscr{S}_\sigma := M \cap \sigma^\vee$.

Remark 2.1.7.
If we think of N as a lattice and M as its dual lattice, $\mathscr{S}_\sigma$ is the set of points of the dual lattice lying in the dual of the cone σ. In this light, the following proposition is rather obvious.

Proposition 2.1.8.

(i) $\mathscr{S}_\sigma$ is a finitely generated additive semigroup of M containing 0, i. e. there exist $m_1, \dots, m_n$ such that $\mathscr{S}_\sigma = \mathbb{Z}_{\geq 0} m_1 + \cdots + \mathbb{Z}_{\geq 0} m_n$.

(ii) $\mathscr{S}_\sigma$ is saturated, i. e. $\forall m \in M\ \forall c \in \mathbb{N}\ :\ cm \in \mathscr{S}_\sigma \implies m \in \mathscr{S}_\sigma$.

(iii) $\mathscr{S}_\sigma$ generates M as a group, i. e. $\mathscr{S}_\sigma + (-\mathscr{S}_\sigma) = M$.

Proof.
See [Oda, Proposition 1.1] □

Notation 2.1.9.
Let $\mathbb{C}[M] := \bigotimes_{m \in M} \mathbb{C} e(m)$ be the group algebra of M over $\mathbb{C}$ where we interpret the $e(m)$ only as symbols with the ring multiplication given by $e(m) \cdot e(m') := e(m + m')$.

Remark 2.1.10.
Since $\mathscr{S}_\sigma$ is an additive subsemigroup of M, its semigroup algebra $\mathbb{C}[\mathscr{S}_\sigma]$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}[M]$. From Proposition 2.1.8 we see that

$$\mathbb{C}[\mathscr{S}_\sigma] = \mathbb{C}[e(m_1), \ldots, e(m_n)].$$

Definition 2.1.11: Affine torus embedding.
The *affine torus embedding* T_σ is the algebraic variety over $\mathbb{C}$ defined by

$$T_\sigma := \left\{ \phi : \mathscr{S}_\sigma \to \mathbb{C} \middle| \phi \text{ is an algebra homomorphism} \right\}.$$

Lemma 2.1.12.
T_σ is indeed an r-dimensional affine variety and can be embedded into $\mathbb{C}^n$ by the map

$$\phi \mapsto \big(\phi(e(m_1)), \ldots, \phi(e(m_n))\big).$$

Proof.
See [Oda, Proposition 1.2]. □

Theorem 2.1.13.

(i) $T_{\{0\}} \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = T_N$

(ii) For each $\tau \prec \sigma$, the inclusion $\mathscr{S}_\sigma \subset \mathscr{S}_\tau$ gives rise to a map $T_\tau \to T_\sigma$ which is an embedding.

(iii) T_σ is non-singular if and only if σ is regular.

Proof.
See [Oda, Proposition 1.3, Theorem 1.4 and Theorem 1.10]. □

We are now in the position to define the main object of this section:

Definition 2.1.14: Toric variety.
The *toric variety* T_Σ associated to the fan Σ in $N_\mathbb{R}$ (also denoted by $T_N\mathrm{emb}(\Sigma)$) is defined to be

$$T_\Sigma := \bigsqcup_{\sigma \in \Sigma} T_\sigma \Big/ \text{glueing}$$

where the glueing is the equivalence relation generated by the embeddings

$$T_{\sigma_1 \cap \sigma_2} \hookrightarrow T_{\sigma_1}, T_{\sigma_2}.$$

2.1.3 Properties

Remark 2.1.15.
T_Σ is an irreducible normal Hausdorff space of dimension $r = \operatorname{rank} N$ containing the algebraic torus $T_N \simeq T_{\{0\}}$ as an open and dense subset.

The operation of T_N on T_Σ is given as follows: let $t \in T_N$ and $\phi \in T_\sigma$. This means we have homomorphisms $t : M \to \mathbb{C}^*$ and $\phi : \mathbb{C}[\mathscr{S}_\sigma] \to \mathbb{C}$. Then

$$t \cdot \phi : \begin{cases} \mathbb{C}[\mathscr{S}_\sigma] \to \mathbb{C} \\ \quad e(m) \mapsto t(m)\phi(e(m)) \end{cases} .$$

Every affine open subset T_σ of T_Σ is stable under this operation and on $T_{\{0\}} \simeq T_N$ this is exactly the usual group multiplication.

If a second fan Σ' is a subset of Σ, i. e. $\sigma' \in \Sigma' \implies \sigma' \in \Sigma$, then $T_{\Sigma'}$ can in a natural way be considered as an open subvariety of T_Σ.

Since we now have an action of T_N on T_Σ, it is natural to ask how we can decompose T_Σ into orbits of this action. This can be done as follows.

Definition 2.1.16: Orbit of a cone.
For a scrp-cone σ define the algebraic torus of group homomorphisms

$$\operatorname{orb}(\sigma) := \operatorname{Hom}_{\mathrm{grp}}(M \cap \sigma^\perp, \mathbb{C}^*).$$

Remark 2.1.17.
Every group homomorphism $\tilde{\varphi} \in \operatorname{orb}(\sigma)$ defines a $\mathbb{C}$-algebra homomorphism $\varphi \in T_\sigma$ by

$$\varphi(e(m)) := \begin{cases} \tilde{\varphi}(m) & \text{for } m \in M \cap \sigma^\perp \\ 0 & \text{for } m \in \mathscr{S}_\sigma \backslash \sigma^\perp \end{cases} .$$

Furthermore, the map $\tilde{\varphi} \mapsto \varphi$ is an embedding of $\operatorname{orb}(\sigma)$ into T_σ. Its image is obviously stable under the action of T_N, and it is even an orbit. All in all, one has the following:

Theorem 2.1.18: Orbit decomposition.
T_Σ has a stratification of T_N-orbits

$$T_\Sigma = \bigsqcup_{\sigma \in \Sigma} \operatorname{orb}(\sigma)$$

satisfying the properties

(i) $\dim(\operatorname{orb}(\sigma)) + \dim(\sigma) = r$

(ii) $\operatorname{orb}(\{0\}) = T_{\{0\}} = T_N$

(iii) $T_\sigma = \bigsqcup_{\tau \prec \sigma} \operatorname{orb}(\tau)$

(iv) $\operatorname{orb}(\sigma) \subseteq \overline{\operatorname{orb}(\tau)} \iff \sigma \succ \tau$

(v) For $\sigma \in \Sigma$, the torus $\operatorname{orb}(\sigma)$ is the unique closed T_N-orbit in T_σ.

Proof.
See [Oda, Proposition 1.6] □

Lemma 2.1.19: Canonical divisor on toric varieties.
If T_Σ is a nonsingular toric variety, and $D_1, \ldots, D_d$ are the irreducible T-divisors on X, then $-\sum D_i$ is a canonical divisor.

Proof.
See [Ful93, Section 4.3, page 85]. □

Another interesting topic is that of maps between toric varieties. Here T. Oda gives the following statement:

Definition 2.1.20: Map of fans.
A *map of fans* $\varphi : (N', \Sigma') \to (N, \Sigma)$ is a $\mathbb{Z}$-linear homomorphism $\varphi : N' \to N$ whose scalar extension $\varphi : N'_\mathbb{R} \to N_\mathbb{R}$ satisfies the property

$$\forall \sigma' \in \Sigma' \; \exists \sigma \in \Sigma \; : \; \varphi(\sigma') \subset \sigma.$$

Theorem 2.1.21: Equivariant Holomorphic Maps.
Let $\varphi : (N', \Sigma') \to (N, \Sigma)$ be a map of fans and let $T_{\Sigma'}$ and T_Σ be the corresponding toric varieties. Then there exists an equivariant[2] holomorphic map $\varphi_* : T_{\Sigma'} \to T_\Sigma$, whose restriction to the open subset $T_{N'}$ coincides with the homomorphism of algebraic tori arising from φ.

This map satisfies the following properties:

(i) If N' is a $\mathbb{Z}$-submodule of N of finite index and $\Sigma' = \Sigma$, then φ_* coincides with the projection to the quotient of $T_{\Sigma'}$ with respect to the natural action of the finite group $\ker[T'_\Sigma \to T_\Sigma] \simeq N/N'$.

(ii) The map φ_* is proper and birational if and only if $\varphi : N' \to N$ is an isomorphism and Σ' is a locally finite subdivision[3] of Σ under the identification $N'_\mathbb{R} = N_\mathbb{R}$.

Proof.
The existence of φ_* as extension of φ is given in [Oda, Theorem 1.13]. The two properties given are stated in [Oda, Corollary 1.16] and [Oda, Corollary 1.17], respectively. □

Corollary 2.1.22: Resolution of Singularities.
Any toric variety admits an equivariant resolution of singularities.

Proof.
This statement is given in [Oda] following Corollary 1.18. □

[2] with respect to the actions of $T_{N'}$ and T_N
[3] i. e. for each $\sigma \in \Sigma$ the set $S := \{\sigma' \in \Sigma' | \sigma' \subset \sigma\}$ is fi nite and $\sigma = \bigcup_S \sigma'$.

2.2 Toroidal compactification

In this section we want to sketch the construction of toroidal compactifications which David Mumford described in [AMRT]. We will not do this in all generality and in particular we will not go into detail on the (mainly Lie theoretic) background for this method. The notation is chosen to follow that in [HKW, Part I][4].

2.2.1 Summary of the construction

Mumford constructed toroidal compactifications starting from bounded symmetric domains[5] $D \subset \mathbb{C}^N$ and the action of an automorphism group Γ on it. Although the Siegel space $\mathfrak{S}_g$ is a symmetric bounded domain the discussion of the boundary 'at infinity' becomes clearer if we precede the construction with a preliminary step:

(i) Use a map Φ to embed $\mathfrak{S}_g$ into $\mathbb{C}^N$ such that $\mathfrak{D}_g := \Phi(\mathfrak{S}_g)$ is bounded. Derive the action of Γ on $\mathfrak{D}_g$ from that on $\mathfrak{S}_g$.

In the case we are concerned with we may assume Γ to be an arithmetic subgroup of $\mathrm{Sp}(\Lambda,\mathbb{R})$. (In fact, it does not make much difference whether we consider $\mathrm{Sp}(J,\mathbb{R})$ or $\mathrm{Sp}(\Lambda,\mathbb{R})$. For a comparison of the two cases see [HKW, p. 56 ff.]. At this point it should be mentioned that in [HKW] and also the later chapters of this thesis we use a tilde on subgroups of $\mathrm{Sp}(\Lambda,\mathbb{R})$ to distinguish them from subgroups of $\mathrm{Sp}(J,\mathbb{R})$. The same tilde is used on objects defined with respect to these groups. But in this chapter this would mean we have a tilde on each F, U, P and so on, which does not help much. Therefore, we chose to leave out the tilde for the section of the toroidal compactification.) Now, start with Mumford's construction:

(ii) Consider the (topological) closure $\overline{\mathfrak{D}}_g$ in $\mathbb{C}^N$. Extend the action of Γ onto $\overline{\mathfrak{D}}_g$. Decompose the boundary $\overline{\mathfrak{D}}_g \backslash \mathfrak{D}_g$ into boundary components (we still have to specify what we mean by this). Define an adjacency relation.

(iii) Select only the rational boundary components[6] F (the others are irrelevant for the construction).

For each of these rational boundary components F we define a partial compactification of $\mathfrak{D}_g/\Gamma$ "in the direction of F" called $Y_\Sigma(F)$ by using a trivial fibre bundle $\mathscr{X}_\Sigma(F)$ whose fibres are toric varieties as follows:

(iv) Associate to F a parabolic subgroup of $\mathrm{Sp}(\Lambda,\mathbb{R})$, namely the stabiliser $\mathscr{P}(F)$.

(v) Intersect $\mathscr{P}(F)$ with Γ to obtain a group $P(F)$ acting on $\mathfrak{D}_g$.

(vi) Split $P(F)$ into $P'(F)$ and $P''(F)$ such that the action of $P'(F)$ on $\mathfrak{D}_g$ is 'nice' and hence we have a nice quotient $X(F) := \mathfrak{D}_g/P'(F)$ with an action of the remaining part $P''(F)$ on it.

(vii) Construct a trivial torus bundle $\mathscr{X}(F)$ with fibres $T \simeq (\mathbb{C}^*)^r$ and regard $X(F)$ as a subset of it.

[4] We will only cite from Part I of this book.

[5] For a definition of *symmetric bounded domain* see [Na, p. 113]

[6] we shall shorten this to 'rbc' where needed

(viii) Compactify the standard fibre T by a toric variety T_Σ. (The fan $\Sigma := \Sigma(F)$ may not be arbitrary but has to satisfy certain properties.)

(ix) Construct a trivial fibre bundle $\mathscr{X}_\Sigma(F)$ with fibres T_Σ that contains $\mathscr{X}(F)$. Hence we have $X(F) \subset \mathscr{X}(F) \subset \mathscr{X}_\Sigma(F)$.

(x) Denote by $X_\Sigma(F)$ the interior of the closure (in the $\mathbb{C}$-topology) of $X(F)$ in $\mathscr{X}_\Sigma(F)$.

(xi) For 'good' fans Σ (which we will call 'admissible') we have an action of $P''(F)$ on $\mathscr{X}_\Sigma(F)$ and hence we can define the partial compactification of $\mathfrak{D}_g/P(F)$ in the direction of F to be $Y_\Sigma(F) := X_\Sigma(F)/P''(F)$.

(xii) Interpret $Y_\Sigma(F)$ as a partial compactification of $\mathfrak{D}_g/\Gamma$.

Now that we have partial compactifications in the directions of all rational boundary components we have to glue all these together to obtain the compactification of $\mathfrak{D}_g/\Gamma$. Note that we have a fan for each boundary component, in other words a collection $\tilde{\Sigma} := \{\Sigma(F)\}$. This collection also has to satisfy some conditions in order to allow the following glueing process:

(xiii) Define the disjoint union $Y := Y(\tilde{\Sigma}) := \bigsqcup_{\text{all rbc } F} Y_\Sigma(F)$.

(xiv) Define an equivalence relation $\sim$ on $Y(\tilde{\Sigma})$ arising from the action of Γ and the adjacency of boundary components. This equivalence is the formal way of saying what is meant by "glueing".

(xv) Finally, define the toroidal compactification $(\mathfrak{D}_g/\Gamma)^*$ of $\mathfrak{D}_g/\Gamma$ to be the quotient space $(\mathfrak{D}_g/\Gamma)^* = Y(\tilde{\Sigma})/\sim$.

Recall that $Y_\Sigma(F)$ is constructed from $X_\Sigma(F)$ using the action of a subgroup of Γ. Now, the aforementioned equivalence relation also arises from the action of Γ. Therefore, we could also use the following construction to arrive at the same result:

(xiii) Define the disjoint union $X := X(\tilde{\Sigma}) := \bigsqcup_{\text{all rbc } F} X_\Sigma(F)$.

(xiv) Define an equivalence relation $\sim$ on $X(\tilde{\Sigma})$ arising from the action of Γ and the adjacency of boundary components.

(xv) Finally, define the toroidal compactification $(\mathfrak{D}_g/\Gamma)^*$ of $\mathfrak{D}_g/\Gamma$ to be the quotient space $(\mathfrak{D}_g/\Gamma)^* = X(\tilde{\Sigma})/\sim$.

2.2.2 The preliminary step and boundary components

Having completed this summary we will now go through the whole process in some more detail. In particular, we have to give precise definitions of all the objects mentioned and list the conditions on the fans. For most propositions we refer to [HKW] for proofs although there most proofs are only given for the case $g = 2$. The general case may, however, be obtained by simply replacing the appropriate indices.

In step (i) we need a map Φ. This can be chosen to be

$$\Phi : \begin{cases} \mathfrak{S}_g \to \mathrm{Sym}(g, \mathbb{C}) \\ \tau \mapsto (\tau - i\mathbb{1})(\tau + i\mathbb{1})^{-1} \end{cases}$$

which maps $\mathfrak{S}_g$ isomorphically onto the $g(g+1)/2$-dimensional bounded domain

$$\mathfrak{D}_g = \left\{ Z \in \mathrm{Sym}(g,\mathbb{C}) \middle| \mathbb{1} - Z\overline{Z} > 0 \right\}.$$

This is called the Cayley transformation, or, in the case of $g = 2$ the Harish-Chandra embedding. It is a generalization of the map from the ordinary upper half plane onto the unit disc.

For this $\mathfrak{D}_g$ the (canonical) closure $\overline{\mathfrak{D}}_g \subset \mathrm{Sym}(g,\mathbb{C})$ is the compact set

$$\overline{\mathfrak{D}}_g = \left\{ Z \in \mathrm{Sym}(g,\mathbb{C}) \middle| \mathbb{1} - Z\overline{Z} \geq 0 \right\}$$

and the action of Γ on $\overline{\mathfrak{D}}_g$ can be defined as in [HKW, Proposition 3.3]. Since we are never actually concerned with the details of this action and its precise statement requires some more definitions, we refer the interested reader to this book.

Definition 2.2.1: Boundary components.

(i) A *boundary component* of $\mathfrak{D}_g$ is an equivalence class of points in $\overline{\mathfrak{D}}_g$ where two points $p,q \in \overline{\mathfrak{D}}_g$ are equivalent if and only if they can be connected by finitely many holomorphic curves. (See [HKW, Definition 3.5]) Denote the set of boundary components by $\mathscr{F}$.

(ii) A boundary component lying in $\overline{\mathfrak{D}}_g \backslash \mathfrak{D}_g$ is called *proper*.

(iii) A boundary component F is said to be *congruent* to a boundary component F' under the action of Γ if there exists an $a \in \Gamma$ such that $F = a(F')$. This fact shall be denoted by $F \sim_\Gamma F'$.

(iv) A boundary component F is said to be *adjacent* to a boundary component F' if $F \neq F'$ and $F \subset \overline{F'}$. This fact shall be denoted by $F \prec F'$.

Note that the points (iii) and (iv) introduce two relations between boundary components. Whenever we map the boundary components onto some other objects we want these relations to be represented in the new set.

Definition 2.2.2: Representing sets.
A partially ordered set $(\mathscr{E}, \prec')$ with an action of Γ is said to *represent the set of boundary components* if we have a surjective map $\varphi : \mathscr{F} \to \mathscr{E}$ such that

- $F \sim_\Gamma F' \quad \Longrightarrow \quad \varphi(F) \sim_\Gamma \varphi(F')$ and
- $F \prec F' \quad \Longleftrightarrow \quad \varphi(F) \prec' \varphi(F')$.

It is called *strongly representing* if φ is a bijection and we have equivalence in both conditions.

Remark 2.2.3.
Before we continue with the next step of the summary, let us introduce the most important example of a representing set. Recall that we are dealing with a group $\Gamma \subset \mathrm{Sp}(\Lambda,\mathbb{R})$. First, we can associate to each point $Z \in \overline{\mathfrak{D}}_g$ a Λ-isotropic subspace[7] $U(Z) \subset \mathbb{R}^{2g}$ such that

[7] A subspace $U \subset \mathbb{R}^{2g}$ is called *(totally) isotropic* if for all $u,v \in U$: $\langle u,v \rangle = u\Lambda\,{}^t v = 0$.

- the action of $a \in \Gamma$ on Z is equivariant to that of a on $U(Z)$ by matrix multiplication in the sense that $U(a(Z)) = U(Z)a^{-1}$ and

- the subspace is an invariant of the boundary components, i. e. for two points Z_1, Z_2 lying on the same boundary component we have $U(Z_1) = U(Z_2)$.

The precise definition of this map may be found in [HKW, Proposition 3.6]. But even from the two properties stated here we see that we have a well defined map

$$U : \mathscr{F} \to \{\text{isotropic subspaces of } \mathbb{R}^{2g}\}$$

by $U(F) := U(Z)$ for a $Z \in F$. This leads to the important statement

Lemma 2.2.4.
The set $(\{U(F)\}, \supsetneq)$ with Γ acting by inverse matrix multiplication as above strongly represents the boundary components.

Proof.
The map $\varphi : \mathscr{F} \to \{U(F)\}$ is a bijection according to [HKW, Proposition 3.12]. Equivariance of the action of Γ follows from the equivariance on the $U(Z)$ mentioned above. The equivalence of the two order relations is given in [HKW, Proposition 3.16]. □

2.2.3 Partial compactifications

Let us now proceed in the construction of the toroidal compactification. Step (iii) asks for the rational boundary components. To define these, we need the stabilizing subgroup mentioned in step (iv).

Definition 2.2.5: Stabiliser groups, rational boundary components.
To each boundary component we associate the *stabiliser group*

$$\mathscr{P}(F) := \{a \in \mathrm{Sp}(\Lambda, \mathbb{R}) \big| a(F) = F\}.$$

A boundary component F is called *rational* if $\mathscr{P}(F)$ is defined over $\mathbb{Q}$ (i. e. there is a subgroup $\mathscr{P}_{\mathbb{Q}} \subset \mathrm{Sp}(2g, \mathbb{Q})$ such that $\mathscr{P}(F) = \mathscr{P}_{\mathbb{Q}}(\mathbb{R})$, the $\mathbb{R}$-valued points of the algebraic group $\mathscr{P}_{\mathbb{Q}}$. In other words, take the equations defining $\mathscr{P}_{\mathbb{Q}}$ and allow not only rational but also real solutions.).

Denote the set of all rational boundary components by $\mathscr{F}^{\mathrm{rat}}$.

Lemma 2.2.6: Characterization of rational boundary components.
For a boundary component $F \subset \overline{\mathfrak{D}}_g$ the following statements are equivalent:

- F is a rational boundary component.

- $U(F) \subset \mathbb{R}^{2g}$ is a rational subspace, i. e. it can be generated by rational (or equivalently integral) vectors.

- There exists $a \in \mathrm{Sp}(\Lambda, \mathbb{Q})$ such that $F = a(F^{(i)})$ where

$$F^{(i)} := \left\{ \begin{pmatrix} Z & \\ & \mathbb{1}_{g-i} \end{pmatrix} \middle| Z \in \mathfrak{D}_i \right\}$$

  with $\mathfrak{D}_i$ as before and $\mathfrak{D}_0 = \emptyset$. The dimension of $U(F)$ is $g-i$.

Proof.
See [HKW, Proposition 3.19] □

This lemma not only gives a characterization of rational boundary components, but it also names standard representators $F^{(i)}$ with respect to the action of the (whole) group $\mathrm{Sp}(\Lambda, \mathbb{Q})$.

Definition 2.2.7: Corank-c boundary components.
A rational boundary component F with $\dim_{\mathbb{R}}(U(F)) = c$ is called *corank-c boundary component*.

Lemma 2.2.8.
Let e_i denote the ith unit vector. Then

$$U(F^{(i)}) = \mathbb{R}e_{g+i+1} + \cdots + \mathbb{R}e_{2g} \quad \text{and} \quad U(F^{(g)}) = \{0\}.$$

A rational boundary component F is a corank-c boundary component if and only if there exists $a \in \mathrm{Sp}(\Lambda, \mathbb{Q})$ with $F = a(F^{(g-c)})$

Proof.
This follows from the definition of $U(Z)$ which can be found in [HKW, Proposition 3.6]. The correspondence is proved in [HKW, Lemma 3.10, Remark 3.22]. □

So far we have only considered the action of $\mathrm{Sp}(\Lambda, \mathbb{R})$ and its subgroups $\mathscr{P}(F)$. Now we reintroduce the group $\Gamma \subset \mathrm{Sp}(\Lambda, \mathbb{R})$ and split the resulting group into two parts as mentioned in step (vi):

Definition 2.2.9: Subgroups of Γ and $\mathscr{P}(F)$.

- Let $P(F) := \mathscr{P}(F) \cap \Gamma$.
- Let $\mathscr{P}'(F)$ be the center of the unipotent radical[8] $R_{\mathrm{u}}(\mathscr{P}(F))$ of $\mathscr{P}(F)$ and let $\mathscr{P}''(F) := \mathscr{P}(F)/\mathscr{P}'(F)$.
- Let $P'(F) := \mathscr{P}'(F) \cap \Gamma$ and $P''(F) := P(F)/P'(F)$.
- Denote the quotient $X(F) := \mathfrak{D}_g/P'(F)$ and define the *partial quotient map* $e(F) : \mathfrak{D}_g \to X(F)$.

[8] The *unipotent radical* $R_{\mathrm{u}}(G)$ of a Lie group G is by definition the maximal connected unipotent normal subgroup of G.

Remark 2.2.10.
This decomposition of $P(F)$ is based on the following idea: $\mathscr{P}'(F)$ is isomorphic to a real vector space containing the lattice $P'(F)$. One can therefore hope that the partial quotient map $e(F)$ can be given in a nice form. Indeed, we have $X(F) \simeq (\mathbb{C}^*)^r \times U$ where r is the rank of $P'(F)$ and U is open in $(\mathbb{C}^*)^{g(g+1)/2-r}$. (See [HKW, page 68f.].)

Step (vii) is done in the following theorem:

Theorem 2.2.11.
For each rational boundary component F of $\mathfrak{D}_g$ there exists a trivial torus bundle $\mathscr{X}(F)$ with fibre $T = (P'(F) \otimes_{\mathbb{Z}} \mathbb{C})/P'(F) \simeq (\mathbb{C}^*)^r$ over the base $F \times V(F)$ where $V(F)$ is the complex vector space $R_{\mathrm{u}}(\mathscr{P}(F))/\mathscr{P}'(F)$. The set $X(F)$ is naturally isomorphic to an open subset of $\mathscr{X}(F)$. The operation of $P''(F)$ on $X(F)$ extends to $\mathscr{X}(F)$.

Proof.
See [AMRT, Chapter III, Paragraph 4]. □

In step (viii) we use a toric variety to compactify the torus fibre T of $\mathscr{X}(F)$. For this construction we need a lattice $N = \mathrm{Hom}(\mathbb{C}^*, T)$ and a fan Σ in $N_{\mathbb{R}}$. Because of the definition of T we actually have $N = P'(F)$ and hence $N_{\mathbb{R}} = \mathscr{P}'(F)$. We still need to make precise what conditions the fan has to satisfy.

The reasons for the following definitions, which may seem a bit ad-hoc, can be found in [HKW, Paragraph 3D]. For the scope of this thesis we will restrict ourselves to simply stating what turns out to be a good choice.

For $F^{(i)}$ we have

$$\mathscr{P}'(F^{(i)}) = \left\{ \begin{pmatrix} \mathbb{1} & S \\ 0 & \mathbb{1} \end{pmatrix} \middle| S = \begin{pmatrix} 0_i & \\ & S' \end{pmatrix} \text{ and } S' \in \mathrm{Sym}(g-i, \mathbb{R}) \right\}.$$

Here Γ acts by adjungation: if for $a \in \Gamma$ we have $F' = a(F)$ then $\mathscr{P}'(F') = a\mathscr{P}'(F)a^{-1}$. Define the open (self adjoint) cone $C(F) \subset \mathscr{P}'(F)$ to be the subset satisfying the condition that S' be positive definite. Hence we have natural identifications

$$\mathscr{P}'(F^{(i)}) \simeq \mathrm{Sym}(g-i, \mathbb{R}) \qquad \text{and} \qquad C(F^{(i)}) \simeq \mathrm{Sym}_+(g-i, \mathbb{R}).$$

On these spaces the adjoint action of $\mathscr{P}(F^{(i)})$ is equivariant to the action of $\mathrm{GL}(g-i, \mathbb{R})$ defined by $Q \in \mathrm{GL}(g-i, \mathbb{R}) : S' \mapsto {}^tQ^{-1}SQ^{-1}$.

These choices allow the following lemma:

Lemma 2.2.12.
The set of cones $(\{C(F)\}, \supsetneqq)$ represents the rational boundary components. More precisely, for two boundary components F, F' we have:

- if $F = a(F')$ for an $a \in \Gamma$ then $\mathscr{P}(F) = a\mathscr{P}(F')a^{-1}$. This correspondence also applies for the other groups $\mathscr{P}', \mathscr{P}'', P', P''$ and for the cones $C(F) = aC(F')a^{-1}$.

- $F \prec F'$ if and only if $\mathscr{P}'(F') \subset \mathscr{P}'(F)$ and $C(F') = \left(\overline{C(F)} \cap \mathscr{P}'(F')\right)^0$ where the closure is taken in $\mathscr{P}'(F)$ with respect to the $\mathbb{C}$-topology but the interior is taken with respect to $\mathscr{P}'(F')$.

Proof.
See [AMRT, Theorem III.4.3] or [HKW, Proposition 3.60]. □

We use this to define the rational closure of the cones $C(F)$:

Definition 2.2.13: Rational closure.
The *rational closure* $C(F)^{rc}$ *of* $C(F)$ is defined to be the union of all adjoint cones:

$$C(F)^{\mathrm{rc}} := C(F) \bigcup_{\substack{F' \in \mathscr{F}^{\mathrm{rat}}, \\ F' \prec F}} C(F').$$

Definition 2.2.14: Automorphism group $\bar{P}(F)$.
Define the group $\bar{P}(F) \subset \mathrm{Aut}(P'(F))$ to consist of all automorphisms $\mathrm{Ad}(a)$ of $P'(F)$ defined by $b \mapsto aba^{-1}$.

Now we are in a position to state the conditions on a fan to be admissible for our toric construction:

Definition 2.2.15: Admissible fan.
A fan $\Sigma \subset \mathscr{P}'(F)$ is called *admissible* if it satisfies the following three conditions:

(i) It covers $C(F)^{\mathrm{rc}}$, i. e. $\bigcup_{\sigma \in \Sigma} \sigma = C(F)^{\mathrm{rc}}$.

(ii) It is stable under the action of $\bar{P}(F)$, i. e. for all $a \in \bar{P}(F)$ and all $\sigma \in \Sigma$ we have $a(\sigma) \in \Sigma$.

(iii) There are only finitely many orbits, i. e. $\Sigma/\bar{P}(F)$ is a finite set.

Next, we can complete step (viii) by using an admissible fan Σ to construct the toric variety T_Σ. This is done exactly as in the previous section, so we need not give further detail here. Steps (ix) and (x) consist of the following definitions:

Definition 2.2.16: Associated fibre bundle.
For a rational boundary component F and an admissible fan $\Sigma = \Sigma(F)$ let

- $\mathscr{X}_\Sigma(F) := \mathscr{X}(F) \times_T T_\Sigma$ be the associated toric fibre bundle and
- $X_\Sigma(F) := \left(\overline{X(F)}\right)^0$ be the interior of the closure of $X(F)$ in $\mathscr{X}_\Sigma(F)$.

This setting enables us to complete step (xi) as planned:

Lemma 2.2.17.
The induced action of $P''(F)$ on $X(F)$ extends in a unique way to a properly discontinuous action of $P''(F)$ on $X_\Sigma(F)$.

The quotient space $Y_\Sigma(F) := X_\Sigma(F)/P''(F)$ is an analytic variety that contains $\mathfrak{D}_g/P(F)$ as an open and dense analytic subvariety whose complement (called the *boundary*) $\partial Y_\Sigma(F) := Y_\Sigma(F) \backslash (\mathfrak{D}_g/P(F))$ is a purely 1-codimensional analytic subvariety.

If $X_\Sigma(F)$ is smooth, then $Y_\Sigma(F)$ contains at worst finite quotient singularities.

Proof.
See [HKW, Proposition 3.62] □

Remark 2.2.18.
So far, $Y_\Sigma(F)$ can be interpreted as a partial compactification of $\mathfrak{D}_g/P(F)$. However, since the natural quotient map $p(F) : \mathfrak{D}_g/P(F) \to \mathfrak{D}_g/\Gamma$ is an isomorphism (if restricted to a sufficiently small interior neighbourhood of F in $\mathfrak{D}_g$), we can use it to attach $\partial Y_\Sigma(F)$ to the space $\mathfrak{D}_g/\Gamma$ and thus get a partial compactification of $\mathfrak{D}_g/\Gamma$ in the direction of F.

This is step (xii) and completes the procedure that has to be done for each rational boundary component. We may now easily define the objects of both alternatives of step (xiii).

2.2.4 Global compactification

Definition 2.2.19.
Denote the collection of fans by $\tilde{\Sigma} := \{\Sigma(F)\}$ and the disjoint union of partial compactifications by

$$Y(\tilde{\Sigma}) := \bigsqcup_{F \in \mathscr{F}^{\mathrm{rat}}} Y_\Sigma(F) \quad \text{and} \quad X(\tilde{\Sigma}) := \bigsqcup_{F \in \mathscr{F}^{\mathrm{rat}}} X_\Sigma(F).$$

However, to be able to glue these partial compactifications and obtain a global compactification, we need to look at the collection of fans in more detail.

Definition 2.2.20: Admissible collection of fans.
A collection of fans $\tilde{\Sigma} := \{\Sigma(F)\}$ is called *admissible* if it satisfies the following three conditions:

(i) Each $\Sigma(F)$ is an admissible fan for the boundary component F.

(ii) $\tilde{\Sigma}$ is compatible with the action of Γ on the boundary components, i. e. if for $a \in \Gamma$ we have $F = a(F')$, then $\Sigma(F) = a(\Sigma(F')) = a\Sigma(F')a^{-1}$.

(iii) $\tilde{\Sigma}$ is compatible with the adjacency relations between the boundary components, i. e. $F \prec F' \iff \Sigma(F') = \Sigma(F) \cap \mathscr{P}'(F')$.

Remark 2.2.21.
Conditions (ii) and (iii) imply that the set $(\tilde{\Sigma}, \supsetneq)$ represents the rational boundary components.

Note that condition (iii) implies that an admissible collection $\tilde{\Sigma}$ is already determined by the fans associated with minimal rational boundary components[9].

We now need the aforementioned equivalence relation on $Y(\Sigma)$. According to step (xiv) this will be built from two parts: the action of Γ and the adjacency relations

[9]A rational boundary component F is called minimal, if there exists no rational boundary component F' with $F' \prec F$.

of boundary components. To define the equivalence properly we need to look at maps between the partial compactifications for two different boundary components.

Again, let us first consider the action of Γ.

Proposition 2.2.22.
Let F, F' be two rational boundary components with $F = a(F')$ for an $a \in \Gamma$. Then a induces natural isomorphisms $\tilde{a}$, $\bar{a}$ such that the following diagrams commute:

$$\begin{array}{ccc} X_{\Sigma(F')}(F') & \xrightarrow{\tilde{a}} & X_{\Sigma(F)}(F) \\ \cup\uparrow & & \cup\uparrow \\ \mathfrak{D}_g/P'(F') & \xrightarrow{a} & \mathfrak{D}_g/P'(F) \end{array} \qquad \begin{array}{ccc} Y_{\Sigma(F')}(F') & \xrightarrow{\bar{a}} & Y_{\Sigma(F)}(F) \\ \cup\uparrow & & \cup\uparrow \\ \mathfrak{D}_g/P(F') & \xrightarrow{a} & \mathfrak{D}_g/P(F) \end{array}$$

The vertical arrows represent the natural inclusions.

Proof.
See [HKW, Proposition 3.69] □

We now consider maps between adjacent boundary components. Since here the construction is a bit easier for $X(\tilde{\Sigma})$ we will consider this case.

Recall that for two adjacent boundary components $F \prec F'$ we have the inclusion $\mathscr{P}'(F') \subset \mathscr{P}'(F)$ and hence a natural quotient map $\pi_{(F',F)} : X(F') \to X(F)$. Because of condition (iii) in Definition 2.2.20, this map extends to an étale map[10]

$$\overline{\pi}_{(F',F)} : X_{\Sigma(F')}(F') \to X_{\Sigma(F)}(F).$$

We can use this map for the equivalence relation.

Definition 2.2.23: Equivalence relation.
Let the equivalence relation $\sim$ on $X(\tilde{\Sigma})$ be generated by the following relations: two points $x \in X_{\Sigma(F)}(F)$ and $x' \in X_{\Sigma(F')}(F')$ are to be equivalent

(i) if there exists $a \in \Gamma$ such that $F = a(F')$ and $x = \tilde{a}(x')$ or

(ii) if $F \prec F'$ and $\overline{\pi}_{(F',F)}(x') = x$.

Remark 2.2.24.
As to the $Y_\Sigma(F)$, the compositions $X_\Sigma(F) \hookrightarrow X(\tilde{\Sigma}) \to X(\tilde{\Sigma})/\sim$ give rise to natural maps

$$\bar{p}(F) : \bar{Y}(F) \to X(\tilde{\Sigma})/\sim$$

which extend the projections $p(F)$ mentioned in Remark 2.2.18.

We have now reached the end and may define

Definition 2.2.25: Toroidal compactification.
For an admissible collection of fans $\tilde{\Sigma}$ define the *toroidal compactification* $(\mathfrak{D}_g/\Gamma)^*$ *of* $\mathfrak{D}_g/\Gamma$ *determined by* $\tilde{\Sigma}$ to be the quotient space

$$(\mathfrak{D}_g/\Gamma)^* := X(\tilde{\Sigma})/\sim.$$

[10] An *étale* map is a smooth map with discrete fi bres; it need not be surjective.

2.2.5 Properties

The space constructed in this way has the following properties:

Theorem 2.2.26.
Let $\mathscr{A}^* := (\mathfrak{D}_g/\Gamma)^*$ be the toroidal compactification of $\mathscr{A} := \mathfrak{D}_g/\Gamma$. Then we have:

(i) $\mathscr{A}^*$ is compact.

(ii) $\mathscr{A}^*$ contains $\mathscr{A}$ as an open and dense subset.

(iii) The boundary $\mathscr{A}^* \setminus \mathscr{A}$ is purely 1-codimensional, i. e. it is a Weil divisor.

(iv) For each rational boundary component F the map $\overline{p}(F)$ is an isomorphism when restricted to a sufficiently small neighbourhood of the boundary of $Y_\Sigma(F)$.

(v) $\mathscr{A}$ is the union of the images of the maps $\overline{p}(F)$ for every F.

(vi) $\mathscr{A}$ is the union of the images of the maps $\overline{p}(F)$ for every minimal F.

Proof.
See [HKW, Theorem 3.82, Remark 3.77] □

Remark 2.2.27.
Having completed the construction, we see that at the core of this process we have the collection of fans $\tilde{\Sigma}$ that determines the compactification. More precisely, we have the fans $\Sigma(F) \in \tilde{\Sigma}$ for minimal rational boundary components F determining everything. These fans are decompositions of the cone $\mathrm{Sym}_+(g,\mathbb{R})$, which explains why this subject is central in the context of toroidal compactification.

The compactification may, of course, be singular. Singularities may arise from two sources:

- On the one hand, the fans used to construct the toroidal compactification may be non-basic, which leads to singularities on the boundary of the corresponding $X_\Sigma(F)$. We say these singularities come 'from the toroidal construction' or 'from the fan'.

- On the other hand, the group Γ (more precisely, $P''(F) \subset \Gamma$) acts on $X_\Sigma(F)$ and may in this way introduce additional singularities in $Y_\Sigma(F)$. These singularities are said to be coming 'from the group action'.

Definition 2.2.28: Stack-smoothness.
If a variety has no singularities coming from the toroidal construction, we call it *stack-smooth*.

Definition 2.2.29: Neat groups.
A subgroup $\Gamma \subset \mathrm{GL}(n,\mathbb{C})$ is called *neat* if the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of all $a \in \Gamma$ is torsion free.

The importance of neat groups in this context lies in the following statement.

Lemma 2.2.30.
Let Γ be an arithmetic subgroup of the automorphism group of a bounded domain $\mathfrak{D}$. Then

- Γ contains a neat subgroup Γ' of finite index.
- If Γ itself is neat, then Γ operates on $\mathfrak{D}$ without fixed points.

Proof.
See [Na, Theorem 7.18]. □

Applying this to toroidal compactification leads to:

Theorem 2.2.31: Existence of Smooth Compactifications.

- For any Γ-admissible collection of fans $\tilde{\Sigma} = \{\Sigma(F)\}$ there exists a Γ-admissible refinement $\tilde{\Sigma}' = \{\Sigma'(F)\}$ such that all cones are regular. The dominating map is a blowing-up and the compactification with respect to Σ' is stack-smooth.
- Hence if Γ is neat, any toroidal compactification $(\mathfrak{D}/\Gamma)^*$ is dominated by a smooth toroidal compactification.
- In general, if Γ is neat there exists a smooth and projective toroidal compactification of $\mathfrak{D}/\Gamma$.

Proof.
See [Na, Theorem 7.20] and [Na, Theorem 7.26]. □

Remark 2.2.32.
This obviously parallels the statement of Corollary 2.1.22. Note that for $n \geq 3$ the group $\Gamma_{\text{pol}}(n)$ is neat[11].

Recall the following definition.

Definition 2.2.33: Order of branching.
Let M, N be two normal algebraic varieties and let $p : M \to N$ be a dominant rational map. Assume we have two divisors $D \subset M$ and $D' \subset N$ with $p^{-1}(D') = D$ as sets. As divisors, we have $p^*(D') = kD$ for some integer k. Then define the *order of branching of p along D* to be k.

Theorem 2.2.34: Maps of toroidal varieties.
Assume we are given two arithmetic subgroups $\Gamma_1 \subset \Gamma_2$ of $\mathrm{Sp}(2g, \mathbb{Z})$ and a collection of fans $\tilde{\Sigma}$ that is admissible for both groups. Let $\mathscr{A}_i^* := (\mathfrak{S}_g/\Gamma_i)^*$. Then we have a map $\pi : \mathscr{A}_1^* \to \mathscr{A}_2^*$. Furthermore, for a corank-1 boundary component F the order of branching of π on F is given by the index $[P'_{\Gamma_2}(F) : P'_{\Gamma_1}(F)]$.

Proof.
The existence of π follows easily from Theorem 2.1.21 since for every boundary component F we have

$$N' = P'_{\Gamma_1}(F) = P'_{\mathrm{Sp}(2g,\mathbb{Z})}(F) \cap \Gamma_1 \subset P'_{\mathrm{Sp}(2g,\mathbb{Z})}(F) \cap \Gamma_2 = P'_{\Gamma_2}(F) = N$$

[11] This follows from a general result by Serre which says that every algebraic integer which is a unit and which is congruent to 1 mod n for $n \geq 3$ is equal to 1.

where $[N:N'] < \infty$ due to the choice of Γ_i. We can glue the maps $\varphi_{F,*}$ since $\tilde{\Sigma}$ is admissible.

Since F has corank 1, the groups $P'_{\Gamma_j}(F)$ for $j=1,2$ are 1-dimensional lattices. To ease the notation, we only consider the case $F=F_0$, but the construction goes through the same for all other rational corank 1 boundary components. For F_0, the quotient maps $e_j(F_0)$ are given by

$$e_j(F_0) : \begin{cases} \mathfrak{S}_g & \to \quad X_j(F_0) = \mathbb{C}^* \times \mathbb{C}^{g-1} \times \mathfrak{S}_{g-1} \\ (\tau_{1,1}, \tau_{1,2}, \dots, \tau_{g,g}) & \mapsto \quad (t_j, \tau_{1,2}, \dots, \tau_{1,g}, \tau') \end{cases}$$

where $\tau' = (\tau_{m,n})_{m,n\geq 2}$ and $t_j = e^{2\pi i \tau_{1,1}/k_j}$ for some $k_j \in \mathbb{N}$, $j=1,2$. Now we have a map

$$\tilde{\pi} : \begin{cases} X_1(F_0) & \to \quad X_2(F_0) \\ t_1 & \mapsto \quad t_2 = (t_1)^{k_1/k_2} \\ \tau_{m,n} & \mapsto \quad \tau_{m,n} \quad \text{for all } (m,n) \neq (1,1) \end{cases}.$$

This map extends naturally to the boundary $\{0\} \times \mathbb{C}^{g-1} \times \mathfrak{S}_{g-1}$ of $\mathbb{C}^g \times \mathfrak{S}_{g-1}$. Obviously, the order of branching of $\tilde{\pi}$ in $\{0\} \times \mathbb{C}^{g-1} \times \mathfrak{S}_{g-1}$ is $\frac{k_1}{k_2}$.

Now we have to consider the quotient maps q_j

$$X_j(F_0) \hookrightarrow X_{\Sigma,j}(F_0) \xrightarrow{q_j} X_{\Sigma,j}(F_0)/P''_{\Gamma_j}(F_0) \hookrightarrow \mathscr{A}_j^*.$$

According to [HKW, Proposition 3.90 and Proposition 3.91] the group $\mathscr{P}''(F_0)$ can be identified as the group consisting of the block matrices

$$\begin{pmatrix} \varepsilon & m & n \\ 0 & A & B \\ 0 & C & D \end{pmatrix} \in \mathrm{GL}(g+1,\mathbb{R})$$

where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2(g-1),\mathbb{R})$, $\varepsilon \in \mathbb{R}$ and $m,n \in \mathbb{R}^{g-1}$. The action of its generators

$$g''_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix}, \quad g''_2 = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \quad \text{and} \quad g''_3 = \begin{pmatrix} 1 & m & n \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix}$$

on $\tau = (\tau_1, \tau_2) \in \mathbb{C}^{g-1} \times \mathfrak{S}_{g-1}$ is given by

$$\begin{aligned} g''_1(\tau) &= (\tau_1(C\tau_2+D)^{-1}, (A\tau_2+B)(C\tau_2+D)^{-1}) \\ g''_2(\tau) &= (\tau_1\varepsilon, \tau_2) \\ g''_3(\tau) &= (\tau_1 + m\tau_2 + n, \tau_2). \end{aligned}$$

Now suppose that $g'' = g''_1 g''_2 g''_3 \in \mathscr{P}''(F_0)$ is an element that operates like the identity on all of the boundary. Obviously, its action on the second component is determined by the submatrix $M := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, and hence we need to have $M = \pm\mathbb{1}$. If $M = \mathbb{1}$, the factor g''_1 leaves τ_1 invariant, and otherwise changes its sign. It is easy to see that in both cases $m=n=0$ and $\varepsilon = \pm 1$, where $\varepsilon = 1$ if and only if $M = \mathbb{1}$. Hence, the only elements of $\mathscr{P}''(F_0)$ that operate like the identity on all of the boundary are in fact $\mathbb{1} \in \mathscr{P}''(F_0)$ and $-\mathbb{1} \in \mathscr{P}''(F_0)$. The same remains true if we intersect $\mathscr{P}''(F_0)$ with the appropriate group Γ_j. But since $-\mathbb{1}$ operates trivially on all of $\mathbb{C}^g \times \mathfrak{S}_{g-1}$, this shows that the maps q_j are not branched along the boundary divisor.

We obtain that the order of branching of $\pi : \mathscr{A}_1^* \to \mathscr{A}_2^*$ on the rational boundary components of corank 1 is also given by $\frac{k_1}{k_2} = [P'_{\Gamma_2}(F) : P'_{\Gamma_1}(F)]$. □

Lemma 2.2.35: Symmetry.
There is a canonical isomorphism of coarse moduli spaces

$$\mathscr{A}_{(e_1,\dots,e_g)} \xrightarrow{\sim} \mathscr{A}_{(e_1,\frac{e_1 e_g}{e_{g-1}},\dots,\frac{e_1 e_g}{e_2},e_g)}.$$

Proof.
This is the main result of [BL02]. □

Lemma 2.2.36: Canonical divisor on toroidal compactifications.
On a toroidal compactification $\mathscr{A}_g^*$ of genus g the canonical divisor in the smooth part is given by

$$K = (g+1)L - D,$$

where D is the boundary divisor of $\mathscr{A}_g^*$ and L is the line bundle of modular forms of weight 1, i. e. the line bundle given by the automorphy factor $\det(C\tau + D)$.

Proof.
This is well known; the interested reader may find a proof in several articles, e. g. in [HS, Section II.1]. □

Chapter 3

Symplectic Theorems

3.1 Divisors of vectors

Let us now begin the analysis of the corank-1 rational boundary components of a toroidal compactification. Since there is only one admissible fan in the corresponding (1-dimensional) cone, namely the fan consisting of $\{0\}$ and $\mathbb{R}_{\geq 0}$, we do not need to specify the collection Σ of fans[1] but only the group Γ. Our field of interest are, of course, the moduli spaces of abelian varieties with different types of polarisation. As mentioned in Remark 1.2.15, we may chose $e_1 = 1$ without changing the groups $\tilde{\Gamma}_{\mathrm{pol}}$ and $\tilde{\Gamma}_{\mathrm{pol}}(n)$. In addition, since $e_i | e_{i+1}$ for all $i = 1, \dots, g-1$, we have some redundancy and rather large numbers. Therefore, we introduce the

Notation 3.1.1.
Let $d_i := e_{i+1}/e_i$ for $i = 1, \dots, g-1$ and define

$$d_{i:j} := \begin{cases} \frac{e_{j+1}}{e_i} = \prod_{n=i}^{j} d_n & \text{for } i \leq j \\ 1 & \text{for } i > j \end{cases} .$$

Then all d_i are positive integers and the polarisation is given by $(1, d_1, d_{1:2}, \dots, d_{1:g-1})$.

Definition 3.1.2: Types of polarisations.
We call a polarisation $(1, d_1, \dots, d_{1:g-1})$ *square-free* if all d_i are square-free. If a polarisation satisfies $\gcd(d_i, d_j) = 1$ for all $i \neq j$ we call it a *coprime polarisation*.

In Remark 2.2.3 we described a map U between boundary components and isotropic subspaces of $\mathbb{R}^{2g}$. In the light of Lemma 2.2.6 this map restricts to the rational boundary components and rational subspaces of $\mathbb{R}^{2g}$. Here, corank-1 boundary components correspond to 1-dimensional subspaces. On the other hand, for each of these subspaces we have (up to the sign) a primitive integral vector, so that the composition of these two maps leads to a bijection

$$V : \begin{cases} \mathscr{F}^{\mathrm{rat}} \text{ of corank } 1 & \to \quad \{\text{primitive vectors in } \mathbb{Z}^{2g}\}/\pm 1 \\ F & \mapsto \quad V(F) := \text{generating element of } U(F) \end{cases} .$$

The group Γ acts on the image space by matrix multiplication from the right. Note that the corank-1 boundary components are in general *not* the minimal boundary components (whose fans determine the compactification, see Remark 2.2.27). There is,

[1]See also the text following Proposition 3.101 in [HKW, p. 92].

however, reason to believe that they are the ones that determine the order of vanishing of cusp forms on all of the boundary (see section 4). This is why we ask for a complete list of non-congruent rational corank-1 boundary components with respect to a group Γ or, equivalently, for an orbit decomposition of the set of primitive integral vectors (again, up to the sign).

First of all, we define the divisors $D_i(v)$ of a vector $v \in \mathbb{Z}^{2g}$ for $i = 1, \dots, g-1$. These are defined using entries of the vector and the values d_i of the polarisation. To keep the notation easier, we shall drop the vector v where possible and write $D_i := D_i(v)$.

Definition 3.1.3: Divisors.
Define the *divisors* $D_i := D_i(v)$ *of a primitive vector* $v \in \mathbb{Z}^{2g}$ recursively to be the generating element of a principal ideal as follows:

$$(D_1) := (d_1, v_1, v_{g+1}) \subset \mathbb{Z} \qquad \text{and for } 1 < i < g \text{ define}$$
$$(D_i) := \left(d_i, \frac{v_1}{D_1 \cdot \ldots \cdot D_{i-1}}, \frac{v_{g+1}}{D_1 \cdot \ldots \cdot D_{i-1}}, \dots, \frac{v_{i-1}}{D_{i-1}}, \frac{v_{g+i-1}}{D_{i-1}}, v_i, v_{g+i}\right) \subset \mathbb{Z}$$

To make things easier we define some notation

Notation 3.1.4.

- Analogously to $d_{i:j}$ define for $1 \le i, j < g$

  $$D_{i:j} := \begin{cases} \prod_{n=i}^{j} D_n & \text{for } i \le j \\ 1 & \text{for } i > j \end{cases}.$$

  We will also use this notation with other variables instead of d_i or D_i but assume the corresponding definition without further mentioning it.

- For large parts of the following calculations, the entries v_j and v_{g+j} of a vector v can be treated simultaneously. In order to shorten the notation and to make it easier to read the proofs, we introduce the notation $v_{j|g+j}$ (or even $v_{j|\cdot}$) for the pair (v_j, v_{g+j}). So, writing $\gcd(v_{j|g+j})$ actually means $\gcd(v_j, v_{g+j})$.

- For an ideal (or a gcd, lcm and so on) containing a similar term T for multiple indices we define

  $$\big(T(j)\big)_{j=a,\dots,b} := \big(T(a), \dots, T(b)\big).$$

Lemma 3.1.5. For $i = 1, \dots, g-1$ we have

$$(D_i) = \left(d_i, \frac{v_{j|g+j}}{D_{j:i-1}}\right)_{j=1,\dots,i} \subset \mathbb{Z}.$$

Proof.
This is nothing but Definition 3.1.3 in the new notation. □

Lemma 3.1.6: Divisibility properties.
For easier reference we collect the following properties:

$$\forall 1 \le i < g \ : \ D_i | d_i$$
$$\forall 1 \le i \le j \le k < g \ : \ D_{j:k} | v_i \text{ and } D_{j:k} | v_{g+i}$$

Proof.
This follows immediately from the definitions. □

Definition 3.1.7: Ideal of lattice and vector.
Let $\mathfrak{L} \subset \mathbb{C}^g$ be the lattice of a polarisation of type $(1, \ldots, d_{1:g-1})$ and identify $\mathfrak{L}$ with $\mathbb{Z}^{2g}$. Define the pairing on $\mathbb{Z}^{2g} \times \mathbb{Z}^{2g}$ by $\langle v, l \rangle := v\Lambda\, {}^t l$. For a vector $v \in \mathbb{Z}^{2g}$ let $(v, \mathfrak{L}) := \{\langle v, l \rangle \big| l \in \mathfrak{L}\}$ which is an ideal in $\mathbb{Z}$, namely

$$(v, \mathfrak{L}) = (d_{1:j-1} v_{j|g+j})_{j=1,\ldots,g} = (v_1, v_{g+1}, d_1 v_2, d_1 v_{g+2}, \ldots, d_{1:g-1} v_g, d_{1:g-1} v_{2g}) \subset \mathbb{Z}.$$

Proposition 3.1.8.
For $1 \le k \le i < g$ and m such that $\frac{m}{D_{k:i-1}} \in \mathbb{Z}$, the following equivalence holds:

$$\frac{m}{D_{k:i-1}} \in \left(d_i, \frac{(v,\mathfrak{L})}{D_{1:i-1}}\right) \iff \frac{d_{1:k-1}}{D_{1:k-1}} \frac{m}{D_{k:i-1}} \in \left(d_i, \frac{(v,\mathfrak{L})}{D_{1:i-1}}\right). \tag{3.1}$$

Proof.
We prove equivalence by induction.

For $k = 1$ we have $\frac{d_{1:k-1}}{D_{1:k-1}} = \frac{1}{1} = 1$ and there is nothing to show.

Assume now that $k > 1$ and that the statement holds for all $j \le k-1$, i. e.

$$\frac{m}{D_{j:i-1}} \in \left(d_i, \frac{(v,\mathfrak{L})}{D_{1:i-1}}\right) \iff \frac{d_{1:j-1}}{D_{1:j-1}} \frac{m}{D_{j:i-1}} \in \left(d_i, \frac{(v,\mathfrak{L})}{D_{1:i-1}}\right) \qquad \text{is true.} \tag{3.2}$$

We have to show (3.1) as stated. For abbreviation, let

$$A_i := \left(d_i, \frac{(v,\mathfrak{L})}{D_{1:i-1}}\right).$$

"$\Rightarrow$": If $\frac{m}{D_{k:i-1}} \in A_i$ then any integer multiple is also in A_i. Since from Lemma 3.1.6 it follows that $\frac{d_j}{D_j} \in \mathbb{Z}$ for any j, we immediately obtain $\frac{d_{1:k-1}}{D_{1:k-1}} \in \mathbb{Z}$ and thus

$$\frac{d_{1:k-1}}{D_{1:k-1}} \frac{m}{D_{k:i-1}} \in A_i$$

which shows the implication.

"$\Leftarrow$": Let m' be such that $\frac{m'}{D_{k:i-1}} \in \mathbb{Z}$. From Lemma 3.1.5 we know that

$$D_{k-1} = \gcd\left(d_{k-1}, \frac{v_{j|g+j}}{D_{j:k-2}}\right)_{j=1,\ldots,k-1}$$
$$\implies \quad 1 = \gcd\left(\frac{d_{k-1}}{D_{k-1}}, \frac{v_{j|g+j}}{D_{j:k-1}}\right)_{j=1,\ldots,k-1}$$
$$\implies \quad \frac{m'}{D_{k:i-1}} = \gcd\left(\frac{d_{k-1} m'}{D_{k-1:i-1}}, \frac{v_{j|g+j} m'}{D_{j:i-1}}\right)_{j=1,\ldots,k-1} \qquad =: I \tag{3.3}$$

where I is an ideal in $\mathbb{Z}$. We will show that all generators of I are contained in A_i. From Definition 3.1.7 we know that $d_{1:j-1}v_{j|g+j} \in (v,\mathfrak{L})$ and hence

$$\frac{d_{1:j-1}}{D_{1:j-1}}\frac{v_j}{D_{j:i-1}} = \frac{d_{1:j-1}v_j}{D_{1:i-1}} \in A_i \quad \text{and} \quad \frac{d_{1:j-1}}{D_{1:j-1}}\frac{v_{g+j}}{D_{j:i-1}} = \frac{d_{1:j-1}v_{g+j}}{D_{1:i-1}} \in A_i,$$

and since $j \le k-1$ we may use assumption (3.2) with $m = v_j$ and $m = v_{g+j}$ to obtain

$$\frac{v_j}{D_{j:i-1}} \in A_i \quad \text{and} \quad \frac{v_{g+j}}{D_{j:i-1}} \in A_i \quad \Longrightarrow \quad \frac{v_{j|g+j}}{D_{j:i-1}}m' \in A_i.$$

Now the right hand side of (3.1) states that

$$\frac{d_{1:k-1}}{D_{1:k-1}}\frac{m'}{D_{k:i-1}} \in A_i.$$

We can rearrange the factors

$$\frac{d_{1:k-1}}{D_{1:k-1}}\frac{m'}{D_{k:i-1}} = \frac{d_{1:k-2}}{D_{1:k-2}}\frac{d_{k-1}m'}{D_{k-1:i-1}}$$

and can now apply assumption (3.2) for $j = k-1$ and $m = d_{k-1}m'$ to obtain

$$\frac{d_{k-1}m'}{D_{k-1:i-1}} \in A_i.$$

We now know that the ideal I defined in (3.3) is contained in A_i and thus $\frac{m'}{D_{k:i-1}} \in I \subset A_i$. This completes the induction. □

Lemma 3.1.9.
The ideal (D_i) can also be given by

$$(D_i) = \left(d_i, \frac{(v,\mathfrak{L})}{D_{1:i-1}}\right).$$

Proof.
Let $A_i := \left(d_i, \frac{(v,\mathfrak{L})}{D_{1:i-1}}\right)$ as above. We know from the definitions that

$$(3.4) \qquad (D_i) = \left(d_i, \frac{v_{1|g+1}}{D_{1:i-1}}, \frac{v_{2|g+2}}{D_{2:i-1}}, \ldots, \frac{v_{i-1|g+i-1}}{D_{i-1:i-1}}, v_{i|g+i}\right) \qquad \text{and}$$

$$(3.5) \qquad A_i = \left(d_i, \frac{v_{1|g+1}}{D_{1:i-1}}, \frac{d_{1:1}v_{2|g+2}}{D_{1:i-1}}, \ldots, \frac{d_{1:i-2}v_{i-1|\cdot}}{D_{1:i-1}}, \frac{d_{1:i-1}v_{i|g+i}}{D_{1:i-1}}, \frac{d_{1:i}v_{i+1|\cdot}}{D_{1:i-1}}, \ldots\right)$$

$$(3.6) \qquad = \left(d_i, \frac{v_{1|g+1}}{D_{1:i-1}}, \frac{d_{1:1}}{D_{1:1}}\frac{v_{2|g+2}}{D_{2:i-1}}, \ldots, \frac{d_{1:i-2}}{D_{1:i-2}}\frac{v_{i-1|g+i-1}}{D_{i-1:i-1}}, \frac{d_{1:i-1}}{D_{1:i-1}}v_{i|g+i}\right)$$

The last elements in (3.5) that have been omitted in (3.6) are multiples of d_i and are therefore not needed to generate A_i. Since now all elements in (3.6) are multiples of elements in (3.4), we obviously have $A_i \subset (D_i)$. For the other inclusion we apply Proposition 3.1.8 to every element in (3.6) to obtain that all elements in (3.4) are contained in A_i. This completes the proof. □

This lemma now shows that Proposition 3.1.8 also applies to (D_i), i. e.

Corollary 3.1.10.
For $1 \leq k \leq i < g$ and $\frac{m}{D_{k:i-1}} \in \mathbb{Z}$ the following equivalence holds:

$$\frac{m}{D_{k:i-1}} \in (D_i) \iff \frac{d_{1:k-1}}{D_{1:k-1}} \frac{m}{D_{k:i-1}} \in (D_i).$$

More importantly, we also now know the following:

Corollary 3.1.11: Invariance.
The divisors D_i of a vector v are invariant under the action of $\tilde{\Gamma}_{\text{pol}}$ on $\mathbb{Z}^{2g}$.

Proof.
This can be deduced from the invariance of $\mathfrak{L}$ under the action of $\tilde{\Gamma}_{\text{pol}}$ as follows: Let $M \in \tilde{\Gamma}_{\text{pol}}$. Since $\tilde{\Gamma}_{\text{pol}} \subset \mathrm{SL}(2g, \mathbb{Z})$ we have in particular that $({}^tM)^{-1}$ is an integer matrix. Hence, for $v \in \mathbb{Z}^{2g}$ we obtain

$$\begin{aligned}
(vM, \mathfrak{L}) &= \left\{ \langle vM, l \rangle \middle| l \in \mathfrak{L} \right\} \\
&= \left\{ vM\Lambda {}^t l \middle| l \in \mathbb{Z}^{2g} \right\} \quad \text{since } M\Lambda = \Lambda({}^tM)^{-1} \text{ by definition of } \tilde{\Gamma}_{\text{pol}} \\
&= \left\{ v\Lambda({}^tM)^{-1} {}^t l \middle| {}^t l \in \mathbb{Z}^{2g} \right\} \quad \text{and since } ({}^tM)^{-1} \text{ is integer} \\
&= \left\{ v\Lambda {}^t l \middle| l \in \mathbb{Z}^{2g} \right\} \\
&= (v, \mathfrak{L}).
\end{aligned}$$

Using Lemma 3.1.9 we see that

$$D_i(vM) = \gcd\left(d_i, \frac{(vM, \mathfrak{L})}{D_{1:i-1}}\right) = \gcd\left(d_i, \frac{(v, \mathfrak{L})}{D_{1:i-1}}\right) = D_i(v).$$

□

Remark 3.1.12.
Let us point out that the divisors D_i are not independent and therefore not every possible combination of divisors of the polarisation d_i can actually occur. E. g. take $g = 3$ and $d_1 = 4, d_2 = 6$ so that we have a polarisation $(1, 4, 24)$. Now, there is no vector with the divisors $D_1 = D_2 = 2$ because that would mean that

$$D_1 = \gcd(4, v_1, v_4) = 2 \quad \text{and} \tag{3.7}$$

$$D_2 = \gcd(6, \tfrac{v_1}{2}, \tfrac{v_4}{2}, v_2, v_5) = 2, \tag{3.8}$$

where equation (3.8) clearly shows that 4 divides both v_1 and v_4, which is a contradiction to (3.7).

The additional restriction on the divisors D_i is the following:

Theorem 3.1.13: Restrictions on D_i.
For $1 \leq i < j \leq g - 1$ we have

$$\gcd\left(\tfrac{d_i}{D_i}, D_j\right) = 1. \tag{3.9}$$

Moreover, any ordered set of positive integers $\{D_i\} := \{D_1, \dots, D_{g-1}\}$ satisfying $D_i | d_i$ and condition (3.9) does occur as set of divisors of a vector $v \in \mathbb{Z}^{2g}$.

Proof.

Necessity:

Take $i < j$ and assume $n := \gcd(\frac{d_i}{D_i}, D_j) \neq 1$. We claim that any power of n divides d_i in contradiction to $d_i \neq 0$. The proof is by induction.

Define the index set $I := \{1, \dots, i, g+1, \dots, g+i\}$ and let

$$v_k^{(0)} := v_k \text{ for } k \in I, \quad d_i^{(0)} := d_i, \quad D_i^{(0)} := D_i \qquad \text{and}$$
$$v_k^{(r)} := \frac{v_k}{n^r}, \quad d_i^{(r)} := \frac{d_i}{n^r}, \quad D_i^{(r)} := \frac{D_i}{n^r} \qquad \text{for } r \geq 1.$$

We want to show that all values we just defined are integers. For the generation $r = 0$ this is obvious.

Assume that all values of the generation $r-1$ are integers. By definition of n we know that n divides $\frac{d_i}{D_i}$ and hence

$$\frac{d_i}{D_i} = \frac{n^{r-1} d_i^{(r-1)}}{n^{r-1} D_i^{(r-1)}} = \frac{d_i^{(r-1)}}{D_i^{(r-1)}} \quad \Longrightarrow \quad n | d_i^{(r-1)} \quad \Longrightarrow \quad d_i^{(r)} \in \mathbb{Z}.$$

From the definitions of n and D_j we know that n divides

$$\begin{aligned} D_j &= \gcd\left(d_j, \frac{v_{1|g+1}}{D_{1:j-1}}, \dots, \frac{v_{j-1|g+j-1}}{D_{1:j-1}}\right) \\ &= \gcd\left(d_j, \frac{n^{r-1} v_{1|g+1}^{(r-1)}}{n^{r-1} D_{1:j-1}^{(r-1)}}, \dots, \frac{n^{r-1} v_{i|g+i}^{(r-1)}}{n^{r-1} D_{i:j-1}^{(r-1)}}, \frac{v_{i+1|g+i+1}}{D_{i+1:j-1}}, \dots, \frac{v_{j-1|g+j-1}}{D_{j-1}}\right), \\ &= \gcd\left(d_j, \frac{v_{1|g+1}^{(r-1)}}{D_{1:j-1}^{(r-1)}}, \dots, \frac{v_{i|g+i}^{(r-1)}}{D_{i:j-1}^{(r-1)}}, \frac{v_{i+1|g+i+1}}{D_{i+1:j-1}}, \dots, \frac{v_{j-1|g+j-1}}{D_{j-1}}\right), \end{aligned}$$

where by $D_{k:j-1}^{(r-1)}$ we actually mean $D_{k:i-1} D_i^{(r-1)} D_{i+1:j-1}$. This shows that in particular

$$n \text{ divides } \frac{v_{k|g+k}^{(r-1)}}{D_{k:j-1}^{(r-1)}} \text{ for } k = 1, \dots, i \tag{3.10}$$

and hence also $n | v_k^{(r-1)}$ which implies $v_k^{(r)} \in \mathbb{Z}$ for all $k \in I$. Furthermore, since $D_{k:i-1} | D_{k:i-1} D_i^{(r-1)} D_{i+1:j-1} = D_{k:j-1}^{(r-1)}$, statement (3.10) implies that

$$n \Big| \gcd\left(d_i^{(r-1)}, \frac{v_{k|g+k}^{(r-1)}}{D_{k:i-1}}\right)_{k=1,\dots,i} = \frac{1}{n^{r-1}} \gcd\left(d_i, \frac{v_{k|g+k}}{D_{k:i-1}}\right)_{k=1,\dots,i} = \frac{1}{n^{r-1}} D_i = D_i^{(r-1)}.$$

So, we have shown that all values in the rth generation are integers. The contradiction follows as mentioned above.

Sufficiency:

Choose integers D_i satisfying the conditions stated in the lemma. Consider the vector

$$v = (D_{1:g-1}, D_{2:g-1}, \dots, D_{g-1}, 1, 0, \dots, 0) \in \mathbb{Z}^{2g}.$$

For $i = 1,\dots,g-1$ we have

$$\begin{aligned}\frac{1}{D_i}D_i(v) &= \frac{1}{D_i}\gcd\left(d_i, \frac{v_k}{D_{k:i-1}}, \frac{v_{g+k}}{D_{k:i-1}}\right)_{k=1,\dots,i} = \frac{1}{D_i}\gcd\left(d_i, \frac{D_{k:g-1}}{D_{k:i-1}}, 0\right)_{k=1,\dots,i} \\ &= \frac{1}{D_i}\gcd\left(d_i, D_{i:g-1}\right) = \gcd\left(\tfrac{d_i}{D_i}, D_{i+1:g-1}\right) = 1,\end{aligned}$$

since for all $j = i+1,\dots,g-1$ we have $\gcd(\frac{d_i}{D_i}, D_j) = 1$. This equation shows that the divisors $D_i(v)$ are exactly the chosen D_i. □

This lemma has an interesting consequence:

Corollary 3.1.14: Characterising property of $D_{1:g-1}$.
For a given polarisation $(d_1,\dots,d_{1:g-1})$, the value $D_{1:g-1}(v)$ determines all the values $D_i(v)$ uniquely.

Proof.
Let $d_1,\dots,d_{g-1}$ and $D_{1:g-1}$ be given. Then Theorem 3.1.13 leads to the following:

$$\begin{aligned}\gcd(\tfrac{d_1}{D_1}, D_2) = \cdots = \gcd(\tfrac{d_1}{D_1}, D_{g-1}) = 1 &\implies \gcd(\tfrac{d_1}{D_1}, D_{2:g-1}) = 1 \\ &\implies \gcd(d_1, D_{1:g-1}) = D_1\end{aligned}$$

so that we can determine D_1 from d_1 and $D_{1:g-1}$. Divide $D_{1:g-1}$ by D_1 to obtain $D_{2:g-1}$ and apply the same lemma. By iterating this method all values D_i are obtained. □

3.2 Properties of symplectic matrices

3.2.1 Conditions on submatrices and rows

We have defined $\tilde{\Gamma}_{\text{pol}} = \mathrm{Sp}(\Lambda, \mathbb{Z})$ in Definition 1.2.2 to be the group of matrices satisfying $M\Lambda{}^tM = \Lambda$. The action of $\tilde{\Gamma}_{\text{pol}}$ on $\mathfrak{S}_g$ is defined using the subdivision of $M \in \tilde{\Gamma}_{\text{pol}}$ into its four square $g \times g$ submatrices A to D. One can ask for a characterisation of $\tilde{\Gamma}_{\text{pol}}$ in terms of these submatrices, and the answer is as follows:

Lemma 3.2.1.
Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}(2g, \mathbb{Z})$ where the submatrices are all in $\mathbb{Z}^{g\times g}$. We have

$$M \in \tilde{\Gamma}_{\text{pol}} \iff \begin{cases} A\Delta{}^tB \text{ is symmetric} \\ C\Delta{}^tD \text{ is symmetric} \\ A\Delta{}^tD - B\Delta{}^tC = \Delta \end{cases}.$$

For principal polarisations, we have in particular

$$M \in \mathrm{Sp}(2g, \mathbb{Q}) \iff \begin{cases} A{}^tB \text{ is symmetric} \\ C{}^tD \text{ is symmetric} \\ A{}^tD - B{}^tC = \mathbb{1} \end{cases}. \tag{3.11}$$

Proof.
This can be seen by simple calculation. □

Definition 3.2.2: Skew symmetric form on row vectors.
For two vectors $v = (v_1, \dots, v_{2g}), w = (w_1, \dots, w_{2g}) \in \mathbb{Z}^{2g}$ the matrix $\Lambda = \left(\begin{smallmatrix} & \Delta \\ -\Delta & \end{smallmatrix}\right)$ defines the skew-symmetric form

$$\begin{aligned}\langle v, w\rangle := v\Lambda{}^t w = \sum_{i=1}^{g} d_{1:i-1} \begin{vmatrix} v_i & v_{g+i} \\ w_i & w_{g+i}\end{vmatrix} \\ = \begin{vmatrix} v_1 & v_{g+1} \\ w_1 & w_{g+1}\end{vmatrix} + d_1 \begin{vmatrix} v_2 & v_{g+2} \\ w_2 & w_{g+2}\end{vmatrix} + \dots + d_{1:g-1} \begin{vmatrix} v_g & v_{2g} \\ w_g & w_{2g}\end{vmatrix}.\end{aligned}$$

If we denote the row vectors of a matrix $M \in \mathrm{GL}(2g, \mathbb{Z})$ by m_i for $i = 1, \dots, 2g$, the conditions given in Lemma 3.2.1 can be transformed into the following:

Lemma 3.2.3.

$$M \in \tilde{\Gamma}_{\text{pol}} \iff \begin{cases} \langle m_i, m_{g+i}\rangle & = & d_{1:i-1} & \text{for } i = 1, \dots, g \\ \langle m_{g+i}, m_i\rangle & = & -d_{1:i-1} & \text{for } i = 1, \dots, g \\ \langle m_i, m_j\rangle & = & 0 & \text{for all other } i, j \end{cases},$$

where the first and second line are equivalent and only stated for completeness.

Proof.
Again, this is simple computation. □

Remark 3.2.4.
Note that the conditions stated in Lemma 3.2.1 and Lemma 3.2.3 are valid for symplectic matrices over any commutative ring, even finite ones that are not integral domains.

So far we have introduced five groups with two different actions on $\mathfrak{S}_g$, namely

$$\begin{aligned}\Gamma_g, \Gamma_g(n) : \tau \mapsto (A\tau + B)(C\tau + D)^{-1} \qquad \text{and} \\ \tilde{\Gamma}_{\text{pol}}, \tilde{\Gamma}_{\text{pol}}(n), \tilde{\Gamma}^{\text{lev}}_{\text{pol}} : \tau \mapsto (A\tau + B\Delta)(C\tau + D\Delta)^{-1}\Delta.\end{aligned}$$

In later chapters of this thesis we will need a combination of these. Since this can be very confusing we want to restrict ourselves to only one action. In order to achieve this we must allow rational numbers.

Definition 3.2.5: Conjugated symplectic groups.
Let $R := \left(\begin{smallmatrix} \mathbb{1} & \\ & \Delta \end{smallmatrix}\right)$ and recall that $J = \left(\begin{smallmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{smallmatrix}\right)$. Define

$$\Gamma_{\text{pol}} := R^{-1}\tilde{\Gamma}_{\text{pol}} R \subset \mathrm{Sp}(J, \mathbb{Q}) \quad \text{and} \quad \Gamma^{\text{lev}}_{\text{pol}} := R^{-1}\tilde{\Gamma}^{\text{lev}}_{\text{pol}} R \subset \mathrm{Sp}(J, \mathbb{Q}).$$

The groups Γ_{pol} and $\Gamma^{\text{lev}}_{\text{pol}}$ act on $\mathfrak{S}_g$ in the same way as $\Gamma_g \subset \mathrm{Sp}(J, \mathbb{Q})$, namely by $\tau \mapsto (A\tau + B)(C\tau + D)^{-1}$.

We drop the tilde to denote that the operation of the group is now independent of the polarisation. The substitution suggested by this definition changes nothing in the quotient by these groups as the following lemma shows.

Lemma 3.2.6.
Let the action of $\mathrm{Sp}(\Lambda, \mathbb{Q})$ be given by

$$\mu_\Lambda : \begin{cases} \mathrm{Sp}(\Lambda, \mathbb{Q}) \times \mathfrak{S}_g & \to & \mathfrak{S}_g \\ (M, \tau) & \mapsto & (A\tau + B\Delta)(C\tau + D\Delta)^{-1}\Delta \end{cases}.$$

For $\Lambda = J$ we have $\Delta = \mathbb{1}$. Define $\rho : \mathrm{Sp}(\Lambda,\mathbb{Q}) \to \mathrm{Sp}(J,\mathbb{Q})$ by $\rho(M) := R^{-1}MR$. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Sp}(\Lambda,\mathbb{Q}) \times \mathfrak{S}_g & \xrightarrow{\mu_\Lambda} & \mathfrak{S}_g \\ {\scriptstyle \rho\times\mathrm{id}}\downarrow & & \| \\ \mathrm{Sp}(J,\mathbb{Q}) \times \mathfrak{S}_g & \xrightarrow{\mu_J} & \mathfrak{S}_g \end{array}$$

Proof.
Yet again, a simple computation will show this. □

3.2.2 Divisibility of matrix entries

The groups Γ we are dealing with are so far defined by equations involving several entries of the matrices M contained in Γ. In many cases, however, we will also need properties of single matrix entries. We will state these properties by giving subsets of $\mathrm{GL}(2g,\mathbb{Q})$ that still contain the groups we are interested in.

Definition 3.2.7: Triangular polarisation matrices.
Define the set of matrices

$$\mathbb{D}(\Delta) := \left\{(s_{ij}) \in \mathbb{Z}^{g\times g} \middle| j < i \implies d_{j:i-1}|s_{ij}\right\},$$

in which the entries in the lower left triangle satisfy the given divisibility condition, which depends on the polarisation $(1, d_1, \ldots, d_{1:g-1})$. Define its subset

$$\mathbb{SD}(\Delta) := \mathbb{D}(\Delta) \cap \mathrm{SL}(g,\mathbb{Z}) = \left\{S \in \mathbb{D}(\Delta) \middle| \det(S) = 1\right\}.$$

Lemma 3.2.8.
The set $\mathbb{D}(\Delta)$ with the normal matrix operations is a ring with unity. Its subset $\mathbb{SD}(\Delta)$ is a multiplicative group.

Proof.
Let $A, B \in \mathbb{D}(\Delta)$ where $A = (a_{ij})$ and $B = (b_{ij})$. For $j < i$, we have $a_{ij} = a'_{ij} d_{j:i-1}$ and $b_{ij} = b'_{ij} d_{j:i-1}$.

It is obvious that $A + B \in \mathbb{D}(\Delta)$, since for $j \geq i$ there is nothing to show and for $j < i$ we have

$$d_{j:i-1}|a_{ij}, \quad d_{j:i-1}|b_{ij} \quad \implies \quad d_{j:i-1}|(a_{ij} + b_{ij}).$$

Equally obvious are the remaining facts to show that $(\mathbb{D}(\Delta), +)$ is a group.

Now, let $C := AB$ and consider one entry c_{ij}. If $j \geq i$, there is again nothing to show. Therefore, let $j < i$. We have

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} a_{ik} b_{kj} \\ &= \sum_{k=1}^{j} (a'_{ik} d_{k:i-1}) b_{kj} + \sum_{k=j+1}^{i-1} (a'_{ik} d_{k:i-1})(b'_{kj} d_{j:k-1}) + \sum_{k=i}^{n} a_{ik} (b'_{kj} d_{j:k-1}) \\ &= d_{j:i-1} \Big(\sum_{k=1}^{j} a'_{ik} d_{k:j-1} b_{kj} + \sum_{k=j+1}^{i-1} a'_{ik} b'_{kj} + \sum_{k=i}^{n} a_{ik} b'_{kj} d_{i:k-1} \Big). \end{aligned}$$

This shows that $C \in \mathbb{D}(\Delta)$. Since the unit matrix also fulfils all conditions, $\mathbb{D}(\Delta)$ is as claimed.

Since $\mathbb{SD}(\Delta) \subset \mathrm{SL}(g,\mathbb{Z})$ per definition, it is obvious that for any $S \in \mathbb{SD}(\Delta)$ the inverse $T := S^{-1}$ exists, is an integer matrix and has determinant 1. It remains to show that $T = (t_{ij}) \in \mathbb{D}(\Delta)$. Since for $j \geq i$ there is no additional condition, let $j < i$. By Cramer's rule we know $t_{ij} = \frac{1}{|S|}|S^{(j,i)}| = |S^{(j,i)}|$ where $S^{(j,i)}$ is the minor of S constructed by removing the jth row and ith column. In more detail, we have

$$t_{ij} = \det \begin{pmatrix} s_{1,1} & \dots & s_{1,i-1} & s_{1,i+1} & \dots & s_{1,g} \\ \vdots & & \vdots & \vdots & & \vdots \\ d_{1:j-2}s_{j-1,1} & \dots & d_{i-1:j-2}s_{j-1,i-1} & d_{i+1:j-2}s_{j-1,i+1} & \dots & s_{j-1,g} \\ d_{1:j}s_{j+1,1} & \dots & d_{i-1:j}s_{j+1,i-1} & d_{i+1:j}s_{j+1,i+1} & \dots & s_{j+1,g} \\ \vdots & & \vdots & \vdots & & \vdots \\ d_{1:g}s_{g,1} & \dots & d_{i-1:g}s_{g,i-1} & d_{i+1:g}s_{g,i+1} & \dots & s_{g,g} \end{pmatrix}.$$

Fix n in $I := \{j, \dots, i-1\}$. The matrix $S^{(j,i)}$ fulfils the condition of Lemma A.1 for $d = d_n$ if we choose $k = n$. Therefore $d_n | \det(S^{(j,i)})$ and we can successively divide out the factors d_n for all $n \in I$. Hence we have $d_{j:i-1} | t_{ij}$ which completes the proof that $T \in \mathbb{SD}(\Delta)$. □

Lemma 3.2.9.
We have the following congruence conditions:

$$\tilde{\Gamma}_{\mathrm{pol}} \subset \mathbb{D}(\Delta)^{2\times 2}$$

where $\mathbb{D}(\Delta)^{2\times 2} := \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \middle| A, B, C, D \in \mathbb{D}(\Delta) \right\}$.

Proof.
Let $M = (m_{i,j}) \in \tilde{\Gamma}_{\mathrm{pol}}$ and chose $k \in \{1, \dots, g-1\}$. Denote the index set

$$I_k := \{k+1, \dots, g, g+k+1, \dots, 2g\}.$$

Now chose any $i \in I_k$ and let $v := e_i \in \mathbb{Z}^{2g}$ be the ith unit vector. From the invariance of the divisors we know that $D_k(v) = D_k(vM)$. Furthermore,

$$\begin{aligned} D_k(v) &= \gcd(d_k, \frac{v_j}{D_{j:k-1}}, \frac{v_{g+j}}{D_{j:k-1}})_{j=1,\dots,k} \quad \text{since } v = e_i \text{ with } i \in I_k \text{ and } j \leq k \\ &= \gcd(d_k, 0, \dots, 0) \\ &= d_k \qquad\qquad \text{and} \\ D_k(vM) &= \gcd(d_k, \frac{\sum_{n=1}^{2g} v_n m_{n,j}}{D_{j:k-1}}, \frac{\sum_{n=1}^{2g} v_n m_{n,g+j}}{D_{j:k-1}})_{j=1,\dots,k} \quad \text{and since } v = e_i \\ &= \gcd(d_k, \frac{m_{i,j}}{D_{j:k-1}}, \frac{m_{i,g+j}}{D_{j:k-1}}). \end{aligned}$$

This reasoning for all valid combinations of values gives rise to

$$\forall k \in \{1, \dots, g-1\}\ \forall i \in I_k\ \forall 1 \leq j \leq k\ :\ d_k | m_{i,j} \text{ and } d_k | m_{i,g+j},$$

which is exactly the divisibility condition for $M \in \mathbb{D}(\Delta)^{2\times 2}$. □

Lemma 3.2.10.
For the conjugated group we have

$$\Gamma_{\text{pol}} = \text{Sp}(2g,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{D}(\Delta) & \mathbb{D}(\Delta)\Delta \\ \Delta^{-1}\mathbb{D}(\Delta) & \Delta^{-1}\mathbb{D}(\Delta)\Delta \end{pmatrix}$$

Proof.
From Definition 3.2.5 we obviously have

$$N \in \Gamma_{\text{pol}} \iff \exists M \in \tilde{\Gamma}_{\text{pol}} \;:\; N = R^{-1}MR.$$

Because of Lemma 3.2.9 this means

$$N \in \Gamma_{\text{pol}} \iff \exists A,B,C,D \in \mathbb{D}(\Delta) \;:\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{\Gamma}_{\text{pol}} \text{ and } N = \begin{pmatrix} A & B\Delta \\ \Delta^{-1}C & \Delta^{-1}D\Delta \end{pmatrix}.$$

The conditions of Lemma 3.2.1 for $M \in \tilde{\Gamma}_{\text{pol}}$ may be transformed as follows:

$$\begin{aligned}
A\Delta{}^tB = {}^t(A\Delta{}^tB) &\iff A\,{}^t(B\Delta) = {}^t\big(A\,{}^t(B\Delta)\big) \\
C\Delta{}^tD = {}^t(C\Delta{}^tD) &\iff \Delta^{-1}C\Delta{}^tD\Delta^{-1} = \Delta^{-1}\,{}^t(C\Delta{}^tD)\Delta^{-1} \\
&\iff \Delta^{-1}C\,{}^t(\Delta^{-1}D\Delta) = \Delta^{-1}D\Delta{}^tC\Delta^{-1} = {}^t\big(\Delta^{-1}C\,{}^t(\Delta^{-1}D\Delta)\big) \\
A\Delta{}^tD - B\Delta{}^tC = \Delta &\iff A\Delta{}^tD\Delta^{-1} - B\Delta{}^tC\Delta^{-1} = \Delta\Delta^{-1} = \mathbb{1} \\
&\iff A\,{}^t(\Delta^{-1}D\Delta) - (B\Delta)\,{}^t(\Delta^{-1}C) = \mathbb{1}.
\end{aligned}$$

This in turn means

$$N \in \Gamma_{\text{pol}} \iff \exists A,B,C,D \in \mathbb{D}(\Delta) \;:\; N = \begin{pmatrix} A & B\Delta \\ \Delta^{-1}C & \Delta^{-1}D\Delta \end{pmatrix} \in \text{Sp}(2g,\mathbb{Q})$$

which proves the claim. □

For the groups with canonical level structure we obtain additional conditions:

Lemma 3.2.11.

$$\tilde{\Gamma}^{\text{lev}}_{\text{pol}} = \left\{ M \in \tilde{\Gamma}_{\text{pol}} \middle| M \in \left(\begin{pmatrix} {}^t\mathfrak{d} \\ {}^t\mathfrak{d} \end{pmatrix} 1_{2g} \right) \otimes \mathbb{Z} + \mathbb{1} \right\}$$

where $\mathfrak{d} := (1, d_1, \ldots, d_{1:g-1})$ and $1_{2g} := (1,\ldots,1) \in \mathbb{Z}^{2g}$. The tensor denotes that each matrix entry of the rank 1 matrix in brackets may be multiplied by an integer z_{ij}.
Proof.
Denote the basis of $\mathfrak{L} \subset \mathbb{C}^g$ by $\{b_1,\ldots,b_{2g}\}$. Then a basis of the dual lattice $\mathfrak{L}^\vee$ can be given by $\{\frac{1}{d_{1:i-1}} b_{i|g+i}\}_{i=1,\ldots,g}$. According to Definition 1.2.22 a matrix $M \in \tilde{\Gamma}_{\text{pol}}$ is in $\tilde{\Gamma}^{\text{lev}}_{\text{pol}}$ if and only if it satisfies $M_{\mathfrak{L}^\vee/\mathfrak{L}} = \text{id}_{\mathfrak{L}^\vee/\mathfrak{L}}$. This is satisfied if and only if for all $i = 1,\ldots,g$ we have

$$\begin{aligned}
\frac{1}{d_{1:i-1}} b_{i|g+i} M \equiv_{\mathfrak{L}} \frac{1}{d_{1:i-1}} b_{i|g+i} &\iff \frac{1}{d_{1:i-1}} b_{i|g+i}(M - \mathbb{1}) \in \mathfrak{L} \\
&\iff \frac{1}{d_{1:i-1}} e_{i|g+i}(M - \mathbb{1}) \in \mathbb{Z}^{2g}
\end{aligned}$$

where e_i is the ith unit vector. This means that $d_{1:i-1}$ divides every entry in the ith and $g+i$th row of the matrix $M-\mathbb{1}$ which is exactly the condition we wanted to prove. $\square$

Lemma 3.2.12.

$$\Gamma_{\text{pol}}^{\text{lev}} = \{M \in \Gamma_{\text{pol}} \big| M \in \left(\begin{pmatrix} {}^t\mathfrak{d} \\ {}^t1_g \end{pmatrix} (1_g, \mathfrak{d}) \right) \otimes \mathbb{Z} + \mathbb{1}\}$$

where again $\mathfrak{d} := (1, d_1, \ldots, d_{1:g-1})$ and $1_g := (1, \ldots, 1) \in \mathbb{Z}^g$.

Proof.

This follows directly from Lemma 3.2.11 by conjugating with R as in Definition 3.2.5. $\square$

One important result from this lemma is the following observation: Although Γ_{pol} may have rational non-integer entries, this is no longer possible for its subgroup $\Gamma_{\text{pol}}^{\text{lev}}$:

Corollary 3.2.13.

$$\Gamma_{\text{pol}}^{\text{lev}} \subset \mathrm{Sp}(2g, \mathbb{Z}).$$

Proof.

With Lemma 3.2.10 we know $\Gamma_{\text{pol}}^{\text{lev}} \subset \Gamma_{\text{pol}} \subset \mathrm{Sp}(2g, \mathbb{Q})$, and since the condition given in Lemma 3.2.12 implies that all matrix entries must be integers the claim follows immediately. $\square$

3.2.3 Restriction to square-free polarisations

So far, if we want to state a theorem on 'all polarisations', we actually have to consider all polarisations separately. If, on the other hand, we know that a theorem is automatically true for all subgroups $\Gamma' \subset \Gamma$ if it is true for Γ, then we can restrict ourselves to square-free polarisations: the groups associated with other polarisations are conjugate to subgroups of the groups for square-free polarisations. The following lemma will make this precise.

Lemma 3.2.14: Square-free.

Let $(1, e_1, \ldots, e_{1:g-1})$ be the type of a polarisation where $e_i = d_i s_i^2$ and all d_i are square-free. Denote the groups associated with this polarisation by

$$\tilde{\Gamma}_{\text{pol},e}, \quad \Gamma_{\text{pol},e}, \quad \tilde{\Gamma}_{\text{pol},e}^{\text{lev}} \quad \text{and} \quad \Gamma_{\text{pol},e}^{\text{lev}}.$$

Denote the groups associated with the polarisation $(1, d_1, \ldots, d_{1:g-1})$ by

$$\tilde{\Gamma}_{\text{pol},d}, \quad \Gamma_{\text{pol},d}, \quad \tilde{\Gamma}_{\text{pol},d}^{\text{lev}} \quad \text{and} \quad \Gamma_{\text{pol},d}^{\text{lev}}.$$

Let $S := \mathrm{diag}(1, s_1, \ldots, s_{1:g-1})$, $T := \left(\begin{smallmatrix} S & 0 \\ 0 & S^{-1} \end{smallmatrix}\right)$ and $U := \left(\begin{smallmatrix} S & 0 \\ 0 & S \end{smallmatrix}\right)$. Then we have

$$T^{-1}\Gamma_{\text{pol},e}T \subset \Gamma_{\text{pol},d}, \quad T^{-1}\Gamma_{\text{pol},e}^{\text{lev}}T \subset \Gamma_{\text{pol},d}^{\text{lev}}$$

and

$$U^{-1}\tilde{\Gamma}_{\text{pol},e}U \subset \tilde{\Gamma}_{\text{pol},d}, \quad U^{-1}\tilde{\Gamma}_{\text{pol},e}^{\text{lev}}U \subset \tilde{\Gamma}_{\text{pol},d}^{\text{lev}}.$$

Proof.
Let us begin with the relation $T^{-1}\Gamma_{\mathrm{pol},e}T \subset \Gamma_{\mathrm{pol},d}$. Denote the matrices for the two polarisations by

$$\Delta_e := \operatorname{diag}(1, e_1, \ldots, e_{1:g-1}) \quad \text{and} \quad \Delta_d := \operatorname{diag}(1, d_1, \ldots, d_{1:g-1}).$$

We have $\Delta_d = S^{-1}\Delta_e S^{-1}$. Furthermore, for two matrices $M = (m_{ij}) \in \mathbb{D}(\Delta_e)$ and $N = (n_{ij}) := S^{-1}MS$ we have $n_{ij} = s_{i:i-1}^{-1} m_{ij} s_{1:j-1}$ and hence

$$\begin{aligned}\text{for } j < i: \quad n_{ij} &= s_{1:i-1}^{-1}(e_{j:i-1}m'_{ij})s_{1:j-1} = \frac{s_{1:j-1}}{s_{1:i-1}}(d_{j:i-1}s_{j:i-1}^2 m'_{ij})\\ &= \frac{s_{1:j-1}s_{j:i-1}}{s_{1:i-1}} d_{j:i-1}s_{j:i-1}m'_{ij} = d_{j:i-1}(s_{j:i-1}m'_{ij}) \qquad \text{and}\\ \text{for } j \geq i: \quad n_{ij} &= \frac{s_{1:j-1}}{s_{1:i-1}} m_{ij} = s_{i:j-1}m_{ij}\end{aligned}$$

which implies the divisibility conditions from Definition 3.2.7 for $N \in \mathbb{D}(\Delta_d)$. Therefore,

$$S^{-1}\mathbb{D}(\Delta_e)S \subset \mathbb{D}(\Delta_d). \tag{3.12}$$

For $\Gamma_{\mathrm{pol},e}$, Lemma 3.2.10 tells us

$$\begin{aligned}T^{-1}\Gamma_{\mathrm{pol},e}T &\subset \begin{pmatrix} S^{-1} & \\ & S\end{pmatrix}\begin{pmatrix}\mathbb{D}(\Delta_e) & \mathbb{D}(\Delta_e)\Delta_e \\ \Delta_e^{-1}\mathbb{D}(\Delta_e) & \Delta_e^{-1}\mathbb{D}(\Delta_e)\Delta_e\end{pmatrix}\begin{pmatrix} S & \\ & S^{-1}\end{pmatrix}\\ &= \begin{pmatrix} S^{-1}\mathbb{D}(\Delta_e)S & S^{-1}\mathbb{D}(\Delta_e)\Delta_e S^{-1} \\ S\Delta_e^{-1}\mathbb{D}(\Delta_e)S & S\Delta_e^{-1}\mathbb{D}(\Delta_e)\Delta_e S^{-1}\end{pmatrix}\\ &= \begin{pmatrix} S^{-1}\mathbb{D}(\Delta_e)S & (S^{-1}\mathbb{D}(\Delta_e)S)(S^{-1}\Delta_e S^{-1}) \\ (S\Delta_e^{-1}S)(S^{-1}\mathbb{D}(\Delta_e)S) & (S\Delta_e^{-1}S)(S^{-1}\mathbb{D}(\Delta_e)S)(S^{-1}\Delta_e S^{-1})\end{pmatrix}\end{aligned}$$

and with relation (3.12) this implies

$$\subset \begin{pmatrix}\mathbb{D}(\Delta_d) & \mathbb{D}(\Delta_d)\Delta_d \\ \Delta_d^{-1}\mathbb{D}(\Delta_d) & \Delta_d^{-1}\mathbb{D}(\Delta_d)\Delta_d\end{pmatrix}.$$

Since on the other hand $\Gamma_{\mathrm{pol},e} \subset \mathrm{Sp}(2g,\mathbb{Q})$ and $T, T^{-1} \in \mathrm{Sp}(2g,\mathbb{Q})$, we may use Lemma 3.2.10 to conclude

$$T^{-1}\Gamma_{\mathrm{pol},e}T \subset \begin{pmatrix}\mathbb{D}(\Delta_d) & \mathbb{D}(\Delta_d)\Delta_d \\ \Delta_d^{-1}\mathbb{D}(\Delta_d) & \Delta_d^{-1}\mathbb{D}(\Delta_d)\Delta_d\end{pmatrix} \cap \mathrm{Sp}(2g,\mathbb{Q}) = \Gamma_{\mathrm{pol},d}.$$

The relation $U^{-1}\tilde{\Gamma}_{\mathrm{pol},e}U \subset \tilde{\Gamma}_{\mathrm{pol},d}$ follows immediately from the previous, since by definition $\Gamma_{\mathrm{pol}} = R^{-1}\tilde{\Gamma}_{\mathrm{pol}}R$ and

$$\begin{aligned} & T^{-1}\Gamma_{\mathrm{pol},e}T && \subset \Gamma_{\mathrm{pol},d}\\ \iff \quad & T^{-1}(R_e^{-1}\tilde{\Gamma}_{\mathrm{pol},e}R_e)T && \subset R_d^{-1}\tilde{\Gamma}_{\mathrm{pol},d}R_d\\ \iff \quad & (R_dT^{-1}R_e^{-1})\tilde{\Gamma}_{\mathrm{pol},e}(R_eTR_d^{-1}) && \subset \tilde{\Gamma}_{\mathrm{pol},d}\end{aligned}$$

and indeed

$$R_dT^{-1}R_e^{-1} = \begin{pmatrix}\mathbb{1} & \\ & \Delta_d\end{pmatrix}\begin{pmatrix}S^{-1} & \\ & S\end{pmatrix}\begin{pmatrix}\mathbb{1} & \\ & \Delta_e^{-1}\end{pmatrix} = \begin{pmatrix}S^{-1} & \\ & \Delta_d S\Delta_e^{-1}\end{pmatrix} = U^{-1}.$$

For the relation $T^{-1}\Gamma^{\text{lev}}_{\text{pol},e}T \subset \Gamma^{\text{lev}}_{\text{pol},d}$ we first note that the first relation we proved implies that

$$T^{-1}\Gamma^{\text{lev}}_{\text{pol},e}T \subset T^{-1}\Gamma_{\text{pol},e}T \subset \Gamma_{\text{pol},d},$$

so that we only need to show the additional conditions imposed by Lemma 3.2.12. This lemma states that the matrices $M \in T^{-1}\Gamma^{\text{lev}}_{\text{pol},e}T$ are those matrices of $\Gamma_{\text{pol},e}$ that have the form

$$M \in \begin{pmatrix} S^{-1} & \\ & S \end{pmatrix} \left(\begin{pmatrix} {}^t\mathfrak{e} \\ 1_g \end{pmatrix} (1_g, \mathfrak{e}) \otimes \mathbb{Z} + \mathbb{1} \right) \begin{pmatrix} S & \\ & S^{-1} \end{pmatrix} = \begin{pmatrix} S^{-1}\,{}^t\mathfrak{e} \\ S \end{pmatrix} (S, \mathfrak{e}S^{-1}) \otimes \mathbb{Z} + \mathbb{1}$$

where $\mathfrak{e} = (1, e_1, \ldots, e_{1:g-1})$. Since $e_i s_i^{-1} = d_i s_i$ for all $i = 1, \ldots, g-1$ this means

$$M \in \begin{pmatrix} S\,{}^t\mathfrak{d} \\ S \end{pmatrix} (S, \mathfrak{d}S) \otimes \mathbb{Z} + \mathbb{1} \subset \begin{pmatrix} {}^t\mathfrak{d} \\ 1_g \end{pmatrix} (1_g, \mathfrak{d}) \otimes \mathbb{Z} + \mathbb{1}$$

Hence, all these matrices also satisfy the conditions of $\Gamma^{\text{lev}}_{\text{pol},d}$.

The last relation $U^{-1}\tilde{\Gamma}^{\text{lev}}_{\text{pol},e}U \subset \tilde{\Gamma}^{\text{lev}}_{\text{pol},d}$ follows from this with the same argument as the second relation from the first. □

Remark 3.2.15.

Since according to Corollary 3.2.13 the groups $\Gamma^{\text{lev}}_{\text{pol}}$ associated with both polarisations are integer matrices and

$$d_i | e_i \quad \Longrightarrow \quad e_{i:j}\mathbb{Z} \subset d_{i:j}\mathbb{Z},$$

we also have the relation $\Gamma^{\text{lev}}_{\text{pol},e} \subset \Gamma^{\text{lev}}_{\text{pol},d}$.

3.3 Orbits under the group actions

3.3.1 Orbits of isotropic lines under $\tilde{\Gamma}^{\text{lev}}_{\text{pol}}$

We are now interested in the orbits of vectors $v \in \mathbb{Z}^{2g}$ under the action of $\tilde{\Gamma}_{\text{pol}}$ and its subgroup $\tilde{\Gamma}^{\text{lev}}_{\text{pol}}$. In other words, we want to find a standard form ($\tilde{v}$ or $\hat{v}$, respectively) which selects a unique vector from each orbit. In a corollary we will then collect these standard vectors into a set of representatives. Let us first consider the smaller group.

Lemma 3.3.1: Orbits of isotropic lines under $\tilde{\Gamma}^{\text{lev}}_{\text{pol}}$.

(i) Under the action of $\tilde{\Gamma}^{\text{lev}}_{\text{pol}}$, every vector $v \in \mathbb{Z}^{2g}$ can be transformed into

$$\tilde{v} = (D_{1:g-1}(v), *, \ldots, *, 0, *, \ldots, *)$$

where the given 0 is at the $g+1$st place.

(ii) Two vectors $v, w \in \mathbb{Z}^{2g}$ are conjugateunder $\tilde{\Gamma}^{\text{lev}}_{\text{pol}}$ if and only if

$$D_{1:g-1}(v) = D_{1:g-1}(w) \quad \text{and} \quad \forall i = 1, \ldots, 2g : v_i \equiv w_i \mod D_{1:g-1}.$$

This means that we may chose the entries of the vector $\tilde{v}$ such that they satisfy the property $0 \le \tilde{v}_i < D_{1:g-1}$ for all $i = 2, \ldots, 2g$.

Proof.

Part (i):
Let us first prove that v can be transformed into a $\tilde{v}$ of the given form. We shall again use the notation introduced in Notation 3.1.4.

Since v is primitive, not all entries v_i are zero. Hence we can assume (if necessary after a suitable transformation with a matrix in $\tilde{\Gamma}^{\text{lev}}_{\text{pol}}$) that $v_1 \neq 0$. For $i = 1, \dots, g$ define

$$x_i := \frac{d_{1:i-1} v_i}{D_{1:g-1}} \quad \text{and} \quad x_{g+i} := \frac{d_{1:i-1} v_{g+i}}{D_{1:g-1}}.$$

These x_i are all integers, since according to Lemma 3.1.6 $D_{1:i-1} | d_{1:i-1}$ and $D_{i:g-1} | v_{i|g+i}$.

The same lemma tells us that $\frac{v_{j|g+j}}{D_{j:k}} \in \mathbb{Z}$ for $j \leq k < g$ and so it makes sense to say

$$\gcd\left(\frac{v_{j|g+j}}{D_{j:g-1}}, \frac{d_i}{D_i}\right)_{j=1,\dots,i} \text{ divides } \gcd\left(\frac{v_{j|g+j}}{D_{j:i}}, \frac{d_i}{D_i}\right)_{j=1,\dots,i}.$$

Since by Lemma 3.1.5 the second $\gcd(\dots) = 1$, this implies

$$\gcd\left(\frac{v_{j|g+j}}{D_{j:g-1}}, \frac{d_i}{D_i}\right)_{j=1,\dots,i} = 1. \tag{3.13}$$

Now we have

$$I := (x_1, \dots, x_{2g}) = \left(\frac{v_{1|g+1}}{D_{1:g-1}}, \frac{d_1}{D_1}\frac{v_{2|g+2}}{D_{2:g-1}}, \dots, \frac{d_1}{D_1}\frac{d_{2:g-1} v_{g|2g}}{D_{2:g-1}}\right) \tag{3.14}$$

and using (3.13) for $i = 1$, i. e. $\gcd(\frac{v_{1|g+1}}{D_{1:g-1}}, \frac{d_1}{D_1}) = 1$, we may drop the terms $\frac{d_1}{D_1}$ to get

$$I = \left(\frac{v_{1|g+1}}{D_{1:g-1}}, \frac{v_{2|g+2}}{D_{2:g-1}}, \dots, \frac{d_{2:g-1} v_{g-1|2g-1}}{D_{2:g-1}}, \frac{d_{2:g-1} v_{g|2g}}{D_{2:g-1}}\right).$$

Iterating this process using (3.13) for $i = 2, \dots, g-1$ we may cancel all the factors $\frac{d_i}{D_i}$ to obtain

$$I = \left(\frac{v_{1|g+1}}{D_{1:g-1}}, \frac{v_{2|g+2}}{D_{2:g-1}}, \dots, \frac{v_{g-1|2g-1}}{D_{g-1}}, v_{g|2g}\right),$$

and since $\gcd(v_1, \dots, v_{2g}) = 1$ we have $I = (1) = \mathbb{Z}$. With Lemma A.2 we can find λ_i such that

$$\Big(x_1, x_{g+1} + \sum_{\substack{i=2,\dots,2g \\ i \neq g+1}} \lambda_i x_i\Big) = (1) \quad \text{or equivalently}$$

$$\Big(v_1, v_{g+1} + \sum_{i=2,\dots,g} (\lambda_i d_{1:i-1} v_i + \lambda_{g+i} d_{1:i-1} v_{g+i})\Big) = (D_{1:g-1}).$$

The matrix

$$M := \left(\begin{array}{cccc|cccc} 1 & -\lambda_{g+2} & \dots & -\lambda_{2g} & 0 & \lambda_2 & \dots & \lambda_g \\ & 1 & & & d_1\lambda_2 & & & \\ & & \ddots & & \vdots & & & \\ & & & 1 & d_{1:g-1}\lambda_g & & & \\ \hline & & & & 1 & & & \\ & & & & d_1\lambda_{g+2} & 1 & & \\ & & & & \vdots & & \ddots & \\ & & & & d_{1:g-1}\lambda_{2g} & & & 1 \end{array}\right)$$

is in $\tilde{\Gamma}_{\text{pol}}^{\text{lev}}$ according to Lemma 3.2.11. The entries of the vector $v' := vM$ satisfy by construction of λ_i the relation $\gcd(v'_1, v'_{g+1}) = D_{1:g-1}$. Therefore there exist $t_1, t_2 \in \mathbb{Z}$ with $t_1 v'_1 + t_2 v'_{g+1} = D_{1:g-1}$, and the matrix

$$N := \left(\begin{array}{cccc|cccc} t_1 & & & & \frac{-v'_{g+1}}{D_{1:g-1}} & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline t_2 & & & & \frac{v'_1}{D_{1:g-1}} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) \in \tilde{\Gamma}_{\text{pol}}^{\text{lev}}$$

transforms v' into a vector of the form $\tilde{v} = v'N = (D_{1:g-1}, *, \ldots, *, 0, *, \ldots, *)$.

Part (ii):

Necessity:

Since the divisors are invariant under the action of $\tilde{\Gamma}_{\text{pol}}^{\text{lev}}$, we must have $D_i(v) = D_i(w)$ for conjugatevectors v and w which obviously implies $D_{1:g-1}(v) = D_{1:g-1}(w)$.

Now, let $M \in \tilde{\Gamma}_{\text{pol}}^{\text{lev}}$ such that $w = vM$ and let $k \in \{1, \ldots, 2g\}$. We have

$$w_k = \sum_{1}^{2g} m_{ik} v_i.$$

To keep the notation easier we only consider the case $k \leq g$, the other case $g < k \leq 2g$ can be treated similarly. From Lemma 3.1.6 we know that $D_{i:g-1}$ divides v_i and v_{g+i}. Lemma 3.2.11 shows that $d_{1:i-1}$ and thus also $D_{1:i-1}$ divides both m_{ik} and $m_{g+i,k}$ for all $i = 1, \ldots, g$ except for the element of the diagonal which can be written as $m_{kk} = d_{1:k-1}n + 1$ for some $n \in \mathbb{Z}$. Hence we have

$$\begin{aligned} w_k &= \sum_{\substack{i=1,\\ i\neq k}}^{g} m_{ik} v_i + \sum_{i=1}^{g} m_{g+i,k} v_{g+i} + m_{kk} v_k \\ &= \sum_{\substack{i=1,\\ i\neq k}}^{g} (D_{1:i-1} m'_{ik} D_{i:g-1} v'_i) + \sum_{i=1}^{g} (D_{1:i-1} m'_{g+i,k} D_{i:g-1} v'_{g+i}) + (d_{1:k-1}n + 1) v_k \\ &\equiv 0 + d_{1:k-1} n D_{k:g-1} \tfrac{v_k}{D_{k:g-1}} + v_k \mod D_{1:g-1} \quad \text{and since } \tfrac{v_k}{D_{k:g-1}} \in \mathbb{Z} \\ &\equiv v_k \mod D_{1:g-1}. \end{aligned}$$

Sufficiency:

Due to part (i) we can assume v and w to be given in the form

$$\begin{aligned} v &= (D_{1:g-1}(v), v_2, \ldots, v_g, 0, v_{g+2}, \ldots, v_{2g}) \quad \text{and} \\ w &= (D_{1:g-1}(w), w_2, \ldots, w_g, 0, w_{g+2}, \ldots, w_{2g}), \end{aligned}$$

where $v_1 = w_1$. Let $D_i := D_i(v) = D_i(w)$. Since $v_i \equiv w_i \mod D_{1:g-1}$, let n_i be defined by

$$w_i = n_i D_{1:g-1} + v_i \quad \text{for} \quad i = 2, \ldots, g, g+2, \ldots, 2g.$$

The matrix M defined as

$$M := \left(\begin{array}{cccc|cccc} 1 & n_2 & \cdots & n_g & 0 & n_{g+2} & \cdots & n_{2g} \\ & 1 & & & d_{1:1}n_{g+2} & & & \\ & & \ddots & & \vdots & & & \\ & & & 1 & d_{1:g-1}n_{2g} & & & \\ \hline & & & & 1 & & & \\ & & & & -d_{1:1}n_2 & 1 & & \\ & & & & \vdots & & \ddots & \\ & & & & -d_{1:g-1}n_g & & & 1 \end{array}\right) \in \tilde{\Gamma}_{\mathrm{pol}}^{\mathrm{lev}}$$

transforms v into $vM = (w_1, \dots, w_g, \overline{v}_{g+1}, w_{g+2}, \dots, w_{2g})$ where

$$\overline{v}_{g+1} = d_{1:1}n_{g+2}v_2 + \cdots + d_{1:g-1}n_{2g}v_g + 0 - d_{1:1}n_2v_{g+2} - \cdots - d_{1:g-1}n_gv_{2g}.$$

Since $D_{i:g-1}$ divides v_i and v_{g+i} according to Lemma 3.1.6, we know that $D_{1:g-1}$ divides every term of $\overline{v}_{g+1}$ and thus $\gcd(D_{1:g-1}, \overline{v}_{g+1}) = D_{1:g-1}$. This implies that we can find a matrix N as in part (i) of this proof which transforms vM into w and thus v and w are conjugateunder $\tilde{\Gamma}_{\mathrm{pol}}^{\mathrm{lev}}$. □

Corollary 3.3.2: Set of representatives.
A set of representatives for the orbits of $\tilde{\Gamma}_{\mathrm{pol}}^{\mathrm{lev}}$ is given by the vectors

$$\tilde{v} = (D_{1:g-1}, D_{2:g-1}a_2, D_{3:g-1}a_3, \dots, a_g, 0, D_{2:g-1}a_{g+2}, D_{3:g-1}a_{g+3}, \dots, a_{2g})$$

where $\{D_i\}$ runs through the set of all possible divisors (see Theorem 3.1.13) and

$$0 \le a_i < D_{1:i-1}, \quad 0 \le a_{g+i} < D_{1:i-1} \quad \text{for } i = 2, \dots, g.$$

Proof.
This follows easily from the above Lemma 3.3.1 considering Lemma 3.1.6, which states that $D_{i:g-1} | v_{i|g+i}$, and using Theorem 3.1.13 for the restrictions on $\{D_i\}$. □

3.3.2 Orbits of isotropic lines under $\tilde{\Gamma}_{\mathrm{pol}}$

In this section we proceed analogously to the last section: first we give a standard form for a vector under the action of $\tilde{\Gamma}_{\mathrm{pol}}$ which we will denote by $\hat{v}$ and then collect these vectors into a set of representatives.

Definition 3.3.3: Representative vector for $\tilde{\Gamma}_{\mathrm{pol}}$.
For $v = (v_1, \dots, v_{2g}) \in \mathbb{Z}^{2g}$ and $i = 1, \dots, g$, let

$$\hat{v}_i := \gcd(v_{1|g+1}, \dots, v_{i|g+i}, d_i v_{i+1|g+i+1}, \dots, d_{i:g-1}v_{g|2g}) \quad \text{and}$$

$$\hat{v} := (\hat{v}_1, \dots, \hat{v}_g, 0, \dots, 0) \in \mathbb{Z}^{2g}.$$

In this form, adjacent entries are related in the following ways:

Lemma 3.3.4: Properties of $\hat{v}_i$.
For all primitive $v \in \mathbb{Z}^{2g}$, the $\hat{v}_i$ satisfy the following relations:

(i) $\hat{v}_1 = D_{1:g-1}(v)$ and $\hat{v}_g = 1$

(ii) $\forall i = 1, \dots, g-1 \;:\; \hat{v}_i | d_i \hat{v}_{i+1}$

(iii) $\forall i = 2, \dots, g \;:\; \hat{v}_i | \hat{v}_{i-1}$

(iv) $\forall i = 2, \dots, g-1 \;:\; \hat{v}_i = \gcd(\hat{v}_{i-1}, v_{i|g+i}, d_i \hat{v}_{i+1})$

Proof.
Part (i):
From Lemma 3.1.6 we already know that $D_{1:g-1}|\hat{v}_1$. The definition of $\hat{v}_1$ immediately gives $(\frac{\hat{v}}{D_{1:g-1}}) = I$ with I defined as in equation (3.14) on page 55. We have already proved that $I = (1)$ and hence we have $\hat{v}_1 = D_{1:g-1}$ as claimed.

Since v is primitive, we have $\hat{v}_g = \gcd(v_1, \dots, v_{2g}) = 1$.

Part (ii) and (iii):
These follow immediately from comparing the elements of the greatest common divisors in the definitions of $\hat{v}_i$ and $\hat{v}_{i+1}$ or $\hat{v}_{i-1}$, respectively.

Part (iv):
Define $v'_i := \gcd(\hat{v}_{i-1}, v_{i|g+i}, d_i \hat{v}_{i+1})$. We want to show $v'_i = \hat{v}_i$. Obviously,

$$\left.\begin{array}{rcc} \text{part (iii)} & \Longrightarrow & \hat{v}_i | \hat{v}_{i-1} \\ \text{definition of } \hat{v}_i & \Longrightarrow & \hat{v}_i | v_{i|g+i} \\ \text{part (ii)} & \Longrightarrow & \hat{v}_i | d_i \hat{v}_{i+1} \end{array}\right\} \Longrightarrow \hat{v}_i | v'_i.$$

On the other hand, from the definition of v'_i above and $\hat{v}_i$ in Definition 3.3.3 we immediately obtain

$$\begin{array}{rl} v'_i|\hat{v}_{i-1} & \Longrightarrow v'_i|\gcd(v_{1|g+1}, \dots, v_{i-1|g+i-1}) \\ \text{and } v'_i|v_{i|g+i} & \\ \text{and } v'_i|d_i\hat{v}_{i+1} & \Longrightarrow v'_i|d_i \gcd(v_{i+1|g+i+1}, d_{i+1}v_{i+2|g+i+2}, \dots, d_{i+1:g-1}v_{g|2g}) \\ & \Longrightarrow v'_i|\gcd(d_i v_{i+i|g+i+1}, d_{i:i+1}v_{i+2|g+i+2}, \dots, d_{i:g-1}v_{g|2g}) \end{array}$$

which in turn implies

$$v'_i|\gcd(v_{1|g+1}, \dots, v_{i-1|g+i-1}, v_{i|g+i}, d_i v_{i+1|g+i+1}, \dots, d_{i:g-1}v_{g|2g}) = \hat{v}_i.$$

Since both are positive integers, this proves equality. □

We now show that there is a unique $\hat{v}$ in each orbit.

Lemma 3.3.5: Orbits of isotropic lines under $\tilde{\Gamma}_{\text{pol}}$.
Let $\sim$ denote congruence with respect to the action of $\tilde{\Gamma}_{\text{pol}}$. Then

(i) $v \sim \hat{v}$.

(ii) $v \sim w \iff \hat{v} = \hat{w}$ (here we have equality, not only congruence)

(iii) $\widehat{(\hat{v})} = \hat{v}$

Proof.

Part (i):

We prove congruence by giving matrices that transform v into $\hat{v}$ iteratively. In the ith step the ith component of the vector will become $\hat{v}_i$ whereas the $(g+i)$th component will become zero. The existence of such matrices is shown by induction.

For the first step we refer to Lemma 3.3.1 where it has already been done using a matrix $M \in \tilde{\Gamma}_{\mathrm{pol}}^{\mathrm{lev}} \subset \tilde{\Gamma}_{\mathrm{pol}}$. For the other steps we shall now construct matrices in a similar way. Assume that we have completed the first $(i-1)$st steps and hence have a vector of the form

$$v = (\hat{v}_1, \ldots, \hat{v}_{i-1}, v_i, \ldots, v_g, 0, \ldots, 0, v_{g+i}, \ldots, v_{2g}).$$

Lemma A.2 tells us that we can find λ_j such that

$$\begin{aligned}
\gcd\Big(v_{g+i}, v_i + \sum_{\substack{j=1,\ldots,g\\ j\neq i}} \lambda_j d_{i:j-1} v_j + \sum_{\substack{j=1,\ldots,g\\ j\neq i}} \lambda_{g+j} d_{i:j-1} v_{g+j}\Big) \\
&= \gcd\big(v_{1|g+1}, \ldots, v_{i|g+i}, d_i v_{i+1|g+i+1}, \ldots, d_{i:g-1} v_{g|2g}\big) \\
&= \hat{v}_i \quad \text{by definition.}
\end{aligned}$$

Since $v_{g+1} = \cdots = v_{g+i-1} = 0$ we may obviously choose $\lambda_{g+1} = \cdots = \lambda_{g+i-1} = 0$. Now we can define the matrix $M_i := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where

$$A := \begin{pmatrix}
1 & & & \lambda_1 & & & \\
& \ddots & & \vdots & & & \\
& & 1 & \lambda_{i-1} & & & \\
& & & 1 & & & \\
& & & d_i\lambda_{i+1} & 1 & & \\
& & & \vdots & & \ddots & \\
& & & d_{i:g-1}\lambda_g & & & 1
\end{pmatrix},$$

$$B := \begin{pmatrix}
\lambda_1 \frac{d_{1:i-1} v_{g+i}}{\hat{v}_1} & & & & & \\
& \ddots & & & & \\
& & \lambda_{i-1} \frac{d_{i-1} v_{g+1}}{\hat{v}_{i-1}} & & & \\
& & & 0 & & \\
& & & & \ddots & \\
& & & & & 0
\end{pmatrix}$$

which is an integer matrix according to Definition 3.3.3,

$$C := \begin{pmatrix}
& & 0 & & & \\
& & \vdots & & & \\
0 & \ldots & 0 & \lambda_{g+i+1} & \ldots & \lambda_{2g} \\
& & d_i\lambda_{g+i+1} & & & \\
& & \vdots & & & \\
& & d_{i:g-1}\lambda_{2g} & & &
\end{pmatrix}$$

where the zeroes outside the diagonal are $\lambda_{g+1}, \dots, \lambda_{g+i-1}$ and

$$D := \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ -d_{1:i-1}\lambda_1 & \dots & -d_{i-1}\lambda_{i-1} & 1 & -\lambda_{i+1} & \dots & -\lambda_g \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

By Lemma 3.2.1, $M \in \tilde{\Gamma}_{\text{pol}}$. Define $w := vM$. Then obviously $w_j = \hat{v}_j$ and $w_{g+j} = 0$ for $1 \le j < i$. The definition of λ_j makes sure that $\gcd(w_i, w_{g+i}) = \hat{v}_i$. For $i < j \le g$ and $g+i < j \le 2g$ we have $w_j = v_j \pm \lambda_k v_{g+i}$ with the appropriate index k. But nevertheless $\hat{w} = \hat{v}$ since v_{g+i} is part of the gcd for all $\hat{v}_j$ with $j > i$.

We can now find integers t_1, t_2 such that $w_i t_1 + w_{g+i} t_2 = \hat{v}_i$. Define the matrix N to be the unit matrix except for the four entries

$$\begin{pmatrix} n_{i,i} & n_{i,g+i} \\ n_{g+i,i} & n_{g+i,g+i} \end{pmatrix} = \begin{pmatrix} t_1 & -\frac{w_{g+i}}{\hat{\gamma}} \\ t_2 & \frac{w_i}{\hat{\gamma}} \end{pmatrix}.$$

Then $N \in \tilde{\Gamma}_{\text{pol}}$ and $v' := wN$ has the properties that $v'_i = \hat{v}_i$ and $v'_{g+i} = 0$. This completes the induction.

Part (ii):

$\Leftarrow$**:** Using (i), we immediately obtain

$$v \sim \hat{v} = \hat{w} \sim w.$$

$\Rightarrow$**:** Since we know from part (i) that $\hat{v}$ is conjugateto v, we assume v and w to be of the form $\hat{v}$ and $\hat{w}$, respectively. Since $v \sim w$ there exists a matrix $M \in \tilde{\Gamma}_{\text{pol}}$ such that $w = vM$. We will show that $\hat{v}_j$ divides $\hat{w}_j$ for all $j = 1, \dots, g$.

Fix $j \in \{1, \dots, g\}$. We have

$$\begin{aligned} \hat{v}_j &= \gcd(v_1, \dots, v_j, d_j v_{j+1}, \dots, d_{j:g-1} v_g) \quad \text{and} \\ \hat{w}_j &= \gcd(w_1, \dots, w_j, d_j w_{j+1}, \dots, d_{j:g-1} w_g) \\ &= \gcd\left(\sum_{i=1}^{g} m_{i,1} v_i, \dots, \sum_{i=1}^{g} m_{i,j} v_i, d_j \sum_{i=1}^{g} m_{i,j+1} v_i, \dots, d_{j:g-1} \sum_{i=1}^{g} m_{i,g} v_i\right). \end{aligned}$$

Consider a single entry in this gcd and denote it by

$$W_k = \begin{cases} \sum_{i=1}^{g} m_{ik} v_i & \text{for } 1 \le k \le j \\ d_{j:k-1} \sum_{i=1}^{g} m_{ik} v_i & \text{for } j < k \le g \end{cases}.$$

Lemma 3.2.9 tells us that $m_{ik} = d_{k:i-1} m'_{ik}$ if $k < i$ and hence we can rewrite this as follows: For $1 \le k \le j$ we have

$$W_k = \sum_{i=1}^{k} m_{ik} v_i + \sum_{i=k+1}^{j} (d_{k:i-1} m'_{ik}) v_i + \sum_{i=j+1}^{g} (d_{k:j-1} d_{j:i-1} m'_{ik}) v_i.$$

The summands in the first two sums each contain the factor v_i with $i \leq j$; the summands of the last sum the factors $d_{j:i-1}v_i$ with $i > j$. On the other hand, for $j < k \leq g$ we have

$$W_k = d_{j:k-1}\sum_{i=1}^{j} m_{ik}v_i + \sum_{i=j+1}^{k}(d_{j:i-1}d_{i:k-1})m_{ik}v_i + \sum_{i=k+1}^{g} d_{j:k-1}(d_{k:i-1}m'_{ik})v_i.$$

Again, the summands of the first sum contain the factor v_i with $i \leq j$ and the other summands contain $d_{j:i-1}v_i$ with $i > j$. Thus, each W_k is a multiple of $\gcd(v_1,\ldots,v_j,d_{j:j}v_{j+1},\ldots,d_{j:g-1}v_g) = \hat{v}_j$ and therefore $\hat{v}_j|\hat{w}_j$.

Since $M^{-1} \in \tilde{\Gamma}_{\text{pol}}$ and $v = wM^{-1}$ we now also know that $\hat{w}_j$ divides $\hat{v}_j$ for all $j = 1,\ldots,g$, thus $\hat{v} = \hat{w}$.

Part (iii):
From part (i) we know that $v \sim \hat{v}$, and so setting $w = \hat{v}$ in part (ii) proves the statement.
□

Again, the interesting consequence is the set of vectors representing all orbits.

Corollary 3.3.6: Set of representatives.
A set of representatives for the orbits of $\tilde{\Gamma}_{\text{pol}}$ is given by the vectors

$$\hat{v} = (D_{1:g-1}, D_{2:g-1}a_2, D_{3:g-1}a_3, \ldots, D_{g-1}a_{g-1}, 1, 0, \ldots, 0) \in \mathbb{Z}^{2g}$$

where $\{D_i\}$ runs through the set of possible divisors (see Theorem 3.1.13) and $a_i \geq 0$ with

$$a_i|\gcd(D_{i-1}a_{i-1}, \tfrac{d_i}{D_i}a_{i+1}) \quad \text{for } i = 2,\ldots,g-1$$

where we let $a_1 = a_g = 1$.

Proof.
We first show that the vectors $\hat{v}$ defined in Definition 3.3.3 can indeed be given in the form stated above.

The factors $D_{i:g-1}$ must be present because of the divisibility conditions in Lemma 3.1.6. Define $a_i := \frac{\hat{\gamma}}{D_{1:g-1}}$. The values for a_1 and a_g follow from Lemma 3.3.4 part (i). Then Lemma 3.3.4 part (iv) shows for $i = 2,\ldots,g-1$

$$\begin{aligned} &D_{i:g-1}a_i|\gcd(D_{i-1:g-1}a_{i-1}, d_iD_{i+1:g-1}a_{i+1}) \\ \iff\ &a_i|\gcd(D_{i-1}a_{i-1}, \tfrac{d_i}{D_i}a_{i+1}). \end{aligned}$$

The fact that this is indeed a set of representatives follows from the just established Lemma 3.3.5. □

Corollary 3.3.7: Coprime polarisations.
If the polarisation is coprime then $a_i = 1$ for all $i = 1,\ldots,g$. In particular, the orbits can be represented by the vectors

$$\begin{aligned} \hat{v} &= (D_{1:g-1}, D_{2:g-1}, \ldots, D_{g-1}, 1, 0, \ldots, 0) \quad \text{or even} \\ \hat{\hat{v}} &= (D_{1:g-1}, 0, \ldots, 0, 1, 0, \ldots, 0) \end{aligned}$$

where D_i divides d_i.

Proof.
We claim that

$$a_i \mid \gcd(D_{1:i-1}, a_{i+1}) \qquad \text{for } i = 2, \dots, g-1 \tag{3.15}$$

and prove this by induction over i.

For $i = 2$ Corollary 3.3.6 states that

$$\begin{aligned} a_2 \mid \gcd(D_1 a_1, \frac{d_2}{D_2} a_3) \quad & \text{and } a_1 = 1 \text{ leads to} \\ = \gcd(D_1, \frac{d_2}{D_2} a_3) \quad & \text{and since } \gcd(d_1, d_2) = 1 \text{ we have} \\ = \gcd(D_1, a_3) & \end{aligned}$$

so statement (3.15) is true.

Assume now that (3.15) is true for a given i. Then Corollary 3.3.6 states

$$\begin{aligned} & a_{i+1} \mid \gcd(D_i a_i, \tfrac{d_{i+1}}{D_{i+1}} a_{i+2}) \\ \Longrightarrow\ & a_{i+1} \mid \gcd(D_i \gcd(D_{1:i-1}, a_{i+1}), \tfrac{d_{i+1}}{D_{i+1}} a_{i+2}) \\ & \qquad = \gcd(D_{1:i}, D_i a_{i+1}, \tfrac{d_{i+1}}{D_{i+1}} a_{i+2}) \\ \Longrightarrow\ & a_{i+1} \mid \gcd(D_{1:i}, \tfrac{d_{i+1}}{D_{i+1}} a_{i+2}) \quad \text{and since } \gcd(D_{1:i}, d_{i+1}) = 1 \\ \Longrightarrow\ & a_{i+1} \mid \gcd(D_{1:i}, a_{i+2}). \end{aligned}$$

This completes the induction and thus (3.15) holds for all $i = 2, \dots, g-1$.

Now we can use the fact that $a_g = 1$ which implies recursively that $a_i = 1$ for all $i = g-1, \dots, 2$. This is what we wanted to show.

The second form now follows easily from the fact that, according to Corollary 3.1.14, the value $D_{1:g-1}(v)$ determines all $D_i(v)$ uniquely, and so $\hat{\hat{v}}$ and $\hat{v}$ are in the same orbit under the action of $\tilde{\Gamma}_{\text{pol}}$.

Note that Theorem 3.1.13 does not imply any restrictions on the D_i since for all $D_i | d_i$ and $i \neq j$

$$\gcd(\tfrac{d_i}{D_i}, d_j) \text{ divides } \gcd(d_i, d_j) = 1.$$

□

3.3.3 Orbits of isotropic g-spaces under $\tilde{\Gamma}_{\text{pol}}$

Let us now consider only polarisations that are square-free and coprime. For these polarisations we prove that $\tilde{\Gamma}_{\text{pol}}$ acts transitively on the g-dimensional isotropic subspaces of $\mathbb{Q}^{2g}$.

In order to do this we consider primitive integer vectors $v^1, \dots, v^g$ that generate an isotropic subspace $h = v^1 \wedge \cdots \wedge v^g \subset \mathbb{Q}^{2g}$. We may restrict the discussion to those sets of vectors that form a $\mathbb{Z}$-basis of $h_{\mathbb{Z}} := h \cap \mathbb{Z}^{2g}$, in other words $h_{\mathbb{Z}} = \bigoplus \mathbb{Z} v^i$. In this case primitivity with respect to $h_{\mathbb{Z}}$ implies primitivity with respect to $\mathbb{Z}^{2g}$.

The main point of the proof is that any $h_{\mathbb{Z}}$ of rank g has a basis satisfying the following property:

$$D_{1:g-1}(v^i) = d_{1:i-1} \quad \text{for all } i = 1, \dots, g. \tag{3.16}$$

To construct such a basis for any given h we use two basic transformations:

- The operation of $\gamma \in \tilde{\Gamma}_{\text{pol}}$ on all of the v^i, so that $\tilde{v}^i := \gamma(v^i)$ for all $i = 1, \dots, g$. A priori we may not assume $\tilde{h} := \gamma(h)$ to be equal to h. However, since the D_i are invariant under the operation of $\tilde{\Gamma}_{\text{pol}}$, we can find a basis of h satisfying property (3.16) if and only if we can find such a basis of $\tilde{h}$.

- A linear combination of basis vectors of h given by a unimodular matrix. According to Lemma A.3 we have $\tilde{h}_{\mathbb{Z}} = h_{\mathbb{Z}}$. Furthermore, we have the following property: assume that the basis transformation only involves $2 \le n \le g$ of the vectors and assume further without loss of generality that these are the vectors $v^1, \dots, v^n$. Then Lemma A.4 shows that

$$\gcd\left(D_k(\tilde{v}^1), \dots, D_k(\tilde{v}^n)\right) = \gcd\left(D_k(v^1), \dots, D_k(v^n)\right)$$

  for any $1 \le k \le g - 1$.

- Note that we can regard the transformation $\tilde{v}^i := v^i + \lambda v^j$ and $\tilde{v}^j := v^j$ with $\lambda \in \mathbb{Z}$ as a special case of the aforementioned linear combination.

We first state a property on n-tuples of the basis vectors.

Proposition 3.3.8.
Let $h \subset \mathbb{Q}^{2g}$ be an isotropic subspace and $v^1, \dots, v^g$ a $\mathbb{Z}$-basis of $h_{\mathbb{Z}}$. Let $2 \le n \le g$ and $1 \le i_1, \dots, i_n \le g$ a set of n distinct indices. Then

$$\gcd\left(D_k(v^{i_1}), \dots, D_k(v^{i_n})\right) = 1 \quad \text{for all } k \ge g - n + 1.$$

Remark 3.3.9.
For $g = 2$, this proposition is the same as [FS, Lemma 2.4], and indeed the following arguments are also given by Friedland and Sankaran in this case. For $g \le 9$, a proof is given below. For large g, although we still conjecture the proposition to be true, neither proof nor counterexample have been found yet.

Proof of Proposition 3.3.8 for $g \le 9$.
Before we explain the idea of the proof let us begin with a few observations that are true for any g. Since the order of the vectors is irrelevant for the gcd, we may assume $i_j = j$ for all $j = 1, \dots, n$. The claim of the lemma is obviously implied by the statement

$$m_k := \gcd\left(D_k(v^1), \dots, D_k(v^n)\right) = 1 \quad \text{for } k = g - n + 1 \tag{3.17}$$

since higher values for k mean smaller values for n and hence we have that a set of fewer D_k is already coprime.

Now, the basic idea is to show that we can construct a basis vector w with the property that m_k divides every entry. Since basis vectors are primitive, this implies that $m_k = 1$ as claimed. To construct this vector w we use the basic transformations mentioned above, none of which change the value m_k. The proof we give here is in fact independent of g. The first part where we bring roughly half of the basis vectors into a standard form is proved in full generality but the second part requires n to be small enough. It has not yet been possible to apply the same techniques to higher n. Therefore, Proposition 3.3.8 is only proved in full for $g \le 9$.

We shall write the basis vectors as row vectors of a matrix, where $*$ is to stand for any value in $\mathbb{Z}$ and $\bullet_k \in m_k\mathbb{Z}$.

Part I:

Claim 1: Let $g \in \mathbb{N}$ and $n = 2,\dots,g$. Let $q := \lfloor \frac{n+1}{2} \rfloor$ and $j = 1,\dots,q$. Then we can transform the basis $v^1,\dots,v^n$ into the following form:

$$\text{for } 1 \le i \le j-1\text{: } v^i = (\underbrace{*,\dots,*}_{g-j},\underbrace{0,\dots,0}_{j-i},1,\underbrace{0,\dots,0}_{i-1};\underbrace{*,\dots,*}_{g-j},\underbrace{\bullet_k,\dots,\bullet_k}_{j-i},\underbrace{0,\dots,0}_{i})$$

$$\text{for } i = j\text{: } v^j = (*,\underbrace{0,\dots,0}_{g-j-1},1,\underbrace{0,\dots,0}_{j-1};\underbrace{0,\dots,0}_{g})$$

$$\text{for } j+1 \le i \le n-j\text{: } v^i = (\underbrace{*,\dots,*}_{g-j},\underbrace{0,\dots,0}_{j};0,\underbrace{*,\dots,*}_{g-j-1},\underbrace{0,\dots,0}_{j})$$

$$\text{for } n-j+1 \le i \le n-1\text{: } v^i = (\underbrace{*,\dots,*}_{g-j},\underbrace{0,\dots,0}_{j};0,\underbrace{*,\dots,*}_{g-j-1},\underbrace{0,\dots,0}_{n-i},\underbrace{\bullet_k,\dots,\bullet_k}_{j-n+i})$$

$$\text{for } i = n\text{: } v^n = (\underbrace{*,\dots,*}_{g-j},\underbrace{0,\dots,0}_{j};\underbrace{*,\dots,*}_{g-j},\underbrace{\bullet_k,\dots,\bullet_k}_{j})$$

We fix g and prove claim 1 by considering the values $n = 2,\dots,g$ separately, using induction over j. For $j = 1$, the first and fourth condition are empty and the second one is implied by Corollary 3.3.7. We transform the basis such that v^1 has the given form. To fulfil conditions three (if $n \ge 3$) and five we proceed as follows:

For $i = 2,\dots,n$ replace v^i by $\tilde{v}^i := v^i - v^i_g v^1$ and denote this new basis (by abuse of notation) again by $v^1,\dots,v^n$. It has the form

$$\begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} = \left(\begin{array}{ccccc|ccc} D_{1:g-1}(v^1) & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ * & * & \dots & * & 0 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & \dots & * & 0 & * & \dots & * \end{array}\right).$$

Since $v^1 \wedge \dots \wedge v^n$ is an isotropic space, we know that for $i = 2,\dots,n$

$$0 = \langle v^1, v^i \rangle = D_{1:g-1}(v^1) v^i_{g+1} + d_{1:g-1} v^i_{2g} \quad \Longrightarrow \quad v^i_{g+1} = -\tfrac{d_{1:g-1}}{D_{1:g-1}(v^1)} v^i_{2g}. \tag{3.18}$$

If all $v^i_{2g} = 0$ we already have a basis satisfying conditions three and five. Otherwise we may assume that $v^n_{2g} \neq 0$. For all $i = 2,\dots,n-1$ where $v^i_{2g} \neq 0$ we fulfil condition three iteratively the following way: we know from number theory that there exist integers λ, μ such that

$$\lambda v^i_{2g} + \mu v^n_{2g} = \gcd(v^i_{2g}, v^n_{2g}).$$

If we now replace

$$\begin{aligned} u^i &:= \tfrac{v^n_{2g}}{\gcd(v^i_{2g}, v^n_{2g})} v^i - \tfrac{v^i_{2g}}{\gcd(v^i_{2g}, v^n_{2g})} v^n \\ u^n &:= \lambda v^i + \mu v^n \end{aligned}$$

we obtain a new basis $v^1,\dots,v^{i-1},u^i,v^{i+1},\dots,v^{n-1},u^n$ where

$$\begin{aligned} u^i_{2g} &= \tfrac{v^n_{2g}}{\gcd(v^i_{2g},v^n_{2g})}v^i_{2g} - \tfrac{v^i_{2g}}{\gcd(v^i_{2g},v^n_{2g})}v^n_{2g} = 0 \quad \text{and} \\ u^i_{g+1} &= \tfrac{v^n_{2g}}{\gcd(v^i_{2g},v^n_{2g})}v^i_{g+1} - \tfrac{v^i_{2g}}{\gcd(v^i_{2g},v^n_{2g})}v^n_{g+1} \\ &\overset{(3.18)}{=} -\tfrac{v^n_{2g}}{\gcd(v^i_{2g},v^n_{2g})}\tfrac{d_{1:g-1}}{D_{1:g-1}(v^1)}v^i_{2g} + \tfrac{v^i_{2g}}{\gcd(v^i_{2g},v^n_{2g})}\tfrac{d_{1:g-1}}{D_{1:g-1}(v^1)}v^n_{2g} \\ &= -\tfrac{d_{1:g-1}}{D_{1:g-1}(v^1)}u^i_{2g} = 0 \end{aligned}$$

and hence (again by abuse of notation letting $v^i := u^i$ and $v^n := u^n$) we have achieved that v^i satisfies condition three. Note that $v^n_{2g} = \gcd(v^i_{2g}, v^n_{2g}) \neq 0$ and so we may proceed with the next i.

For condition five we use the isotropy

$$0 = \langle v^1, v^n\rangle = D_{1:g-1}(v^1)v^n_{g+1} + d_{1:g-1}v^n_{2g}. \tag{3.19}$$

Since m_k divides both $D_k(v^1)$ and $D_k(v^n)$ and the latter in turn divides v^n_{g+1}, this implies that

$$d_{1:g-1}v^n_{2g} \equiv 0 \mod (m_k)^2.$$

On the other hand, $\gcd(d_r, m_k) = 1$ for all $r \neq k$ since d_r and d_k are coprime, and $\gcd(\frac{d_k}{m_k}, m_k) = 1$ since d_k is square-free. Hence, we must have $m_k | v^n_{2g}$. This completes the proof of condition five for $j = 1$.

Now we continue the induction over j by assuming that claim 1 is true for some $j = 1,\dots,q-1$ and establish it for $j+1$. This is done by essentially the same methods we have used for $j = 1$.

The vector v^{j+1} has the form

$$v^{j+1} = (\underbrace{*,\dots,*}_{g-j},\underbrace{0,\dots,0}_{j};0,\underbrace{*,\dots,*}_{g-j-1},\underbrace{0,\dots,0}_{j})$$

since $j \leq q-1$ and hence

$$2j+1 \leq 2(q-1)+1 = 2(\lfloor\tfrac{n+1}{2}\rfloor - 1)+1 \leq 2(\tfrac{n+1}{2}-1)+1 = n \quad\implies\quad j+1 \leq n-j.$$

(If $j = q$, as happens at the end of the induction, the third condition is empty and the vector v^{j+1} is governed by condition four.) In particular, we may use Corollary 3.3.7 for genus $g-j$ to find a matrix $\gamma \in \tilde{\Gamma}_{\text{pol}}$ of the form

$$\gamma = \left(\begin{array}{cc|cc} A & & B & \\ & \mathbb{1} & & \\ \hline C & & D & \\ & & & \mathbb{1} \end{array}\right) \quad \text{where } A,B,C,D \in \mathbb{Z}^{(g-j)\times(g-j)}$$

that transforms v^{j+1} into

$$\tilde{v}^{j+1} = (D_{1:g-j}(v^{j+1}),\ \underbrace{0,\dots,0}_{g-(j+1)-1},1,\underbrace{0,\dots,0}_{j};\underbrace{0,\dots,0}_{g})$$

which already satisfies condition two for $j+1$. It also (possibly) alters the entries $v^i_1,\dots,v^i_{g-j},v^i_{g+1},\dots,v^i_{2g-j}$ for all other vectors but this does not present a problem. We now replace v^i by $\tilde{v}^i := v^i - v^i_{g-j}\tilde{v}^{j+1}$ for $i = 1,\dots,j,j+2,\dots,n$ and denote the new basis again by $v^1,\dots,v^n$. The first halves of all these vectors (i. e. the entries $v^i_1,\dots,v^i_g$) are in accordance with the five conditions of claim 1 for $j+1$.

To obtain the zeroes at v^i_{g+1} and v^i_{2g-j} of condition three and four we proceed similar to (3.18). Let $i = j+2,\dots,n-1$. If $v^i_{g+1} = 0$ then the isotropy $0 = \langle v^{j+1}, v^i\rangle$ shows that $v^i_{2g-j} = 0$ and nothing has to be done. If $v^{i+1}_{g+1} = 0$ we swap the vectors v^i and v^{i+1} and otherwise replace both by linear combinations as described above. By this method, both conditions three and four are fulfilled.

It remains to show that in condition one and five the entry v^i_{2g-j} is indeed a multiple of m_k. This is done in the same way as (3.19) using

$$0 = \langle v^{j+1}, v^i\rangle = D_{1:g-(j+1)}(v^{j+1})v^i_{g+1} + d_{1:g-(j+1)}v^i_{2g-j}$$

and the appropriate divisibility, coprimality and square-freeness properties. This completes the induction and hence the proof of claim 1.

Part II:

We are now in a position to try and transform the basis such that we obtain a basis vector w having the property that m_k divides every entry of w. Recall that this proves the proposition since basis vectors are primitive and hence m_k must be equal to 1.

From the definition of m_k we know that it divides D_k and hence the entries $v^i_1,\dots,v^i_k,v^i_{g+1},\dots,v^i_{g+k}$ for all $i = 1,\dots,n$. Because of the entry 1 in the vectors $v^1,\dots,v^q$ where all other vectors have zeroes, it does not make sense to use them in the construction of w.

The vectors v^i for $q+1 \le i \le n$ are constructed such that m_k divides all entries $v^i_{g-q+1},\dots,v^i_{g-1},v^i_{2g-q+1},\dots,v^i_{2g}$ since they are either zero or $\bullet_k$. This shows that we only need to consider the entries v^i_r where $k < r < g-q+1$ or $g+k < r < 2g-q+1$. If we count these entries, we obtain the number

$$2(g-q-k) = 2(g - \lfloor \tfrac{n+1}{2}\rfloor - (g-n+1)) = 2(n - \lfloor \tfrac{n+1}{2}\rfloor - 1) = 2(\lfloor \tfrac{n}{2}\rfloor - 1).$$

Let $\delta_n := \lfloor \frac{n}{2}\rfloor - 1$.

For $g \le 3$ we have $n \le 3$ which gives $\delta_n = 0$. Hence, for this case the proof is complete.

For $3 < g \le 5$ we also have to consider $n = 4,5$ and here $\delta_n = 1$. In these cases, $n-q = 2$ and hence we have two vectors of the form

$$\begin{pmatrix} v^{n-1} \\ v^n \end{pmatrix} = \left(\begin{array}{ccccc|cccccc} * & \dots & * & 0 & 0 & 0 & * & \dots & * & 0 & \bullet_k \\ * & \dots & * & 0 & 0 & * & * & \dots & * & \bullet_k & \bullet_k \end{array}\right) \quad \text{if } n \text{ is even and}$$

$$\begin{pmatrix} v^{n-1} \\ v^n \end{pmatrix} = \left(\begin{array}{cccccc|ccccccc} * & \dots & * & 0 & 0 & 0 & 0 & * & \dots & * & 0 & \bullet_k & \bullet_k \\ * & \dots & * & 0 & 0 & 0 & * & * & \dots & * & \bullet_k & \bullet_k & \bullet_k \end{array}\right) \quad \text{if } n \text{ is odd.}$$

Since both cases can be treated absolutely the same and everything is independent of g, we only give the details for $g = 4$ and $n = 4$, which means we are considering $k = 1$. Here,

$$\begin{pmatrix} v^3 \\ v^4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} v^3_1 & v^3_2 & 0 & 0 & 0 & v^3_6 & 0 & v^3_8 \\ v^4_1 & v^4_2 & 0 & 0 & v^4_5 & v^4_6 & v^4_7 & v^4_8 \end{array}\right). \tag{3.20}$$

If $v_6^3 = 0$ we already have the situation of equation 3.21. Otherwise, we can find integers λ, μ such that $\lambda v_2^3 + \mu v_6^3 = \gcd(v_2^3, v_6^3)$. Now the matrix that differs from the unit matrix only in the entries

$$\begin{pmatrix} a_{22} & a_{26} \\ a_{62} & a_{66} \end{pmatrix} = \begin{pmatrix} \lambda & -\frac{v_6^3}{\gcd(v_2^3,v_6^3)} \\ \mu & \frac{v_2^3}{\gcd(v_2^3,v_6^3)} \end{pmatrix}$$

is in $\tilde{\Gamma}_{\text{pol}}$ and transforms the basis into

$$\begin{pmatrix} v^3 \\ v^4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} v_1^3 & v_2^3 & 0 & 0 & 0 & 0 & 0 & v_8^3 \\ v_1^4 & v_2^4 & 0 & 0 & v_5^4 & v_6^4 & v_7^4 & v_8^4 \end{array} \right). \tag{3.21}$$

We know that m_1 divides v_8^3 from the construction of part I, and it divides v_1^3 since it divides $D_1(v^3)$ by definition. So, since v^3 is a basis vector and hence primitive, we must have $\gcd(v_2^3, m_1) = 1$ and in particular either $m_1 = 1$ and we are done or $v_2^3 \neq 0$. Note that at this point the method relies on the fact that we have only a single entry to consider, in other words, that $\delta_n = 1$. We proceed as in (3.19) and obtain

$$0 = \langle v^3, v^4 \rangle = v_1^3 v_5^4 + d_1 v_2^3 v_6^4 \equiv_{(m_1)^2} d_1 v_2^3 v_6^4 \quad \Longrightarrow \quad m_1 | v_6^4. \tag{3.22}$$

Now we find integers α, β such that $\alpha v_2^3 + \beta v_2^4 = \gcd(v_2^3, v_2^4)$ and transform the basis into

$$\begin{pmatrix} \frac{v_2^4}{\gcd(v_2^3,v_2^4)} v^3 - \frac{v_2^3}{\gcd(v_2^3,v_2^4)} v^4 \\ \alpha v^3 + \beta v^4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} \bullet_k & 0 & 0 & 0 & \bullet_k & \bullet_k & \bullet_k & \bullet_k \\ \bullet_k & * & 0 & 0 & \bullet_k & \bullet_k & \bullet_k & \bullet_k \end{array} \right) \tag{3.23}$$

which clearly shows that we have constructed a basis vector with the property that m_1 divides every entry. This is the contradiction we wanted to reach and hence the proof for $g \leq 5$ is complete.

Since no general proof for all g is yet known to us, we have developed a short-hand notation for these constructions that will make the proofs easier to read. First of all, recall that we only need to consider the entries v_k^i with $k < r < g - q + 1$ and $g + k < r < 2g - q + 1$ and $q + 1 \leq i \leq n$ where $q = \lfloor \frac{n+1}{2} \rfloor$, assuming that g is large enough such that these ranges are not empty. Since these are always $2\delta_n$ entries in $\delta_n + 1$ vectors, we can work independently of g. Note, however, that for the proposition to hold for a given g, we need to consider all $\delta_n \leq \lfloor \frac{g}{2} \rfloor - 1$. Conversely, if we have completed the construction for a fixed δ_n, this proves the proposition for $g \leq 2\delta_n + 3$.

We denote the entries again by 0, $* \in \mathbb{Z}$, $\bullet_k \in m_k\mathbb{Z}$ and $X \in \mathbb{Z} \backslash m_k\mathbb{Z}$. Furthermore, we denote a transformation concerning the columns $x + 1$ and $g + x + 1$ (for example from (3.20) to (3.21) with $x = 1$ – note that the first column will be excluded from the short notation) by $\overset{|x|}{\rightsquigarrow}$ and a transformation combining the ith and jth vector (for example the transformation that leads to (3.23)) by $\overset{i,j}{\rightsquigarrow}$. When we use the fact that the ith vector has to be primitive we denote this by $\overset{(i)=1}{\rightsquigarrow}$, and when we consider the product of the ith and jth vector as in (3.22) we denote this by $\overset{\langle i,j \rangle}{\rightsquigarrow}$.

For some steps to be possible we need certain entries of the basis vectors to be non-zero. We assume this to be the case where needed. If these entries would vanish, we could either alter the order of the basis vectors, skip the step in question or even arrive directly at a contradiction proving our claim.

Using this new notation, we rewrite the above proof for $\delta_n = 1$ starting with the entries $\left(\begin{array}{c|c} v_2^3 & v_6^3 \\ v_2^4 & v_6^4 \end{array}\right)$ as follows:

$$\left(\begin{array}{c|c} * & * \\ * & * \end{array}\right) \overset{|1|}{\rightsquigarrow} \left(\begin{array}{c|c} * & 0 \\ * & * \end{array}\right) \overset{(1)=1}{\rightsquigarrow} \left(\begin{array}{c|c} X & 0 \\ * & * \end{array}\right) \overset{\langle 1,2\rangle}{\rightsquigarrow} \left(\begin{array}{c|c} X & 0 \\ * & \bullet_k \end{array}\right) \overset{1,2}{\rightsquigarrow} \left(\begin{array}{c|c} 0 & \bullet_k \\ * & \bullet_k \end{array}\right)$$

where now all entries of the first row vector are divisible by m_k while it is supposed to be a primitive vector, giving the contradiction.

For $\delta_n \geq 2$ we need to use Corollary A.7 choosing v to be the ith vector which we will denote by $\overset{\vec{i}}{\rightsquigarrow}$. The gcd thus constructed will be denoted by $\cdot$ and all its multiples by $\star$. The proof for $\delta_n = 2$ can now be given by

$$\left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \overset{|1|\ |2|}{\rightsquigarrow\rightsquigarrow} \left(\begin{array}{cc|cc} * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{array}\right) \overset{\vec{1}}{\rightsquigarrow} \left(\begin{array}{cc|cc} \star & \cdot & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{array}\right)$$

$$\overset{(1)=1}{\rightsquigarrow} \left(\begin{array}{cc|cc} * & X & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{array}\right) \overset{2,3}{\rightsquigarrow} \left(\begin{array}{cc|cc} * & X & 0 & 0 \\ * & * & 0 & * \\ * & * & * & * \end{array}\right) \overset{\langle 1,2\rangle}{\rightsquigarrow} \left(\begin{array}{cc|cc} * & X & 0 & 0 \\ * & * & 0 & \bullet_k \\ * & * & * & * \end{array}\right)$$

$$\overset{1,2}{\rightsquigarrow} \left(\begin{array}{cc|cc} 0 & * & 0 & \bullet_k \\ * & * & 0 & \bullet_k \\ * & * & * & * \end{array}\right) \overset{(1)=1}{\rightsquigarrow} \left(\begin{array}{cc|cc} 0 & X & 0 & \bullet_k \\ * & * & 0 & \bullet_k \\ * & * & * & * \end{array}\right) \overset{\langle 1,3\rangle}{\rightsquigarrow} \left(\begin{array}{cc|cc} 0 & X & 0 & \bullet_k \\ * & * & 0 & \bullet_k \\ * & * & * & \bullet_k \end{array}\right)$$

$$\overset{1,2}{\rightsquigarrow} \left(\begin{array}{cc|cc} * & 0 & 0 & \bullet_k \\ * & * & 0 & \bullet_k \\ * & * & * & \bullet_k \end{array}\right) \overset{(1)=1}{\rightsquigarrow} \left(\begin{array}{cc|cc} X & 0 & 0 & \bullet_k \\ * & * & 0 & \bullet_k \\ * & * & * & \bullet_k \end{array}\right) \overset{\langle 1,3\rangle}{\rightsquigarrow} \left(\begin{array}{cc|cc} X & 0 & 0 & \bullet_k \\ * & * & 0 & \bullet_k \\ * & * & \bullet_k & \bullet_k \end{array}\right)$$

$$\overset{2,3}{\rightsquigarrow} \left(\begin{array}{cc|cc} X & 0 & 0 & \bullet_k \\ * & 0 & \bullet_k & \bullet_k \\ * & * & \bullet_k & \bullet_k \end{array}\right) \overset{1,2}{\rightsquigarrow} \left(\begin{array}{cc|cc} 0 & 0 & \bullet_k & \bullet_k \\ * & 0 & \bullet_k & \bullet_k \\ * & * & \bullet_k & \bullet_k \end{array}\right)$$

and again the first row gives the contradiction as before.

For $\delta_n = 3$ the proof runs as follows:

$$\left(\begin{array}{ccc|ccc} \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{|1|}{\rightsquigarrow}\overset{|2|}{\rightsquigarrow}\overset{|3|}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & \ast & \ast & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{\vec{1}}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \star & \star & \cdot & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right)$$

$$\overset{(1)=1}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & \ast & X & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{2,4}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & \ast & X & 0 & 0 & 0 \\ \ast & \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{3,4}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & \ast & X & 0 & 0 & 0 \\ \ast & \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right)$$

$$\overset{2,3}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & \ast & X & 0 & 0 & 0 \\ \ast & \ast & \ast & 0 & 0 & \ast \\ \ast & \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{\langle 1,2\rangle}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & \ast & X & 0 & 0 & 0 \\ \ast & \ast & \ast & 0 & 0 & \bullet_k \\ \ast & \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{1,2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & \ast & \ast & 0 & 0 & \bullet_k \\ \ast & \ast & \ast & 0 & 0 & \bullet_k \\ \ast & \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right)$$

$$\overset{\vec{1}}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & \star & \cdot & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{(1)=1}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & \ast & X & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{3,4}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & \ast & X & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & 0 & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right)$$

$$\overset{\langle 1,3\rangle}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & \ast & X & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & 0 & \bullet_k \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{1,2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & \ast & 0 & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & 0 & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & 0 & \bullet_k \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{2,3}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & \ast & 0 & 0 & \bullet_k & \bullet_k \\ \ast & \ast & 0 & \ast & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right)$$

$$\overset{1,2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & 0 & 0 & \ast & \bullet_k & \bullet_k \\ \ast & \ast & 0 & \ast & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{|1|}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} \ast & 0 & 0 & 0 & \bullet_k & \bullet_k \\ \ast & \ast & 0 & \ast & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right) \overset{(1)=1}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} X & 0 & 0 & 0 & \bullet_k & \bullet_k \\ \ast & \ast & 0 & \ast & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right)$$

$$\overset{\langle 1,2\rangle}{\rightsquigarrow}\overset{\langle 1,3\rangle}{\rightsquigarrow}\overset{\langle 1,4\rangle}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} X & 0 & 0 & 0 & \bullet_k & \bullet_k \\ \ast & \ast & 0 & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \ast & \ast \end{array}\right) \overset{1,2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & \ast & 0 & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & 0 & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \ast & \ast \end{array}\right)$$

$$\overset{(1)=1}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & X & 0 & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & 0 & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \ast & \ast \end{array}\right) \overset{\langle 1,4\rangle}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & X & 0 & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & 0 & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \ast \end{array}\right) \overset{2,3}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & X & 0 & \bullet_k & \bullet_k & \bullet_k \\ 0 & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \ast \end{array}\right)$$

$$\overset{1,2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & 0 & \ast & \bullet_k & \bullet_k & \bullet_k \\ 0 & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \ast \end{array}\right) \overset{(1)=1}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & 0 & X & \bullet_k & \bullet_k & \bullet_k \\ 0 & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \ast \end{array}\right) \overset{\langle 1,4\rangle}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & 0 & X & \bullet_k & \bullet_k & \bullet_k \\ 0 & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \end{array}\right)$$

$$\overset{3,4}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & 0 & X & \bullet_k & \bullet_k & \bullet_k \\ 0 & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ 0 & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \end{array}\right) \overset{2,3}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & 0 & X & \bullet_k & \bullet_k & \bullet_k \\ 0 & 0 & \ast & \bullet_k & \bullet_k & \bullet_k \\ 0 & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \end{array}\right) \overset{1,2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & \bullet_k & \bullet_k & \bullet_k \\ 0 & 0 & \ast & \bullet_k & \bullet_k & \bullet_k \\ 0 & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \\ \ast & \ast & \ast & \bullet_k & \bullet_k & \bullet_k \end{array}\right)$$

and again the first row vector gives the contradiction.

This completes the proof for g up to $2\delta_n + 3 = 9$. $\square$

Lemma 3.3.10.
Assume Proposition 3.3.8 is true for g. Then, in any rank-n-sublattice $\tilde{h}_{\mathbb{Z}} \subset h_{\mathbb{Z}}$ with $2 \leq n \leq g$ we find a vector v satisfying $D_{g-n+1}(v) = 1$.

Proof.
Let $k := g-n+1$ and denote a basis of $\tilde{h}_Z$ by $\tilde{u}^1, \dots, \tilde{u}^n$. Let

$$m := \min\{D_k(u) \big| u \in \tilde{h}_{\mathbb{Z}}\}.$$

Now, let $\hat{u}^1 \in \tilde{h}_{\mathbb{Z}}$ be a primitive vector with $D_k(\hat{u}^1) = m$. We can obviously always find such a vector. Our aim is to show that $m = 1$. Since $\hat{u}^1$ is primitive, Lemma A.3 tells us that we can find $\hat{u}^2, \dots, \hat{u}^n$ such that $\hat{u}^1, \dots, \hat{u}^n$ is a basis of $\tilde{h}_{\mathbb{Z}}$. According to Corollary 3.3.7 we can find a transformation γ such that in the basis $u^i := \gamma\hat{u}^i$ of $\tilde{h}_{\mathbb{Z}}$ the kth entry of u^1 is

$$u^1_k = D_{k:g-1}(u^1) = mD_{k+1:g-1}(u^1).$$

Note that due to the invariance of the divisors we have the equality $m = \min\{D_k(u) \big| u \in \gamma\tilde{h}_{\mathbb{Z}}\}$.

We modify the basis as follows: Let $i = 2, \dots, n$. If the kth entry of u^i is equal to zero, we leave u^i unchanged. Otherwise, we replace both u^1 and u^i by linear combinations of these two vectors, such that the kth entry of $\overline{u}^1$ is $\overline{u}^1_k = \gcd(u^1_k, u^i_k)$ and $\overline{u}^i_k = 0$. This is possible without changing $\tilde{h}_{\mathbb{Z}}$ according to the considerations at the beginning of this section. By abuse of notation, after all but the last transformation we denote the new basis again by u^i.

After repeating this procedure for $i = 2, \dots, n$ we have a basis of $\tilde{h}_{\mathbb{Z}}$ where

$$\overline{u}^1_k = \gcd(u^1_k, \dots, u^n_k) \neq 0 \tag{3.24}$$

and $\overline{u}^i_k = 0$ for all $i = 2, \dots, n$. We modify the basis one more time by letting

$$v^1 := \overline{u}^1, \quad \text{and} \quad v^i := \overline{u}^i + \overline{u}^1 \quad \text{for } i \geq 2,$$

so that now the kth entries of all vectors $v^1, \dots, v^n$ are equal to $\overline{u}^1_k$. Therefore, for all $i = 1, \dots, n$ we have

$$\begin{aligned} D_k(v^i) &= \gcd(d_k, \tfrac{v^i_{s|g+s}}{D_{s:k-1}(v^i)})_{s=1,\dots,k} \quad \text{and considering only the } k\text{th entry} \\ &= \gcd(d_k, \dots, \overline{u}^1_k, \dots) \quad \text{which, according to (3.24), is equal to} \\ &= \gcd(d_k, \dots, \gcd(u^1_k, \dots, u^n_k), \dots). \end{aligned}$$

Hence we know that $D_k(v^i)$ divides $\gcd(d_k, u^1_k) = \gcd(d_k, mD_{k+1:g-1}(u^1)) = m$ which implies $D_k(v^i) \leq m$. On the other hand, from the definition of m we know $D_k(v^i) \geq m$ since $v^i \in \tilde{h}_{\mathbb{Z}}$ and m is minimal. Therefore, $D_k(v^i) = m$. This shows that, using Proposition 3.3.8,

$$m = \gcd\left(D_k(v^1), \dots, D_k(v^n)\right) = 1$$

which shows that $D_k(u^1) = m = 1$ as claimed. □

Theorem 3.3.11.
Assume Proposition 3.3.8 is true for g. Then $\tilde{\Gamma}_{\text{pol}}$ acts transitively on the g-dimensional isotropic subspaces of $\mathbb{Q}^{2g}$

Proof.
Let $e_k := (0,\dots,0,1,0,\dots,0)$ where the 1 is the kth entry. We want to show that, given any g-dimensional isotropic subspace $h \subset \mathbb{Q}^{2g}$ we can find a basis $u^1,\dots,u^g$ of $h_{\mathbb{Z}}$ such that there exists a transformation $\gamma \in \tilde{\Gamma}_{\text{pol}}$ satisfying $\gamma u^i = e^i$ for $i = 1,\dots,g$. The proof is by induction.

More precisely, we want to show the following for any $k \in \{0,\dots,g\}$:
Claim 1: We can transform the basis $u^1,\dots,u^g$ of $h_{\mathbb{Z}}$ such that

$$\begin{aligned} &u^i = e^i \quad \text{for } i = 1,\dots,k \text{ and} \\ &u^i = (\underbrace{0,\dots,0}_{k},\underbrace{*,\dots,*}_{g-k},\underbrace{0,\dots,0}_{k},\underbrace{*,\dots,*}_{g-k}) \quad \text{for } i = k+1,\dots,g. \end{aligned} \tag{3.25}$$

For $k = 0$ this is trivially true and hence we may use this as start for the induction. Assume that claim 1 is true for some $k \in \{0,\dots,g-2\}$. Denote the isotropic subspace generated by $u^{k+1},\dots,u^g$ by $\tilde{h}$. Note that we may apply Lemma 3.3.10 for this subspace without losing the property (3.25): of the basic transformations mentioned at the beginning of this section only the operation of $\gamma \in \tilde{\Gamma}_{\text{pol}}$ could cause problems since we have to apply it to all basis vectors. However, we may restrict ourselves to using transformations of the form

$$\gamma = \left(\begin{array}{cc|cc} \mathbb{1}_k & & & \\ & A & & B \\ \hline & & \mathbb{1}_k & \\ & C & & D \end{array}\right) \in \tilde{\Gamma}_{\text{pol}} \tag{3.26}$$

and these leave the property (3.25) valid.

Hence, Lemma 3.3.10 tells us that we may assume (if necessary after suitable transformations) that the basis $u^{k+1},\dots,u^g$ of $\tilde{h}_{\mathbb{Z}}$ is such that $D_i(u^i) = 1$ for $i = k+1,\dots,g-1$.

If $k = g-2$, the vector $v := u^{g-1}$ already has the property that $D_{k+1:g-1}(v) = 1$. Otherwise, we let

$$v := \sum_{n=k+1}^{g-1} d^{(n)}_{k+1:g-1} u^n,$$

where $d^{(c)}_{a:b} := d_{a:c-1} d_{c+1:b}$. Since $\gcd(d^{(n)}_{k+1:g-1})_{n=k+1,\dots,g-1} = 1$, Lemma A.3 shows that we can find a basis $v^{k+1},\dots,v^g$ of $\tilde{h}_{\mathbb{Z}}$ where $v^{k+1} = v$. We want to show that $D_{k+1:g-1}(v) = 1$ for $0 \le k < g-1$. Again, we use induction to prove
Claim 2: For $j = k,\dots,g-1$ we have $D_{k+1:j}(v) = 1$.

Again, for $j = k$ the claim is trivially true and we have a start for the induction. Assume now that claim 2 is true for some $j \in \{k,\dots,g-2\}$. Then

$$\begin{aligned} D_{j+1}(v) &= \gcd\left(d_{j+1}, \frac{v_{s|g+s}}{D_{s:j}(v)}\right)_{s=1,\dots,j+1} \quad \text{and since } v_{1|g+1} = \dots = v_{k|g+k} = 0, \\ &= \gcd\left(d_{j+1}, \frac{v_{s|g+s}}{D_{s:j}(v)}\right)_{s=k+1,\dots,j+1}. \end{aligned}$$

By assumption $D_{k+1:j}(v) = 1$, which implies $D_{s:j}(v) = 1$ since $s \geq k+1$. Hence

$$
\begin{aligned}
&= \gcd\left(d_{j+1}, v_{s|g+s}\right)_{s=k+1,\ldots,j+1} \\
&= \gcd\left(d_{j+1}, \sum_{n=k+1}^{g-1} d^{(n)}_{k+1:g-1} u^{n}_{s|g+s}\right)_{s=k+1,\ldots,j+1} \quad \text{leaving out multiples of } d_{j+1} \\
&= \gcd\left(d_{j+1}, d^{(j+1)}_{k+1:g-1} u^{j+1}_{s|g+s}\right)_{s=k+1,\ldots,j+1} \quad \text{and coprimality of the } d_i \text{ gives} \\
&= \gcd\left(d_{j+1}, u^{j+1}_{s|g+s}\right)_{s=k+1,\ldots,j}
\end{aligned}
$$

and since the polarisation is coprime we have $\gcd(d_{j+1}, D_{1:j}(u^{j+1})) = 1$ and therefore

$$
\begin{aligned}
&= \gcd\left(d_{j+1}, \frac{u^{j+1}_{s|g+s}}{D_{s:j}(u^{j+1})}\right)_{s=k+1,\ldots,j+1} \quad \text{and since } u^{j+1}_{1|g+1} = \cdots = u^{j+1}_{k|g+k} = 0, \\
&= \gcd\left(d_{j+1}, \frac{u^{j+1}_{s|g+s}}{D_{s:j}(u^{j+1})}\right)_{s=1,\ldots,j+1} \\
&= D_{j+1}(u^{j+1}) \\
&= 1.
\end{aligned}
$$

This shows that claim 2 is true for $j+1$, completing the proof that $D_{k+1:g-1}(v) = 1$ for any $k \in \{0, \ldots, g-1\}$.

Hence, we can find $\gamma \in \tilde{\Gamma}_{\text{pol}}$ of the form (3.26) such that $\gamma v = e_{k+1}$. Under this operation the basis $v^{k+1}, \ldots, v^{g}$ of $\tilde{h}_{\mathbb{Z}}$ is transformed into a basis of $\gamma \tilde{h}_{\mathbb{Z}}$ which we shall, by abuse of notation, again denote by $v^{k+1}, \ldots, v^{g}$. Note that now $v^{k+1} = e_{k+1}$. Since γh is again an isotropic subspace, we have for $j = k+2, \ldots, g$:

$$
0 = \langle v^{k+1}, v^{j} \rangle = d_{1:k} \cdot 1 \cdot v^{j}_{g+k+1} \quad \Longrightarrow \quad v^{j}_{g+k+1} = 0. \tag{3.27}
$$

Thus, we obtain a basis $\tilde{u}^{k+1} := v^{k+1}, \tilde{u}^{i} := v^{i} - v^{i}_{k+1} v^{k+1}$ satisfying claim 1 for $k+1$. This completes the induction.

Now that we have reached (3.25) for $k = g-1$ it is easy to see that we only need one more transformation of the form (3.26) (where the matrices A to D are just integers) to prove claim 1 for $k = g$.

Since we have now shown that for any g-dimensional isotropic subspace h we can find a basis of $h_{\mathbb{Z}}$ that can be transformed into $e_1, \ldots, e_g$ by the action of an element in $\tilde{\Gamma}_{\text{pol}}$, we have proved the transitivity of the group action. □

Chapter 4

Vanishing on the boundary of higher codimension

4.1 The result by Barnes and Cohn

We have already mentioned in section 3.1 that the corank-1 boundary components play a crucial part in determining the order of vanishing of a cusp form on all of the boundary. In the principally polarised case this is shown using the result by Barnes and Cohn in [BC] which we restate in Theorem 4.1.2.

For the non-principally polarised case this theorem unfortunately cannot be established; in fact, there is a counterexample which we will give in Example 4.2.6. Nevertheless, a generalisation of the result by Barnes and Cohn provides a weaker bound which may be used instead.

Following the paper [BC] we generalise their theorem 3 to some more general lattices which correspond to the non-principally polarised case. We first establish our notation which is closely related to that of [BC].

Notation 4.1.1.
Let $f(x) := xA{}^t x$ and $h(x) := xB{}^t x$ be two quadratic forms with real symmetric matrices A and B, and define their inner product as $(f,h) := \mathrm{tr}(AB) := \sum_{i,j} a_{ij}b_{ij}$. For positive definite f denote by $M(f)$ its arithmetic minimum, i. e. the minimum of $f(x)$ with integral $x \neq 0$.

Barnes and Cohn show the following

Theorem 4.1.2.
Let f be a real positive definite n-ary form and h an integral positive definite or semi-definite n-ary form with $h \not\equiv 0$. Then

$$(f,h) \geq M(f),$$

and equality can hold only if h has rank 1 and so has the form

$$h(x) = (x{}^t m)^2 = \sum_{i,j} m_i m_j x_i x_j.$$

Proof.
See [BC, Theorem 3]. □

Remark 4.1.3.
We always find a h_0 of rank 1 for which equality holds, namely $h_0(x) := (x^t m)^2$ where we chose the m satisfying $f(m) = M(f)$.

This theorem is used in the context of moduli of principally polarised abelian varieties in form of the following

Corollary 4.1.4.
Let f be a real positive definite n-ary form and denote by $\mathbb{L}^0$ the lattice of all positive definite or positive semi-definite integral forms and by $\mathbb{L}_1 \subset \mathbb{L}^0$ the sublattice of forms of rank 1. Then

$$\min_{h\in\mathbb{L}^0\setminus\{0\}}(f,h) \geq \min_{h\in\mathbb{L}_1}(f,h).$$

The main connection between extending pluricanonical forms to a toroidal compactification and this corollary is a theorem by Y.-S. Tai.

Theorem 4.1.5.
Consider the bounded symmetric domain $\mathfrak{D} = \mathfrak{S}_g$. Let Γ be a neat, arithmetic subgroup of the connected group of automorphisms $\mathrm{Aut}(\mathfrak{D})^0$, χ be an automorphic form of weight $l(g+1)$ with respect to Γ, $\omega = \bigwedge_{i\leq j} d\tau_{i,j}$, $\chi\omega^{\otimes l} \in \Omega^N(\mathfrak{D}/\Gamma)^{\otimes l}$, and let $\overline{\mathfrak{D}/\Gamma}$ be the compactification of $\mathfrak{D}/\Gamma$ corresponding to a Γ-admissible collection of fans $\{\Sigma(F)\}$ where each $\sigma_\alpha(F) \in \Sigma(F)$ is basic (i. e. it can be generated by a part of a basis of $P'(F)$). Then

$$\chi\omega^{\otimes l} \text{ extends to } \overline{\mathfrak{D}/\Gamma} \iff \begin{cases} \text{for every rational boundary component } F, \\ \quad \text{in the Fourier expansion of } \chi \text{ at } F: \\ \quad \chi(z) = \sum_{f\in(P'(F))^\vee} a_f^F(u,t)\mathbf{e}[\langle f,z\rangle] \\ \quad a_f^F \not\equiv 0 \text{ implies } \min_{\substack{h\in P'(F)\cap\overline{C(F)},\\ h\neq 0}} (f,h) \geq l. \end{cases}$$

Proof.
See [AMRT, Chapter IV, paragraph 1, Theorem 1]. □

We can now prove the correspondence between the vanishing on the corank-1 boundary components and on the rest of the boundary[1]:

Corollary 4.1.6.
Suppose $\mathfrak{D} = \mathfrak{S}_g, \Gamma \subset \mathrm{Sp}(2g,\mathbb{Z})$ and let χ and ω be as in Theorem 4.1.5. Then

$$\chi\omega^{\otimes l}\text{extends to } \overline{\mathfrak{S}_g/\Gamma} \iff \begin{cases} \chi \text{ vanishes on all} \\ \text{rational corank-1 boundary components} \\ \text{of order at least } l. \end{cases}$$

Proof.
Since we have a principal polarisation, $(P'(F))^\vee$ consists of integer matrices for all

[1]See also [Tai, Theorem 1.1].

rational boundary components F. Therefore, according to Corollary 4.1.4, the minimum of (f,h) with $f \in (P'(F))^\vee$ over all $h \in P'(F) \cap \overline{C(F)}$ is obtained for a form h of rank 1. For any such h we can find a corank-1 boundary component $F_1 \prec F$ with $h \in P'(F_1) \cap \overline{C(F_1)}$. Because the coefficients a_f^F of the Fourier-Jacobi expansion are the same for every pair $F \succ F_1$ we can now bound the minimum over all h for all F by the minimal order of vanishing on all rational corank-1 boundary components. □

4.2 Non-principal polarisations

Corollary 4.1.4 depends heavily on the fact that we consider the minimum over *all* integral forms h. However, in our situation this is only the case if we apply it to principal polarisations. Otherwise the matrix of the bilinear form h is no longer simply an element of $\mathrm{Sym}(g,\mathbb{Z})$ but of a sublattice. To make things precise we define the relevant lattices as follows.

Definition 4.2.1: Tits Lattice.
By the *Tits lattice* we mean the lattice $\mathbb{L} = P'(F^{(0)}) \cap \overline{C(F^{(0)})}$ for the standard corank-g boundary component $F^{(0)}$, where we identify the containing space $\mathscr{P}'(F^{(0)})$ with the space of symmetric matrices as in section 2.2.3. If the polarisation is given by $(1,d_1,\dots,d_{1:g-1})$ and we have no level structure we also write $\mathbb{L}(1,d_1,\dots,d_{1:g-1})$.

Remark 4.2.2.
Although the definition above only considers the standard corank-g boundary component, this is no restriction, since according to Theorem 3.3.11 all other corank-g boundary components are conjugate to this one under the action of $\tilde{\Gamma}_{\mathrm{pol}}$ for those g where Proposition 3.3.8 is true.

Definition 4.2.3: Characteristic values of a lattice.

- Let $\mathbb{L} \subset \mathrm{Sym}(n,\mathbb{Z})$ be a sublattice of the lattice of symmetric matrices and define the subsets $\mathbb{L}^0 \subset \mathbb{L}$ and $\mathbb{L}^+ \subset \mathbb{L}^0$ of positive semi-definite (including the zero matrix) and positive definite matrices, respectively. Let $\mathbb{L}_1 \subset \mathbb{L}$ be the subset of rank 1 matrices.

- If $\mathbb{L}$ is of maximal rank, define two characteristic values for the lattice, namely the greatest common divisor of all (non-zero) determinants

  $$\mu(\mathbb{L}) := \max\{\lambda \in \mathbb{N} \big| \forall B \in \mathbb{L}^+ \ : \ \lambda | \det(B)\}$$

  and the least value ν that makes sure that all matrices νC are members of the lattice

  $$\nu(\mathbb{L}) := \min\{\lambda \in \mathbb{N} \big| \forall C \in \mathrm{Sym}(n,\mathbb{Z}), C \text{ positive semi-definite} \ : \ \lambda C \in \mathbb{L}^0\}.$$

Lemma 4.2.4.
The Tits lattice of a polarisation of type $(1, d_1, \ldots, d_{1:n-1})$ without level-structure is

$$\mathbb{L}(1,\ldots,d_{1:g-1}) = \left\{ M \in \begin{pmatrix} \mathbb{Z} & d_1\mathbb{Z} & \ldots & d_{1:n-1}\mathbb{Z} \\ d_1\mathbb{Z} & d_1\mathbb{Z} & & d_{1:n-1}\mathbb{Z} \\ \vdots & & \ddots & \vdots \\ d_{1:n-1}\mathbb{Z} & d_{1:n-1}\mathbb{Z} & \ldots & d_{1:n-1}\mathbb{Z} \end{pmatrix} \middle| M \text{ symmetric} \right\}$$

and it has the characteristics $\mu(\mathbb{L}) = \prod_i d_i^{n-i}$ and $\nu(\mathbb{L}) = d_{1:n-1}$.

Proof.
In section 2.2.3 we stated

$$\mathscr{P}'(F^{(0)}) \simeq \{ \left(\begin{smallmatrix} \mathbb{1} & S \\ & \mathbb{1} \end{smallmatrix}\right) \big| S \in \mathrm{Sym}(g,\mathbb{R}) \} \simeq \mathrm{Sym}(g,\mathbb{R})$$

for the standard rational boundary component $F^{(0)}$. This isomorphism maps a matrix $M \in \mathscr{P}'(F^{(0)})$ onto its upper right quarter. Since in Definition 2.2.9 we defined $P'(F^{(0)}) = \mathscr{P}'(F^{(0)}) \cap \Gamma_{\mathrm{pol}}$ we are only interested in the symmetric $g \times g$ matrices satisfying the conditions on the upper right quarter of the matrices in Γ_{pol}. Lemma 3.2.10 gives the condition claimed. □

Remark 4.2.5.
We shall make no difference between a form f and its corresponding matrix A, so that by $f \in \mathbb{L}$ we actually mean $f(x) = xA\,{}^t\!x$ with $A \in \mathbb{L}$.

Now we want to give the aforementioned counterexample to the inequality in Corollary 4.1.4:

Example 4.2.6.
Let $\mathbb{L} = \mathbb{L}(1,17)$ and

$$f(x) = x \begin{pmatrix} 3 & -\frac{14}{17} \\ -\frac{14}{17} & \frac{4}{17} \end{pmatrix} {}^t\!x \in \mathbb{L}^\vee.$$

We claim that $\min_{h\in\mathbb{L}_1}(f,h) = 3$. To show this, define h_0 to be a rank 1 form realizing the minimum and let the form be given by the matrix $\left(\begin{smallmatrix} a^2 & ab \\ ab & b^2 \end{smallmatrix}\right)$. For $h_0 \in \mathbb{L}_1$ we need $17|ab$ and $17|b^2$.

Since the rank of h_0 is 1, we cannot have $a = b = 0$. If $a = 0$ or $b = 0$ we obtain

$$(f,h_0) = \mathrm{tr}(fh_0) = \frac{4}{17}b^2 = 4 \quad \text{or} \quad (f,h_0) = \mathrm{tr}(fh_0) = 3a^2 = 3,$$

respectively, since 17 divides b^2 and the minimality of h_0. This shows that

$$\min_{h\in\mathbb{L}_1}(f,h) \leq 3.$$

Now assume that $ab \neq 0$ and $\mathrm{tr}(fh_0) < 3$. Since h_0 is positive semi-definite, we have $a^2, b^2 \in \mathbb{N}$ and hence $a, b \in \mathbb{R}$. Fix $a \in \mathbb{R}$ and define

$$f_a(b) := \mathrm{tr}(fh_0) = 3a^2 - \tfrac{28}{17}ab + \tfrac{4}{17}b^2 = \tfrac{4}{17}(b - \tfrac{7}{2}a)^2 + \tfrac{2}{17}a^2.$$

Then f_a has no zeroes and assumes its minimum over $\mathbb{R}$ at $b = \frac{7}{2}a$. Since the assumption that $f_a(\frac{7}{2}a) = \frac{2}{17}a^2 < 3$ leads to $a^2 < \frac{51}{2}$ and we have seen that $a^2 \in \mathbb{N}$, this leaves only 10 possible values for a.

If $a = \pm 1, \pm 2$ then the condition $ab \in 17\mathbb{Z}$ leads to $b = 17b'$ with $b' \in \mathbb{Z}$. Easy calculation shows that $f_a(17b') = 3a^2 - 28ab' + 68b'^2 \geq 3$. If $a = \pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{5}$ the condition $ab \in 17\mathbb{Z}$ leads to $b = 17ab'$ with $b' \in \mathbb{Z}$. But now $f_a(17ab') = a^2 f_1(17b') \geq 3a^2 > 3$. Hence, $min_{h\in\mathbb{L}_1}(f,h) \geq 3$, which shows the claim.

On the other hand, for the rank 2 form h with matrix $\left(\begin{smallmatrix} 6 & 17 \\ 17 & 51 \end{smallmatrix}\right)$ we calculate $(f,h) = \mathrm{tr}(fh) = 2$, so obviously

$$\min_{h\in\mathbb{L}}(f,h) \leq 2 < 3 = \min_{h\in\mathbb{L}_1}(f,h)$$

which shows that the analogue to the inequality of Corollary 4.1.4 cannot be established for $p = 17$.

4.3 Barnes and Cohn generalised

4.3.1 Retracing Barnes and Cohn

In the following propositions we shall retrace some steps of the paper by Barnes and Cohn and then give the generalisation of their main theorem to non-principal polarisations.

Lemma 4.3.1.
For all positive definite forms f,h we have

$$(f,h) \geq n\sqrt[n]{\det(f)}\sqrt[n]{\det(h)}.$$

Proof.
See [BC, Theorem 1]. □

We now get

Lemma 4.3.2.
For all positive definite forms f,h with $h \in \mathbb{L}^+$ we have

$$(f,h) \geq \frac{n}{\gamma_n}\sqrt[n]{\mu(\mathbb{L})}M(f)$$

where γ_n is Hermite's constant.

Proof.
Let ϕ be a positive definite form in n variables. From the definition of Hermite's constant

$$\gamma_n := \sup_{\text{pos. def. }\phi} \frac{M(\phi)}{\sqrt[n]{\det(\phi)}}$$

we obtain $\sqrt[n]{\det(f)} \geq M(f)/\gamma_n$. On the other hand, from the definition of $\mu(\mathbb{L})$ we have $\mu(\mathbb{L})|\det(h)$ and so $\sqrt[n]{\det(h)} \geq \sqrt[n]{\mu(\mathbb{L})}$. Inserting this into Lemma 4.3.1 gives the result. □

Corollary 4.3.3.
If $h \in \mathbb{L}^+$ and $n \geq 2$, then

$$(f,h) \geq \sqrt{3}\sqrt[n]{\mu(\mathbb{L})}M(f).$$

Proof.
We want to use Lemma 4.3.2 and therefore need

$$\frac{n}{\gamma_n} \geq \sqrt{3}. \tag{4.1}$$

For $2 \leq n \leq 8$ the values for γ_n are known and given in [Cas, p. 332] to be

$$\gamma_2^2 = \frac{4}{3}, \quad \gamma_3^3 = 2, \quad \gamma_4^4 = 4,$$
$$\gamma_5^5 = 8, \quad \gamma_6^6 = \frac{64}{3}, \quad \gamma_7^7 = 64, \quad \gamma_8^8 = 2^8.$$

These all fulfil inequality (4.1). For $n > 8$ (actually, even for $n \geq 3$) we may use Minkowski's bound (which can easily be derived from [Cas, p. 247])

$$\gamma_n < \tfrac{4}{\pi}\big(\Gamma(1+\tfrac{n}{2})\big)^{2/n},$$

where Γ is the usual gamma function. It remains to show that for $n \geq 3$

$$\Gamma(1+\tfrac{n}{2}) < \big(\tfrac{\pi}{4\sqrt{3}}n\big)^{\frac{n}{2}}, \tag{4.2}$$

which immediately establishes inequality (4.1). First of all, recall that

$$\forall n \in \mathbb{N} \ : \ \Gamma(1+\tfrac{n+2}{2}) = \Gamma(1+\tfrac{n}{2}+1) = (1+\tfrac{n}{2})\Gamma(1+\tfrac{n}{2}).$$

We will need the following inequality, which can be verified by easy calculation:

$$\forall n \in \mathbb{N} \ : \ \tfrac{\pi}{2\sqrt{3}}(n+1) > 1+\tfrac{n}{2}. \tag{4.3}$$

Now, consider the even integers. We show inequality (4.2) by induction. For $n = 4$ we have

$$\big(\tfrac{\pi}{4\sqrt{3}} \cdot 4\big)^{\frac{4}{2}} \approx 3.29 > 2 = \Gamma(1+\tfrac{4}{2}).$$

Assume the inequality holds for n. For $n+2$ we obtain

$$\begin{aligned}
\big(\tfrac{\pi}{4\sqrt{3}}(n+2)\big)^{\frac{n+2}{2}} &= \big(\tfrac{\pi}{4\sqrt{3}}n + \tfrac{\pi}{2\sqrt{3}}\big)^{\frac{n}{2}+1} \quad \text{the binomial theorem gives} \\
&\geq \big(\tfrac{\pi}{4\sqrt{3}}n\big)^{\frac{n}{2}+1} + \binom{\frac{n}{2}+1}{1}\big(\tfrac{\pi}{4\sqrt{3}}n\big)^{\frac{n}{2}}\big(\tfrac{\pi}{2\sqrt{3}}\big) \\
&= \big(\tfrac{\pi}{4\sqrt{3}}n\big)^{\frac{n}{2}}\big(\tfrac{\pi}{4\sqrt{3}}n + (\tfrac{n}{2}+1)\tfrac{\pi}{2\sqrt{3}}\big) \\
&= \big(\tfrac{\pi}{4\sqrt{3}}n\big)^{\frac{n}{2}}(2 \cdot \tfrac{\pi}{4\sqrt{3}}n + \tfrac{\pi}{2\sqrt{3}}) \quad \text{and with (4.3)} \\
&> \big(\tfrac{\pi}{4\sqrt{3}}n\big)^{\frac{n}{2}}(1+\tfrac{n}{2}) \quad \text{now using the assumption for } n \\
&> \Gamma(1+\tfrac{n}{2})(1+\tfrac{n}{2}) \\
&= \Gamma(1+\tfrac{n+2}{2}).
\end{aligned} \tag{4.4}$$

This completes the case of even integers. The inequality for $n = 3,5,7$ can be shown by simple calculation. For the other odd integers we use the following trick: the distance between the two sides of the inequality grows so rapidly that we may compare terms for $n = 8$ and $n = 10$, namely

$$\left(\tfrac{\pi}{4\sqrt{3}} \cdot 8\right)^{\frac{8}{2}} \approx 173,17 > 120 = \Gamma(1+\tfrac{10}{2}).$$

Using this as a start for the induction and noting that for $n \geq 3$ we may replace (4.3) by $\frac{\pi}{2\sqrt{3}}(n+1) > 1 + \frac{n+2}{2}$ we may use the same reasoning as in (4.4) to show

$$\forall n \geq 8, n \text{ even } : \left(\tfrac{\pi}{4\sqrt{3}} n\right)^{\frac{n}{2}} > \Gamma(1+\tfrac{n+2}{2}). \tag{4.5}$$

We may now use the monotony of both sides of the inequality to complete the proof for odd $n \geq 9$ by

$$\left(\tfrac{\pi}{4\sqrt{3}} n\right)^{\frac{n}{2}} \overset{\text{monotony}}{\geq} \left(\tfrac{\pi}{4\sqrt{3}}(n-1)\right)^{\frac{n-1}{2}} \overset{(4.5)}{>} \Gamma(1+\tfrac{n+1}{2}) \overset{\text{monotony}}{\geq} \Gamma(1+\tfrac{n}{2}).$$

□

Theorem 4.3.4.
Let f be a real positive definite n-ary form where $n \geq 2$. Then

$$\min_{h \in \mathbb{L}^+}(f,h) \geq \frac{\sqrt{3}\sqrt[n]{\mu(\mathbb{L})}}{\nu(\mathbb{L})} \min_{h \in \mathbb{L}_1}(f,h).$$

Proof.
According to Remark 4.1.3 we can find h_0 of rank 1 with $(f,h_0) = M(f)$. Now, Corollary 4.3.3 gives

$$\begin{aligned}
\min_{h \in \mathbb{L}^+}(f,h) &> \sqrt{3}\sqrt[n]{\mu(\mathbb{L})} M(f) \quad \text{from the definition of } h_0 \\
&= \sqrt{3}\sqrt[n]{\mu(\mathbb{L})}(f,h_0) \\
&= \frac{\sqrt{3}\sqrt[n]{\mu(\mathbb{L})}}{\nu(\mathbb{L})}(f,\nu(\mathbb{L})h_0) \\
&\geq \frac{\sqrt{3}\sqrt[n]{\mu(\mathbb{L})}}{\nu(\mathbb{L})} \min_{h \in \mathbb{L}_1}(f,h)
\end{aligned}$$

since $\nu(\mathbb{L})h_0 \in \mathbb{L}$ from the definition of $\nu(\mathbb{L})$ and since h_0 has rank 1 we also have $\nu(\mathbb{L})h_0 \in \mathbb{L}_1$. □

Corollary 4.3.5: Dimension 2.
Let f be a real positive definite binary form, i. e. $n = 2$. Then

$$\min_{h \in \mathbb{L}^0 \setminus \{0\}}(f,h) \geq \min\left\{\frac{\sqrt{3\mu(\mathbb{L})}}{\nu(\mathbb{L})}, 1\right\} \min_{h \in \mathbb{L}_1}(f,h).$$

Proof.
This follows directly from Theorem 4.3.4 since for $n = 2$ we have $\mathbb{L}^0 \setminus \{0\} = \mathbb{L}^+ \cup \mathbb{L}_1$.
□

4.3.2 Application to Tits lattices

Let us now apply the results to the case we are interested in, namely when $\mathbb{L}$ is a Tits lattice.

Corollary 4.3.6: $(1,t)$-polarisation.
Let $t \in \mathbb{N}, t \geq 3$ and $\mathbb{L} = \mathbb{L}(1,t)$. Then

$$\min_{h\in\mathbb{L}^0\setminus\{0\}}(f,h) \geq \sqrt{\frac{3}{t}} \min_{h\in\mathbb{L}_1}(f,h).$$

Proof.
This follows from Theorem 4.3.4 using the values given in Lemma 4.2.4. □

Unfortunately, for a general lattice of higher dimension it is not as easily possible to compare $\min_{h\in\mathbb{L}^0}$ and $\min_{h\in\mathbb{L}_1}$. For a Tits lattice, i. e. a lattice that comes from a polarisation, we can however give the following theorem:

Theorem 4.3.7.
Let f be a real positive n-ary form with $n \geq 2$ and let $\mathbb{L} = \mathbb{L}(1,d_1,\dots,d_{1:n-1})$. Then

$$\min_{h\in\mathbb{L}^+}(f,h) \geq \frac{\sqrt{3}}{\sqrt[n]{\prod_{i=1}^{n-1} d_i^i}} \min_{h\in\mathbb{L}_1}(f,h) \quad \text{and} \tag{4.6}$$

$$\min_{h\in\mathbb{L}^0\setminus\{0\}}(f,h) \geq C(\mathbb{L}) \min_{h\in\mathbb{L}_1}(f,h) \quad \text{where} \tag{4.7}$$

$$C(\mathbb{L}) := \min\left\{1, \min_{2\leq r\leq n} \frac{\sqrt{3}}{\sqrt[r]{\prod_{i=1}^{r-1} d_i^i}}\right\}.$$

Proof.
If h is positive definite, we may use Theorem 4.3.4 with the values given in Lemma 4.2.4 to obtain

$$\begin{aligned}\min_{h\in\mathbb{L}^+}(f,h) &\geq \frac{\sqrt{3}\sqrt[n]{\prod_{i=1}^{n-1} d_i^{n-i}}}{d_{1:n-1}} \min_{h\in\mathbb{L}_1}(f,h) \\ &= \frac{\sqrt{3}}{\sqrt[n]{\prod_{i=1}^{n-1} d_i^i}} \min_{h\in\mathbb{L}_1}(f,h)\end{aligned}$$

which proves (4.6).

The value $C(\mathbb{L})$ is constructed from terms that give valid bounds for the different possible cases $r := \operatorname{rank}(h) = 1,\dots,n$. The first term, which is 1, obviously covers for h of rank $r = 1$. The term for $r = n$ has already been established in (4.6).

For positive semi-definite h of rank r with $1 < r < n$, we proceed along the lines of Theorem 3 in [BC].

We can give h as $h(x) = {}^t xBx$ where B is a rational singular matrix; the equation

$$Bx = 0$$

hence has a rational solution $x \neq 0$. Multiplying by a suitable rational number, we obtain a primitive integral vector $v = (v_1, \dots, v_n)$ with

$$Bv = 0.$$

According to Lemma A.8 we can find an integral unimodular matrix T of the form

$$T = \begin{pmatrix} * & d_1 & d_{1:2} & \dots & d_{1:n-2} & v_1 \\ * & * & d_2 & & d_{2:n-2} & v_2 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ * & & \dots & * & d_{n-2} & v_{n-2} \\ * & & \dots & & * & v_{n-1} \\ * & & \dots & & * & v_n \end{pmatrix}.$$

We now replace f and h by ${}^tT^{-1}f$ and Th, respectively; this leaves $M(f)$ unchanged and also (f,h), by

$$({}^tT^{-1}f, Th) = \mathrm{tr}(T^{-1}A{}^tT^{-1}{}^tTBT) = \mathrm{tr}(T^{-1}ABT) = \mathrm{tr}(AB) = (f,h).$$

The matrix B of h is replaced by the matrix ${}^tTBT$ and, since $Bv = 0$, the integral form h has been replaced by an integral form in the $n-1$ variables $x_1, \dots, x_{n-1}$. Furthermore, the special form of T guarantees that ${}^tTBT \in \mathbb{L}$. We may clearly repeat this procedure until $h(x)$ is expressed as a positive definite integral form in the variables $x_1, \dots, x_r$.

Let

$$\begin{aligned} \overline{h}(x_1, \dots, x_r) &:= h(x) = h(x_1, \dots, x_r, 0, \dots, 0), \\ \overline{f}(x_1, \dots, x_r) &:= f(x_1, \dots, x_r, 0, \dots, 0). \end{aligned}$$

Then $\overline{f}, \overline{h}$ are positive definite forms in r variables, and $\overline{h}$ is integral. Clearly we have $M(\overline{f}) \geq M(f)$ and $(\overline{f}, \overline{h}) = (f,h)$. With respect to the sublattice

$$\overline{\mathbb{L}} := \left\{ \begin{pmatrix} \mathbb{Z} & \dots & d_{1:r-1}\mathbb{Z} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \\ d_{1:r-1}\mathbb{Z} & \dots & d_{1:r-1}\mathbb{Z} & 0 & & \vdots \\ 0 & \dots & 0 & 0 & & \\ \vdots & & & & \ddots & \\ 0 & & \dots & & & 0 \end{pmatrix} \right\} \cap \mathbb{L} \subset \mathbb{L}$$

which contains $\overline{h}$ we may therefore use (4.6) to obtain

$$\min_{\overline{h} \in \overline{\mathbb{L}} \text{ of rank } r} (\overline{f}, \overline{h}) \geq \frac{\sqrt{3}}{\sqrt[r]{\prod_{i=1}^{r-1} d_i^i}} \min_{\overline{h} \in \overline{\mathbb{L}}_1} (\overline{f}, \overline{h}).$$

Note, that $\overline{\mathbb{L}}_1 \subset \mathbb{L}_1$ and thus

$$\min_{\overline{h} \in \overline{\mathbb{L}}_1} (\overline{f}, \overline{h}) \geq \min_{h \in \mathbb{L}_1} (f,h).$$

Hence, we have

$$
\begin{aligned}
(f,h) = (\overline{f},\overline{h}) &\geq \min_{\overline{h}\in\overline{\mathbb{L}}^{+}}(\overline{f},\overline{h}) \\
&\geq \frac{\sqrt{3}}{\sqrt[r]{\prod_{i=1}^{r-1} d_i^i}} \min_{\overline{h}\in\overline{\mathbb{L}}_1}(\overline{f},\overline{h}) \\
&\geq \frac{\sqrt{3}}{\sqrt[r]{\prod_{i=1}^{r-1} d_i^i}} \min_{h\in\mathbb{L}_1}(f,h).
\end{aligned}
$$

This construction supplies all the other terms in $C(\mathbb{L})$ and thus ends the proof. □

Corollary 4.3.8: Dimension 3.
For $g=3$ and $\mathbb{L}=\mathbb{L}(1,m,mn)$ with $m,n\in\mathbb{N}$ we have

$$
\min_{h\in\mathbb{L}^0\setminus\{0\}}(f,h) \geq C(m,n) \min_{h\in\mathbb{L}_1}(f,h)
$$

where

$$
C(m,n) = \begin{cases} 1 & \text{if } mn\leq 2 \\ \sqrt{3/m} & \text{if } m\geq 3 \text{ and } m\geq n^4 \\ \frac{\sqrt{3}}{\sqrt[3]{mn^2}} & \text{otherwise} \end{cases} .
$$

Proof.
This follows easily from Theorem 4.3.7 by explicitely determining the minimum. □

Remark 4.3.9.
Theorem 4.3.7 can now be used as a substitute for Corollary 4.1.4. This leads to the following generalisation of Corollary 4.1.6:

Theorem 4.3.10.
Assume a (non-principal) polarisation $(1,d_1,\ldots,d_{1:g-1})$ and let $\mathbb{L}$ be its Tits lattice. Suppose $\mathfrak{D}=\mathfrak{S}_g, \Gamma\subset\Gamma_{\text{pol}}$ and let χ and ω be as in Theorem 4.1.5. Then

$$
\chi\omega^{\otimes l} \text{extends to } \overline{\mathfrak{D}/\Gamma} \iff \begin{cases} \chi \text{ vanishes on all} \\ \text{rational corank-1 boundary components} \\ \text{of order at least } {}^{l}/_{C(\mathbb{L})}. \end{cases}
$$

Remark 4.3.11.
This theorem can now be used in the case of interest as a substitute for Corollary 4.1.4. We shall now present the construction how we combine the known facts to reach $\mathscr{A}_{\text{pol}}(n)$.

Chapter 5

How to get from $\mathscr{A}_g$ to $\mathscr{A}_{\mathrm{pol}}(n)$

5.1 Geometric layout

Our main goal is to investigate the Kodaira dimension of $\mathscr{A}_{\mathrm{pol}}(n)$. For this, we need pluricanonical forms on a compactification $(\mathscr{A}_{\mathrm{pol}}(n))^*$ which we can construct from, for example, cusp forms. However, we do not know a non-trivial cusp form on $\mathscr{A}_{\mathrm{pol}}(n)$ yet. But there is a cusp form on $\mathscr{A}_g$, as we have stated in Theorem 1.2.35. For the time being, assume we have a cusp form χ with respect to $\mathrm{Sp}(2g,\mathbb{Z})$ of weight w_χ that vanishes on the cusp of $\mathscr{A}_g$ of degree v_χ. How can we use χ to construct a cusp form on $(\mathscr{A}_{\mathrm{pol}}(n))^*$?

5.1.1 Maps, cusps and branching

Assume the following situation:

$$\begin{array}{ccccc} & (\mathscr{A}_{\mathrm{pol}}^{\mathrm{lev}})' & & (\mathscr{A}_{\mathrm{pol}}(n))' & \\ \pi_1 \swarrow & & \searrow \pi_2 \quad \pi_3 \swarrow & & \\ (\mathscr{A}_g)' & & (\mathscr{A}_{\mathrm{pol}})' & & \end{array}$$

where by $\mathscr{A}'$ we denote Mumford's partial compactification of $\mathscr{A}$. This is constructed from $\mathscr{A}$ by adding only the corank-1 boundary components. Note that this construction is well defined since it does not depend on a fan. Due to Lemma 3.2.6 we may define the spaces by $\mathscr{A}_g := \mathfrak{S}_g/\mathrm{Sp}(2g,\mathbb{Z}), \mathscr{A}_{\mathrm{pol}} := \mathfrak{S}_g/\Gamma_{\mathrm{pol}}, \mathscr{A}_{\mathrm{pol}}^{\mathrm{lev}} := \mathfrak{S}_g/\Gamma_{\mathrm{pol}}^{\mathrm{lev}}$ and $\mathscr{A}_{\mathrm{pol}}(n) := \mathfrak{S}_g/\Gamma_{\mathrm{pol}}(n)$ and have all groups acting on $\mathfrak{S}_g$ in the same way. We know that $\Gamma_{\mathrm{pol}}^{\mathrm{lev}} \subset \mathrm{Sp}(2g,\mathbb{Z})$ – which, of course, is the reason for using this intermediate step.

What do we know about the partial compactifications of these spaces? First of all, we know[1] that $(\mathscr{A}_g)'$ has only a single cusp which we shall call C_0.

In $(\mathscr{A}_{\mathrm{pol}})'$ there are several rational corank-1 boundary components which we shall denote by $C_1,\dots,C_u$. Fix i in $1,\dots,u$ and denote the irreducible components of the reduction of $\pi_2^* C_i$ by $C_i^1,\dots,C_i^{v_i} \subset (\mathscr{A}_{\mathrm{pol}}^{\mathrm{lev}})'$. We have already seen that to each C_i^j we can associate a unique[2] primitive vector in $\mathbb{Z}^{2g}$ that we shall, by abuse of notation, also denote by C_i^j. Let $\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)$ be a set of vectors that is a full system of representatives

[1] See [HKW, Part I, Lemma 3.11]

[2] up to multiplication with -1

for these boundary components. Denote the order of branching of π_1 in C_i^j by $m_1(i,j)$ and that of π_2 by $m_2(i,j)$.

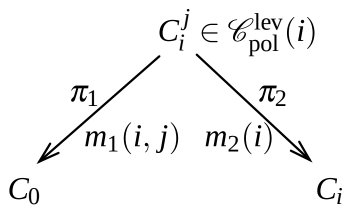
We know that $\Gamma_{\mathrm{pol}}^{\mathrm{lev}}$ is a normal subgroup of Γ_{pol} and so $\pi_2 : \mathscr{A}_{\mathrm{pol}}^{\mathrm{lev}} \to \mathscr{A}_{\mathrm{pol}}$ is a Galois cover. The Galois group Γ_G of this cover will be discussed in more detail in section 5.2.2. It operates transitively on $\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)$, so that for any fixed i the order of the stabiliser $\mathrm{Stab}_{\Gamma_G}(C) := \{g \in \Gamma_G \big| g(C) = C\}$ is the same for all $C \in \mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)$. If $-1 \notin \Gamma_G$ (as we will see in Lemma 5.2.7 this is implied by $d_{1:g-1} > 2$), it can be given by

$$|\,\mathrm{Stab}_{\Gamma_G}(C_i^j)| = \frac{|\Gamma_G|}{|\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)|}. \tag{5.1}$$

Furthermore, the values $m_2(i,j)$ are the same for all $C \in \mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)$ and we can denote them by $m_2(i)$. Then we have

$$\pi_2^* C_i = \sum_j m_2(i,j) C_i^j = m_2(i) \sum_j C_i^j.$$

All in all, we have considered the following cusps and maps with order of branching:



5.1.2 Modular forms

From Corollary 3.2.13 we know that $\Gamma_{\mathrm{pol}}^{\mathrm{lev}} \subset \mathrm{Sp}(2g,\mathbb{Z})$ and hence χ is also a cusp form with respect to $\Gamma_{\mathrm{pol}}^{\mathrm{lev}}$. On C_i^j it vanishes of order $\mathrm{ord}(\chi, C_i^j) = v_\chi m_1(i,j)$.

For a modular form F and a matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ the slash-operator is defined by

$$F|_k M(\tau) := \det(C\tau + D)^{-k} F(M\tau).$$

We use this operator to construct the symmetrisation of χ

$$\chi^{\mathrm{sym}} := \prod_{M \in \Gamma_G} \chi|_{w_\chi} M$$

where w_χ is the weight of χ. This is a cusp form with respect to $\Gamma_{\mathrm{pol}}^{\mathrm{lev}}$ (and also with respect to Γ_{pol}) of weight $w_{\mathrm{sym}} = |\Gamma_G| w_\chi$. Because of the symmetrisation, we may choose any one cusp C_i^1 and have

$$\forall C \in \mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i) \ : \ \mathrm{ord}(\chi^{\mathrm{sym}}, C) = \mathrm{ord}(\chi^{\mathrm{sym}}, C_i^1).$$

To be precise, we have

$$\begin{aligned}\operatorname{ord}(\chi^{\mathrm{sym}}, C_i^1) &= \sum_{a\in\Gamma_G} \operatorname{ord}(\chi, a^{-1}(C_i^1))\\ &= \sum_{C_i^j\in\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)} |\operatorname{Stab}_{\Gamma_G}(C_i^j)| \operatorname{ord}(\chi, C_i^j)\\ &= \sum_{C_i^j\in\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)} \frac{|\Gamma_G|}{|\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)|} v_\chi m_1(i,j)\\ &= v_\chi \frac{|\Gamma_G|}{|\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)|} \sum_{C_i^j\in\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)} m_1(i,j).\end{aligned}$$

To ease the notation define

$$M_1(i) := \sum_{C_i^j\in\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)} m_1(i,j).$$

We have already mentioned that χ^{sym} is also a cusp form with respect to Γ_{pol}. To make clear which group we are referring to we use the notation $\overline{\chi}$ in case of this second group. The weight obviously stays the same, i. e. we have $w_{\overline{\chi}} = w_{\mathrm{sym}} = |\Gamma_G| w_\chi$. On $(\mathscr{A}_{\mathrm{pol}})'$ we now have

$$\begin{aligned}\operatorname{ord}(\overline{\chi}, C_i) &= \operatorname{ord}(\chi^{\mathrm{sym}}, C_i^1)/m_2(i)\\ &= v_\chi \frac{|\Gamma_G|}{m_2(i)|\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)|} M_1(i).\end{aligned}$$

5.1.3 Vanishing on higher codimension

So far we are able to control the order of vanishing on the corank-1 boundary components of a compactification of $\mathscr{A}_{\mathrm{pol}}$. This compactification may, however, be singular. Assume we are given a Γ_{pol}-admissible collection of fans Σ and obtain the corresponding compactification $(\mathscr{A}_{\mathrm{pol}})^*$. According to Theorem 2.2.31, there exists a refinement $\tilde{\Sigma}$ of the collection Σ, which is also Γ_{pol}-admissible, such that the corresponding compactification $(\mathscr{A}_{\mathrm{pol}})^\sim$ is stack-smooth. Furthermore, we also get that the map $(\mathscr{A}_{\mathrm{pol}})^\sim \to (\mathscr{A}_{\mathrm{pol}})^*$ is a blowing-up and hence $(\mathscr{A}_{\mathrm{pol}})^\sim$ is constructed from $(\mathscr{A}_{\mathrm{pol}})^*$ by inserting new boundary divisors.

We are now ready to proceed to the map π_3. Assume that the level n is such that π_3 is branched of order n along all boundary components. (It will turn out that the assumption $\gcd(d_{1:g-1}, n) = 1$ does already imply this.)

For any cusp C in the pullback $\pi_3^* C_i$ we then have

$$\operatorname{ord}(\overline{\chi}, C) = n \operatorname{ord}(\overline{\chi}, C_i) = n v_\chi \frac{|\Gamma_G|}{m_2(i)|\mathscr{C}_{\mathrm{pol}}^{\mathrm{lev}}(i)|} M_1(i). \tag{5.2}$$

Now we use the generalised Barnes and Cohn Theorem 4.3.7 on $(\mathscr{A}_{\mathrm{pol}}(n))^\sim$ which states that $\overline{\chi}$ vanishes on all of the boundary at least of order $\operatorname{ord}(\overline{\chi}, C)C(\mathbb{L})$. (Recall that $C(\mathbb{L}) \leq 1$.) On the other hand, $\overline{\chi}$ is a modular form of weight $w_{\overline{\chi}} = |\Gamma_G| w_\chi$ with respect to $\tilde{\Gamma}_{\mathrm{pol}}(n) \subset \tilde{\Gamma}_{\mathrm{pol}}$. This leads to the following equation for $(\mathscr{A}_{\mathrm{pol}}(n))^{\sim}$:

$$w_\chi |\Gamma_G| L = \mathrm{ord}(\overline{\chi}, C) C(\mathbb{L}) D + D_{\mathrm{eff}}$$

where L is the divisor corresponding to the ($\mathbb{Q}$-)line bundle[3] of modular forms of weight 1 on $\mathscr{A}_{\mathrm{pol}}(n)$, D is the boundary divisor of $(\mathscr{A}_{\mathrm{pol}}(n))^{\sim}$ and D_{eff} is some effective divisor that we do not need to specify more precisely. This implies

$$-D \geq -\frac{w_\chi |\Gamma_G|}{\mathrm{ord}(\overline{\chi}, C) C(\mathbb{L})} L + D'_{\mathrm{eff}}.$$

Assume now that $n \geq 3$ such that $\tilde{\Gamma}_{\mathrm{pol}}(n)$ is neat. For any smooth toroidal compactification of $\mathscr{A}_{\mathrm{pol}}(n)$ we may use Lemma 2.2.36 to obtain

$$\begin{aligned} K &= (g+1)L - D \\ &\geq \left[(g+1) - \frac{w_\chi |\Gamma_G|}{\mathrm{ord}(\overline{\chi}, C) C(\mathbb{L})}\right] L + D'_{\mathrm{eff}}. \end{aligned}$$

We know from Mumford's extension of Hirzebruch proportionality (see [Mu2, Corollary 3.5]) that $h^0(L^k) \sim k^{\frac{1}{2}g(g+1)}$. We can therefore conclude that $h^0(K^k) \sim h^0(L^k) \sim k^{\frac{1}{2}g(g+1)}$ and hence that the Kodaira dimension is maximal if the coefficient of L is positive. This means we want

$$\begin{aligned} & \mathrm{ord}(\overline{\chi}, C) C(\mathbb{L}) > \frac{w_\chi |\Gamma_G|}{g+1} \\ \iff & n \frac{v_\chi |\Gamma_G| M_1(i) C(\mathbb{L})}{m_2(i) |\mathscr{C}^{\mathrm{lev}}_{\mathrm{pol}}(i)|} > \frac{w_\chi |\Gamma_G|}{g+1} \\ \iff & n > \frac{w_\chi m_2(i) |\mathscr{C}^{\mathrm{lev}}_{\mathrm{pol}}(i)|}{(g+1) v_\chi M_1(i) C(\mathbb{L})}. \end{aligned} \tag{5.3}$$

5.2 Properties of the Construction

5.2.1 The geometry of $\mathscr{A}^{\mathrm{lev}}_{\mathrm{pol}} \to \mathscr{A}_g$ and $\mathscr{A}^{\mathrm{lev}}_{\mathrm{pol}} \to \mathscr{A}_{\mathrm{pol}}$

We shall now give more details on the geometry of the maps π_1 and π_2. In particular, we shall state a lemma on the order of branching for these maps in each corank-1 boundary component of $(\mathscr{A}^{\mathrm{lev}}_{\mathrm{pol}})'$.

Lemma 5.2.1: Order of branching.
For a corank-1 boundary component $F \subset (\mathscr{A}^{\mathrm{lev}}_{\mathrm{pol}})'$ the orders of branching of the maps between the partial compactifications $\pi_1 : (\mathscr{A}^{\mathrm{lev}}_{\mathrm{pol}})' \to (\mathscr{A}_g)'$ and $\pi_2 : (\mathscr{A}^{\mathrm{lev}}_{\mathrm{pol}})' \to (\mathscr{A}_{\mathrm{pol}})'$ are given by

$$\begin{aligned} m_1(C) &:= [P'_{\mathrm{Sp}(2g,\mathbb{Z})}(C) : P'_{\Gamma^{\mathrm{lev}}_{\mathrm{pol}}}(C)] \quad \text{and} \\ m_2(C) &:= [P'_{\Gamma_{\mathrm{pol}}}(C) : P'_{\Gamma^{\mathrm{lev}}_{\mathrm{pol}}}(C)], \end{aligned}$$

[3] For $n \geq 3$ this is in fact a line bundle.

respectively, where $P'_\Gamma(C) := \mathscr{P}'(F) \cap \Gamma \subset \mathscr{P}(F)$ is the relevant lattice part of the stabiliser of F with $C = V(F)$ as in Definition 2.2.9.

Proof.
This is a specialisation of Theorem 2.2.34. □

Let us now give the general outline of how we want to perform this calculation in both cases. We do the calculations that are the same for all cases over the rationals, and only then intersect with the four different groups.

The group $\mathrm{Sp}(2g,\mathbb{Q})$ has only a single corank-1 boundary component, namely $C_0 \widehat{=} (0,\dots,0,1) \in \mathbb{Z}^{2g}$, and for this cusp we have shown in section 2.2.3

$$P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C_0) = \left\{ \begin{pmatrix} \mathbb{1} & S \\ 0 & \mathbb{1} \end{pmatrix} \text{ where } S = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & s \end{pmatrix} \text{ and } s \in \mathbb{Q} \right\}.$$

From this information we calculate the groups $P'_\Gamma(C)$ for the other $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Q})$ and any cusp C as follows: First, we calculate $P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C)$ by conjugating with a suitable matrix $M \in \mathrm{Sp}(2g,\mathbb{Q})$. Since C_0 is the only cusp with respect to $\mathrm{Sp}(2g,\mathbb{Q})$ and C is given by a primitive vector, we can always find a matrix $M \in \mathrm{Sp}(2g,\mathbb{Q})$ satisfying

$$C = C_0 M. \tag{5.4}$$

This implies

$$M^{-1} \mathscr{P}_{\mathrm{Sp}(2g,\mathbb{Q})}(C_0) M = \mathscr{P}_{\mathrm{Sp}(2g,\mathbb{Q})}(C)$$

which, in turn, implies by intersecting with $\mathrm{Sp}(2g,\mathbb{Q})$

$$M^{-1} P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C_0) M = P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C).$$

Now, since $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Q})$, we can calculate the group we are interested in by

$$P'_\Gamma(C) = P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C) \cap \Gamma.$$

This leads to the following lemma:

Lemma 5.2.2.
Let $P'_\mathbb{Q} := P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C_0)$. Then we have

$$\begin{aligned} m_1(C) &= [M^{-1} P'_\mathbb{Q} M \cap \mathrm{Sp}(2g,\mathbb{Z}) : M^{-1} P'_\mathbb{Q} M \cap \Gamma^{\mathrm{lev}}_{\mathrm{pol}}] \text{ and} \\ m_2(C) &= [M^{-1} P'_\mathbb{Q} M \cap \Gamma_{\mathrm{pol}} : M^{-1} P'_\mathbb{Q} M \cap \Gamma^{\mathrm{lev}}_{\mathrm{pol}}]. \end{aligned}$$

Note that the matrices $Q_0 \in P'_\mathbb{Q}$ have the form $Q_0 = \left(\begin{smallmatrix} 0 & S \\ 0 & 0 \end{smallmatrix}\right) + \mathbb{1}$, so that for all $Q_C \in P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C) = M^{-1} P'_\mathbb{Q} M$ we have

$$Q_C = M^{-1} Q_0 M = M^{-1} \left(\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} + \mathbb{1} \right) M = M^{-1} \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} M + \mathbb{1} = Q + \mathbb{1}$$

where obviously

$$Q := M^{-1}\begin{pmatrix}0 & S\\ 0 & 0\end{pmatrix}M. \tag{5.5}$$

To intersect the group $P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C)$ with Γ we only need to consider the conditions imposed on Q by the appropriate lemma from section 3.2.2.

Because of the properties of $M \in \mathrm{Sp}(2g,\mathbb{Q})$ stated in Lemma 3.2.1 we know that the inverse of a matrix $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is given by $M^{-1} = \left(\begin{smallmatrix}{}^t\delta & -{}^t\beta\\ -{}^t\gamma & {}^t\alpha\end{smallmatrix}\right)$ where $\alpha,\beta,\gamma,\delta \in \mathbb{Q}^{g\times g}$. Split the vector representing the cusp into two vectors of length g such that $C = (\mathfrak{c}_1,\mathfrak{c}_2)$. Then equation (5.4) implies that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are the last rows of the matrices γ and δ, respectively. Since the matrix S has only one non-zero entry $s \in \mathbb{Q}$ we see that

$$Q = s\begin{pmatrix}{}^t\mathfrak{c}_2\\ -{}^t\mathfrak{c}_1\end{pmatrix}(\mathfrak{c}_1,\mathfrak{c}_2). \tag{5.6}$$

We shall now give the explicit calculation in the two cases separately.

Lemma 5.2.3: Branching of π_1.
For a cusp $C_i^j = (D_{1:g-1}, D_{2:g-1}a_2,\dots,a_g,0,D_{2:g-1}a_{g+2},\dots,a_{2g}) \in \mathscr{A}_{\mathrm{pol}}^{\mathrm{lev}}$ given with respect to $\tilde{\Gamma}_{\mathrm{pol}}^{\mathrm{lev}}$ the order of branching of $\pi_1 : \mathscr{A}_{\mathrm{pol}}^{\mathrm{lev}} \to \mathscr{A}_g$ is given by

$$m_1(C_i^j) = \gcd(D_{1:g-1}, D_{2:g-1}a_2,\dots,D_{g-1}a_{g-1},a_g)^2.$$

Proof.
Recall from Lemma 3.3.1 that any cusp can be represented in the form given in the statement. Since we want to work with $\Gamma_{\mathrm{pol}}^{\mathrm{lev}}$ rather than with $\tilde{\Gamma}_{\mathrm{pol}}^{\mathrm{lev}}$ we have to multiply by R and obtain

$$C_i^j R = (D_{1:g-1}, D_{2:g-1}a_2,\dots,a_g,0,d_1D_{2:g-1}a_{g+2},\dots,d_{1:g-1}a_{2g}).$$

In case this is not a primitive vector we divide by $k := \gcd(D_{1:g-1},\dots,d_{1:g-1}a_{2g})$ to obtain as representative of the cusp

$$C := \Big(\frac{D_{1:g-1}}{k},\frac{D_{2:g-1}a_2}{k},\dots,\frac{a_g}{k},0,\frac{d_1D_{2:g-1}a_{g+2}}{k},\dots,\frac{d_{1:g-1}a_{2g}}{k}\Big) \sim_{\mathbb{Q}} C_i^j R.$$

We have already stated above that a matrix M satisfying equation (5.4) always exists. So, we define Q as in (5.5) and can now proceed by asking when $Q+\mathbb{1}$ is in $P'_\Gamma(C) = P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C)\cap\Gamma$ for $\Gamma = \mathrm{Sp}(2g,\mathbb{Z})$ or $\Gamma = \tilde{\Gamma}_{\mathrm{pol}}^{\mathrm{lev}}$, respectively.

When taking the intersection of $P'_{\mathrm{Sp}(2g,\mathbb{Q})}(C)$ with $\mathrm{Sp}(2g,\mathbb{Z})$ the only condition is that the matrix Q be integer. We can use

$$\mathfrak{c}_1 = \Big(\frac{D_{1:g-1}}{k},\frac{D_{2:g-1}a_2}{k},\dots,\frac{a_g}{k}\Big) \quad \text{and}$$
$$\mathfrak{c}_2 = \Big(0,\frac{d_1D_{2:g-1}a_{g+2}}{k},\dots,\frac{d_{1:g-1}a_{2g}}{k}\Big).$$

in equation (5.6) to see that the first entry of the $g+1$st row is given by

$$q_{g+1,1} = -s(-(\mathfrak{c}_1)_1)(\mathfrak{c}_1)_1 = -\tfrac{D^2_{1:g-1}}{k^2}s.$$

We substitute $t := \frac{D^2_{1:g-1}}{k^2}s$ and obtain the necessary condition $t \in \mathbb{Z}$ for Q to be integer. With this substitution,

$$Q = \frac{k^2 t}{D^2_{1:g-1}} \begin{pmatrix} {}^t\mathfrak{c}_2 \\ -{}^t\mathfrak{c}_1 \end{pmatrix} (\mathfrak{c}_1, \mathfrak{c}_2).$$

Obviously, the k^2 cancels in every entry. Furthermore, it is easy to see that $D_{1:g-1}$ divides every entry of $\mathfrak{c}_2$. Define

$$\tilde{\mathfrak{c}}_1 := \tfrac{k}{D_{1:g-1}}\mathfrak{c}_1 = \left(1, \tfrac{a_2}{D_1}, \tfrac{a_3}{D_{1:2}}, \dots, \tfrac{a_g}{D_{1:g-1}}\right).$$

Then the lower left quarter of Q is given by $-\frac{k^2 t}{D^2_{1:g-1}} {}^t\mathfrak{c}_1\mathfrak{c}_1 = -t\,{}^t\tilde{\mathfrak{c}}_1\tilde{\mathfrak{c}}_1$. The diagonal elements of this quarter give rise to the necessary condition

$$(5.7) \qquad -t\left(\frac{a_i}{D_{1:i-1}}\right)^2 \in \mathbb{Z} \quad \Longleftrightarrow \quad D^2_{1:i-1} | a_i^2 t \quad \text{for all } i = 2, \dots, g.$$

If this is given, Lemma A.9 tells us that both

$$D_{1:i-1}|a_i t \quad \text{and} \quad D_{1:i-1}D_{1:j-1}|a_i a_j t$$

are satisfied for all $2 \le i, j \le g$. This means that $t\,{}^t\tilde{\mathfrak{c}}_1 \frac{\mathfrak{c}_2}{D_{1:g-1}}$ and $t\,{}^t\tilde{\mathfrak{c}}_1\mathfrak{c}_1$ are integer matrices, and hence so is Q. Therefore, condition (5.7) is also sufficient. Since for $t \in \mathbb{Z}$ we have the equivalence

$$D^2_{1:i-1}|a_i^2 t \iff \left(\frac{D_{1:i-1}}{\gcd(D_{1:i-1}, a_i)}\right)^2 | t,$$

we have the condition $t \in n^2\mathbb{Z}$ where

$$n := \operatorname{lcm}\left[\frac{D_{1:i-1}}{\gcd(D_{1:i-1}, a_i)}\right]_{i=2,\dots,g} = \operatorname{lcm}\left[\frac{D_{1:g-1}}{\gcd(D_{1:g-1}, D_{i:g-1}a_i)}\right]_{i=2,\dots,g}.$$

Using Lemma A.10 (and letting $a_1 = 1$ to simplify the notation) we therefore obtain

$$(5.8) \qquad Q \in \operatorname{Sp}(2g, \mathbb{Z}) \quad \Longleftrightarrow \quad t \in \left(\frac{D_{1:g-1}}{\gcd(D_{i:g-1}a_i)_{i=1,\dots,g}}\right)^2 \mathbb{Z}.$$

Since $\Gamma^{\text{lev}}_{\text{pol}} \subset \operatorname{Sp}(2g, \mathbb{Z})$ we also get this condition for $P'_{\Gamma^{\text{lev}}_{\text{pol}}}(C)$ but in addition we have to consider Lemma 3.2.12. The conditions of the upper right quarter of Q state that

$$\frac{k^2 t}{D^2_{1:g-1}} {}^t\mathfrak{c}_2\mathfrak{c}_2 \in {}^t\mathfrak{d}\mathfrak{d} \otimes \mathbb{Z}$$

where $\mathfrak{d} := (1, d_1, \dots, d_{1:g-1})$ as in Lemma 3.2.12. From the elements on the diagonal of this quarter we obtain the necessary condition

$$\frac{k^2 t}{D^2_{1:g-1}} \left(\frac{d_{1:i}D_{i+1:g-1}a_{g+i+1}}{k}\right)^2 = \frac{d^2_{1:i}}{D^2_{1:i}} a^2_{g+1+i} t \in d^2_{1:i}\mathbb{Z} \quad \Longleftrightarrow \quad D^2_{1:i}|a^2_{g+1+i}t$$

for all $i = 1, \dots, g-1$. Again, Lemma A.9 tells us that this implies all other conditions on Q and hence the condition is also sufficient. We proceed exactly as before and obtain (letting $a_{g+1} = 0$ again to simplify the notation)

$$Q \in \Gamma_{\text{pol}}^{\text{lev}} \iff t \in \left(\frac{D_{1:g-1}}{\gcd(D_{i:g-1}a_i, D_{i:g-1}a_{g+i})_{i=1,\dots,g}}\right)^2 \mathbb{Z} = D_{1:g-1}^2 \mathbb{Z}$$

since the vector C is primitive and hence the denominator is 1. Combining this with equation (5.8) gives $m_1(C) = \gcd(D_{i:g-1}a_i)^2_{i=1,\dots,g}$ as claimed. □

Lemma 5.2.4: Branching of π_2.
For a cusp $C_i^j = (D_{1:g-1}, D_{2:g-1}a_2, \dots, a_g, 0, D_{2:g-1}a_{g+1}, \dots, a_{2g}) \in \mathscr{A}_{\text{pol}}^{\text{lev}}$ given with respect to $\tilde{\Gamma}_{\text{pol}}^{\text{lev}}$ the order of branching of $\pi_2 : \mathscr{A}_{\text{pol}}^{\text{lev}} \to \mathscr{A}_{\text{pol}}$ is given by

$$m_2(C_i^j) = D_{1:g-1}.$$

Proof.
Since $\Gamma_{\text{pol}}^{\text{lev}}$ is a normal subgroup of Γ_{pol}, the map π_2 induces a Galois covering. This means that we may consider any cusp C_i^0 in the orbit of C_i^j under the action of the Galois group. All those cusps map to the same cusp $C_i \in \mathscr{A}_{\text{pol}}$ which, according to Corollary 3.3.6, can be given in the form $C_i = (D_{1:g-1}, D_{2:g-1}a_2, \dots, D_{g-1}a_{g-1}, 1, 0, 0, 0)$. Let us therefore choose $C_i^0 = (D_{1:g-1}, D_{2:g-1}a_2, \dots, D_{g-1}a_{g-1}, 1, 0, 0, 0)$ which after multiplying by R remains unchanged. Furthermore, it is obviously a primitive vector.

As before we define M and Q by (5.4) and (5.5), respectively. Again, we obtain conditions on Q by intersecting $P'_{\text{Sp}(2g,\mathbb{Q})}(C)$ with Γ.

Since $\mathfrak{c}_2 = 0$, according to (5.6) the only non-zero entries of Q are in the lower left quarter. Lemma 3.2.12 states that for $Q + \mathbb{1} \in \Gamma_{\text{pol}}^{\text{lev}}$ these entries need to be integers. In particular, $q_{2g,g} = s \cdot 1 \cdot 1 = s \in \mathbb{Z}$. Since now $s\,{}^t\mathfrak{c}_1\mathfrak{c}_1$ is obviously an integer matrix we obtain the equivalence

$$Q + \mathbb{1} \in \Gamma_{\text{pol}}^{\text{lev}} \iff s \in \mathbb{Z}.$$

For $Q + \mathbb{1} \in \Gamma_{\text{pol}}$ we consider Lemma 3.2.10 where for the lower left quarter we find the condition

$$-s\,{}^t\mathfrak{c}_1\mathfrak{c}_1 \in \Delta^{-1}\mathbb{D}(\Delta) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \dots & \mathbb{Z} \\ \mathbb{Z} & \frac{1}{d_1}\mathbb{Z} & \frac{1}{d_1}\mathbb{Z} & & \frac{1}{d_1}\mathbb{Z} \\ \mathbb{Z} & \frac{1}{d_1}\mathbb{Z} & \frac{1}{d_{1:2}}\mathbb{Z} & & \frac{1}{d_{1:2}}\mathbb{Z} \\ \vdots & & & \ddots & \vdots \\ \mathbb{Z} & \frac{1}{d_1}\mathbb{Z} & \frac{1}{d_{1:2}}\mathbb{Z} & \dots & \frac{1}{d_{1:g-1}}\mathbb{Z} \end{pmatrix}.$$

The condition on the top right matrix entry reads $q_{g+1,g} = sD_{1:g-1} \in \mathbb{Z}$ and hence we know that $s \in \frac{1}{D_{1:g-1}}\mathbb{Z}$ is a necessary condition. We claim that it is in fact also sufficient.

To show this, choose one matrix entry $q_{g+i,j}$ in the lower left quarter of Q. Since both $s\,{}^t\mathfrak{c}_1\mathfrak{c}_1$ and $\Delta^{-1}\mathbb{D}(\Delta)$ are symmetric, we may assume $i \le j$. The condition for this entry is

$$\begin{aligned} & s \cdot D_{i:g-1}a_i \cdot D_{j:g-1}a_j \in \tfrac{1}{d_{1:i-1}}\mathbb{Z} \\ \iff \quad & sd_{1:i-1}D_{i:g-1}D_{j:g-1}a_ia_j \in \mathbb{Z} \end{aligned} \tag{5.9}$$

On the other hand,

$$sd_{1:i-1}D_{i:g-1} \in \tfrac{1}{D_{1:g-1}}d_{1:i-1}D_{i:g-1}\mathbb{Z} = \frac{d_{1:i-1}}{D_{1:i-1}}\mathbb{Z} \subset \mathbb{Z}$$

and hence relation (5.9) is true. Therefore,

$$m_2(C_j^0) = [P'_{\Gamma_{\text{pol}}}(C_j^0) : P'_{\Gamma_{\text{pol}}^{\text{lev}}}(C_j^0)] = [\tfrac{1}{D_{1:g-1}}\mathbb{Z} : \mathbb{Z}] = D_{1:g-1}$$

which completes the proof. □

subsectionBranching of $\mathscr{A}_{\text{pol}}(n) \to \mathscr{A}_{\text{pol}}$

Lemma 5.2.5.
Assume $\gcd(n, d_{1:g-1}) = 1$. Then $\pi_3 : \mathscr{A}_{\text{pol}}(n) \to \mathscr{A}_{\text{pol}}$ is branched of order n on all corank-1 boundary components.

Proof.
Let D be a corank-1 boundary divisor. Denote the stabilisers of the corresponding isotropic line in the groups $\tilde{\Gamma}_{\text{pol}}$ and $\tilde{\Gamma}_{\text{pol}}(n)$ by $\text{Stab}_{\tilde{\Gamma}_{\text{pol}}}(D)$ and $\text{Stab}_{\tilde{\Gamma}_{\text{pol}}(n)}(D)$, respectively. Since D has corank 1, these stabilisers are one-dimensional lattices and can therefore be given by $\text{Stab}_{\tilde{\Gamma}_{\text{pol}}}(D) \simeq k_1\mathbb{Z}$ and $\text{Stab}_{\tilde{\Gamma}_{\text{pol}}(n)}(D) \simeq k_2\mathbb{Z}$. Since $\tilde{\Gamma}_{\text{pol}}(n) \subset \tilde{\Gamma}_{\text{pol}}$ by definition, we know that $k_1|k_2$. Since $\gcd(n, d_{1:g-1}) = 1$, the congruence condition imposed by $\tilde{\Gamma}_{\text{pol}}(n)$ implies that $k_2/k_1 = n$ for every such pair of lattices. But this index is exactly the order of branching, which proves the claim. □

5.2.2 The Galois group of $\mathscr{A}_{\text{pol}}^{\text{lev}} \to \mathscr{A}_{\text{pol}}$

In this section we give a result by Brasch [Bra93], only slightly adjusting the notation, and then give a more explicit description of the Galois group.

Definition 5.2.6: Symplectic group of $K(\Delta)$.
Let

$$K(\Delta) := (\mathbb{Z}^g/\Delta\mathbb{Z}^g)^2 \text{ with standard generators } f_1, \ldots, f_{2g}.$$

Define a (multiplicative) alternating form $e^\Delta : K(\Delta)^2 \to \mathbb{C}^*$ by

$$e^\Delta(f_\nu, f_\mu) := \begin{cases} \exp\left(+\frac{2\pi i}{e_\nu}\right) & \text{if } \mu = g+\nu \\ \exp\left(-\frac{2\pi i}{e_\mu}\right) & \text{if } \nu = g+\mu \\ 1 & \text{otherwise.} \end{cases}$$

Denote by $\text{Sp}(\Delta)$ the group of symplectic transformations of $K(\Delta)$.

Lemma 5.2.7: Galois group.
The embedding $\tilde{\Gamma}_{\text{pol}}^{\text{lev}} \hookrightarrow \tilde{\Gamma}_{\text{pol}}$ as normal subgroup induces a Galois covering $\mathscr{A}_{\text{pol}}^{\text{lev}} \to \mathscr{A}_{\text{pol}}$ of finite degree. The effective Galois group Γ_G of this Galois covering is isomorphic to

$$\Gamma_G \cong \begin{cases} \text{Sp}(\Delta) & \text{for } d_{1:g-1} = 1,2 \\ \text{Sp}(\Delta)/<\pm\mathbb{1}_{2g}> & \text{otherwise.} \end{cases}$$

Proof.
See [Bra93]. □

Lemma 5.2.8: Factorisation of the Galois group.
Define

$$S := \bigoplus_{n=1}^{g-1} \mathrm{Sp}\left(2(g-n), \mathbb{Z}_{d_n}\right).$$

For a coprime polarisation we have an isomorphism

$$\Gamma_G \cong \begin{cases} S & \text{for } d_{1:g-1} = 1,2 \\ S/ < \pm \mathbb{1}_{2g} > & \text{otherwise} \end{cases} .$$

Proof.
With Lemma 5.2.7 we know that we can determine Γ_G from S as claimed. This means that we need to calculate the group of automorphisms of

$$K(\Delta) = (\mathbb{Z} \times \cdots \times \mathbb{Z})^2 / (\mathbb{Z} \times d_1\mathbb{Z} \times \cdots \times d_{1:g-1}\mathbb{Z})^2$$

that leave the form e^Δ invariant, i. e. for $\varphi \in \Gamma_G$ we have $e^\Delta \circ (\varphi \times \varphi) = e^\Delta$. Since no confusion can occur we will simply write $e^\Delta \circ \varphi = e^\Delta$. Since the d_n are coprime, the Chinese remainder theorem tells us that

$$K(\Delta) \cong \mathbb{Z}_{d_1}^{2(g-1)} \times \cdots \times \mathbb{Z}_{d_{g-1}}^2 =: K.$$

For $n = 1, \ldots, g-1$ denote the standard generators of the factors

$$\mathbb{Z}_{d_{1:n}}^2 \subset K(\Delta) \quad \text{and} \quad (\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_n})^2 \subset K$$

by $\{f_1^n, f_2^n\}$ and $\{g_1^n, \ldots, g_{2n}^n\}$, respectively. With respect to these generators the alternating form e^Δ on $K(\Delta)$ is given by

$$e^\Delta(f_a^n, f_b^m) = \begin{cases} \exp(\frac{2\pi i}{d_{1:n}}) & \text{for } n = m \text{ and } a = 1, b = 2 \\ \exp(-\frac{2\pi i}{d_{1:n}}) & \text{for } n = m \text{ and } a = 2, b = 1 \\ 1 & \text{otherwise} \end{cases} .$$

On K, define the alternating form e' by

$$e'(g_a^n, g_b^m) := \begin{cases} \exp(\frac{2\pi i}{d_n}) & \text{for } n = m \text{ and } a = b - n \\ \exp(-\frac{2\pi i}{d_n}) & \text{for } n = m \text{ and } a = b + n \\ 1 & \text{otherwise} \end{cases} .$$

Obviously, $e' = \bigoplus e'_n$ where e'_n are the alternating forms on $\mathbb{Z}_{d_n}^{2(g-n)}$ that induce the standard symplectic structure. This shows that $\mathrm{Sp}(K) = \bigoplus \mathrm{Sp}(2(g-n), \mathbb{Z}_{d_n})$.

We claim that there is an isomorphism $\overline{\alpha} : \mathrm{Aut}(K(\Delta)) \to \mathrm{Aut}(K)$ with the property $\overline{\alpha}(\mathrm{Sp}(\Delta)) = \mathrm{Sp}(K)$. This is proved by explicitely giving an isomorphism $\alpha : K(\Delta) \to K$ which induces such an $\overline{\alpha}$:

Let $d_{1:n}^{(j)} := d_{1:j-1}d_{j+1:n}$ for $j = 1,\dots,n$. Since the d_j are pairwise coprime and hence $\gcd(d_{1:n}^{(j)})_{j=1,\dots,n} = 1$, there exist $\lambda_j^n \in \mathbb{Z}$ such that

$$\sum_{j=1}^{n} \lambda_j^n d_{1:n}^{(j)} = 1. \tag{5.10}$$

This also means that $\lambda_j^n d_{1:n}^{(j)} \equiv 1 \mod d_j$ for all j.

For the subspaces mentioned above define α_n by

$$\alpha_n : \begin{cases} \mathbb{Z}_{d_{1:n}}^2 \to (\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_n})^2 \\ (a_1,a_2) \mapsto (\lambda_1^n a_1,\dots,\lambda_n^n a_1, a_2,\dots,a_2) \end{cases}.$$

Then the inverse map is given by

$$\alpha_n^{-1} : \begin{cases} (\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_n})^2 \to \mathbb{Z}_{d_{1:n}}^2 \\ (b_1,\dots,b_{2n}) \mapsto (\sum_{j=1}^{n} d_{1:n}^{(j)} b_j, \sum_{j=1}^{n} \lambda_j d_{1:n}^{(j)} b_{n+j}) \end{cases}.$$

These maps are well defined homomorphisms and an easy calculation shows that they are indeed inverse maps, so $\alpha := \bigoplus \alpha_n$ is a bijection. Furthermore, this bijection is equivariant with respect to the forms defined above, i. e. we have $e' \circ \alpha = e^{\Delta}$. This can be seen because e^{Δ} and e' are both multiplicative and for the generators f_a^n we have

$$\begin{aligned}(e' \circ \alpha)(f_a^n, f_b^m) &= e'(\delta_{a,1} \sum_{k=1}^{n} \lambda_k^n g_k^n + \delta_{a,2} \sum_{k=n+1}^{2n} g_k^n, \delta_{b,1} \sum_{l=1}^{m} \lambda_l^m g_l^m + \delta_{b,2} \sum_{l=m+1}^{2m} g_l^m) \\ &= 1 = e^{\Delta}(f_1^n, f_2^m)\end{aligned}$$

for $n \neq m$ and any $a,b \in \{1,2\}$. This follows from $e'(g_k^n, g_l^m) = 1$ for all k,l. Here, $\delta_{x,y}$ is the Kronecker symbol, which is 1 if $x = y$ and 0 otherwise. For the remaining combinations of generators we obtain

$$\begin{aligned}(e' \circ \alpha)(f_1^n, f_1^n) &= e'(\sum_{k=1}^{n} \lambda_k^n g_k^n, \sum_{l=1}^{n} \lambda_l^n g_l^n) = 1 = e^{\Delta}(f_1^n, f_1^n), \\ (e' \circ \alpha)(f_2^n, f_2^n) &= e'(\sum_{k=n+1}^{2n} g_k^n, \sum_{l=n+1}^{2n} g_l^n) = 1 = e^{\Delta}(f_2^n, f_2^n) \quad \text{and} \\ (e' \circ \alpha)(f_1^n, f_2^n) &= e'(\lambda_1^n g_1^n + \cdots + \lambda_n^n g_n^n, g_{n+1}^n + \cdots + g_{2n}^n) \\ &= \exp(\lambda_1^n \tfrac{2\pi i}{d_1}) \cdot \ldots \cdot \exp(\lambda_n^n \tfrac{2\pi i}{d_n}) \\ &= \exp\big(\tfrac{2\pi i}{d_{1:n}}(\lambda_1^n d_{1:n}^{(1)} + \cdots + \lambda_n^n d_{1:n}^{(n)})\big) \quad \text{and with (5.10)} \\ &= \exp(\tfrac{2\pi i}{d_{1:n}}) = e^{\Delta}(f_1^n, f_2^n).\end{aligned}$$

By $\overline{\alpha}(\varphi) := \alpha \circ \varphi \circ \alpha^{-1}$ we now have an isomorphism between the groups $\mathrm{Aut}(K(\Delta))$ and $\mathrm{Aut}(K)$. Let $\varphi \in \mathrm{Sp}(\Delta)$ and $\psi := \overline{\alpha}(\varphi) = \alpha \circ \varphi \circ \alpha^{-1}$. Then

$$e' \circ \psi = e' \circ \alpha \circ \varphi \circ \alpha^{-1} = e^{\Delta} \circ \varphi \circ \alpha^{-1} = e^{\Delta} \circ \alpha^{-1} = e'$$

which means that $\psi \in \mathrm{Sp}(K)$, hence $\overline{\alpha}(\mathrm{Sp}(\Delta)) \subset \mathrm{Sp}(K)$. Analogously, for $\psi \in \mathrm{Sp}(K)$ and $\varphi := \overline{\alpha}^{-1}(\psi) = \alpha^{-1} \circ \psi \circ \alpha$ we obtain

$$e^{\Delta} \circ \varphi = e^{\Delta} \circ \alpha^{-1} \circ \psi \circ \alpha = e' \circ \psi \circ \alpha = e' \circ \alpha = e^{\Delta}$$

and hence $\varphi \in \mathrm{Sp}(\Delta)$, which proves equality. $\square$

Chapter 6

Putting it all together

6.1 Main Theorem

We shall now combine the facts collected so far to prove our main theorem.

Theorem 6.1.1: General type for general genus.
For any genus $3 \leq g \leq 9$ and coprime $d_1, \ldots, d_{g-1} \in \mathbb{N}$ with $d_{1:g-1} \neq 2$, the moduli space $\mathscr{A}_{\text{pol}}(n)$ of $(1, d_1, \ldots, d_{1:g-1})$-polarised Abelian varieties with a full level-n structure is of general type, provided $\gcd(n, d_{1:g-1}) = 1, n \geq 3$ and

$$n > \frac{(2^g+1)d_{2:g-2}}{(g+1)2^{g-3}} \min\left\{\frac{d_1}{C(\mathbb{L}(d_1,\ldots,d_{g-1}))}, \frac{d_{g-1}}{C(\mathbb{L}(d_{g-1},\ldots,d_1))}\right\}$$

where

$$C(\mathbb{L}(x_1,\ldots,x_{g-1})) = \min\left\{1, \min_{2\leq r\leq g}\left\{\frac{\sqrt{3}}{\sqrt[r]{\prod_{i=1}^{r-1} x_i^i}}\right\}\right\}.$$

Proof.
First of all, if $d_{1:g-1} = 1$ we are in the principally polarised case and much weaker bounds than the one given are already known. Hence, we may assume $d_{1:g-1} > 2$. Furthermore, we may assume the d_i to be square-free. Otherwise, we may write $d_i = s_i^2 e_i$ where the e_i are square-free. Then, according to Lemma 3.2.14, we can conjugate $\Gamma_{\text{pol},d}(n)$ such that it becomes a subgroup of $\Gamma_{\text{pol},e}(n)$. This means that we have a map $\pi_4 : (\mathscr{A}_{\text{pol},d}(n))^\sim \to (\mathscr{A}_{\text{pol},e}(n))^\sim$ and after some blowing-up this map becomes a morphism. By this morphism each form on $(\mathscr{A}_{\text{pol},e}(n))^\sim$ gives rise to a form on a suitable blow-up of $(\mathscr{A}_{\text{pol},d}(n))^\sim$ which implies that, if we can show general type for the (square-free) polarisation e, we also have general type for the polarisation d.

We consider the construction given in section 5.1. For $\mathscr{A}_g$, Theorem 1.2.35 tells us we have a cusp form χ of weight $w_\chi = (2^g+1)2^{g-2}$ that vanishes of order $v_\chi = 2^{2g-5}$ according to Theorem 1.2.36.

Since we have $n \geq 3$ we know that $\tilde{\Gamma}_{\text{pol}}(n)$ is neat[1] and hence operates without fixed points. This implies that the quotient by P'' introduces no singularities, and since $(\mathscr{A}_{\text{pol}})^\sim$ is stack-smooth we know that $(\mathscr{A}_{\text{pol}}(n))^\sim$ is smooth.

The map π_3 needs to be branched of order n. According to Lemma 5.2.5 this is implied by the condition $\gcd(n, d_{1:g-1}) = 1$. We can now calculate a bound for the

[1] See Remark 2.2.32

level n by the construction described in section 5.1, which gives

$$n > \frac{w_\chi m_2(i)|\mathscr{C}_{\text{pol}}^{\text{lev}}(i)|}{(g+1)v_\chi M_1(i)C(\mathbb{L})}. \tag{6.1}$$

Here we need that Proposition 3.3.8 holds and hence we can only prove the theorem for $g \leq 9$. Let us now calculate this value explicitely.

From Lemma 3.3.5 we know that the cusps of $\mathscr{A}_{\text{pol}}$ are given by vectors of the form

$$C_i = (D_{1:g-1}, D_{2:g-1}, \ldots, D_{g-1}, 1, 0, \ldots, 0).$$

Let us consider such a cusp and the set $\mathscr{C}_{\text{pol}}^{\text{lev}}(i)$ consisting of the primitive vectors of the form

$$C_i^j = (D_{1:g-1}, D_{2:g-1}a_2, \ldots, a_g, 0, D_{2:g-1}a_{g+2}, \ldots, a_{2g})$$

with $0 \leq a_k, a_{g+k} < D_{1:k-1}$ for $k = 2, \ldots, g$. From Lemma 5.2.3 and Lemma 5.2.4 we know that

$$m_1(i,j) = \gcd(D_{1:g-1}, D_{2:g-1}a_2, \ldots, a_g)^2 \quad \text{and} \quad m_2(i) = D_{1:g-1}.$$

Define $B_k | d_k$ for $k = 1, \ldots, g-1$ by $B_{1:k}^2 = m_1(i,j)$. This definition is unique because the d_k are coprime. We now have

$$B_{1:g-1} = \gcd(D_{1:g-1}, D_{2:g-1}a_2, \ldots, a_g). \tag{6.2}$$

We need to count these vectors, and we can do this using Lemma A.11. We let both the d_i and c_i of the lemma to be equal to D_i and the b_i of the lemma to be equal to B_i. Then we obtain that the number of $(g-1)$-tuples $(a_2, \ldots, a_g)$ satisfying equation (6.2) is $\prod_{j=1}^{g-1} \varphi_{g-j}(\frac{D_j}{B_j})$.

On the other hand, C_i^j is a primitive vector, so we have

$$\begin{aligned} 1 &= \gcd(D_{1:g-1}, D_{2:g-1}a_2, \ldots, a_g, 0, D_{2:g-1}a_{g+2}, \ldots, a_{2g}) \\ &= \gcd(B_{1:g-1}, D_{2:g-1}a_{g+2}, \ldots, a_{2g}) \end{aligned}$$

and Lemma A.11 (this time by letting also the c_i of the lemma to be equal to B_i) states that we have a choice of $\prod_{j=1}^{g-1} \varphi_{g-j}(B_j)\left(\frac{D_j}{B_j}\right)^{g-j}$ values for the $(g-1)$-tuple $(a_{g+2}, \ldots, a_{2g})$. So all in all

$$\begin{aligned} \left| \left\{ C_i^j \in \mathscr{C}_{\text{pol}}^{\text{lev}}(i) : m_1(i,j) = B_{1:g-1}^2 \right\} \right| &= \prod_{j=1}^{g-1} \varphi_{g-j}\left(\tfrac{D_j}{B_j}\right) \prod_{j=1}^{g-1} \varphi_{g-j}(B_j)\left(\tfrac{D_j}{B_j}\right)^{g-j} \\ &= \prod_{j=1}^{g-1} \varphi_{g-j}(D_j)\left(\tfrac{D_j}{B_j}\right)^{g-j} \end{aligned}$$

where we use the property that the d_k and hence the D_k are square-free, because this implies that in the cases considered the functions φ_{g-j} are multiplicative. Taking the unweighted and weighted sum over all $B_k|D_k$ we therefore get

$$
\begin{aligned}
|\mathscr{C}_{\text{pol}}^{\text{lev}}(i)| &= \sum_{B_1|D_1} \cdots \sum_{B_{g-1}|D_{g-1}} \prod_{j=1}^{g-1} \varphi_{g-j}(D_j)\big(\tfrac{D_j}{B_j}\big)^{g-j} \\
&= \Big(\prod_{j=1}^{g-1} \varphi_{g-j}(D_j)\Big) \sum_{B_1|D_1} \cdots \sum_{B_{g-1}|D_{g-1}} \prod_{j=1}^{g-1} \big(\tfrac{D_j}{B_j}\big)^{g-j} \\
&= \Big(\prod_{j=1}^{g-1} \varphi_{g-j}(D_j)\Big) \prod_{j=1}^{g-1} \sum_{B_j|D_j} \big(\tfrac{D_j}{B_j}\big)^{g-j} \\
&= \prod_{j=1}^{g-1} \Big(\varphi_{g-j}(D_j) \sum_{B'_j|D_j} (B'_j)^{g-j}\Big) \\
&= \prod_{j=1}^{g-1} \varphi_{g-j}(D_j)\sigma_{g-j}(D_j)
\end{aligned}
$$

and analogously

$$
\begin{aligned}
M_1(i) &= \sum_{B_1|D_1} \cdots \sum_{B_{g-1}|D_{g-1}} \prod_{j=1}^{g-1} \varphi_{g-j}(D_j)\big(\tfrac{D_j}{B_j}\big)^{g-j} \cdot B_{1:g-1}^2 \\
&= \prod_{j=1}^{g-1} \varphi_{g-j}(D_j) \prod_{j=1}^{g-1} \sum_{B_j|D_j} \frac{D_j^{g-j}}{B_j^{g-j-2}} \\
&= \prod_{j=1}^{g-1} \varphi_{g-j}(D_j)\Big[\Big(\prod_{j=1}^{g-2} \sum_{B_j|D_j} \big(\tfrac{D_j}{B_j}\big)^{g-j-2} D_j^2\Big) \sum_{B_{g-1}|D_{g-1}} D_{g-1}B_{g-1}\Big] \\
&= \prod_{j=1}^{g-1} \varphi_{g-j}(D_j)\Big[D_{1:g-2}^2 D_{g-1}\Big(\prod_{j=1}^{g-2} \sigma_{g-j-2}(D_j)\Big)\sigma_1(D_{g-1})\Big].
\end{aligned}
$$

Inserting this into condition (6.1) (using $m_2(i) = D_{1:g-1}$) the product of the φ_{g-j} cancels and we are left with

$$
\begin{aligned}
n &> \frac{w_\chi}{(g+1)v_\chi C(\mathbb{L})} \frac{D_{1:g-1} \cdot \prod_{j=1}^{g-1} \sigma_{g-j}(D_j)}{D_{1:g-2}^2 D_{g-1}\sigma_1(D_{g-1}) \prod_{j=1}^{g-2} \sigma_{g-j-2}(D_j)} \\
&= \frac{w_\chi}{(g+1)v_\chi C(\mathbb{L})} \frac{\prod_{j=1}^{g-1} \sigma_{g-j}(D_j)}{D_{1:g-2}\sigma_1(D_{g-1}) \prod_{j=1}^{g-2} \sigma_{g-j-2}(D_j)}
\end{aligned}
$$

and since $\sigma_{a+b}(D) = \sum_{B|D} B^{a+b} \le \sum_{B|D} B^a D^b = D^b \sigma_a(D)$ this is implied by

$$
\begin{aligned}
\Longleftarrow \quad n &> \frac{w_\chi}{(g+1)v_\chi C(\mathbb{L})} \frac{\prod_{j=1}^{g-2} \sigma_{g-j-2}(D_j) D_j^2 \cdot \sigma_{g-(g-1)}(D_{g-1})}{D_{1:g-2}\sigma_1(D_{g-1}) \prod_{j=1}^{g-2} \sigma_{g-j-2}(D_j)} \\
&= \frac{w_\chi}{(g+1)v_\chi C(\mathbb{L})} D_{1:g-2}.
\end{aligned}
$$

This condition has to hold true for all valid $D_k|d_k$ which obviously gives the condition

$$\begin{aligned} n &> \frac{w_\chi d_{1:g-2}}{(g+1)v_\chi C(\mathbb{L})} \\ &= \frac{(2^g+1)2^{g-2}d_{1:g-2}}{(g+1)2^{2g-5}C(\mathbb{L})} \\ &= \frac{(2^g+1)d_{1:g-2}}{(g+1)2^{g-3}C(\mathbb{L})} \\ &= \frac{(2^g+1)d_{1:g-2}}{(g+1)2^{g-3}\min\{\min_{2\le r\le g}\sqrt{3}\sqrt[r]{\prod_{i=1}^{r-1}d_i^i},1\}}. \end{aligned}$$

Finally, we may use the symmetry given in Lemma 2.2.35 to obtain the other term of the statement. □

To conclude this thesis, we give the bound for some special kinds of polarisations as corollaries:

Corollary 6.1.2.
For any genus $3 \le g \le 9$ and $d \in \mathbb{N}$, $d \ge 3$, the moduli space $\mathscr{A}_{\text{pol}}(n)$ of $(1,\dots,1,d)$-polarised Abelian varieties with a full level-n structure is of general type, provided $\gcd(n,d) = 1$, $n \ge 3$ and

$$n > \frac{2^g+1}{(g+1)2^{g-3}\sqrt{3}}\sqrt[g]{d^{g-1}}.$$

The same bound for the level applies for the moduli space of $(1,d,\dots,d)$-polarised abelian varieties with a full level-n structure.

If the polarisation is of type $(1,\dots,1,d,\dots,d)$ where $1 < i < g-1$ is the number of 1's, the bound is

$$n > \frac{2^g+1}{(g+1)2^{g-3}\sqrt{3}}d\min\{1,\sqrt[g]{d^{\min\{i,g-i\}}}\}.$$

Proof.
If the polarisation is of type $(1,\dots,1,d)$ we have $d_1 = \cdots = d_{g-2} = 1$ and $d_{g-1} = d$. Therefore,

$$\begin{aligned} C(\mathbb{L}(d_1,\dots,d_{g-1})) &= \min\{1,\min\{\sqrt{3},\dots,\sqrt{3},\tfrac{\sqrt{3}}{\sqrt[g]{d^{g-1}}}\}\} \\ &= \min\{1,\tfrac{\sqrt{3}}{\sqrt[g]{d^{g-1}}}\} \quad \text{and} \\ C(\mathbb{L}(d_{g-1},\dots,d_1)) &= \min\{1,\min\{\tfrac{\sqrt{3}}{\sqrt{d}},\dots,\tfrac{\sqrt{3}}{\sqrt[g]{d}}\}\} \\ &= \min\{1,\tfrac{\sqrt{3}}{\sqrt[g]{d}}\}. \end{aligned}$$

Hence, Theorem 6.1.1 gives the bound

$$\begin{aligned} n &> \frac{2^g+1}{(g+1)2^{g-3}}\min\left\{\frac{1}{\min\{1,\frac{\sqrt{3}}{\sqrt[g]{d^{g-1}}}\}},\frac{d}{\min\{1,\frac{\sqrt{3}}{\sqrt[g]{d}}\}}\right\} \\ &= \frac{2^g+1}{(g+1)2^{g-3}}\min\left\{\max\left\{1,\frac{\sqrt[g]{d^{g-1}}}{\sqrt{3}}\right\},\max\left\{d,\frac{\sqrt[g]{d^{g+1}}}{\sqrt{3}}\right\}\right\}. \end{aligned} \tag{6.3}$$

Let

$$M_1 := \max\{1, 3^{-1/2}\sqrt[g]{d^{g-1}}\} \quad \text{and} \quad M_2 := \max\{d, 3^{-1/2}\sqrt[g]{d^{g+1}}\}.$$

Since $g \geq 3$ we have $g < 2(g-1)$ which implies

$$3^{-1/2}\sqrt[g]{d^{g-1}} \leq 1 \quad \Longleftrightarrow \quad \sqrt[g]{d^{g-1}} \leq \sqrt{3} \quad \Longleftrightarrow \quad d \leq 3^{\frac{g}{2(g-1)}} < 3$$

and therefore $d \geq 3$ leads to $M_1 = 3^{-1/2}\sqrt[g]{d^{g-1}}$. On the other hand,

$$M_2 \geq 3^{-1/2}\sqrt[g]{d^{g+1}} \geq 3^{-1/2}\sqrt[g]{d^{g-1}} = M_1$$

and hence the minimum in equation (6.3) is always equal to M_1. This shows the first statement.

If the polarisation is of type $(1, d, \ldots, d)$ we have $d_1 = d$ and $d_2 = \cdots = d_{g-1} = 1$ and the same reasoning applies with the roles of M_1 and M_2 exchanged.

For the polarisation of type $(1, \ldots, 1, d, \ldots, d)$ we have $d_i = d$ and $d_j = 1$ for $j \neq i$. Therefore,

$$\begin{aligned} C(\mathbb{L}(d_1, \ldots, d_{g-1})) &= \min\{1, \min\{\sqrt{3}, \ldots, \sqrt{3}, \tfrac{\sqrt{3}}{\sqrt[i+1]{d^i}}, \ldots, \tfrac{\sqrt{3}}{\sqrt[g]{d^i}}\}\} \\ &= \min\{1, \tfrac{\sqrt{3}}{\sqrt[g]{d^i}}\} \quad \text{and similarly} \\ C(\mathbb{L}(d_{g-1}, \ldots, d_1)) &= \min\{1, \tfrac{\sqrt{3}}{\sqrt[g]{d^{g-i}}}\}. \end{aligned}$$

Hence, Theorem 6.1.1 gives the bound

$$\begin{aligned} n &> \frac{(2^g+1)d}{(g+1)2^{g-3}} \min\left\{ \frac{1}{\min\{1, \frac{\sqrt{3}}{\sqrt[g]{d^i}}\}}, \frac{1}{\min\{1, \frac{\sqrt{3}}{\sqrt[g]{d^{g-i}}}\}} \right\} \\ &= \frac{(2^g+1)d}{(g+1)2^{g-3}} \min\left\{ \max\{1, 3^{-1/2}\sqrt[g]{d^i}\}, \max\{1, 3^{-1/2}\sqrt[g]{d^{g-i}}\} \right\} \\ &= \frac{(2^g+1)d}{(g+1)2^{g-3}} \min\left\{1, 3^{-1/2}\sqrt[g]{d^{\min\{i, g-i\}}}\right\}. \end{aligned}$$

□

Remark 6.1.3.
To make this result more accessible, we give a table for the lower bounds for n in the case of polarisations of type $(1, \ldots, 1, d)$. Note that we have disregarded the condition $\gcd(d, n) = 1$ to make make the pattern more obvious.

$g \backslash d$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	3	4	4	5	5	6	6	7	7	7	8	8	8	9	9	9	10	10
4	3	3	4	4	5	5	6	6	6	7	7	8	8	8	9	9	9	10
5	3	3	3	4	4	5	5	6	6	6	7	7	7	8	8	9	9	9
6	3	3	3	3	4	4	5	5	5	6	6	7	7	7	8	8	8	9
7	3	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8
8	3	3	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8
9	3	3	3	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7

Corollary 6.1.4.
Let $s,t > 1$ be integers with $\gcd(s,t) = 1$. Then the moduli space $\mathscr{A}_{\mathrm{pol}}(n)$ of $(1,s,st)$-polarised Abelian varieties with a full level-n structure is of general type provided $\gcd(n,st) = 1$ and

$$n > \tfrac{3}{4}\sqrt{3}\sqrt[3]{s^2t^2\min\{s,t\}^2}.$$

Proof.
From Theorem 6.1.1 we obtain the bound

$$n > \frac{9}{4}\min\left\{\frac{s}{\min\{1,\frac{\sqrt{3}}{\sqrt{s}},\frac{\sqrt{3}}{\sqrt[3]{st^2}}\}},\frac{t}{\min\{1,\frac{\sqrt{3}}{\sqrt{t}},\frac{\sqrt{3}}{\sqrt[3]{s^2t}}\}}\right\}.$$

Due to the symmetry we may assume $s < t$ without loss of generality. Let M_1 be the denominator of the first fraction and M_2 the denominator of the second.

First, we consider M_1. Since $1 < s < t$ we have $s^2t^4 > 27$ which implies $\frac{\sqrt{3}}{\sqrt[3]{st^2}} < 1$. Furthermore, since $s < t$ implies $s < t^4$ we also have $\frac{\sqrt{3}}{\sqrt[3]{st^2}} < \frac{\sqrt{3}}{\sqrt{s}}$. Hence, $M_1 = \frac{\sqrt{3}}{\sqrt[3]{st^2}}$.

Now, consider M_2. Again, $1 < s < t$ implies $s^4t^2 > 27$ and hence $\frac{\sqrt{3}}{\sqrt[3]{s^2t}} < 1$. Contrary to M_1, we cannot exclude one of the other two possibilities, so we have

$$M_2 = \begin{cases} \frac{\sqrt{3}}{\sqrt{t}} & \text{if } s^4 \le t \\ \frac{\sqrt{3}}{\sqrt[3]{s^2t}} & \text{if } s^4 > t \end{cases}.$$

The bound for n can now be given by

$$n > \frac{9}{4}\min\left\{\frac{\sqrt[3]{s^4t^2}}{\sqrt{3}},\frac{t}{M_2}\right\}.$$

For $s^4 \le t$ we have $\frac{t}{M_2} = \frac{\sqrt{t^3}}{\sqrt{3}}$ which is greater than $\frac{\sqrt[3]{s^4t^2}}{\sqrt{3}}$ since otherwise we had

$$\sqrt{t^3} \le \sqrt[3]{s^4t^2} \iff t^9 \le s^8t^4 \iff t^5 \le s^8 \overset{s^4 \le t}{\Longrightarrow} t^5 \le s^8 \le t^2,$$

which is a contradiction to $t > 1$. So in this case the minimum is given by the first fraction.

For $s^4 > t$ we have $\frac{t}{M_2} = \frac{\sqrt[3]{s^2t^4}}{\sqrt{3}}$ which is obviously greater than the first fraction since $s < t$. Hence, in both cases we obtain the bound

$$n > \frac{9}{4}\frac{\sqrt[3]{s^4t^2}}{\sqrt{3}} = \frac{3}{4}\sqrt{3}\sqrt[3]{s^4t^2}.$$

Note that $1 < s < t$ implies that the minimal value of n satisfying this inequality is $n = 7$ so that we do not need to state the condition $n \ge 3$ seperately. □

Remark 6.1.5.
To give an impression of the case $g = 3$, we give the following table of minimal values for n for a fixed polarisation of type $(1,s,st)$ for arbitrary $s,t \in \mathbb{N}$. Where the level had to be increased to satisfy the condition $\gcd(n,st) = 1$ this is denoted by a pair

of brackets around the increased value. The empty spaces result from the conditions $st \neq 2$ and $\gcd(s,t) = 1$.

$s \backslash t$	1	2	3	4	5	6	7	8	9	10
1	3		(4)	(5)	4	5	5	(7)	(7)	7
2			7		(11)		(13)		(17)	
3	(4)	7		(17)	17		(22)	23		(29)
4	(5)		(17)		(27)		31		(37)	
5	4	(11)	17	(27)		37	41	(47)	49	
6	5				37		(53)			
7	5	(13)	(22)	31	41	(53)		(71)	76	81
8	(7)		23		(47)		(71)		(91)	
9	(7)	(17)		(37)	49		76	(91)		113
10	7		(29)				81		113	

Remark 6.1.6.
We need the constant condition $n \geq 3$ to know that $\Gamma(n)$ is neat and so we have no singularities coming from the group action. However, if we consider only principal polarisations, Y.-S. Tai showed in [Tai] that for $g \geq 5$ all singularities that occur on a suitable toroidal compactification are canonical. An argument by Salvetti Manni[2] shows that a similar reasoning can be applied to $g = 4$. Since this means that we can extend pluricanonical forms to a smooth model we may drop the condition $n \geq 3$ in this case. The same reasoning we employed for Theorem 6.1.1 now leads to the bound

$$n > \frac{2^g + 1}{(g+1)2^{g-3}}$$

which gives (including the known results for $g = 1,2$)

g	1	2	3	4	5	6	7	8	9
n	7	4	3	2	2	2	2	1	1

These are exactly the numbers given in [HS, p. 17], except for $g = 7$ where $n = 1$ is known to be sufficient. Note that for $g = 1,2$ the above formula remains true and even gives a sharp bound. Note also that this gives the known result that $\mathscr{A}_g$ is of general type for $g \geq 8$. This was originally proved by E. Freitag [Fre83], respectively D. Mumford [Mu3] and is better by 1 that the result by Y.-S. Tai [Tai].

[2] given in [HS, p. 19]

Appendix A

Technical lemmata

In the appendix we want to state the lemmata that were used in the text but not yet proved, since their nature is mainly technical.

Lemma A.1.
Let $g, d \in \mathbb{N}$ and $A = (a_{ij}) \in \mathbb{Z}^{g \times g}$. If there exists $k \in \{1, \ldots, g\}$ such that for all i, j satisfying $1 \leq j \leq k \leq i \leq g$ we have $d | a_{ij}$, then $d | \det(A)$.

Proof.
We prove this by induction over g. For $g = 1$ the claim is trivial.

Assume the claim is true for some g, i. e. we have

$$(\exists 1 \leq k' \leq g \; \forall 1 \leq j' \leq k' \leq i' \leq g \; : \; d | a'_{i'j'}) \implies d | \det(A'). \tag{A.1}$$

Let $A \in \mathbb{Z}^{(g+1) \times (g+1)}$ and $k \in \{1, \ldots, g+1\}$ such that it fulfils the condition

$$\forall 1 \leq j \leq k \leq i \leq g+1 \; : \; d | a_{ij}. \tag{A.2}$$

We want to calculate $\det(A)$ using the expansion by minors along the kth column. Denote the minors by $A^{(i,k)}$. If k is odd we have

$$\begin{aligned} \det(A) &= a_{1,k} |A^{(1,k)}| \mp \cdots + a_{k-1,k} |A^{(k-1,k)}| - a_{k,k} |A^{(k,k)}| \pm \cdots + a_{g+1,k} |A^{(g+1,k)}| \\ &\equiv a_{1,k} |A^{(1,k)}| \mp \cdots + a_{k-1,k} |A^{(k-1,k)}| \mod d \end{aligned}$$

since d divides the other $a_{i,k}$ by (A.2). Now, the $A^{(i,k)}$ are $g \times g$-matrices and by construction satisfy the condition of (A.1) for $k' = k - 1$. The assumption now provides $d | \det(A^{(i,k)})$ for all $i = 1, \ldots, k-1$ and hence $d | \det(A)$. For even k only the signs change but the same reasoning may be employed. □

Lemma A.2.
Let x_1, x_2 and $y_1, \ldots, y_i$ be integers with $x_1 \neq 0$ and $\gcd(x_1, x_2, y_1, \ldots, y_i) = d \in \mathbb{N}$. Then there exist integers $\alpha_1, \ldots, \alpha_i \in \mathbb{Z}$ such that $\gcd(x_1, x_2 + \alpha_1 y_1 + \cdots + \alpha_i y_i) = d$.

Proof.
If $y_1 = \cdots = y_i = 0$ there is nothing to prove. Assume this is not the case. Let $z := \gcd(y_1, \ldots, y_i)$. Since any multiple of z can be expressed as an integral linear combination of $y_1, \ldots, y_i$, is suffices to show that there exists an integer α with $\gcd(x_1, x_2 + \alpha z) = d$.

Let $s := \gcd(\frac{x_2}{d}, \frac{z}{d})$ and write $x_2 = dsm, z = dsn$. By definition of s we have $\gcd(m,n) = 1$. Let $t := \frac{x_1}{d}$. By assumption we have $\gcd(s,t) = 1$. To find an α with the property stated above, it therefore suffices to find α with $u := m + \alpha n$ satisfying $\gcd(t,u) = 1$, because we obtain

$$\begin{aligned} & 1 = \gcd(t,u) = \gcd(t, m+\alpha n) \\ \Longrightarrow\ & \gcd(t, sm + \alpha sn) = 1 \\ \Longrightarrow\ & d = \gcd(dt, dsm + \alpha dsn) \\ & \quad = \gcd(x_1, x_2 + \alpha z) \\ & \quad = \gcd(x_1, x_2 + \alpha(\lambda_1 y_1 + \cdots + \lambda_i y_i)) \end{aligned}$$

for suitable $\lambda_1, \ldots, \lambda_i$.

Now, let p_j be the primes dividing t for $j = 1, \ldots, k$. If p_j does not divide m set $a_j = 0$, while if p_j divides m set $a_j = 1$, so that in any case p_j does not divide $m + a_j n$. Now apply the Chinese remainder theorem to obtain α satisfying the congruences $\alpha \equiv a_j (\text{mod } p_j)$ for $j = 1, \ldots, k$. □

Lemma A.3.
Assume we are given a lattice $h \subset \mathbb{Z}^N$ with a basis $v_1, \ldots, v_n$ and a vector $u \in h$ which is either

- known to be primitive with respect to h or
- given as a linear combination $u = \sum_{i=1}^n \lambda_i v_i$ with $\gcd(\lambda_1, \ldots, \lambda_n) = 1$.

Then we can find a basis $u_1, \ldots, u_n$ of h such that $u_1 = u$.

Proof.
Since $u \in h$ we can always write $u = \sum_{i=1}^n \lambda_i v_i$. If u is primitive this also implies $\gcd(\lambda_1, \ldots, \lambda_n) = 1$, so we may assume this condition in both cases. Hence, we can find a unimodular integer matrix $A = (a_{ij})$ with first row vector $(a_{11}, \ldots, a_{1n}) = (\lambda_1, \ldots, \lambda_n)$. This matrix can act as basis transformation matrix

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} := A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Due to the construction of the first row of A we have $u_1 = u$. Obviously, $\bigoplus u_i\mathbb{Z} \subset \bigoplus v_i\mathbb{Z}$. Since A is unimodular, A^{-1} is again an integer matrix, and thus $\bigoplus v_i\mathbb{Z} \subset \bigoplus u_i\mathbb{Z}$. This shows that $u_1, \ldots, u_n$ is indeed a basis of h. □

Lemma A.4.
Assume we are given a coprime polarisation, vectors $v^1, \ldots, v^n \in \mathbb{Z}^{2g}$ and a unimodular integer matrix A. Consider the basis transformation

$$\begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} := A \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}.$$

Then

$$\gcd\big(D_k(u^1),\dots,D_k(u^n)\big) = \gcd\big(D_k(v^1),\dots,D_k(v^n)\big)$$

for any $1 \le k \le g-1$.

Proof.
Assume the notation $A = (a_{il})$. The jth entry of the ith vector is given by

$$u^i_j = \sum_{l=1}^n a_{il} v^l_j.$$

Recall that by definition

$$D_k(u^i) = \gcd\big(d_k, \tfrac{u^i_{j|g+j}}{D_{j:k-1}(u^i)}\big)_{j=1,\dots,k}$$

and since the polarisation is coprime and $D_r | d_r$ we have $\gcd(d_k, D_r(u^i)) = 1$ for $r \ne k$ which implies

$$D_k(u^i) = \gcd(d_k, u^i_{j|g+j})_{j=1,\dots,k}$$

and so

$$\begin{aligned}\gcd\big(D_k(v^1),\dots,D_k(v^n)\big) &= \gcd(d_k, v^1_{j|g+j},\dots,v^n_{j|g+j})_{j=1,\dots,k}\\ &= \gcd(d_k, v^i_j, v^i_{g+j})_{i=1,\dots,n;\, j=1,\dots,k}\end{aligned}$$

divides

$$\begin{aligned}\gcd(d_k, \sum_{l=1}^n a_{il} v^l_j, \sum_{l=1}^n a_{il} v^l_{g+j})_{i=1,\dots,n;\, j=1,\dots,k} &= \gcd(d_k, u^i_j, u^i_{g+j})_{i=1,\dots,n;\, j=1,\dots,k}\\ &= \gcd(d_k, u^1_{j|g+j},\dots,u^n_{j|g+j})_{j=1,\dots,k}\\ &= \gcd\big(D_k(u^1),\dots,D_k(u^n)\big).\end{aligned}$$

Since A^{-1} is also a unimodular integer matrix we also obtain divisibility in the other direction, and since both numbers are positive integers this implies equality. □

Lemma A.5.
Assume $a, b, d \in \mathbb{Z}$ given. Let $\Delta = \operatorname{diag}(1, d)$. Then there exists a matrix $G \in \mathbb{SD}(\Delta)$ such that $(a, b)G = (u, v)$ with $v = \gcd(a, b)$.

Proof.
Denote $x := \gcd(a, b)$. Then there exist integers $\alpha, \beta \in \mathbb{Z}$ such that

$$\begin{aligned}\alpha a + \beta b = x &\implies \alpha \tfrac{a}{x} + \beta \tfrac{b}{x} = 1\\ &\implies \gcd(\tfrac{a}{x}, \beta) = 1.\end{aligned} \tag{A.3}$$

Furthermore, we have for all $t \in \mathbb{Z}$:

$$\begin{aligned}(\alpha + t\tfrac{b}{x})a + (\beta - t\tfrac{a}{x})b = x &\implies (\alpha + t\tfrac{b}{x})\tfrac{a}{x} + (\beta - t\tfrac{a}{x})\tfrac{b}{x} = 1\\ &\implies \gcd(\alpha + t\tfrac{b}{x}, \beta - t\tfrac{a}{x}) = 1.\end{aligned} \tag{A.4}$$

Now chose

$$t = \prod_{\substack{p \text{ prime} \\ p|d, p\nmid\beta}} p.$$

Then we claim

$$\gcd(\beta - t\tfrac{a}{x}, d) = 1. \tag{A.5}$$

This can be seen as follows: Let p be a prime dividing d. Then either $p|\beta$ which implies both $\gcd(p, \frac{a}{x}) = 1$ because of equation (A.3) and $\gcd(p,t) = 1$ by choice of t; this in turn implies $\gcd(\beta - t\frac{a}{x}, p) = 1$. Or we have $p \nmid \beta$. Then $p|t$ by choice of t and hence again $\gcd(\beta - t\frac{a}{x}, p) = 1$. Equation (A.5) follows.

Combining equations (A.5) and (A.4) we obtain $\gcd\left(\beta - t\frac{a}{x}, d(\alpha + t\frac{b}{x})\right) = 1$ and hence there exist integers $\lambda, \mu \in \mathbb{Z}$ with

$$\lambda(\beta - t\tfrac{a}{x}) - \mu d(\alpha + t\tfrac{b}{x}) = 1.$$

Therefore, the matrix

$$G = \begin{pmatrix} \lambda & \alpha + t\frac{b}{x} \\ d\mu & \beta - t\frac{a}{x} \end{pmatrix}$$

satisfies the properties claimed. □

Corollary A.6.
Assume $g \geq 2$, let $(1, d_1, \ldots, d_{1:g-1})$ be any polarisation and $\Delta = \operatorname{diag}(1, d_1, \ldots, d_{1:g-1})$. For any $v = (v_1, \ldots, v_g) \in \mathbb{Z}^g$ we can find a matrix $G \in \mathbb{SD}(\Delta)$ such that $u := vG$ satisfies $u_g = \gcd(v_1, \ldots, v_g)$.

Proof.
The proof is by induction and shows that G can be chosen to be of the form

$$G = \begin{pmatrix} \beta_1 & 0 & \ldots & 0 & \alpha_1 \\ 0 & \beta_2 & & 0 & \alpha_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \beta_{g-1} & \alpha_{g-1} \\ d_{1:g-1}\gamma_1 & d_{2:g-1}\gamma_2 & \ldots & d_{g-1}\gamma_{g-1} & \alpha_g \end{pmatrix}. \tag{A.6}$$

For $g = 2$ this is exactly the statement of Lemma A.5. For the induction, fix any $g \geq 2$ and assume we can find G_g of the form (A.6) satisfying

$$\det G_g = 1 \quad \text{and} \quad \sum_{i=1}^{g} \alpha_i v_i = \gcd(v_1, \ldots, v_g). \tag{A.7}$$

Now let the polarisation for $g+1$ be given by $(1, d_0, d_{0:1}, \ldots, d_{0:g-1})$ and $v = (v_0, v_1, \ldots, v_g)$.

We use Lemma A.5 with $a = v_0, b = \gcd(v_1, \ldots, v_g)$ and $d = d_{0:g-1} \prod_{i=1}^{g-1} \beta_i$ to obtain a matrix $G' = \left(\begin{smallmatrix} \mu_0 & \lambda_0 \\ d\mu_1 & \lambda_1 \end{smallmatrix}\right)$ satisfying

$$\det G' = 1 \quad \text{and} \quad \lambda_0 v_0 + \lambda_1 \gcd(v_1, \ldots, v_g) = \gcd(v_0, \ldots, v_g). \tag{A.8}$$

Now define the matrix G_{g+1} to be

$$G_{g+1} := \begin{pmatrix} \mu_0 & 0 & 0 & \dots & 0 & \lambda_0 \\ 0 & \beta_1 & 0 & & 0 & \lambda_1\alpha_1 \\ 0 & 0 & \beta_2 & & 0 & \lambda_1\alpha_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & \beta_{g-1} & \lambda_1\alpha_{g-1} \\ d_{0:g-1}\mu_1 & d_{1:g-1}\gamma_1 & \dots & d_{g-2:g-1}\gamma_{g-2} & d_{g-1}\gamma_{g-1} & \lambda_1\alpha_g \end{pmatrix}.$$

Then we have for $u := vG_{g+1}$ with $u = (u_0, \dots, u_g)$

$$\begin{aligned} u_g &= \lambda_0 v_0 + \lambda_1\alpha_1 v_1 + \dots + \lambda_1\alpha_g v_g \\ &= \lambda_0 v_0 + \lambda_1 \sum_{i=1}^{g} \alpha_i v_i \\ &\overset{(A.7)}{=} \lambda_0 v_0 + \lambda_1 \gcd(v_1, \dots, v_g) \\ &\overset{(A.8)}{=} \gcd(v_0, \dots, v_g). \end{aligned}$$

Furthermore, by developing along the first column

$$\begin{aligned} \det G_{g+1} = \mu_0 \det &\begin{pmatrix} \beta_1 & \dots & \lambda_1\alpha_1 \\ \vdots & \ddots & \vdots \\ d_{1:g-1}\gamma_1 & \dots & \lambda_1\alpha_g \end{pmatrix} + \\ &+ (-1)^g d_{0:g-1}\mu_1 \det \begin{pmatrix} 0 & \dots & 0 & \lambda_0 \\ \beta_1 & \ddots & \vdots & \lambda_1\alpha_1 \\ 0 & \ddots & 0 & \vdots \\ 0 & \dots & \beta_{g-1} & \lambda_1\alpha_{g-1} \end{pmatrix}. \end{aligned}$$

Apart from the factor λ_1 which we can take out of the last column, the first matrix is exactly G_g. The second determinant can be developed along the first row to

$$\begin{aligned} \det G_{g+1} &= \mu_0\lambda_1 \det G_g + (-1)^g d_{0:g-1}\mu_1 \cdot (-1)^{g-1}\lambda_0 \det\big(\operatorname{diag}(\beta_1, \dots, \beta_{g-1})\big) \\ &\overset{(A.7)}{=} \mu_0\lambda_1 \cdot 1 - \mu_1\lambda_0 (d_{0:g-1} \prod_{i=1}^{g-1} \beta_i) \quad \text{and by definition of } d \\ &= \mu_0\lambda_1 - d\mu_1\lambda_0 \\ &= \det G' \overset{(A.8)}{=} 1 \end{aligned}$$

and hence G_{g+1} is as claimed. □

Corollary A.7.
Assume $g \geq 2$ with any polarisation and $v = (v_1, \dots, v_g, 0, \dots, 0) \in \mathbb{Z}^{2g}$. Then there exists a matrix $M \in \tilde{\Gamma}_{\text{pol}}$ such that for $u = (u_1, \dots, u_{2g}) := vM \in \mathbb{Z}^{2g}$ we have $u_g = \gcd(v_1, \dots, v_g)$. Furthermore, M can be chosen such that it is an automorphism of the sublattices $\mathbb{Z}^g \times \{0\}^g \subset \mathbb{Z}^{2g}$ and $\{0\}^g \times \mathbb{Z}^g \subset \mathbb{Z}^{2g}$.

If we choose a set of indices $1 \le i_1 < \cdots < i_n \le g$ then there exists $M \in \tilde{\Gamma}_{\mathrm{pol}}$ such that $u_{i_n} = \gcd(v_{i_1}, \dots, v_{i_n})$ and M is an automorphism of the sublattices $\bigoplus_j e_{i_j}\mathbb{Z}$ and $\bigoplus_j e_{g+i_j}\mathbb{Z}$ where e_{i_j} is the i_jth unit vector.

Proof.

Use Corollary A.6 to obtain a matrix $G \in \mathbb{SD}(\Delta)$ satisfying $u'_g = \gcd(v_1, \dots, v_g)$ for $u' := (v_1, \dots, v_g)G$. Since $\mathbb{SD}(\Delta)$ is a multiplicative group, $G^{-1} \in \mathbb{SD}(\Delta)$. Now, $M = \left(\begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix}\right)$ satisfies the properties claimed.

The proof goes through the same if we restrict everything to the sublattice $\bigoplus_j (e_{i_j}\mathbb{Z} \oplus e_{g+i_j}\mathbb{Z})$. □

Lemma A.8.

Let $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$. Then we can find an integer matrix T of the form

$$T = \begin{pmatrix} * & \bullet & \bullet & \dots & \bullet & v_1 \\ * & * & \bullet & & \bullet & v_2 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ * & & \dots & * & \bullet & v_{n-2} \\ * & & \dots & & * & v_{n-1} \\ * & & \dots & & * & v_n \end{pmatrix}$$

(where the $\bullet$ are arbitrary fixed integer values) such that $\det(T) = \gcd(v_1, \dots, v_n)$.

Proof.

We prove the claim by induction.

For $n = 2$ we have the matrix $T = \left(\begin{smallmatrix} t_{11} & v_1 \\ t_{21} & v_2 \end{smallmatrix}\right)$. We can choose t_{11}, t_{21} such that

$$\det(T) = t_{11}v_2 - t_{21}v_1 = \gcd(v_1, v_2)$$

which completes this case.

Let $n \in \mathbb{N}$ be arbitrary and assume the claim holds for $n-1$. Let $T^{(i)}$ and $T^{(i,j)}$ denote the submatrices of T that consist of the columns 2 to $n-1$ with the ith or ith and jth rows removed. Expansion of the determinant along the 1st column shows that

$$\det(T) = t_{11} \begin{vmatrix} & v_2 \\ T^{(1)} & \vdots \\ & v_n \end{vmatrix} - t_{21} \begin{vmatrix} & v_1 \\ & v_3 \\ T^{(2)} & \vdots \\ & v_n \end{vmatrix} \pm \cdots - (-1)^n t_{n1} \begin{vmatrix} & v_1 \\ T^{(n)} & \vdots \\ & v_{n-1} \end{vmatrix}.$$

In particular, $t_{11}, \dots, t_{n1}$ can be chosen such that the claim holds if

$$F := \gcd\left(\begin{vmatrix} & v_2 \\ T^{(1)} & \vdots \\ & v_n \end{vmatrix}, \begin{vmatrix} & v_1 \\ & v_3 \\ T^{(2)} & \vdots \\ & v_n \end{vmatrix}, \dots, \begin{vmatrix} & v_1 \\ T^{(n)} & \vdots \\ & v_{n-1} \end{vmatrix} \right) \stackrel{?}{=} \gcd(v_1, \dots, v_n). \tag{A.9}$$

We now prove this equality. The first matrix is a $n-1 \times n-1$ matrix of the special form needed for the induction. We may therefore assume that

$$\begin{vmatrix} & v_2 \\ T^{(1)} & \vdots \\ & v_n \end{vmatrix} = \gcd(v_2, \dots, v_n) =: f. \tag{A.10}$$

Furthermore, expansion of this determinant along the last column gives

$$
\begin{aligned}
& v_2|T^{(1,2)}| \mp \cdots + (-1)^n v_n |T^{(1,n)}| = -(-1)^n f \\
\iff & \tfrac{v_2}{f}|T^{(1,2)}| \mp \cdots + (-1)^n \tfrac{v_n}{f} |T^{(1,n)}| = -(-1)^n 1 \\
\Longrightarrow & \gcd\left(|T^{(1,2)}|, \ldots, |T^{(1,n)}|\right) = 1.
\end{aligned} \tag{A.11}
$$

Now we can simplify F by using expansion of the determinants along the last columns. Almost all terms of these expansions are multiples of f because they contain one of $v_2, \ldots, v_n$. Since according to (A.10) the first term in the gcd of (A.9) is equal to f these terms are not needed to determine the value of F. Hence, we are left with

$$
\begin{aligned}
F &= \gcd\left(f, v_1|T^{(1,2)}|, \ldots, v_1|T^{(1,n)}|\right) \\
&= \gcd\left(f, v_1 \gcd\left(|T^{(1,2)}|, \ldots, |T^{(1,n)}|\right)\right)
\end{aligned}
$$

and because of equation (A.11) this gives

$$
= \gcd(f, v_1) = \gcd(v_1, \ldots, v_n).
$$

So, the equation in (A.9) is true and hence we can find T as claimed. □

Lemma A.9.
Let $a_1, a_2, b_1, b_2, c \in \mathbb{Z}$. Then

$$
a_1^2 | b_1^2 c \text{ and } a_2^2 | b_2^2 c \quad \Longrightarrow \quad a_i | b_i c \text{ for } i = 1,2 \quad \text{and } a_1 a_2 | b_1 b_2 c.
$$

Proof.
Let p be any prime number and define $e(p,n) = m :\iff p^m | n$ and $p^{m+1} \nmid n$.

Then we have

$$
\begin{aligned}
a_i^2 | b_i^2 c &\iff \forall p \,:\, 2e(p,a_i) \le 2e(p,b_i) + e(p,c) \\
&\iff \forall p \,:\, e(p,a_i) \le e(p,b_i) + \tfrac{1}{2} e(p,c) \\
&\Longrightarrow \forall p \,:\, e(p,a_i) \le e(p,b_i) + e(p,c) \\
&\Longrightarrow a_i | b_i c
\end{aligned} \tag{A.12}
$$

and by adding inequality (A.12) for $i = 1,2$ we obtain

$$
\forall p \,:\, e(p,a_1) + e(p,a_2) \le e(p,b_1) + e(p,b_2) + \tfrac{2}{2} e(p,c) \iff a_1 a_2 | b_1 b_2 c
$$

as claimed. □

Lemma A.10.
Let $n \in \mathbb{N}$ and $a, b_1, \ldots, b_n \in \mathbb{Z}$. Then

$$
\operatorname{lcm}\left[\frac{a}{\gcd(a,b_1)}, \ldots, \frac{a}{\gcd(a,b_n)}\right] = \frac{a}{\gcd(a,b_1,\ldots,b_n)}.
$$

Proof.
We prove this by induction over n. For $n = 1$ the claim is trivial. Let $n = 2$. Then we have

$$\text{lcm}\left[\frac{a}{\gcd(a,b_1)}, \frac{a}{\gcd(a,b_2)}\right] = \frac{\frac{a^2}{\gcd(a,b_1)\gcd(a,b_2)}}{\gcd\left(\frac{a}{\gcd(a,b_1)}, \frac{a}{\gcd(a,b_2)}\right)} = \frac{a}{\gcd(\gcd(a,b_2),\gcd(a,b_1))} = \frac{a}{\gcd(a,b_1,b_2)} \tag{A.13}$$

which proves this case. Assume the claim holds for some $n \in \mathbb{N}$. Then we obtain

$$\begin{aligned}
&\text{lcm}\left[\frac{a}{\gcd(a,b_1)}, \dots, \frac{a}{\gcd(a,b_{n+1})}\right] \\
&= \text{lcm}\left[\text{lcm}\left[\frac{a}{\gcd(a,b_1)}, \dots, \frac{a}{\gcd(a,b_n)}\right], \frac{a}{\gcd(a,b_{n+1})}\right] \quad \text{by assumption} \\
&= \text{lcm}\left[\frac{a}{\gcd(a,b_1,\dots,b_n)}, \frac{a}{\gcd(a,b_{n+1})}\right] \\
&= \text{lcm}\left[\frac{a}{\gcd(a,\gcd(b_1,\dots,b_n))}, \frac{a}{\gcd(a,b_{n+1})}\right] \quad \text{now, with equation (A.13)} \\
&= \frac{a}{\gcd(a,\gcd(b_1,\dots,b_n),b_{n+1})} \\
&= \frac{a}{\gcd(a,b_1,\dots,b_{n+1})}
\end{aligned}$$

as claimed. □

Lemma A.11.
Let $k \in \mathbb{N}$ and let $d_1, \dots, d_k \in \mathbb{N}$ be coprime integers. Chose integers $c_1, \dots, c_k$ and $b_1, \dots, b_k$ satisfying $b_i | c_i | d_i$ for all $i = 1, \dots, k$. Then

$$\left| \left\{ (x_1, \dots, x_k) \middle| 0 \le x_i < d_{1:i}, \gcd(x_i c_{i+1:k})_{i=0,\dots,k} = b_{1:k} \right\} \right| = \prod_{i=1}^{k} \varphi_{k+1-i}\left(\frac{c_i}{b_i}\right)\left(\frac{d_i}{c_i}\right)^{k+1-i}$$

where we let $x_0 = 1$ to ease the notation of the gcd.

Proof.
Define $d_{i:k}^{(j)} := d_{i:j-1} d_{j+1:k}$. Since the d_i are coprime we can rewrite the x_i as

$$x_i \equiv \sum_{j=1}^{i} y_{i,j} d_{1:i}^{(j)} \mod d_{1:i} \quad \text{for } i = 2, \dots, k$$

where we may chose $0 \le y_{i,j} < d_j$ (and let $y_{1,1} := x_1$). This, according to the Chinese Remainder Theorem, makes the $y_{i,j}$ unique. Now the condition we have to consider is

$$\begin{aligned}
b_{1:k} &= \gcd\left(c_{1:k}, c_{2:k} y_{1,1}, c_{3:k} \sum_{j=1}^{2} y_{2,j} d_{1:2}^{(j)}, \dots, c_k \sum_{j=1}^{k-1} y_{k-1,j} d_{1:k-1}^{(j)}, \sum_{j=1}^{k} y_{k,j} d_{1:k}^{(j)}\right) \\
&= \gcd\left(c_{1:k}, c_{2:k} y_{1,1}, \sum_{j=1}^{2} c_{1:k}^{(j)} y_{2,j}, \dots, \sum_{j=1}^{k} c_{1:k}^{(j)} y_{k,j}\right)
\end{aligned}$$

since $c_i|d_i$ and the d_i are coprime. Furthermore, since $b_i|c_i$ and the c_i are coprime we obtain that $b_j|y_{i,j}$ for all $1 \leq j \leq i$. Now the above condition is equivalent to

$$\iff \quad 1 = \gcd\left(\tfrac{c_{1:k}}{b_{1:k}}, \tfrac{c_{2:k}}{b_{2:k}}\tfrac{y_{1,1}}{b_1}, \sum_{j=1}^{2} \tfrac{c_{1:k}^{(j)}}{b_{1:k}^{(j)}}\tfrac{y_{2,j}}{b_j}, \ldots, \sum_{j=1}^{k} \tfrac{c_{1:k}^{(j)}}{b_{1:k}^{(j)}}\tfrac{y_{k,j}}{b_j}\right).$$

Now let $\tilde{y}_{i,j} := y_{i,j}/b_j$. Then the equality above is equivalent to

$$\iff \quad 1 = \gcd\left(\tfrac{c_1}{b_1}, \tilde{y}_{1,1}, \ldots, \tilde{y}_{k,1}\right) \cdot \ldots \cdot \gcd\left(\tfrac{c_k}{b_k}, \tilde{y}_{k,k}\right)$$

$$\iff \quad 1 = \gcd\left(\tfrac{c_j}{b_j}, \tilde{y}_{j,j}, \ldots, \tilde{y}_{k,j}\right) \quad \forall j = 1, \ldots, k.$$

Since we have chosen $0 \leq y_{i,j} < d_j$ we know $0 \leq \tilde{y}_{i,j} < \frac{d_j}{b_j}$. In the restricted range $0 \leq \tilde{y}_{i,j} < \frac{c_j}{b_j}$ the number of possible $(k-j+1)$-tuples $(\tilde{y}_{j,j}, \ldots, \tilde{y}_{k,j})$ satisfying the conditions is given by $\varphi_{k-j+1}(\frac{c_j}{b_j})$. Since $c_j|d_j$ we have exactly $(\frac{d_j}{c_j})^{k-j+1}$ copies of this range. This gives the value claimed. □

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